D-brane sources in supergravity and gauge/string duality at finite temperature.

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D-brane sources in supergravity and gauge/string duality at finite temperature

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Submitted to the University of Wales in fulfilment of the requirements for the degree of Doctor of Philosophy

Swansea University

2010

Abstract

This thesis deals with two independent aspects of gauge/string duality: The inclusion of fundamental matter in string duals via backreaction and the study of quark-gluon plasma physics using the duality. Concerning the flavoring procedure, we focus on the role of source terms for D-branes. Here, we are especially interested in various technical issues such as the construction of suitable source densities, their relation to generalized calibrated geometry and the M-theory lift of such sources in the special case where these appear as smeared KK-monopoles.

In this context we will study several examples of flavored supergravity duals, such as the flavored Maldacena-Núñez and Klebanov-Witten solutions, and further examples based on D5-D5, D6-D6 and D3-D7 intersections in 2+1 and 3+1 dimensions, all of which preserve some supersymmetry.

The parts focussing on QGP physics will exhibit an attempt at constructing a type IIA background based on D6-branes wrapped on three-cycles that is dual to a super Yang-Mills theory with four supercharges at finite temperature. Studying thermodynamic properties, deconfinement as well as parton energy loss, we come to the puzzling conclusion that the standard approach to constructing such a solution does not provide the searched for dual. We are able to give some explanation for this by comparing the eleven-dimensional background with the Schwarzschild and Reissner-Nordstrom black holes in four dimensions.

Declaration

This work has not previously been accepted in substance for any degree and is not being concurrently submitted in candidature for any degree.

Signed ____ (candidate)

Date August 22, 2010

First statement

This thesis is the result of my own investigations, except where otherwise stated. Where correction services have been used, the extent and nature of the correction is clearly marked in a footnote(s).

Other sources are acknowledged by endnotes giving explicit references. A bibliography is appended.

Signed ____ (candidate)

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Second statement

I hereby give consent for my thesis, if accepted, to be available for photocopying and for inter-library loan, and for the title and abstract to be made available to outside organisations.

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Date August 22, 2010

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Acknowledgements

First and foremost, I would like to thank my supervisor, Carlos Núñez, for his advice and guidance as well as our many discussions. I am especially grateful to him as he did not only teach me about the intricacies of contemporary high energy physics, but also gave me the freedom to let my work follow my own curiosities, even at times when this took me on very long detours. I have cherished this freedom a lot.

I am indebted to many other people who have been at Swansea University in the past four years and so would like to thank Gert Aarts, Adi Armoni, Gaetano Bertoldi, David Dunbar, Daniel Elander, Jérôme Gaillard, Simon Hands, Tim Hollowood, Carlos Hoyos, Emiliano Imeroni, Harald Ita, Prem Kumar, Biagio Lucini, Dario Martelli, Gregory Moraitis, Asad Naqvi, Ioannis Papadimitriou, Maurizio Piai and Graham Shore.

Since 2004 I have been a fellow of the *Studienstiftung des deutschen Volkes* (German National Academic Foundation), whose support extended to virtually all aspects of my academic career and has been a great help as well as an inspiration.

Finally, I would like to thank my family.

To my parents

Ich habe den Verdacht, daß alles viel schöner ist, als man darüber spricht. Alles ist viel schöner, als man bisher es sagen kann. Und sagen kann man bisher schon sehr viel, denn wir haben ja schon viel geschaffen, um auszudrücken, wie schön es ist. Wir machen neue Anläufe und versuchen immer neu, auszudrücken, wie schön alles ist. Aber schöner ist es trotzdem noch immer, als man es sagen kann.

Johanna Walser, "Verdacht"

Chapter 1

Introduction

Gauge/string duality takes its origin in the celebrated AdS/CFT correspondence [1]-[3] relating $\mathcal{N} = 4, SU(N_c)$ super Yang-Mills to type IIB string theory on $AdS_5 \times S^5$. Soon after its discovery, further dualities were postulated relating various gauge theories to their respective string duals [4] - [8]. As the duality provides an approach to both strongly coupled gauge theories as well as quantum gravity, applications are numerous and range from black holes to superconductors. In this thesis, we will focus on two issues: The gauge/string duality for gauge theories with fundamental matter, as well as the study of quark-gluon plasmas as observed in heavy ion collisions at RHIC or ALICE.

1.1 Flavors and D-brane sources

In order to study gauge theories with fundamental matter, one needs to introduce additional modes to the string theory side that transform under the fundamental representations of the gauge group and an additional, global flavor symmetry group. These modes are provided by the addition of further D-branes to the string theory background [9]. In a widespread abuse of terminology that we will adopt as well, they are referred to as flavors or quarks, and so the branes are known as flavor-branes. Of course there is a further gauge theory living on the world volume of the flavor-branes and if one had solely added N_f additional, space-time filling branes to the background, one would have simply changed the gauge group, by adding an $SU(N_f)$ factor or by enhancing it to $SU(N_c + N_f)$ for example, with details depending on the embedding and background. Thus, the flavor-branes are embedded in the geometry in such a way that they extend also along a non-compact cycle transverse to the color-branes. The effective Yang-Mills coupling of the new gauge degrees of freedom depends on the volume of the wrapped transverse cycle, and since this is infinite, the gauge symmetry appears as a global one from the point of view of the original gauge theory. Strings stretching between the color- and flavor-branes transform both under the original gauge symmetry as well as the new global one and do thus constitute the fundamental matter. It is customary to refer to these modes as quarks, but one should keep in mind that their link to their namesakes in QCD is rather tenuous.

The appearance of fundamental matter via additional D-branes in the string theory links also nicely with considerations made using diagrammatic expansions at large N_c in the 't Hooft double line notation. Here, fundamental matter adds boundaries to the diagrams, from which it follows that there has to be an open string sector in the dual string theory. This open string sector is provided for by the addition of the D-branes.

It is sufficient to study these additional D-branes in their probe limit, as long as the number of flavors is small compared to that of colors. Or more precisely, as long as the theory is in the 't Hooft limit

$$\lambda = g_{\rm YM}^2 N_c = \text{const} \quad g_{\rm YM}^2 \to 0 \quad N_c \to \infty \quad N_f = \text{const} \tag{1.1}$$

For an application of flavor-branes in the probe limit, see the study of mesons spectroscopy in [9], [10] and [11].

Of course, it is desirable to go beyond the probe limit. Not least of all because $N_f \sim N_c$ in the case of QCD. Further problems of interest are charge screening or Seiberg duality. The appropriate scaling limit here is the Veneziano limit

$$\lambda = g_{\rm YM}^2 N_c = \text{const} \quad g_{\rm YM}^2 \to 0 \quad N_c \to \infty \quad N_f \to \infty \quad \frac{N_c}{N_f} = \text{const} \tag{1.2}$$

Here it is not possible to ignore the backreaction of the flavor branes, and one turns to studying the system

$$S = S_{\rm M,IIA/B} + S_{\rm flavor} \tag{1.3}$$

where $S_{\rm M,IIA/B}$ is the action of $d = 11$ or type IIA/B supergravity and the source term $S_{\rm flavor}$ is essentially the standard brane action – we will comment on the precise form of S_{flavor} extensively in chapters 2 and 3. This line of research originated in [12] and [13] and has been under continuous development since then ([14] - [29]). For a recent review, see [30].

When constructing flavored supergravity duals, the standard approach is to start with an existing gauge/gravity dual relating an unflavored gauge theory with a certain supergravity background. While this is strictly speaking not necessary, it makes the search for a solutions as well as their interpretation in terms of a gauge dual considerably easier. Naturally, the additional flavor-branes will deform the background, and so the first step towards a flavored dual lies then in studying deformations of the unflavored dual. Following this, one searches for probe embeddings that are stable[1] for all deformations considered. The final step is to pick a physically suitable distribution of flavor branes and then solve the equations of motion derived from (1.3).

While the process is conceptually straightforward, there are numerous technical difficulties. If the flavor-branes are taken to be coincident, they contribute δ-function terms to the equations of motion, making them in all but the simplest cases intractable. Furthermore, one might argue that the action describing N_f conincident D-branes should have g_s corrections, which are not known, making it impossible to find a suitable source term S_{flavor}. These issues are addressed simultaneously by the smearing process. As the flavor-branes are space-time filling and extend along a further non-compact cycle, the directions transverse to them are usually compact. One then distributes the branes over this transverse cycle and takes the large N_f limit. Since the transverse cycle is compact, the separation between individual flavor branes shrinks until it is smaller than the string scale $\sqrt{\alpha'}$ and it is possible to approximate the brane distribution by a continuous function; the branes have been smeared over the compact cycle. Note that smearing is by far not unique to gauge/string duality with flavor. As a matter of fact, smeared branes appear in T-duality in the supergravity limit. Performing a T-duality along an internal direction of a Dp-brane leads to a D$p-1$-brane smeared along the T-dual coordinate. Hence it is not necessary to be in the large N_f limit for smearing to be a sensible process. In any

[1]The most convenient way to ensure stability is to restrict the search to supersymmetric backgrounds and embeddings. One then uses κ-symmetry or a calibration condition as the creterium for stability. See section 2.3.

case, after smearing, the δ-function sources are replaced by continuous source terms, so that the equations of motion are considerably easier to solve. In many cases it is also possible to smear the branes in such a way that PDEs become ODEs. Furthermore, as the flavor branes are still separated, the strings stretching between them are massive and we can ignore g_s corrections. However, all these simplifications come at a price. Upon smearing, the flavor group is broken $SU(N_f) \rightarrow U(1)^{N_f}$, and one has to keep in mind that one is studying a different theory.

In this thesis, we will be not so much concerned with the physics of gauge- and string-theories that can be gleaned from the duality, but with a series of issues related to the construction of duals. Even with the procedure and simplifications outlined above, the task is in general quite complicated and one does often rely heavily on supersymmetry.

Supersymmetry features in the construction of unflavored backgrounds using lower dimensional gauged supergravities [8], [31], in the study of brane embeddings via κ-symmetry [32] or calibrations [33] - [36] and especially in the integrability theorems [37] - [40] used to find explicit solutions. According to these it is sufficient to study the supersymmetry conditions as well as the equations of motion and Bianchi identities for the p-form fields; the second order Einstein and dilaton equations will then be implied. This of course further simplifies the search for solutions, as one studies the first order BPS-equations opposed to the second order Einstein or dilaton ones.

The role of the Ramond-Ramond equations of motion and Bianchi identities in these theorems is of special interest to us. After all, one might wonder why supersymmetry alone is not a sufficient condition for the equations of motion to be satisfied. The answer lies in the close relation between the Bianchi identities of the magnetic field strengths and the presence of sources. As we will discuss in great detail later, a violation of the magnetic Bianchi identity goes hand in hand[2] with the presence of D-brane sources. Hence these identities – as well as their dual equations of motion – need to be part of the integrability theorems, as the supersymmetry relations by themselves are not sensitive of the presence

[2] Recall that Chern-Simons terms in the supergravity actions do also modify the Bianchi identities. However, in all the cases studied in this thesis these terms can be dropped, leading to a one-to-one correspondence between source terms and Bianchi identities.

of additional source terms.[3]

The source-term violating the Bianchi identity should be thought of as a distribution density for the new sources and its explicit construction had been a problem in the early days of the flavoring program. Due to its importance in smearing, we will often refer to the source distribution as the smearing form. As the sources are smeared, one is not only dealing with a single brane embedding – and its associated source term – but with a family of mutually supersymmetric embeddings, parametrized by the transverse coordinates. In some cases it is also not possible to find global coordinates allowing for an explicit split of transverse and world-volume coordinates, complicating the situation even further.

Today, the construction of the smearing form is approached via two complementary *macroscopic* and *microscopic* approaches. We we will discuss the macroscopic perspective in detail in chapter 3. Here we will see how, for supersymmetric embeddings, the most generic form of the source term can be inferred from the specific form of the Ansatz for the background, the crucial link being the concept of the calibration form. The application of generalized calibrated geometry to the problem of brane physics has its origin in [36] yet had up to [28] not been applied to the flavoring procedure. Calibration forms are a property of the background space-time and do not depend on the individual brane embedding considered – hence the moniker "macroscopic". Earlier work had focussed on a specific family of embeddings, and in the context of this microscopic approach the construction of the source term was not necessarily straightforward – see however [20] for quite sophisticated technology used in this microscopic ansatz. Two two approaches have been linked in [41] and [42].

Let us now give a further outline of the chapters of this thesis dealing with backreacting flavors. As we have seen, the problem focusses on finding solutions to supergravity with sources, so section 2 reviews the standard $1/2$-BPS flat p-brane solutions of supergravity with their source terms. While the material is purely a review, it is often ignored and so we present it in some detail.

[3] One should be aware of the following distinction however: While the above holds in the context of integrability, it changes once one makes a specific ansatz for the background. As the sources modify all the fields in the background – especially the p-forms – they do appear in the supersymmetry variations once one substitutes a given ansatz into the variations. E.g. the backgrounds studied in [14] - [16] exhibit Seiberg duality, that is, invariance under $N_c \rightarrow N_f - N_c$, at the level of the supergravity variations once one has substituted the ansatz into them.

The remainder of chapter 2 gives then a further introduction to the various techniques and concepts encountered in flavoring that we have mentioned so far. In chapter 3 we will then turn to the actual problem of flavoring using the macroscopic perspective outlined before. Following this introduction of the flavoring procedure, we will give some further comments on the role of the source terms in chapter 4. Here we will especially focus on the distinction between color- and flavor-branes and the fact that usually only the latter appear in the action (1.3).

Chapter 5 deals with an issue inspired by some observations made when adding flavor-branes to a D6-system in type IIA that has a simple M-theory lift as pure gravity on a G_2-holonomy manifold ([43], [44] and [45]). The problem concerns the duality between type IIA string theory and eleven-dimensional supergravity – or other Kaluza-Klein setups in general. It can be easily summarized in by asking the following questions: What is the eleven-dimensional origin of the D6-brane's source term? And: What is the Kaluza-Klein lift of a monopole condensate? To illustrate the relation between these questions, one should recall that D6-branes couple magnetically to the Ramond-Ramond two-form $F_{(2)}$. While standard KK-formulas relate the associated one-form potential $A_{(1)}$ to the higher dimensional metric

$$\mathrm{d}s_{\mathrm{M}}^2 = e^{-\frac{2}{3}\Phi}\mathrm{d}s_{\mathrm{IIA}}^2 + e^{\frac{4}{3}\Phi}(A_{(1)} + \mathrm{d}z)^2 \tag{1.4}$$

monopole condensation in the lower dimensional theory – captured by $\mathrm{d}F_{(2)} \neq 0$ – implies that the relation between $F_{(2)}$ and the higher dimensional metric cannot be $F_{(2)} = \mathrm{d}A_{(1)}$. As the D6-branes couple to $\mathrm{d}F_{(2)}$, the Bianchi identity is violated by D6-sources, which is why we can compare the presence of such-soures with monopole condensation. While a first, partial solution was given in [46], the problem becomes truly apparent once the D6-branes are smeared, as the Bianchi identity is now violated on an open set. In [29] it was suggested studying the involved supersymmetries and G-structures, that one might resolve the issue by adding torsion terms to the higher dimensional theory.

Of course, we will encounter a series of examples of flavored backgrounds. In section 3.1.1 we will review some quite well known examples; flavored versions of the Maldacena-Núñez [8] and Klebanov-Witten [5] solutions as well as a $\mathcal{N} = 1$ example in $d = 2 + 1$ dimensions based on D5-branes. Following this we will then turn to a further D5-D5 system dual to a $d = 2+1$ dimensional gauge theory with $\mathcal{N} = 2$ supersymmetries and fundamental matter (section 3.2). The discussion of color- vs. flavor-branes in chapter 4 is supplemented by the very simple example of D3-D7 branes with eight supercharges; neither of the branes are wrapped, making embeddings, the action and the smearing form very simple. A further example in $d = 3+1$ is given by the case of the D6-D6 system in section 5.1.

1.2 Quark-gluon plasma physics

In the final chapter of this thesis – chapter 6 – we will take a look at an entirely different aspect of gauge/string duality: Its application to the study of quark-gluon plasma physics. This is the domain of non-perturbative QCD at finite temperature – an area where conventional methods such as Lattice-QCD encounter their limitations, albeit being still the method of choice for making predictions for real world physics. The highly fruitful relation between QGP physics and string theory started with studies of the shear-viscosity to entropy ratio $\frac{\eta}{s}$. Previous experiments at the Relativistic Heavy Ion Collider (RHIC) had shown that such plasmas behave as liquids of suprprisingly low viscosity, rather than the predicted gas-like behavior. This was confirmed by string theory calculations which introduced the bound $\frac{\eta}{s} \geq \frac{1}{4\pi}$, showing considerably better agreement with results from RHIC than estimates based on conventional methods. See [47] - [49].

The field enjoys to this day a very high level of activity. We will not follow most recent developments, but focus on an attempt to model QGP physics using a finite temperature variant of the background presented in chapter 5, that was studied extensively in [31], [45], [50] and [51]; that is, D6-branes in type IIA or equivalently eleven-dimensional supergravity on a manifold with G_2 holonomy[4]. Studying thermodynamic properties as well as jet-quenching and various Wilson lines, we will see that the finite temperature geometry exhibits physical properties very much unlike those of QGP physics in $d = 3+1$. This realization is slightly puzzling, as the possibilities of generalizing the G_2 holonomy solution to non-extremality are highly constrained as long as one wants to avoid the

[4]For a review of special holonomy manifolds in string- and M-theory see [52].

technical difficulties of adding further $G_{(4)}$ flux or deforming the internal geometry. Ultimately we will be able to partially explain our findings by linking the thermodynamic properties of the solution to those exhibited by Schwarzschild and Reissner-Nordstrom black holes and arguing that our eleven-dimensional new solution is essentially of Schwarzschild type.

Chapter 2

D-brane sources in supergravity

We now review some of the concepts essential to the appearance of D-branes in gauge/string duality, with a view towards the flavoring process. Starting with some basics regarding supergravity, we turn to a review of the flat 1/2-BPS p-brane solutions of supergravity and their source terms (section 2.2).

2.1 The supergravities

In the bulk of this thesis, we will be working with ten dimensional IIA/B or eleven dimensional supergravity. The bosonic parts of their relevant actions in Einstein frame are

$$
\begin{aligned}
S_{\mathrm{M}} &= \frac{1}{16\pi G_{11}}\int [\mathrm{d}^{11}x\sqrt{-g}R - \frac{1}{2}G_{(4)}\wedge *G_{(4)} + \frac{1}{6}A_{(3)}\wedge G_{(4)}\wedge G_{(4)}] \\
S_{\mathrm{IIA}} &= \frac{1}{16\pi G_{10}}\int [\mathrm{d}^{10}x\sqrt{-g}(R - \frac{1}{2}\partial^{\mu}\Phi\partial_{\mu}\Phi) - \frac{1}{2}e^{-\Phi}H_{(3)}\wedge *H_{(3)} \\
&\quad - \frac{1}{2}(e^{\frac{3}{2}\Phi}F_{(2)}\wedge *F_{(2)} + e^{\frac{\Phi}{2}}F_{(4)}\wedge *F_{(4)}) \\
&\quad + \frac{1}{2}B_{(2)}\wedge \mathrm{d}C_{(3)}\wedge \mathrm{d}C_{(3)}] \qquad (2.1) \\
S_{\mathrm{IIB}} &= \frac{1}{16\pi G_{10}}\int [\mathrm{d}^{10}x\sqrt{-g}(R - \frac{1}{2}\partial^{\mu}\Phi\partial_{\mu}\Phi) - \frac{1}{2}e^{-\Phi}H_{(3)}\wedge *H_{(3)} \\
&\quad - \frac{1}{2}(e^{2\Phi}F_{(1)}\wedge *F_{(1)} + e^{\Phi}F_{(3)}\wedge *F_{(3)} + F_{(5)}\wedge *F_{(5)}) \\
&\quad + C_{(4)}\wedge F_{(3)}\wedge H_{(3)}]
\end{aligned}
$$

with

$$
\begin{aligned}
F_{(5)} &= \mathrm{d}C_{(4)} + C_{(2)} \wedge H_{(3)} \\
F_{(4)} &= \mathrm{d}C_{(3)} + C_{(1)} \wedge H_{(3)} \\
F_{(3)} &= \mathrm{d}C_{(2)} - C_{(0)} \wedge H_{(3)}
\end{aligned}
\tag{2.2}
$$

In the context of smeared branes in chapters 3, 4 and 5 however, we will frequently encounter backgrounds with $H_{(3)} = 0$ and violated Bianchi identities $\mathrm{d}F_{(p+2)} = \rho_{(p+3)}$. Here one either has to work in terms of the magnetic dual $F_{(d-p-2)}$ or modify (2.2).

