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ON ADJOINT ADDITIVE PROCESSES

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Abstract. Starting with an additive process $(Y_t)_{t\geqslant 0}$, it is in certain cases possible to construct an adjoint process $(X_t)_{t\geqslant 0}$ which is itself additive. Moreover, assuming that the transition densities of $(Y_t)_{t\geqslant 0}$ are controlled by a natural pair of metrics $\mathrm{d}_{\psi,t}$ and $\delta_{\psi,t}$, we can prove that the transition densities of $(X_t)_{t\geqslant 0}$ are controlled by the metrics $\delta_{\psi,1/t}$ replacing $\mathrm{d}_{\psi,t}$ and $\mathrm{d}_{\psi,1/t}$ replacing $\delta_{\psi,t}$.

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INTRODUCTION

The origin of this investigation is the paper [7] where it was suggested to understand the transition density $p_t(x)$ of a symmetric Lévy process $(Y_t)_{t\geqslant 0}$ with characteristic exponent ψ in terms of two in general t-dependent metrics $d_{\psi,t} = \sqrt{t}\,\mathrm{d}_\psi$, where $d_\psi(\xi,\eta) = \psi^{1/2}(\xi-\eta)$, and $\delta_{\psi,t}$, i.e.,

$$p_t(x-y) = p_t(0)e^{-\delta^2_{\psi,t}(x,y)} \tag{0.1}$$

and

$$p_t(0) = (2\pi)^{-n}\int_0^\infty \lambda^{(n)}(B^{\mathrm{d}_\psi}(0,\sqrt{r/t}))e^{-r}\,\mathrm{d}r. \tag{0.2}$$

The term (0.2) has already been considered in [9]. While the metric $\mathrm{d}_{\psi,t}$ is, under mild conditions, always at our disposal, the existence of $\delta_{\psi,t}$ is in general an open problem. Examples in [7] suggest that in some cases $x\mapsto \delta^2_{\psi,t}(x,0)$ for $t>0$ fixed is itself the characteristic exponent of a Lévy process, i.e. a continuous negative definite function, and that $(t,x)\mapsto \delta^2_{\psi,1/t}(x,0)$ is the characteristic exponent of an additive process $(X_t)_{t\geqslant 0}$. An example is of course Brownian motion, a further one is the Cauchy process $(Y_t)_{t\geqslant 0}$ where the corresponding additive process

$(X_t)_{t\geqslant 0}$ is the Laplace process. In [4], the relations between the transition densities of $(Y_t)_{t\geqslant 0}$ and $(X_t)_{t\geqslant 0}$ were studied in more detail when $(Y_t)_{t\geqslant 0}$ is a Lévy process and when $(X_t)_{t\geqslant 0}$ exists, i.e. $x \mapsto \delta^2_{\psi,t}(x,0)$ is a continuous negative definite function and $\delta^2_{\psi,1/t}(x,0)$ is the characteristic exponent of an additive process. A natural question is whether it is possible to already start with an additive process $(Y_t)_{t\geqslant 0}$ with generator $-q(t,D)$, where $q(t,D)$ is a pseudo-differential operator with symbol $q(t,\xi)$, and for $t>0$ fixed $\xi \mapsto q(t,\xi)$ is the characteristic exponent of a Lévy process, and to obtain a new additive process $(X_t)_{t\geqslant 0}$ similar to the construction when starting with a Lévy process. Additive processes can be traced back to P. Lévy and this notion was further clarified by K. Itô as well as A. V. Skorokhod; we refer to the notes in [14].

While pursuing these ideas, we learned about the work initiated by T. Lewis [12] who was (to the best of our knowledge) the first to consider probability distributions which are characteristic functions themselves. Such distributions he called adjoint. In the monograph [11], adjoint distributions were discussed in more detail. Thus in light of these investigations and the discussion in [7] and [4], we consider our paper as a further step to understand adjoint additive processes with densities Φ_t. Here we call $(X_t)_{t\geqslant 0}$ *adjoint* to $(Y_t)_{t\geqslant 0}$ if there exists a mapping $j:(0,\infty)\to(0,\infty)$ such that for all $t\in(0,\infty)$ we have

$$\hat{p}_t = \Phi_{j(t)}, \tag{0.3}$$

where $\hat{p}_t$ is the Fourier transform of p_t. Often $j(t)=1/t$ will be a suitable choice.

Our approach is essentially an analytic one, namely to construct, with the help of p_t, a symbol of an operator $A(t,D)$ which admits a fundamental solution such that this fundamental solution allows us to construct the transition densities Φ_t of an additive process. Given p_t, with $\sigma_t(\xi) := \frac{p_{1/t}(\xi)}{p_{1/t}(0)}$ we have to take $A(t,\xi) = -\frac{\partial}{\partial t}\ln\sigma_t(\xi)$. Beside some more or less standard technical assumptions we need the crucial, but restrictive *Basic Assumption I*: $\xi\mapsto A(t,\xi)$ is a continuous negative definite function, i.e. for fixed $t>0$ it has a Lévy–Khinchin representation.

We then turn to the question of understanding the structure of transition densities, and for this we add *Basic Assumption II*: $\mathrm{d}_\psi(\xi,\eta) := \sqrt{\psi(\xi-\eta)}$ is a metric on $\mathbb{R}^n$ generating the Euclidean topology and $(\mathbb{R}^n, \mathrm{d}_\psi, \lambda^{(n)})$ is a metric measure space having the volume doubling property. Under these two basic assumptions and, as previously mentioned, some standard assumptions on the symbol $q(t,\xi)$ of the generator of the additive process $(Y_t)_{t\geqslant 0}$ we start with, we can show that $(Y_t)_{t\geqslant 0}$ admits an adjoint process $(X_t)_{t\geqslant 0}$. In addition, with $Q_{t,0}(\xi) = \int_0^t q(\tau,\xi)\,\mathrm{d}\tau$ and $\mathrm{d}_{Q_{t,0}}(\xi,\eta) = Q^{1/2}_{t,0}(\xi-\eta)$, for the transition density $p_t(x-y)$ of Y_t we have

$$p_t(x-y) = (2\pi)^{-n}\int_{\mathbb{R}^n} \lambda^{(n)}(B^{\mathrm{d}_{Q_{t,0}}}(0,\sqrt{r}))e^{-r}\,\mathrm{d}r\, e^{-\delta^2_{Q_{t,0}}(x,y)}, \tag{0.4}$$

and for the transition density Φ_t of X_t we find

$$\Phi_t(x-y) = (2\pi)^{-n} \int_{\mathbb{R}^n} \lambda^{(n)}(B^{\delta_{Q_{1/t,0}}}(0,\sqrt{r}))e^{-r}\,\mathrm{d}r\, e^{-d^2_{Q_{1/t,0}}(x,y)}. \tag{0.5}$$

Of importance, of course, are examples and they are provided with the help of the symbols $q_1(t,\xi) = h_1(t)|\xi|^2$, $q_2(t,\xi) = h_2(t)|\xi|$ and $q_3(t,\xi) = h_3(t)\ln\cosh\xi$ (here we require $\xi \in \mathbb{R}$). Clearly certain combinations such as direct sums lead to more examples. As indicated in [7], in particular Theorem 7.1, subordination in the sense of Bochner (see [16] for the general theory), shall lead to further examples. Readers with an interest in state of the art results of the theory of Markov processes related to pseudo-differential operators are referred to Schilling et al. [3] as well as to F. Kühn [10] and the forthcoming survey [8]. Whether it is possible to extend our considerations to the classes of processes constructed in [2] using the symbolic calculus of Hoh [5] and in [18] using the ideas of [6] with the help of x and t dependent negative definite symbols remains an open question.

