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Enumeration degrees and topology

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Abstract. The study of enumeration degrees appears prima facie to be far removed from topology. Work by Miller, and subsequently recent work by Kihara and the author has revealed that actually, there is a strong connection: Substructures of the enumeration degrees correspond to σ-homeomorphism types of second-countable topological spaces. Here, a gentle introduction to the area is attempted.

1 Enumeration reducibility

Enumeration reducibility is a computability-theoretic reducibility for subsets of $\mathbb{N}$ introduced by Friedberg and Rogers [10].

Definition 1. *$A \leq_e B$ iff there is a c.e.-set W such that:*

$$A = \{n \in \mathbb{N} \mid \exists k \in \mathbb{N}\ \exists m_0, \dots, m_k \in B\ \langle n, m_0, \dots, m_k \rangle \in W\}$$

where $\langle\ \rangle : \mathbb{N}^* \to \mathbb{N}$ is some standard coding for tuples of arbitrary length. We write $\mathcal{E}$ for the collection of enumeration degrees.

The intuitive idea is that the elements of W are rules, that upon observing the presence of some finite collection of numbers in B let us conclude the presence of some number in A. Contrasted to Turing reducibility, we note that the symmetry between presence and absence is broken: We only receive positive information about B, and only need to provide position information about A.

Looking at individual sets, we find that enumeration reducibility and Turing reducibility appear very different: For any combination of enumeration and Turing reductions between A and B, we can construct sets realizing that combination. On the level of the degree structures, both $\mathcal{E}$ and the Turing degrees (to be denoted by $\mathcal{T}$) are join-semilattices with the usual tupling function $\oplus$ acting as join. We can make the following observation:

Observation 2 *The map $A \mapsto A \oplus A^C$ induces a join-semilattice embedding of $\mathcal{T}$ into $\mathcal{E}$.*

A different intuitive interpretation of what enumeration reducibility is about is connected to this observation: Using the usual oracle Turing machines, we obtain a reducibility not only between total functions, but between partial functions, too (e.g. [7, Section 11.3]). We obtain the degree structure $\mathcal{E}$ like that, and the embedding just becomes the natural inclusion of the total functions into the partial functions.

A third notion of reducibility relevant for us is Medvedev reducibility. Rather than being about subsets of $\mathbb{N}$, we are dealing with subsets of $2^{\mathbb{N}}$ or $\mathbb{N}^{\mathbb{N}}$ here. We recall that computability for functions of type $F :\subseteq 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ can be defined via Type-2 Turing machines (which run forever, and have a write-once-only output tape), or equivalently via Turing functionals. In the latter case, we consider an oracle Turing machine M. We say that M computes F, if given oracle access to $p \in 2^{\mathbb{N}}$, it computes $n \mapsto F(p)(n)$.

Definition 3. *$A \leq_M B$ if there is a computable $F : B \to A$. We denote the collection of Medvedev degrees by $\mathcal{M}$.*

Both the Turing degrees and the enumeration degrees embed in natural ways into the Medvedev degrees: For the former, map $p \in 2^{\mathbb{N}}$ to $\{p\} \subseteq 2^{\mathbb{N}}$. For the latter, map $A \subseteq \mathbb{N}$ to $\{p \in \mathbb{N}^{\mathbb{N}} \mid \forall n \in \mathbb{N}\ n \in A \Leftrightarrow \exists k \in \mathbb{N}\ p(k) = n + 1\}$.[1] In words, to embed an enumeration degree into the Medvedev degrees, we move from a set $A \subseteq \mathbb{N}$ to the set of all enumerations of A. If we start with a Turing degree, embed it as enumeration degree, and then embed that as a Medvedev degree, we obtain the same degree (although not the same set) as when we move from Turing degrees directly to Medvedev degrees.

2 Represented spaces and generalized Turing reductions

If we wish to delve deeper into the idea that the difference between Turing reducibility and enumeration reducibility is about having different information about the sets available, we are led to the notion of a *represented space*. Represented spaces are the means by which we introduce computability to the spaces of interest in computable analysis [44].

