Cronfa - Swansea University Open Access Repository

_____

This is an author produced version of a paper published in:
Applicable Analysis

_____

Cronfa URL for this paper:

http://cronfa.swan.ac.uk/Record/cronfa48055

_____

Paper:

Litvinov, W. & Lytvynov, E. (2019). Construction of solutions to parabolic and hyperbolic initial–boundary value problems. _Applicable Analysis_, 1-33.

http://dx.doi.org/10.1080/00036811.2019.1566528

_____

http://www.swansea.ac.uk/library/researchsupport/ris-support/

Construction of solutions to parabolic and hyperbolic initial-boundary value problems

William G. Litvinov

Institute of Mathematics, University of Augsburg, Universitätsstr. 14,
D-86159 Augsburg, Germany
e-mail: `william.litvinov@`gmail.com

Eugene Lytvynov

Department of Mathematics, Swansea University, Singleton Park, Swansea SA2 8PP,
U.K.
e-mail: `e.lytvynov@swansea.ac.uk`

Abstract

Assume that, in a parabolic or hyperbolic equation, the right-hand side is analytic in time and the coefficients are analytic in time at each fixed point of the space. We show that the infinitely differentiable solution to this equation is also analytic in time at each fixed point of the space. This solution is given in the form of the Taylor expansion with respect to time t with coefficients depending on x. The coefficients of the expansion are defined by recursion relations, which are obtained from the condition of compatibility of order $k = \infty$. The value of the solution on the boundary is defined by the right-hand side and initial data, so that it is not prescribed. We show that exact regular and weak solutions to the initial-boundary value problems for parabolic and hyperbolic equations can be determined as the sum of a function that satisfies the boundary conditions and the limit of the infinitely differentiable solutions for smooth approximations of the data of the corresponding problem with zero boundary conditions. These solutions are represented in the form of the Taylor expansion with respect to t. The suggested method can be considered as an alternative to numerical methods of solution of parabolic and hyperbolic equations.

Key words: Parabolic equation, hyperbolic equation, smooth solution, regular solution, Taylor expansion.

1 Introduction

Initial-boundary value (mixed) problems for parabolic and hyperbolic equations have since long ago led to a great number of works; see e.g. the monographs [21, 22, 28, 41, 46] and the references therein.

This paper is devoted to construction of infinitely differentiable solutions to parabolic and hyperbolic equations, and its applications to construction of regular and weak solutions to initial-boundary problems for these equations.

We consider problems with regular coefficients, right-hand sides, and boundary data. It should noted that there is an extensive literature that makes clear what happens for parabolic, hyperbolic, and also elliptic equations in the case where the coefficients, the right-hand side

and boundary data are not regular enough. For this, see the following recent works, which also contain references to previous investigations: [1, 3, 4, 6, 8, 11–17, 19, 20, 25–27, 29–37, 44].

It is well known that, for the existence of a smooth solution to parabolic or hyperbolic equation, the compatibility condition of an order $k \in \mathbb{N}$, corresponding to the smoothness of the solution to the problem, should be satisfied.

The compatibility condition of order k means that the functions $\frac{\partial^i u}{\partial t^i}\big|_{t=0}$, $i = 0, 1, 2, \ldots, k$ (u being the solution, t time), which are determined from the equation, initial data, and the right-hand side, should be equal on the boundary to $\frac{\partial^i u_b}{\partial t^i}\big|_{t=0}$, $i = 0, 1, 2, \ldots, k$, where u_b is the given function of values of the solution on the boundary. In the case where the solution is infinitely differentiable, one has $k = \infty$.

We consider problems in a bounded domain Ω in $\mathbb{R}^n$ with a boundary S of the C^∞ class on the time interval $(0, T)$, $T < \infty$.

We suppose that the coefficients of the equation, the right-hand side, and the initial data are infinitely differentiable, and furthermore the coefficients of the equation and the right–hand side are given in the form of the Taylor expansion with respect to time t with the origin at the point $t = 0$ and with coefficients depending on x, where x is a point in the space. Then the solution to the problem under consideration is informally given in the form of the Taylor expansion with respect to t in which coefficients depend on x, i.e.,

$$u(x, t) = \sum_{i=0}^{\infty} \frac{1}{i!} \frac{\partial^i u}{\partial t^i}(x, 0) t^i. \tag{1.1}$$

The coefficients $\frac{\partial^i u}{\partial t^i}(\cdot, 0)$ are determined by recurrence relations, more exactly, they are determined by the derivatives with respect to time t at $t = 0$ of the right-hand side f, the coefficients of the equation, and by the initial data u_0 for a parabolic equation and u_0, u_1 for a hyperbolic equation.

We prove converges of the series (1.1) in the space $C^\infty(\overline{Q})$, $Q = \Omega \times (0, T)$ by using the existence of an infinitely differentiable solution to the problem. So that, the value of the solution u on the boundary $u\big|_{S \times (0,T)} = u_b$ is uniquely determined by f and u_0 for a parabolic equation, and by f, u_0 and u_1 for a hyperbolic equation.

This peculiarity is for the first time shown in our work. In the usual, accepted approach, one prescribes for parabolic and hyperbolic equations a right-hand side, initial, and boundary conditions.

For the zero Dirichlet boundary condition, we assume that u_0 and u_1 are elements of $\mathcal{D}(\Omega)$ and $f \in C^\infty([0, T]; \mathcal{D}(\Omega))$. Then the compatibility condition of order $k = \infty$ is satisfied, and the solution to parabolic and hyperbolic equations can be represented in the form of (1.1).

It is known that the space $C^\infty(\overline{Q})$ is dense both in $W_q^l(Q)$ and in the space $(W_g^l(Q))^*$, $1/q + 1/g = 1$, that is the dual of $W_q^l(Q)$ for any $l \in \mathbb{N}$, $q \geq 2$. By the corollary to the Weierstrass–Stone theorem, the set of products of polynomials with respect to x and polynomials with respect to t is dense in $C^\infty(\overline{Q})$. Therefore, the set of functions that are represented in the form of the Taylor expansion with respect to t with coefficients

which are elements of the space $C^\infty(\overline{\Omega})$, is dense in $C^\infty(\overline{Q})$, in $W_q^l(Q)$, and in $(W_g^l(Q))^*$.

Because of these properties, one can approximate smooth and non-smooth data of the problem and the coefficients of equation by corresponding infinitely differentiable functions with an arbitrary accuracy.

We apply the Taylor representation (1.1) to construction of regular and weak solutions to parabolic and hyperbolic equations for which we prescribe the right-hand side, initial, and boundary conditions. We consider well-posed parabolic and hyperbolic problems for which the solution depends continuously on the data of the problem. The problems with inhomogeneous boundary conditions are reduced to problems with homogeneous boundary conditions. The data of these problems are approximated by corresponding infinitely differentiable functions for which the compatibility condition of order $k = \infty$ is satisfied. The solution to the problem with homogeneous boundary condition is constructed in the form (1.1). The solution to the problem with non-smooth data is determined as a limit of solutions for smooth approximated data.

The convergence of the Taylor series in the corresponding spaces is proved on the basis of the existence result for corresponding data.

Numerical solution of a parabolic problem with large convection, when one of the coefficients of the equation by the derivative with respect to some x_i is large for the norm of $L^\infty(Q)$, is a very difficult problem. There are many publications dealing with these problems. Many methods s where developed for numerical solution of such problems, see e.g. [2, 7, 18]. However, for significantly large convection, this problem is practically not solved.

The method proposed in this paper permits one to construct exact solutions to such problems for infinitely differentiable approximations of the right-hand side f and initial data u_0. Moreover, if an approximation of f is represented in the form of a finite sum of terms in the Taylor expansion in t with coefficients depending on x, then the exact solution for this approximation of f is also represented in the form of a finite sum of the Taylor expansion. The exact solution to the problem for given data is the limit of solutions for smooth approximations of f and u_0.

Thus, the suggested method of construction of solutions to parabolic and hyperbolic equations is an alternative to methods of numerical solution of parabolic and hyperbolic equations.

Below in Section 2, we consider problems for linear and nonlinear parabolic equations. Regular solutions to these equations with homogeneous and nonhomogeneous boundary conditions are constructed. In the case of a nonhomogeneous boundary condition, the solution is represented as a sum of a function satisfying the boundary condition and a limit of solutions to the this problem with zero boundary condition for infinitely differentiable data. These solutions are represented in the form (1.1)

In much the same way, we construct regular solutions to a system of parabolic equations in Section 3.

In Section 4, we consider an initial boundary value problem for a system of hyperbolic equations for homogeneous and nonhomogeneous boundary conditions. Solutions

3

to these problems are constructed.

In Section 5, we formulate a nonlinear problem on vibration of an orthotropic plate in a viscous medium. We show that there exists a unique solution to this problem, and this solution is obtained as a limit of solutions u^n to this problem for corresponding approximations of the data of the problem; the functions u^n are computed in the form of Taylor expansion.

In Section 6, we consider a 3-dimensional problem for Maxwell equations and a problem on diffraction of electromagnetic wave by a superconductor, i.e., a slotted antenna's problem. Solutions to these problems are constructed.

The authors are grateful to the anonymous referee for suggesting the related bibliography.

2 Parabolic equations

2.1 Linear problem and Taylor expansion

Let Ω be a bounded domain in $\mathbb{R}^n$ with a boundary S of the class C^∞. Let $Q = \Omega \times (0, T)$, where $T \in (0, \infty)$. Consider the problem: Find u such that

$$\frac{\partial u}{\partial t} - a_{ij}(x, t) \frac{\partial^2 u}{\partial x_i \partial x_j} + a_i(x, t) \frac{\partial u}{\partial x_i} + a(x, t)u = f \quad \text{in } Q, \tag{2.1}$$

$$u|_{t=0} = u_0 \quad \text{in } \Omega, \quad u(\cdot, 0)|_S = u_0|_S. \tag{2.2}$$

Here and below the Einstein convention on summation over repeated index is applied. As seen from (2.2), we prescribe the value of the function u on the boundary at the point $t = 0$ only.

Since the boundary S is of the class C^∞, we can assume that the coefficients of equation (2.1) and the right-hand side f are given in a bounded domain $Q_1 = \Omega_1 \times (0, T)$, where $\Omega_1 \supset \overline{\Omega}$, and u_0 is prescribed in Ω_1, see [40], Theorem 9.1, Chapter 1.

We denote the space of infinitely differentiable functions with support in Ω_1 by $\mathcal{D}(\Omega_1)$, and the space of infinitely differentiable functions on $\Omega_1 \times [0, T]$ with support in Ω_1 for each $t \in [0, T]$ by $C^\infty([0, T]; \mathcal{D}(\Omega_1))$.

Topologies in both $\mathcal{D}(\Omega_1))$ and $C^\infty([0, T]; \mathcal{D}(\Omega_1))$ are defined by the families of corresponding seminorms.

We assume that

$$(f, u_0) \in U, \tag{2.3}$$

where

$$U = \Big\{ (f, u_0) \mid f \in C^\infty([0, T]; \mathcal{D}(\Omega_1)), \ f(x, t) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\partial^k f}{\partial t^k}(x, 0) t^k,$$

$$(x, t) \in \overline{\Omega}_1 \times [0, T] = \overline{Q}_1, \ u_0 \in \mathcal{D}(\Omega_1) \Big\}, \tag{2.4}$$

$$a_{ij} \in C^\infty(\overline{Q}_1), \quad a_{ij}(x,t) = \sum_{k=0}^\infty \frac{1}{k!} \frac{\partial^k a_{ij}}{\partial t^k}(x,0)t^k, \quad i,j = 1,\ldots,n,$$

$$a_{ij}(x,t)\xi_i\xi_j \geq \mu\xi^2, \quad \mu > 0, \quad (x,t) \in Q_1, \quad \xi_i,\xi_j \in \mathbb{R}, \quad \xi^2 = \xi_1^2 + \cdots + \xi_n^2, \quad (2.5)$$

$$a_i \in C^\infty(\overline{Q}_1), \quad a_i(x,t) = \sum_{k=0}^\infty \frac{1}{k!} \frac{\partial^k a_i}{\partial t^k}(x,0)t^k,$$

$$a \in C^\infty(\overline{Q}_1), \quad a(x,t) = \sum_{k=0}^\infty \frac{1}{k!} \frac{\partial^k a}{\partial t^k}(x,0)t^k. \tag{2.6}$$

A topology on U is defined by the product of the topologies of $C^\infty([0,T]; \mathcal{D}(\Omega_1))$ and $\mathcal{D}(\Omega_1)$.

Denote

$$A\left(x,t,\frac{\partial}{\partial x}\right)u = a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} - a_i(x,t)\frac{\partial u}{\partial x_i} - a(x,t)u. \tag{2.7}$$

Then equation (2.1) can be represented in the form

$$\frac{\partial u}{\partial t} - A\left(x,t,\frac{\partial}{\partial x}\right)u = f \quad \text{in } Q. \tag{2.8}$$

We differentiate equation (2.8) in t $k-1$ times and set $t = 0$. This gives the following recurrence relation:

$$\frac{\partial^k u}{\partial t^k}(\cdot,0) = \left(\frac{\partial^{k-1}}{\partial t^{k-1}}\left(A\left(x,t,\frac{\partial}{\partial x}\right)u\right)\right)(\cdot,0) + \frac{\partial^{k-1} f}{\partial t^{k-1}}(\cdot,0)$$

$$= \sum_{j=0}^{k-1} C_{k-1}^j \left(\frac{\partial^j A}{\partial t^j}\left(x,t,\frac{\partial}{\partial x}\right)\right)(\cdot,0)\left(\frac{\partial^{k-1-j}}{\partial t^{k-1-j}}u\right)(\cdot,0) + \frac{\partial^{k-1} f}{\partial t^{k-1}}(\cdot,0), \quad k = 1,2,\ldots$$

$$\tag{2.9}$$

Here $u(\cdot,0) = u_0$, C_{k-1}^j are the binomial coefficients, $\frac{\partial^j A}{\partial t^j}(x,t,\frac{\partial}{\partial x})$ is the operator obtained from the operator A by differentiation of its coefficients in t j times.

A smooth solution u satisfies the condition

$$\frac{\partial^m u}{\partial t^m}(x,0) = \frac{\partial^m u_b}{\partial t^m}(x,0), \quad x \in S. \tag{2.10}$$

For $m = 0$, we get $u_0(x) = u_b(x,0)$, $x \in S$.

We say that the compatibility condition of order k is satisfied if (2.10) holds for $m = 0,1,2,\ldots,k$.

For infinitely differentiable solutions the compatibility condition of order $k = \infty$ is satisfied.

5

Theorem 2.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ with a boundary S of the class C^∞ and $T \in (0, \infty)$. Suppose that the conditions (2.3)–(2.6) are satisfied. Then there exists a unique solution to the problem (2.1), (2.2) such that $u \in C^\infty(\overline{Q})$, and this solution is represented in the form of a Taylor expansion*

$$u(x,t) = u_0(x) + \sum_{k=1}^{\infty} \frac{1}{k!} \frac{\partial^k u}{\partial t^k}(x,0)\, t^k, \quad (x,t) \in \overline{Q}. \tag{2.11}$$

The coefficients $\frac{\partial^k u}{\partial t^k}(\cdot, 0)$ are defined by the recurrence relation (2.9). Furthermore, the boundary condition function $u_b = u|_{S_T}$, $S_T = S \times [0,T]$, is determined as follows:

$$u_b(x,t) = u_0(x) + \sum_{k=1}^{\infty} \frac{1}{k!} \frac{\partial^k u}{\partial t^k}(x,0)\, t^k, \quad x \in S, \ t \in [0,T]. \tag{2.12}$$

The function $(f, u_0) \mapsto u$ defined by the solution to the problem (2.1), (2.2) is a continuous mapping of U into $C^\infty(\overline{Q})$.

