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Designing a curriculum based on four purposes: let mathematics speak for itself

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Trends in curriculum reform recognise the need to develop skills and competencies in addition to specifying what knowledge should be taught and when. However, a balance between skills and knowledge is sometimes difficult to achieve. In this paper which takes mathematics as the focus, we consider reform currently underway in Wales, from the perspective of a 'knowledge approach' and from the perspective of the *Successful Futures* report which is, we argue, driven more by a skills approach at the heart of which are 'four purposes': developing young people as *ambitious, capable learners; enterprising, creative contributors; ethical, informed citizens;* and *healthy, confident individuals*. Our interest is in the contribution that mathematics makes to the four purposes; and contribution that the four purposes make (or do not make) to the development of a school mathematics curriculum. After outlining the background and context, the paper consults literature and experts to consider what mathematics is and how the learning of mathematics can be seen to fulfil the four purposes. The study contributes to understanding the difficulties of re-contextualising school subjects from the academic disciplines and proposes that operating with a curriculum driven by big ideas or overarching statements places higher demands on teacher knowledge.

Keywords: big ideas; mathematics; skills; knowledge; curriculum

Subject classification codes: include these here if the journal requires them

Introduction: reforming school curricula

School curricula get reformed for two main, related, reasons. The first is that students are not performing well enough according to some criteria. In terms of mathematics, these criteria have for many years included international tests such as the 1985 Second International Mathematics Study (see, e.g. Garden, 1987), and more recently the TIMSS and PISA studies. The second is that students leave school inadequately prepared to contribute to the economic prosperity of the nation. For example, their skills do not meet the expectations of employers or higher education (ACME, 2011; North & Christiansen, 2015; Whitehouse & Burdett, 2013) or they are not attracted to higher study, or work in, disciplines that are seen to be necessary for the growth of the country (such as mathematics) (Vorderman, Porkess, Budd, Dunne, & Rahman-hart, 2011).

Although the causes of the problems outlined above could, in different contexts, include poor teaching, poor student attitudes (Royal Society, 2011), societal undervaluing of education (Jerrim & Choi, 2013), high-stakes assessment regimes and lack of resources, very often the curriculum is cited as a cause. The argument appears to be that the curriculum, in other words what is taught, is somehow deficient and not fit for purpose (Schoenfeld, 1994b). In different contexts, there are some who claim the curriculum is outdated, overcrowded or fragmented (Bethell, 2016; Wilson-Thompson, 2008), and others who claim it is colonial and unfit for post-colonial contexts (Semali, Hristova, & Owiny, 2015), irrelevant to children (Tikly et al., 2016), dumbed-down (Whitehouse & Burdett, 2013) or elitist (Tikly et al., 2016; Vorderman et al., 2011). In some contexts, the curriculum is seen as over-prescriptive thus limiting the autonomy of teachers and in others it is seen as too vague. There is also a view that the curriculum could be too broad at the expense of depth (see, e.g. Hoyles, Newman, & Noss, 2001).

Whatever the deficiencies of the curriculum, however, generally reforming it could be seen as a 'quick-fix'. This is not to say that curriculum reform is a quick or easy process, but perhaps it is quicker and easier, and more feasible, than reforming many of the other causes of the perceived problems.

The current educational reform in Wales can be seen as an example. In response to the perceived relatively poor performance of schools, and concerns about the performance of the education system, as identified by the OECD (2014), Wales began reforming aspects of their education system, and the school curriculum in particular

(Donaldson, 2015). The question at the heart of this paper is the extent to which the guidance included in the reform could help expert groups (teachers, government advisors, consultants) to determine the content of the curriculum and pedagogic approaches they are designing. To do this one subject area, mathematics, is taken in focus.

Before discussing the arrangements of the reform and exploring how the learning of specific disciplines contributes to the aims or purposes of the curriculum, we briefly review current trends in curriculum development.

Trends in curriculum development: skills and knowledge

Many of the new curricula being developed around the world, such as for example in New Zealand, Australia, Canada, USA, Macedonia, Croatia, Zambia and Scotland, claim to be driven by an increased focus on citizenship and employability (see, e.g. Yates & Young, 2010) and tend to have moved from explicitly specifying content towards skills-based approaches (Priestley & Sinnema, 2014; Sinnema, 2017; Young, 2013). These curricula can sometimes be interpreted as 'learning to' rather than 'learning about' in a given subject area (Davies, 2016) and they tend to focus on what learners are expected to be or be able to do, rather than what they need to know (Watson, 2010). For example, the Scottish curriculum states that it aims to foster four capacities in young people: successful learners, confident individuals, effective contributors and responsible citizens (Biesta, 2008).

The move towards emphasising skills, or 'learning to', rather than knowledge or 'learning about' is one of several trends in curriculum reform. Others include: moving away from subject areas towards areas of experience or 'learning areas' such as, for example, creative and performing arts comprising dance, music, visual arts, media arts; making the lived experience of students more explicit (Koyama, 2012; Lui & Li, 2012);

and the use of 'big ideas' such as overarching statements, key concepts, core concepts, threshold ideas or principles, as a way to structure the curriculum (Chalmers, Carter, Cooper, & Nason, 2017; Harlen, 2015; Sinnema, 2017). There is no universal agreement about what these big ideas are, either within a single discipline or across disciplines but one way of looking at them is outlined by Chalmers and her colleagues (2017), who discuss big ideas both of and about the interdisciplinary area of study known as STEM (science, technology, engineering and mathematics). They suggest that big ideas of STEM include both content and process big ideas. For them, content big ideas might be, for example, concepts (e.g. "space, time and force"), theories (e.g. "atomic theory, chaos theory"), principles (e.g. "inverse principle") and models (e.g. "probabilistic models") whereas process big ideas might be formulating hypotheses or interpreting data (Chalmers et al, 2017, p. 27).