In the case of type IIB, the equations of motion have to be supplemented by the self-duality condition

$$
F_{(5)} = *F_{(5)} \tag{2.3}
$$

Also note that Newton's constants in eleven and ten dimensions are related to the relevant gravitational constants, the Planck length and the string scale $l_s = \sqrt{\alpha'}$ as

$$
\begin{aligned}
16\pi G_{11} &= 2\kappa_{11}^2 = \frac{(2\pi l_p)^9}{2\pi} \\
16\pi G_{10} &= 2\kappa_{10}^2 = \frac{(2\pi l_s)^8}{2\pi} g_s^2
\end{aligned}
\tag{2.4}
$$

Let us for a moment ignore the issue of solving the equations of motion of (2.1) and simply focus on the supersymmetry of the backgrounds. As we are always interested in purely bosonic solutions, we assume the fermionic fields to vanish. Hence the SUSY variations of the bosonic fields vanish and we can restrict to those of the fermionic ones. For $d = 11$, there is only the gravitino. Its variation is given by

$$
\delta_\epsilon \psi_\mu = D_\mu \epsilon + \frac{1}{4! \times 24} G_{\nu_1 \ldots \nu_4} (\Gamma_\mu{}^{\nu_1 \ldots \nu_4} - 8 \delta_\mu^{\nu_1} \Gamma^{\nu_2 \ldots \nu_4}) \epsilon \tag{2.5}
$$

where we made use of the covariant derivative for spinors $D_\mu \epsilon$ with spin connection $\omega_{\mu ab}$

$$
\begin{aligned}
D_\mu \epsilon &= \partial_\mu \epsilon + \frac{1}{4} \omega_{\mu ab} \Gamma^{ab} \epsilon \\
\omega_{abc} &= \frac{1}{2} (\Omega_{cab} - \Omega_{bac} - \Omega_{abc}) \\
\Omega_{abc} &= -(\mathrm{d}e_a)_{bc} = -\eta_{ad} (\partial_\mu e_\nu^d - \partial_\nu e_\mu^d) E_b^\mu E_c^\nu
\end{aligned}
\tag{2.6}
$$

For type IIA and type IIB supergravity, we have – in addition to the gravitino

– a dilatino. For IIA

$$
\begin{aligned}
\delta_\epsilon\lambda &= \frac{1}{4}\sqrt{2}D_\mu\Phi\Gamma^\mu\Gamma^{11}\epsilon + \frac{3}{16}\frac{1}{\sqrt{2}}e^{\frac{3}{4}\Phi}F_{\mu_1\mu_2}\Gamma^{\mu_1\mu_2}\epsilon \\
&\quad + \frac{1}{24}\frac{\imath}{\sqrt{2}}e^{-\frac{\Phi}{2}}H_{\mu_1\mu_2\mu_3}\Gamma^{\mu_1\mu_2\mu_3}\epsilon - \frac{1}{192}\frac{\imath}{\sqrt{2}}e^{\frac{\Phi}{4}}F_{\mu_1\mu_2\mu_3\mu_4}\Gamma^{\mu_1\mu_2\mu_3\mu_4}\epsilon \\
\delta_\epsilon\psi_\mu &= D_\mu\epsilon + \frac{1}{64}e^{\frac{3}{4}\Phi}F_{\mu_1\mu_2}({\Gamma_\mu}^{\mu_1\mu_2} - 14\delta_\mu^{\mu_1}\Gamma^{\mu_2})\Gamma^{11}\epsilon \\
&\quad + \frac{1}{96}e^{-\frac{\Phi}{2}}H_{\mu_1\mu_2\mu_3}({\Gamma_\mu}^{\mu_1\mu_2\mu_3} - 9\delta_\mu^{\mu_1}\Gamma^{\mu_2\mu_3})\Gamma^{11}\epsilon \\
&\quad + \frac{\imath}{256}e^{\frac{\Phi}{4}}F_{\mu_1\mu_2\mu_3\mu_4}({\Gamma_\mu}^{\mu_1\mu_2\mu_3\mu_4} - \frac{20}{3}\delta_\mu^{\mu_1}\Gamma^{\mu_2\mu_3\mu_4})\Gamma^{11}\epsilon
\end{aligned}
\tag{2.7}
$$

While for IIB[1]

$$
\begin{aligned}
\delta_\epsilon\lambda &= \frac{\imath}{2}(\partial_\mu\Phi + \imath e^\Phi\partial_\mu C_{(0)})\Gamma^\mu\epsilon^* - \frac{\imath}{24}(e^{-\frac{\Phi}{2}}H_{\mu_1\mu_2\mu_3} + \imath e^{\frac{\Phi}{2}}F_{\mu_1\mu_2\mu_3})\Gamma^{\mu_1\mu_2\mu_3}\epsilon \\
\delta_\epsilon\psi_\mu &= D_\mu\epsilon - \frac{\imath}{1920}F_{\mu_1\dots\mu_5}\Gamma^{\mu_1\dots\mu_5}\Gamma_\mu\epsilon \\
&\quad + \frac{1}{96}(e^{-\frac{\Phi}{2}}H_{\mu_1\mu_2\mu_3} + \imath e^{\frac{\Phi}{2}}F_{\mu_1\mu_2\mu_3})({\Gamma_\mu}^{\mu_1\mu_2\mu_3} - 9\delta_\mu^{\mu_1}\Gamma^{\mu_2\mu_3})\epsilon^*
\end{aligned}
\tag{2.8}
$$

The search for supersymmetric backgrounds is now considerably simplified by the integrability theorems of [38] and [39].[2] According to these, the dilaton and Einstein equations are implied by the combination of SUSY equations, Bianchi identities and p-form equations of motion.

A few comments on G-structures The first order equations one obtains by setting the supergravity variations (2.5), (2.7) or (2.8) to zero can be recast in the language of G-structures. The uses of G-structures are manifold, as they provide a very economic way of dealing with the supersymmetry conditions, even in the presence of fluxes – this is especially the case if they are used in conjunction with pure spinors, see [40] for an example of this and [53] for a review. Thus they provide an excellent means for studying problems such as the classification of supergravity backgrounds or the derivation of new solutions [54]. We will give a very informal yet hopefully intuitive introduction to this vast subject in the next paragraph and refer the reader to [55], [56], [57] and references therein for a more formal discussion.

[1] We write the two IIB spinors ϵ_L, ϵ_R as one complex spinor $\epsilon = \epsilon_L + \imath\epsilon_R$. To change to the notation using $\epsilon = \binom{\epsilon_L}{\epsilon_R}$ use

$$\epsilon^* \leftrightarrow \sigma_3\epsilon \qquad \imath\epsilon^* \leftrightarrow \sigma_1\epsilon \qquad \imath\epsilon \leftrightarrow -\imath\sigma_2\epsilon$$

[2] Note that all these integrability theorems require certain, mild assumptions. Most notable among these is the existence of a space/time split. They will all be satisfied in the following, so we refer the reader to the original papers for details [38, 39, 40].

In the context of this thesis, it is sufficient to think of the language of G-structures as generalizations of the special holonomy arguments used for string compactifications. In the absence of fluxes, the gravitino variation demands that the SUSY spinor has to be covariantly constant and the existence of such a spinor implies special holonomy. One should think of M-theory on G_2-holonomy manifolds or heterotic string theory on Calabi-Yau spaces. Crucially, one can decompose all fields in representations of the special holonomy group, a standard method in string phenomenology.

Now, when turning on the fluxes, the supergravity variations can still be interpreted as connections. As these are not metric-compatible, the existence of a covariantly constant spinor under these does not imply special holonomy. It does however imply the existence of a principle sub-bundle of the frame-bundle with structure group G. In other words, it is still possible to decompose fields under a group G. In the absence of fluxes, G is naturally the holonomy group of the preceding paragraph.

The existence of this principle sub-bundle and that of G-invariant forms are equivalent definitions of G-structures. In practice as well as in the examples encountered in the following chapters the more useful one is the latter definition. As we will see, these invariant tensors that can be constructed as bilinears of the SUSY spinor ϵ along the lines of[3]

$$(\bar{\epsilon}\Gamma_{\mu_1\dots\mu_p}\epsilon)\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_p} \tag{2.9}$$

The supergravity equations are then equivalent to first order differential equations satisfied by these forms. One can make this connection obvious as follows: Acting on (2.9) with the exterior differential d is equivalent to the action of $\nabla\wedge$, where ∇ is the Levi-Civita connection. This again can be rewritten in terms

[3]As a further note on conventions, note that

$$\Gamma_{\mu_1\dots\mu_p}=\frac{1}{p!}\sum_{\sigma\in S_p}(-1)^\sigma\Gamma_{\mu_{\sigma(1)}}\cdots\Gamma_{\mu_{\sigma(p)}}$$

denotes the fully antisymmetrized product of Γ-matrices. We will use an identical notation for differential forms. S_p is the group of permutations of p elements, $(-1)^\sigma$ is the sign of a given permutation.

We use an identical notation for wedge products of one-forms, i.e. for a set of differential forms ω^i, $i=1,\dots,k$ we define

$$\omega^{1\dots k}=\omega^1\wedge\cdots\wedge\omega^k$$

of the spin connection acting on ϵ (the Γ_μ are invariant under ∇). Using the supergravity variations one obtains first order equations involving the fluxes.

2.2 1/2-BPS branes in flat space

Let us now begin discussing p-brane sources, starting with the simplest solutions possible, flat 1/2-BPS p-branes. These are based on consistent truncations of equations (2.1). They are very well known, yet the discussion of their source terms is often neglected, so we will go over this in some detail. This section is based on [58] and [59]. One might also want to refer to the original papers [60] - [63] or reviews [64]. The connection to gauge/string duality is discussed in [4].

In all cases of interest to us one is able to drop the Chern-Simons terms, neglect (2.2) (as $H = 0$) and truncate and generalize the system to the form

$$S_{\mathrm{SUGRA}} = \frac{1}{16\pi G_D}\int [\mathrm{d}^D x\sqrt{-g}(R - \frac{1}{2}\partial^\mu\Phi\partial_\mu\Phi) - \frac{1}{2}e^{a\Phi}F_{(p+2)}\wedge *F_{(p+2)}] \tag{2.10}$$

where $F_{(p+2)}$ can either stand for the Neveux-Schwarz three form, for one of the Ramond-Ramond p-form field strengths or for the four form in eleven dimensions. For type II Dp-branes we have $a = \frac{3-p}{2}$ and $a = -1$ for the NS5-brane. In $d = 11$ there is no dilaton and so $a = 0$. The equations of motion are[4]

$$\begin{aligned} R_{\mu\nu} &= \frac{1}{2}\partial_\mu\Phi\partial_\nu\Phi + \frac{1}{2(p+2)!}e^{a\Phi}(pF_{\mu\lambda_1\ldots\lambda_{p+1}}F_\nu{}^{\lambda_1\ldots\lambda_{p+1}} - \frac{p+1}{d-2}g_{\mu\nu}F^2_{(p+2)}) \\ 0 &= \nabla^2\Phi - \frac{a}{2(p+2)!}e^{a\Phi}F^2_{(p+2)} \\ 0 &= \mathrm{d}(e^{a\Phi} * F_{(p+2)}) \end{aligned} \tag{2.11}$$

The simplest ansatz solving (2.11) leads to the electrically charged, extremal, black p-brane solutions [60]:

$$\begin{aligned} \mathrm{d}s^2 &= H^{-2\frac{d-2}{\Delta}}\mathrm{d}x^2_{1,p} + H^{2\frac{p+1}{\Delta}}(\mathrm{d}y_q^2 + \mathrm{d}z_r^2) \\ F_{(p+2)} &= e^{-\frac{a}{2}\Phi_\infty}\sqrt{2\frac{D-2}{\Delta}}\mathrm{d}(H^{-1} - 1)\wedge \mathrm{d}x^0\wedge\cdots\wedge\mathrm{d}x^p \\ e^\Phi &= e^{\Phi_\infty}H^{a\frac{D-2}{\Delta}} \end{aligned} \tag{2.12}$$

[4] Recall that the Hodge dual satisfies

$$\lambda_{(p)}\wedge *\mu_{(p)} = \frac{1}{p!}\lambda_{\mu_1\ldots\mu_p}\mu^{\mu_1\ldots\mu_p}\eta = \frac{1}{p!}\lambda_{\mu_1\ldots\mu_p}\mu^{\mu_1\ldots\mu_p}\sqrt{-g}\mathrm{d}^d x$$

with Φ_∞ constant and

$$\begin{aligned}
\Delta =&(p+1)(d-2)+\frac{1}{2}a^2(D-2) \\
D =& (p+1)+q+r \\
d =& q+r \\
H =& 1+\begin{cases} h|y| & q=1 \\ h\log y & q=2 \\ \frac{h}{y^{q-2}} & q\geq 3 \end{cases}
\end{aligned} \tag{2.13}$$

$H(y)$ is a harmonic function on $\mathbb{R}^q$. Note that we could absorb the slightly awkward factor $\sqrt{2\frac{D-2}{\Delta}}$ in the constant Φ_∞.

The branes extend along the $p+1$ dimensional space spanned by the x^μ. While they are localized at $y=0$ for $r=0$, the brane charge cannot be localized in the z^σ directions for $r>0$. One either thinks of a single brane having been smeared over the z^σ, or a superposition of a large (infinite) number of parallel branes distributed evenly over z^σ, which becomes indistinguishable from a continuous distribution in the limit where the separation between the branes is smaller than the string scale $\sqrt{\alpha'}$.

As a matter of fact, the solution (2.12) does not solve the equations of motion (2.11) everywhere. After all, the functions H are the fundamental solutions to the Laplace equations[5] on $\mathbb{R}^q$. I.e. for

$$\Psi_q(y)=\begin{cases} \frac{1}{2}|y| & q=1 \\ \frac{1}{2\pi}\log y & q=2 \\ -\frac{1}{q(q-2)\omega_q y^{q-2}} & q\geq 3 \end{cases} \tag{2.14}$$

we have

$$\Box_{(\mathbb{R}^q)}\Psi_q=\delta^{(q)}(y) \tag{2.15}$$

where $\Box_{(\mathbb{R}^q)}=\delta^{ab}\partial_a\partial_b$ is the Laplacian on the space parametrized by the y^a and $\omega_q=\frac{\pi^{q/2}}{\Gamma(q/2)}$ is the volume of a unit q-sphere. Our earlier solutions H agree with the functions Ψ_q up to normalization and the overall additive constant and it follows that all terms in the supergravity equations of motion (2.11) including the Laplacian $\Box_{(\mathbb{R}^q)}$ generate δ-function singularities, indicating the presence of sources. One needs to match these singularities with the contribution of additional source terms that we will add to the action shortly.

[5] A discussion of this and a proof of (2.14) can be found in most textbooks on partial differential equations. E.g. [65].

It is quite interesting to study the origin and cancellation of these source terms in some detail. We will do so in the case of localized ($d = q, r = 0$) p-branes, with $d \geq 3$. In the case of the Einstein equations, it is only the Ricci tensor (and hence also the Ricci scalar) that contains second order derivatives. As a matter of fact, for the ansatz chosen, all first order terms cancel algebraically and the equation of motion reduces to those terms in the Einstein tensor that contain the Laplacian. I.e.

$$(R_{mn} - \frac{1}{2}g_{mn}R)_{\mathcal{O}(\Box H)} = \eta_{mn}\frac{D-2}{\Delta}H^{\alpha-\beta-1}\Box_{(\mathbb{R}^d)}H \tag{2.16}$$

with $\alpha = -2\frac{d-2}{\Delta}, \beta = 2\frac{p+1}{\Delta}$ and $m, n \in \{0, \ldots, p\}$ and

$$\Box_{\mathbb{R}^d}H = -d(d-2)\omega_d h\delta^{(d)}(y) \tag{2.17}$$

Note that (2.16) vanishes for $m, n \in \{p+1, \ldots, D\}$. In the case of the Dilaton equation,

$$(\nabla^2\Phi)_{\mathcal{O}(\Box H)} = \frac{D-2}{\Delta}\frac{a}{H}\Box_{\mathbb{R}^d}H \tag{2.18}$$

A suitable source term complements the action (2.10) as in (1.3) to

$$S = S_{\text{SUGRA}} + S_{\text{src}} \tag{2.19}$$

The contribution of S_{src} will cancel the terms on the right hand side of (2.16).

In the case at hand, the source term will be given by a suitable brane action. Later on, we will be focussing on Dp-branes with action

$$\begin{aligned} S_{\text{src}} = &- T_p \int \mathrm{d}^{p+1}\xi e^{\frac{p-3}{4}\Phi}\sqrt{-\det(X^*[g+B] + 2\pi\alpha' F)} \\ &+ (-1)^{p+1}\mu_p \int_{p+1} (\sum_n X^*[C_n] \wedge e^{X^*[B]+2\pi\alpha' F})_{p+1} \end{aligned} \tag{2.20}$$

For differential forms, X^* denotes the pull-back, while X^*g refers to the induced metric $\partial_i X^\mu \partial_j X^\nu g_{\mu\nu}$. To find the source for (2.10) and (2.12), we modify and truncate (2.12) slightly, dropping the world-volume gauge field F as well as the coupling to the Neveux-Schwarz two-form B and introducing an auxiliary metric γ_{ij}.

$$S = -\frac{T_p}{2}\int \mathrm{d}^{p+1}\xi\sqrt{-\gamma}e^{b\Phi}[\gamma^{ij}\partial_i X^\mu \partial_j X^\nu g_{\mu\nu} + (p-1)] + \mu_p \int X^* C_{(p+1)} \tag{2.21}$$

As in the case of the Polyakov action for strings, the role of the world-volume metric is that it allows us to rewrite an action that is non-linear in $\partial_i X^\mu \partial_j X^\nu g_{\mu\nu}$

as a linear one. The worldvolume fields γ_{ij}, X^μ need to satisfy the equations of motion[6]

$$\begin{aligned}
0 &= e^{b\Phi}\partial_i X^\mu \partial_j X^\nu g_{\mu\nu} - \frac{1}{2}\gamma_{ij}(e^{b\Phi}\gamma^{kl}\partial_k X^\mu \partial_l X^\nu g_{\mu\nu} + p - 1) \\
0 &= \partial_j(-T_p\sqrt{-\gamma}e^{b\Phi}\gamma^{ij}\partial_i X^\mu g_{\mu\lambda} + \frac{\mu_p}{p!}\epsilon^{ji_1\dots i_p}C_{\lambda i_1\dots i_p}) \\
&+ \{\frac{T_p}{2}\sqrt{-\gamma}e^{b\Phi}[b\partial_\mu\Phi(\gamma^{ij}\partial_i X^\kappa \partial_j X^\lambda - p + 1) + \gamma^{ij}\partial_i X^\kappa \partial_j X^\lambda \partial_\mu g_{\kappa\lambda}] \\
&- \frac{\mu_p}{(p+1)!}\epsilon^{i_0\dots i_p}\partial_{i_0}X^{\mu_0}\dots\partial_{i_p}X^{\mu_p}\partial_\mu C_{\mu_0\dots\mu_p}\}
\end{aligned} \tag{2.22}$$

The first of these is solved by

$$\gamma_{ij} = e^{b\Phi}\partial_i X^\mu \partial_j X^\nu g_{\mu\nu} \tag{2.23}$$

which, upon substitution into (2.21), yields an action of the standard form (2.20). We will turn to the second equation in (2.22) later. Let us investigate instead the relation between the source term and the space-time action. (2.21) modifies the equations of motion of the background space-time to

$$\begin{aligned}
0 &= R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \frac{1}{2}(\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}g_{\mu\nu}\partial^\lambda\Phi\partial_\lambda\Phi) \\
&- \frac{1}{2(p+2)!}e^{a\Phi}[(p+2)F_{\mu\lambda_1\dots\lambda_{p+1}}F_\nu{}^{\lambda_1\dots\lambda_{p+1}} - \frac{1}{2}g_{\mu\nu}F^2] \\
&+ \frac{8\pi G_D T_p}{\sqrt{-g}}\int \mathrm{d}^{p+1}\xi\sqrt{-\gamma}e^{b\Phi}\gamma^{ij}\partial_i X^\kappa \partial_j X^\lambda g_{\kappa\mu}g_{\lambda\nu}\delta^{(D)}(x - X(\xi)) \\
0 &= \partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi) - \frac{a}{2(p+2)!}\sqrt{-g}e^{a\Phi}F^2 \\
&- 8\pi G_D T_p b\int \mathrm{d}^{p+1}\xi\sqrt{-\gamma}e^{b\Phi}[\gamma^{ij}\partial_i X^\mu \partial_j X^\nu g_{\mu\nu} - (p-1)]\delta^{(D)}(x - X(\xi)) \\
0 &= \partial_\mu(\sqrt{-g}e^{a\Phi}F^{\mu\nu_0\dots\nu_p}) \\
&+ 16\pi G_D\mu_p\int \mathrm{d}^{p+1}\xi\epsilon^{i_0\dots i_p}\partial_{i_0}X^{\nu_0}\dots\partial_{i_p}X^{\nu_p}\delta^{(D)}(x - X(\xi))
\end{aligned} \tag{2.24}$$

Matching the solution (2.12) with the source terms arising from (2.14) will fix some relations between the various constants, $G_D, T_p, \mu_p, e^{\Phi_\infty}$ that we have introduced up to this point. It is easiest to do so in the case of the Dilaton where we need to match (2.18) with the source in (2.24).

$$\begin{aligned}
-\frac{D-2}{\Delta}\frac{ad(d-2)\omega_d h}{H}\delta^{(d)}(y) &\overset{!}{=} \frac{16\pi G_D T_p e^{b\Phi_\infty}b}{H}\delta^{(d)}(y) \\
h &= \frac{16\pi G_D T_p e^{-\frac{a}{2}\Phi_\infty}}{d(d-2)\omega_d}\frac{\Delta}{2(D-2)}
\end{aligned} \tag{2.25}$$

[6]The relative complexity of the second term in the equation of motion for the X^μ stems from the fact that Φ and $C_{(p+1)}$ depend on the X^μ explicitly, while γ_{ij} does only after we solve some of the equations of motion. Hence $\delta_{X^\mu}\gamma_{ij} = 0$.

Here we fixed $b = -\frac{1}{2}a$.