Definition 4. *A represented space is a set X together with a partial surjection $\delta_{\mathbf{X}} :\subseteq \mathbb{N}^{\mathbb{N}} \to X$.*

A function between represented spaces $\mathbf{X} = (X, \delta_{\mathbf{X}})$ and $\mathbf{Y} = (Y, \delta_{\mathbf{Y}})$ is just a set-theoretic function on the underlying sets. We say that $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is a *realizer* of $f :\subseteq \mathbf{X} \to \mathbf{Y}$, if $\delta_{\mathbf{Y}}(F(p)) = f(\delta_{\mathbf{X}}(p))$ for all $p \in \mathrm{dom}(f \circ \delta_{\mathbf{X}})$. We then call a function between represented spaces *computable*, if it has a computable realizer. More on the theory of represented spaces is presented in [35].

A potential way to introduce Turing reducibility is to say that for $p, q \in \mathbb{N}^{\mathbb{N}}$ we have $p \leq_T q$ if there is a computable $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ with $F(q) = p$. Not only does this approach align well with our embedding of $\mathcal{T}$ into $\mathcal{M}$, it immediately suggests a generalization. Given that we have available to us a notion of partial computable functions between represented spaces in general, we can introduce:

[1] Using $p(k) = n + 1$ rather than $p(k) = n$ is necessary to deal with the empty set in a uniform way.

Definition 5 (Kihara & P. [24]). *For represented spaces* $\mathbf{X}$, $\mathbf{Y}$, *and elements* $x \in \mathbf{X}$, $y \in \mathbf{Y}$, *say that* x *is reducible to* y *(written* $x^{\mathbf{X}} \leq_T y^{\mathbf{Y}}$*), if there is a computable* $F :\subseteq \mathbf{Y} \to \mathbf{X}$ *with* $F(y) = x$.

Unraveling the definitions, we see that $x^{\mathbf{X}} \leq_T y^{\mathbf{Y}}$ is just a fancy way of saying $\delta_{\mathbf{X}}^{-1}(x) \leq_M \delta_{\mathbf{Y}}^{-1}(y)$. Considering the degree structure for arbitrary represented spaces just gives back $\mathcal{M}$ again. However, we do gain two advantages: On the one hand, we can now deal with the degrees of meaningful mathematical objects, rather than faceless subsets of Baire space. On the other hand, we can look at the degrees of all points in more restricted classes of represented spaces than all of them at once, and investigate which degrees are present in those.

Two particular classes of represented spaces have received significant attention, computable metric spaces and countably-based spaces[2]. In both, representations can be constructed in a canonical way from other data about the space:

Definition 6. *A computable metric space is a metric space* (X, d) *together with a dense sequence* $(a_i)_{i\in\mathbb{N}}$ *such that* $(i, j) \mapsto d(a_i, a_j) : \mathbb{N}^2 \to \mathbb{R}$ *is computable. Its associated representation is defined via* $\delta(p) = x$ *iff* $\forall i \in \mathbb{N}\ d(x, a_{p(i)}) < 2^{-i}$.

Definition 7. *Given an enumeration* $(B_n)_{n\in\mathbb{N}}$ *of a basis of a* T_0 *topological space, its induced representation is given by* $\delta(p) = x$ *iff:*

$$\{n \mid x \in B_n\} = \{n \mid \exists i\ p(i) = n\}$$

In effective descriptive set theory [32], a slightly different effectivization of metric spaces is used, namely recursively presented metric spaces. The difference disappears if we add a completeness-requirement and forget the specific choice of metric [14]. We then arrive at computable Polish spaces. The complete version of countably-based T_0 spaces are the quasi-Polish spaces [4].

The generalized reducibility restricted to computable metric spaces was studied by Miller [30]. He showed that there are indeed degrees of points in computable metric spaces (he called these *continuous* degrees) that are not Turing degrees, and obtained many results about them. Since there are universal Polish spaces (for example, the Hilbert cube $[0,1]^\omega$ or the space of continuous functions $\mathcal{C}([0,1],[0,1])$), we can in particular define:

Definition 8. *We call* $\mathrm{Spec}([0,1]^\omega)$ *the* continuous degrees.

For countably-based spaces, we find that there too exists a universal space. Let $\mathcal{O}(\mathbb{N})$ be the space of open subsets of $\mathbb{N}$, i.e. of all subsets of $\mathbb{N}$ represented via $\psi(p) = \{n \mid \exists i\ p(i) = n+1\}$. Alternatively, $\mathcal{O}(\mathbb{N})$ can be seen as derived from the countably-based topology generated by the subbasis $\{\{U \subseteq \mathbb{N} \mid n \in U\} \mid n \in \mathbb{N}\}$, i.e. carrying the Scott topology.