The new reforms are not without their critics, however. Although 'big ideas' are considered to be effective for aiding students' learning at the discipline level (Chalmers et al., 2017), little evidence exists on how big ideas/overarching statements help when designing a curriculum or what lessons are to be learnt from how these get implemented in classroom practice (Sinnema, 2017). There are also many who criticise the new skills-focused curricula for downgrading knowledge in different subject areas (Rata, 2012; Wheelahan, 2012; Yates & Young, 2010; Young, 2013). For example, when analysing the implications of the Welsh reform for teaching primary humanities Jones and Whitehouse (2017) noticed the difficulty of balancing skills and knowledge in the context of teaching humanities (see also an argument in Davies (2016) about science in the context of the Welsh reform) . Using an argument in Lambert (2011) they maintained that while a lot of emphasis is placed on employability, humanities teachers should not assume that knowledge will be "picked up along the way and such knowledge does not matter" (Lambert (2011) cited in Jones and Whitehouse (2017)). This argument holds, in our view, for mathematics and other disciplines as well.

More generally, Young argues strongly for a knowledge-based approach in any subject, stating that any curriculum study "must begin not from the learner but the learners' entitlement to knowledge" (2013, p.101; see also Young, 2010a). It is this view that we take for developing our argument for mathematics in this paper. Interestingly, some studies and policy documents propose that 'big ideas' define core knowledge and thus could serve as a mechanism for a greater inclusion of knowledge that cannot easily be utilised for everyday life in school syllabi (see, e.g. Hirsch Jr, Kett, & Trefil, 1988; Millar, Osborne, & Nott, 1998) .

The Wales example: the *Successful Futures* report

It was in the context of the curricular development trends outlined above that Professor Graham Donaldson was invited to lead a review of curriculum and assessment arrangements in Wales. Donaldson's review, *Successful Futures* (Donaldson, 2015), was published in 2015 and all 68 recommendations of the review were accepted. Overall the review recommends that Wales should develop a new curriculum and a new assessment and evaluation framework. While some propositions of the *Successful Futures* report, such as the teaching of the Welsh language, are specific to the country of Wales, the ideas underpinning the proposed reform have much in common with curricular policy worldwide, such as overarching principles, which was discussed above.

Donaldson (2015), in line with the trend to define a curriculum in terms of what young people should be (rather than what they should know) makes a strong case for clarity about the overall purposes of a school curriculum and proposes that children and young people should develop as (p. 29):

- Ambitious, capable learners, ready to learn throughout their lives;
- Enterprising, creative contributors, ready to play a full part in life and work;
- Ethical, informed citizens of Wales and the world;
- Healthy, confident individuals, ready to lead fulfilling lives as valued members of society.

The report provides “illustrative examples” of the ways in which school experiences in different curriculum areas contribute to the four purposes. For example, in mathematics, personal satisfaction from “gaining a sense of achievement by solving tricky mathematical puzzles and problems” (Donaldson, 2015, p. 50) as well as understanding mathematics in a wide range of contexts is seen as contributing to developing ambitious, capable learners.

The review strongly articulates the role of the four purposes as central in organising the whole of the curriculum and states that “the structure of the curriculum should reflect directly and promote the curriculum purposes” (p. 35). While acknowledging that maintaining an overall focus on these can be challenging, the report calls for the four purposes to serve as a key reference point for developing each aspect of the curriculum. For example, subject areas, which the *Successful Futures* report conceives of as ‘Areas of Learning and Experiences’ (AoLE) and which tend to group more traditional subjects (such as, for example, Expressive Arts; Mathematics and Numeracy), are required to have a clear strategy of how they would promote the four purposes (Donaldson, 2015, pp. 94-95). This is mentioned as the first of ten elements to be included in the design of each AoLE. Similarly, the first of the twelve pedagogical principles that teachers should apply for planning their teaching and their students’ learning states that “good teaching and learning maintains a consistent focus on the overall purposes of the curriculum” (Donaldson, 2015, p. 64). The report makes similar statements about assessment arrangements (Donaldson, 2015, pp. 74-75) and achievement outcomes for learners (Donaldson, 2015, p. 114).

Apart from referencing (or contributing to) the four purposes of the curriculum, each Area of Learning and Experience should describe its scope and boundaries and its distinct features and importance to learners. It should also provide "a decision about how best to present the various component subjects and/or strands within the Area of Learning and Experience"; and advice on teaching and learning strategies that are particular to the AoLE.

In his argument, Donaldson discusses challenges of developing a curriculum. For example, the report acknowledges that the current curriculum is viewed as "unwieldy, overcrowded and atomistic" and as "inhibiting opportunities to apply learning more holistically in 'real life' situations, or to use that learning creatively to address issues that cross subject boundaries" (2015, p. 35). *Successful Futures* calls for a review of the content and implies relevance as one of the criteria to be reviewed when it acknowledges that "a reluctance to let go of some of the aspects of the curriculum that are of limited relevance". Relevance is strongly emphasised in terms of the demands of employers and the workplace which are considered as vital "if young people are to move smoothly and successfully into employment" (p. 7, but see also other examples throughout the report, such as pp. 13, 23 and 59).

The report argues for greater ownership of the reform to be given to teachers and calls for the establishment of "mechanisms designed to secure the active involvements of the teaching profession" (Donaldson, 2015, p. 92). As a response, several groups of teachers were tasked with the development of the curriculum and at the date of writing, some detail of the content of the curriculum has been developed. There appears to be an intention to include some guidance related to pedagogical approaches, alongside the content of the curriculum. As part of this work the groups were tasked with producing an interpretation of the four purposes for each Area of Learning and Experience.

As discussed above, one of the concerns associated with the "new" reforms is the "de-emphasis" on knowledge in the school curriculum. Within his discussion, the *Successful Futures* report addresses the 'polarisation' between subjects and skills or competencies as unhelpful and calls for balancing both because "both make important contributions to fulfilling the purposes of the curriculum" (p. 36, see also p. 6). However, it can be argued that the review leans towards emphasising the skills/competency approach throughout the report. For example, the word "skills" is mentioned more than 180 times and "competence" more than 80 times in the report while "knowledge" appears approximately 70 times.