Proof. We consider the problem: Find $\breve{u}$ satisfying

$$\frac{\partial \breve{u}}{\partial t} - A\left(x, t, \frac{\partial}{\partial x}\right)\breve{u} = f \ \text{ in } Q_1,$$

$$\breve{u}|_{t=0} = u_0 \ \text{ in } \Omega_1, \quad \breve{u}|_{S_{1T}} = 0, \tag{2.13}$$

where $S_{1T} = S_1 \times [0,T]$, S_1 is the boundary of Ω_1. By (2.4), S_1 is of the class C^∞. It follows from (2.3), (2.9) and (2.13) that the compatibility condition of any order $k \in \mathbb{N}$ is satisfied, and by [28], Theorem 5.2, Chapter IV and [46], Theorem 5.4, Chapter V, there exists a unique solution to the problem (2.13) such that $\breve{u} \in W_q^{2(k+1),k+1}(Q_1)$, $k \in \mathbb{N} = \{0, 1, 2, \dots\}$, $q \geq 2$. Therefore, $\breve{u} \in C^\infty(\overline{Q}_1)$.

The functions $\frac{\partial^k \breve{u}}{\partial t^k}(\cdot, 0)$ are defined by formula (2.9) in which Ω is replaced by Ω_1 and u by $\breve{u}$. (2.4) implies $\frac{\partial^k \breve{u}}{\partial t^k}(\cdot, 0) \in \mathcal{D}(\Omega_1)$, $k \in \mathbb{N}$.

Informally, the solution to the problem (2.13) is represented in the form of a Taylor expansion

$$\breve{u}(x,t) = u_0(x) + \sum_{k=1}^{\infty} \frac{1}{k!} \frac{\partial^k \breve{u}}{\partial t^k}(x,0)t^k, \quad (x,t) \in Q_1. \tag{2.14}$$

The function $\breve{u}$ defined by (2.14) represents a smooth solution to the problem (2.13) for all points $t \in [0,T]$ such that the series (2.14) converges at t in $\mathcal{D}(\Omega_1)$.

Let us prove this. Denote

$$\breve{u}_m(x,t) = u_0(x) + \sum_{k=1}^{m} \frac{1}{k!} \frac{\partial^k \breve{u}}{\partial t^k}(\cdot, 0)\, t^k. \tag{2.15}$$

(2.9) and (2.15) imply that the function $\check{u}_m$ is a solution to the problem

$$\frac{\partial \check{u}_m}{\partial t} - A\left(x, t, \frac{\partial}{\partial x}\right) \check{u}_{m-1} = f_{m-1} \quad \text{in } Q_1,$$

$$\check{u}_m|_{t=0} = u_0 \quad \text{in } \Omega_1, \quad \check{u}_m|_{S_{1T}} = 0, \tag{2.16}$$

where

$$f_{m-1}(x, t) = \sum_{k=0}^{m-1} \frac{1}{k!} \frac{\partial^k f}{\partial t^k}(x, 0) \, t^k, \quad (x, t) \in Q_1. \tag{2.17}$$

It follows from (2.4) that

$$f_m \to f \quad \text{in } C^\infty([0, T]; \mathcal{D}(\Omega_1)). \tag{2.18}$$

It is known that the solution of a parabolic problem depends continuously on the data of the problem f, u_b, u_0, see [28], Theorem 5.2, Chapter IV and [46], Theorem 5.4, Chapter V. Because of this, (2.13) and (2.16) yield

$$\|\check{u} - \check{u}_m\|_{W_q^{2(k+1),k+1}(Q_1)} \leq c \|f_m - f\|_{W_q^{2k,k}(Q_1)}, \quad k \in \mathbb{N}, \ q \geq 2. \tag{2.19}$$

Therefore

$$\check{u}_m \to \check{u} \quad \text{in } W_q^{2(l+1),l+1}(Q_1), \quad l \in \mathbb{N}, \ q \geq 2,$$

and $\check{u}_m \to \check{u}$ in $C^\infty(\overline{Q}_1)$. The function $u = \check{u}|_Q$ is a solution to the problem (2.1), (2.2), and it is determined by (2.9) and (2.11). This solution is unique.

It follows from [28] and [46] that the function $(f, u_0) \mapsto \check{u}$ defined by the solution to the problem (2.13), is a continuous mapping of U into $W_q^{2(l+1),l+1}(Q_1)$ for any $l \in \mathbb{N}$, $q \geq 2$. Therefore, the function $(f, u_0) \mapsto u$, where u is the solution to the problem (2.1), (2.2), is a continuous mapping of U into $C^\infty(\overline{Q})$. $\qquad \square$

Remark 2.1. It is customary to prescribe for a parabolic equation the functions f, u_0 and the boundary condition u_b. However, it follows from Theorem 2.1 that under the conditions of this theorem, one prescribes only f and u_0. In this case, there exists a unique solution to the problem (2.1), (2.2) that is represented in the form (2.11) and the function u_b is determined by f and u_0.

Corollary 2.1. *Let f be a function in Q that is represented in the form of the Taylor expansion in t with coefficients depending on x. Let u be a solution to the problem (2.1) such that $u \in C^\infty(\overline{Q})$. Then, for any fixed point $x \in \overline{\Omega}$, the partial function $t \mapsto u(x, t)$ is analytical, and u is represented in the form (2.11).*

Proof. Indeed, in this case, $f \in C^\infty(\overline{Q})$, $u_0 \in C^\infty(\overline{\Omega})$, the compatibility condition of order infinity is satisfied, and it follows from the Theorem 2.1 that u represented in the form (2.11). $\qquad \square$

Consider the following problem on existence of an infinitely differentiable solution to a parabolic problem with given boundary and initial conditions:

$$\frac{\partial u}{\partial t} - A\left(x, t, \frac{\partial}{\partial x}\right) u = f \quad \text{in } Q. \tag{2.20}$$

$$u|_{S_T} = u_b, \quad u|_{t=0} = u_0. \tag{2.21}$$

Here $A\left(x, t, \frac{\partial}{\partial x}\right)$ is defined by (2.7).

We define the following spaces:

$$X = \{u \mid u \in C^\infty(\overline{Q}_1), \text{ supp } u(\cdot, t) \subset \Omega_1, \ t \in [0, T],$$

$$u(x, t) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\partial^k u}{\partial t^k}(x, 0) \, t^k, \ (x, t) \in \overline{Q}_1\},$$

$$X_0 = \{u \mid u \in X, \ u\big|_{S_T} = 0, \ u(\cdot, 0) = 0\},$$

$$Z = \{(v, w) \mid v = h(\cdot, 0)|_{\overline{\Omega}}, \ w = h|_{S_T}, \ h \in X\}. \tag{2.22}$$

We define an operator $\gamma : X \to Z$ by

$$\gamma(u) = (u(\cdot, 0)|_{\overline{\Omega}}, \, u|_{S_T}).$$

Note that X_0 is the kernel of the operator γ.

Let X/X_0 be the factor space. If u^1 and u^2 are elements of X such that $u^1 - u^2 \in X_0$, then u^1 and u^2 belong to the same class in X/X_0, say $\overline{u}$. We say that $\overline{u}$ is of class $(u_0, u_b) \in Z$ if $\gamma(u) = (u_0, u_b)$ for all $u \in \overline{u}$. The result bellow follows from Theorem 2.1.

Corollary 2.2. *Let $(u_0, u_b) \in Z$ and let $\overline{u}$ be the class (u_0, u_b) from X/X_0. Then any function $u \in \overline{u}\big|_{\overline{Q}}$ is the solution to the problem (2.20), (2.21) for u_0, u_b, and f that is determined as follows:*

$$f(x, t) = \sum_{k=1}^{\infty} \frac{1}{(k-1)!} \frac{\partial^{k-1} f}{\partial t^{k-1}}(x, 0) \, t^{k-1}, \quad (x, t) \in \overline{Q}, \tag{2.23}$$

where

$$\frac{\partial^{k-1} f}{\partial t^{k-1}}(x, 0) = \frac{\partial^k u}{\partial t^k}(x, 0) - \sum_{j=0}^{k-1} C_{k-1}^j \left(\frac{\partial^j A}{\partial t^j}\left(x, t, \frac{\partial}{\partial x}\right)\right)(x, 0)$$

$$\times \left(\frac{\partial^{k-1-j} u}{\partial t^{k-1-j}}\right)(x, 0), \quad x \in \Omega. \tag{2.24}$$

Define the following set:

$$U_1 = \left\{ (f, u_0, u_b) \mid f \in C^\infty([0, T]; \mathcal{D}(\Omega)), \ f(x, t) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\partial^k f}{\partial t^k}(x, 0) t^k, \right.$$

$$\left. (x, t) \in \overline{\Omega} \times [0, T] = \overline{Q}, \ u_0 \in \mathcal{D}(\Omega), \ u_b = 0 \right\}. \tag{2.25}$$

We consider the problem: Given $(f, u_0, u_b) \in U_1$, find u such that

$$u \in C^\infty(\overline{Q}),$$
$$\frac{\partial u}{\partial t} - A\left(x, t, \frac{\partial u}{\partial t}\right) u = f,$$
$$u|_{t=0} = u_0, \quad u|_{S_T} = u_b. \tag{2.26}$$

The following result follows from the proof of Theorem 2.1:

Corollary 2.3. *Let Ω be a bounded domain in $\mathbb{R}^n$ with a boundary S of class C^∞. Suppose that the conditions (2.5), (2.6) are satisfied, and $(f, u_0, u_b) \in U_1$. Then there exists a unique solution to the problem (2.26) that is represented in form (2.11), and the function $(f, u_0, 0) \mapsto u$ is a continuous mapping of U_1 into $C^\infty([0, T]; \mathcal{D}(\Omega))$.*

2.2 Solution of initial-boundary value problems in Sobolev space

We consider the problem (2.20), (2.21) in which we are given f, u_0, u_b. We suppose that

$$f \in L^2(Q), \quad u_0 \in H^1(\Omega), \quad u_b \in H^{\frac{3}{2}, \frac{3}{4}}(S_T), \quad u_0(x) = u_b(x, 0) \ x \in S. \tag{2.27}$$

In this case, the compatibility condition of order zero is satisfied.

For the sake of simplicity, we assume that the coefficients of the equation (2.20) are elements of $C^\infty(\overline{Q})$, and they are represented in the form of Taylor expansion in t with coefficients depending on x, i.e., (2.5), (2.6) hold, and the boundary S is of the class C^∞.

It follows from the corollary to the Stone–Weierstrass theorem that the set of tensor products of polynomials in x and polynomials in t is dense in $C^\infty(\overline{Q})$. Therefore, the solution to the equation with non-smooth coefficients is obtained as the limit of solutions of equations with smooth coefficients as above.

By analogy, a non-smooth boundary S can be approximated by boundaries of the class C^∞. In this case, solutions for smooth boundaries converge to the solution for non-smooth boundary in the corresponding space, see [42].

It follows from the known results, see e.g. [41], Chapter 4, Theorems 2.3 and 6.2, [46], Chapter V, Theorem 5.4 that, under the above conditions, there exists a unique solution to the problem (2.20), (2.21) such that

$$u \in H^{2,1}(Q). \tag{2.28}$$

We define the following function:

$$w(x,t) = \begin{cases} u_b(Px,t)e^{1-\frac{a^2}{a^2-(x-Px)^2}} & \text{if } |x - Px| < a, \\ 0 & \text{if } |x - Px| \geq a. \end{cases} \tag{2.29}$$

Here $x \in \Omega$, $t \in (0,T)$, P is the operator of projection of points of Ω onto S, a is a small positive constant. Note that $w \in H^{2,1}(Q)$.

Let

$$\tilde{u} = u - w. \tag{2.30}$$

(2.28), (2.29) imply

$$\tilde{u} \in H^{2,1}(Q), \quad \tilde{u}|_{S_T} = 0, \quad \tilde{u}|_{t=0} = u_0 - w|_{t=0} \in H_0^1(\Omega). \tag{2.31}$$

The function $\tilde{u}$ is the solution to the following problem:

$$\frac{\partial \tilde{u}}{\partial t} - A\left(x,t,\frac{\partial}{\partial x}\right)\tilde{u} = \tilde{f},$$

$$\tilde{u}|_{S_T} = 0, \quad \tilde{u}|_{t=0} = u_0 - w|_{t=0} \in H_0^1(\Omega), \tag{2.32}$$

where

$$\tilde{f} = f - \frac{\partial w}{\partial t} + A\left(x,t,\frac{\partial}{\partial x}\right)w \in L^2(Q). \tag{2.33}$$

(2.27), (2.30), (2.32) imply

$$\tilde{u}(x,0) = u_b(x,0) - w(x,0) = 0, \quad x \in S. \tag{2.34}$$

Therefore, the compatibility condition of order zero is satisfied, and there exists a unique solution to the problem (2.32) such that $\tilde{u} \in H^{2,1}(Q)$, $u|_{S_T} = 0$.

Let $\{\tilde{f}_m, \tilde{u}_{0m}\}$ be a sequence such that

$$\tilde{f}_m \in C^\infty([0,T],\mathcal{D}(\Omega)), \quad \tilde{f}_m(x,t) = \sum_{k=0}^\infty \frac{1}{k!}\frac{\partial^k \tilde{f}_m}{\partial t^k}(x,0)t^k, \quad \tilde{f}_m \to \tilde{f} \text{ in } L^2(Q),$$

$$\tilde{u}_{0m} \in \mathcal{D}(\Omega), \quad \tilde{u}_{0m} \to u_0 - w|_{t=0} \text{ in } H_0^1(\Omega). \tag{2.35}$$

Consider the problem: Find $\tilde{u}_m$ such that

$$\tilde{u}_m \in C^\infty([0,T];\mathcal{D}(\Omega)),$$

10

$$\frac{\partial \tilde{u}_m}{\partial t} - A\left(x, t, \frac{\partial}{\partial x}\right)\tilde{u}_m = \tilde{f}_m,$$
$$\tilde{u}_m(\cdot, 0) = \tilde{u}_{0m}. \tag{2.36}$$

We can assume that the functions $\tilde{f}_m$ and $\tilde{u}_{0m}$ are extended by zero to domains Q_1 and Ω_1 so that $(\tilde{f}_m, \tilde{u}_{0m}) \in U$, see (2.4). Then by Theorem 2.1, there exists a unique solution to our problem in Q_1, and it is determined by (2.14), where the functions $\check{u}$ and $\check{u}_0$ are replaced by $\tilde{u}_m$ and $\tilde{u}_{0m}$. Thus, the solution to the problem (2.36) belongs to $C^\infty(\overline{Q})$ and it is represented in the form

$$\tilde{u}_m(x, t) = \tilde{u}_{0m}(x) + \sum_{k=1}^\infty \frac{1}{k!}\frac{\partial^k \tilde{u}_m}{\partial t^k}(x, 0)t^k, \quad (x, t) \in Q, \tag{2.37}$$

where $\frac{\partial^k \tilde{u}_m}{\partial t^k}$ are determined by (2.9).

(2.32), (2.36) and [46], Theorem 5.4, Chapter V imply

$$\|\tilde{u}_m - \tilde{u}\|_{H^{2,1}(Q)} \leq c(\|\tilde{f}_m - \tilde{f}\|_{L^2(Q)} + \|\tilde{u}_{0m} - \tilde{u}_0 + w|_{t=0}\|_{H_0^1(\Omega)}. \tag{2.38}$$

Now, (2.35) yields

$$\tilde{u}_m \to \tilde{u} \text{ in } H^{2,1}(Q). \tag{2.39}$$

Thus, we have proved the following result:

Theorem 2.2. *Let Ω be a bounded domain in $\mathbb{R}^n$ with a boundary S of the class C^∞, $T \in (0, \infty)$, and let the conditions (2.5), (2.6) be satisfied. Let also the conditions (2.27) be satisfied. Then, there exists a unique solution to the problem (2.20), (2.21) that satisfies (2.28), and it is represented in the form $u = \tilde{u} + w$, where w is given by (2.29), and $\tilde{u}$ is determined by (2.39).*

Remark 2.2. When applying Theorem 2.2 in practice, one can compute the solution to the problem (2.36) directly by (2.37) and (2.9), without the extension that was used to prove converges of the series.

Remark 2.3. Theorem 2.2 also holds in the case where $u_b = 0$. Indeed, we just need to take $w = 0$ in the above computations.