[2] There are some variations here regarding what aspects are required to be effective. Typical names used in the literature are *effective topological space*, *computable topological space* or *effectively enumerable topological space*, see e.g. [25,45]. These details do not matter for our purposes.

Observation 9 $\mathrm{Spec}(\mathcal{O}(\mathbb{N}) = \mathcal{E}$

Thus, we see that the enumeration degrees are the degrees of points in countably-based spaces.

3 σ-homeomorphisms

While we have seen that enumeration degrees can naturally be conceived of as the degrees of points in countably-based spaces, we have not yet discussed how topological properties of a space interact with the degrees of its points. The relevant notion here is that of a σ-homeomorphism. We have both a computable and a continuous version of σ-homeomorphism.

Definition 10. *A represented space* $\mathbf{X}$ *σ-embeds into a represented space* $\mathbf{Y}$*, if there is a countable partition* $\mathbf{X} = \bigcup_{i\in\mathbb{N}} \mathbf{X}_i$ *such that any* $\mathbf{X}_i$ *embeds into* $\mathbf{Y}$*.* [3] *If* $\mathbf{X}$ *σ-embeds into* $\mathbf{Y}$ *and vice versa, we call* $\mathbf{X}$ *and* $\mathbf{Y}$ *σ-homeomorphic.*

Definition 11 (Kihara & P. [24]). *For a represented space* $\mathbf{X}$*, let* $\mathrm{Spec}(\mathbf{X})$ *be the set of degrees of points in* $\mathbf{X}$*.*

Theorem 12 (Kihara & P. [24]). *The following are equivalent for represented spaces* $\mathbf{X}$*,* $\mathbf{Y}$*:*

1. $\mathrm{Spec}(\mathbf{X}) \subseteq \mathrm{Spec}(\mathbf{Y})$
2. $\mathbf{X}$ *computably σ-embeds into* $\mathbf{Y}$

Thus, we see that the degrees present in a space (above some oracle) just characterize its σ-homeomorphism type. For Polish spaces, the question of their σ-homeomorphism types has received significant attention. Here, the partition of $\mathbf{X}$ in Definition 10 can even be chosen to be Π^0_2. Intricate arguments from descriptive set theory [13, 33, 38] then show that for Polish spaces, σ-homeomorphism agrees with second-level Borel isomorphism. Jayne had explored second-level Borel isomorphism of Polish spaces in 1974 [20] motivated by applications in Banach space theory. While it is well-known that $2^{\mathbb{N}}$ and $[0,1]^\omega$ are not σ-homeomorphic[4], it remained open whether there were more σ-homeomorphism types of uncountable Polish spaces. This question was reraised by Motto-Ros [33] and by Motto-Ros, Schlicht and Selivanov [34]. The context of [34] is the generalization of Wadge degrees. The question was answered using recursion-theoretic methods by Kihara and P., yielding:

Theorem 13 (Kihara & P. [24]). *The poset* $(\omega_1^{\leq\omega}, \subseteq)$ *of countable subsets of the first uncountable ordinal* ω_1 *ordered by set-inclusion embeds into the poset of uncountable Polish spaces ordered by σ-embeddability.*

[3] We have slightly deviated from the usual definition here. In topology, we would typically demand that the $\mathbf{X}_i$ can be *disjointly* embedded into $\mathbf{Y}$. The difference can be removed by replacing $\mathbf{Y}$ with $\mathbf{Y} \times \mathbb{N}$. The results we need from topology hold for either version, and the present one makes the connection to degree theory more elegant.

[4] We now realize that this means that there are continuous degrees which are not Turing degrees!

4 Non-total continuous degrees

Miller had observed in [30] that all continuous degrees share a peculiar property, namely being *almost-total* (the name was coined later, in [2]). The name is explained by noting that an enumeration degree is called *total*, if it is in the range of the embedding of the Turing degrees.

Definition 14. *An enumeration degree d is almost-total, if for every total degree p with $p \nleq_e d$ we find that $p \oplus d$ is total.*

The question of existence of non-total almost-total degrees prima facie is purely recursion-theoretic question. Miller's result shows that the existence of continuous degrees which are not Turing degrees gives a positive answer. Miller's proof of the existence of continuous non-total degrees in [30] proceeds by constructing a multi-valued function on $[0, 1]^\omega$ whose fixed-points are non-total, and invoking a generalization of Brouwer's fixed point theorem to conclude existence. It is, in particular, relying heavily on topological arguments.