Research Questions and Theoretical Positioning

In the paper we aim to consider two questions: what is the contribution that mathematics makes to the four purposes; and what is the contribution that the four purposes make (or do not make) to the development of a school mathematics curriculum. To investigate these, we follow the knowledge-based approach as proposed by Young who argues that curriculum study "must begin not from the learner but the learners' entitlement to knowledge" (2013, p. 101; see also Young, 2010a).

Young (2013) bases his argument on the theories of Lev Vygotsky (1986) who envisioned education as a process of interaction between everyday experiences of learners and theoretical concepts that are understood as fundamentally distinct. While students bring their everyday concepts with them to school, it is only at school that they learn theoretical/intellectual/scientific concepts and theories otherwise not available to them. This new knowledge extends and transforms students' everyday concepts which in turn transform the learners' relationship with the world thus ensuring that they do not remain fixed in their experiences (Young, 2010b). The relationship between student's experience and intellectual concepts is thus dialectic.

Young (2010a) argues that as theoretical concepts are based in subject knowledge, "the knowledge stipulated by the curriculum must be based on specialist knowledge developed by the communities of researchers" (p. 25). However, to form school subjects, this specialist knowledge necessarily undergoes a process of curriculum re-conceptualization. School subjects while being quite different from academic disciplines offer guarantees that students are offered access to the best knowledge available in a particular field and incorporate pedagogy as bridges to access the knowledge.

We see two main reasons to adopt the knowledge approach in the context of Donaldson. Firstly, the approach may help to avoid the polarisation between skills and knowledge as recommended by Donaldson (2015) in the *Successful Futures* report, and compensate for our view that there is not enough emphasis on knowledge in the original report as discussed above. The approach also helps to address the arguments in Jones and Whitehouse (2017) and Davies (2016) who called for paying more attention to the place of knowledge in the context of the Welsh report, as stated earlier.

Secondly, the knowledge-based approach allows one to adopt the perspective that mathematics knowledge exists as 'an objective reality' (cf. Davis, Hersh, & Marchisotto, 2011) and the main purpose of the curriculum is in facilitating access to this knowledge. It thus allows one to start from mathematics rather than follow "some analytic or rationally derived prescribed mission" when defining mathematics education, that, according to Ernest, would lead to a misrepresentation (1998, p. 75). We see this particularly relevant to the *Successful Futures* report because the report can be seen as encouraging a (re)definition of what should constitute school mathematics, such as, for example, when it states that each AoLE should give the scope and boundaries of disciplines "including its central concerns, how it is distinct from other areas and why it is important for the education of each child and young person" (Donaldson, 2015, p. 95). To accomplish this, one needs to ask and answer the questions 'What is the nature of mathematics?' and 'What does it mean to do mathematics?' that provide the basis that for defining aspects of what mathematics should be taught and how it should be taught (see, e.g. Khait, 2005; Voskoglou, 2018). We see answering these two questions as central to answering the question about the contribution of mathematics to the four purposes.

Methods

Our main chosen method for answering the research questions was to consult academic mathematics researchers. This is grounded in the idea of seeing education as a process of socialisation or enculturation (Resnick, 1988) where learning occurs through acquiring dispositions and habits "true of members of communities of practice, groups of people engaged in common endeavours within their own culture" (Schoenfeld, 2016, p. 7). Under such a view, it has been argued, mathematics learners should experience doing mathematics "the ways mathematicians do" and "consistent with the way mathematics is done" (Schoenfeld, 2016, pp. 6–7). However, we felt that to engage with the mathematicians, we ourselves needed to get a deeper insight into the way mathematicians see mathematics. As a result, we consulted two sources of "specialist knowledge developed by the community of researchers" (Young, 2010a, p. 25)

Firstly, as one source of specialist knowledge, we explored literature on mathematics written by mathematicians and, especially, the work of a contemporary mathematician and Fields Medallist Timothy Gowers (Gowers, 2000a, 2000b, 2002, 2006). In particular, we referred to the book “Mathematics. A Very Short Introduction” (Gowers, 2002). Importantly for us, mathematics as presented in the book has undergone some processes of re-contextualisation to make it accessible while, arguably, preserving the features of advanced mathematics. For example, the book claims that only the knowledge of British GCSE or equivalent is required from the readers. Yet, Gowers aimed for “depth rather than breadth” when explaining mathematics and deliberately avoided topics “that have a hold on the public imagination out of proportion to their impact on current mathematics research” thus “letting [mathematics] speak for itself” rather than employing other conventions (2002, preface) which suited our goals. Although the works of other distinguished mathematicians such as George Pólya and Hermann Weyl were considered (see for example, Pólya, 1948, 1977; Weyl, 2013), their arguments assume a rather broad knowledge of advanced mathematics that may be difficult to re-contextualise for school mathematics. Consulting the literature allowed us to choose key themes about mathematics and doing mathematics.

Then, as a second source of specialist knowledge, we consulted the opinions of twelve academics across three Higher Education Institutions’ mathematics departments. All mathematicians were active researchers, represented various areas of pure and applied mathematics, had recent experience of working with children such as through mathematics outreach or mathematics camps, and the group included representatives who had experienced mathematics education, that is, learnt or taught mathematics, in different countries. The academics were either known to the authors of the paper or were suggested by a colleague as someone interested in the issues of school education.

We first asked two mathematicians to 'sanity check' the key themes we had previously identified. We then used the themes as starting points for semi-structured interviews with the academics where they were asked to comment on the relationship between mathematics as a discipline and the four purposes of Donaldson (2015) in view of the themes selected in the first stage. The themes thus created a bridge between mathematics as practised by the researchers and thinking about the nature of mathematics when defining mathematics education. Some academics were given a day or two to think and write their responses, other preferred to give their ideas on the spot. Selected ideas were shared with other academics and in some cases a second interview was held to help us clarify ideas/statements. A smaller self-selected group of academics including one of the authors of the paper was formed that refined the ideas produced by the wider group of academics for consistency. After a few iterations, the group produced a table of statements that it felt satisfied with (presented in Table 1 in Findings 2). The resulting statements were seen by the group as consistent, that is describing how learning or doing mathematics contributes to learners becoming ambitious capable learners (purpose 1), enterprising creative contributors (purpose 2), ethical informed citizens (purpose 3) and healthy, confident individuals (purpose 4). They were also seen as true to mathematics, that is the statements were seen as : a) holding at the level of doing/learning advanced mathematics as well as at school level; b) describing mathematics rather than other academic disciplines that use mathematics, such as physics or computer science; and c) giving a picture of mathematics learning that goes beyond the widespread view that "mathematics is not useful beyond simple calculation" (Stengel, 1997, p. 597). The group also felt that the statements were accessible in language and could be understood by mathematics practitioners who teach school mathematics and not only those involved in mathematics research.