Remark 2.4. In practical applications the data of the problem are usually not accurately given, often they are determined by intuition, or even are plucked out of thin air. Therefore, in such a case, it makes no sense to solve the problem (2.36) for a series of functions $\{\tilde{f}_m, \tilde{u}_{0m}\}$ which satisfy the condition (2.35). It is sufficient to solve problem (2.36) for one or two pairs $\tilde{f}_m, \tilde{u}_{0m}$ which are close to $\tilde{f}$ and $\tilde{u}_0$ with not a high precision. Moreover, here $\tilde{f}_m$ can be taken in a form of a finite sum of the Taylor expansion, which is suitable for a given T. In this case, if $\tilde{f}_m = \tilde{f}_{ml}$, where

$$\tilde{f}_{ml}(x, t) = \sum_{k=0}^l \frac{1}{k!}\frac{\partial^k \tilde{f}_m}{\partial t^k}(x, 0)t^k, \tag{2.40}$$

then exact solution to the problem (2.36) is the function $\tilde{u}_m(x,t) = \tilde{u}_{m(l+1)}(x,t)$ where

$$\tilde{u}_{m(l+1)}(x,t) = \tilde{u}_{0m}(x) + \sum_{k=1}^{l+1} \frac{1}{k!} \frac{\partial^k \tilde{u}_m}{\partial t^k}(x,0) t^k, \quad (x,t) \in Q, \tag{2.41}$$

see (2.15), (2.16), (2.17).

Remark 2.5. Numerical solution of the problem (2.20), (2.21) in the case of a large convection, when the norm of one of the coefficients a_i of the operator A is large in $L^\infty(Q)$ is a very hard problem. Our method permits one to construct the exact solution to the problem (2.36) in the form (2.37). Moreover, if $\tilde{f}_m = \tilde{f}_{ml}$ is represented in the form (2.40), then the solution to the problem (2.36) is represented in the form (2.41).

2.3   Nonlinear parabolic equation

We consider the following problem:

$$\frac{\partial u}{\partial t} - A\left(x,t,\frac{\partial}{\partial x}\right)u + \left(b_0(x,t)u^2 + b_i(x,t)\frac{\partial u}{\partial x_i}u\right.$$
$$\left. + b_{ij}(x,t)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\right)\lambda = f \quad \text{in } Q, \quad i,j=1,2,\dots,n, \tag{2.42}$$

$$u|_{t=0} = u_0, \ u|_{S_T} = 0. \tag{2.43}$$

As before, $Q = \Omega \times (0,T)$, Ω is a bounded domain in $\mathbb{R}^n$ with a boundary S of the class C^∞, $T < \infty$.

We suppose that $A\left(x,t,\frac{\partial}{\partial x}\right)$ is defined by (2.7) and the conditions (2.5), (2.6) are satisfied. Furthermore,

$$f \in C^\infty([0,T]; \mathcal{D}(\Omega)), \ f(x,t) = \sum_{k=0}^\infty \frac{1}{k!} \frac{\partial^k f}{\partial t^k}(x,0) t^k, \ u_0 \in \mathcal{D}(\Omega),$$

$$b_0 \in C^\infty(\overline{Q}), \ b_0(x,t) = \sum_{k=0}^\infty \frac{1}{k!} \frac{\partial^k b_0}{\partial t^k}(x,0) t^k,$$

$$b_i \in C^\infty(\overline{Q}), \ b_i(x,t) = \sum_{k=0}^\infty \frac{1}{k!} \frac{\partial^k b_i}{\partial t^k}(x,0) t^k, \ i=1,2,\dots,n,$$

$$b_{ij} \in C^\infty(\overline{Q}), \ b_{ij}(x,t) = \sum_{k=0}^\infty \frac{1}{k!} \frac{\partial^k b_{ij}}{\partial t^k}(x,0) t^k, \ i,j=1,2,\dots,n, \tag{2.44}$$

and λ is a small positive parameter, $\lambda \in (0,\check{\lambda}]$, $\check{\lambda} > 0$. We define the following mapping:

$$M(u) = b_0 u^2 + b_i \frac{\partial u}{\partial x_i} u + b_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}. \tag{2.45}$$

12

Equation (2.42) can be represented in the form

$$\frac{\partial u}{\partial t} - A\left(x, t, \frac{\partial}{\partial x}\right) u + \lambda M(u) = f. \tag{2.46}$$

We differentiate equation (2.46) in t $k-1$ times and set $t=0$. We obtain the relations

$$\frac{\partial^k u}{\partial t^k}(\cdot, 0) = \left(\frac{\partial^{k-1}}{\partial t^{k-1}}\left(A\left(x, t, \frac{\partial}{\partial x}\right) u\right)\right)(\cdot, 0) - \lambda\left(\frac{\partial^{k-1}}{\partial t^{k-1}} M(u)\right)(\cdot, 0) + \frac{\partial^{k-1}}{\partial t^{k-1}} f(\cdot, 0),$$

$$k = 1, 2, \dots, \tag{2.47}$$

where

$$\frac{\partial^{k-1}}{\partial t^{k-1}} M(u) = \sum_{l=0}^{k-1} C_{k-1}^l \left(\frac{\partial^l b_0}{\partial t^l} \frac{\partial^{k-1-l} u^2}{\partial t^{k-1-l}} + \frac{\partial^l b_i}{\partial t^l} \frac{\partial^{k-1-l}}{\partial t^{k-1-l}}\left(\frac{\partial u}{\partial x_i} u\right)\right)$$

$$+ \frac{\partial^l b_{ij}}{\partial t^l} \frac{\partial^{k-1-l}}{\partial t^{k-1-l}}\left(\frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}\right)\right). \tag{2.48}$$

Theorem 2.3. *Let Ω be a bounded domain in $\mathbb{R}^n$ with a boundary S of the class C^∞. Suppose that the conditions (2.5), (2.6), (2.44) are satisfied. Then for any l, q such that $l \in \mathbb{N}$, $(n+2)/2q < l$, $q \geq 2$, there is $\lambda_0 > 0$ such that, for any $\lambda \in (0, \lambda_0)$, there exists a unique solution $u = u_\lambda$ to the problem (2.42), (2.43) such that $u_\lambda \in W_q^{2l+2,l+1}(Q)$ and*

$$u_\lambda(x, t) = u_0(x) + \sum_{k=1}^{\infty} \frac{1}{k!} \frac{\partial^k u}{\partial t^k}(x, 0)\, t^k, \quad (x, t) \in \overline{Q}, \tag{2.49}$$

where $\frac{\partial^k u}{\partial t^k}(x, 0)$ is determined by (2.47) and (2.48). Furthermore, $\lambda \mapsto u_\lambda$ is s continuous mapping of $(0, \lambda_0)$ into $W_q^{2l+2,l+1}(Q)$.

Proof. We consider the problem: Find u_λ satisfying

$$\frac{\partial u_\lambda}{\partial t} - A\left(x, t, \frac{\partial}{\partial x}\right) u_\lambda + \lambda M(u_\lambda) = f \quad \text{in } Q, \tag{2.50}$$

$$u_\lambda|_{t=0} = u_0 \quad \text{in } \Omega, \quad u_\lambda|_{S_T} = 0. \tag{2.51}$$

Denote

$$W_{q,0}^{2l+2,l+1}(Q) = \left\{w \mid w \in W_q^{2l+2,l+1}(Q),\, w(\cdot, t) \in \overset{\circ}{W}_q^{2l+2-\frac{2}{q}}(\Omega) \text{ a.e. in } (0, T),\right.$$

$$\left. l > \frac{n+2}{2q},\, q \geq 2\right\},$$

where $\overset{\circ}{W}_q^{2l+2-\frac{2}{q}}(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $W_q^{2l+2-\frac{2}{q}}(\Omega)$.

The function M maps the space $W_{q,0}^{2l+2,l+1}(Q)$ into $W_{q,0}^{2l+1,l+\frac{1}{2}}(Q)$. Let u, h be elements of $W_{q,0}^{2l+2,l+1}(Q)$. We have

$$\lim_{\gamma \to 0} \frac{M(u+\gamma h) - M(u)}{\gamma} = 2b_0 uh + b_i \left(\frac{\partial u}{\partial x_i} h + u \frac{\partial h}{\partial x_i} \right)$$
$$+ b_{ij} \left(\frac{\partial u}{\partial x_i} \frac{\partial h}{\partial x_j} + \frac{\partial u}{\partial x_j} \frac{\partial h}{\partial x_i} \right) = M'(u)\, h.$$

It is easy to see that

$$\| M(u+h) - M(u) - M'(u)h \|_{W_{q,0}^{2l+1,l+\frac{1}{2}}(Q)} \le c \| h \|_{W_{q,0}^{2l+2,l+1}(Q)}^2.$$

Therefore, the operator M is a Fréchet continuously differentiable mapping of $W_{q,0}^{2l+2,l+1}(Q)$ into $W_{q,0}^{2l+1,l+\frac{1}{2}}(Q)$.

By Corollary 2.4, for $\lambda = 0$, there exists a unique solution to the problem (2.50), (2.51) that belongs to $C^\infty([0,T]; \mathcal{D}(\Omega))$, and it is determined by (2.11).

By applying the implicit function theorem, see e.g. [45], Theorem 25, Chapter III, we obtain that for any l, q such that $\frac{n+2}{2q} < l$, $q \ge 2$, $l \in \mathbb{N}$, there is $\lambda_0 > 0$ such that for any $\lambda \in (0, \lambda_0)$, there exists a unique solution u_λ to the problem (2.50), (2.51) such that $u_\lambda \in W_{q,0}^{2l+2,l+1}(Q)$, and the function $\lambda \mapsto u_\lambda$ is a continuous mapping of $(0, \lambda_0)$ into $W_{q,0}^{2l+2,l+1}(Q)$.

Informally, the solution to the problem (2.42), (2.43) is represented in the form (2.49). Define u_m as follows:

$$u_m(x,t) = u_0(x) + \sum_{k=1}^{m} \frac{1}{k!} \frac{\partial^k u}{\partial t^k}(x,0)\, t^k, \quad (x,t) \in \overline{Q}. \tag{2.52}$$

(2.47) and (2.52) imply that

$$\frac{\partial u_m}{\partial t}(x,t) - A\left(x,t,\frac{\partial}{\partial x}\right) u_{m-1}(x,t) + \lambda \sum_{k=0}^{m-1} \frac{1}{k!} \left(\frac{\partial^k}{\partial t^k}(M(u)) \right)(x,0)\, t^k = f_{m-1}(x,t), \tag{2.53}$$

where

$$f_{m-1}(x,t) = \sum_{k=0}^{m-1} \frac{1}{k!} \frac{\partial^k f}{\partial t^k}(x,0) t^k. \tag{2.54}$$

It follows from (2.44) that

$$f_{m-1} \to f \text{ in } C^\infty([0,T]; \mathcal{D}(\Omega)). \tag{2.55}$$

By (2.44) and (2.47), we have $u_m \in C^\infty([0,T]; \mathcal{D}(\Omega))$.

We apply the implicit function theorem to the case where λ is from a small vicinity of zero in the set of nonnegative numbers, and the right-hand side of (2.42) belongs to a small vicinity of f in $W_{q,0}^{2l,l}(Q)$. Then (2.55) yields

$$u_m \to u_\lambda \text{ in } W_{q,0}^{2l+2,l+1}(Q). \tag{2.56}$$

Therefore, the series (2.49) converges in $W_{q,0}^{2l+2,l+1}(Q)$, and gives the solution to the problem (2.42), (2.43). $\qquad\square$

We remark that, in the case $f \in L^2(Q)$ and $u_0 \in H_0^1(\Omega)$, the solution to the problem (2.42), (2.43) can be defined as the limit of solutions to this problem for $f = \tilde{f}_m$ and $u_0 = \tilde{u}_{0m}$ that are determined by (2.35) with $w = 0$.

2.4 Construction of functions of $\mathcal{D}(\Omega)$

Let Ω_2 be a domain in $\mathbb{R}^n$ such that $\overline{\Omega}_2 \subset \Omega$, and S_2 be the boundary of Ω_2. We suppose that

$$d(x, S) = 2a \text{ for any } x \in S_2, \tag{2.57}$$

where

$$d(x, S) = \min \left(\sum_{i=1}^n (x_i - y_i)^2 \right)^{\frac{1}{2}}, \ y = (y_1, \ldots, y_n) \in S, \tag{2.58}$$

and a is a small positive constant.

Define the following function:

$$g_a(x) = \begin{cases} 1, & \text{if } d(x, S_2) > 0, \ x \in \Omega_2, \\ e^{1 - \frac{a^2}{a^2 - (d(x,S_2))^2}}, & \text{if } d(x, S_2) \in [0, a), \ x \in \Omega \setminus \Omega_2, \\ 0, & \text{if } d(x, S_2) \geq a, \ x \in \Omega \setminus \Omega_2. \end{cases} \tag{2.59}$$

The function g_a belongs to $\mathcal{D}(\Omega)$, and if $f \in C^\infty(\overline{\Omega})$, then $w = f \cdot g_a \in \mathcal{D}(\Omega)$, and

$$\text{the set}\{P_m \cdot g_a\}, \ m \in \mathbb{N}, \ a > 0 \text{ is dense in } H_0^l(\Omega), \ l \in \mathbb{N}, \tag{2.60}$$

where P_m is any polynomial in x such that the order of polynomial in $x_i, i = 1, \cdots, n$, does not exceed m.

In the general case, a smooth boundary S is defined by local cards, i.e., by local coordinate systems $(y_1^k, \cdots, y_n^k)$ and mappings F_k, $k = 1, \cdots, \beta$, such that

$$y_n^k = F_k(y_1^k, \ldots, y_{n-1}^k), \tag{2.61}$$

and by a corresponding partition of unity, see e.g. [40, 45, 46].

For a ball or a paraboloid, the boundary S is defined by

$$\omega_b(x) = \sum_{i=1}^{n} x_i^2 - c^2 = 0, \quad \omega_p = \sum_{i=1}^{n} \frac{x_i^2}{b_i^2} - c^2 = 0, \tag{2.62}$$

where b_i and c are positive constants.

Polyhedral domains are widely used in practical computations. For convex polyhedron, whose faces are defined by equations

$$f_k(x) = \sum_{i=1}^{n} a_{ik} x_i - c_k = 0, \quad k = 1, \ldots, m, \tag{2.63}$$

where a_{ik} and c_k are constants, the boundary S is given as follows:

$$\omega_{cp}(x) = \pm \prod_{k=1}^{m} f_k(x) = \pm \prod_{k=1}^{m} \sum_{i=1}^{n} a_{ik} x_i - c_k = 0, \tag{2.64}$$

where the sign is chosen so that $\omega_{cp}(x) > 0$ in Ω.

The domain Ω_2 of polyhedron is the polyhedron with the boundary S_2 that satisfies the condition (2.57). In this case, (2.60) holds.

The boundary of polyhedron is infinitely differentiable everywhere with exception of angular points, at which it is not differentiable. Nevertheless, in small vicinities of angular points this boundary can be regularized by convolution of the function F_k, see (2.61), with an infinitely differentiable function with a small support, in particular, with the bump function.

If the boundary of a polyhedron is not regularized, then the computation of the solution to the problem in exteriors of any small vicinities of angular points can be fulfilled.

For the case of non-convex polyhedron, one can identify the faces of the polyhedron with local cards, without using a partition of unity. That is, one assumes that f_k are the identity mappings of the sets

$$G_k = \{x \mid f_k(x) = 0, \ f_k(x) \in S\}$$

onto itself, and G_k are defined so that $\bigcup_{k=1}^{m} G_k = S$.

3   System of parabolic equations

Let us consider the following problem for a system of equations that are parabolic in the sense of Petrowski: Find $u = (u_1, u_2, \ldots, u_N)$ such that

$$\frac{\partial u_i}{\partial t} - \frac{\partial}{\partial x_r}\left(a_{ijrm}(x,t)\frac{\partial u_j}{\partial x_m}\right) + b_{jm}^i(x,t)\frac{\partial u_j}{\partial x_m} + g_j^i(x,t)u_j = f_i \quad \text{in } Q,$$

$$i, j = 1, \ldots, N, \quad r, m = 1, \ldots, n, \tag{3.1}$$

$$u|_{t=0} = u_0 \ \text{ in } \Omega, \ \ u|_{S_T} = u_b. \tag{3.2}$$

As before, $Q = \Omega \times (0, T)$, Ω is a bounded domain in $\mathbb{R}^n$ with a boundary S of the class C^∞, $T \in (0, \infty)$.