A different proof follows from the observation by Day and Miller [8] that Levin's neutral measures from [27] (see also [11]) have non-total continuous degrees. A measure μ is called weakly *neutral*, if every point is μ-random – this in particular requires every point computable from μ to have positive measure. The existence of neutral measures is obtained via the Kakutani fixed point theorem.

We already mentioned that by Theorem 12 the existence of non-total continuous degrees is equivalent to saying that $2^{\mathbb{N}}$ and $[0, 1]^\omega$ are not σ-homeomorphic. This is in turn a consequence of a classic result in topological dimension theory [19]: A Polish space is countably-dimensional iff it is σ-homeomorphic to $2^{\mathbb{N}}$ – and the Hilbert cube is not countably-dimensional. We thus have three different proofs of the existence of the recursion-theoretic theorem that non-total almost-total degrees exist – and all of them invoke classic topological theorems.

That the existence of non-total almost-total degrees is proven via a seeming detour through the continuous degrees is not accident: Andrews, Igusa, Miller and Soskova proved that the almost-total degrees are exactly the continuous degrees [2]. Their proof proceeds via a number of characterizations, essentially showing that every almost-total degree has a certain representative, and that the collection of these representatives forms an effective regular topological space. Schröder's effective metrization theorem [16,39] then enables the conclusion that all these representatives have continuous degree.

5 G_δ-spaces and cototal degrees

That enumeration degrees correspond to countably-based spaces, and Turing degrees to Cantor space, or more generally, countably-dimensional spaces could still be put aside as a superficial resemblance. We shall thus present further examples of topological spaces and substructures of the enumeration degrees both studied in their own right, and explain how they link up.

The total enumeration degrees can be characterized as having a representative $A \subseteq \mathbb{N}$ such that $A^C \leq_e A$. We can dualize this to get the *cototal* enumeration degrees as those having a representative such that $A \leq_e A^C$. Every total degree is cototal, but not vice versa. McCarthy [29] revealed various characterizations of the cototal degrees: They are the degrees of complements of maximal antichains in $\mathbb{N}^{<\omega}$, of (uniformly) e-pointed trees and of the languages of minimal subshifts. Here, a (uniformly) e-pointed tree is an infinite binary tree T such that every infinite path through T is (uniformly) $\leq_e$-above T. For more on degrees and minimal subshifts, see [21]. The cototal enumeration degrees are further studied in [1, 31].

A topological space is a G_δ-space, if every closed subset can be written as a countable intersection of open sets. If we consider only countably-based spaces, this is equivalent to saying that every closed subset is equal to the intersection of all open sets containing it. Every Polish space is G_δ, while neither the Sierpiński space $\mathbb{S}$ nor $\mathcal{O}(\mathbb{N})$ are.

Theorem 15 (Kihara, Ng and P. [23]). *The degrees of points in countably-based G_δ-spaces are exactly the co-total degrees.*

In particular, we can conceive of the space of complements of maximal antichains in $\mathbb{N}^{<\omega}$ or the space of languages of minimal subshifts to be a universal G_δ-space, taking into account McCarthy's results.

6 Graph-cototal degrees and the cofinite topology

Call an enumeration degree graph-cototal, if it contains a representative of the form $\text{Graph}(f)^C$ for some $f : \mathbb{N} \to \mathbb{N}$. Graph-cototal enumeration degrees were studied by Solon [42][5] in the context of quasi-minimal enumeration degrees[6].

To find their topological counterpart, we turn to the cofinite topology on $\mathbb{N}$. Here, a subset of $\mathbb{N}$ is open iff it is empty or cofinite. As a representation, we find that $\delta_{\text{cof}}(p) = n$ iff $\{p(i) \mid i \in \mathbb{N}\} = \mathbb{N} \setminus \{n\}$ produces the desired represented space $\mathbb{N}_{\text{cof}}$. The space $\mathbb{N}_{\text{cof}}$ is perhaps the simplest example of a topological space satisfying the T_1 separation axiom (every singleton is closed), but not the T_2 separation axiom (being Hausdorff, i.e. every two points are separated by disjoint open sets).