The rest of the paper is organised as follows. In Findings 1 and 2 we consider what mathematics has to offer to the four purposes of the curriculum. In Findings 1 we lay out selected themes of what is mathematics and what it means to do mathematics, which is mainly a re-contextualisation of the work of Gowers. In Findings 2 we present an interpretation of the four purposes which is given as a table of statements that we co-constructed with the academics. There we also outline the difficulties met when composing the statements. In Findings 3 we speculate about what guidance the result, that is the table of statements, offers to the development of a mathematics curriculum. This is followed by a discussion section where we talk about whether there are any benefits or constraints of applying the knowledge approach in the context of the *Successful Futures* reform in view of our findings. The paper concludes with the implications of our findings for teacher training, outlined in the last section.

Findings 1: Let mathematics speak for itself

After the initial discussions with academic colleagues, referred to earlier, about what makes mathematics itself, we wrote down that mathematics was “a unique and universal scientific language of interdisciplinary scientific communication and cooperation”. However, it seems to us that such a statement allows different interpretations. While for a mathematical "outsider" it may seem that learning and applying mathematics in a context as opposed to learning abstract mathematics is better linked to its interdisciplinary and cross-curriculum aspects, mathematical “insiders” may agree with Timothy Gowers who emphasised that it is precisely the abstract nature of mathematics that makes mathematics a very powerful tool of cooperation (for pedagogical implications of defining and interpreting mathematics by mathematical ‘insiders’ and ‘outsiders’, see, for example Khait, 2005). We therefor start with discussing what abstraction means in mathematics.

Abstraction

Using abstraction, we manipulate with objects that we may not know (do not know the meanings of them or the context), yet, the power of mathematical abstraction is that we CAN manipulate and derive conclusions which then can be interpreted for a specific problem (Gowers, 2002, pp. 15–16). Through mathematical abstraction, methods invented in, say, finance, become used in engineering or physics and vice versa. One may further argue that abstraction itself often makes the task of comprehending mathematics easier and not only at advanced levels. Indeed, while in mathematics we try to visualise (that is to imagine it in two or three dimensional space) and/or measure mathematical objects, it may not always be possible. Yet, one should learn to think abstractly to become comfortable with, say, the idea of infinity or an unmeasurable set. Similarly, at a less advanced level, the difficulty of linking fraction division with real life (Tanner & Jones, 2003) simply vanishes if we accept that division is an inverse operation to multiplication which is there for mathematical completeness rather than real life necessity. As Gowers (2002) summarises, some mathematical concepts "are puzzling if you try to understand them concretely, but they lose their mystery when you relax, stop worrying about what they are, and use the abstract method" (p. 34). One may further extend the argument to propose that the difficulties in communicating mathematics between mathematical "insiders" and "outsiders" may simply disappear if one learns to think abstractly.

Anecdotally, one of the commonly held views is that abstraction is irrelevant at primary level teaching. However, this is very far from the truth. When we learn to count without physically adding several objects together but by using numbers then we already use the concept of a number as an abstraction (Gowers, 2000b). Interestingly, the difficulties of comprehending the number concept as an abstraction were so many that it was only fully understood as an abstract concept in the 20th century. Yet, it did not prevent mathematicians and non-mathematicians using numbers for several thousand years or longer. This example highlights that one can, and is often forced to, work with abstract concepts in mathematics without fully comprehending them. As Gowers (2002) stresses throughout his book, this contradiction can be happily ignored when doing mathematics by simply accepting that "a mathematical object is what is does" (p. 18, see also p. 34 and p. 132). For example, zero is not just a point on a number line equidistant from numbers 1 and -1 or a symbol that defines "nothingness". Rather it is a number such that when added to any other number it does not change the value of that number (a+0=a for any number a).

Is learning how to think abstractly in mathematics necessary for active and lifelong sustenance as an informed citizen in today's society? Some may argue that the goal of mathematics education should be the preparation of its citizens in its common culture and language that fosters the interpretation of quantitative information for personal and professional decision-making. Such a belief is well captured in mathematical literacy or numeracy being recognised as an important and standalone part of learning mathematics. OECD (2013) defines mathematical literacy as:
"…an individual's capacity to formulate, employ, and interpret mathematics in a variety of contexts. It includes reasoning mathematically and using mathematical concepts, procedures, facts, and tools to describe, explain, and predict phenomena. It assists individuals in recognising the role that mathematics plays in the world and to make the well-founded judgements and decisions needed by constructive, engaged and reflective citizens." (p. 17). To address the question of whether the definition above largely leaves the learning of abstract mathematics out of its equation, let us consider what exactly one may mean by using mathematical concepts to study real life phenomena.

Abstraction and mathematical modelling

While science uses observations and experiments as a basis for their theory when searching for a solution and, in this sense, deals directly with the real world, "[mathematicians] do not apply scientific theories directly to the world but rather to models." (Gowers, 2002, p. 4) One way to think about a model is to consider it as "an imaginary, simplified version of the part of the world being studied, one in which exact calculations are possible" (Gowers, 2002, p. 4). Arguably, the very aim of mathematical modelling is to make exact mathematical calculations possible while retaining some important features of a real-world problem. An engineer may aim to create a model that can be accurately built, which may serve as an assessment criterion for the whole task. However, one may need a different criterion for creating a good mathematical model: "when choosing a [mathematical] model, one priority is to make its behaviour correspond closely to the actual, observed behaviour of the world. However, other factors, such as simplicity and mathematical elegance, can often be more important." (Gowers, 2002, p. 5).