1. ADJOINT PROCESSES

Let $(\Omega, \mathcal{A}, P^x, (X_t)_{t\geqslant 0})_{x\in\mathbb{R}^n}$ be a stochastic process (adapted to a suitable filtration). Following K. Sato [14], we call $(X_t)_{t\geqslant 0}$ an *additive process in law* if $(X_t)_{t\geqslant 0}$ has independent increments and if it is stochastically continuous. If, in addition, the increments are also stationary, we call $(X_t)_{t\geqslant 0}$ a *Lévy process*. For the distribution $\gamma_{t,s}$ of the increments $X_t - X_s$, $0 \leqslant s < t$, of an additive process, the following conditions are satisfied:

$$\gamma_{s,s} = \epsilon_0, \quad 0 \leqslant s, \tag{1.1}$$

$$\gamma_{t,r} * \gamma_{r,s} = \gamma_{t,s}, \quad 0 \leqslant s \leqslant r \leqslant t, \tag{1.2}$$

$$\gamma_{t,s} \to \epsilon_0 \quad \text{weakly for } s \to t,\ s < t, \tag{1.3}$$

$$\gamma_{t,s} \to \epsilon_0 \quad \text{weakly for } t \to s,\ s < t. \tag{1.4}$$

In the case of a Lévy process we have $\gamma_{t,s} = \mu_{t-s}$ and $(\mu_t)_{t\geqslant 0}$ is a convolution semigroup of probability measures on $\mathbb{R}^n$, i.e.,

$$\mu_0 = \epsilon_0,$$
$$\mu_t * \mu_s = \mu_{t+s},$$
$$\mu_t \to \epsilon_0 \quad \text{weakly as } t \to 0.$$

A continuous function $\psi : \mathbb{R}^n \to \mathbb{C}$ is called a *continuous negative definite function* if $\psi(0) \geqslant 0$ and if for all $t > 0$ the function $\xi \mapsto e^{-t\psi(\xi)}$ is positive definite in the sense of Bochner. Given a convolution semigroup of probability measures on $\mathbb{R}^n$, there exists a unique continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{C}$ such that

$$\hat{\mu}_t(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\,\mu_t(\mathrm{d}x) = (2\pi)^{-n/2} e^{-t\psi(\xi)}. \tag{1.5}$$

A remark about the normalisation of the Fourier transform is in order. In our normalisation the convolution theorem reads as

$$(\mu_t * \mu_s)^\wedge(\xi) = (2\pi)^{n/2}\hat{\mu}_t(\xi)\hat{\mu}_s(\xi)$$

and the inverse Fourier transform is given by

$$(F^{-1}u)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} u(\xi)\,\mathrm{d}\xi.$$

If $\mu_t = p_t(\cdot)\lambda^{(n)}$ then we have of course $\hat{\mu}_t = \hat{p}_t$ and from (1.5) it follows that

$$\begin{aligned} p_t(x) &= F^{-1}(\hat{\mu}_t)(x) = F^{-1}((2\pi)^{-n/2}e^{-t\psi(\cdot)})(x) \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi}e^{-t\psi(\xi)}\,\mathrm{d}\xi. \end{aligned}$$

Here and in the following, $\hat{\mu}$ denotes the Fourier transform of μ and $F^{-1}u$ is the inverse Fourier transform of u. If the continuous negative definite function ψ is real-valued, the measures μ_t are symmetric and in this note we are only interested in the symmetric case. Moreover, we do not allow a killing or diffusion part and therefore the Lévy–Khinchin representation of ψ is given by

$$\psi(\xi) = \int_{\mathbb{R}\backslash\{0\}} (1 - \cos(y\cdot\xi))\,\nu(\mathrm{d}y) \tag{1.6}$$

with Lévy measure ν.

A probability measure μ on $\mathbb{R}^n$ is called *infinitely divisible* if for every $k \in \mathbb{N}$ there exists a probability measure μ_k on $\mathbb{R}^n$ such that

$$\mu = \mu_k * \cdots * \mu_k \quad (k \text{ terms}). \tag{1.7}$$

It is known (see [1]) that every infinitely divisible measure μ can be embedded into a convolution semigroup $(\mu_t)_{t\geqslant 0}$, $\mu_1 = \mu$.

Following T. Lewis [12], we call a probability distribution p on $\mathbb{R}^n$ *adjoint* to a probability distribution Φ if

$$\hat{p} = \Phi. \tag{1.8}$$

We call p *self-adjoint* if

$$\hat{p} = p, \tag{1.9}$$

i.e. if p is a fixed point of the Fourier transform. Note that at this point the choice of the normalisation of the Fourier transform must be taken into account. Examples of adjoint distributions are (see [11])

$$\begin{aligned} p(x) &= \frac{2x}{\pi^2\sinh x}, &\quad \Phi(x) &= \frac{\pi}{4\cosh(\pi x/2)}, \\ p(x) &= \frac{1}{\pi}\Big(\frac{\sin x}{x}\Big)^2, &\quad \Phi(x) &= \frac{1}{2}\max\Big(1 - \frac{|x|}{2}, 0\Big), \end{aligned}$$

and in addition to the normal distribution we find that

$$p(x) = \frac{1}{\sqrt{2\pi}\cosh(\sqrt{\pi/2}\,x)}, \tag{1.10}$$

$$p(x) = \frac{1}{\sqrt{2\pi}}\frac{\cos(\sqrt{\pi/2}\,x)}{\cosh(\sqrt{\pi}\,x)}, \tag{1.11}$$

$$p_k(x) = C_k(H_{4k}(\sqrt{2}\,x) - m_{4k})e^{x^2/2}, \tag{1.12}$$

where H_l is the lth Hermite polynomial, are self-adjoint distributions.

If a distribution p has an adjoint distribution Φ which is infinitely divisible, the corresponding convolution semigroup $(\Phi_t)_{t\geqslant 0}$ gives rise to a Lévy process. We call two stochastic processes with distribution $(p_t)_{t\geqslant 0}$ and $(\Phi_t)_{t\geqslant 0}$ *adjoint processes* if for a bijective mapping $j : (0,\infty) \to (0,\infty)$ we have

$$\hat{p}_t = \Phi_{j(t)},$$

where we will often use $j(t) = 1/t$. One aim of the paper is to study this notion for Lévy and additive processes.