The Maxwell equation reduces to

$$0 = \sqrt{2\frac{D-2}{\Delta}} e^{-\frac{a}{2}\Phi_\infty} \Box_{\mathbb{R}^q} H + 16\pi G_D \mu_p \delta^{(d)}(x) \tag{2.26}$$

which imposes

$$h = \frac{16\pi G_D \mu_p e^{\frac{a}{2}\Phi_\infty}}{d(d-2)\omega_d} \sqrt{\frac{\Delta}{2(D-2)}} \tag{2.27}$$

Similarly, one matches (2.16) with the contribution of (2.21) to the Einstein equation

$$(R_{mn} - \frac{1}{2} g_{mn} R)_{\mathcal{O}(\Box H)} \stackrel{!}{=} -8\pi G_D T_p e^{b\Phi_\infty} H^{\alpha - \frac{d}{2}\beta + ab\frac{D-2}{\Delta}} \eta_{mn} \delta^{(d)}(y) \tag{2.28}$$

Here

$$\begin{aligned} 8\pi G_D T_p e^{-\frac{a}{2}\Phi_\infty} \delta^{(d)}(y) &= \frac{D-2}{\Delta} d(d-2)\omega_d h \delta^{(d)}(y) \\ h &= \frac{16\pi G_D T_p e^{-\frac{a}{2}\Phi_\infty}}{d(d-2)\omega_d} \frac{\Delta}{2(D-2)} \end{aligned} \tag{2.29}$$

in agreement with (2.25). Comparing (2.27) and (2.29) fixes the relation between T_p and μ_p

$$\frac{T_p}{\mu_p} = \sqrt{2\frac{D-2}{\Delta}} e^{a\Phi_\infty} \tag{2.30}$$

As we remarked after (2.13), we can absorb the square root into Φ_∞.

Let us finish this section with a discussion of the equation of motion of the embedding fields $X^\mu(\xi)$ – see again (2.22). As no fields in our ansatz depend on the world-volume coordinates ξ^i, they reduce to

$$\begin{aligned} 0 = &\frac{T_p}{2}\sqrt{-\gamma} e^{b\Phi}[b\partial_\mu \Phi(\gamma^{ij}\partial_i X^\kappa \partial_j X^\lambda - p + 1) + \gamma^{ij}\partial_i X^\kappa \partial_j X^\lambda \partial_\mu g_{\kappa\lambda}] \\ &- \frac{\mu_p}{(p+1)!} \epsilon^{i_0 \ldots i_p} \partial_{i_0} X^{\mu_0} \ldots \partial_{i_p} X^{\mu_p} \partial_\mu C_{\mu_0 \ldots \mu_p} \end{aligned} \tag{2.31}$$

which vanishes identically for $\mu \in \{0, \ldots, p\}$. Let us however generalize this part of the discussion to include non-extremal p-branes. That is, we assume the metric to take the form

$$\mathrm{d}s^2 = H(y)^{-2\frac{d-2}{\Delta}}[-f(y)\mathrm{d}t^2 + \mathrm{d}x_p^2] + \ldots \tag{2.32}$$

where we dropped the transverse directions, which include off-diagonal elements in our choice of coordinates, but do not contribute to the following discussion. (2.31) reduces to

$$0 = [\frac{1}{2}a^2\frac{D-2}{\Delta} + \frac{(p+1)(d-2)}{\Delta} f] H^{-2}\partial_\mu H - \frac{p+1}{2} H^{-1}\partial_\mu f \tag{2.33}$$

Setting $f = 1 - \frac{Q}{y^{d-2}}$, it follows that the above is only solved if $Q = 0$, i.e. if the brane is extremal. Note further that in the non-extremal case, the term $H^{-1}\partial_\mu f$ diverges as $y \to 0$, while $H^{-2}\partial_\mu H \to 0$. One can interpret this behavior in the light of supersymmetry. By introducing f, we only modify the part of the brane action coupling to the metric, but not that coupling to the $p+2$-form. The X^μ equation of motion can be thought of as a balancing between these two sectors (it imposes a relation between T_p and μ_p), so it is no surprise that it holds no longer once we have perturbed this balance. This might simply indicate an instability of the embedding or might indicate that it is not possible to find a source term for the non-extremal solution. One should take into account [66] however, where the authors constructed a finite temperature background including flavor branes. In opposite to our discussion in the previous paragraph however, this background's non-extrmality is due to a horizon associated with the color-branes, while only the flavor-branes are represented by a source.

Dropping the 1 in the harmonic function $H(y)$ in (2.13) yields the near-horizon limit of the extremal p-brane considered. From (2.14) it follows however that the source-terms are still necessary in this limit – the argument does not depend on the asymptotic value of H.[7]

2.3 Supersymmetric branes

In the previous section we were able to study the p-brane solutions as well as their source terms using the equations of motion of the supergravity-plus-brane action (2.19) alone. While this procedure can be extended to slightly more complicated cases such as branes on tori or branes at the tip of singularities, it quickly reaches the limits of what is feasible for more complicated solutions. As it is so often the case in string theory, supersymmetry is the method of choice to deal with this issue.

The integrability theorems of [38] and [39] were generalized to the case of backgrounds with sources in [40]. And again we only need to impose supersymmetry as well as the Bianchi identities and equations of motion for the p-form fields for the Einstein and dilaton equations to be satisfied. The main difference

[7] After all, no matter whether in the near-horizon limit or not, H is harmonic everywhere except at the origin. See e.g. chapter 2.2 of [65].

to the source-free case lies in the fact that the sources appear in the p-form equations. We already encountered this in the (2.24), where the Maxwell equation took the form $\mathrm{d} * (e^{a\Phi}F) \sim \delta^{(d)}(y)$. We will discuss in detail in chapter 3 how the right hand side can be interpreted as a source-density for the branes – it is in general the smearing form.

In this section, we shall give a short introduction to κ-symmetry [32] and generalized calibrations, the methods of choice when discussing the supersymmetry of branes. There are several definitions of a generalized calibration in the literature, yet for our purposes it is sufficient to think of them in their original form in [36]. Generalizations can be found in [67] or [68]. The discussion given ignores the case of world-volume fluxes and follows that of the review [69].

From an intuitive point of view, a given brane configuration is supersymmetric if it is a minimum energy configuration. In the absence of fluxes, brane actions for static branes are essentially volume integrals, and the minimum energy condition translates to a minimum volume one. I.e. in the absence of fluxes, branes wrap minimum volume cycles in a given homology class. Turning on fluxes deforms these cycles. The embeddings are still supersymmetric, but no longer of minimum volume.

κ-symmetry and calibration forms A p-brane embedding consists of a map

$$\begin{aligned} X : \mathbb{R}^{p+1} &\longrightarrow \mathbb{R}^D \\ \xi^i &\mapsto X^\mu(\xi) \end{aligned} \tag{2.34}$$

as well as the gauge-invariant combination of NS and world-volume gauge field $\mathcal{F} = X^*[B] + 2\pi\alpha' F$. It is considered κ- and hence supersymmetric iff the associated κ-symmetry matrix $\Gamma_\kappa = \Gamma_\kappa[X, \mathcal{F}]$ satisfies

$$\Gamma_\kappa \epsilon = \epsilon \tag{2.35}$$

where ϵ are the SUSY spinors of the background. The generic form of Γ_κ is quite involved, so we restrict to the case $\mathcal{F} = 0$:

$$\Gamma_\kappa = \frac{1}{(p+1)!} \frac{1}{\sqrt{-\det X^*[g]}} \epsilon^{i_0 \dots i_p} \begin{cases} (\Gamma^{11})^{\frac{p-2}{2}} \gamma_{i_0 \dots i_p} & \text{(IIA)} \\ \sigma_3^{\frac{p-3}{2}} \imath\sigma_2 \otimes \gamma_{i_0 \dots i_p} & \text{(IIB)} \end{cases} \tag{2.36}$$

$\gamma_i = \partial_i X^\mu \Gamma_\mu$ are the induced Γ-matrices on the brane world-volume, $\Gamma^{11} = \Gamma^{0 \dots 10}$ is the chirality matrix in type IIA.

Γ_κ is hermitian and squares to one. It follows that[8]

$$\epsilon^\dagger \frac{1-\Gamma_\kappa}{2}\epsilon = \epsilon^\dagger \frac{1-\Gamma_\kappa}{2}\frac{1-\Gamma_\kappa}{2}\epsilon = \left\| \frac{1-\Gamma_\kappa}{2}\epsilon \right\|^2 \geq 0 \tag{2.37}$$

Which implies that $\epsilon^\dagger\epsilon \geq \epsilon^\dagger\Gamma_\kappa\epsilon$ with equality if and only if the embedding is supersymmetric. Normalizing the spinor such that $\epsilon^\dagger\epsilon = 1$ and using (2.36), we may rephrase this as

$$\sqrt{-\hat{g}_{(p+1)}} \geq \frac{1}{(p+1)!}\epsilon^{\alpha_0\ldots\alpha_p} \begin{cases} \epsilon^\dagger(\Gamma^{11})^{\frac{p-2}{2}}\gamma_{\alpha_0\ldots\alpha_p}\epsilon & \text{(IIA)} \\ \epsilon^\dagger\sigma_3^{\frac{p-3}{2}}\imath\sigma_2\otimes\gamma_{\alpha_0\ldots\alpha_p}\epsilon & \text{(IIB)} \end{cases} \tag{2.38}$$

Equality holds if and only if the embedding is supersymmetric. Now the right hand side of (2.38) may be written as the pull-back of a differential form defined in space-time.

$$\mathcal{K} = \frac{1}{(p+1)!}e^{a_0\ldots a_p} \begin{cases} \epsilon^\dagger(\Gamma^{11})^{\frac{p-2}{2}}\Gamma_{a_0\ldots a_p}\epsilon & \text{(IIA)} \\ \epsilon^\dagger\sigma_3^{\frac{p-3}{2}}\imath\sigma_2\otimes\Gamma_{a_0\ldots a_p}\epsilon & \text{(IIB)} \end{cases} \tag{2.39}$$

$\mathcal{K}$ is known as the calibration form. A criterion for supersymmetry of an embedding that is alternative to (2.35) is then given by the following

$$X^*\mathcal{K} = \sqrt{-\hat{g}_{(p+1)}}\mathrm{d}^{p+1}\xi \tag{2.40}$$

that is, the pull-back of the calibration form onto the world-volume is equal to the induced volume form.

One may obtain $\mathcal{K}$ directly from its definition (2.39) and the knowledge of the projections imposed onto the SUSY spinors. We shall give an example of this in appendix 3.A.

The BPS-bound Formally one defines a calibration on a Riemannian manifold as a $(p+1)$-form $\mathcal{K}$ satisfying

$$\mathrm{d}\mathcal{K} = 0 \qquad\qquad \mathcal{K}|_{\xi^{p+1}} \leq \eta_{(p+1)}|_{\xi^{p+1}} \tag{2.41}$$

Here ξ^p is a set of vectors specifying a tangent $(p+1)$-plane to a $(p+1)$-cycle Σ_{p+1} while $\eta_{(p+1)} = \sqrt{-\hat{g}_{(p+1)}}\mathrm{d}^{p+1}\xi$ is the volume form induced onto that cycle. The cycle Σ_{p+1} is calibrated if the above bound is saturated, i.e. if $\mathcal{K}|_{\xi^{p+1}} = \eta_{(p+1)}|_{\xi^{p+1}}$.

[8]In general equations (2.37), (2.38) and (2.39) should be formulated in terms of a suitable conjugate spinor $\bar{\epsilon}$, in all cases that we will study though it is possible to identify $\bar{\epsilon} = \epsilon^\dagger$.

As we have seen above (2.40), κ-symmetric brane embeddings satisfy the volume bound, which can be thought of as a BPS-bound. In this and in the next paragraph we shall turn to the issue of the closure of (2.39). For a background without fluxes, the issue is rather easily resolved. From the gravitino variation

$$\delta_\epsilon \psi_M = D_M \epsilon = 0 \tag{2.42}$$

it follows that the SUSY spinor ϵ is covariantly constant. As the covariant derivative of both the vielbein and the tangent-space Γ-matrices does also vanish it follows that

$$\mathrm{d}\mathcal{K} = \nabla \wedge \mathcal{K} = 0 \tag{2.43}$$

$\nabla \wedge \mathcal{K}$ is to be taken as a formal expression. The wedge product antisymmetrizes over the relevant indices and, as the Levi-Civita connection is symmetric in two of its indices, it follows that the first equality holds. As all the ingredients of (2.39) are covariantly constant, it follows that the exterior derivative is closed.

There is a nice interpretation of the closure of the calibration form. Let us assume that we deform the calibrated cycle Σ_{p+1} to Σ'_{p+1}. The two cycles differ by a boundary $\Sigma_{p+1} - \Sigma'_{p+1} = \delta\Xi_{p+2}$. More formally we would not consider Σ'_{p+1} as a deformation, yet as a cycle within the homology class defined by Σ_{p+1}. We use Stokes theorem to establish

$$\mathrm{Vol}(\Sigma_{p+1}) = \int_{\Sigma_{p+1}} \mathcal{K} = \int_{\Xi_{p+2}} \mathrm{d}\mathcal{K} + \int_{\Sigma'_{p+1}} \mathcal{K} = \int_{\Sigma'_{p+1}} \mathcal{K} \leq \mathrm{Vol}(\Sigma'_{p+1}) \tag{2.44}$$

The final inequality uses (2.38). It follows that the calibrated cycle Σ_{p+1} is a minimal volume cycle. This matches our experience from string theory. In the absence of fluxes branes wrap minimal volume cycles.

Generalized calibrations The κ-symmetry matrix (2.36) does not change in the presence of Ramond-Ramond background fields and thus neither does the definition of the calibration form or the supersymmetry condition (2.40). Background fluxes however deform branes such that they do not longer wrap minimal volume cycles. For a background with fluxes we do therefore not expect the calibration form (2.39) to be closed. Rather, it's exterior differential should be related to the flux. Indeed, in all the examples studied in chapters 3 to 5, the calibration will satisfy

$$\mathrm{d}(e^{\frac{p-3}{4}\Phi}\mathcal{K}) = F_{(p+2)} \tag{2.45}$$

and one speaks of a generalized calibration. The original proof [36] verifying (2.45) showed that the expression $(e^{\frac{p-3}{4}\Phi}\mathcal{K} - C_{(p+1)})$ appears as the central charge of a supersymmetry algebra and must therefore be topological and thus exact. It is also possible to verify (2.45) in terms of the dilatino and gravitino supersymmetry transformations.

There is a generalization of (2.44) for the presence of background fluxes. Let us assume that both the brane and the background fields are static. It follows that the energy of the system is proportional to its action – with the proportionality constant being infinity. Moreover, minimum energy configurations will therefore minimize the brane action. Let Σ_{p+1} be the supersymmetric cycle wrapped by the brane and $\Sigma'_{p+1} = \Sigma_{p+1} + \delta\Xi_{(p+2)}$ a deformation. Then (setting $T_p = 1$)

$$\begin{aligned}
\Delta E &\propto S_{\Sigma'_{p+1}} - S_{\Sigma_{p+1}} \\
&= \int_{\Sigma'_{p+1}} (e^{\frac{p-3}{4}\Phi}\eta - C_{(p+1)}) - \int_{\Sigma_{p+1}} (e^{\frac{p-3}{4}\Phi}\mathcal{K} - C_{(p+1)}) \\
&\geq \int_{\delta\Xi_{p+2}} (e^{\frac{p-3}{4}\Phi}\mathcal{K} - C_{(p+1)}) \\
&= \int_{\Xi_{p+2}} \mathrm{d}(e^{\frac{p-3}{4}\Phi}\mathcal{K} - C_{(p+1)}) = 0
\end{aligned} \tag{2.46}$$

The inequality in the second line used again (2.38). It follows that supersymmetric, static embeddings are minimum energy configurations.

Chapter 3

Geometric aspects

The focus of this chapter is the construction of a suitable source density[1] Ω when flavoring supergravity backgrounds. The strategy is to use generalized calibrated geometry. Recall from section 2.3 that a brane is supersymmetric iff the pull-back of the calibration form onto the world-volume, $X^*\mathcal{K}$, is equal to the induced volume form (2.40). It follows immediately that one can write the DBI action of any supersymmetric brane in terms of this pull-back.

Let us briefly state the central argument. Starting point of the flavoring procedure is the action (1.3). In the case of type IIA/B backgrounds with Ramond-Ramond flux $F_{(p+2)}$, we can write the source term in Einstein frame as (see [70])

$$S_{\text{flavor}} = -T_p \int_{\mathcal{M}_{10}} (e^{\frac{p-3}{4}\Phi}\mathcal{K}_{(p+1)} - C_{(p+1)}) \wedge \Omega_{(9-p)} \tag{3.1}$$

which is a truncation of (2.20). As we will see, it is always possible to relate the smearing form to the calibration form using supersymmetry and the equations of motion:

$$\mathrm{d}[*e^{\frac{10-2p-4}{4}\Phi}\mathrm{d}(e^{\frac{p-3}{4}\Phi}\mathcal{K})] = \pm 2\kappa_{10}^2 T_p \Omega \tag{3.2}$$

This imposes a constraint on the distribution density. In the following we shall study how equations (3.1) and (3.2) can be applied to address the problem of smeared flavors.

Proceeding rather pedagogically, section 3.1.1 exhibits these methods for three different, well-known examples. We will contrast the macroscopic perspective gained by the use of calibrations against that of the original papers.

[1] We will use the expressions source density, distribution density, brane density or smearing form interchangeably when referring to Ω.

In this way we will see that our methods are not only capable of reproducing the known results, yet also provide some new, interesting ones. The examples studied are the $\mathcal{N}=1$ sQCD-like dual of [14, 15, 16], the $d=2+1$ dimensional $\mathcal{N}=1$ theory of [71] and the Klebanov-Witten theory [5] with massless [17] and massive [20] flavors. Following this we turn to the generic case (section 3.1.2), showing how the action (3.1) can be constructed from purely geometric considerations and proving its equivalence with other actions used in the field of smeared flavors.

In section 3.2 we shall apply our methods to the problem of flavoring a background dual to an $\mathcal{N}=2$ super Yang-Mills-like theory first studied in [72, 73]. We will see that we are able to do so without an explicit knowledge of the brane embeddings used. We find new analytic and asymptotic solutions to the flavored and unflavored equations of motion and discuss various properties of these backgrounds.

Following [56] we will also show for the examples considered, how all constraints imposed by supersymmetry upon space-time can be understood and recovered from geometric grounds using methods such as G-structures.

In section 2.3 we gave a short review of the required background in generalized calibrations. Appendix 3.A contains a detailed example of how to calculate a calibration form.

3.1 The geometry of smeared branes

In the following we shall investigate what generalized calibrated geometry can teach us about string theory duals with backreacting, smeared flavor branes. First we will take a detailed look at three examples [14, 17, 71]. For each of these we will briefly summarize the conventional approach to flavoring and will then show explicitly that it can be nicely understood in terms of a suitable calibration form. In section 3.1.2 we will turn to the case of a generic supergravity dual.

3.1.1 Three examples

The string dual to an $\mathcal{N}=1$ sQCD-like theory

Review of the $\mathcal{N}=1$ sQCD-like string dual As a first example we shall turn to the string dual to an $\mathcal{N}=1$ sQCD-like theory [14, 15, 16]. It is based on the background of [8] which is given by the following solution of the type IIB equations of motion (Einstein frame):

$$
\begin{aligned}
\mathrm{d}s^2 &= \alpha' g_s N_c e^{\frac{\Phi}{2}} \left[\frac{1}{\alpha' g_s N_c} \mathrm{d}x_{1,3}^2 + \mathrm{d}r^2 + e^{2h}(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) + \frac{1}{4}(\tilde{\omega}_i - A^i)^2 \right] \\
F_{(3)} &= -\frac{1}{4} \bigwedge_a (\tilde{\omega}_a - A^a) + \frac{1}{4} \sum_a F^a \wedge (\tilde{\omega}_a - A^a)
\end{aligned}
\tag{3.3}
$$

with

$$
\begin{aligned}
A^1 &= -a(r)\mathrm{d}\theta & \tilde{\omega}_1 &= \cos\psi \mathrm{d}\tilde{\theta} + \sin\psi \sin\tilde{\theta} \mathrm{d}\tilde{\phi} \\
A^2 &= a(r)\sin\theta \mathrm{d}\phi & \tilde{\omega}_2 &= -\sin\psi \mathrm{d}\tilde{\theta} + \cos\psi \sin\tilde{\theta} \mathrm{d}\tilde{\phi} \\
A^3 &= -\cos\theta \mathrm{d}\phi & \tilde{\omega}_3 &= \mathrm{d}\psi + \cos\tilde{\theta} \mathrm{d}\tilde{\phi}
\end{aligned}
\tag{3.4}
$$

The metric describes a space with topology $\mathbb{R}^{1,3} \times \mathbb{R} \times S^2 \times S^3$, where the three-sphere is parametrized by the Maurer-Cartan forms $\tilde{\omega}_i$ and the one-forms A^i describe the fibration between the two spheres. It is interpreted as the near-horizon geometry of a stack of N_c D5-branes wrapping an S^2, thus describing the dynamics of $d = 3+1$ dimensional $\mathcal{N}=1, SU(N_c)$ super Yang-Mills theory coupled to some extra matter. To keep the discussion as simple as possible, we shall focus on the so-called singular solution which is obtained from the assumption $a(r) = 0$.

The possibility of adding probe flavor branes to the above background (3.3) was studied in [74]. Using κ-symmetry the authors found several classes of flavor D5-branes; the simplest of these is given by branes extending along (x^μ, r) and wrapping ψ. They are pointlike on the four-dimensional submanifold given by $(\theta, \phi, \tilde{\theta}, \tilde{\phi})$ and extend to $r = 0$, thus describing massless flavors. In what follows, the most important feature of this embedding is that we are able to identify world-volume coordinates ξ^α with space-time ones, (x^μ, r, ψ). So even at the level of the space-time coordinates X^M there is a very well defined notion of coordinates tangential and transverse to the brane.