We suppose that

$$f = (f_1, \ldots, f_N) \in L^2(Q)^N, \ \ u_0 = (u_{01}, \ldots, u_{0N}) \in H^1(\Omega)^N,$$

$$u_b = (u_{b1}, \ldots, u_{bN}) \in H^{\frac{3}{2}, \frac{3}{4}}(S_T)^N, \ \ u_0(x) = u_b(x, 0), \ x \in S, \tag{3.3}$$

and

$$a_{ijrm} \in C^\infty(\overline{Q}_1), \ a_{ijrm}(x, t) = \sum_{k=0}^\infty \frac{1}{k!} \frac{\partial^k a_{ijrm}}{\partial t^k}(x, 0) \, t^k, \ (x, t) \in \overline{Q}_1,$$

$$a_{ijrm}(x, t) \xi_r \xi_m \nu_j \nu_i \geq \mu \sum_{r=1}^n \xi_r^2 \sum_{i=1}^N \nu_i^2, \ (x, t) \in \overline{Q}_1, \ \xi_r \in \mathbb{R}, \ \nu_i \in \mathbb{R}, \ \mu > 0, \tag{3.4}$$

$$b_{jm}^i \in C^\infty(\overline{Q}_1), \ b_{jm}^i(x, t) = \sum_{k=0}^\infty \frac{1}{k!} \frac{\partial^k b_{jm}^i}{\partial t^k}(x, 0) \, t^k, \ (x, t) \in \overline{Q}_1, \tag{3.5}$$

$$g_j^i \in C^\infty(\overline{Q}_1), \ g_j^i(x, t) = \sum_{k=0}^\infty \frac{1}{k!} \frac{\partial^k g_j^i}{\partial t^k}(x, 0) \, t^k, \ (x, t) \in \overline{Q}_1, \tag{3.6}$$

(3.3) yields that, the compatibility condition of order zero is satisfied. It follows from [46] that there exists a unique solution to the problem (3.1), (3.2) such that $u \in H^{2,1}(Q)^N$.

Informal differentiation of (3.1) in t gives the following relations:

$$\frac{\partial^k u_i}{\partial t^k}(\cdot, 0) = \left(\frac{\partial^{k-1}}{\partial t^{k-1}} \left(B_i \left(x, t, \frac{\partial}{\partial x} \right) u \right) \right)(\cdot, 0) + \frac{\partial^{k-1} f_i}{\partial t^{k-1}}(\cdot, 0)$$

$$= \sum_{j=0}^{k-1} C_{k-1}^j \left(\frac{\partial^j B_i}{\partial t^j} \left(x, t, \frac{\partial}{\partial x} \right) \right)(\cdot, 0) \left(\frac{\partial^{k-1-j} u}{\partial t^{k-1-j}} \right)(\cdot, 0)$$

$$+ \frac{\partial^{k-1} f_i}{\partial t^{k-1}}(\cdot, 0), \quad k = 1, 2, \ldots, \tag{3.7}$$

where

$$B \left(x, t, \frac{\partial}{\partial x} \right) u = \left\{ B_i \left(x, t, \frac{\partial}{\partial x} \right) u \right\}_{i=1}^N,$$

$$B_i \left(x, t, \frac{\partial}{\partial x} \right) u = \frac{\partial}{\partial x_r} \left(a_{ijrm}(x, t) \frac{\partial u_j}{\partial x_m} \right) - b_{jm}^i(x, t) \frac{\partial u_j}{\partial x_m} - g_j^i u_j, \quad i = 1, \ldots, N.$$

$$\tag{3.8}$$

17

We mention that the inequality for a_{ijrm} in (3.4) is the condition of strong ellipticity of the operator B.

Equations (3.1) are represented in the form

$$\frac{\partial u_i}{\partial t} - B_i\left(x, t, \frac{\partial}{\partial x}\right) u = f_i \text{ in } Q, \ i = 1, \ldots, N. \tag{3.9}$$

The existence of a unique solution to the problem (3.1), (3.2) such that $u \in H^{2,1}(Q)^N$ follows from [46].

By analogy with (2.29), we define the following vector-function $w = (w_1, \ldots, w_N)$:

$$w_i(x, t) = \begin{cases} u_{bi}(Px, t) \exp\left(1 - \frac{a^2}{a^2 - (x - Px)^2}\right), & \text{if } |x - Px| < a, \ i = 1, \ldots, N, \\ 0, & \text{if } |x - Px| \geq a. \end{cases} \tag{3.10}$$

Then $w \in H^{2,1}(Q)^N$.

Let

$$\tilde{u} = u - w. \tag{3.11}$$

The function $\tilde{u}$ is the solution to the problem

$$\tilde{u} \in H^{2,1}(Q)^N,$$
$$\frac{\partial \tilde{u}_i}{\partial t} - B_i\left(x, t, \frac{\partial}{\partial x}\right) \tilde{u} = \tilde{f}_i, \text{ in } Q,$$
$$\tilde{u}|_{S_T} = 0, \quad \tilde{u}|_{t=0} = u_0 - w|_{t=0} \in H_0^1(Q)^N, \tag{3.12}$$

where

$$\tilde{f}_i = f_i - \frac{\partial w_i}{\partial t} + B_i\left(x, t, \frac{\partial}{\partial x}\right) w \in L^2(Q^N). \tag{3.13}$$

Let $(\tilde{f}_m, \tilde{u}_{0m})$ be a sequence such that

$$\tilde{f}_m \in C^\infty([0, T]; \mathcal{D}(\Omega))^N, \quad \tilde{f}_m(x, t) = \sum_{k=0}^\infty \frac{1}{k!} \frac{\partial^k \tilde{f}_m}{\partial t^k}(x, 0) t^k, \quad \tilde{f}_m \to \tilde{f} \text{ in } L^2(Q)^N,$$

$$\tilde{u}_{0m} \in \mathcal{D}(\Omega)^N, \quad \tilde{u}_{0m} \to u_0 - w|_{t=0} \text{ in } H_0^1(\Omega)^N. \tag{3.14}$$

Consider the problem: Find $\tilde{u}_m$ such that

$$\tilde{u}_m \in C^\infty([0, T]; \mathcal{D}(\Omega))^N,$$
$$\frac{\partial \tilde{u}_{mi}}{\partial t} - B_i(x, t, \frac{\partial}{\partial x}) \tilde{u}_m = \tilde{f}_{mi} \text{ in } Q,$$
$$\tilde{u}_m(\cdot, 0) = \tilde{u}_{0m}. \tag{3.15}$$

It follows from [46] that there exists a unique solutions to the problem (3.15). By analogy with the above, we get that

$$\tilde{u}_m(x,t) = \tilde{u}_{0m}(x) + \sum_{k=1}^{k=\infty} \frac{1}{k!} \frac{\partial^k \tilde{u}_m}{\partial t^k}(x,0)t^k, \quad (x,t) \in Q, \tag{3.16}$$

where $\frac{\partial^k \tilde{u}_m}{\partial t^k}$ are determined by (3.7) with u and f being replaced by $\tilde{u}_m$ and $\tilde{f}_m$, respectively.

Since the solution to the problem (3.15) depends continuously on $\tilde{f}_m, \tilde{u}_{0m}$ formulas (3.12), (3.14), and (3.15) imply

$$\tilde{u}_m \to \tilde{u} \text{ in } H^{2,1}(Q)^N. \tag{3.17}$$

Thus, we have proved

Theorem 3.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ with a boundary S of the class C^∞ and $T \in (0,\infty)$. Suppose that the conditions (3.3)–(3.6) are satisfied. Then there exists a unique solution to the problem (3.1), (3.2) such that $u \in H^{2.1}(Q)^N$, and this solution is represented in the form $u = \tilde{u} + w$, where $\tilde{u} = \lim \tilde{u}_m$ and w is given by (3.10).*

4 System of hyperbolic equations

4.1 Problem with boundary condition at $t = 0$

We consider the problem: Find $u = (u_1, u_2, \ldots, u_N)$ such that

$$\frac{\partial^2 u_i}{\partial t^2} - B_i\left(x,t,\frac{\partial}{\partial x}\right)u = f_i \text{ in } Q, \quad i = 1,2,\ldots,N, \tag{4.1}$$

$$u(x,0) = u_0(x), \ \frac{\partial u}{\partial t}(x,0) = u_1(x), \ x \in \overline{\Omega}, \ u_0(x) = u_b(x,0), \ x \in S. \tag{4.2}$$

Here $B_i\left(x,t,\frac{\partial}{\partial x}\right)$ are the components of the operator $B\left(x,t,\frac{\partial}{\partial x}\right)$ that are defined in (3.8).

We assume that the coefficients of the operator $B\left(x,t,\frac{\partial}{\partial x}\right)$ satisfy the conditions (3.4)–(3.6) and

$$(f, u_0, u_1) \in U_2,$$
$$U_2 = \big\{(f, u_0, u_1) \mid f = (f_1, \ldots, f_N) \in C^\infty([0,T]; \mathcal{D}(\Omega_1))^N,$$
$$f(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\partial^k f}{\partial t^k}(x,0)\, t^k, \ (x,t) \in Q_1,$$
$$u_0 = (u_{01}, \ldots, u_{0N}) \in \mathcal{D}(\Omega_1)^N, \ u_1 = (u_{11}, \ldots, u_{1N}) \in \mathcal{D}(\Omega_1)^N \big\}. \tag{4.3}$$

We differentiate equations (4.1) in t $k-2$ times, $k \geq 3$, and set $t = 0$. This gives the following recurrence relation:

$$\frac{\partial^k u_i}{\partial t^k}(\cdot, 0) = \frac{\partial^{k-2} f_i}{\partial t^{k-2}}(\cdot, 0) + \left(\frac{\partial^{k-2}}{\partial t^{k-2}} \left(B_i \left(x, t, \frac{\partial}{\partial x} \right) u \right) \right)(\cdot, 0)$$

$$= \frac{\partial^{k-2} f_i}{\partial t^{k-2}}(\cdot, 0) + \sum_{j=0}^{k-2} C_{k-2}^j \left(\frac{\partial^j B_i}{\partial t^j} \left(x, t, \frac{\partial}{\partial x} \right) \right)(\cdot, 0) \frac{\partial^{k-2-j} u}{\partial t^{k-2-j}}(\cdot, 0). \qquad (4.4)$$

Here $u(\cdot, 0)$ and $\frac{\partial u}{\partial t}(\cdot, 0)$ are prescribed.

Theorem 4.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ with a boundary S of the class C^∞ and $T \in (0, \infty)$. Suppose that the conditions (4.3) (3.4)-(3.6) are satisfied. Then there exists a unique solution to the problem (4.1), (4.2) such that $u \in C^\infty(\overline{Q})^N$, and this solution is represented in the form of the Taylor expansion*

$$u(x, t) = u_0(x) + u_1(x)t + \sum_{k=2}^{\infty} \frac{1}{k!} \frac{\partial^k u}{\partial t^k}(x, 0) t^k, \quad (x, t) \in \overline{Q}. \qquad (4.5)$$

The coefficients $\frac{\partial^k u}{\partial t^k}(\cdot, 0)$ are determined by the recurrence relations (4.4). Furthermore, the boundary condition function $u_b = u|_{S_T}$ is determined as follows:

$$u_b(x, t) = u_0(x) + u_1(x)t + \sum_{k=2}^{\infty} \frac{1}{k!} \frac{\partial^k u}{\partial t^k}(x, 0) t^k, \ (x, t) \in S_T. \qquad (4.6)$$

The function $(f, u_0, u_1) \mapsto u$ that is defined by the solution to the problem (4.1), (4.2) in the form (4.5) is a continuous mapping of U_2 into $C^\infty(\overline{Q})^N$.

Proof. We consider the problem: Find $\check{u}$ satisfying

$$\frac{\partial^2 \check{u}_i}{\partial t^2} - B_i \left(x, t, \frac{\partial}{\partial x} \right) \check{u} = f_i \quad \text{in } Q_1, \quad i = 1, 2, \ldots, N, \qquad (4.7)$$

$$\check{u}(x, 0) = u_0(x), \quad \frac{\partial \check{u}}{\partial t}(x, 0) = u_1(x), \quad x \in \overline{\Omega}_1, \ \check{u}|_{S_{1T}} = 0, \qquad (4.8)$$

where $(f, u_0, u_1) \in U_2$.

It follows from [41], Chapter 5, Theorem 2.1, that under the conditions

$$f \in H^{0,1}(Q_1)^N, \ u_0 \in H^2(\Omega_1)^N \cap H_0^1(\Omega_1)^N, \ u_1 \in H^1(\Omega_1)^N, \qquad (4.9)$$

there exists a unique solution to the problem (4.7), (4.8) such that

$$\check{u} \in L^2(0, T; H^2(\Omega_1))^N, \ \frac{\partial^2 \check{u}}{\partial t^2} \in L^2(Q_1)^N, \qquad (4.10)$$

i.e. $\check{u} \in H^{2,2}(Q_1)^N$, and the function $(f, u_0, u_1) \mapsto \check{u}$ is a continuous mapping of $H^{0,1}(Q_1)^N \times H^2(\Omega_1)^N \cap H_0^1(\Omega_1)^N \times H^1(\Omega_1)^N$ into $H^{2,2}(Q_1)^N$.

Informally, the solution to the problem (4.7), (4.8) is represented in the form

$$\check{u}(x,t) = u_0(x) + u_1(x)t + \sum_{k=2}^{\infty} \frac{1}{k!} \frac{\partial^k \check{u}}{\partial t^k}(x,0)\, t^k, \quad (x,t) \in \overline{Q}_1. \tag{4.11}$$

The function $\check{u}$ defined by (4.11) and the formula (4.4) with u replaced by $\check{u}$ is a solution to the problem (4.7), (4.8) for all $t \in [0,T]$ such that the series (4.11) converges at t in the corresponding space.

Taking that into account, we conclude by analogy with the above that the series (4.11) converges in $H^{2,2}(Q_1)^N$.

Consider the problem: Find a function $\hat{u} = (\hat{u}_1, \ldots, \hat{u}_N)$ given in Q_1 that solves the problem

$$\frac{\partial^2 \hat{u}}{\partial t^2} - B\left(x, t, \frac{\partial}{\partial x}\right)\hat{u} = \frac{\partial^2 f}{\partial t^2} \quad \text{in } Q_1, \tag{4.12}$$

$$\hat{u}(x,0) = \frac{\partial^2 \check{u}}{\partial t^2}(x,0), \quad \frac{\partial \hat{u}}{\partial t}(x,0) = \frac{\partial^3 \check{u}}{\partial t^3}(x,0), \quad x \in \Omega_1,$$

$$\hat{u}(x,t) = 0, \quad (x,t) \in S_{1T}, \tag{4.13}$$

where $\frac{\partial^2 \check{u}}{\partial t^2}(x,0)$ and $\frac{\partial^3 \check{u}}{\partial t^3}(x,0)$ are determined by (4.4).