2. SOME ADDITIVE PROCESSES

In the following, let $q : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that for every $t \geqslant 0$ the function $q(t,\cdot) : \mathbb{R}^n \to \mathbb{R}$ is a continuous negative definite function. It follows that $q(t,\xi) \geqslant 0$ and for $0 \leqslant s < t$,

$$\xi \mapsto \int_s^t q(\tau,\xi)\,\mathrm{d}t \tag{2.1}$$

is a continuous negative definite function too. We assume, in addition, that for a fixed continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{R}$ we have $\lim_{|\xi|\to\infty}\psi(\xi) = \infty$, $e^{-t\psi} \in L^1(\mathbb{R}^n)$, and for $0 < \kappa_0 < \kappa$,

$$\kappa_0\nu_0(A) \leqslant \nu(t,A) \leqslant \kappa_1\nu_0(A), \qquad A \in \mathcal{B}^{(n)}(\mathbb{R}^n \setminus \{0\}), \tag{2.2}$$

where ν_0 is the Lévy measure corresponding to ψ and $\nu(t,\mathrm{d}y)$ is the Lévy measure corresponding to $q(t,\xi)$. We refer to [9] and [7] where the condition $e^{-t\psi} \in L^1(\mathbb{R}^n)$ is related to growth conditions on ψ or the doubling property. The estimate (2.2) induces of course

$$\kappa_0\psi(\xi) \leqslant q(t,\xi) \leqslant \kappa_1\psi(\xi) \tag{2.3}$$

for all $\xi \in \mathbb{R}^n$. Estimates such as (2.2) or (2.3) have the interpretation that corresponding pseudo-differential operators have the same continuity properties in an intrinsic scale of generalised Bessel potential spaces. Their origin is of course classical ellipticity estimates. We set

$$Q(t,\xi) := \int_0^t q(\tau,\xi)\,\mathrm{d}\tau \tag{2.4}$$

and we find

$$\int_s^t q(\tau,\xi)\,\mathrm{d}\tau = Q(t,\xi) - Q(s,\xi) \geqslant 0, \tag{2.5}$$

and by

$$\hat{\mu}_{t,s}(\xi) := (2\pi)^{-n/2} e^{-(Q(t,\xi)-Q(s,\xi))} = (2\pi)^{-n/2} e^{-\int_s^t q(\tau,s)\,\mathrm{d}\tau} \tag{2.6}$$

a family $(\mu_{t,s})_{0\leqslant s\leqslant t}$ of probability measures is defined. From our assumption it follows immediately that

$$\hat{\mu}_{s,s}(\xi) = (2\pi)^{-n/2} = \hat{\epsilon}_0(\xi), \tag{2.7}$$

where ϵ_0 is the Dirac measure at 0, and

$$\mu_{t,r} * \mu_{r,s} = \mu_{t,s}, \quad s \leqslant r \leqslant t. \tag{2.8}$$

Moreover, we have

$$\lim_{\substack{s\to t\\ s<t}} \hat{\mu}_{t,s}(\xi) = \hat{\epsilon}_0(\xi), \tag{2.9}$$

$$\lim_{\substack{t\to s\\ s<t}} \hat{\mu}_{t,s}(\xi) = \hat{\epsilon}_0(\xi), \tag{2.10}$$

which implies the corresponding weak convergence of the measures. It follows that the family $(\mu_{t,s})_{0\leqslant s\leqslant t}$ forms the family of distributions of the increments of an additive process in law (see [14]).

From (2.3) we deduce that $\mu_{t,s}$ has a density given by

$$\begin{aligned} p_{t,s}(x) &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-\int_s^t q(\tau,\xi)\,\mathrm{d}\tau}\,\mathrm{d}\xi \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-(Q(t,\xi)-Q(s,\xi))}\,\mathrm{d}\xi, \quad 0 < s < t. \end{aligned}$$

As it is the inverse Fourier transform of an L^1-function, we have $p_{t,s} \in C_\infty(\mathbb{R}^n)$. For $t > 0$ and $s = 0$ we write p_t for $p_{t,0}$, i.e.

$$\begin{aligned} p_t(x) &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-\int_0^t q(\tau,\xi)\,\mathrm{d}\tau}\,\mathrm{d}\xi \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-Q(t,\xi)}\,\mathrm{d}\xi. \end{aligned} \tag{2.11}$$

3. ON FUNDAMENTAL SOLUTIONS

Let q, Q and $\mu_{t,s}$ and $p_{t,s}$ be as in Section 2. On the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ we may define the operators

$$q(t,D)u(x) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} q(t,\xi)\hat{u}(\xi)\,\mathrm{d}\xi \tag{3.1}$$

as well as

$$H_{t,s}u(x) := \int_{\mathbb{R}^n} u(x-y)\,\mu_{t,s}(\mathrm{d}y), \qquad 0 \leqslant s \leqslant t. \tag{3.2}$$

Applying the convolution theorem, we obtain

$$(H_{t,s}u)^\wedge(\xi) = (u*\mu)^\wedge_{t,s}(\xi) = (2\pi)^{n/2}\hat{u}(\xi)\hat{\mu}_{t,s}(\xi) = e^{-(Q(t,\xi)-Q(s,\xi))}\hat{u}(\xi),$$

or

$$H_{t,s}u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-(Q(t,\xi)-Q(s,\xi))}\hat{u}(\xi)\,\mathrm{d}\xi.$$

We want to study the operators $(H_{t,s})_{0<s<t}$ in $L^2(\mathbb{R}^n)$ and $C_\infty(\mathbb{R}^n)$. The properties of $(\mu_{t,s})_{0\leqslant s\leqslant t}$ imply immediately, on $\mathcal{S}(\mathbb{R}^n)$,

$$H_{s,s}u = u, \quad \text{or} \quad H_{s,s} = \mathrm{id}, \tag{3.3}$$

and

$$(H_{t,r}\circ H_{r,s})u = H_{t,r}(H_{r,s}u) = H_{t,s}u, \tag{3.4}$$

or

$$H_{t,r}\circ H_{r,s} = H_{t,s}. \tag{3.5}$$

Moreover, we have

$$\|H_{t,s}u\|_\infty \leqslant \|u\|_\infty \tag{3.6}$$

and by Plancherel's theorem

$$\|H_{t,s}u\|_{L^2} \leqslant \|u\|_{L^2}. \tag{3.7}$$

The weak convergence properties of $(\mu_{t,s})_{0<s<t}$ also yield

$$\lim_{\substack{s\to t\\ s<t}} \|H_{t,s}u - u\|_\infty = \lim_{\substack{t\to s\\ s<t}} \|H_{t,s}u - u\|_\infty = 0, \tag{3.8}$$

and since by Plancherel's theorem

$$\|H_{t,s}u - u\|_0^2 = \int_{\mathbb{R}^n} \left|e^{Q(t,\xi)-Q(s,\xi)} - 1\right|^2 |\hat{u}(\xi)|^2\,\mathrm{d}\xi, \tag{3.9}$$

we deduce

$$\lim_{\substack{s\to t\\ s<t}} \|H_{t,s}u - u\|_0 = \lim_{\substack{t\to s\\ s<t}} \|H_{t,s}u - u\|_0 = 0. \tag{3.10}$$