Observation 16 (Kihara, Lempp, Ng, P. [23]) $\text{Spec}(\mathbb{N}^\omega_{\text{cof}})$ *contains exactly the graph-cototal enumeration degrees.*

The question has been raised[7] whether all almost-total degrees are graph-cototal. Via the aforementioned results and Theorem 12, we can rephrase this question to a topological one:

[5] Solon uses the name cototal instead of graph-cototal, which we have already used for a different concept above.
[6] An enumeration degree is quasi-minimal, if it is non-computable, but every total degree below is computable.
[7] This open question was brought to the author's attention by Joe Miller.

Question 17. Does $[0,1]^\omega$ σ-embed into $\mathbb{N}_{\mathrm{cof}}^\omega$?

What we can rule out easily is an actual embedding of $[0,1]^\omega$ into $\mathbb{N}_{\mathrm{cof}}^\omega$, due to the following:

Theorem 18 (Sierpiński, see e.g. [9, Theorem 6.1.27]). *Let $\mathbf{X}$ be a connected compact metric space. Then every continuous $f : \mathbf{X} \to \mathbb{N}_{\mathrm{cof}}$ is constant.*

By a classic theorem from topological dimension theory (see [19]), the Hilbert cube cannot be the countable union of disconnected spaces. At first glance, this may appear to answer Question 17. However, there are connected metric spaces containing no non-trivial connected subspaces. Spaces with the latter property are called *punctiform*, and a construction of a connected punctiform space is found as [26, Example 1.4.8]. For a further discussion of punctiform spaces and additional references, see [28].

7 The lower reals and semirecursive sets

The lower reals $\mathbb{R}_<$ are the real numbers, where $x \in \mathbb{R}$ is represented via an increasing sequence $(q_i)_{i\in\mathbb{N}}$ of rationals with $\sup_{i\in\mathbb{N}} q_i = x$. Equivalently, they are the reals equipped with the (countably-based) topology $\{\{y \in \mathbb{R} \mid y < x\} \mid x \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$. This spaces appears naturally when performing the Dedekind-construction of the reals in a constructive setting, and has a central role in the development of measure theory via valuations (e.g. [6, 37]).

Recall from [22] that a set $A \subseteq \mathbb{N}$ is called *semirecursive*, if there is a computable function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for all $n, m \in \mathbb{N}$ we find $f(n,m) \in \{n,m\}$, and if $n \in A$ or $m \in A$, then $f(n,m) \in A$. Combining results by Jockusch [22] and by Ganchev and Soskova [12] shows:

Theorem 19. *The semirecursive enumeration degrees are precisely* $\mathrm{Spec}(\mathbb{R}_<)$.

For most natural spaces, taking finite products does not change their σ-homeomorphism type. In other words, the products of any two degrees of their corresponding spectra will lie in the spectrum again. The situation is different for semirecursive sets. It is readily seen that $2^\mathbb{N}$ does not embed into $\mathbb{R}_<$, whereas $\mathbb{R}_< \times \mathbb{R}_>$ contains a copy of $\mathbb{R}$ (and thus of $2^\mathbb{N}$). In degree language, almost all semirecursive degrees are not total, whereas any total degree can be written as a product of two semirecursive degrees. Using geometric reasoning, one can obtain the following general result:

Theorem 20 (Kihara & P. [24]). *Let $\mathbf{X}$ be uncountable, countably-based and T_1. Then the spectra of $\mathbf{X} \times \mathbb{R}_<^n$ and $\mathbb{R}_<^{n+1}$ are incomparable.*

One recursion-theoretic corollary is that for any n, there is a degree arising as the product of $n+1$ semirecursive degrees, but not of n semirecursive degrees and one graph-cototal degree, and vice versa. Studying the spectrum of spaces $\mathbf{X} \times \mathbb{R}_<$ also led to a generalization ([24, Lemma 8.2]) of Arslanov, Kalimullin and Cooper's result [3] that if a real x is neither left-c.e. nor right-c.e., then $x \in \mathbb{R}_<$ is quasi-minimal.

8 And more...

In [23], several countably-based spaces from "Counterexamples in topology" [43] had their spectra classified in recursion-theoretic terms, including the Arens square, the Gandy-Harrington space, Roy's space and the relatively-coprime topology on the integers. One can lift the notion of quasi-minimality to spaces: a non-computable point $x \in \mathbf{X}$ is called $\mathbf{Y}$-quasi-minimal, if x computes no non-computable point in $\mathbf{Y}$. Various existence result for such quasi-minimal points are provided in [23].