From the perspective of type IIB string theory, it is clear that the addition of a large number of such branes to the system (3.3) will deform the geometry of the background. Given the form of the brane embeddings it follows that a suitable ansatz for the deformed background should be of the form

$$\begin{aligned}
\mathrm{d}s^2 &= e^{2f(r)}[\mathrm{d}x_{1,3}^2 + \mathrm{d}r^2 + e^{2h(r)}(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) \\
&\quad + \frac{e^{2g(r)}}{4}(\tilde{\omega}_1^2 + \tilde{\omega}_2^2) + \frac{e^{2k(r)}}{4}(\tilde{\omega}_3 + \cos\theta \mathrm{d}\phi)^2] \\
F_{(3)} &= -2N_c e^{-3f-2g-k} e^{123} + \frac{N_c}{2} e^{-3f-2h-k} e^{\theta\phi 3}
\end{aligned} \tag{3.5}$$

as the flavor branes are points on the four-dimensional transverse manifold while singling out the $U(1) \subset S^3$ parametrized by ψ. When writing (3.5) we introduced a vielbein

$$\begin{aligned}
&e^{x^i} = e^f \mathrm{d}x^i \quad e^1 = \frac{e^{f+g}}{2}\tilde{\omega}_1 \quad e^2 = \frac{e^{f+g}}{2}\tilde{\omega}_2 \qquad e^3 = \frac{e^{f+k}}{2}(\tilde{\omega}_3 + \cos\theta \mathrm{d}\phi) \\
&e^r = e^f \mathrm{d}r \quad\; e^\theta = e^{f+h}\mathrm{d}\theta \quad e^\phi = e^{f+h}\sin\theta \mathrm{d}\phi
\end{aligned} \tag{3.6}$$

One can also interpret the ansatz (3.5) from the gauge theory point of view. The $U(1)$ describes the R-symmetry of the flavored theory, which one demands not to be broken classically by the addition of massless flavors.

Studying the dilatino and gravitino variations of the deformed background one obtains the projections satisfied by the SUSY spinor ϵ,

$$\Gamma_{r123}\epsilon = \epsilon \quad \Gamma_{r\theta\phi 3}\epsilon = \epsilon \quad \epsilon = \sigma_3 \epsilon \tag{3.7}$$

as well as the BPS equations

$$\begin{aligned}
4f &= \Phi \\
h' &= \frac{1}{4}N_c e^{-2h-k} + \frac{1}{4}e^{-2h+k} = \frac{1}{2}e^{3f}F_{\theta\phi 3} + \frac{1}{4}e^{-2h+k} \\
g' &= -N_c e^{-2g-k} + e^{-2g+k} = \frac{1}{2}e^{3f}F_{123} + e^{-2g+k} \\
k' &= \frac{1}{4}N_c e^{-2h-k} - N_c e^{-2g-k} - \frac{1}{4}e^{-2h+k} - e^{-2g+k} + 2e^{-k} \\
&= \frac{1}{2}e^{3f}(F_{\theta\phi 3} + F_{123}) - \frac{1}{4}e^{-2h+k} - e^{-2g+k} + 2e^{-k} \\
\Phi' &= -\frac{1}{4}N_c e^{-2h-k} + N_c e^{-2g-k} = -\frac{1}{2}e^{3f}(F_{\theta\phi 3} + F_{123})
\end{aligned} \tag{3.8}$$

It is a priori not obvious that the flavor branes mentioned earlier are still supersymmetric brane embeddings for the deformed background for arbitrary functions g, h, k. One therefore has to check again that probes with world-volume directions as before, $\xi^\alpha = (x^\mu, r, \psi)$, still preserve all of the backgrounds supersymmetries.

Having deformed the original background one turns to the system given by the combined action (1.3) with S_{flavor} given by by the source term (2.20). One can anticipate that the brane action will contribute to the energy-momentum tensor in the Einstein equation, add a source term for the 3-form field strength and modify the dilaton equation by a contribution related to the DBI action.

For the case of N_f flavor branes localized at $(\theta_0, \phi_0, \tilde{\theta}_0, \tilde{\phi}_0)$, the brane action is ($X^*$ denoting the pull-back onto the world-volume)

$$S_{\text{flavor}} = T_5 \sum_{N_f} \left(-\int_{\mathcal{M}_6} \mathrm{d}^6\xi e^{\frac{\Phi}{2}} \sqrt{-\hat{g}_{(6)}} + \int_{\mathcal{M}_6} X^* C_{(6)} \right) \Bigg|_{(\theta_0, \phi_0, \tilde{\theta}_0, \tilde{\phi}_0)} \tag{3.9}$$

As these branes are localized in the four transverse directions, the equations of motion will contain δ-function sources, making the search for solutions a difficult endeavour. The idea is therefore to smoothly distribute the branes over the transverse directions. If one assumes a transverse brane distribution with density

$$\Omega = \frac{N_f}{(4\pi)^2} \sin\theta \sin\tilde{\theta} \mathrm{d}\theta \wedge \mathrm{d}\phi \wedge \mathrm{d}\tilde{\theta} \wedge \mathrm{d}\tilde{\phi} \tag{3.10}$$

the action (3.9) may be generalized to

$$\begin{aligned} S_{\text{flavor}} &= T_5 \left(-\frac{N_f}{(4\pi)^2} \int_{\mathcal{M}_{10}} \mathrm{d}^{10}x e^{\frac{\Phi}{2}} \sin\theta \sin\tilde{\theta} \sqrt{-\hat{g}_{(6)}} + \int_{\mathcal{M}_{10}} C_{(6)} \wedge \Omega \right) \\ &= T_5 \left(-\int_{\mathcal{M}_{10}} \mathrm{d}^{10}x e^{\frac{\Phi}{2}} \sqrt{-g_{(10)}} |\Omega| + \int_{\mathcal{M}_{10}} C_{(6)} \wedge \Omega \right) \end{aligned} \tag{3.11}$$

where we have defined the modulus of a p-form Ω as

$$|\Omega| \equiv \sqrt{\frac{1}{p!} \Omega_{M_1 \dots M_p} \Omega^{M_1 \dots M_p}} \tag{3.12}$$

and have checked the equality of the first and second lines by explicit calculation.

Let us take a look at how the brane action modifies the second order equations of motion, starting with the Ramond-Ramond field strength. Here the relevant part of the total action is

$$S = \int_{\mathcal{M}_{10}} -\frac{1}{2\kappa_{10}^2} \frac{e^{-\Phi}}{2} (F_{(7)} \wedge *F_{(7)}) + T_5 C_{(6)} \wedge \Omega \tag{3.13}$$

If we vary the potential $C_{(6)}$,

$$\begin{aligned} \delta_C S &= \int_{\mathcal{M}_{10}} -\frac{1}{2\kappa_{10}^2} \frac{e^{-\Phi}}{2} (\mathrm{d}\delta C_{(6)} \wedge *F_{(7)} + F_{(7)} \wedge *\mathrm{d}\delta C_{(6)}) + T_5 \int \delta C_{(6)} \wedge \Omega \\ &= \int_{\mathcal{M}_{10}} \delta C_{(6)} \wedge \left(\frac{1}{2\kappa_{10}^2} \mathrm{d} * e^{-\Phi} F_{(7)} + T_5 \Omega \right) \\ &\Rightarrow \mathrm{d}F_{(3)} = 2\kappa_{10}^2 T_5 \Omega \end{aligned} \tag{3.14}$$

The change in the dilaton and Einstein equations does not take such a nice geometric form. Choosing $T_5 = \frac{1}{(2\pi)^5}, 2\kappa_{10}^2 = (2\pi)^7$, the complete equations of motion are

$$
\begin{aligned}
0 &= \mathrm{d}F_{(3)} - (2\pi)^2\Omega \\
0 &= \frac{1}{\sqrt{-g_{(10)}}}\partial_\mu(g^{\mu\nu}\sqrt{-g_{(10)}}\partial_\nu\Phi) - \frac{1}{12}e^{\Phi}F_{(3)}^2 - \frac{N_f}{8}e^{\frac{\Phi}{2}}\frac{\sqrt{-\hat{g}_{(6)}}}{\sqrt{-g_{(10)}}}\sin\theta\sin\tilde{\theta} \\
0 &= R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \frac{1}{2}\left(\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}g_{\mu\nu}\partial_\lambda\Phi\partial^\lambda\Phi\right) \\
&\quad - \frac{1}{12}e^{\Phi}\left(3F_{\mu\kappa\lambda}F_\nu{}^{\kappa\lambda} - \frac{1}{2}g_{\mu\nu}F_{(3)}^2\right) - T_{\mu\nu}^{\mathrm{flvr}} \\
T_{\mu\nu}^{\mathrm{flvr}} &= -\frac{N_f}{4}\sin\theta\sin\tilde{\theta}\frac{1}{2}e^{\frac{\Phi}{2}}g_{\mu\alpha}g_{\nu\beta}\hat{g}_{(6)}^{\alpha\beta}\frac{\sqrt{-\hat{g}_{(6)}}}{\sqrt{-g_{(10)}}}
\end{aligned}
\tag{3.15}
$$

The search for solutions of (3.15) is simplified considerably by a powerful result due to Koerber and Tsimpis [40] who showed that any solution to the BPS equations satisfying the modified Bianchi identity of (3.14) solves also the Einstein and dilaton equations and is therefore a solution of (3.15).

So we turn again to the issue of the BPS equations. As the brane embeddings are supersymmetric, the projections (3.7) imposed on the spinor ϵ remain the same. However, the three-form field strength $F_{(3)}$ is modified by the appearance of the source term in (3.15). To incorporate this one makes a new ansatz for the field strength of (3.5)

$$
F_{(3)} = -\frac{N_c}{4}e^{-3f-2g-k}e^{123} - \frac{N_f - N_c}{4}e^{-3f-2h-k}e^{\theta\phi 3} \tag{3.16}
$$

It follows that the BPS equations (3.8) change to

$$
\begin{aligned}
4f &= \Phi \\
h' &= \frac{1}{4}(N_c - N_f)e^{-2h-k} + \frac{1}{4}e^{-2h+k} = \frac{1}{2}e^{3f}F_{\theta\phi 3} + \frac{1}{4}e^{-2h+k} \\
g' &= -N_c e^{-2g-k} + e^{-2g+k} = \frac{1}{2}e^{3f}F_{123} + e^{-2g+k} \\
k' &= \frac{1}{4}(N_c - N_f)e^{-2h-k} - N_c e^{-2g-k} - \frac{1}{4}e^{-2h+k} - e^{-2g+k} + 2e^{-k} \\
&= \frac{1}{2}e^{3f}(F_{\theta\phi 3} + F_{123}) - \frac{1}{4}e^{-2h+k} - e^{-2g+k} + 2e^{-k} \\
\Phi' &= -\frac{1}{4}(N_c - N_f)e^{-2h-k} + N_c e^{-2g-k} = -\frac{1}{2}e^{3f}(F_{\theta\phi 3} + F_{123})
\end{aligned}
\tag{3.17}
$$

It is curious to note that when written in terms of $F_{\theta\phi 3}$ and F_{123} the BPS equations of the deformed and flavored systems are the same – see (3.8) and

(3.17). The change in the BPS equations stems solely from the modification of the field strength. This should not come as a surprise, as the brane embeddings are supersymmetric.[2]

By construction $F_{(3)}$ satisfies the modified Bianchi identity. Thus any solution of (3.17) solves the flavoring problem for the Maldacena-Núñez background. For a discussion of these solutions and their physical interpretation see [14, 15, 16].

In the above background, the generalization of the action (3.9) to (3.11) is fairly intuitive and simple, because there is only one stack of flavor branes with world-volume coordinates that can be globally identified with space-time coordinates. However we can already anticipate the shortcomings of this definition. On a technical level, the first line of (3.11) is inherently dependent on the coordinate split while the second is non-linear in the smearing form Ω. From a more formal point of view it is also unsatisfying that the formalism of those equations treats the DBI and Wess-Zumino contributions to the brane action on an unequal footing. One should recall that, roughly speaking, the DBI action defines the tree level couplings of the brane to the NS sector of the background while the couplings to Ramond-Ramond fields are contained in the Wess-Zumino term. A standard string theory calculation shows the cancellation of the effects of closed strings from the two sectors on supersymmetric branes. So it would be desirable to see an explicit symmetry between the two terms even after smearing. Adopting once again a more physics centered perspective we might also wonder if there are any constraints on the choice of the smearing form. E.g. one should note that the smearing form does not agree with the volume form induced on the four-cycle $(\theta, \phi, \tilde{\theta}, \tilde{\phi})$. At first glance it might appear that there are none. After all, the cancellations between parallel BPS branes allow us to place them at arbitrary separations. As we will soon see, however, there are constraints on Ω which can be traced back to the geometric structure of the background.

The perspective of generalized calibrated geometry The properties of generalized calibrations and their relation to supersymmetry were discussed in detail in section 2.3. As the backgrounds considered are not fully generic,

[2] We commented on this issue in footnote 3 on page 11.

yet only include dilaton and Ramond-Ramond fields in type IIB supergravity, we will not make use of the most general concept of a generalized calibration. Again we refer to [67] and [68]. For our purposes it is sufficient to recall that a p-brane with embedding $X^M(\xi)$ is supersymmetric if and only if it satisfies (2.40). Using this, we write the DBI action in (3.9) as

$$S_{\text{DBI}} = -T_p \int_{\mathcal{M}_{p+1}} e^{\frac{p-3}{4}\Phi} X^* \mathcal{K} \tag{3.18}$$

Furthermore, if the p-brane couples electrically to the flux given by $F_{(p+2)}$, supersymmetry in the Einstein frame requires [36]

$$\mathrm{d}(e^{\frac{p-3}{4}\Phi}\mathcal{K}) = F_{(p+2)} \tag{3.19}$$

In the case at hand, the calibration six-form is given by

$$\mathcal{K} = \frac{1}{6!}(\epsilon^\dagger \sigma_3 \otimes \Gamma_{a_0 \dots a_5} \epsilon) e^{a_0 \dots a_5} \tag{3.20}$$

As explained in appendix 3.A, evaluation of the calibration form requires only the chirality of the type IIB spinors, $\epsilon = \Gamma^{11}\epsilon$ and knowledge of the projections imposed on the SUSY spinors (3.7). From the last of these it follows that one of the Majorana-Weyl spinors of type IIB is fixed to zero, $\epsilon = \binom{\epsilon}{0}$. Thus there is only one calibration six-form and we may use ϵ instead of ϵ. In section 3.2 we will encounter an example with two calibration forms. Combining the SUSY projections (3.7) with the definition (3.20) yields

$$\mathcal{K}_{x^0 x^1 x^2 x^3 \theta\phi} = \epsilon^\dagger \Gamma_{x^0 x^1 x^2 x^3 \theta\phi} \epsilon = -\epsilon^\dagger \Gamma_{r123} \epsilon = -1 \tag{3.21}$$

The second equality makes use of chirality, the third of the SUSY projections and the normalization $\epsilon^\dagger \epsilon = 1$. When calculating calibration forms it is actually more difficult to show that certain components vanish. However, the process is rather straightforward and discussed in considerable detail in appendix 3.A. When the dust settles, we are left with

$$\mathcal{K} = e^{x^0 x^1 x^2 x^3} \wedge (e^{r3} - e^{\theta\phi} - e^{12}) \tag{3.22}$$

As e^3 is the only part of the vielbein containing $\mathrm{d}\psi$, it is obvious that equation (2.40) is satisfied and we recover the result of [74] that the embedding in question is supersymmetric. Noting that

$$\begin{aligned} \sqrt{-\hat{g}_{(6)}}\mathrm{d}^6\xi &= e^{x^0 x^1 x^2 x^3 r3} \\ \Omega &= 4\, N_f e^{-4f-2g-2h} e^{\theta\phi 12} \end{aligned} \tag{3.23}$$

it is easy to see that we may write the smeared brane action (3.11) as

$$S_{\text{flavor}} = T_5 \int_{\mathcal{M}_{10}} (-e^{\frac{\Phi}{2}}\mathcal{K} + C_{(6)}) \wedge \Omega \tag{3.24}$$

In opposite to (3.11) this is independent of coordinates, linear in the smearing form, and treats the DBI and Wess-Zumino contributions to the brane action on an equal footing.

Concerning the supersymmetry condition (3.19), we find

$$\begin{aligned} \mathrm{d}(e^{\frac{\Phi}{2}}\mathcal{K}) = e^{-f+\frac{\Phi}{2}} e^{x^0x^1x^2x^3} \wedge [&e^{-2g}(2e^k - 6e^{2g}f' - 2e^{2g}g' - e^{2g}\Phi')e^{r12} \\ &+ e^{-2h}(\frac{1}{2}e^k - 6e^{2h}f' - 4e^{2h}h' - e^{2h}\Phi')e^{r\theta\phi}] \end{aligned} \tag{3.25}$$

Using the BPS equations (3.8) or (3.17), one may verify for the three-form field strength with (3.16) and without sources (3.5) that $\mathrm{d}(e^{\frac{\Phi}{2}}\mathcal{K}) = F_{(7)}$ is satisfied. We can exploit the calibration form even further. From $e^{-\Phi} * F_{(7)} = F_{(3)}$ and $\mathrm{d}F_{(3)} = (2\pi)^2\Omega$ it follows that

$$\begin{aligned} e^{-\Phi} * \mathrm{d}(e^{\frac{\Phi}{2}}\mathcal{K}) &= F_{(3)} \\ \mathrm{d}[e^{-\Phi} * \mathrm{d}(e^{\frac{\Phi}{2}}\mathcal{K})] &= (2\pi)^2\Omega \end{aligned} \tag{3.26}$$

Again note that these equations hold with or without the backreaction of the source terms – in the latter case with $\Omega = 0$. One should think of them rather as a characteristic of the supersymmetries preserved by the background than a property of the branes.

When we first introduced the smearing form in (3.10) it appeared that its choice was rather arbitrary. After all supersymmetry allows us to place branes at arbitrary separations. However, (3.26) is not a result of supersymmetry alone yet rather an interplay of supersymmetry and the Einstein equations, as the following illustrates.

$$\mathrm{d}(e^{\frac{\Phi}{2}}\mathcal{K}) \overset{\text{SUSY}}{=} F_{(7)}, \quad *e^{-\Phi}F_{(7)} \equiv F_{(3)}, \quad \mathrm{d}F_{(3)} \overset{\text{EOM}}{=} (2\pi)^2\Omega \tag{3.27}$$

BPS equations and G-structures We showed before that the requirement of supersymmetry is related to geometry, notably with the calibration form. As supersymmetry gives us the BPS equations of the system, it is logical to think that one can retrieve those equations through geometric considerations, namely G-structures. When looking at the supersymmetric gravitino equation, we can identify $F_{(3)}$ with a torsion (straightforward in string frame), defining a new covariant derivative $\tilde{\nabla}_\mu$ such that

$$\tilde{\nabla}_\mu \epsilon = \epsilon \tag{3.28}$$

This means that we have a covariantly constant spinor satisfying the projections (3.7). From these it follows that in the six-dimensional internal manifold, there is a covariantly constant complex chiral spinor η verifying

$$\gamma_{r123}\eta = \eta \quad \gamma_{r\theta\phi 3}\eta = \eta \tag{3.29}$$

where γ_i are the gamma matrices of the six-dimensional internal manifold. We can choose the chirality of η to be

$$\imath\gamma_{r123\theta\phi}\eta = -\eta \tag{3.30}$$

Then we recognize that the six-dimensional manifold is a generalized Calabi-Yau. It has a Kähler two-form J and a holomorphic three-form Ω defined as

$$J_{mn} = \imath\eta^\dagger\gamma_{mn}\eta \tag{3.31}$$

$$\Omega_{mnp} = \eta^T\gamma_{mnp}\eta \tag{3.32}$$

Supersymmetry imposes the following conditions on the forms (see [56]):

$$\mathrm{d}(e^\Phi *_6 J) = 0 \tag{3.33}$$

$$\mathrm{d}(e^{\frac{5}{4}\Phi}\Omega) = 0 \tag{3.34}$$

From those equations, plus the generalized calibration condition (3.26), we can retrieve the BPS equations of the system, imposing $4f = \Phi$. Indeed, this last condition, describing how the internal manifold is embedded in space-time, cannot be captured by those geometric properties that concern only the six-dimensional manifold. It can however easily be found using the supersymmetric variations of the dilatino and the gravitino.