LEMMA 3.1. *For $u \in \mathcal{S}(\mathbb{R}^n)$ and $t > s > 0$ we have*

$$\frac{\partial}{\partial t} H_{t,s}u(x) = -q(t,D)H_{t,s}u(x), \tag{3.11}$$

$$\frac{\partial}{\partial s} H_{t,s}u(x) = -H_{t,s}(-q(s,D)u)(x). \tag{3.12}$$

Proof. Using the definitions, for $u \in \mathcal{S}(\mathbb{R}^n)$ and $0 < s < t$ we obtain

$$\begin{aligned}
\frac{\partial}{\partial t} H_{t,s}u(x) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \frac{\partial}{\partial t}\big(e^{-(Q(t,\xi)-Q(s,\xi))}\big)\hat{u}(\xi)\,\mathrm{d}\xi \\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \left(-\frac{\partial}{\partial t} Q(t,\xi)\right) e^{-(Q(t,\xi)-Q(s,\xi))}\hat{u}(\xi)\,\mathrm{d}\xi \\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} (-q(t,\xi)) e^{Q(t,\xi)-Q(s,\xi)}\hat{u}(\xi)\,\mathrm{d}\xi \\
&= -q(t,D)H_{t,s}u(x),
\end{aligned}$$

which proves (3.11). Further we get

$$\begin{aligned}
\frac{\partial}{\partial s} H_{t,s}u(x) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \left(\frac{\partial}{\partial s} e^{-(Q(t,\xi)-Q(s,\xi))}\right)\hat{u}(\xi)\,\mathrm{d}\xi \\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-(Q(t,\xi)-Q(s,\xi))} q(s,\xi)\hat{u}(\xi)\,\mathrm{d}\xi \\
&= -H_{t,s}(-q(s,D)u)(x),
\end{aligned}$$

and the lemma is proved. ∎

By (3.6) we can extend $H_{t,s}$ continuously to $C_\infty(\mathbb{R}^n)$, and by (3.7) we can extend it continuously to $L^2(\mathbb{R}^n)$. In each case, we will use $H_{t,s}$ to denote the extension. It is clear that (3.6)–(3.9) also hold for the extension. More care is needed for extending Lemma 3.1 to $C_\infty(\mathbb{R}^n)$. The L^2-case is however not too difficult to deal with. Using ψ from (2.3), we introduce the space

$$H^{\psi,2}(\mathbb{R}^n) := \{v \in L^2(\mathbb{R}^n) \mid \|u\|_{\psi,2} < \infty\} \tag{3.13}$$

where

$$\|v\|^2_{\psi,2} = \int_{\mathbb{R}^n} (1+\psi(\xi))^2 |\hat{v}(\xi)|^2\,\mathrm{d}\xi. \tag{3.14}$$

The uniformity of (2.3) with respect to t implies that $(-q(t,D), H^{\psi,2}(\mathbb{R}^n))$ is a closed L^2-operator and that (3.11) as well as (3.12) hold as equations in $L^2(\mathbb{R}^n)$. In order to interpret this observation, we recall (see [17])

DEFINITION 3.1. Let $(X, \|\cdot\|_X)$ be a Banach space. Suppose that for every $t > 0$ an operator $(A(t), D(A(t)))$ on X is given which for each $t_0 > 0$ fixed generates a strongly continuous contraction semigroup on X. Suppose that $D(A(t))$ is independent of t. We call a strongly continuous family $(U(t,s))_{0\leqslant s\leqslant t}$, $0 \leqslant s \leqslant t$, $0 \leqslant t \leqslant T$, of bounded operators $U(t,s) : X \to X$ an *X-fundamental solution* to the initial value problem

$$\frac{\partial u(t)}{\partial t} = A(t)u(t) = f(t), \qquad 0 \leqslant t \leqslant T, \tag{3.15}$$

$$u(0) = u_0, \tag{3.16}$$

where $u_0 \in X$, $u(\cdot) \in D(A(t))$, $f \in C([0,T]; X)$, if

$$U(t,r)U(r,s) = U(t,s) \quad \text{for } 0 \leqslant s \leqslant r \leqslant t \leqslant T, \tag{3.17}$$

$$U(s,s) = \mathrm{id} \quad \text{for } 0 \leqslant s \leqslant T, \tag{3.18}$$

$$\frac{\partial}{\partial t}U(t,s) = -A(t)U(t,s), \qquad 0 \leqslant s \leqslant t \leqslant T, \tag{3.19}$$

$$\frac{\partial}{\partial s}U(t,s) = U(t,s)A(s), \qquad 0 \leqslant s \leqslant t \leqslant T. \tag{3.20}$$

Thus, by the calculations from the proof of Lemma 3.1 we have

THEOREM 3.1. *The family $(H_{t,s})_{0\leqslant s\leqslant t\leqslant T}$ is an L^2-fundamental solution to the problem*

$$\frac{\partial}{\partial t}u(t,x) + q(t,D)u(t,x) = f(t,x), \qquad u(0,x) = u_0(x), \tag{3.21}$$

where the domain of $q(t,D)$ is $H^{\psi,2}(\mathbb{R}^n)$, and ψ is taken from (2.3).

The situation for $C_\infty(\mathbb{R}^n)$ is (as we must expect) more complicated. Using the Lévy measure $\nu(t, \mathrm{d}y)$ and representation (3.2), we can prove that $C^2_\infty(\mathbb{R}^n) \cap C_\infty(\mathbb{R}^n)$ will be in the domain of the generator of the Feller semigroup $(T_t^{q(t_0,\cdot)})_{t\geqslant 0}$ associated with $q(t_0,\cdot)$ and that this domain is independent of t. Then Theorem 3.1 can be extended to the case where $L^2(\mathbb{R}^n)$ is replaced by $C_\infty(\mathbb{R}^n)$. For our purposes, it is sufficient to note that by (2.2) the domain of the generator of $(T_t^{q(t_0,\cdot)})_{t\geqslant 0}$ is independent of t_0 and that $\mathcal{S}(\mathbb{R}^n)$ is a subspace of the domain on which (3.17)–(3.20) hold.

4. ON ADJOINT DISTRIBUTIONS

We use the notation and assumptions of the previous sections and introduce the probability measures

$$\rho_t := \tilde{\rho}(\cdot)\lambda^{(n)} := \frac{e^{-Q(t,\cdot)}}{(2\pi)^{n/2}p_t(0)}, \qquad t > 0. \tag{4.1}$$

From (4.1) we obtain

$$\hat{\rho}_t(y) = \frac{p_t(y)}{p_t(0)}. \tag{4.2}$$

Our assumptions on $q(t,\cdot)$, in particular (2.2) and (2.3), imply that for every $\delta > 0$,

$$\inf_{|\xi|\geqslant\delta} q(\tau,\xi) \geqslant \kappa_0 \inf_{|\xi|\geqslant\delta} \psi(\xi) =: M_\delta > 0, \tag{4.3}$$

where the last estimate follows from the fact that $\psi(\xi) > 0$ for $\xi \neq 0$.

Following [9, proof of Lemma 5.6], we find

$$\int\limits_{|\xi|\geqslant\delta} e^{-Q(t,\xi)}\,\mathrm{d}\xi = \int\limits_{|\xi|\geqslant\delta} e^{-\int_0^t q(\tau,\xi)\,\mathrm{d}\tau}\,\mathrm{d}\xi \leqslant \int\limits_{|\xi|\geqslant\delta} e^{-t\kappa_0\psi(\xi)}\,\mathrm{d}\xi$$

or for $0 < t_0 < t$,

$$\int\limits_{|\xi|\geqslant\delta} e^{-Q(t,\xi)}\,\mathrm{d}\xi \leqslant e^{-(t-t_0)M_\delta} \int\limits_{|\xi|\geqslant\delta} e^{-t_0\kappa_0\psi(\xi)}\,\mathrm{d}\xi. \tag{4.4}$$