The question of what characteristics to retain when deciding on the mathematical model further reveals the role mathematics plays in problem solving. Examples are known where oversimplifying or ignoring different features of the phenomena at different stages of investigation can lead to important discoveries. Another good reason to make mathematical models simple is: "if you are lucky the same model can be used for studying many phenomena at once" (Gowers, 2002, p. 16). For example, the four-colour map theorem can help to solve a time-tabling problem. One mathematical symbol such as a simple dot for example, may mean countries or subjects depending on context (see Chapter 1 in Gowers (2002) for more examples of various features of mathematical modelling).

Even more fascinatingly, mathematical models can be useful even if they have almost nothing in common with the real world at all. Some models provide a useful framework for thinking about a physical phenomenon that can never be checked in practice. Throwing a coin two times is unlikely to return one head and one tail and yet, the probabilistic model of a 50 per cent chance of getting either proves to be useful. Conversely, mathematically unlikely events still happen.

In summary, mathematical abstraction is central to investigating real-life phenomena in two ways. Mathematics "abstracts the important features from a problem and deals with objects that are not concrete and tangible" (Gowers, 2002, p. 16).

Abstraction and mathematics as a linguistic activity

The other form of abstraction in mathematics is normally referred to as the one that deals with formal definitions of mathematical objects (see, e.g. Mitchelmore & White, 2004). This is one of the reasons why some refer to mathematics as an essentially linguistic activity and many call mathematics a language. While one may argue that too much emphasis on learning mathematical abstraction undermines practising wider skills contributing to employability, we find it informative to refer to an argument in Khait (2005). Teaching mathematics as an essentially linguistic activity operating with abstract objects assigned precise meanings, as opposed to teaching algorithmic procedures, helps create a mind-set essential for living in the computerised and industrialised society of today (Khait, 2005). This is especially true if we acknowledge that "children and young people need to learn how to be more than consumers of technology and to develop the knowledge and skills required to use that technology creatively as learners and future members of a technologically competent workforce" (Donaldson, 2015, p. 8).

Proof

Rigour in mathematics is often associated with mathematical proof as well as abstraction. By proof of a mathematical statement we mean “an argument that puts a statement beyond all possible doubt” (Gowers, 2002, p. 36). It follows from the argument above that there is indeed a need for a special mechanism in mathematics to validate a mathematical concept or argument by means other than a physical experiment. In mathematics “proving is a way of thinking, exploring, of coming to understand” (Schoenfeld, 1994a, p. 74). As such, a proof is a major vehicle of communicating mathematics. However, learning how to prove mathematical statements is also known to improve understanding of mathematics as “one will understand them in a completely different and much more interesting way” (Gowers, 2002, p. 48; see also Schoenfeld, 1994b). Some proofs are more valuable than others if they provide a more economical way of understanding the statement, a better visualisation for the mathematical phenomena, reveal a new connection between various area of mathematics or if they can be utilised to prove more statements or create a new practical application (Gowers, 2002, Chapters 3 and 6). As a reward for that hard work, mathematical proofs can provide great satisfaction “with sudden revelations, unexpected yet natural ideas, and intriguing hints that there is more to discover” (Gowers, 2002, p. 51), thus contributing to increased intrinsic motivation.

The discoveries of 20th century mathematicians and philosophers give us an understanding that “any dispute about a validity of a mathematical proof can always be resolved” (Gowers, 2002, p. 40). This makes mathematics particularly distinct from other disciplines where subjectivity is an allowed form of reasoning. This however should not lead to seeing mathematics as a “very clean, exact subject” (Gowers, 2002, p. 112) as we outline in the next subsection.

Doing mathematics

Many problems in engineering, physics or computer science as well as in other disciplines will not have a clean, exact, unambiguous solution because they are very difficult and sometime impossible to find. Rather we would normally be satisfied with an approximation or even a rough estimate thus making mathematics look quite "messy". This may explain why students who are used to mathematical problems expressed succinctly and expecting to find a short solution from a simple formula find it difficult to embark on mathematical problems set in contexts, such as STEM (Science, technology, mathematics and engineering) project based learning. Understandably such tasks are rarely used in the classroom. Yet, there is an opinion that estimates, and approximations are prevalent in modern mathematics and its applications.

Maryam Mirzakhani, another Fields Medallist, compared doing mathematics with "being lost in a jungle and trying to use all the knowledge that you can gather to come up with some new tricks" (Zagrebnov, 2017, p. 39) to find a way out. Arguably, many mathematics learners would agree with Maryam, when she said: "There are times when I feel like I'm in a big forest and don't know where I'm going." (Svoboda, 2005, p. 58). But not many would be able to testify that: "But then somehow I come to the top of a hill and can see everything more clearly. When that happens it's really exciting" (Svoboda, 2005, p. 58). This encourages one to ask if the difficulties that a mathematics learner faces when learning mathematics at school are essentially similar to the difficulties of an academic conducting research in mathematics. What makes one's experience different is resilience in learning mathematics.

Learning mathematics assumes a lot of practice which leads to becoming technically fluent but also grasping difficult mathematical concepts. Timothy Gowers explicitly describes the contrasts between the two, yet highlights that "almost everybody who is good at one is good at the other" (Gowers, 2002, p. 132). He further outlines one important pedagogical implication of this on the teaching and learning of mathematics. If "understanding a mathematical object is largely a question of learning the rules it obeys rather than grasping its essence, then this is exactly what one would expect [learning mathematics] to be". (Gowers, 2002, p. 132). He concludes that "the distinction between technical fluency and mathematical understanding is less clear-cut than one might imagine" (Gowers, 2002, pp. 132–133).