We can also leave behind the realm of enumeration degrees and countably-based spaces, and study degrees in non-countably-based spaces. The spectrum of $\mathcal{O}(\mathbb{N}^{\mathbb{N}})$ exceeds the enumeration degrees; we can show this by lifting a diagonalization argument from Hinman [17] from partial functions on $2^{\mathbb{N}}$ to partial functions on $\mathcal{O}(\mathbb{N})$.

Non-countably-based spaces can be very resistant to the usual descriptive set theoretic methods. Hoyrup [18] has shown that already the lowest levels of the Borel hierarchy behave very differently for $\mathcal{O}(\mathbb{N})$ than their usual behaviour on quasi-Polish spaces[8]. In [5, 41] various hierarchies of non-countably-based represented topological spaces are explored. It is an open task to explore how these align with hierarchies of spectra.

Two further approaches to non-countably-based spaces in sight are to generate examples via the sequential de Groot dual [15] (essentially, given a T_1-space, consider the space of singletons as a subspace of its space of closed subsets); or via countably cs-networks. It was shown by Schröder that the existence of these characterize the topological spaces than can be obtained as represented spaces in [40].

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References

1. Andrews, U., Ganchev, H.A., Kuyper, R., Lempp, S., Miller, J.S., Soskova, A.A., Soskova, M.I.: On cototality and the skip operator in the enumeration degrees. preprint, http://www.math.wisc.edu/ msoskova/preprints/cototal.pdf

[8] This raises the question how exactly one ought to define the Borel hierarchy in these spaces. One approach is found in [36].

2. Andrews, U., Igusa, G., Miller, J.S., Soskova, M.I.: Characterizing the continuous degrees. preprint (2017), http://www.math.wisc.edu/ jmiller/Papers/codable.pdf
3. Arslanov, M.M., Kalimullin, I.S., Cooper, S.B.: Splitting properties of total enumeration degrees. Algebra and Logic 42(1) (2003)
4. de Brecht, M.: Quasi-Polish spaces. Annals of Pure and Applied Logic 164(3), 354–381 (2013)
5. de Brecht, M., Schröder, M., Selivanov, V.: Base-complexity classifications of QCB_0-spaces. In: Beckmann, A., Mitrana, V., Soskova, M. (eds.) Evolving Computability: CiE 2015, Lecture Notes in Computer Science, vol. 9136, pp. 156–166. Springer (2015), http://dx.doi.org/10.1007/978-3-319-20028-6_16
6. Collins, P.: Computable stochastic processes. arXiv:1409.4667 (2014)
7. Cooper, S.B.: Computability Theory. Chapman and Hall/CRC (2004)
8. Day, A., Miller, J.: Randomness for non-computable measures. Transactions of the AMS 365 (2013)
9. Engelking, R.: General Topology. Heldermann, Berlin (1989)
10. Friedberg, R., Rogers, H.: Reducibility and completeness for sets of integers. Zeitschrift für mathematische Logik und Grundlagen der Mathematik 5, 117–125 (1959)
11. Gács, P.: Uniform test of algorithmic randomness over a general space. Theoretical Computer Science 341(1), 91 – 137 (2005), http://www.sciencedirect.com/science/article/pii/S030439750500188X
12. Ganchev, H.A., Soskova, M.I.: Definability via Kalimullin pairs in the structure of the enumeration degrees. Trans. Amer. Math. Soc. 367(7), 4873–4893 (2015)
13. Gregoriades, V., Kihara, T., Ng, K.M.: Turing degrees in Polish spaces and decomposability of Borel functions. preprint (2016)
14. Gregoriades, V., Kispéter, T., Pauly, A.: A comparison of concepts from computable analysis and effective descriptive set theory. Mathematical Structures in Computer Science (2016), http://arxiv.org/abs/1403.7997
15. de Groot, J., Strecker, G., Wattel, E.: The compactness operator in general topology. In: General Topology and its Relations to Modern Analysis and Algebra. pp. 161–163. Academia Publishing House of the Czechoslovak Academy of Sciences (1967), http://eudml.org/doc/221016
16. Grubba, T., Schröder, M., Weihrauch, K.: Computable metrization. Mathematical Logic Quarterly 53(4-5), 381–395 (2007)
17. Hinman, P.G.: Degrees of continuous functionals. J. Symbolic Logic 38, 393–395 (1973)
18. Hoyrup, M.: Results in descriptive set theory on some represented spaces. arXiv 1712.03680 (2017)
19. Hurewicz, W., Wallman, H.: Dimension Theory, Princeton Mathematical Series, vol. 4. Princeton University Press (1948)
20. Jayne, J.E.: The space of class α Baire functions. Bull. Amer. Math. Soc. 80, 1151–1156 (1974)
21. Jeandel, E., Vanier, P.: Turing degrees of multidimensional sfts. Theoretical Computer Science 505, 81 – 92 (2013), http://www.sciencedirect.com/science/article/pii/S0304397512008031
22. Jockusch, C.: Semirecursive sets and positive reducibility. Transactions of the AMS 131(2), 420–436 (1968)
23. Kihara, T., Ng, K.M., Pauly, A.: Enumeration degrees and non-metrizable topology. in preparation (201X)
24. Kihara, T., Pauly, A.: Point degree spectra of represented spaces. arXiv:1405.6866 (2014)