Since

$$\psi(\xi) \leqslant C_R^\psi|\xi|^2 + a_R^\psi, \tag{4.5}$$

where $C_R^\psi \asymp \int_{|y|\leqslant R} |y|^2\,\nu(\mathrm{d}y)$ and $a_R^\psi \asymp \nu_0(B_R(0)^\complement)$, it follows that

$$\begin{aligned}\int\limits_{\mathbb{R}^n} e^{-Q(t,\xi)}\,\mathrm{d}\xi &= \int\limits_{\mathbb{R}^n} e^{-\int_0^t q(\tau,\xi)\,\mathrm{d}\tau}\,\mathrm{d}\xi \geqslant \int\limits_{\mathbb{R}^n} e^{-t\kappa_1\psi(\xi)}\,\mathrm{d}\xi \\ &\geqslant \int\limits_{\mathbb{R}^n} e^{-t\kappa_1 C_R^\psi|\xi|^2}\,\mathrm{d}\xi\, e^{-ta_R^\psi},\end{aligned} \tag{4.6}$$

where $a \asymp b$ means that $0 < \gamma_1 \leqslant b/a \leqslant \gamma_2$. Combining (4.4) with (4.6) we obtain (compare with [9])

$$\begin{aligned}\frac{\int_{|\xi|>\delta} e^{-Q(t,\xi)}\,\mathrm{d}\xi}{(2\pi)^{-n/2}p_t(0)} &\leqslant \frac{e^{-(t-t_0)M_\delta}\int_{|\xi|>\delta} e^{-t_0\kappa_0\psi(\xi)}\,\mathrm{d}\xi}{(2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-t\kappa_1 C_R^\psi|\xi|^2}\,\mathrm{d}\xi\, e^{-ta_R^\psi}} \\ &= t^{n/2}e^{-t(M_\delta - a_R^\psi)}e^{t_0M_\delta}\,\frac{\int_{|\xi|>\delta} e^{-t_0\kappa_0\psi(\xi)}\,\mathrm{d}\xi}{(2\pi)^{n/2}\int_{\mathbb{R}^n} e^{-\kappa_1 C_R^\psi|\eta|^2}\,\mathrm{d}\eta}.\end{aligned}$$

We may choose for a given $\delta > 0$ the value of $R > 0$ such that $M_\delta > a_R^\psi$ and we have proved

LEMMA 4.1. *For $\delta > 0$ and $t > 0$, we have*

$$\lim_{t\to\infty} \frac{\int_{|\xi|>\delta} e^{-Q(t,\xi)}\,\mathrm{d}\xi}{(2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-Q(t,\xi)}\,\mathrm{d}\xi} = 0. \tag{4.7}$$

Now, for $t > 0$ and $\eta \in \mathbb{R}^n$ it follows that for $u \in C_\infty(\mathbb{R}^n)$,

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^n} \tilde{\rho}_t(\xi)(u(\eta-\xi)-u(\eta))\,\mathrm{d}\xi\Big| \\
&\qquad \leqslant \int_{|\xi|\leqslant\delta} \tilde{\rho}_t(\xi)|u(\eta-\xi)-u(\eta)|\,\mathrm{d}\xi + 2\int_{|\xi|>\delta} \tilde{\rho}_t(\xi)\,\mathrm{d}\xi\,\|u\|_\infty \\
&\qquad \leqslant \sup_{|\xi|\leqslant\delta} |u(\eta-\xi)-u(\eta)| + 2\int_{|\xi|\geqslant\delta} \tilde{\rho}_t(\xi)\,\mathrm{d}\xi\,\|u\|_\infty,
\end{aligned}$$

and Lemma 4.1 now implies

LEMMA 4.2. *For $u \in C_\infty(\mathbb{R}^n)$ we have*

$$\lim_{t\to\infty} \int_{\mathbb{R}^n} \tilde{\rho}_t(\xi)u(\eta-\xi)\,\mathrm{d}\xi = u(\eta). \tag{4.8}$$

For $u \in \mathcal{S}(\mathbb{R}^n)$ we define

$$(S_t u)(x) := (\rho_{1/t} * u)(x) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\xi}(\rho_{1/t} * u)^\wedge(\xi)\,\mathrm{d}\xi. \tag{4.9}$$

Since by the convolution theorem

$$(\rho_{1/t} * u)^\wedge(\xi) = (2\pi)^{n/2}\hat{\rho}_{1/t}(\xi)\hat{u}(\xi) \tag{4.10}$$

and $\hat{\rho}_{1/t}(\xi) = \frac{p_{1/t}(\xi)}{p_{1/t}(0)}$, we get (at least on $\mathcal{S}(\mathbb{R}^n)$)

$$(S_t u)(x) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\frac{p_{1/t}(\xi)}{p_{1/t}(0)}\hat{u}(\xi)\,\mathrm{d}\xi. \tag{4.11}$$

With

$$\sigma_t(\xi) := \frac{p_{1/t}(\xi)}{p_{1/t}(0)} \tag{4.12}$$

we have

$$(S_t u)(x) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\sigma_t(\xi)\hat{u}(\xi)\,\mathrm{d}\xi. \tag{4.13}$$

Since $p_{1/t}(\xi) \leqslant p_{1/t}(0)$ for $t > 0$, our construction yields

$$\|S_t u\|_\infty \leqslant \|u\|_\infty, \tag{4.14}$$
$$\|S_t u\|_{L^2} \leqslant \|u\|_{L^2}, \tag{4.15}$$

and from Lemma 4.2 and its proof we now deduce

$$\lim_{t\to 0} \|S_t u - u\|_\infty = \lim_{t\to\infty} \|S_t v - v\|_{L^2} = 0 \tag{4.16}$$

for all $u \in C_\infty(\mathbb{R}^n)$ and $v \in L^2(\mathbb{R}^n)$. We note further that

$$\frac{\partial}{\partial t}\sigma_t(\xi) = \frac{\partial}{\partial t}\frac{p_{1/t}(\xi)}{p_{1/t}(0)} = \sigma_t(\xi)\frac{\partial}{\partial t}\ln\sigma_t(\xi).$$

We set

$$A(t,\xi) := -\frac{\partial}{\partial t}\ln\sigma_t(\xi) \tag{4.17}$$

and consider on $\mathcal{S}(\mathbb{R}^n)$ the operator

$$A(t,D)u(x) := (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\cdot\xi}A(t,\xi)\hat{u}(\xi)\,\mathrm{d}\xi. \tag{4.18}$$

We first observe that

$$\begin{aligned}
\frac{\partial}{\partial t}S_t u(x) &= \frac{\partial}{\partial t}\left((2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\frac{p_{1/t}(\xi)}{p_{1/t}(0)}\hat{u}(\xi)\,\mathrm{d}\xi\right)\\
&= \frac{\partial}{\partial t}\left((2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\sigma_t(\xi)\hat{u}(\xi)\,\mathrm{d}\xi\right)\\
&\quad - (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\left(\frac{\partial}{\partial t}\ln\sigma_t(\xi)\right)\sigma_t(\xi)\hat{u}(\xi)\,\mathrm{d}\xi\\
&= -A(t,D)(S_t u)(x),
\end{aligned}$$

or

$$\frac{\partial}{\partial t}S_t u + A(t,D)S_t u = 0. \tag{4.19}$$

We now introduce the family of operators $V(t,s)$, $0 < s < t$, by

$$(V(t,s)u)^\wedge(\xi) = e^{-\int_s^t A(\tau,\xi)\,\mathrm{d}\tau}\hat{u}(\xi), \qquad u \in \mathcal{S}(\mathbb{R}^n). \tag{4.20}$$

The condition $A(t,\xi) \geqslant 0$ will already lead to a satisfactory L^2-theory for the operator $V(t,s)$, $0 < s < t$. However, since we eventually want to investigate adjoint processes, we add here

BASIC ASSUMPTION I. For all $t > 0$ the function $\xi \mapsto A(t,\xi)$ is a real continuous negative definite function.