In summary, the above arguments gave us the following themes for our discussion with mathematics researchers on how learning mathematics could contribute to fulfilling the four purposes of the curriculum proposed by Donaldson (2015). Firstly, abstraction as part of mathematical modelling is the key tool for applying mathematics to studying real life phenomena (mathematics is not applied "directly to the world but rather to models" (Gowers, 2002, p. 4)). Secondly, resilience in learning mathematics is of paramount importance. This is both in terms of learning mathematics requiring a lot of practice, being forced to operate with mathematical abstractions we may not yet fully understand and not knowing an exact way in which to apply mathematics when solving real world problems. Thirdly, mathematics has its own specific way of communicating and establishing truths. And, finally, mathematics is functional and action-oriented ("a mathematical object is what is does" (Gowers, 2002, p. 18)).

Findings 2: An interpretation of the four purposes for mathematics

This section presents the response to the question concerning how mathematics contributes to the four purposes. We start with outlining the issues we encountered with the group of academics in interpreting the four purposes for mathematics.

In Purpose 1 the word "ambitious" was questioned as being a personal quality rather than a result of learning. The difficulty was resolved when it was suggested to interpret "ambitious" as "empowered" and to consider how learners become empowered by learning mathematics. This was interpreted in view of the argument in Gowers, as discussed in a previous section, about how through abstraction and modelling mathematicians strive to find optimal methods that would solve several problems at once. Moreover, mathematics aims to do this on a piece of paper and without any equipment which makes it distinct from science or engineering. This way of reasoning resulted in statements 1.3 and 1.4 in the table below. The academics proposed to make a further emphasis on how learning arguing and thinking in mathematics develops one's communication skills. Arguably this can be seen as relevant to each of the four purposes. However, the group felt that this quality helps first of all to become capable learner, hence statement 1.1. While the group felt that developing resilience in one’s self contributes to becoming both "empowered capable learners" (purpose 1) and "healthy confident individuals" (purpose 4), it was decided when composing the statements for purpose 1 to stress what in learning mathematics empowers one to approach complex problems, hence statement 1.2.

When composing statements for the second purpose, the word "creative" attracted considerable attention. Although mathematics was thought of essentially “being all creativity”, mathematicians do not talk much about creativity as fish do not talk much about water, as one of the academics noted. It was felt that the argument in Gowers also did not mention creativity specifically, possibly for the same reason. By association "creative" was extended to "imaginary" and "beautiful". The first of these immediately offered a link to operating with abstract objects through the argument in Gowers, hence a reference to abstraction in statements 2.1 and 2.2. Beautiful (or elegant) were linked to an idea of a proof when such adjectives are sometimes used to refer to particularly interesting (whatever that means) ways of getting results in mathematics (cf. with an argument in Gelfert (2017)), hence statements 2.3 and 2.4. On the contrary, the group saw no difficulties in unpacking how learning mathematics serves becoming "enterprising" contributors. The argument in Gowers stating that "in mathematics objects are what they do" which was broadly acknowledged by academics as pedagogically important, offered a straightforward interpretation of mathematics as an essentially enterprising activity.

Developing a vision of how learning mathematics could contribute to purposes 3 and 4 was more difficult. Firstly, some academics questioned the relationship between purpose 3 and purpose 4. For example, it was discussed whether being "ethical informed citizens" is a necessary condition for becoming "valued members of the society". To help us distinguish between the two purposes it was decided to interpret the third purpose as the one when our actions are directed towards others and the external world, while the fourth purpose was the one directed toward self.

Secondly, the word ethical was initially proposed as not relevant to mathematics as mathematics is “non-judgemental”. However, the problem also offered a solution: constructing arguments in mathematics and especially distinguishing between a demonstration and a proof, was seen by the group as a key feature of doing mathematics that furthers one to want to establish an objective truth and not to submit to a subjective opinion, see statements 3.1-3.5. The group did not see that it was reinforced well enough in the themes selected by us in the previous chapter.

Lastly, the fourth purpose was interpreted in view of resilience developed in a dialectic fashion that is seen as essential for succeeding with mathematics while developing as a result of doing mathematics, see statement 4.1.

Table 1: How mathematics promotes each of the four purposes

Purpose 1: Ambitious capable learners who are ready to learn throughout their lives	
1.1	Learning mathematics provides practice for communicating ideas in a clear,

logical and conscious way. 1.2 Mathematics teaches us not to be afraid of complex problems as they can be reduced to a succession of simpler problems and, eventually, to basic computations. 1.3 Mathematics provides opportunities to use modelling and abstraction to aid problem solving beyond simply manipulating quantitative information, thus making learners even more capable problem solvers. 1.4 Mathematics teaches us to be ambitious as it allows for a minimum of equipment to solve complex problems using thinking and abstraction as a main vehicle, and it strives for universality when one method or model would solve various problems in different fields.
Purpose 2: Enterprising creative contributors who are ready to play a full part in life and work
2.1 Mathematics is fundamentally abstract, it teaches learners to operate with things that do not physically exist, using and developing learners' creativity to transcend what we see and what has existed hitherto, to imagine/develop/discover new realities. 2.2 Mathematics learners learn to consider the new realities as a field for enterprise. Mathematics teaches us to overcome obstacles by creating new mathematical objects (such as negative numbers or *i*) which always come with a set of operations assigned to them. This allows us to continue our enterprise. 2.3 In a mathematical proof we rule out all possibilities other than the given conclusion, and this entails thinking about, mapping out, and considering all possibilities, not simply following an established path. 2.4 Many mathematical problems will have not one but many methods for getting to the final answer. By exploring various approaches, learners learn to be creative in their own way. As one method may solve many problems in various disciplines, it creates opportunities for mathematics learners to collaborate in interdisciplinary

fields.
Purpose 3: Ethical informed citizens of Wales and beyond
3.1 Doing mathematics encourages clarity of thinking and, as such, enables learners to know the reasons for making one or other decision or adapting one or other view. 3.2 Mathematics develops a habit in learners to wish to fathom to impersonal truths while distinguishing between objective data and subjective opinions. 3.3 A study of mathematical hypotheses starts from considering evidence in terms of examples and counterexamples. It thus develops a habit of searching for evidence when studying any phenomenon. However, in mathematics, truth may be independent from our subjective views and not what it seems at first glance. Thus, mathematics demands from learners a constant questioning of their understanding of every step of their thinking process. 3.4 Mathematics cultivates a habit of paying careful attention to which properties hold universally and which do not. Knowing what pertains to the whole world and knowing what is special about a given situation or country, makes one judge what makes Wales unique and in what ways a citizen of Wales is the same as a citizen of the world. 3.5 Mathematics is highly interconnected and does not study a single phenomenon but interconnections between various phenomena. Looking at any problem or issue from different angles translates into seeing somebody else's point of view, and making decisions in the light of this, that can benefit all and not just one person.
Purpose 4: Healthy, confident individuals who are ready to lead fulfilling lives as valued members of the society
4.1 Learning mathematics is difficult and everyone sooner or later hits mathematics they find challenging. Moreover, one cannot succeed in mathematics without a lot of practice and mastering fluency. Mathematics learning thus develops resilience and encourages learners to recognise their ability to learn, and want to continue to learn,

throughout their lives.