25. Korovina, M.V., Kudinov, O.V.: Towards computability over effectively enumerable topological spaces. Electr. Notes Theor. Comput. Sci. 221, 115–125 (2008)
26. Krupka, D.: Introduction to Global Variational Geometry. Elsevier (2000)
27. Levin, L.A.: Uniform tests of randomness. Soviet Math. Dokl. 17(2), 337–340 (1976)
28. Lipham, D.: Widely-connected sets in the bucket-handle continuum. arXiv:1608.00292 (2016)
29. McCarthy, E.: Cototal enumeration degrees and their application to computable matheamtics. Proceedings of the AMS (to appear)
30. Miller, J.S.: Degrees of unsolvability of continuous functions. Journal of Symbolic Logic 69(2), 555 – 584 (2004)
31. Miller, J.S., Soskova, M.I.: Density of the cototal enumeration degrees. Annals of Pure and Applied Logic (2018), http://www.sciencedirect.com/science/article/pii/S0168007218300010
32. Moschovakis, Y.N.: Descriptive Set Theory, Studies in Logic and the Foundations of Mathematics, vol. 100. North-Holland (1980)
33. Motto-Ros, L.: On the structure of finite level and omega-decomposable Borel functions. Journal of Symbolic Logic 78(4), 1257–1287 (2013)
34. Motto Ros, L., Schlicht, P., Selivanov, V.: Wadge-like reducibilities on arbitrary quasi-polish spaces. Mathematical Structures in Computer Science pp. 1–50 (11 2014), http://journals.cambridge.org/article_S0960129513000339, arXiv 1204.5338
35. Pauly, A.: On the topological aspects of the theory of represented spaces. Computability 5(2), 159–180 (2016), http://arxiv.org/abs/1204.3763
36. Pauly, A., de Brecht, M.: Descriptive set theory in the category of represented spaces. In: 30th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS). pp. 438–449 (2015)
37. Pauly, A., Fouché, W.: How constructive is constructing measures? Journal of Logic & Analysis 9 (2017), http://logicandanalysis.org/index.php/jla/issue/view/16
38. Pawlikowski, J., Sabok, M.: Decomposing Borel functions and structure at finite levels of the Baire hierarchy. Ann. Pure Appl. Logic 163(12), 1748–1764 (2012)
39. Schröder, M.: Effective metrization of regular spaces. In: Ko, K.I., Nerode, A., Pour-El, M.B., Weihrauch, K., Wiedermann, J. (eds.) Computability and Complexity in Analysis. Informatik Berichte, vol. 235. FernUniversität Hagen (1998)
40. Schröder, M.: Extended admissibility. Theoretical Computer Science 284(2), 519–538 (2002)
41. Schröder, M., Selivanov, V.L.: Some hierarchies of QCB_0-spaces. Mathematical Structures in Computer Science pp. 1–25 (11 2014), arXiv 1304.1647
42. Solon, B.: Co-total enumeration degrees. In: Beckmann, A., Berger, U., Löwe, B., Tucker, J.V. (eds.) Proc. of CiE 2006: Logical Approaches to Computational Barriers. pp. 538–545. Springer Berlin Heidelberg, Berlin, Heidelberg (2006)
43. Steen, L.A., Seebach, Jr., J.A.: Counterexamples in topology. Springer-Verlag, New York-Heidelberg, second edn. (1978)
44. Weihrauch, K.: Computable Analysis. Springer-Verlag (2000)
45. Weihrauch, K., Grubba, T.: Elementary computable topology. Journal of Universal Computer Science 15(6), 1381–1422 (2009)