This is clearly a substantial and restrictive assumption and it is open to characterise those symbols $q(\tau,\xi)$ which will eventually lead to a symbol $A(t,\xi)$ satisfying this assumption. Non-trivial examples will be provided in Section 6.

Under Basic Assumption I, it follows that $e^{-\int_s^t A(\tau,\xi)\,\mathrm{d}\tau}$ is a positive definite function in the sense of Bochner, hence by

$$\hat{\gamma}_{t,s}(\xi) := (2\pi)^{-n/2} e^{-\int_s^t A(\tau,\xi)\,\mathrm{d}\tau} \tag{4.21}$$

a family of probability measures $\gamma_{t,s}$, $0 < s < t$, is defined. From (4.21) we deduce immediately

$$\gamma_{s,s} = \epsilon_0, \quad 0 \leqslant s, \tag{4.22}$$

$$\gamma_{t,r} * \gamma_{r,s} = \gamma_{t,s}, \quad 0 < s < r < t, \tag{4.23}$$

$$\gamma_{t,s} \to \epsilon_0 \quad \text{weakly for } s \to t,\ s < t, \tag{4.24}$$

$$\gamma_{t,s} \to \epsilon_0 \quad \text{weakly for } t \to s,\ s < t. \tag{4.25}$$

Following [14, Theorem 9.7], we can associate with $(\gamma_{t,s})_{0<s<t<\infty}$ a canonical additive process in law with state space $\mathbb{R}^n$. Thus we have proved

THEOREM 4.1. *Let $q : [0,\infty) : \mathbb{R}^n \to \mathbb{R}$ and $\psi : \mathbb{R}^n \to \mathbb{R}$ satisfy the assumptions of Section 2 and suppose that $A(t,\xi)$ defined by (4.17) fulfils Basic Assumption I. Then we can associate with $q(t,\xi)$ an additive process in law $(Y_t)_{t\geqslant 0}$, and with $A(t,\xi)$ an additive process in law $(X_t)_{t\geqslant 0}$. The distributions of the increments are given by*

$$P_{Y_t - Y_s} = \mu_{t,s}, \tag{4.26}$$

$$P_{X_t - X_s} = \gamma_{t,s}. \tag{4.27}$$

DEFINITION 4.1. We call $(Y_t)_{t\geqslant 0}$ and $(X_t)_{t\geqslant 0}$ a pair of *adjoint additive processes in law*.

Using (4.22)–(4.25), or (4.21), it is straightforward to see that we can extend the family $(V(t,s))_{0<s<t}$ as an X-fundamental solution to $-A(t,D)$ for $X \in \{C_\infty(\mathbb{R}^n), L^2(\mathbb{R}^n)\}$. However, even in the case $X = L^2(\mathbb{R}^n)$ it is not obvious how to characterise $D(A(t))$ in terms of ψ, one of the data characterising our construction.

5. SOME GEOMETRIC INTERPRETATIONS OF THE DENSITIES

The measures $\mu_{t,s}$ and $\gamma_{t,s}$ have densities given by

$$P_{Y_t - Y_s} = \mu_{t,s} = F^{-1}\big(e^{-(Q(t,\cdot)-Q(s,\cdot))}\big)\lambda^{(n)} = p_{t,s}(\cdot)\lambda^{(n)} \tag{5.1}$$

and

$$P_{X_t-X_s}=\gamma_{t,s}=F^{-1}\big((2\pi)^{-n/2}e^{-\int_s^t A(\tau,\xi)\,\mathrm{d}\tau}\big)\lambda^{(n)} \tag{5.2}$$
$$=F^{-1}\big((2\pi)^{-n/2}e^{-\int_s^t \frac{\partial}{\partial\tau}\ln\sigma_\tau(\xi)\,\mathrm{d}\tau}\big)\lambda^{(n)}$$
$$=F^{-1}\big((2\pi)^{-n/2}e^{\ln\sigma_t(\cdot)-\ln\sigma_s(\cdot)}\big)\lambda^{(n)}$$
$$=F^{-1}\left((2\pi)^{-n/2}\frac{p_{1/t}(\cdot)}{p_{1/t}(0)}\cdot\frac{p_{1/s}(0)}{p_{1/s}(\cdot)}\right)\lambda^{(n)},$$

Some care is needed with (5.2). Since by Basic Assumption I, $\int_s^t A(\tau,\xi)\,\mathrm{d}\tau$ is a continuous negative definite function, it follows that $\int_s^t A(\tau,\xi)\,\mathrm{d}\tau \geqslant 0$ and at least in the sense of $\mathcal{S}'(\mathbb{R}^n)$ we can calculate the inverse Fourier transform of $e^{-\int_s^t A(\tau,\xi)\,\mathrm{d}\tau}$. In fact we know more, namely that $e^{-\int_s^t A(\tau,\xi)\,\mathrm{d}\tau}$ is a positive definite function. Thus (5.2) is justified. However, while we can guarantee that $p_{1/t}(\cdot)/p_{1/t}(0)$ belongs to $L^1(\mathbb{R}^n)$, we cannot a priori guarantee that $p_{1/s}(0)/p_{1/s}(\cdot)$ belongs to $\mathcal{S}'(\mathbb{R}^n)$, and we cannot a priori apply the convolution theorem to (5.2).

For the case $s=0$, however, we obtain

$$\mu_t := P_{Y_t-Y_0}=\mu_{t,0}=F^{-1}(e^{-Q(t,\cdot)})=p_t(\cdot)\lambda^{(n)}$$

and using a consequence of Lemma 4.2, namely that $\lim_{s\to 0}\sigma_{1/s}=1$, we obtain

$$\gamma_t := P_{X_t-X_0}=\gamma_{t,0}=F^{-1}\big((2\pi)^{-n/2}e^{-\int_0^t A(\tau,\xi)\,\mathrm{d}\tau}\big)\lambda^{(n)} \tag{5.3}$$
$$=F^{-1}\big((2\pi)^{-n/2}e^{\ln\sigma_t(\cdot)}\big)\lambda^{(n)}=\frac{1}{(2\pi)^{n/2}}F^{-1}(\sigma_t(\cdot))\lambda^{(n)}$$
$$=\frac{1}{(2\pi)^{n/2}}F^{-1}\left(\frac{p_{1/t}(\xi)}{p_{1/t}(0)}\right)\lambda^{(n)},$$

i.e.

$$\gamma_t=\Phi_t(\cdot)\lambda^{(n)}:=\frac{e^{-Q(1/t,\cdot)}}{(2\pi)^{n/2}p_{1/t}(0)}\lambda^{(n)}. \tag{5.4}$$

Our aim is to give geometric interpretations for p_t as well as for Φ_t and for this we follow closely the ideas of [4] which are based on [7]. For this we add

BASIC ASSUMPTION II. For the continuous negative definite function ψ from (2.3) by $\mathrm{d}_\psi(\xi,\eta):=\sqrt{\psi(\xi-\eta)}$ a metric is defined on $\mathbb{R}^n$ which generates the Euclidean topology. Moreover, $(\mathbb{R}^n,\mathrm{d}_\psi,\lambda^{(n)})$ has the *volume doubling property*, i.e.

$$\lambda^{(n)}(B^{\mathrm{d}_\psi}(x,2r))\leqslant c_0\lambda^{(n)}(B^{\mathrm{d}_\psi}(x,r)) \tag{5.5}$$

for all $x \in \mathbb{R}^n$ and $r > 0$ where $B^{\mathrm{d}_\psi}(x,r) = \{y \in \mathbb{R}^n \mid \mathrm{d}_\psi(x,y) < r\}$ is the open ball with respect to d_ψ.