Findings 3: Mathematics, the Four Purposes and a Mathematics Curriculum for Wales

Let us first examine what guidance our statements in Table 1 offer to what mathematics topics should be included in the curriculum. One striking observation we make is that none of the statements referred to a specific area of mathematics such as number, algebra, shape and measure or data handling. However, adapting the classification proposed by Chalmers et al. (2017) (see above), some of the statements may be seen as referencing 'big ideas' *of* mathematics related to mathematical processes (i.e., "the intellectual skill associated with the acquisition and effective use of content knowledge" (Chalmers et al., 2017, p. 27)), such as statements 1.3, 2.3 and 4.1 in Table 1. Others can be called 'big ideas' *about* mathematics which are more abstract and global in nature and focus on habits of mind in mathematics, such as statements 1.2, 1.4, 2.1, 2.2, 3.3 and 3.5 in Table 1.

Is there a way to organise a curriculum using our statements? Although little guidance exists about how to construct a curriculum using big ideas or how effective such attempts are when implemented, some studies hypothesise how it could be done. We speculate that what we have at hand would not be enough to follow the approach outlined by Chalmers and her colleagues (2017), who designed curriculum units based on different types of 'big ideas'. At least big ideas related to content-specific concepts and theories should be added to exemplify what material will be taught.

Another striking observation we make is that Table 1 suggest that the outcome of learning mathematics is a special way of thinking or a mindset, rather than a toolkit of skills or a list of known facts or procedures (cf. Schoenfeld, 1994b). Moreover, statements 1.1, 1.4, 2.2, 3.1, 3.2, 3.4 and 4.1 suggest that such habits of mind are achieved through practising mathematical abstraction and rigour. Furthermore, practising abstraction is further linked with fostering creativity which is underlined in statements 2.2, 2.3 and 2.4. A reference to resilience is especially interesting as our argument is that resilience develops in a dialectic fashion. Resilience is necessary for succeeding mathematics but is also develops as an outcome of practising mathematics.

In summary we propose that the main outcome emerging from Table 1might be considered as “big ideas” *of* mathematics related to processes, that is abstraction and rigour and big ideas *about* mathematics, that is resilience and creativity.

Discussion

Should the emphasis on mathematical abstraction and proof in our interpretation of the four purposes thus be viewed as problematic, as these are perceived as challenging to teach and learn? Or, more generally, should topics or methods that may be perceived as alien to learners’ experiences, or difficult to teach, not enter a school curriculum? The knowledge approach presupposes that difficulties like these are inherent although not unresolvable. An essential underlying epistemological assumption is that “although knowledge can be experienced as oppressive and alienating, this is not a property of knowledge itself” (Young, 2013, p. 107) and the tensions can be resolved if appropriate pedagogical strategies are employed. Under the knowledge approach, to facilitate a learner’s access to “the best knowledge we have in any field of study” (Young, 2013, p. 115), teachers would need to start from considering what are the meanings to which this curriculum provides their students access, rather than whether their students could *a priori* relate to the curriculum (Young, 2013). Indeed, examples provided by Schoenfeld (1994a) demonstrate that mathematical proof, for instance, can be experienced by learners in a meaningful way if appropriate pedagogy is employed (see also Schoenfeld, 2016). In fact, the knowledge approach places an emphasis on deciding on pedagogical approaches being the major role of the teachers. In contrast with the above argument, Donaldson (2015), while recognising the challenging nature of teaching and learning, suggests that learner engagement is achieved when mathematics is "taught through relevant contexts" (p. 49) rather than stressing the role of the teacher and pedagogy in engaging learners.

Another tension between the knowledge approach and the one proposed by *Successful Futures* is the fact that the report does not distinguish between learners' everyday experiences and theoretical concepts. This de-emphasises the role of schooling as challenging, transforming and extending learners' everyday experiences (as explained by Young (2013), for example). For instance, proposing to make education more relevant to the interests of students, the needs of employers and the workplace of today, may potentially be fixed too much on current and local needs without looking into the future. In mathematics in particular, it may create a serious problem. As the history of mathematics shows, predicting which mathematics will be used in the future is difficult if not impossible. Even those topics that looked highly abstract and inapplicable in the past later developed (sometime with a significant delay) into fascinating applications (Gowers, 2000a). The examples of the American reforms in the 60s when specific topics then considered relevant entered the mathematics curriculum warn us against a "quick fix". Furthermore, as Hoyles and her colleagues (2001) argued, using utility as a deciding factor on mathematics topics may make studying mathematics less valuable and, quite ironically, reduce its perceived utility. The knowledge approach, at least in theory, allows one to avoid this as it sees relevance being built through teaching and in a dialectic fashion.