An $\mathcal{N} = 1$, $d = 2+1$ example We turn now to the string dual of a $d = 2+1$ dimensional $\mathcal{N} = 1$ theory that was discussed in [71]. We will keep the discussion rather brief, only exhibiting the equivalence of the actions (3.11) and (3.24) for this example. In comparison to the $\mathcal{N} = 1$ sQCD-like dual of the previous section the situation is complicated by the fact that there are three stacks of branes. While it is possible to find coordinates such that the worldvolume of one of these stacks may be identified with space-time coordinates, it is not possible to do so for all three stacks simultaneously. The system has the topology $\mathbb{R}^{1,2} \times \mathbb{R} \times S^3 \times S^3$. As in section 3.1.1, we shall work with a simplification, the truncated system, for which the background is given by

$$\begin{aligned} e^{x^i} &= e^f \mathrm{d}x^i \quad e^r = e^f \mathrm{d}r \quad e^i = \frac{e^{f+h}}{2}\sigma^i \quad e^{\hat{i}} = \frac{e^{f+g}}{2}(\omega^i - \frac{1}{2}\sigma^i) \\ F_{(3)} &= -2N_c e^{-3g-3f} e^{\hat{1}\hat{2}\hat{3}} + \frac{1}{2}N_c e^{-g-2h-3f}(e^{13\hat{2}} - e^{12\hat{3}} - e^{23\hat{1}}) \end{aligned} \tag{3.35}$$

σ^i and ω^i are sets of Maurer-Cartan forms parametrizing the two three-spheres. The projections satisfied by the SUSY spinor η are

$$\Gamma_{1\hat{1}2\hat{2}}\eta = -\eta \quad \Gamma_{1\hat{1}3\hat{3}}\eta = -\eta \quad \Gamma_{2\hat{2}3\hat{3}}\eta = -\eta \quad \Gamma_{r\hat{1}\hat{2}\hat{3}}\eta = \eta \quad \eta = \sigma_3\eta \tag{3.36}$$

And the BPS equations take the form

$$\begin{aligned} \Phi' &= N_c e^{-3g} - \frac{3}{4}N_c e^{-g-2h} \\ h' &= \frac{1}{2}e^{g-2h} + \frac{1}{2}N_c e^{-g-2h} \\ g' &= e^{-g} - \frac{1}{4}e^{g-2h} + \frac{N_c}{4}e^{-g-2h} - N_c e^{-3g} \\ \Phi &= 4\, f \end{aligned} \tag{3.37}$$

Once more, it follows from $\eta = \sigma_3\eta = \binom{\eta}{0}$ that there is only one calibration six-form which is given by (assuming $\Gamma^{11}\eta = -\eta$)

$$\mathcal{K} = e^{012} \wedge (e^{r1\hat{1}} + e^{r2\hat{2}} + e^{r3\hat{3}} - e^{123} + e^{3\hat{1}\hat{2}} - e^{2\hat{1}\hat{3}} + e^{1\hat{2}\hat{3}}) \tag{3.38}$$

From the calibration condition for supersymmetric branes, $X^*\mathcal{K} = \mathrm{d}\xi^6\sqrt{-\hat{g}_{(6)}}$, one can see immediately that there are supersymmetric 5-brane embeddings with tangent vectors[3] $(\partial_{x^0}, \partial_{x^1}, \partial_{x^2}, E_r, E_i, E_{\hat{i}})$, $i \in \{1,2,3\}$. We also learn from

[3] When labeling brane embeddings in terms of their tangent vectors one should think of the brane being along the submanifold spanned by the integral curves of the tangent vector fields. That is, if one were to find coordinates y^M such that

$$\partial_{x^0} = \partial_{y^0} \quad \partial_{x^1} = \partial_{y^1} \quad \partial_{x^2} = \partial_{y^2} \quad E_r = \partial_{y^3} \quad E_1 = \partial_{y^4} \quad E_{\hat{1}} = \partial_{y^5}$$

the corresponding $1\hat{1}$ brane embedding would be given by

$$Y^\alpha(\xi) = \xi^\alpha \qquad Y^a = \text{const} \quad \alpha \in \{0,\ldots,5\} \quad a \in \{6,\ldots,9\}$$

One should note however, that it is necessary to verify, that the distribution given by the tangent vectors is integrable, i.e. to verify that the coordinates y^M exist. One can do so using Frobenius theorem, which states that a distribution given by vectors T_a is integrable iff it is in involution, that is iff $[T_a, T_b] = f_{abc}T_c$.

(3.38) that these embeddings are absolutely equivalent. They were originally derived in [71] using κ-symmetry. There the authors introduced a standard set of Maurer-Cartan forms ω, σ to parametrize the two S^3s, and then found a coordinate representation of the $(\partial_\mu, \partial_r, E_3, E_{\hat{3}})$ branes given by $(x^\mu, r, \psi_1, \psi_2)$. Subsequently they argued from the symmetries of the space that there are also $1\hat{1}$ and $2\hat{2}$ embeddings, whose coordinate representation would become apparent upon using different Maurer-Cartan forms. As we mentioned earlier, it does not seem to be possible to find global coordinates for this system in which all three flavor brane embeddings have good coordinate representations – thus this is an ideal setting for using the calibration form (3.38).

Our analysis here shall start with the $3\hat{3}$ embeddings. In [71] their smeared action was given by

$$
\begin{aligned}
S_{D5} &= T_5 \left(-\int \mathrm{d}^{10}x e^{\frac{\Phi}{2}} \sqrt{-G_{10}} |\Omega^{(1)}| + \int_{\mathcal{M}_{10}} C_{(6)} \wedge \Omega^{(1)} \right) \\
\Omega^{(1)} &= -\frac{N_f}{\pi^2} e^{-4f-2h-2g} e^{12\hat{1}\hat{2}} \\
|\Omega^{(1)}| &= \frac{N_f}{\pi^2} e^{-4f-2h-2g} \\
\sqrt{-G_{10}} &= \frac{1}{64} e^{10f+3g+3h} \sin\theta \sin\tilde{\theta} \\
\sqrt{-\hat{G}_6} &= \frac{1}{4} e^{6f+g+h}
\end{aligned}
\tag{3.39}
$$

Now

$$
\mathcal{K} \wedge \Omega^{(1)} = -\frac{N_f}{\pi^2} e^{-4f-2h-2g} \sqrt{-G_{10}} \mathrm{d}^{10}x = \mathrm{d}^{10}x \sqrt{-G_{10}} |\Omega^{(1)}| \tag{3.40}
$$

Thus again, we may write the action of one stack of $(3\hat{3})$ branes as

$$
S_{D5} = T_5 \int_{\mathcal{M}_{10}} \left(-e^{\frac{\Phi}{2}} \mathcal{K} + C_{(6)} \right) \wedge \Omega^{(1)} \tag{3.41}
$$

The above may be easily generalized to the case of three stacks of D5-branes as the expression is linear in Ω.

$$
\begin{aligned}
S_{D5} &= T_5 \int_{\mathcal{M}_{10}} \left(-e^{\frac{\Phi}{2}} \mathcal{K} + C_{(6)} \right) \wedge \Omega \\
\Omega &= \Omega^{(1)} + \Omega^{(2)} + \Omega^{(3)} \\
\Omega^{(2)} &= -\frac{N_f}{\pi^2} e^{-4f-2h-2g} e^{13\hat{1}\hat{3}} \\
\Omega^{(3)} &= -\frac{N_f}{\pi^2} e^{-4f-2h-2g} e^{23\hat{2}\hat{3}}
\end{aligned}
\tag{3.42}
$$

Where $\Omega^{(2)}$ is the smearing form for branes extending along $2\hat{2}$ and $\Omega^{(3)}$ smears the $1\hat{1}$ embedding. The linearity of the above expression gives a good motivation for the use of $\sum_i |\Omega^{(i)}|$ instead of $|\Omega|$ in the original action of [71]

$$S_{D5} = T_5 \left(-\int \mathrm{d}^{10}x e^{\frac{\Phi}{2}} \sqrt{-G_{10}} \sum_{i=1}^{3} |\Omega^{(i)}| + \int_{\mathcal{M}_{10}} C_{(6)} \wedge \Omega \right) \tag{3.43}$$

Independently of whether one uses the action (3.42) or (3.43) the Bianchi identity is modified to $\mathrm{d}F_{(3)} = -2\kappa_{10}^2 T_5 \Omega$ – the minus sign being due to the convention $e^{\Phi} F_{(3)} = - * F_{(7)}$ used in [71]. Accordingly one changes the ansatz for the field-strength by adding a term $f_{(3)}$ which is not closed,

$$\begin{aligned} F_{(3)} &\mapsto F_{(3)} + f_{(3)} \\ f_{(3)} &= 2N_f e^{-g-2h-3f} (e^{12\hat{3}} + e^{23\hat{1}} - e^{13\hat{2}}) \end{aligned} \tag{3.44}$$

The BPS equations (3.37) change to

$$\begin{aligned} \Phi' &= N_c e^{-3g} - \frac{3}{4}(N_c - N_f) e^{-g-2h} \\ h' &= \frac{e^{g-2h}}{2} + \frac{N_c - 4N_f}{2} e^{-g-2h} \\ g' &= e^{-g} - \frac{1}{4} e^{g-2h} - N_c e^{-3g} + \frac{N_c - 4N_F}{4} e^{-g-2h} \\ \Phi &= 4\, f \end{aligned} \tag{3.45}$$

Let us now turn to the SUSY condition (3.19). A straightforward calculation yields

$$\begin{aligned} \mathrm{d}(e^{\frac{\Phi}{2}} \mathcal{K}) = e^{\frac{\Phi}{2} - f} e^{012} \wedge \{ &(2e^{-g} - 6f' - 2g' - h' - \Phi')(e^{r\hat{1}\hat{2}\hat{3}} - e^{r2\hat{1}\hat{3}} + e^{r3\hat{1}\hat{2}}) \\ &+ \frac{e^{-2h}}{2} (-3e^{g} + 12e^{2h} f' + 6e^{2h} h' + e^{2h} \Phi') e^{r123} \} \end{aligned} \tag{3.46}$$

Using the BPS equations (3.37) or (3.45) respectively one can verify that $-e^{-\Phi} * \mathrm{d}(e^{\frac{\Phi}{2}} \mathcal{K}) = F_{(3)}$ is satisfied in both the deformed and flavored case. Furthermore we know that $\mathrm{d}F_{(3)} = (2\pi)^2 \Omega$, thus we are again able to obtain a constraint on the smearing form as

$$(2\pi)^2 \Omega = \mathrm{d}[-e^{-\Phi} * \mathrm{d}(e^{\frac{\Phi}{2}} \mathcal{K})] \tag{3.47}$$

We immediately see why there have to be three stacks of flavor D5-branes in the backreacted solution – the calibration form respects the symmetries of the two three-spheres and from (3.47) it follows that the same holds true for the smearing form. It would therefore not be possible to obtain a smeared system with only one or two of the three stacks.

We can again use G-structures to derive the BPS equations for the system. In this case the internal manifold is seven-dimensional, with a covariantly constant spinor which satisfies

$$\gamma_{1\hat{1}2\hat{2}}\eta = -\eta \quad \gamma_{1\hat{1}3\hat{3}}\eta = -\eta \quad \gamma_{r\hat{1}\hat{2}\hat{3}}\eta = \eta \tag{3.48}$$

We recognize here a generalized G_2 holonomy manifold with the associative three-form $\mathcal{K}$ defined as

$$\mathcal{K}_{mnp} = -\imath\bar{\eta}\gamma_{mnp}\eta \tag{3.49}$$

The condition imposed by supersymmetry is

$$\mathrm{d}(e^{\Phi} *_7 \mathcal{K}) = 0 \tag{3.50}$$

Together with the generalized calibration condition, and assuming $\Phi = 4f$, this condition provides us with a method to rederive the BPS equations (3.37), (3.45).

The Klebanov-Witten model Finally we take a look at the Klebanov-Witten model for the cases of massless [17] and massive flavors [20]. The Klebanov-Witten model [5] is based on D3-branes at the tip of the conifold and is dual to a certain $\mathcal{N} = 1$ super Yang-Mills theory. So apart from the dilaton and the metric there is self-dual $F_{(5)}$ flux due to the D3s. In contrast to the previous two examples, one uses D7s to introduce flavor degrees of freedom into the system. These source $F_{(1)}$, so the suitable ansatz for the relevant deformed, flavored background is

$$\begin{aligned}
\mathrm{d}s^2 &= h^{-\frac{1}{2}}\mathrm{d}x_{1,3}^2 \\
&\quad + h^{\frac{1}{2}}\left[e^{2f}\mathrm{d}\rho^2 + \frac{e^{2g}}{6}\sum_{i=1,2}(\mathrm{d}\theta_i^2 + \sin^2\theta_i\mathrm{d}\phi_i^2) + \frac{e^{2f}}{9}(\mathrm{d}\psi + \sum_{i=1,2}\cos\theta_i\mathrm{d}\phi_i)^2\right] \\
F_{(5)} &= 27\pi N_c e^{-4g-f}h^{-5/4}(e^{x^0x^1x^2x^3\rho} - e^{\theta_1\phi_1\theta_2\phi_2\psi}) \\
F_{(1)} &= \frac{N_f(\rho)}{4\pi}(\mathrm{d}\psi + \cos\theta_1\mathrm{d}\phi_1 + \cos\theta_2\mathrm{d}\phi_2)
\end{aligned} \tag{3.51}$$

with $\psi \in [0, 4\pi], \theta_i \in [0, \pi], \phi_i \in [0, 2\pi], \rho \in \mathbb{R}$. There is an obvious choice of vielbein

$$\begin{aligned} e^{x^i} &= h^{-1/4}\mathrm{d}x^i & e^{\rho} &= h^{1/4}e^f\mathrm{d}\rho \\ e^{\theta_i} &= \frac{1}{\sqrt{6}}h^{1/4}e^g\mathrm{d}\theta_i & e^{\phi_i} &= \frac{1}{\sqrt{6}}h^{1/4}e^g\sin\theta_i\mathrm{d}\phi_i \\ e^{\psi} &= \frac{1}{3}h^{1/4}e^f(\mathrm{d}\psi + \cos\theta_1\mathrm{d}\phi_1 + \cos\theta_2\mathrm{d}\phi_2) \end{aligned} \tag{3.52}$$

The flavor branes behave differently in the massless or massive case. In the former, the authors of [17] used two stacks of branes whose world-volume coordinates may once more be identified with space-time ones,

$$\begin{aligned} \xi_1^\alpha &= (x^\mu, \rho, \theta_2, \phi_2, \psi) \quad \theta_1 = \text{const.} \quad \phi_1 = \text{const.} \\ \xi_2^\alpha &= (x^\mu, \rho, \theta_1, \phi_1, \psi) \quad \theta_2 = \text{const.} \quad \phi_2 = \text{const.} \end{aligned} \tag{3.53}$$

So prior to smearing the system has a global $U(N_f) \times U(N_f)$ flavor symmetry – one for each set of D7s. This is obviously a four-parameter family of embeddings, which can be smeared over the transverse (θ_i, ϕ_i) directions. In the massive case the embeddings are more complicated. In the field theory, the mass term breaks the global symmetry to the diagonal $U(N_f) \times U(N_f) \mapsto U(N_f)$, which corresponds the two stacks joining into one on the string theory side. There is again a four-parameter family of brane embeddings, yet as the generic embedding is much more complicated than those of (3.53), we shall only look at one representative, trusting that the calibration form will ensure that we make use of the whole family of branes. Choosing world-volume coordinates $\xi = (x^\mu, \theta_1, \phi_1, \theta_2, \phi_2)$, this is given by

$$\begin{aligned} X^M(\xi) &= \left(x^\mu, \rho_q - \frac{2}{3}\log\sin\frac{\theta_1}{2} - \frac{2}{3}\log\sin\frac{\theta_2}{2}, \theta_1, \phi_1, \theta_2, \phi_2, \phi_1 + \phi_2 + 2\beta\right) \\ &\rho_q, \beta = \text{const} \end{aligned} \tag{3.54}$$

The constant ρ_q denotes the minimal radius reached by the brane and may therefore be identified as the mass.

The branes have an $(7+1)$-dimensional world-volume and we therefore need to construct the calibration 8-form. In the case at hand this requires the knowledge of the supersymmetric spinors on the conifold. These were discussed in [75]. Our conventions however are those of [17]. The SUSY spinor ϵ is related to a constant spinor η as $\epsilon = h^{-1/8} e^{-\frac{i}{2}\psi}\eta$. Both satisfy the projections

$$\begin{aligned} i\sigma_2 \otimes \Gamma_{x^0x^1x^2x^3}\eta = \eta \qquad & \Gamma_{r\psi} = i\sigma_2\eta \\ \Gamma_{\theta_1\phi_1} = -i\sigma_2\eta \quad & \Gamma_{\theta_2\phi_2} = -i\sigma_2\eta \end{aligned} \tag{3.55}$$

From equation (2.39) it follows that the calibration form for D7-branes is given by

$$\mathcal{K} = \frac{1}{8!}(\eta^\dagger i\sigma_2 \otimes \Gamma_{a_0\dots a_7}\eta)e^{a_0\dots a_7} \tag{3.56}$$

which we may evaluate using (3.55) to be

$$\mathcal{K} = e^{x^0x^1x^2x^3} \wedge \left(e^{\rho\theta_1\phi_1\psi} + e^{\rho\theta_2\phi_2\psi} - e^{\theta_1\phi_1\theta_2\phi_2}\right) \tag{3.57}$$

At this point we may calculate the pull-backs $X^*\mathcal{K}$ for both embeddings (3.53) and (3.54). Finding $X^*\mathcal{K} = \sqrt{-\hat{g}_{(8)}}\mathrm{d}^8\xi$ we do thus verify that the brane embeddings are indeed supersymmetric.

In Einstein frame, the integrand of the DBI action is $e^\Phi\sqrt{-\hat{g}_{(8)}}\mathrm{d}^8\xi = e^\Phi X^*\mathcal{K}$. As before, supersymmetry requires this to satisfy $\mathrm{d}(e^\Phi\mathcal{K}) = F_{(9)}$. Making use of the definition $F_{(1)} = -e^{-2\Phi} * F_{(9)}$ and the equation of motion $\mathrm{d}F_{(1)} = -\Omega$, we arrive at the following

$$\begin{aligned} F_{(9)} &= \mathrm{d}(e^\Phi\mathcal{K}) = 3h^{-\frac{1}{4}}e^{-f}\frac{N_f(\rho)}{4\pi}e^{x^0x^1x^2x^3\rho\theta_1\phi_1\theta_2\phi_2} \\ F_{(1)} &= -e^{-2\Phi} * F_{(9)} = -3h^{-\frac{1}{4}}e^{-f}\frac{N_f(\rho)}{4\pi}e^\psi \\ \Omega &= -\mathrm{d}F_{(1)} = \frac{N_f(\rho)}{4\pi}(\sin\theta_1\mathrm{d}\theta_1 \wedge \mathrm{d}\phi_1 + \sin\theta_2\mathrm{d}\theta_2 \wedge \mathrm{d}\phi_2) \\ &\quad + \frac{N_f'(\rho)}{4\pi}\mathrm{d}\rho \wedge (\mathrm{d}\psi + \cos\theta_1\mathrm{d}\phi_1 + \cos\theta_2\mathrm{d}\phi_2) \\ N_f(\rho) &= \frac{4\pi}{3}e^{-2g-\Phi}(4e^{2g}g' + e^{2g}\Phi' - 4e^{2f}) \end{aligned} \tag{3.58}$$

The name for the function $N_f(\rho)$ has been chosen in anticipation of what is to come – it will denote the effective number of flavors at a given energy scale. It should not be confused with N_f, the number of flavor branes.

One should notice that the only assumptions made in deriving (3.58) are the form of $F_{(5)}$ and the vielbein describing the deformed background (3.52). That is, the above relations hold for all types of D7-branes one might want to smear, massless or massive. They allow us to write down the BPS equations of the system which can be derived from the SUSY variations [20] or using geometric methods.

$$\begin{aligned} g' &= e^{2f-2g} & f' &= 3 - 2e^{2f-2g} - \frac{3N_f(\rho)}{8\pi}e^{\Phi} \\ \Phi' &= \frac{3N_f(\rho)}{4\pi}e^{\Phi} & h' &= -27\pi N_c e^{-4g} \end{aligned} \tag{3.59}$$

Note that there are four first-order equations for the five functions Φ, f, g, h, N_f. Furthermore, the smearing procedure always uses the same action,

$$S_{\text{flavor}} = T_7 \int_{\mathcal{M}_{10}} \left(-e^{\Phi}\mathcal{K} + C_{(8)}\right) \wedge \Omega \tag{3.60}$$

The authors of [17, 20] used an action of the type encountered in (3.11) and (3.43), yet once more the equivalence with (3.60) may be shown explicitly – we will also present a general proof of the validity of (3.60) in section 3.1.2.

Given that the discussion up to this point is completely independent of the type of brane one wants to smear, one might ask how to distinguish between the different classes of potential flavor branes. The answer to that question lies in the choice of the function $N_f(\rho)$.

However, even before looking at specific choices of $N_f(\rho)$ the generic form of Ω in (3.58) tells us quite a bit about possible smeared-brane configurations. Firstly, it is not possible to break the $SU(2) \times SU(2) \times U(1) \times \mathbb{Z}_2$ symmetry of the background, as this is the inherent symmetry of Ω (The $\mathbb{Z}_2$ describes the exchange of the two spheres). So for massless branes we will only be able to smear both stacks simultaneously.

The massless branes may be identified with the coordinates given by (3.53). Thus they are smeared by the terms proportional to $d\theta_i \wedge d\phi_i$. As the smearing form is symmetric under the exchange $(\theta_1, \phi_1) \leftrightarrow (\theta_2, \phi_2)$ it is clear that we will have to smear both stacks of branes. I.e. one cannot assume $\Omega_{\theta_1\phi_1}$ to vanish without $\Omega_{\theta_2\phi_2}$ vanishing as well. The term involving $d\rho$ on the other hand is not transverse to the world-volume defined by (3.53). In order to smear only massless branes, one needs this term to vanish. I.e. massless branes require

$$N_f'(\rho) = 0 \tag{3.61}$$

Using this constraint the system (3.59) is fully determined and can be solved. In that case, we can see from (3.60) that the last term in (3.57) – which does not contain e^{ρ} – does not contribute. Interpreting the smearing form as a brane-density, we may identify the overall factor with the number of flavors,

$$N_f = 4\pi N_f(\rho) \tag{3.62}$$

That is, our decision to smear N_f massless branes with a constant number of flavors imposes two constraints into the system, namely (3.61) and (3.62).