Note that if $\psi : \mathbb{R}^n \to \mathbb{R}$ is a continuous negative definite function such that $\psi(\xi) = 0$ if and only if $\xi = 0$, then d_ψ is always a metric on $\mathbb{R}^n$. In [7], in particular Lemma 3.2, conditions are proved for d_ψ to generate the Euclidean topology, and the volume doubling property of d_ψ is discussed in more detail.

Since in (2.3) we can replace ψ by $q(t_0, \cdot)$ for a fixed $t_0 > 0$ (with a change of the constants κ_0 and κ_1), we can transfer the results of [4, Section 4]. Thus, it follows that under Basic Assumption II with

$$Q_{t,s}(\xi) = \int_s^t q(\tau, \xi)\,\mathrm{d}\tau \tag{5.6}$$

a new metric is given by

$$d_{Q_{t,s}}(\xi, \eta) := Q_{t,s}^{1/2}(\xi - \eta), \qquad 0 \leqslant s < t, \tag{5.7}$$

and this metric generates the Euclidean topology on $\mathbb{R}^n$ and has the volume doubling property. This applies, in particular, to $d_{Q_{t,0}}$. The proof of Theorem 4.1 in [4] (compare also with [7, Theorem 4.1]) yields, under Basic Assumptions I and II,

$$p_{t,s}(0) = (2\pi)^{-n} \int_{\mathbb{R}^n} \lambda^{(n)}(B^{\mathrm{d}_{Q_{t,s}}}(0, \sqrt{r}))e^{-r}\,\mathrm{d}r, \tag{5.8}$$

and using the volume doubling property, as well as (2.3), we get

$$p_{t,s}(0) \asymp \lambda^{(n)}(B^{\mathrm{d}_{Q_{t,s}}}(0, \sqrt{\kappa_1/\kappa_0})). \tag{5.9}$$

We now consider the case $s = 0$ and write $p_t = p_{t,0}$ etc. It follows that

$$\begin{aligned} p_t(x) &= p_t(0)\frac{p_t(x)}{p_t(0)} = p_t(0)e^{\ln(p_t(x)/p_t(0))} \\ &= p_t(0)e^{-(-\ln\sigma_{1/t}(x))} = p_t(0)e^{-((-\ln\sigma_{1/t}(x))^{1/2})^2}, \end{aligned}$$

and by our assumptions, for $t > 0$ fixed, a metric is given by

$$\delta_{Q_{t,0}}(x,y) := (-\ln\sigma_{1/t}(x-y))^{1/2}, \tag{5.10}$$

which allows us to write

$$p_t(x-y) = p_t(0)e^{-\delta^2_{Q_{t,0}}(x,y)} \tag{5.11}$$

with $p_t(0) \asymp \lambda^{(n)}(B^{\mathrm{d}_{Q_{t,0}}}(0, \sqrt{\kappa_1/\kappa_0}))$. On the other hand, we have

$$\Phi_t(x) = \Phi_t(0)\frac{\Phi_t(x)}{\Phi_t(0)} = \Phi_t(0)e^{-Q_{1/t,0}(x,0)} \tag{5.12}$$

or

$$\Phi_t(x-y) = \Phi_t(0) e^{-\mathrm{d}^2_{Q_{1/t,0}}(x,y)}. \tag{5.13}$$

For $\Phi_t(0)$ we have

$$\Phi_t(0) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-\int_0^t A(\tau,\xi)\,\mathrm{d}\tau} \mathrm{d}\xi, \tag{5.14}$$

but

$$\ln \sigma_t(\xi) = -\int_0^t A(\tau,\xi)\,\mathrm{d}\tau. \tag{5.15}$$

It follows from the definition of σ_t that we can write

$$\Phi_t(0) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-(-\ln \sigma_t(\xi))} \mathrm{d}\xi \tag{5.16}$$

and $-\ln \sigma_t$ is the square of a metric, namely $-\ln \sigma_t = \delta^2_{Q_{1/t,0}}$. We can now use the arguments in [4] to obtain

$$\Phi_t(0) = (2\pi)^{-n} \int_0^\infty \lambda^{(n)}(B^{\delta_{Q_{1/t,0}}}(0,\sqrt{r}))e^{-r}\,\mathrm{d}r \tag{5.17}$$

and eventually we have the dual formulae

$$p_t(x-y) = (2\pi)^{-n} \int_{\mathbb{R}^n} \lambda^{(n)}(B^{\mathrm{d}_{Q_{t,0}}}(0,\sqrt{r}))e^{-r}\,\mathrm{d}r\, e^{-\delta^2_{Q_{t,0}}(x,y)} \tag{5.18}$$

and

$$\Phi_t(x-y) = (2\pi)^{-n} \int_{\mathbb{R}^n} \lambda^{(n)}(B^{\delta_{Q_{1/t,0}}}(0,\sqrt{r}))e^{-r}\,\mathrm{d}r\, e^{-\mathrm{d}^2_{Q_{1/t,0}}(x,y)}. \tag{5.19}$$

Thus, under our assumptions of Section 2, Basic Assumptions I and II and the assumption that p_t is unimodal, we obtain for the two additive processes $(Y_t)_{t\geqslant 0}$ generated by $-q(t,D)$ and $(X_t)_{t\geqslant 0}$ generated by $-A(t,D) = \big(\frac{\partial}{\partial t}\ln\sigma_t\big)(D)$ the dual formulae (5.18) and (5.19) for the transition densities of Y_t and X_t respectively.

6. EXAMPLES

EXAMPLE 6.1. In this example we consider the case where $Q(t,\xi) = h(t)|\xi|^2$, $h(t) > 0$ for $t > 0$, $h(0) = 0$ and h is strictly increasing. We first consider the transition densities $p_{t,0}(x)$ for $t > 0$,

$$p_{t,0}(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-h(t)|\xi|^2}\,\mathrm{d}\xi = \frac{1}{(4\pi h(t))^{n/2}} e^{-\frac{|x|^2}{4h(t)}}.$$

Now, for the adjoint process we find using the fact that $h(1/t) \geqslant 0$ and that $t \mapsto h(1/t)$ is strictly decreasing that

$$\begin{aligned}\Phi_t(x) &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \frac{p_{1/t}(\xi)}{p_{1/t}(0)} \,\mathrm{d}\xi = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{\frac{-|\xi|^2}{4h(1/t)}} \,\mathrm{d}\xi \\ &= \pi^{-n/2} h^{n/2}(1/t) e^{-|x|^2 h(1/t)}.\end{aligned}$$

EXAMPLE 6.2. We next consider the case where $Q(t,\xi) = h(t)|\xi|$, again with $h(t) > 0$ for $t > 0$, $h(0) = 0$, and h strictly increasing. The transition densities for $t > 0$ are given by

$$\begin{aligned}p_{t,0}(x) &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-h(t)|\xi|} \,\mathrm{d}\xi \\ &= \pi^{(-n-1)/2} \Gamma\left(\frac{n+1}{2}\right) \frac{h(t)}{\left(h^2(t) + \left|\frac{x}{h(t)}\right|^2\right)^{(n+1)/2}}.\end{aligned}$$