Again, (4.3) and [41] imply that there exists a unique solution to the problem (4.12), (4.13) such that $\hat{u} \in H^{2,2}(Q_1)^N$. As $\hat{u} = \frac{\partial^2 \check{u}}{\partial t^2}$, by (4.11) it is represented in the form

$$\hat{u}(x,t) = \sum_{k=2}^{\infty} \frac{1}{(k-2)!} \frac{\partial^k \check{u}}{\partial t^k}(x,0)\, t^{k-2}, \quad (x,t) \in Q_1, \tag{4.14}$$

and

$$\frac{\partial^4 \check{u}}{\partial t^4} \in L^2(Q_1)^N, \qquad \frac{\partial^4 \check{u}}{\partial t^2 \partial x_i^2} \in L_2(Q_1)^N, \quad i = 1, \ldots, N. \tag{4.15}$$

Now consider the problems: Find functions $\tilde{u}_j = (\tilde{u}_{j1}, \ldots, \tilde{u}_{jN})$ given in Q_1 such that

$$\frac{\partial^2 \tilde{u}_{ji}}{\partial t^2} - B_i\left(x, t, \frac{\partial}{\partial x}\right)\tilde{u}_j = \frac{\partial^2 f_i}{\partial x_j^2} \quad \text{in } Q_1, \ j = 1, \ldots, n, \ i = 1, \ldots, N, \tag{4.16}$$

$$\tilde{u}_j(x,0) = \frac{\partial^2 \check{u}}{\partial x_j^2}(x,0) = \frac{\partial^2 u_0}{\partial x_j^2}(x), \quad \frac{\partial \tilde{u}_j}{\partial t}(x,0) = \frac{\partial^3 \check{u}}{\partial t \partial x_j^2}(x,0) = \frac{\partial^2 u_1}{\partial x_j^2}(x),$$

$$j = 1, \ldots, n, \quad x \in \Omega_1, \quad \tilde{u}_j(x,t) = 0, \quad (x,t) \in S_{1T}. \tag{4.17}$$

The preceding arguments show the existence of a unique solution to this problem such that $\tilde{u}_j \in H^{2,2}(Q_1)^N$. Since $\tilde{u}_j = \frac{\partial^2 \check{u}}{\partial x_j^2}$, $j = 1, \ldots, n$, we obtain

$$\frac{\partial^4 \check{u}}{\partial x_j^4} \in L^2(Q_1)^N, \quad j = 1, \ldots, n.$$

21

From here and (4.15), we get $\check{u} \in H^{4,4}(Q_1)^N$.

By analogy, we obtain that $\check{u} \in H^{2k,2k}(Q_1)^N$ for any $k \in \mathbb{N}$, and $\check{u} \in C^\infty(\overline{Q}_1)^N$, and the series (4.11) converges to $\check{u}$ in $C^\infty(\overline{Q}_1)^N$. The function $(f, u_0, u_1) \mapsto \check{u}$ is a continuous mapping of U_2 into $C^\infty(\overline{Q}_1)^N$, and $u = \check{u}|_Q$. $\qquad\square$

4.2 Problem with given boundary conditions

We first consider the following problem with homogeneous boundary conditions:

$$\frac{\partial^2 u_i}{\partial t^2} - B_i\left(x, t, \frac{\partial}{\partial x}\right) u = f_i \quad \text{in } Q, \quad i = 1, 2, \dots, N,$$

$$u|_{t=0} = u_0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = u_1,$$

$$u|_{S_T} = 0. \tag{4.18}$$

We suppose

$$f \in L^2(Q)^N, \ u_0 \in H_0^1(\Omega)^N, \ u_1 \in L^2(\Omega)^N. \tag{4.19}$$

Theorem 4.2. *Let Ω be a bounded domain in $\mathbb{R}^n$ with a boundary S of the class C^∞ and $T \in (0, \infty)$. Suppose that the conditions (3.4)–(3.6) and (4.19) are satisfied. Then there exists a unique solution to the problem (4.18) and furthermore*

$$(f, u_0, u_1) \to \left(u, \frac{\partial u}{\partial t}\right) \text{ is a linear continuous mapping of}$$

$$(L^2(Q)^N \times H_0^1(\Omega)^N \times L^2(Q)^N) \text{ into } L^2(0, T; H_0^1(\Omega)^N \times L^2(Q)^N. \tag{4.20}$$

Let $\{f^m, u_0^m, u_1^m\}_{m=1}^\infty$ be a sequence that satisfies the following conditions:

$$f^m \in C^\infty([0, T]; \mathcal{D}(\Omega))^N, \ f^m(x, t) = \sum_{k=0}^\infty \frac{1}{k!} \frac{\partial^k f^m}{\partial t^k}(x, 0) t^k, \quad (x, t) \in Q,$$

$$f^m \to f \quad \text{in } L^2(Q)^N, \quad u_0^m \in \mathcal{D}(\Omega)^N, \quad u_0^m \to u_0 \quad \text{in } H_0^1(\Omega)^N,$$

$$u_1^m \in \mathcal{D}(\Omega)^N, \quad u_1^m \to u_1 \quad \text{in } L^2(\Omega)^N. \tag{4.21}$$

Let also u^m be the solution to the problem

$$\frac{\partial^2 u_i^m}{\partial t^2} - B_i\left(x, t, \frac{\partial}{\partial x}\right) u^m = f_i^m, \ i = 1, 2, \dots, N,$$

$$u^m(x, 0) = u_0^m(x), \quad \frac{\partial u^m}{\partial t}(x, 0) = u_1^m(x), \quad x \in \Omega. \tag{4.22}$$

Then $u^m \in C^\infty([0, T]; \mathcal{D}(\Omega))^N$ and $u^m \to u$ in $L^2([0, T]; H_0^1(\Omega))^N$, $\frac{\partial u^m}{\partial t} \to \frac{\partial u}{\partial t}$ in $L^2(Q)^N$, where u is the solution to the problem (4.18).

Proof. The existence of a unique solution to the problem (4.18) such that $u \in L^2([0,T]; H_0^1(\Omega))^N$, $\frac{\partial u}{\partial t} \in L^2(Q)^N$ and (4.20) holds follows from [39], Chapter 4, Theorem 1.1. Informally, the solution to the problem (4.22) is represented in the form

$$u^m(x,t) = u_0^m(x) + u_1^m(x)t + \sum_{k=2}^{\infty} \frac{1}{k!} \frac{\partial^k u^m}{\partial t^k}(x,0)t^k, \quad (x,t) \in Q. \tag{4.23}$$

Let $(\hat{f}^m, \hat{u}_0^m, \hat{u}_1^m)$ be an extension of (f^m, u_0^m, u_1^m) to Q_1 and Ω_1, respectively, such that $(\hat{f}^m, \hat{u}_0^m, \hat{u}_1^m) \in U_2$. Then, by using Theorem 4.1, we obtain that $u^m \in C^\infty(\overline{Q})^N$ and the series (4.23) converges in $C^\infty(\overline{Q})^N$.

Since the solution to the problem (4.18) depends continuously on the data of the problem, we obtain from (4.21) that $u^m \to u$ in $L^2([0,T]; H_0^1(\Omega))^N$ and $\frac{\partial u^m}{\partial t} \to \frac{\partial u}{\partial t}$ in $L^2(Q)^N$. $\square$

Consider now the problem with inhomogeneous boundary conditions: Find u satisfying

$$\frac{\partial^2 u_i}{\partial t^2} - B_i\left(x,t,\frac{\partial}{\partial x}\right)u = f_i \text{ in } Q, \quad i = 1,2,\ldots,N,$$

$$u|_{t=0} = u_0, \quad \frac{\partial u}{\partial t}\bigg|_{t=0} = u_1, \quad u|_{S_T} = u_b. \tag{4.24}$$

We suppose that

$$f \in L^2(Q)^N, \quad u_b \in H^{\frac{3}{2},\frac{3}{2}}(S_T)^N, \quad u_0 \in H^1(\Omega)^N,$$
$$u_1 \in L^2(\Omega)^N, \quad u_0(x) = u_b(x,0), \quad x \in S. \tag{4.25}$$

We use the function w defined in (3.10). Since $u_b \in H^{\frac{3}{2},\frac{3}{2}}(S_T)^N$, we have $w \in H^{2,2}(\Omega)^N$. We set

$$\hat{u} = u - w. \tag{4.26}$$

Then

$$\frac{\partial^2 \hat{u}_i}{\partial t^2} - B_i\left(x,t,\frac{\partial}{\partial x}\right)\hat{u} = \hat{f}_i \text{ in } Q, \quad i = 1,2,\ldots,N,$$

$$\hat{u}|_{t=0} = \hat{u}_0 = u_0 - w]_{t=0}, \quad \frac{\partial \hat{u}}{\partial t}\bigg|_{t=0} = \hat{u}_1 = u_1 - \frac{\partial w}{\partial t}\bigg|_{t=0}, \quad \hat{u}|_{S_T} = 0, \tag{4.27}$$

where

$$\hat{f}_i = f_i - \frac{\partial^2 w_i}{\partial t^2} + B_i\left(x,t,\frac{\partial}{\partial x}\right)w. \tag{4.28}$$

Then

$$\hat{f} \in L^2(Q)^N, \quad \hat{u}_0 \in H_0^1(\Omega)^N, \quad \hat{u}_1 \in L^2(\Omega)^N.$$

It follows from Theorem 4.2 that, there exists a unique solution to the problem (4.27) such that

$$\hat{u} \in L^2(0,T; H_0^1(\Omega))^N, \quad \frac{\partial \hat{u}}{\partial t} \in L^2(Q)^N, \tag{4.29}$$

and

$$(\hat{f}, \hat{u}_0, \hat{u}_1) \rightarrow (\hat{u}, \frac{\partial \hat{u}}{\partial t}) \text{ is a linear continuous mapping of}$$
$$L^2(Q)^N \times H_0^1(\Omega)^N, \times L^2(\Omega)^N \text{ into } L^2(0,T; H_0^1(\Omega))^N \times L^2(Q)^N. \tag{4.30}$$

Let $(\hat{f}_m, \hat{u}_{0m}, \hat{u}_{1m})$ be a sequence such that

$$\hat{f}_m \in C^\infty([0,T], \mathcal{D}(\Omega))^N, \ \hat{f}_m(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\partial^k \hat{f}_m}{\partial t^k}(x,0)t^k, \ \hat{f}_m \rightarrow \hat{f} \text{ in } L^2(Q)^N,$$

$$\hat{u}_{0m} \in \mathcal{D}(\Omega)^N, \ \hat{u}_{0m} \rightarrow \hat{u}_0 - w|_{t=0} \text{ in } H_0^1(\Omega)^N,$$

$$\hat{u}_{1m} \in \mathcal{D}(\Omega)^N, \ \hat{u}_{1m} \rightarrow u_1 - \frac{\partial w}{\partial t}\Big|_{t=0} \text{ in } L^2(\Omega)^N. \tag{4.31}$$

Consider the problem: Find $\hat{u}_m$ satisfying

$$\frac{\partial^2 \hat{u}_{mi}}{\partial t^2} - B_i\left(x, t, \frac{\partial}{\partial x}\right)\hat{u}_m = \hat{f}_{mi} \text{ in } Q, \quad i = 1, 2, \ldots, N,$$

$$\hat{u}_m|_{t=0} = \hat{u}_{0m}, \quad \frac{\partial \hat{u}_m}{\partial t}\Big|_{t=0} = \hat{u}_{1m}. \tag{4.32}$$

It follows from Theorem 4.2 that, there exists the unique solution to the problem (4.32) such that $\hat{u}_m \in C^\infty(\overline{Q})^N$. This solution is presented in the form

$$\hat{u}_m(x,t) = \hat{u}_{0m}(x) + \hat{u}_{1m}(x)t + \sum_{k=2}^{\infty} \frac{1}{k!} \frac{\partial^k \hat{u}_m}{\partial t^k}(x,0)t^k, \quad (x,t) \in Q, \tag{4.33}$$

and by (4.31)

$$\hat{u}_m \rightarrow \hat{u} \text{ in } L^2(0,T; H_0^1(\Omega))^N, \quad \frac{\partial \hat{u}_m}{\partial t} \rightarrow \frac{\partial \hat{u}}{\partial t} \text{ in } L^2(Q)^N. \tag{4.34}$$

Thus, we have proved the following result:

Theorem 4.3. *Let Ω be a bounded domain in $\mathbb{R}^n$ with a boundary S of the class C^∞ and $T \in (0, \infty)$. Suppose that the conditions (3.4)–(3.6) and (4.25) are satisfied. Then there exists the unique solution to the problem (4.24) such that $u \in L^2(0,T; H^1(\Omega))^N$, $\frac{\partial u}{\partial t} \in L^2(Q)^N$. This solution is presented in the form $u = \hat{u} + w$, where w is defined in (3.10) and $\hat{u}$ is determined by (4.32) and (4.34).*

5 Problem on vibration of an orthotropic plate in a viscous medium.

Plates fabricated from composite materials are used in modern constructions. Such plates are orthotropic. The strain energy of the orthotropic plate is defined by the following formula, see [42]

$$\Phi(u) = \frac{1}{2} \int_{\Omega} \left(D_1 \left(\frac{\partial^2 u}{\partial x_1^2} \right)^2 + 2 D_{12} \frac{\partial^2 u}{\partial x_1^2} \frac{\partial^2 u}{\partial x_2^2} + D_2 \left(\frac{\partial^2 u}{\partial x_2^2} \right)^2 + 2 D_3 \left(\frac{\partial^2 u}{\partial x_1 \partial x_2} \right)^2 \right) dx. \quad (5.1)$$

Here Ω is the midplane of the plate, Ω is a bounded domain in $\mathbb{R}^2$ with a boundary S,

$$dx = dx_1 \, dx_2, \; D_i = \frac{h^3 E_i}{12(1 - \mu_1 \mu_2)}, \; i = 1, 2, \; D_{12} = \mu_2 D_1 = \mu_1 D_2, \; D_3 = \frac{h^3 G}{6},$$

E_1, E_2, G, μ_1, μ_2 being the elasticity characteristics of the material, h the thickness of the plate,

$$E_1, \, E_2, \, G \text{ are positive constants, } \mu_1 \text{ and } \mu_2 \text{ are constants, } 0 \le \mu_i < 1, \, i = 1, 2, \quad (5.2)$$

u is the function of deflection, i.e., the function of displacements of points of the midplane in the direction perpendicular to the midplane.

We suppose that

$$h \in C^{\infty}(\overline{\Omega}), \quad e_1 \le h \le e_2, \quad e_1, \, e_2 \text{ are positive constants.} \quad (5.3)$$

Variation of the strain energy of the plate determines the following bilinear form

$$a(u, v) = \int_{\Omega} \left[D_1 \frac{\partial^2 u}{\partial x_1^2} \frac{\partial^2 v}{\partial x_1^2} + D_2 \frac{\partial^2 u}{\partial x_2^2} \frac{\partial^2 v}{\partial x_2^2} + D_{12} \left(\frac{\partial^2 u}{\partial x_1^2} \frac{\partial^2 v}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_2^2} \frac{\partial^2 v}{\partial x_1^2} \right) \right.$$
$$\left. + 2 D_3 \frac{\partial^2 u}{\partial x_1 \partial x_2} \frac{\partial^2 v}{\partial x_1 \partial x_2} \right] dx. \quad (5.4)$$

In our case $a(u, u) = 2\Phi(u)$.

We assume that the plate is clamped. Thus,

$$u\big|_S = 0, \quad \frac{\partial u}{\partial \nu}\Big|_S = 0, \quad (5.5)$$

where ν is the unit outward normal to S.

One can easily see that, on the set of smooth functions which satisfy the condition (5.5), the following equality holds.

$$a(u, v) = (Au, v) = (u, Av). \quad (5.6)$$

Here $(\cdot, \cdot)$ is the scalar product in $L^2(\Omega)$, and the operator A given as follows:

$$
\begin{aligned}
Au = & \frac{\partial^2}{\partial x_1^2}\left(D_1 \frac{\partial^2 u}{\partial x_1^2}\right) + \frac{\partial^2}{\partial x_2^2}\left(D_2 \frac{\partial^2 u}{\partial x_2^2}\right) + \frac{\partial^2}{\partial x_2^2}\left(D_{12} \frac{\partial^2 u}{\partial x_1^2}\right) \\
& + \frac{\partial^2}{\partial x_1^2}\left(D_{12} \frac{\partial^2 u}{\partial x_2^2}\right) + 2\frac{\partial^2}{\partial x_1 \partial x_2}\left(D_3 \frac{\partial^2 u}{\partial x_1 \partial x_2}\right) = -F_{\text{re}},
\end{aligned} \tag{5.7}
$$

$F_{\text{re}} = -Au$ is the resistance force induced by the elasticity for the function of displacement u.