Our choice for $N_f(\rho)$ may also be interpreted using the local geometry of the brane embeddings instead of their global coordinates. The vectors

$$(\partial_{x^\mu}, \partial_\rho, \partial_\psi) \tag{3.63}$$

are tangent to either stack of branes. As the smearing form should – locally – define a volume orthogonal to these vectors, we demand[4]

$$\imath_{\partial_{x^\mu}}\Omega = \imath_{\partial_\rho}\Omega = \imath_{\partial_\psi}\Omega = 0 \tag{3.64}$$

It follows that $4\pi N_f(\rho) = \text{const} = N_f$.

Turning to the massive case, the authors of [20] used

$$N_f'(\rho) = 3N_f e^{3\rho_q - 3\rho}(3\rho - 3\rho_q) \tag{3.65}$$

In principle one would expect that one can combine the knowledge of the embedding (3.54) together with the general form for Ω in order to derive this form for $N_f(\rho)$, as we did for massless branes, yet in [28] we were unable to do so. Our analysis contributes to the construction of $N_f(\rho)$ in so far, however, as the derivation in [20] requires the assumption that the $SU(2) \times SU(2) \times U(1) \times \mathbb{Z}_2$ symmetry cannot be broken, while we have shown that this is not an assumption, but an innate property of the background. As we mentioned in the introduction, these limitations of the approach are resolved when merging it with the microscopic perspective as in [41] and [42].

Once more one invokes [40] and needs only to study the BPS equations (3.59) together with the modified Bianchi identity to find solutions of the second order equations. We refer to the original papers for a discussion of the solutions.

Anticipating the possibility of using the formalism presented up to this point in order to smear branes whose coordinate representation is unknown, we shall now discuss the problem of correctly interpreting the smearing form Ω. Using the vielbein it takes the form

$$\Omega = \frac{6N_f(\rho)}{\sqrt{h}} e^{-2g}(e^{\theta_1\phi_1} + e^{\theta_2\phi_2}) + \frac{6N_f'(\rho)}{\sqrt{h}} e^{-2f} e^{\rho\psi} \tag{3.66}$$

[4] Interior multiplication of forms with vectors is defined as

$$(\imath_X\omega)_{N_1\ldots N_{p-1}} = X^M \omega_{MN_1\ldots N_{p-1}}$$

In the case for the massless embeddings (3.53) the second term disappeared and it is straightforward to interpret the first as a distribution on the space transverse to the two stacks of D7s. If we did not know about the massive embeddings (3.54) it would be tempting to interpret the term including N_f' as the distribution of a third stack of branes extending along x^μ, wrapping $(\theta_1, \phi_1, \theta_2, \phi_2)$ and positioned at fixed (ρ, ψ). That is we would think of this term as a contribution of compact, smeared D7 branes. The presence of such branes is potentially disastrous as the gauge theory in their world-volume could remain dynamic from a four-dimensional point of view. In the case at hand, the eight-dimensional gauge coupling behaves as $g_{\text{YM}} \sim g_s \alpha'^2$, which vanishes for $\alpha' \to 0$, the decoupling limit of the D3s. When using D5 branes on the other hand this does not have to happen. For the massive Klebanov-Witten model we know that our interpretation in terms of compact D7 branes is wrong as we are smearing a single stack of massive ones. Keeping this in mind we conclude that it is not straightforward to know which branes have been smeared by simply investigating Ω.

3.1.2 The generic case

The three examples of the previous section provide us with all the intuition needed to understand the relation between generalized calibrated geometry and supergravity duals with backreacted, smeared flavors. For a type IIA/B background with Ramond-Ramond flux $F_{(p+2)}$ and arbitrary dilaton we expect that we should always be able to write the action in terms of the calibration and smearing form as

$$S_{\text{flavor}} = -T_p \int_{\mathcal{M}_{10}} (e^{\frac{p-3}{4}\Phi}\mathcal{K} - C_{(p+1)}) \wedge \Omega \tag{3.67}$$

Now as we discussed in section 2.3, supersymmetry imposes

$$\mathrm{d}(e^{\frac{p-3}{4}\Phi}\mathcal{K}) = F_{(p+2)} \tag{3.68}$$

Combining this with the modified p-form equation of motion $\mathrm{d}F_{(10-p-2)} = 2\kappa_{10}^2 T_p \Omega$, as derived in (3.14), we may link the calibration and the smearing form

$$\mathrm{d}[*e^{\frac{10-2p-4}{4}\Phi}\mathrm{d}(e^{\frac{p-3}{4}\Phi}\mathcal{K})] = \pm 2\kappa_{10}^2 T_p \Omega \tag{3.69}$$

The overall sign depends on the conventions used when relating the field strength $F_{(p+2)}$ to its dual. In what follows, we shall give a more formal argument why the action (3.67) is appropriate to describe smeared branes, show that it is equivalent to the actions previously used in the literature and finally examine some of the consequences of the above relations.

The smeared brane action The problem of smearing a generic DBI+Wess-Zumino system takes a rather simple form from a mathematical point of view. Here we are dealing with two spaces, the world-volume $\mathcal{M}_{p+1}$ and space-time $\mathcal{M}_{10}$, which are related by the embedding map

$$\begin{aligned} X : \mathcal{M}_{p+1} &\to \mathcal{M}_{10} \\ \xi^\alpha &\mapsto X^M(\xi) \end{aligned} \tag{3.70}$$

As integrals of scalars are ill-defined on manifolds, it is mandatory for this discussion to think of the brane action as an integral of differential forms. For the Wess-Zumino term, the integrand is the pull-back of the relevant electrically coupled gauge-potential onto the world-volume, $\int_{\mathcal{M}_{p+1}} X^* C_{(p+1)}$. Whereas we integrate over the induced volume form and the dilaton in the case of the DBI action,[5] $\int_{\mathcal{M}_{p+1}} \mathrm{d}^{p+1}\xi e^{\frac{p-3}{4}\Phi}\sqrt{-\hat{g}_{(p+1)}}$. The crucial point is that there is no way to a priori identify the DBI integrand with a $(p+1)$-form in space-time, as the induced volume form is usually not thought of as the pull-back of a differential form. Indeed, we were rather careless in section 3.1.1 as we did not discriminate between the set of form-fields in the world-volume of the brane, $\Omega(\mathcal{M}_{p+1})$, and that defined on all of space-time, $\Omega(\mathcal{M}_{10})$.

One might argue that we should be able to somehow push the induced volume form forward onto space-time. This is certainly the case if we are able to identify world-volume with space-time coordinates. In the case of the string dual of the $\mathcal{N} = 1$ sQCD-like theory this was strikingly obvious. As a matter of fact, the action written in the first line of (3.11) is exactly of the form (3.67). In a generic situation however, we cannot expect to be able to find such a set of global coordinates. Moreover the natural operations induced by maps between manifolds are push-forwards of vectors and pull-backs of forms. And as they connect spaces of different dimensions, they cannot be assumed to be invertible.

[5] The discussion in this section considers branes without world-volume gauge fields or the NS potential B. See however [76, 68, 67].

This is where calibrated geometry comes in. As we have seen before, supersymmetric branes satisfy $X^*\mathcal{K} = \sqrt{-\hat{g}_{(p+1)}}\mathrm{d}^{p+1}\xi$. Making use of this fact allows us to treat the DBI and Wess-Zumino terms on a democratic footing, as both integrands can now be written as pull-backs of $(p+1)$-forms defined on space-time.

We shall now show that the action (3.67) can always be written in the form used in [17, 71]. Essentially the whole discussion boils down to the fact that we may locally choose nice coordinates. Let us assume that we have a single stack of supersymmetric p-branes. Locally, we may choose coordinates $x^M = (z^\mu, y^m)$ such that the branes extend along the z^μ; that is for world-sheet coordinates ξ^μ and embeddings $X^M(\xi)$ we have

$$\partial_\nu X^M = \begin{cases} \delta^M_\nu & M \in \{0, \dots, p\} \\ 0 & M \notin \{0, \dots, p\} \end{cases} \tag{3.71}$$

The vectors ∂_μ are tangent to the brane. They span a subset of $T\mathcal{M}_{10}$ which may be thought of as the embedding of the tangent space $T\mathcal{M}_{p+1}$ of the brane into that of space-time. Orthonormalizing the ∂_μ we obtain a new basis of $T\mathcal{M}_{p+1}$ given by some E_α. I.e. $span(E_\alpha) = T\mathcal{M}_{p+1} \subset T\mathcal{M}_{10}$. It follows from the construction that the E_α are closed under the Lie bracket, i.e. $[E_\alpha, E_\beta] \in span(E_\gamma)$. Therefore $E^m_\alpha = 0$ and the matrix E^μ_α is invertible. We may complete the set E_α to a basis of the whole tangent space, $E_A = (E_\alpha, E_a)$. Naturally, there is a dual basis of covectors, $e^A = (e^\alpha, e^a)$ which we may use as a vielbein.

Having constructed a vielbein suitable for our purposes we shall now express the DBI action in terms of that vielbein. As the two bases are dual we have

$$0 = E_\alpha e^b = E^M_\alpha e^b_M \tag{3.72}$$

Contracting with $(E^\mu_\alpha)^{-1} = e^\alpha_\mu$, we obtain

$$e^b_\mu = 0 \tag{3.73}$$

This is quite important. It means that the components e^a of the vielbein are not pulled back onto the brane world-volume whereas all the e^α are. After all, the pull-back acts as $X^*(\omega_M \mathrm{d}x^M) = \omega_\mu \mathrm{d}\xi^\mu$. It follows that the volume form induced onto the brane world-volume is given by the pull-back of the forms e^α

$$\sqrt{-\hat{g}_{(p+1)}}\mathrm{d}^{p+1}\xi = \bigwedge_\alpha (X^* e^\alpha) \tag{3.74}$$

The DBI action in this frame is therefore given by

$$S_{DBI} = -T_p \int_{\mathcal{M}_{p+1}} e^{\frac{p-3}{4}\Phi} \bigwedge_\alpha (X^* e^\alpha) \tag{3.75}$$

In the final part of our discussion, we will impose some constraints on the calibration and smearing form, and show that an action of the form (3.1) can always be rewritten in the form (3.11). For the calibration form to satisfy $X^*\mathcal{K} = \sqrt{-\hat{g}_{(p+1)}}\mathrm{d}\xi^0 \wedge \cdots \wedge \mathrm{d}\xi^p$, it has to include $\bigwedge_\alpha e^\alpha$. So we may assume it to be of the form $\mathcal{K} = \bigwedge_\alpha e^\alpha + K$, where K is a $(p+1)$-form which does not depend on all the indices α simultaneously and therefore includes some of the e^a. It follows that $X^*K = 0$. The smearing form is defined on the space transverse to the branes. This space has a one-form basis given by $\mathrm{d}y^m$. As we saw above $e^a_\mu = 0$ and it follows that we may write the smearing form in this basis as

$$\begin{aligned}\Omega &= \frac{1}{(10-p-1)!}\Omega_{m_1 \dots m_{10-p-1}} \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^{10-p-1} \\ &= \frac{1}{(10-p-1)!}\Omega_{a_1 \dots a_{10-p-1}} e^{a_1 \dots a_{10-p-1}} = \Omega_{(p+2)\dots 9} e^{(p+2)\dots 9}\end{aligned} \tag{3.76}$$

That is, locally the smearing form is defined by a single scalar function $\Omega_{(p+2)\dots 9}$ and includes the wedge product over all the transverse components of the vielbein, $\bigwedge_a e^a$. We see immediatly that $K \wedge \Omega = 0$. Moreover

$$\mathcal{K} \wedge \Omega = e^{0\dots 9}\Omega_{(p+2)\dots 9} \tag{3.77}$$

The trick is now to associate the indices of the function $\Omega_{(p+2)\dots 9}$ with something other than those of the relevant components of the vielbein, as we need those for the overall volume form $e^{0\dots 9} = \sqrt{-g_{(10)}}\mathrm{d}^{10}x$. As the form reduces to a function and we are working in flat indices, we may resolve this as follows:

$$\begin{aligned}\mathcal{K} \wedge \Omega &= e^{0\dots 9}\Omega_{(p+2)\dots 9} = e^{0\dots 9}\sqrt{\Omega_{(p+2)\dots 9}\Omega^{(p+2\dots 9)}} \\ &= \sqrt{-g_{(10)}}\mathrm{d}^{10}x|\Omega|\end{aligned} \tag{3.78}$$

with the modulus of the smearing form defined as in (3.12). As the wedge product is linear, one may immediately generalize our argument here for multiple stacks of branes, thus proving our initial assertion.

As an immediate application of the results of this section we shall take a brief look at central extensions of SUSY algebras. From the equations of motion (3.15) it follows that the smearing form is exact, $\mathrm{d}F_{(10-p-2)} = 2\kappa_{10}^2 T_p \Omega$. Supersymmetry requires that $(e^{\frac{p-3}{4}\Phi}\mathcal{K} - C_{(p+1)})$ is closed. It follows that we may write the smeared brane action (3.67) as a surface integral at infinity,

$$S_{\text{flavor}} = -\frac{1}{2\kappa_{10}^2}\int_{S^9_\infty}(e^{\frac{p-3}{4}\Phi}\mathcal{K} - C_{(p+1)}) \wedge F_{(10-p-2)} \tag{3.79}$$

This takes the form of a charge. From the original discussion of generalized calibrated geometry in [36] we recall the fact that probe-brane actions relate to central charges in supersymmetry algebras – as one would expect for BPS objects. We conjecture that the charge defined by (3.79) has the same interpretation.

3.2 $\mathcal{N} = 2$ gauge-string duality in $d = 2 + 1$

Let us now apply the methods described in the previous section to the flavoring of an $\mathcal{N} = 2$ super Yang-Mills-like dual in $d = 2 + 1$. A string dual can be found in the unflavored case by constructing a domain-wall solution in $d = 7$ gauged supergravity and then lift it to ten dimensions. It then describes a stack of NS5-branes wrapping a three-sphere. Details and physical interpretation of this solution can be found in [72] and [73]. We are first going to describe the unflavored solution using notations from [73] before studying the addition of flavors.

3.2.1 The unflavored solution

In the unflavored case, we consider only NS5-branes wrapping a three-sphere. So the non-zero fields in type IIB supergravity are the metric $g_{\mu\nu}$, the dilaton Φ and the NS-NS 3-form field strength H. The solution found in [73] is, in string frame

$$\mathrm{d}s^2 = \mathrm{d}\xi_{1,2}^2 + \frac{2z}{g^2}\mathrm{d}\Omega_3^2 + \frac{e^{2x}}{g^2}(\mathrm{d}z^2 + \mathrm{d}\psi^2) + \frac{1}{g^2\Omega}\sin^2\psi(E_1^2 + E_2^2) \tag{3.80}$$

$$e^{2\Phi} = \left(\frac{2z}{g^2}\right)^{3/2}\frac{e^{-2A+x}}{\Omega} \tag{3.81}$$

$$\begin{aligned} H = & \frac{ge^{-2x}}{2z\Omega^{1/2}}\left[\cos\psi(e^{124} - e^{236} - e^{135}) - e^{2x}\sin\psi e^{127}\right] \\ & - \frac{ge^{-2x}\sin\psi}{\Omega^{3/2}}\left[e^{6x}\sin^2\psi + e^{2x}(4\cos^2\psi + 1) - 3e^{-2x}\cos^2\psi - \frac{\cos^2\psi}{z}\right]e^{567} \end{aligned}$$

$$-\frac{ge^{-2x}\cos\psi}{\Omega^{3/2}}\left[e^{4x}\sin^2\psi-3+e^{-4x}\cos^2\psi-\frac{e^{2x}\sin^2\psi}{z}\right]e^{456} \tag{3.82}$$

A and x are functions of z defined as

$$e^{-2x} = \frac{I_{3/4}(z)-cK_{3/4}(z)}{I_{-1/4}(z)+cK_{1/4}(z)} \tag{3.83}$$

$$e^{A+3x/2} = z\left(I_{-1/4}(z)+cK_{1/4}(z)\right) \tag{3.84}$$

where I_α and K_α are the modified Bessel functions and c is an integration constant. In the previous equations, we used the vielbein

$$\begin{aligned}
e^a &= \frac{\sqrt{2z}}{g}S^a \quad a=1,2,3 & e^7 &= \frac{1}{g\Omega^{1/2}}(\cos\psi \mathrm{d}z - e^{2x}\sin\psi\mathrm{d}\psi)\\
e^4 &= \frac{1}{g\Omega^{1/2}}(e^{2x}\sin\psi\mathrm{d}z+\cos\psi\mathrm{d}\psi) & e^8 &= \mathrm{d}\xi^1\\
e^5 &= \frac{1}{g\Omega^{1/2}}\sin\psi E_1 & e^9 &= \mathrm{d}\xi^2\\
e^6 &= \frac{1}{g\Omega^{1/2}}\sin\psi E_2 & e^0 &= \mathrm{d}\xi^0
\end{aligned} \tag{3.85}$$

with

$$\begin{aligned}
\sigma^1 &= \cos\tilde\beta\mathrm{d}\tilde\theta+\sin\tilde\beta\sin\tilde\theta\mathrm{d}\tilde\phi\\
\sigma^2 &= \sin\tilde\beta\mathrm{d}\tilde\theta-\cos\tilde\beta\sin\tilde\theta\mathrm{d}\tilde\phi\\
\sigma^3 &= \mathrm{d}\tilde\beta+\cos\tilde\theta\mathrm{d}\tilde\phi\\
S^1 &= \cos\phi\frac{\sigma^1}{2}-\sin\phi\frac{\sigma^2}{2}\\
S^2 &= \sin\theta\frac{\sigma^3}{2}-\cos\theta\left(\sin\phi\frac{\sigma^1}{2}+\cos\phi\frac{\sigma^2}{2}\right)\\
S^3 &= -\cos\theta\frac{\sigma^3}{2}-\sin\theta\left(\sin\phi\frac{\sigma^1}{2}+\cos\phi\frac{\sigma^2}{2}\right)\\
E_1 &= \mathrm{d}\theta+\cos\phi\frac{\sigma^1}{2}-\sin\phi\frac{\sigma^2}{2}\\
E_2 &= \sin\theta\left(\mathrm{d}\phi+\frac{\sigma^3}{2}\right)-\cos\theta\left(\sin\phi\frac{\sigma^1}{2}+\cos\phi\frac{\sigma^2}{2}\right)\\
\Omega &= e^{2x}\sin^2\psi+e^{-2x}\cos^2\psi\\
&\theta,\tilde\theta,\psi\in[0,\pi] \qquad \phi,\tilde\phi\in[0,2\pi[\qquad \tilde\beta\in]0,4\pi]
\end{aligned} \tag{3.86}$$

and $\mathrm{d}\Omega_3^2 = \sigma^i\sigma^i$. We know that type IIB supergravity contains thirty-two supercharges that can be described by an $SO(2)$ doublet of chiral spinors $\epsilon = (\epsilon^-,\epsilon^+)$. Their chirality is expressed as

$$\Gamma_{11}\epsilon = \Gamma_{1234567890}\epsilon = -\epsilon \tag{3.87}$$

This background preserves four supercharges, corresponding to $\mathcal{N} = 2$ in $d = 2+1$ dimensions. This means that ϵ has to verify the projections

$$\begin{aligned}
\Gamma^{1256}\epsilon &= \epsilon \\
\Gamma^{1346}\epsilon &= \epsilon \\
\Gamma^{4567}\epsilon &= \sigma_3\epsilon
\end{aligned} \tag{3.88}$$

where σ_3 is the third Pauli matrix.

3.2.2 Deformation of the solution

We are now working again in Einstein frame. We first notice that, in the solution of the previous section, e^4 and e^7 are mixing the z and ψ coordinates. In order to simplify this, we make a common change of coordinates, first proposed in [77]:

$$\begin{aligned}
\rho &= \sin\psi \frac{e^{A-x/2}}{(2zg^2)^{1/4}} \\
\sigma &= \sqrt{g}\frac{\cos\psi}{(2z)^{3/4}}e^{A+3x/2}
\end{aligned} \tag{3.89}$$

We then get that $e^4 = h_1(\rho,\sigma)\mathrm{d}\rho$ and $e^7 = h_2(\rho,\sigma)\mathrm{d}\sigma$. Let us now deform the metric by modifying the vielbein in (3.85)

$$\begin{aligned}
e^a &= e^{-f/2}\sqrt{j(\rho,\sigma)}S^a \quad a=1,2,3 & e^7 &= e^{-f/2}\sqrt{h_2(\rho,\sigma)}\mathrm{d}\sigma \\
e^4 &= e^{-f/2}\sqrt{h_1(\rho,\sigma)}\mathrm{d}\rho & e^8 &= e^{-f/2}\mathrm{d}\xi^1 \\
e^5 &= e^{-f/2}\sqrt{h_1(\rho,\sigma)k(\rho,\sigma)}E_1 & e^9 &= e^{-f/2}\mathrm{d}\xi^2 \\
e^6 &= e^{-f/2}\sqrt{h_1(\rho,\sigma)k(\rho,\sigma)}E_2 & e^0 &= e^{-f/2}\mathrm{d}\xi^0
\end{aligned} \tag{3.90}$$

It gives us the following ansatz for the metric:

$$\begin{aligned}
\mathrm{d}s^2 = e^{-f(\rho,\sigma)}\Big(&\mathrm{d}\xi_{1,2}^2 + j(\rho,\sigma)\mathrm{d}\Omega_3^2 + h_1(\rho,\sigma)[\mathrm{d}\rho^2 + k(\rho,\sigma)(E_1^2+E_2^2)] \\
&+ h_2(\rho,\sigma)\mathrm{d}\sigma^2\Big)
\end{aligned} \tag{3.91}$$

It is straightforward to see that this ansatz leaves the topology of the previous solution invariant.