The knowledge approach allowed us to highlight a few other aspects of the *Successful Futures* report which do not sufficiently capture the nature of the discipline. If other AoLEs are recognised as contributing to children developing resilience, perseverance, attention to detail or personal effectiveness in the *Successful Futures* report (Donaldson, 2015), these are not mentioned in the context of learning mathematics. This is probably unintentional, but it highlights a substantial problem. Resilience in learning mathematics is crucial for making progress in mathematics, as suggested by our early arguments and the statements in Table 1, yet it only happens in mathematics classrooms by accident if happens at all (Johnston-Wilder & Lee, 2010; Lee & Johnston-Wilder, 2012; Ward-Penny, Johnston-Wilder, & Lee, 2011). As Gowers' (2002) arguments imply, a learner's progress results in increased competence and subject knowledge as well as in a learner's realisation that "there is more to discover" (p. 51). This appears to relate to the emphasis on personal satisfaction and enjoyment from doing mathematics stated in the Donaldson report, e.g., "gaining a sense of achievement by solving tricky mathematical puzzles and problems" (Donaldson, 2015, p. 50). Similarly, while there is an emphasis on creativity throughout the *Successful Futures* report, it is not sufficiently linked with mathematics. As can be seen from our statements, mathematics is intrinsically creative (cf. Bicer et al., 2017). By carefully highlighting the creative element in mathematics one can enhance the learning of mathematics (Oner, Nite, Capraro, & Capraro, 2016).

What can serve as a mechanism to decide which topics should be seen as important if one attempts to develop a curriculum with the knowledge approach in mind? We propose that instead of looking elsewhere, mathematics itself should suggest it. "[M]athematics is very interconnected, far more so than it appears on the surface" (Gowers, 2000a). While some topics in mathematics may be seen as less valuable because of their low utility, lack of straightforward real-world applications or high abstractness, the connections between particular topics will expose the consequences of removing them from the curriculum or deemphasising them. In contrast, analysing connections between the topics can provide relevant criteria for deciding which topics are more crucial for fluency (technically or conceptually) in further learning. Revealing the structure of mathematics connections can also characterise the shifts in learners' reasoning that teachers should facilitate over time, thus revealing teaching strategies and pedagogical approaches to teaching the topics (Cooper & Carter, 2016; Shifter & Fosnot, 1993). This perspective may also suggest an approach to addressing concerns about an overcrowded mathematics curriculum.

While the knowledge approach attempts to ease such a burden when it refers to the specialist knowledge developed by the communities of researchers as a source to stipulate the content of the curriculum, the *Successful Futures* reform does not. Thus, it would be up to the teachers involved in the development of the new curriculum as to argue their case using some other sources. In our argument we often referred to *Mathematics. A Very Short Introduction* by the distinguished mathematician, Timothy Gowers (2002). As we attempted to demonstrate, the book can be used as a framework for considering the implications for teaching mathematics arising from the nature of the discipline and, as such, has a potential to be utilised more effectively by mathematics education practitioners.

Wider implications and conclusions

There is a view (Deng, 2013; Deng & Luke, 2008) that knowing the subject matter of a secondary school subject involves understanding epistemological (what knowing means in the context of the discipline and how these principles of knowing have been formed), psychological (how the content to be taught can be developed out of interest, experience and prior knowledge of students), pedagogical (concerning the effective ways of representing and reformulating knowledge) and socio-cultural dimensions (how the knowledge relates and interacts with society and culture), as well as the knowledge of how the discipline is organised (what concepts, principles and facts are embedded in it and how they are related). We do not wish to claim that the ideas presented in Table 1 fully and truly describe the nature of the mathematics, but some aspects of the argument attempt to touch upon the dimensions above. As such, the four purposes of the curriculum create a framework that allows the introduction of these dimensions into the curriculum. However, one could be sceptical about these dimensions being helpful without knowing the mathematical structure of the curriculum (see, for example, an argument in Lampert, 1991).

The knowledge approach makes several important presuppositions in terms of teacher knowledge and attitude. As Lerman (1983) suggested, "one's perspective of mathematics teaching is a logical consequence of one's epistemological commitment in relation to mathematical knowledge" (p. 59) alongside other factors. In view of this, constructing a curriculum with the knowledge approach in mind will have implications for initial teacher education as well as teacher professional development and the role that subject knowledge plays in it. Indeed, some of the academics involved in constructing Table 1 explicitly mentioned that they wished they had a better knowledge of psychology or philosophy of mathematics and mathematics education to help them to express the role of mathematics more effectively. This in turn encourages one to ask if these should be better emphasised in teacher education to allow teachers to answer the demands of the "new" reforms driven by big ideas or overarching statements effectively. This is especially pertinent if reforms demand that teachers derive the curriculum "from the first principles".

In conclusion, while the task of interpreting the four purposes of the curriculum for mathematics was initially questioned as meaningful, the approach chosen in the study allowed us to produce an interesting result. This is despite the tensions between the knowledge-based approach and the one proposed by Donaldson (2015) as discussed earlier. Initially suspecting that the four purposes would only emphasise the quantitative aspect of mathematics, the result, quite unexpectedly, highlighted mathematical abstraction, rigour and reasoning instead. Our findings imply that these aspects of mathematics remain important and powerful, so to deny learners access to the fulfilment of grasping mathematical concepts and abstract ideas would be to impoverish their education. In view of this, Table 1 can be seen as a (clear) "statement of how the Area of Learning and Experience (of Mathematics and Numeracy) promotes the four purposes of the curriculum" and offers some insight into how mathematics "is distinct from other areas and why it is important for the education of each child and young person" (Donaldson, 2015, p. 95). However, the statements do not set the "scope and boundaries" of the subject (Donaldson, 2015, p. 95) nor do they suggest how to organise and sequence the subject matter.

Finally, building on the observation of one of the academics about mathematics being non-judgemental, we would like to make a concluding remark. While any academic discipline (and certainly mathematics itself) aims to be above moral issues striving for universal objectivity, this fact may serve as the very reason for the need to integrate the moral/ethical/existential dimension into the curriculum by other means. Gudmundsdottir (1990) recommends the use of narrative as a port of entry for such concerns and the four purposes of *Successful Futures* in Wales or the four capacities of the Scottish curriculum can be viewed as vehicles for such an integration. Moreover, Stengel (1997) suggests that the "moral may be hidden or even mystified, but it is always present just beneath the surface of academic knowledge" (p. 599). Certainly, mathematics as a way of thinking suggests a specific "way" of moral thinking as reflected in the summary of the statements.

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