The viscous medium resists the vibration of the plate. The resistance force F_{rm} that it induces is opposite in direction to the velocity $\frac{\partial u}{\partial t}$, $F_{\text{rm}} = -\varphi \frac{\partial u}{\partial t}$, where φ is the resistance coefficient which is an increasing function of $|\frac{\partial u}{\partial t}|$ that takes positive values.

We take the resistance force in the form

$$
F_{\text{rm}} = -\left(a_0 + a_1 \left(\frac{\partial u}{\partial t}\right)^2\right)\frac{\partial u}{\partial t}, \tag{5.8}
$$

where a_0 and a_1 are positive constants.

The D'Alembert inertia force is given by

$$
F_{\text{in}} = -\rho h \frac{\partial^2 u}{\partial t^2}, \tag{5.9}
$$

ρ being the density, a positive constant.

Let K be an exterior transverse force that acts on the plate. According to the D'Alembert principle, the sum of an active force that is applied at any point at each instant of time and the internal and inertia forces which it induces is equal to zero. Therefore,

$$
F_{\text{re}} + F_{\text{rm}} + F_{\text{in}} + K = 0. \tag{5.10}
$$

From here, we obtain the following equation on vibration of the orthotropic plate in a viscous medium:

$$
\rho h \frac{\partial^2 u}{\partial t^2} + Au + \left(a_0 + a_1 \left(\frac{\partial u}{\partial t}\right)^2\right)\frac{\partial u}{\partial t} = K. \tag{5.11}
$$

Dividing both sides of equation (5.11) by ρh gives

$$
\frac{\partial^2 u}{\partial t^2} + Mu + \alpha_0 \frac{\partial u}{\partial t} + \alpha_1 \left(\frac{\partial u}{\partial t}\right)^2 \frac{\partial u}{\partial t} = f, \tag{5.12}
$$

where

$$
Mu = \frac{1}{\rho h} Au, \quad \alpha_0 = \frac{a_0}{\rho h}, \quad \alpha_1 = \frac{a_1}{\rho h}, \quad f = \frac{K}{\rho h}.
$$

According to (5.5), the boundary conditions have the form

$$
u\big|_{S_T} = 0, \quad \frac{\partial u}{\partial \nu}\Big|_{S_T} = 0. \tag{5.13}
$$

We set the initial conditions in the form

$$u\big|_{t=0} = u_0, \quad \frac{\partial u}{\partial t}\bigg|_{t=0} = u_1. \tag{5.14}$$

We suppose

$$f \in L^2(Q), \quad \frac{\partial f}{\partial t} \in L^2(Q), \text{ i.e, } f \in H^{0,1}(Q), \tag{5.15}$$

$$u_0 \in H_0^4(\Omega), \quad u_1 \in H_0^2(\Omega). \tag{5.16}$$

Theorem 5.1. *Let Ω be a bounded domain in $\mathbb{R}^2$ with a boundary S of the class C^5, $T \in (0, \infty)$. Suppose that the conditions (5.2), (5.3), (5.15), (5.16) are satisfied. Then there exists a unique solution u to the problem (5.12), (5.13), (5.14) such that $u \in W$, where*

$$W = \left\{ v \mid v \in L^\infty(0, T; H^4(\Omega) \cap H_0^2(\Omega)), \ \frac{\partial v}{\partial t} \in L^\infty(0, T; H_0^2(\Omega)), \right.$$
$$\left. \frac{\partial^2 v}{\partial t^2} \in L^\infty(0, T; L^2(\Omega)) \right\}, \tag{5.17}$$

and

$$(f, u_0, u_1) \mapsto u \text{ is a continuous mapping of } H^{0,1}(Q) \times H_0^4(\Omega) \times H_0^2(\Omega) \text{ into } W. \tag{5.18}$$

Let $\{f_m, u_{0m}, u_{1m}, h_m\}$ be a sequence such that

$$f_m \in C^\infty([0, T]; \mathcal{D}(\Omega)), \ f_m(x, t) = \sum_{k=0}^\infty \frac{1}{k!} \frac{\partial^k f_m}{\partial t^k}(x, 0)t^k, \quad (x, t) \in \overline{Q},$$

$$f_m \to f \quad in \ H^{0,1}(Q), \quad u_{0m} \in \mathcal{D}(\Omega) \quad u_{0m} \to u_0 \quad in \ H_0^4(\Omega)$$
$$u_{1m} \in \mathcal{D}(\Omega), \quad u_{1m} \to u_1 \quad in \ H_0^2(\Omega),$$
$$h_m \in C^\infty(\overline{\Omega}), \quad e_1 \leq h_m \leq e_2, \quad h_m \to h \quad in \ C^3(\overline{\Omega}). \tag{5.19}$$

Let u_m be the solution to the problem

$$\frac{\partial^2 u_m}{\partial t^2} + M_m u_m + \alpha_{0m} \frac{\partial u_m}{\partial t} + \alpha_{1m} \left(\frac{\partial u_m}{\partial t} \right)^2 \frac{\partial u_m}{\partial t} = f_m,$$

$$u_m\big|_{S_T} = 0, \quad \frac{\partial u_m}{\partial \nu}\bigg|_{S_T} = 0, \tag{5.20}$$

where $M_m = \frac{1}{\rho h_m} A_m$, $\alpha_{0m} = \frac{a_0}{\rho h_m}$, $\alpha_{1m} = \frac{a_1}{\rho h_m}$, A_m is defined by (5.7), where h is replaced by h_m. Then

$$u_m \to u \quad in \ L^\infty(0, T; H^4(\Omega) \cap H_0^2(\Omega)),$$

27

$$\frac{\partial u_m}{\partial t} \to \frac{\partial u}{\partial t} \quad in \ \ L^\infty(0,T;H_0^2(\Omega)),$$

$$\frac{\partial^2 u_m}{\partial t^2} \to \frac{\partial^2 u}{\partial t^2} \quad in \ \ L^\infty(0,T;L^2(\Omega)). \tag{5.21}$$

Proof. The existence of a unique solution u to the problem (5.12), (5.13), (5.14) such that $u \in W$ and (5.18) holds is proved by a small modification of the proofs of Theorem 2.1, Chapter 5 in [41] or Theorem 3.1, Chapter 1 in [38]. In this case, we take into account that

$$c\|u\|_{H_0^2(\Omega)}^2 \geq a(u,u) \geq c_1\|u\|_{H_0^2(\Omega)}^2, \ \ u \in H_0^2(\Omega),$$

use the Faedo–Galerkin approximations, and the theorem on compactness, see Theorem 5.1, Chapter 1 in [38], is applied to pass to the limit in the nonlinear term of (5.12).

Informally, the solution to the problem (5.20) is represented in the form

$$u_m(x,t) = u_{0m}(x) + u_{1m}(x)t + \sum_{k=2}^\infty \frac{1}{k!}\frac{\partial^k u_m}{\partial t^k}(x,0)t^k, \quad (x,t) \in \overline{Q}, \tag{5.22}$$

where $\frac{\partial^k u_m}{\partial t^k}(x,0)$ are determined by the following recurrence relations

$$\frac{\partial^k u_m}{\partial t^k}(\cdot,0) = \frac{\partial^{k-2}f_m}{\partial t^{k-2}}(\cdot,0) - \frac{\partial^{k-2}M_m u_m}{\partial t^{k-2}}(\cdot,0) - \alpha_0\frac{\partial^{k-1}u_m}{\partial t^{k-1}}(\cdot,0)$$

$$- \alpha_1 \sum_{j=0}^{k-2} C_{k-2}^j \Big[\frac{\partial^j}{\partial t^j}\Big(\frac{\partial u_m}{\partial t}\Big)^2\Big](\cdot,0)\frac{\partial^{k-j-1}u_m}{\partial t^{k-j-1}}(\cdot,0), \quad k = 2,3,\dots$$

The convergence of the series (5.22) is proved by analogy with the proof of Theorem 2.3. In this case, we consider the functions

$$u_{me}(x,t) = u_{0m}(x) + u_{1m}(x)t + \sum_{k=2}^e \frac{1}{k!}\frac{\partial^k u_m}{\partial t^k}(x,0)t^k, \quad (x,t) \in \overline{Q} \tag{5.23}$$

and apply the infinite function theorem. Then we obtain that $u_{me} \to u_m$ in W as $e \to \infty$.

Since the solution to the problem (5.12), (5.13), (5.14) depends continuously on the data of the problem, (5.21) follows from (5.19). $\qquad\square$

6 Maxwell's equations.

6.1 General problem.

We consider the following problem of electromagnetism: Find functions D and B such that, see [10], [43]

$$\frac{\partial D}{\partial t} - \text{curl}(\hat{\mu}B) + \sigma\hat{\xi}D = G_1 \quad in \ Q, \tag{6.1}$$

28

$$\frac{\partial B}{\partial t} + \operatorname{curl}(\hat{\xi}D) = G_2 \quad \text{in } Q, \tag{6.2}$$

$$\nu \wedge D = 0 \quad \text{on } S_T, \tag{6.3}$$

$$D\Big|_{t=0} = D_0, \quad B\Big|_{t=0} = B_0 \quad \text{in } \Omega. \tag{6.4}$$

Here $Q = \Omega \times (0, T)$, $T < \infty$, Ω is a bounded domain in $\mathbb{R}^3$ with a boundary S, $S_T = S \times (0, T)$, D is the electric induction, B is the magnetic induction, $\hat{\mu}$, $\hat{\xi}$, and σ are scalar functions of $x \in \Omega$ that take positive values, ν is the unit outward normal to S.

We define the following spaces

$$\begin{aligned} V &= \{v \mid v \in L^2(\Omega)^3, \ \operatorname{curl} v \in L^2(\Omega)^3\}, \\ V_1 &= \{v \mid v \in V, \ v \wedge \nu = 0\}. \end{aligned} \tag{6.5}$$

The space V_1 is the closure of $\mathcal{D}(\Omega)^3$ with respect to the norm of V,

$$\|v\|_V = \left(\|v\|_{L^2(\Omega)^3}^2 + \|\operatorname{curl} v\|_{L^2(\Omega)^3}^2 \right)^{\frac{1}{2}}. \tag{6.6}$$

For further detail about the spaces V and V_1, see [23], Chapter 1, Sections 2,3, and [10], Chapter 7.

Let also

$$X = \left\{ h \mid h \in L^\infty(0, T; V), \ \frac{\partial h}{\partial t} \in L^\infty(0, T; L^2(\Omega)^3) \right\},$$

$$X_1 = \left\{ h \mid h \in L^\infty(0, T; V_1), \ \frac{\partial h}{\partial t} \in L^\infty(0, T; L^2(\Omega)^3) \right\}.$$

The norm in X and X_1 is defined by

$$\|h\|_X = \|h\|_{L^\infty(0,T;V)} + \left\| \frac{\partial h}{\partial t} \right\|_{L^\infty(0,T;L^2(\Omega)^3)}.$$

We suppose

$$G_1 \in H^{0,1}(Q)^3, \quad G_2 \in H^{0,1}(Q)^3, \quad D_0 \in V_1, \ B_0 \in V, \tag{6.7}$$

$$\hat{\mu} \in C^1(\overline{\Omega}), \quad \mu_1 \geq \hat{\mu} \geq \mu_2, \quad \hat{\xi} \in C^1(\overline{\Omega}), \quad \xi_1 \geq \hat{\xi} \geq \xi_2, \tag{6.8}$$

$$\sigma \in L^\infty(\Omega), \quad \tilde{\sigma}_1 \geq \sigma \geq \tilde{\sigma}_2. \tag{6.9}$$

Here μ_1, μ_2, ξ_1, ξ_2, $\tilde{\sigma}_1$, $\tilde{\sigma}_2$ are positive constants.

Theorem 6.1. *Let Ω be a bounded domain in $\mathbb{R}^3$ with a boundary S of the class C^∞. Suppose that the conditions (6.7)–(6.9) are satisfied. Then, there exists a unique solution to the problem (6.1)–(6.4) such that.*

$$D \in X_1, \quad B \in X. \tag{6.10}$$

Theorem 6.1 is proved in [10], Chapter 7, by using Galerkin approximations.

Let us discuss construction of the solution to the problem (6.1)–(6.4). In order to apply our method to this problem, we should somewhat change the formulation of this problem.

We present the function B in the form

$$B = B^1 + B^2, \quad B^1 \in X_1, \quad B^2 \in X, \quad \frac{\partial^2 B^2}{\partial t^2} \in L^2(Q)^3. \tag{6.11}$$

We consider that B^1 is unknown, while B^2 is given and satisfies the condition

$$B^2 \wedge \nu = B \wedge \nu \quad \text{in} \ \ L^\infty(0, T; H^{-\frac{1}{2}}(S)^3). \tag{6.12}$$

Here B is the solution to the problem (6.1)–(6.4) together with D. Equality (6.12) has sense for elements of X, see [10], Lemma 4.2, Chapter 7.

According to (6.11), (6.12), we set

$$\begin{aligned} B_0 &= B_0^1 + B_0^2, \quad B_0^1 \in V_1, \quad B_0^2 \in V, \\ B^1|_{t=0} &= B_0^1, \quad B^2|_{t=0} = B_0^2. \end{aligned} \tag{6.13}$$

Now the problem (6.1)–(6.4) is represented as follows:

$$\frac{\partial D}{\partial t} - \operatorname{curl}(\hat{\mu} B^1) + \sigma \hat{\xi} D = G_1 + \operatorname{curl}(\hat{\mu} B^2) \quad \text{in} \ Q, \tag{6.14}$$

$$\frac{\partial B^1}{\partial t} + \operatorname{curl}(\hat{\xi} D) = G_2 - \frac{\partial B^2}{\partial t} \quad \text{in} \ Q, \tag{6.15}$$

$$\nu \wedge D = 0 \quad \text{on} \ S_T, \ \nu \wedge B^1 = 0 \quad \text{on} \ S_T, \tag{6.16}$$

$$D\Big|_{t=0} = D_0, \quad B^1\Big|_{t=0} = B_0^1 \quad \text{in} \ \Omega. \tag{6.17}$$

The existence of a unique solution (D, B^1) to the problem (6.14)–(6.17) such that $D \in X_1$, $B^1 \in X_1$ follows from Theorem 6.1.

Thus, if the pair (D, B) is the solution to the problem (6.1)–(6.4), and

$$B^2 \in X, \quad \frac{\partial^2 B^2}{\partial t^2} \in L^2(Q)^3, \tag{6.18}$$

and (6.12) is satisfied, then the pair (D, B^1) with $B^1 = B - B^2$ is the solution to the problem (6.14)–(6.17).

On the contrary, if the couple (D, B^1) is the solution to the problem (6.14)–(6.17), where B^2 meets (6.18), then the couple (D, B) with $B = B^1 + B^2$ is the solution to the problem (6.1)–(6.4), and (6.12) holds.

Therefore, the formulations (6.1)–(6.4) and (6.14)–(6.17) are equivalent in the above sense.