3.2.3 Calibration, smearing and G-structures

We are now interested in adding flavor D5-branes to the background. Following the usual method, we first deform the unflavored solution for D5-branes. Then we find calibrated cycles where we can put supersymmetric D5-branes. We finally smear them and find a solution that includes their backreaction.

The solution in the previous section describes NS5-branes. As we are interested in the IR behaviour of the gauge dual, we want to consider D5-branes. So we first perform an S-duality on the solution. It gives a new solution of type IIB supergravity describing D5-branes, for which non-zero fields are the metric, the dilaton and the Ramond-Ramond 3-form such that

$$g_{\mu\nu}^{NS5} \longrightarrow g_{\mu\nu}^{D5} \tag{3.92}$$

$$\Phi^{NS5} \longrightarrow -\Phi^{D5} \tag{3.93}$$

$$H_{(3)}^{NS5} \longrightarrow F_{(3)}^{D5} \tag{3.94}$$

$$\sigma_3 \longrightarrow \sigma_1 \tag{3.95}$$

As we want to keep the same number of supercharges, and just deform the previous solution, we are imposing the same projections on the SUSY spinors as (3.88). We then define a new $SO(2)$ doublet

$$\boldsymbol{\eta} = \begin{pmatrix} \eta^- \\ \eta^+ \end{pmatrix} = \begin{pmatrix} \epsilon^- + \epsilon^+ \\ \epsilon^- - \epsilon^+ \end{pmatrix} \tag{3.96}$$

such that (3.88) becomes

$$\begin{aligned} \Gamma^{1256}\boldsymbol{\eta} &= \boldsymbol{\eta} \\ \Gamma^{1346}\boldsymbol{\eta} &= \boldsymbol{\eta} \\ \Gamma^{4567}\boldsymbol{\eta} &= \sigma_3\boldsymbol{\eta} \end{aligned} \tag{3.97}$$

Notice that $\boldsymbol{\eta}$ is still a doublet of chiral spinors that satisfies

$$\Gamma_{11}\boldsymbol{\eta} = -\boldsymbol{\eta} \tag{3.98}$$

From the third projection, we see that η^- and η^+ are both non-zero, but behave differently under the action of gamma matrices. So for each spinor we can construct a six-dimensional generalized calibration form

$$\begin{aligned} \mathcal{K}^- &= \eta^{-T}\Gamma_{089abc}\eta^- e^{089abc} \\ \mathcal{K}^+ &= \eta^{+T}\Gamma_{089abc}\eta^+ e^{089abc} \end{aligned} \tag{3.99}$$

Those forms can be written as

$$\begin{aligned} \mathcal{K}^- &= e^{089} \wedge K^- \\ \mathcal{K}^+ &= e^{089} \wedge K^+ \end{aligned} \tag{3.100}$$

where K^+ and K^- are three-forms. Using supersymmetric variations of the gravitino and the dilatino and identifying $F_{(3)}$ with a torsion term, it is possible to define two covariant derivatives $\tilde{\nabla}^+$ and $\tilde{\nabla}^-$ such that

$$\begin{aligned}\tilde{\nabla}^+\eta^+ &= 0\\ \tilde{\nabla}^-\eta^- &= 0\end{aligned} \tag{3.101}$$

So the existence of $\eta^\pm$ imposes that the internal manifold has special holonomy, and thus admits a corresponding G-structure. With each spinor satisfying the projections (3.97), it is possible to define two different G_2 structures in the seven-dimensional space with tangent directions {1,2,3,4,5,6,7}. The corresponding associative three-forms are K^+ and K^-. We want the flavor branes we add to preserve the same supercharges as in the unflavored solution. From [56], we know that there is in fact an $SU(3)$ structure in that space, for which the three-dimensional calibration form is

$$K = \frac{1}{2}(K^- - K^+) \tag{3.102}$$

So the calibration form for D5-branes in this geometry is

$$\mathcal{K} = e^{089} \wedge K \tag{3.103}$$

We have (details of the calculation can be found in Appendix 3.A)

$$\begin{aligned}K^- &= e^{123} + e^{145} - e^{167} + e^{246} + e^{257} + e^{347} - e^{356}\\ K^+ &= -e^{123} - e^{145} - e^{167} - e^{246} + e^{257} + e^{347} + e^{356}\end{aligned} \tag{3.104}$$

So,

$$\mathcal{K} = e^{089} \wedge (e^{123} + e^{145} + e^{246} - e^{356}) \tag{3.105}$$

In order to find solutions for the deformed background, we first need to provide an ansatz for the Ramond-Ramond form $F_{(3)}$:

$$\begin{aligned}F = e^{-3\Phi/4}(&F_{124}(\rho,\sigma)e^{124} + F_{135}(\rho,\sigma)e^{135} + F_{236}(\rho,\sigma)e^{236} + F_{127}(\rho,\sigma)e^{127}\\ &+ F_{456}(\rho,\sigma)e^{456} + F_{567}(\rho,\sigma)e^{567})\end{aligned} \tag{3.106}$$

and we assume the dilaton depends only on ρ and σ. As mentioned previously, we know from [40] that conservation of supersymmetry gives us first order differential equations that, in addition to imposing the Bianchi identity for $F_{(3)}$, will solve the equations of motion. One way to find those equations is to study the type IIB supersymmetry transformations of the dilatino and the gravitino

$$\delta\lambda = \frac{1}{2}\Gamma^\mu\partial_\mu\Phi\boldsymbol{\eta} + \frac{1}{24}e^{\Phi/2}F_{\mu\nu\rho}\Gamma^{\mu\nu\rho}\sigma_3\boldsymbol{\eta} = 0 \tag{3.107}$$

$$\delta\psi_\mu = \nabla_\mu\boldsymbol{\eta} + \frac{1}{96}e^{\Phi/2}F_{\nu\rho\sigma}(\Gamma_\mu{}^{\nu\rho\sigma} - 9\delta_\mu^\nu\Gamma^{\rho\sigma})\sigma_3\boldsymbol{\eta} = 0 \tag{3.108}$$

Another way is to use the geometric properties of the space, using G-structures and generalized calibration conditions. As stated previously, we need to assume that $\Phi = 2f$. Otherwise, we can look at the dilatino variation to get an additional condition. From it we get

$$\partial_\rho \Phi = \frac{e^{(2f-\Phi)/4}\sqrt{h_1}}{2}(F_{127} - F_{567}) \tag{3.109}$$
$$\partial_\sigma \Phi = \frac{e^{(2f-\Phi)/4}\sqrt{h_2}}{2}(F_{135} + F_{236} + F_{456} - F_{124}) \tag{3.110}$$

Then we remember that $\mathcal{K}^-$ is a generalized calibration and K^- defines a G_2 structure. So we get two conditions on those forms

$$\mathrm{d}(e^{\Phi/2}\mathcal{K}^-) = -e^{\Phi} *_{10} F \tag{3.111}$$
$$\mathrm{d}(e^{\Phi} *_7 K^-) = \mathrm{d}(e^{\Phi} *_{10} \mathcal{K}^-) = 0 \tag{3.112}$$

Using the conditions on the dilaton, those two equations give us

$$f = \frac{\Phi}{2} \tag{3.113}$$
$$\partial_\rho \Phi = -\frac{j\sqrt{h_1}F_{567} + h_1\sqrt{k}}{2j} \tag{3.114}$$
$$\partial_\sigma \Phi = \frac{\sqrt{h_2}(F_{456} - 3F_{124})}{2} \tag{3.115}$$
$$\partial_\rho j = 2h_1\sqrt{k} \tag{3.116}$$
$$\partial_\sigma j = 2j\sqrt{h_2}F_{124} \tag{3.117}$$
$$\partial_\rho k = 2\sqrt{k} - \frac{h_1 k^{3/2}}{j} + k\frac{h_1^{3/2}F_{567} - \partial_\rho h_1}{h_1} \tag{3.118}$$
$$\partial_\sigma k = 0 \tag{3.119}$$
$$\partial_\rho h_2 = h_2\frac{j\sqrt{h_1}F_{567} + h_1\sqrt{k}}{j} \tag{3.120}$$
$$\partial_\sigma h_1 = h_1\sqrt{h_2}(F_{124} - F_{456}) \tag{3.121}$$
$$F_{127} = -\frac{\sqrt{h_1}k}{j} \tag{3.122}$$
$$F_{135} = -F_{124} \tag{3.123}$$
$$F_{236} = -F_{124} \tag{3.124}$$

Moreover, we must have

$$\begin{aligned} \partial_\rho\partial_\sigma\Phi &= \partial_\sigma\partial_\rho\Phi \\ \partial_\rho\partial_\sigma j &= \partial_\sigma\partial_\rho j \end{aligned} \tag{3.125}$$

So we get

$$\partial_\rho F_{124} = -\frac{j\sqrt{h_1}F_{124}F_{567} + h_1\sqrt{k}(3F_{124}+2F_{456})}{2j} \tag{3.126}$$

$$\frac{\partial_\rho F_{456}}{\sqrt{h_1}} = -\frac{\partial_\sigma F_{567}}{\sqrt{h_2}} - \frac{\sqrt{h_1 k}(4F_{124}+5F_{456}) + jF_{124}F_{567}}{2j} \tag{3.127}$$

Let us now eliminate components of F in (3.114) to (3.121) and try to solve those equations. We get

$$h_1 = \frac{e^{-2\Phi}}{j}e^{a(\rho)} \tag{3.128}$$

$$h_2 = e^{-2\Phi}e^{b(\sigma)} \tag{3.129}$$

$$e^{2\Phi} = \frac{2\sqrt{k}}{j\partial_\rho j}e^{a} \tag{3.130}$$

$$F_{124} = \frac{e^{(a-b)/2}k^{1/4}\partial_\sigma j}{\sqrt{2\partial_\rho j}j^{3/2}} \tag{3.131}$$

$$F_{456} = \frac{e^{(a-b)/2}k^{1/4}(\partial_\sigma j\partial_\rho j - 2j\partial_\sigma\partial_\rho j)}{\sqrt{2}(j\partial_\rho j)^{3/2}} \tag{3.132}$$

$$F_{567} = \frac{\sqrt{k}(\partial_\rho j)^2 - j((2+\sqrt{k}a')\partial_\rho j - 2\sqrt{k}\partial_\rho^2 j)}{\sqrt{2}k^{1/4}j(\partial_\rho j)^{3/2}} \tag{3.133}$$

$$\partial_\rho k = 2\sqrt{k} - ka' \tag{3.134}$$

We notice that $b(\sigma)$ is arbitrary, which corresponds to the fact that it is always possible to redefine the σ coordinate. To simplify the problem, we are taking $b = 0$ in the following sections.

3.2.4 Addition and smearing of flavor branes

In order to add and smear flavor branes, one needs to find the smearing form Ω. Following the prescription presented in the first part of this article, we know that this form is related to the calibration form of our background $\mathcal{K}$ (see (3.105)) through

$$\Omega = \mathrm{d}\, F = -\mathrm{d}(e^{-\Phi} * \mathrm{d}(e^{\Phi/2}\mathcal{K})) \tag{3.135}$$

Using this, the ansatz for the metric and for F and the equations found in the previous section ((3.113) to (3.127)), we can deduce that the most general form of Ω is

$$\Omega = e^{\Phi}\Big(N_{f1}(\rho,\sigma)[e^{2367} + e^{1357} - e^{1247}] + N_{f2}(\rho,\sigma)e^{4567}\Big) \tag{3.136}$$

with

$$\partial_\sigma F_{124} = \sqrt{h_2}\frac{j(F_{124}F_{456} - 5F_{124}^2 + 2N_{f1}e^{2\Phi}) - 2 - 2\sqrt{h_1 k}F_{567}}{2j} \tag{3.137}$$

$$\frac{\partial_\sigma F_{456}}{\sqrt{h_2}} = \frac{\partial_\rho F_{567}}{\sqrt{h_1}} + \frac{3F_{567}^2}{2} + \frac{F_{567}(4j - h_1 k)}{2j\sqrt{h_1 k}} + \frac{3F_{456}(F_{456} - F_{124})}{2} - e^{2\Phi}N_{f2} \tag{3.138}$$

Consistency between those equations and (3.128) to (3.134) imposes that

$$N_{f2} = N_{f1} + \frac{j}{h_1\sqrt{k}}\partial_\rho N_{f1} \tag{3.139}$$

$$0 = 2j^2\partial_\rho^2 j + 2e^a j\partial_\sigma^2 j + j(\partial_\rho j)^2 - e^a(\partial_\sigma j)^2 - j^2(a'\partial_\rho j + 4e^a N_{f1}) \tag{3.140}$$

We now see that the only unknown we have is N_{f1}. Any function of ρ and σ is possible and will give first order differential equations that will solve the modified equations of motion for type IIB supergravity plus flavor embeddings. Finding a solution then consists only on solving the second-order differential equation (3.140). However, while the choice of the function N_{f1} determines which branes are smeared, we are unable to derive the embedding of the supersymmetric branes that have been smeared. One might want to recall the discussion at the end of section 3.1.1.

Different possibilities for the smearing form As it was stated before, the starting point of adding smeared flavors is to choose a smearing form, which, in the case we are currently studying, corresponds to choosing a function $N_{f1}(\rho, \sigma)$.

A first possibility would be to take N_{f1} independent of ρ. It follows from (3.139) that

$$N_{f1} = N_{f2} = N_f(\sigma) \tag{3.141}$$

Then we can try to solve (3.140) by making the following ansatz for j:

$$j(\rho, \sigma) = G(\rho)^{2/3}H(\sigma)^2 \tag{3.142}$$

We obtain

$$G' = c_1 e^{a/2} \tag{3.143}$$

$$\frac{H''}{H} = N_f \tag{3.144}$$

where c_1 is a constant. In the case where $a = 0$ and N_f is a constant, we can solve this and find

$$k = (\rho + \rho_0)^2 \tag{3.145}$$

and

$$j = (c_1\rho + c_2)^{2/3}\cos(\sqrt{-N_f}\sigma + c_3)^2 \qquad \text{if } N_f \leq 0 \tag{3.146}$$

$$j = (c_1\rho + c_2)^{2/3}\cosh(\sqrt{N_f}\sigma + c_3)^2 \qquad \text{if } N_f \geq 0 \tag{3.147}$$

with c_1, c_2 and c_3 are integration constants. These provide analytic solutions to the equations of motion of type IIB supergravity with modified Bianchi identity. When looking at the dilaton behavior, we find

$$e^{2\Phi} = \frac{3(\rho+\rho_0)}{c_1(c_2+c_1\rho)^{1/3}\cos(c_3+\sqrt{-N_f}\sigma)^4} \qquad \text{if } N_f \leq 0 \tag{3.148}$$

$$e^{2\Phi} = \frac{3(\rho+\rho_0)}{c_1(c_2+c_1\rho)^{1/3}\cosh(c_3+\sqrt{N_f}\sigma)^4} \qquad \text{if } N_f \geq 0 \tag{3.149}$$

When $N_f \leq 0$, in (3.148), it is remarkable that there are singularities for $c_3 + \sqrt{-N_f}\sigma = \frac{\pi}{2} \bmod (2\pi)$. Those singularities may be a sign of the presence of the smeared flavor branes.

Another possibility would be to try to have a smearing form independent of one of the radial coordinates, instead of just the function N_{f1} as in the previous paragraph. For Ω to be independant of σ, we have to take

$$N_{f1} = \frac{N(\rho)}{\sqrt{j}} \tag{3.150}$$

Then (3.140) becomes

$$0 = 2j^2\partial_\rho^2 j + 2e^a j\partial_\sigma^2 j + j(\partial_\rho j)^2 - e^a(\partial_\sigma j)^2 - j^2 a'\partial_\rho j - 4e^a N(\rho) j^{3/2} \tag{3.151}$$

Taking here $N(\rho)$ to be constant, we get $N_{f2} = 0$ which suppresses one of the terms in the smearing form. Nevertheless, it is not obvious how to find a solution to the equation for j.

For Ω to be independent of ρ, one needs to impose k to be a constant. Then

$$a(\rho) = 2a_1\rho \tag{3.152}$$

$$N_{f1} = \frac{e^{-a_1\rho}}{\sqrt{j}} N(\sigma) \tag{3.153}$$

where a_1 is a strictly positive constant. We now have to solve:

$$0 = 2j^2\partial_\rho^2 j + 2e^{2a_1\rho} j\partial_\sigma^2 j + j(\partial_\rho j)^2 - e^{2a_1\rho}(\partial_\sigma j)^2 - 2j^2 a_1\partial_\rho j - 4e^{a_1\rho} N(\sigma) j^{3/2} \tag{3.154}$$

In the case where $N(\sigma) = N_f$ is a constant, the smearing form is independent of any radial dependence. In that case we can find asymptotic solutions, considering ρ as the energy scale. One interesting fact is that it seems it is not possible to ignore the term involving N_f in the IR, that is when ρ goes to zero. In the IR ($\rho \to 0$), we find that

$$j = e^{2a_1\rho/3}\left(\frac{3N_f}{a_1^2 - a_1} + c_1 e^{(1-a_1)\rho}\right)^{2/3} \qquad \text{if } a_1 \neq 1 \tag{3.155}$$

$$j = e^{2a_1\rho/3}\left(3N_f\rho + c_2 e^{-\rho}\right)^{2/3} \qquad \text{if } a_1 = 1 \tag{3.156}$$

In the UV, we have two possibilities: we can decide that the term in N_f is suppressed or plays a role. The two cases give

$$j = c_3 e^{2a_1\rho/3}\sigma^2 \qquad \text{if we neglect the term in } N_f \tag{3.157}$$

$$j = e^{-2a_1\rho}\frac{N_f^2\sigma^4}{4} \tag{3.158}$$

Comments on the solution Firstly one can notice that none of the solutions presented in the previous section goes to the solution found in [73] in the limit N_{f1}, N_{f2} goes to zero, as expected from the dual gauge theory point of view.

We are trying to find a solution that describes a stack of N_c color branes plus one or several stacks of smeared flavor branes. The number of color branes is related to the Ramond-Ramond field $F_{(3)}$ through

$$\int_{S^3} F_{(3)} = 2\kappa_{10}^2 T_5 N_c \tag{3.159}$$

where S^3 is a three-sphere around the point where the color branes are placed in the four-dimensional space transverse to their world-volume. We were not able to find a constant when calculating the previous integral for the solutions of the previous section. It means that either we did not find the right transverse four-dimensional space, or these results cannot have the usual interpretation of stacks of branes.

This relates to the most prominent problem of the method presented in this section. As we mentioned in footnote 3, it is necessary to verify the existence of a cycle wrapped by the branes. As we explicitly avoided the issue of considering the embedding smeared, one cannot be certain that the above solutions do describe smeared branes. In simple cases when the smearing form does not have a term along the radial direction of the space, each component in the vielbein basis can usually be interpreted as the volume form of the space orthogonal to the brane smeared. In the case studied above, Ω has to have a term in $d\rho$. So in comparison to Klebanov-Witten, it seems that we are smearing massive flavor branes. But we were not able to determine their embedding. However, the form of Ω tells us it is not possible to smear massless flavor branes in this background. Moreover, knowing the explicit embedding of the flavor branes is not necessary to look at some properties of the gauge theory dual.

3.3 Discussion

In this chapter we have taken a first, detailed look at the flavoring procedure and its relation to calibrated geometry and G-structures. In section 3.1, we showed that the process is equivalent to those used previously in the literature, but makes the symmetry with the Wess-Zumino term apparent and the linearity in the smearing form Ω manifest. The crucial point is that this macroscopic perspective allows us to impose strong constraints on Ω by relating it to the calibration form. While the explicit form of Ω depends on the embedding smeared, the methodology allowed us to explain various features of the examples in section 3.1.1; in particular why the smearing has to preserve certain symmetries, which again implies that it is often only possible to smear several stacks of branes at once.

We exhibited the potential of the methods not only by studying known examples, yet by also flavoring a background dual to a $d = 2 + 1$, $\mathcal{N} = 2$ super Yang-Mills-like theory (See section 3.2). Here we found several solutions and some interesting features, notably the fact that it is not possible to smear massless flavors – a property which would be nice to understand from the point of view of the dual gauge theory.

The formalism unifies the treatment of different possible embeddings for any single background, allowing for a general study of the smearing procedure in a given background, instead of the case by case methods previously used. Even if it remains necessary to verify the existence of the cycles wrapped by the branes, their knowledge is not necessary for the actual calculation. However, as we have seen in the case of the $d = 2 + 1$, $\mathcal{N} = 2$ duality and will see again in chapter 5, backgrounds constructed without knowledge of the embeddings might be very difficult to interpret.

A similar study for a type IIA background is in chapter 5, the extension to a background with world-volume gauge fields or the Kalb-Ramond field should be straightforward using the results of [67]. While we did impose strong mathematical constraints onto the smearing form Ω, we did not link it to the physical interpretation of a brane density. In other words, we are not providing a general way of knowing from the smearing form and the ansatz for $N_f(\rho)$ what the embeddings of the smeared flavor branes are. Even if such knowledge is not required to study some aspects of the gauge theory dual, it would give a better understanding of the way the duality is working. This has been addressed in [20] [41] [42] though. One might also wonder how much one can learn about the various dual gauge theories from the generic form of Ω prior to selecting one of them by making an ansatz for $N_f(\rho)$.