Then for the adjoint we get

$$\begin{aligned}\Phi_t(x) &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \frac{p_{1/t}(\xi)}{p_{1/t}(0)} \,\mathrm{d}\xi \\ &= (2\pi)^{-n/2} F^{-1}\left(\frac{h^{n+1}(1/t)}{\left(h^2(1/t) + \left|\frac{\xi}{h(1/t)}\right|^2\right)^{(n+1)/2}}\right) \\ &= \frac{2^{-n/2}(2\pi)^{-n/2}\sqrt{\pi}\, h^n(1/t)}{\Gamma\left(\frac{n+1}{2}\right)} e^{-h(1/t)|x|}.\end{aligned}$$

EXAMPLE 6.3. Here we consider the case where ξ belongs to $\mathbb{R}$, i.e. $n = 1$, and $Q(t,\xi) = h(t) \ln\cosh\xi$, $h(t) > 0$ for $t > 0$, $h(0) = 0$ and h is strictly increasing. The transition densities for $t > 0$ are given by

$$\begin{aligned}p_{t,0}(x) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\cdot\xi} \left(\frac{1}{\cosh\xi}\right)^{h(t)} \mathrm{d}\xi = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\cdot\xi} \frac{2^{h(t)} e^{-h(t)\xi}}{(1+e^{-2\xi})^{h(t)}} \,\mathrm{d}\xi \\ &= \frac{1}{2\pi} 2^{h(t)-1} \int_{\mathbb{R}} \frac{2e^{-2q(t,x)\xi}}{(1+e^{-2\xi})^{p(t,x)+q(t,x)}} \,\mathrm{d}\xi,\end{aligned}$$

where $p + q = h(t)$ with $q(x,t) = \frac{h(t)-ix}{2}$ and $p(x,t) = \frac{h(t)+ix}{2}$. Then

$$\begin{aligned}p_{t,0}(x) &= \frac{1}{2\pi} 2^{h(t)-1} \int_{\mathbb{R}} \frac{2(e^{-2\xi})^q}{(1+e^{-2\xi})^{p+q}} \,\mathrm{d}\xi = \frac{1}{2\pi} 2^{h(t)-1} \int_0^1 u^{p-1}(1-u)^{q-1} \,\mathrm{d}u \\ &= \frac{1}{2\pi} 2^{h(t)-1} B\left(\frac{h(t)+ix}{2}, \frac{h(t)-ix}{2}\right) = \frac{2^{h(t)-2}}{\pi} \left|\Gamma\left(\frac{h(t)+ix}{2}\right)\right|^2.\end{aligned}$$

In summary,

$$p_{t,0}(x) = \frac{2^{h(t)-2}}{\pi}\left|\Gamma\left(\frac{h(t)+ix}{2}\right)\right|^2,$$

and

$$\delta_t^2(x,0) = -\ln\left|\frac{\Gamma\left(\frac{h(t)+ix}{2}\right)}{\Gamma\left(\frac{h(t)}{2}\right)}\right|^2 = \sum_{j=1}^{\infty}\ln\left(1+\frac{x^2}{(h(t)+2j)^2}\right).$$

Our calculation made use of the one in [13] where the case $q(\xi) = \ln\cosh\xi$ was treated. Further, we note that $A(t,\xi) := \sum_{j=1}^{\infty}\ln\left(1+\frac{x^2}{(h(1/t)+2j)^2}\right)$ fulfils our basic assumptions for $t > 0$.

REMARK 6.1. We may also combine the previous examples to form new examples, for instance, we could consider

$$Q(t,\xi,\eta) = h_1(t)|\xi|^2 + h_2(t)|\eta|,$$

where $h_i(t) > 0$ for $t > 0$, $h_i(0) = 0$ and h_i is strictly increasing, $i = 1, 2$.

REMARK 6.2. In the case of a Lévy process, the symbol can be used to obtain results with direct probabilistic interpretations, e.g. estimates for passage times. Results of this type had been extended to Feller processes generated by pseudo-differential operators with state space dependent symbols (see R. Schilling [15]). In [8] it was pointed out that with the help of the metric $d_\psi(\xi,\eta) = \psi^{1/2}(\xi-\eta)$ these results admit a geometric interpretation. For additive processes we are not aware of explicit results of this type, but by a standard procedure we can consider additive processes with state space $\mathbb{R}^n$ as time-homogeneous Markov processes with state space $\mathbb{R}^{n+1}$: see for example [2] in the context of pseudo-differential operators. Hence a transfer obtained for Lévy processes to certain additive processes should be possible.

REFERENCES

[1] H. Bauer, *Probability Theory*, de Gruyter, Berlin, 1996.

[2] B. Böttcher, *Construction of time inhomogeneous Markov processes via evolution equations using pseudo-differential operators*, J. London Math. Soc. 78 (2008), 605–621.

[3] B. Böttcher, R. L. Schilling and J. Wang, *Lévy-type Processes: Construction, Approximation and Sample Path Properties*, Lecture Notes in Math. 2099, Springer, Berlin, 2013.

[4] L. Bray and N. Jacob, *Some considerations on the structure of transition densities of symmetric Lévy processes*, Comm. Stoch. Anal. 10 (2016), 405–420.

[5] W. Hoh, *A symbolic calculus for pseudo-differential operators generating Feller semigroups*, Osaka J. Math. 35 (1998), 798–820.

[6] N. Jacob, *A class of Feller semigroups generated by pseudo-differential operators*, Math. Z. 215 (1994), 151–166.

[7] N. Jacob, V. Knopova, S. Landwehr and R. Schilling, *A geometric interpretation of the transition density of a symmetric Lévy process*, Sci. China Math. 55 (2012), 1099–1126.

[8] N. Jacob and E. O. T. Rhind, *Aspects of micro-local analysis and geometry in the study of Lévy-type generators*, in: Open Quantum Systems, Birkhäuser/Springer, 2019, 77–140.

[9] V. Knopova and R. Schilling, *A note on the existence of transition probability densities for Lévy processes*, Forum Math. 25 (2013), 125–149.

[10] F. Kühn, *Lévy-Type Processes: Moments, Construction and Heat Kernel Estimates*, Lecture Notes in Math. 2187, Springer, Berlin, 2017.

[11] G. Laue, M. Riedel and H.-J. Roßberg, *Unimodale und positiv definite Dichten*, B.G. Teubner, Stuttgart, 1999.

[12] T. Lewis, *Probability functions which are proportional to characteristic functions and the infinite divisibility of the von Mises distribution*, in: Perspectives in Probability and Statistics, Papers in honour of M. S. Bartlett on the occasion of his sixty-fifth birthday (ed. by J. Gani), Academic Press, London, 1976, 19–28.

[13] J. Pitman and M. Yor, *Infinitely divisible laws associated with hyperbolic functions*, Canad. J. Math. 55 (2003), 292–330.

[14] K. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge Univ. Press, Cambridge, 1999.

[15] R. L. Schilling, *Growth and Hölder conditions for the sample paths of Feller processes*, Probab. Theory Related Fields 112 (1998), 565–611.

[16] R. L. Schilling, R. Song and Z. Vondraček, *Bernstein Functions*, 2nd ed., de Gruyter, Berlin, 2012.

[17] H. Tanabe, *Equations of Evolution*, Pitman, Boston, MA, 1979.

[18] R. Zhang, *Fundamental solutions of a class of pseudo-differential operators with time-dependent negative definite symbols*, PhD thesis, Swansea Univ., Swansea, 2011.

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