Let $\{G_{1n}, G_{2n}, D_{0n}, B^1_{0n}, \hat{\mu}_n, \hat{\xi}_n, \sigma_n\}$ be a sequence such that

$$G_{in} \in C^\infty([0,T]; \mathcal{D}(\Omega)^3),$$

$$G_{in}(x,t) = \sum_{k=0}^\infty \frac{1}{k!} \frac{\partial^k G_{in}}{\partial t^k}(x,0) t^k, \quad (x,t) \in Q, \quad i = 1,2,$$

$$G_{1n} \to G_1 + \mathrm{curl}(\hat{\mu} B^2) \quad \text{in } H^{0,1}(Q)^3,$$

$$G_{2n} \to G_2 - \frac{\partial B^2}{\partial t} \quad \text{in } H^{0,1}(Q)^3, \tag{6.19}$$

$$D_{0n} \in \mathcal{D}(\Omega)^3, \; D_{0n} \to D_0 \text{ in } V_1, \; B^1_{0n} \in \mathcal{D}(\Omega)^3, \; B^1_{0n} \to B^1_0 \text{ in } V_1, \tag{6.20}$$

$$\hat{\mu}_n \in C^\infty(\overline{\Omega}), \; \hat{\mu}_n \to \hat{\mu} \text{ in } C^1(\overline{\Omega}), \; \hat{\xi}_n \in C^\infty(\overline{\Omega}), \; \hat{\xi}_n \to \hat{\xi} \quad \text{in } C^1(\overline{\Omega}),$$

$$\sigma_n \in C^\infty(\overline{\Omega}), \; \sigma_n \to \sigma \quad \text{in } L^\infty(\Omega). \tag{6.21}$$

Consider the problem: Find functions D_n and B^1_n such that

$$\frac{\partial D_n}{\partial t} - \mathrm{curl}(\hat{\mu}_n B^1_n) + \sigma_n \hat{\xi}_n D_n = G_{1n} \quad \text{in } Q, \tag{6.22}$$

$$\frac{\partial B^1_n}{\partial t} + \mathrm{curl}(\hat{\xi}_n D_n) = G_{2n} \quad \text{in } Q, \tag{6.23}$$

$$\nu \wedge D_n = 0 \quad \text{on } S_T, \; \nu \wedge B^1_n = 0 \quad \text{on } S_T, \tag{6.24}$$

$$D_n \Big|_{t=0} = D_{0n}, \quad B^1_n \Big|_{t=0} = B^1_{0n} \quad \text{in } \Omega. \tag{6.25}$$

Theorem 6.2. *Let Ω be a bounded domain, in $\mathbb{R}^3$ with a boundary S of the class C^∞ and $T \in (0, \infty)$. Suppose that the conditions of Theorem 6.1 and (6.11), (6.12), (6.13) are satisfied. Let also (6.19)–(6.21) hold. Then for any $n \in \mathbb{N}$ there exists a unique solution (D_n, B^1_n) to the problem (6.22)–(6.25) that is represented in the form*

$$D_n(x,t) = D_{0n}(x) + \sum_{k=1}^\infty \frac{1}{k!} \frac{\partial^k D_n}{\partial t^k}(x,0) t^k, \tag{6.26}$$

$$B^1_n(x,t) = B^1_{0n}(x) + \sum_{k=1}^\infty \frac{1}{k!} \frac{\partial^k B^1_n}{\partial t^k}(x,0) t^k, \tag{6.27}$$

where

$$\frac{\partial^k D_n}{\partial t^k}(x,0) = \mathrm{curl}\left(\hat{\mu}_n(x) \frac{\partial^{k-1} B^1_n}{\partial t^{k-1}}\right)(x,0) - \sigma_n(x) \hat{\xi}_n(x) \frac{\partial^{k-1} D_n}{\partial t^{k-1}}(x,0)$$

$$+ \frac{\partial^{k-1} G_{1n}}{\partial t^{k-1}}(x,0), \quad k = 1,2,\ldots,$$

$$\frac{\partial^k B^1_n}{\partial t^k}(x,0) = - \mathrm{curl}\left(\hat{\xi}_n(x) \frac{\partial^{k-1} D_n}{\partial t^{k-1}}\right)(x,0) + \frac{\partial^{k-1} G_{2n}}{\partial t^{k-1}}(x,0), \quad x \in \Omega, \quad k = 1,2,\ldots. \tag{6.28}$$

31

The series for D_n and B_n^1 converge in X_1 and

$$D_n \to D \quad in \ X_1, \quad B_n^1 \to B^1 \quad in \ X_1, \tag{6.29}$$

where (D, B^1) is the solution to the problem (6.14)–(6.17).

Proof. The existence of the unique solution to the problem (6.22)–(6.25) follows from Theorem 6.1. The condition of compatibility of order infinity for this problem is satisfied. Because of this, informally, the solution to the problem (6.22)–(6.25) is represented in the form (6.26), (6.27) .

It follows from the proofs of Theorems 5.1 and 4.1 in [10], Chapter 7 that, in the case where $\hat{\xi}$, $\hat{\mu}$, and σ are fixed functions that satisfy conditions (6.8), (6.9), the following inequality for the solution to the problem (6.14)–(6.17) holds:

$$\|D\|_{X_1} + \|B^1\|_{X_1} \le C(\|G_1\|_{H^{0,1}(Q)^3} + \|G_2\|_{H^{0,1}(Q)^3} + \|D_0\|_{V_1} + \|B_0^1\|_{V_1}), \tag{6.30}$$

where C depends on $\hat{\xi}$, $\hat{\mu}$, and σ.

The converges of the series (6.26) and (6.27) in X_1 is proved analogously to the above by using (6.19)–(6.21), and (6.30).

Taking (6.19)–(6.21) into account in the same way as it is done in [10], Theorems 4.1 and 5.1, Chapter 7, we get

$$\|D_n\|_{X_1} \le C_1, \quad \|B_n^1\|_{X_1} \le C_2. \tag{6.31}$$

Therefore, we can extract a subsequence $\{D_m, B_m^1\}$ such that

$$\begin{aligned}
D_m &\to D \quad *\text{-weakly in } X_1, \\
B_m^1 &\to B^1 \quad *\text{-weakly in } X_1.
\end{aligned} \tag{6.32}$$

Let w_1 and w be arbitrary elements of $L^2(\Omega)^3$. We take the scalar products of (6.22) and (6.23) for $n = m$ with w_1 and w, respectively, in $L^2(\Omega)^3$. This gives

$$\left(\frac{\partial D_m}{\partial t}, w_1\right) - \left(\text{curl}(\hat{\mu}_m B_m^1), w_1\right) + (\sigma_m \hat{\xi}_m D_m, w_1) = (G_{1m}, w_1) \quad \text{a.e. in } (0, T), \tag{6.33}$$

$$\left(\frac{\partial B_m^1}{\partial t}, w\right) + \left(\text{curl}(\hat{\xi}_m D_m), w\right) = (G_{2m}, w) \quad \text{a.e. in } (0, T). \tag{6.34}$$

Taking (6.19)–(6.21) and (6.32) into account, we pass to the limit as $m \to \infty$ in (6.33), (6.34), and (6.24), (6.25). We conclude that the pair (D, B^1) determined in (6.32) is a solution to the problem (6.14)–(6.17). Since the solution to this problem is unique in $X_1 \times X_1$, (6.32) is also valid when m is replaced by n.

It remains to prove (6.29).

We subtract equalities (6.22)–(6.25) from (6.14)–(6.17), respectively. This gives

$$\frac{\partial}{\partial t}(D - D_n) - \mathrm{curl}(\hat{\mu}B^1 - \hat{\mu}_n B_n^1) + \sigma\hat{\xi}D - \sigma_n\hat{\xi}_n D_n = G_1 + \mathrm{curl}(\hat{\mu}B^2) - G_{1n}, \quad (6.35)$$

$$\frac{\partial}{\partial t}(B^1 - B_n^1) + \mathrm{curl}(\hat{\xi}D - \hat{\xi}_n D_n) = G_2 - \frac{\partial B^2}{\partial t} - G_{2n}, \quad (6.36)$$

$$\nu \wedge (D - D_n) = 0 \text{ on } S_T, \quad \nu \wedge (B^1 - B_n^1) = 0 \quad \text{on } S_T, \quad (6.37)$$

$$(D - D_n)\big|_{t=0} = D_0 - D_{0n} \quad \text{in } \Omega, \quad (B^1 - B_n^1)\big|_{t=0} = B_0^1 - B_{0n}^1 \quad \text{in } \Omega. \quad (6.38)$$

We have

$$\mathrm{curl}(\hat{\mu}B^1 - \hat{\mu}_n B_n^1) = \mathrm{curl}(\hat{\mu}(B^1 - B_n^1)) + \mathrm{curl}((\hat{\mu} - \hat{\mu}_n)B_n^1),$$
$$\sigma\hat{\xi}D - \sigma_n\hat{\xi}_n D_n = \sigma\hat{\xi}(D - D_n) + D_n(\sigma\hat{\xi} - \sigma_n\hat{\xi}_n),$$
$$\mathrm{curl}(\hat{\xi}D - \hat{\xi}_n D_n) = \mathrm{curl}(\hat{\xi}(D - D_n)) + \mathrm{curl}((\hat{\xi} - \hat{\xi}_n)D_n). \quad (6.39)$$

We denote

$$\gamma_{1n} = -\mathrm{curl}((\hat{\mu} - \hat{\mu}_n)B_n^1) + D_n(\sigma\hat{\xi} - \sigma_n\hat{\xi}_n),$$
$$\gamma_{2n} = \mathrm{curl}((\hat{\xi} - \hat{\xi}_n)D_n). \quad (6.40)$$

(6.21) and (6.31) yield

$$\gamma_{1n} \to 0 \quad \text{in } L^\infty(0, T; L_2(\Omega)^3), \ \gamma_{2n} \to 0 \quad \text{in } L^\infty(0, T; L_2(\Omega)^3). \quad (6.41)$$

By (6.39)–(6.41) equations (6.35), (6.36) take the form

$$\frac{\partial}{\partial t}(D - D_n) - \mathrm{curl}(\hat{\mu}(B^1 - B_n^1)) + \sigma\hat{\xi}(D - D_n) + \gamma_{1n} = G_1 + \mathrm{curl}(\hat{\mu}B^2) - G_{1n}$$

$$\frac{\partial}{\partial t}(B^1 - B_n^1) + \mathrm{curl}(\hat{\xi}(D - D_n)) + \gamma_{2n}. = G_2 - \frac{\partial B^2}{\partial t} - G_{2n}.$$

From here and (6.30), taking (6.19), (6.20), and (6.41) into account, we obtain (6.29).
□

According to the theory of electromagnetism, the function B should satisfy the condition

$$\mathrm{div}\, B = 0 \quad \text{in } Q. \quad (6.42)$$

Theorem 6.3. *Suppose that the conditions of Theorem 6.1 are satisfied and, in addition,*

$$\mathrm{div}\, G_2 = 0 \quad \text{in } Q, \quad \mathrm{div}\, B_0 = 0 \quad \text{in } \Omega. \quad (6.43)$$

Then the function B of the solution (D, B) to the problem (6.1)–(6.4) also meets the condition $\mathrm{div}\, B = 0$ *in Q.*

Indeed, applying the operator div, in the sense of distributions, to both sides of equation (6.2), we obtain

$$\frac{\partial}{\partial t}(\operatorname{div} B) = \operatorname{div} G_2 \quad \text{in } Q.$$

That is

$$\operatorname{div} B(\cdot, t) = \operatorname{div} B_0 + \int_0^t \operatorname{div} G_2(\cdot, \tau) d\tau = 0 \quad \text{in } (0, T).$$

Theorem 6.4. *Suppose that the conditions of Theorem 6.2 are satisfied and, in addition,*

$$G_2 = \operatorname{curl} F, \quad F \in L^2(0, T; V), \quad \frac{\partial F}{\partial t} \in L^2(0, T; V),$$

$$B^2 = \operatorname{curl} P, \ P \in L^\infty(0, T; H^2(\Omega)^3), \ \frac{\partial P}{\partial t} \in L^\infty(0, T; H^1(\Omega)^3),$$

$$\frac{\partial^2 P}{\partial t^2} \in L^2(0, T; H^1(\Omega)^3),$$

$$B_0^1 = \operatorname{curl} M^1, \ M^1 \in H_0^2(\Omega)^3, \ B_0^2 = \operatorname{curl} M^2, \ M^2 \in H^2(\Omega)^3, \ P|_{t=0} = M^2. \quad (6.44)$$

The corresponding functions G_{2n} and B_{0n}^1 are given as follows

$$G_{2n} = \operatorname{curl} F_n, \quad F_n \in C^\infty([0, T]; \mathcal{D}(\Omega)^3),$$

$$F_n(x, t) = \sum_{k=0}^\infty \frac{1}{k!} \frac{\partial^k F_n}{\partial t^k}(x, 0) t^k, \quad (x, t) \in Q, \quad (6.45)$$

$$\operatorname{curl} F_n \to \operatorname{curl}\left(F - \frac{\partial P}{\partial t}\right) \quad \text{in } L^2(Q)^3,$$

$$\operatorname{curl} \frac{\partial F_n}{\partial t} \to \operatorname{curl}\left(\frac{\partial F}{\partial t} - \frac{\partial^2 P}{\partial t^2}\right) \quad \text{in } L^2(Q)^3, \quad (6.46)$$

$$B_{0n}^1 = \operatorname{curl} M_n^1, \quad M_n^1 \in \mathcal{D}(\Omega)^3, \quad M_n^1 \to M^1 \quad \text{in } H_0^2(\Omega)^3. \quad (6.47)$$

Then the solution D_n, B_n^1 to the problem (6.22)–(6.25) *also meets the condition* $\operatorname{div} B_n^1 = 0$, (6.29) *holds and* $\operatorname{div} B = 0$.

Theorem 6.4 follows from results of Theorems 6.2 and 6.3.

6.2   Slotted antenna

We consider the problem on diffraction of electromagnetic wave by a superconductor, see [10], Chapter 7, Section 3.4. Let Ω_1 be a bounded domain in $\mathbb{R}^3$, of a superconductor, the boundary S of Ω_1 is of the class C^∞. We consider a problem in a domain Ω in $\mathbb{R}^3$ with an internal boundary S. We assume that Ω is a bounded domain.

We seek a solution to the following problem: Find vector functions D and B such that

$$\frac{\partial D}{\partial t} - \operatorname{curl}(\hat{\mu}B) + \sigma\hat{\xi}D = G_1 \quad \text{in } Q, \tag{6.48}$$

$$\frac{\partial B}{\partial t} + \operatorname{curl}(\hat{\xi}D) = G_2 \quad \text{in } Q, \tag{6.49}$$

$$\operatorname{div} D = 0 \quad \text{in } Q, \quad \nu \wedge D = 0 \quad \text{on } S_T, \tag{6.50}$$

$$\operatorname{div} B = 0 \quad \text{in } Q, \quad \nu \cdot B = 0 \quad \text{on } S_T, \tag{6.51}$$

$$D\big|_{t=0} = D_0, \quad B\big|_{t=0} = B_0 \quad \text{in } \Omega. \tag{6.52}$$

We introduce the following spaces:

$$X_2 = \left\{ h \mid h = \operatorname{curl} w, w \in L^2(0,T;V), \quad \frac{\partial w}{\partial t} \in L^2(0,T;V) \right\},$$

$$X_3 = \left\{ h \mid h = \operatorname{curl} w, w \in L^2(0,T;H^1(\Omega)^3), \tag{6.53} \right.$$

$$\left. h \cdot \nu = 0 \text{ in } L^2(0,T;H^{-\frac{1}{2}}(S)), \frac{\partial w}{\partial t} \in L^2(0,T;H^1(\Omega)^3) \right\}. \tag{6.54}$$

We assume

$$G_1 = \operatorname{curl} w \in X_2, \quad G_2 = \operatorname{curl} u \in X_3, \quad D_0 = \operatorname{curl} p, \quad p \in H_0^2(\Omega)^3,$$
$$B_0 = \operatorname{curl} v, \quad v \in H^2(\Omega)^3, \quad \operatorname{curl} v \cdot \nu = 0 \quad \text{on } S. \tag{6.55}$$

Theorem 6.5. *Let Ω be a bounded domain in $\mathbb{R}^3$ with a boundary S of the class C^∞. Suppose that the conditions (6.55) are satisfied. Let also $\hat{\xi}$, $\hat{\mu}$, σ be positive constants. Then, there exists a unique solution to the problem (6.48)–(6.52) such that*

$$D \in L^\infty(0,T;H_0^1(\Omega)^3), \quad \frac{\partial D}{\partial t} \in L^\infty(0,T;L^2(\Omega)^3),$$

$$B \in L^\infty(0,T;H^1(\Omega)^3), \quad \frac{\partial B}{\partial t} \in L^\infty(0,T;L^2(\Omega)^3). \tag{6.56}$$

Indeed, the existence of a unique solution to the problem (6.48), (6.49), (6.51), such that $\nu \wedge D = 0$ on S_T and $\operatorname{div} B = 0$ in Q, follows from Theorems 6.1 and 6.3. The conditions $\nu \cdot B = 0$ on S_T, $\operatorname{div} D = 0$ in Q, and (6.56) follow from Theorems 5.3, 6.3 and 6.4 in [10], Chapter 7.

As before, we represent the function B in the form $B = B^1 + B^2$, where B^2 is a given function such that

$$B^2 = \operatorname{curl} \alpha^2, \quad \alpha^2 \in L^\infty(0,T;H^2(\Omega)^3), \quad \operatorname{curl} \alpha^2 \cdot \nu = 0 \quad \text{on } S_T,$$

35

$$\frac{\partial \alpha^2}{\partial t} \in L_\infty(0, T; H^1(\Omega)^3), \quad \frac{\partial^2 \alpha^2}{\partial t^2} \in L^2(0, T; H^1(\Omega)^3),$$

$$\text{curl}\, \frac{\partial \alpha^2}{\partial t} \cdot \nu = 0 \quad \text{on } S_T. \tag{6.57}$$

Let B_τ be the tangential component of the vector B on S_T. It is determined as $B_\tau = B|_{S_T} - B \cdot \nu$. Since $B \cdot \nu = 0$. we get $B|_{S_T} = B_\tau$ and the following boundary condition for B^2:

$$B^2|_{S_T} = B|_{S_T} \quad \text{in } H^{\frac{1}{2}}(S_T)^3. \tag{6.58}$$

According to (6.57), (6.58), we set

$$\begin{aligned} B_0 &= B_0^1 + B_0^2, \quad B_0^2 = B^2|_{t=0} = \text{curl}\, \alpha^2|_{t=0} \in H^1(\Omega)^3, \\ B_0^1 &= \text{curl}\, v - \text{curl}\, \alpha^2|_{t=0} \in H_0^1(\Omega)^3. \end{aligned} \tag{6.59}$$

Now for the functions D, B^1, we obtain the following problem:

$$\begin{aligned} &\frac{\partial D}{\partial t} - \text{curl}(\hat{\mu} B^1) + \sigma \hat{\xi} D = G_1 + \text{curl}(\hat{\mu} B^2) \quad \text{in } Q, \\ &\frac{\partial B^1}{\partial t} + \text{curl}(\hat{\xi} D) = G_2 - \frac{\partial B^2}{\partial t} \quad \text{in } Q, \\ &\text{div}\, D = 0 \quad \text{in } Q, \quad \nu \wedge D = 0 \quad \text{on } S_T, \\ &\text{div}\, B^1 = 0 \quad \text{in } Q, \quad \nu \wedge B^1 = 0 \quad \text{on } S_T, \\ &D\Big|_{t=0} = D_0, \quad B^1\Big|_{t=0} = B_0^1 = B_0 - B_0^2 \quad \text{in } \Omega. \end{aligned} \tag{6.60}$$

By analogy with the above, we get the next result.

Theorem 6.6. *Let Ω be a bounded domain in $\mathbb{R}^3$ with a boundary S of the class C^∞. Suppose that the conditions (6.55) and (6.57)–(6.59) are satisfied. Let also $\hat{\xi}$, $\hat{\mu}$, σ be positive constants. Then, there exists a unique solution (D, B^1) to the problem (6.60) that meets the conditions*

$$D \in L^\infty(0, T; H_0^1(\Omega)^3), \quad \frac{\partial D}{\partial t} \in L^\infty(0, T; L^2(\Omega)^3),$$

$$B^1 \in L^\infty(0, T; H_0^1(\Omega)^3), \quad \frac{\partial B^1}{\partial t} \in L^\infty(0, T; L^2(\Omega)^3). \tag{6.61}$$

Thus, if the pair (D, B) is the solution to the problem (6.48)–(6.51), and B^2 meets (6.57), (6.58), then the pair (D, B^1) with $B^1 = B - B^2$ is the solution to the problem (6.60).

On the contrary, if the couple (D, B^1) is the solution to the problem (6.60), where B^2 meets (6.57), then the couple (D, B) with $B = B^1 + B^2$ is the solution to the problem (6.48)–(6.51), and (6.58) holds.

Therefore, the formulations (6.1)–(6.4) and (6.14)–(6.17) are equivalent in the above sense.

Let $\{G_{1n},\, G_{2n},\, D_{0n},\, B^1_{0n}\}$ be a sequence such that

$$G_{in} = \operatorname{curl} w_{in}, \quad w_{in} \in C^\infty([0,T]; \mathcal{D}(\Omega)^3),$$

$$w_{in}(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\partial^k w_{in}}{\partial t^k}(x,0)t^k, \quad (x,t) \in Q, \quad i = 1,2,$$

$$G_{1n} \to G_1 + \operatorname{curl}(\hat{\mu}B^2) \quad \text{in } H^{0,1}(Q)^3,$$

$$G_{2n} \to G_2 - \frac{\partial B^2}{\partial t} \quad \text{in } H^{0,1}(Q)^3, \tag{6.62}$$

$$D_{0n} = \operatorname{curl} p_n, \ p_n \in \mathcal{D}(\Omega)^3. \ \operatorname{curl} p_n \to \operatorname{curl} p \quad \text{in } H^1_0(Q)^3,$$

$$B^1_{0n} = \operatorname{curl} e_n, \ e_n \in \mathcal{D}(\Omega)^3, \ \operatorname{curl} e_n \to \operatorname{curl} v - \operatorname{curl} \alpha^2|_{t=0} \in H^1_0(\Omega)^3. \tag{6.63}$$

We consider the problem: Find functions D_n and B^1_n such that

$$\frac{\partial D_n}{\partial t} - \operatorname{curl}(\hat{\mu}B^1_n) + \sigma\hat{\xi}D_n = G_{1n} \quad \text{in } Q,$$

$$\frac{\partial B^1_n}{\partial t} + \operatorname{curl}(\hat{\xi}D_n) = G_{2n} \quad \text{in } Q,$$

$$\operatorname{div} D_n = 0 \quad \text{in } Q, \quad \nu \wedge D_n = 0 \quad \text{on } S_T,$$

$$\operatorname{div} B^1_n = 0 \quad \text{in } Q, \quad \nu \wedge B^1_n = 0 \quad \text{on } S_T,$$

$$D_n\Big|_{t=0} = D_{0n}, \quad B^1_n\Big|_{t=0} = B^1_{0n} \quad \text{in } \Omega. \tag{6.64}$$

Theorem 6.7. *Let Ω be a bounded domain in $\mathbb{R}^3$ with a boundary S of the class C^∞. Suppose that the conditions (6.62), (6.63) are satisfied, and let $\hat{\xi}$, $\hat{\mu}$, σ be positive constants. Then for any $n \in \mathbb{N}$, there exists a unique solution D_n, B^1_n to the problem (6.64) that is represented in the form (6.26)–(6.28) and*

$$D_n \to D \quad \text{in } L^\infty(0,T; H^1_0(\Omega)^3), \quad \frac{\partial D_n}{\partial t} \to \frac{\partial D}{\partial t} \quad \text{in } L^\infty(0,T; L^2(\Omega)^3),$$

$$B^1_n \to B^1 \quad \text{in } L^\infty(0,T; H^1_0(\Omega)^3), \quad \frac{\partial B^1_n}{\partial t} \to \frac{\partial B^1}{\partial t} \quad \text{in } L^\infty(0,T; L^2(\Omega)^3). \tag{6.65}$$

The proof of this theorem is analogous to the proof of Theorem 6.2.

Remark 6.1. The problem (6.48)–(6.51) is connected with finding functions y such that $\operatorname{div} y = 0$ in Ω, $y \cdot \nu = 0$ on S, see (6.54), (6.55). These functions can be determined in the form

$$y = \operatorname{curl} v + \operatorname{grad} h, \ v \in H^1(\Omega)^3, \ h \in H^1(\Omega), \tag{6.66}$$

where h is the solution to the problem

$$\operatorname{div} \operatorname{grad} h = \Delta h = 0$$

$$\operatorname{grad} h \cdot \nu = \frac{\partial h}{\partial \nu}\Big|_S = -\operatorname{curl} v \cdot \nu. \tag{6.67}$$

We mention that the suggested method based on the Taylor expansion with respect to t can also be used to construct solutions to other equations and system of equations, which contain derivatives with respect to time for all unknown functions.

References

[1] Acerbi E., Mingione, G.: Gradient estimates for a class of parabolic systems. *Duke Math. J.* **136** (2007), 285–320.

[2] Adak D., Natarajan E., Kumar S.: A new nonconforming finite-element method for convection dominated diffusion-reaction equations. *Int. J. Adv. Eng. Sci. Appl. Math.* **8** (2016), 274–283.

[3] Baroni, P.: Marcinkiewicz estimates for degenerate parabolic equations with measure data. *J. Funct. Anal.* **267** (2014), 3397–3426.

[4] Boccardo, L., Orsina, L., Porzio, M.M.: Existence results for quasilinear elliptic and parabolic problems with quadratic gradient terms and sources. *Adv. Calc. Var.* **4** (2011), 397–419.

[5] Besov, O.V., Il'in, V.P., Nikolsky, S.M.: Integral representation of functions and embedding theorems. Nauka, Moskow, 1975 (in Russian).

[6] Bögelein, V., Duzaar, F., Mingione, G.: The boundary regularity of non-linear parabolic system. I. *Ann. Inst. H. Poincaré Anal. Non Lináire* **27** (2010), 201–255.

[7] Burman E., Hansbo F.: Edge stabilization for Galerkin approximation of convection-diffusion-reaction problems. *Comput. Methods Appl. Mech. Engrg.* **193** (2004), 1437–1453.

[8] Cianci, P., Cirmi, G.R., D'Asero, S., Leonardi, S.: Morrey estimates for solutions of singular quadratic nonlinear equations. *Ann. Mat. Pura Appl. (4)* **196** (2017), 1739–1758.

[9] Ciarlet, P.: The finite element method for elliptic problems. North-Holland, Amsterdam, 1978.

[10] Duvaut, G., Lions, J.-L.: Les inéquations en mécanique et en physique. Dunod, Paris, 1972.

[11] Cirmi, G.R., D'Asero, S., Leonardi, S.: Fourth-order nonlinear elliptic equations with lower order term and natural growth conditions. *Nonlinear Anal.* **108** (2014), 66–86.

[12] Cirmi, G.R., D'Asero, S., Leonardi, S.: Gradient estimate for solutions of nonlinear singular elliptic equations below the duality exponent. *Math. Methods Appl. Sci.* **41** (2018), 261–269.

[13] Cirmi, G.R., Leonardi, S.: Regularity results for the gradient of solutions linear elliptic equations with $L^{1,\lambda}$ data. *Ann. Mat. Pura Appl. (4)* **185** (2006), 537–553.

[14] Cirmi, G.R., Leonardi, S.: Regularity results for solutions of nonlinear elliptic equations with $L^{1,\lambda}$ data. *Nonlinear Anal.* **69** (2008), 230–244.

[15] Cirmi, G.R., Leonardi, S.: Higher differentiability for solutions of linear elliptic systems with measure data. *Discrete Contin. Dyn. Syst.* **26** (2010), 89–104.

[16] Cirmi, G.R., Leonardi, S.: Higher differentiability for the solutions of nonlinear elliptic systems with lower-order terms and $L^{1,\theta}$-data. *Ann. Mat. Pura Appl. (4)* **193** (2014), 115–131.

[17] Cirmi, G.R., Leonardi, S., Stará, J.: Regularity results for the gradient of solutions of a class of linear elliptic systems with $L^{1,\lambda}$ data. *Nonlinear Anal.* **68** (2008), 3609–3624.

[18] Douglas, J.Jr., Russell, T.F.: Numerical methods for convection-dominated diffusion problems based on combining the method of characteristics with finite element or finite difference procedures. *SIAM J. Numer. Anal.* **19** (1982), 871–1090.

[19] Duzaar, F., Mingione, G.: Second order parabolic systems, optimal regularity, and singular sets of solutions. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **22** (2005), 705–751.

[20] Duzaar, F., Mingione, G., Steffen, K.: Parabolic systems with polynomial growth and regularity. *Mem. Amer. Math. Soc.* **214** (2011), no. 1005.

[21] Eidelman, S.D.: Parabolic systems. Nauka, Moscow, 1964.

[22] Friedman, A.: Partial differential equations of parabolic type. Prentice Hall, New York, 1964.

[23] Girault, V., Raviart, P.-A.: Finite element approximation of the Navier–Stokes equations. Lecture Notes in Mathemathics 749, Springer, Berlin, 1981.

[24] Heinrichk, J.C., Pepper, D.W.: Intermediate finite element method. Taylor and Francis, Philadelphia, 1999.

[25] Hou, S. Xiao, J.: Cordes-Nirenberg's imbedding and restricting with application to an elliptic equation. *Commun. Contemp. Math.* **20** (2018), 1750080, 20 pp.

[26] Koshelev, A.: Regularity problem for quasilinear elliptic and parabolic systems. Lecture Notes in Mathematics, 1614. Springer-Verlag, Berlin, 1995.

[27] Kuusi, T., Mingione, G.: The Wolff gradient bound for degenerate parabolic equations. *J. Eur. Math. Soc. (JEMS)* **16** (2014), 835–892.

[28] Ladyzhenskaya, O.A., Solonnikov, V.A, Uraltseva, N.N.: Linear and quasilinear equations of parabolic type. Amer. Math. Soc., Providence, RI, 1968

[29] Leonardi, S.: Remarks on the regularity of solutions of elliptic systems. Applied nonlinear analysis, pp. 325–344, Kluwer/Plenum, New York, 1999.

[30] Leonardi, S.: Gradient estimates below duality exponent for a class of linear elliptic systems. *NoDEA Nonlinear Differential Equations Appl.* **18** (2011), 237–254.

[31] Leonardi, S.: Fractional differentiability for solutions of a class of parabolic systems with $L^{1,\theta}$-data. *Nonlinear Anal.* **95** (2014), 530–542.

[32] Leonardi, S.: Morrey estimates for some classes of elliptic equations with a lower order term. *Nonlinear Anal.* **177**, Part B, (2018), 611–627.

[33] Leonardi, S., Kottas, J., Stará, J.: Hölder regularity of the solutions of some classes of elliptic systems in convex nonsmooth domains. *Nonlinear Anal.* **60** (2005), 925–944.

[34] Leonardi, S., Stará, J.: Regularity results for the gradient of solutions of linear elliptic systems with VMO-coefficients and $L^{1,\lambda}$ data. *Forum Math.* **22** (2010), 913–940.

[35] Leonardi, S., Stará, J.: Regularity up to the boundary for the gradient of solutions of linear elliptic systems with VMO-coefficients and $L^{1,\lambda}$ data. *Complex Var. Elliptic Equ.* **56** (2011), 1085–1098.

[36] Leonardi, S., Stará, J.: Regularity results for solutions of a class of parabolic systems with measure data. *Nonlinear Anal.* **75** (2012), 2069–2089.

[37] Leonardi, S., Stará, J.: Higher differentiability for solutions of a class of parabolic systems with $L^{1,\theta}$-data. *Q. J. Math.* **66** (2015), 659–676.

[38] Lions, J.-L.: Quelques méthodes de résolution des problèmes aux limites non linéairies. Dunod, Paris, 1969.

[39] Lions, J.-L.: Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles. Dunod, Paris, 1968.

[40] Lions, J.-L., Magenes, E.: Problémes aux limites non homogènes et applications, Vol. 1. Dunod, Paris, 1968.

[41] Lions, J.-L., Magenes E.: Non-homogeneous boundary value problems and applications, Vol. 2. Springer, Berlin, 1972.

[42] Litvinov, W.G.: Optimization in elliptic problems with applications to mechanics of deformable bodies and fluid mechanics. Birkhäuser, Basel, 2000.

[43] Maugin, G.A.: Continuum mechanics of electromagnetic solids. North-Holland, Amsterdam, 1988.

[44] Mingione, G.: The Calderón–Zygmund theory for elliptic problems with measure data. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **6** (2007), 195–261.

[45] Schwartz, L.: Analyse mathématique, I. Hermann, Paris, 1967

[46] Solonnikov, V.A.: On boundary value problems for linear parabolic systems of differential equations of general form. *Trudy MIAN SSSR* **83** (1965), 3–162 (in Russian). English translation: Proceedings of the Steklov Institute of Mathematics, no. 83 (1965), Boundary Value Problems of Mathematical Physics III, edited by O.A. Ladyzhenskaya.