3.A Finding the calibration form – an explicit example

As an example we will calculate the calibration form for the theory of section 3.2. Apart from the definition (2.39) we will need the projections imposed on the background SUSY spinors. To simplify things we perform a change of basis on the spinors taking $\sigma_1 \mapsto \sigma_3$. As a result of this transformations, the two Majorana-Weyl spinors in $\zeta = \binom{\zeta^-}{\zeta^+}$ decouple

$$\Gamma^{1256}\zeta^{\mp} = \zeta^{\mp} \quad \Gamma^{1346}\zeta^{\mp} = \zeta^{\mp} \quad \Gamma^{4567}\zeta^{\mp} = \pm\zeta^{\mp} \tag{3.160}$$

We will also need the fact that IIB supergravity is chiral, with the chirality chosen so that

$$\Gamma_{11}\zeta^{\mp} = \Gamma_{123\ldots890}\zeta^{\mp} = -\zeta^{\mp} \tag{3.161}$$

Note that our change of basis does also affect the definition of the calibration form (2.39) – we obtain two calibration forms, $\mathcal{K}^{\mp}$. Note also that we will work in flat indices.

Before looking at the most generic case, we shall look at a few examples of how to calculate components of $\mathcal{K}^{\mp}$

$$\begin{aligned}
\mathcal{K}^{\mp}_{089123} &= \zeta^{\mp T}\Gamma_{089123}\zeta^{\mp} = \zeta^{\mp T}\Gamma_{4567}\zeta^{\mp} = \pm 1 \\
\mathcal{K}^{\mp}_{089145} &= \zeta^{\mp T}\Gamma_{2367}\zeta^{\mp} = \zeta^{\mp T}\Gamma_{4567}\zeta^{\mp} = \pm 1 \\
\mathcal{K}^{\mp}_{089167} &= \zeta^{\mp T}\Gamma_{2345}\zeta^{\mp} = 1
\end{aligned} \tag{3.162}$$

These examples show nicely that the two forms disagree on those cycles making use of the Γ^{4567} projection. As the two forms need to disagree by an overall sign for a cycle to be supersymmetric, it follows that cycles involving the 7 direction cannot be supersymmetric. One can arrive at the same result directly from the κ-symmetry condition.

The more difficult step is to show why components such as

$$\mathcal{K}^{\mp}_{089567} = \zeta^{\mp T}\Gamma_{1234}\zeta^{\mp} = \zeta^{\mp T}\Gamma_{26}\zeta^{\mp} \tag{3.163}$$

vanish. Starting from the projections

$$\frac{1-\Gamma^{1346}}{2}\zeta^{\mp} = 0 \quad \frac{1-\Gamma^{1256}}{2}\zeta^{\mp} = 0 \quad \frac{1\mp\Gamma^{4567}}{2}\zeta^{\mp} = 0 \tag{3.164}$$

we define orthogonal projectors

$$\frac{1+\Gamma^{1346}}{2} \quad \frac{1+\Gamma^{1256}}{2} \quad \frac{1\pm\Gamma^{4567}}{2} \tag{3.165}$$

which may be used to project an arbitrary spinor ψ onto the subspace of spinors satisfying (3.164) because

$$\left(\frac{1-\Gamma^{1346}}{2}\right)\left(\frac{1+\Gamma^{1346}}{2}\right)\psi = 0 \tag{3.166}$$

independently of the choice of ψ. This is simply the defining property of orthogonal projections. Note that ζ may be assumed to be invariant under the orthogonal projections, as it satisfies (3.164). Applying this to the question of $\mathcal{K}^{\mp}_{089567}$,

$$\begin{aligned}
\mathcal{K}^{\mp}_{089567} &= \zeta^{\mp T}\Gamma_{26}\zeta^{\mp} \\
&= \frac{1}{4}\left[(1+\Gamma^{1346})\zeta\right]^T\Gamma^{26}(1+\Gamma^{1346})\zeta \\
&= \frac{1}{4}\zeta^{\mp T}(\Gamma^{26} - \Gamma^{1234} + \Gamma^{1234} - \Gamma^{26})\zeta^{\mp} \\
&= 0
\end{aligned} \tag{3.167}$$

Note however that it does not appear to be obvious which of the projections (3.165) one has to choose to show that a particular component of $\mathcal{K}$ vanishes.

To give an example of this, let's look at

$$
\begin{aligned}
\mathcal{K}^{\mp}_{089124} &= \pm\zeta^{\mp T}\Gamma_{34}\zeta^{\mp} \\
&= \pm\frac{1}{4}\zeta^{\mp T}(1+\Gamma^{1346})\Gamma^{34}(1+\Gamma^{1346})\zeta^{\mp} \\
&= \pm\frac{1}{4}\zeta^{\mp T}(\Gamma^{34}-\Gamma^{16}-\Gamma^{16}+\Gamma^{34})\zeta^{\mp} \\
\mathcal{K}^{\mp}_{089124} &= \pm\frac{1}{4}\zeta^{\mp T}(1\pm\Gamma^{4567})\Gamma^{34}(1\pm\Gamma^{4567})\zeta^{\mp} \\
&= \frac{1}{4}\zeta^{\mp T}(\pm\Gamma^{34}-\Gamma^{3567})(1\pm\Gamma^{4567})\zeta^{\mp} \\
&= \frac{1}{4}\zeta^{\mp T}(\pm\Gamma^{34}-\Gamma^{3567}+\Gamma^{3567}\mp\Gamma^{34})\zeta^{\mp} = 0
\end{aligned}
\tag{3.168}
$$

Let's try to look at a generic case. There is no summation in the following. Instead the indices $(a,b,c,d,e,f,g) \in \{1,\dots,7\}$ are all independent and mutually non-equal, $a \neq b, a \neq c, \dots, f \neq g$.

$$
\begin{aligned}
\pm 4\mathcal{K}^{\mp}_{089abc} &= \zeta^{\mp T}(1\pm\Gamma^{cdef})\Gamma^{defg}(1\pm\Gamma^{cdef})\zeta^{\mp} \\
&= \zeta^{\mp T}(\Gamma^{defg}\mp\Gamma^{cg})(1\pm\Gamma^{cdef})\zeta^{\mp} \\
&= \zeta^{\mp T}(\Gamma^{defg}\mp\Gamma^{cg}\pm\Gamma^{cg}-\Gamma^{defg})\zeta^{\mp} = 0
\end{aligned}
\tag{3.169}
$$

In the first line we used the chirality matrix Γ_{11} to change Γ_{089abc} into Γ^{defg}. In the process we might have picked up an overall minus sign, which we moved together with the factor 4 to the left hand side. In the projection matrices we have Γ-matrices assumed to be of the form Γ^{cdef}. Here there is again a sign ambiguity, as we have moved c to the left and as the projection might involving Γ^7. Note that we have the same sign in both parentheses, so in the following lines we will always have either the upper signs or the lower signs, never a mixture of the two – which is why $\pm\mp = -$ in the second to last equality. Similarly we shall now take a look at

$$
\begin{aligned}
\pm 4\mathcal{K}^{\mp}_{089abc} &= \zeta^{\mp T}(1\pm\Gamma^{abcd})\Gamma^{defg}(1\pm\Gamma^{abcd})\zeta^{\mp} \\
&= \zeta^{\mp T}(\Gamma^{defg}\pm\Gamma^{abcefg}\mp\Gamma^{abcefg}-\Gamma^{defg})\zeta^{\mp} = 0
\end{aligned}
\tag{3.170}
$$

Equation (3.169) is a very potent result. It follows immediately that

$$
\begin{array}{llllll}
\mathcal{K}_{124}=0 & \mathcal{K}_{125}=0 & \mathcal{K}_{126}=0 & \mathcal{K}_{127}=0 & \mathcal{K}_{134}=0 & \mathcal{K}_{135}=0 \\
\mathcal{K}_{136}=0 & \mathcal{K}_{137}=0 & \mathcal{K}_{147}=0 & \mathcal{K}_{157}=0 & \mathcal{K}_{234}=0 & \mathcal{K}_{235}=0 \\
\mathcal{K}_{236}=0 & \mathcal{K}_{237}=0 & \mathcal{K}_{245}=0 & \mathcal{K}_{247}=0 & \mathcal{K}_{256}=0 & \mathcal{K}_{267}=0 \\
\mathcal{K}_{345}=0 & \mathcal{K}_{346}=0 & \mathcal{K}_{357}=0 & \mathcal{K}_{367}=0 & \mathcal{K}_{457}=0 & \mathcal{K}_{467}=0
\end{array}
\tag{3.171}
$$

Similarly we gather from (3.170)

$$\mathcal{K}_{146} = 0 \quad \mathcal{K}_{156} = 0 \quad \mathcal{K}_{456} = 0 \tag{3.172}$$

All these things considered we are able to reproduce the two calibration forms exhibited in [73],

$$\begin{aligned} \mathcal{K}^{-} &= e^{089} \wedge \left(e^{123} + e^{145} - e^{167} + e^{246} + e^{257} + e^{347} - e^{356}\right) \\ \mathcal{K}^{+} &= e^{089} \wedge \left(-e^{123} - e^{145} - e^{167} - e^{246} + e^{257} + e^{347} + e^{356}\right) \end{aligned} \tag{3.173}$$

There is a second result following immediately from equations (3.169) and (3.170). For the SUSY projections not to be mutually exclusive they have to have pairwise two indices in common. Note that this may be easily generalized to arbitrary dimensions. In general one finds that if the SUSY projections take the form of antisymmetrized Gamma matrices with four indices, $\Gamma^{abcd}\zeta = \zeta$, different projections have to have an even number of indices in common (zero or two; four means that the projections are equal) in order to be compatible. Compatible means that this requirement is necessary for a spinor ζ satisfying all projections to exist. This result simply requires the properties of the Dirac algebra.

Chapter 4

Color- vs. flavor-branes

Having discussed the flavoring procedure in some detail in the previous chapter, we will now take a further look at the role played by the source terms. More precisely, we will question the form of the action (1.3) and the way it distinguishes between color- and flavor-branes.

In the context of the flavoring problem one often argues, that the physics of the pure Yang-Mills sector (e.g. glueballs) are captured by the supergravity action, those of the open strings describing the fundamental matter (mesons) by the brane action and interactions between the two by the fact that the background fields as well as world-volume fields couple in the brane action. In this chapter, we will critically investigate the above statements. Working in the supergravity limit, our observations will be based on a series of examples signifying the relevance of source-terms such as S_{flavor} in (1.3) for various brane-solutions.

The discussion is based on chapters 2 and 3, as well as the study of 1/4-BPS D3-D7 systems in section 4.2. In contrast to the backgrounds studied so far, these take a very simple form and are thus ideally suited for the discussion of more conceptual issues. The material presented in this section is based on [78]. Earlier work on localized D3-D7 solutions can be found in [79], [80], [81], [82] and [83]. The novelty of the solutions presented in 4.2 lies in the fact that we smear the D7 branes over part of their transverse $\mathbb{R}^2$ maintaining a $U(1)$ isometry.

4.1 General remarks

Let us begin our discussion by recalling some results from section 2.2. Here we saw that the flat p-brane solutions, which are dual to pure super Yang-Mills theories [4], do not solve the equations of motion of the relevant supergravities at the locus of the branes. Instead, one has to add a source term. However, we saw also that one is able to derive the correct solutions using the supergravities alone. I.e. from a technical point of view, the source terms were only required to fix various constants as in (2.27) and (2.29). As we pointed out at the end of our discussion of flat p-branes, the issue of the source term is independent of whether one is in the near-horizon limit or not. Hence one can argue that to fully solve the equations of motion, one should add the source term for both the p-brane solutions as well as their near-horizon limit. It follows that at least for the pure super Yang-Mills theories as considered in [4], it woule be appropriate to add a source term for the color-branes to the supergravity action. While this is technically not necessary, one needs to see this in the light of our remarks concerning the role of open- and closed-string modes that are dual to mesons and glueballs made in the introduction to this chapter.

While it is possible to ignore the source terms even if the branes are smeared along some of their transverse directions, one is not able to do so as soon as the branes are smeared over an open subset of space-time; as was the case for the more involved backgrounds of chapter 3. This can be easily seen when considering the Maxwell (Bianchi) equations when smearing sources.

To allow for smearing, we include a distribution density $\Omega_{(d)}$ in the source term, which is formally a d-form on the space transverse to the additional branes. Using the calibration form $\mathcal{K}_{(p+1)}$, we write the source term for *supersymmetric* sources as in chapter 3:

$$S_{\text{src}} = -T_p \int (e^{b\Phi}\mathcal{K}_{(p+1)} - C_{(p+1)}) \wedge \Omega_{(d)} \tag{4.1}$$

Calculating the resulting equations of motion, the Maxwell equation takes the form

$$\mathrm{d}(*e^{a\Phi}F_{(p+2)}) = 16\pi G_D T_p \Omega_{(d)} \tag{4.2}$$

– a straightforward generalization of the corresponding equation in (2.24). In contrast to the localized case of section 2.2, we would not have been able to derive suitable equations of motion without the source term, after all, the exact form of the distribution density $\Omega_{(d)}$ can in general not be inferred from the behavior of the branes away from the sources. So in the context of smearing (over an open subset), the source term is essential. These observations imply that in the generic case (1.3) implicitly contains a source-term for the color-branes and should be replaced with

$$S = S_{\mathrm{M,IIA/B}} + S_{\mathrm{color}} + S_{\mathrm{flavor}} \tag{4.3}$$

4.2 D3-D7 solutions

With all this in mind, let us take a look at D3-D7 solutions with 8 supercharges. This has previously been studied in [79] - [83] in the case where the D7-branes are localized. Note that the authors of [79] - [81] did not include any source terms in their actions working with $S = S_{\mathrm{IIB}}$, while [82] and [83] do include source terms for color- and flavor-branes. From our remarks in sections 2.2 and 4.1 we suspect that this is not necessary (as their sources are localized), but we will see so explicitly. First, let us briefly summarize the background of [81] (in string frame):

$$\begin{aligned} \mathrm{d}s^2 &= H^{-1/2}\mathrm{d}x^2 + H^{1/2}(\mathrm{d}z_1\mathrm{d}\bar{z}_1 + \mathrm{d}z_2\mathrm{d}\bar{z}_2 + e^{\Psi(z_3,\bar{z}_3)}\mathrm{d}z_3\mathrm{d}\bar{z}_3) \\ e^{\Psi(z_3,\bar{z}_3)} &= \tau_2(z_3)|\eta(\tau)|^4|z_3|^{-N_f/6} \\ \tau &= C_{(0)} + \imath e^{-\Phi} \\ F_{(5)} &= -\frac{1}{2\sqrt{2}(2\pi)^{7/2}g_s(\alpha')^2}(1+*)\mathrm{d}H^{-1}\wedge \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^3 \end{aligned} \tag{4.4}$$

The complex structure (or axio-dilaton) τ is fixed by the presence of *localized* D7 branes. Crucial for us is that the warp factor $H(z_i, \bar{z}_i)$ must satisfy a deformation of the Laplace equation on the transverse space,

$$(\partial_1\bar{\partial}_1 + \partial_2\bar{\partial}_2 + e^{-\Psi}\partial_3\bar{\partial}_3)H = 0 \tag{4.5}$$

Strictly speaking, we are not interested in solutions to a modified Laplace equation, but a modified Poisson equation, as D3-branes will appear as a singularity, just as in the p-brane case

$$(\partial_1\bar{\partial}_1 + \partial_2\bar{\partial}_2 + e^{-\Psi}\partial_3\bar{\partial}_3)H = \delta^{(6)}(z) \tag{4.6}$$

In the following we will encounter different examples of H with different kinds of δ-functions appearing on the right hand side of equations like (4.6). To simplify the notation, we shall always drop the δ-function, write the equations as (4.5) but keep in mind that H is usually singular at some point.

Looking for new solutions and working in the spirit of the flavoring program, we study $S = S_{\mathrm{IIB}} + S_{\text{flavor}}$ with the source term being a superposition of D7 actions. Then we make the Ansatz (Einstein frame)

$$\begin{aligned} \mathrm{d}s^2 &= e^{-\frac{1}{2}\Phi}[e^{2f}\mathrm{d}x_{1,3}^2 + e^{2g}\mathrm{d}v_4^2 + e^{2h}(\mathrm{d}w^2 + w^2\mathrm{d}\phi^2)] \\ F_{(5)} &= (1 + *_{10})(\mathrm{d}f_5 \wedge \mathrm{d}x^{0123}) \\ F_{(1)} &= f_1(w)w\mathrm{d}\phi \\ \Phi &= \Phi(v, w) \end{aligned} \tag{4.7}$$

Where f, g, h, f_5, Φ depend on $w, v = \sqrt{v^i v^i}$ while f_1 depends on w alone. The most striking difference between (4.7) and (4.4) is that our choice for $F_{(1)}$ is in general not exact and can thus not be understood in terms of a 0-form potential $C_{(0)}$ and the relation $F_{(1)} = \mathrm{d}C_{(0)}$. In contrast, the appearance of $C_{(0)}$ in (4.4) implies $\mathrm{d}F_{(1)} = 0$, except at isolated singularities.[1] This is why the former ansatz will not allow for smeared D7 branes. Of course we study the action $S_{\mathrm{IIB}} + S_{\mathrm{src}}$, so there will be $\Omega_{(2)}$ such that $\mathrm{d}F_{(1)} = \Omega_{(2)}$. In other words, we will not need to impose a Bianchi identity for $F_{(1)}$, but are on the contrary rather interested in its explicit violation. Note also that our choice for $F_{(1)}$ implies that all D7 sources will be smeared along ϕ.

Demanding the existence of a SUSY spinor ϵ satisfying $\imath\Gamma^{0123}\epsilon = -\epsilon$ and $\Gamma^{4567}\epsilon = \epsilon$ we study the BPS-system given by

$$\begin{aligned} 0 &\stackrel{!}{=} \delta_\epsilon\lambda = \frac{1}{2}(\partial_\mu\Phi - \imath e^{\Phi}F_\mu)\Gamma^\mu\epsilon \\ 0 &\stackrel{!}{=} \delta_\epsilon\psi_\mu = \partial_\mu\epsilon + \frac{1}{4}\omega_{\mu ab}\Gamma^{ab}\epsilon + \frac{\imath}{4}e^{\Phi}F_\mu + \frac{\imath}{16}\frac{1}{5!}F_{\nu\rho\sigma\tau\upsilon}\Gamma^{\nu\rho\sigma\tau\upsilon}\Gamma_\mu\epsilon \end{aligned} \tag{4.8}$$

as well as the Bianchi identity for $F_{(5)}$. As we mentioned earlier, integrability ensures that the remaining equations of motion will be satisfied. One then sees quickly that any solution of the original ansatz can be rewritten in terms of only

[1] The non-exactness of $F_{(1)}$ explains also why in opposite to the earlier papers we do not rely on holomorphy of the axio-dilaton in the (w, ϕ) plane. If $F_{(1)}$ is exact, the supergravity variations can be phrased in terms of $C_{(0)}$ and the dilatino variation quickly takes the form of Cauchy-Riemann equations for $e^{-\Phi} + \imath C_{(0)}$.

two functions, $H(v,w), \Delta_{gf}(w)$, and a set of integration constants

$$\begin{aligned}
\mathrm{d}s^2 &= e^{-\frac{c_\Phi}{2}}\{H^{-1/2}\mathrm{d}x_{1,3}^2 + H^{1/2}[\mathrm{d}v_4^2 + e^{-2(\Delta_{gf}-c_h)}(\mathrm{d}w^2 + w^2\mathrm{d}\phi^2)]\} \\
F_{(5)} &= (1+*)\mathrm{d}[(e^{-2c_\Phi}H^{-1} + c_{f_5})\mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3] \\
F_{(1)} &= w(\partial_w e^{-2\Delta_{gf}})\mathrm{d}\phi \\
\Phi &= 2\Delta_{gf} + c_\Phi
\end{aligned} \tag{4.9}$$

subject to the modified Laplace/Poisson equation[2]

$$0 = \left[(\partial_v^2 + \frac{3}{v}\partial_v) + e^{\Delta_{gf}(w)-c_h}(\partial_w^2 + \frac{1}{w}\partial_w)\right]H(v,w) \tag{4.10}$$

which can be more succinctly summarized as

$$0 = (\Box_v + e^{\Delta_{gf}(w)-c_h}\Box_w)H(v,w) \tag{4.11}$$

Apart from the w and $z,\bar{z}$ dependence, this is the same equation as (4.5). However, while (4.4) was derived without use of an additional source term, the derivation of (4.10) was based on $S_{\mathrm{IIB}} + S_{\mathrm{D7}}$. As we found previously, as long as the sources are localized, one is free not to include the source term. Note that working in the spirit of gauge/string duality with flavor, we did not include a source term for the D3 *color* branes – yet of course, we could have.

4.2.1 An aside: T-dualities

It is instructive to take a look at various T-dualities. There are two cases of interest – performing four T-dualities along the v^i, or performing two in the (w,ϕ) plane. In the latter case it is appropriate to change coordinates to Cartesian ones – (w^1,w^2) – to perform the dualities. The first case gives

$$\begin{aligned}
\mathrm{d}s^2 &= e^{-\frac{c_\Phi}{2}}\{e^{\Delta_{gf}}\mathrm{d}x_{1,3}^2 + e^{-\Delta_{gf}}[\mathrm{d}v_4^2 + e^{2c_h}H(\mathrm{d}w^2 + w^2\mathrm{d}\phi^2)]\} \\
\Phi &= c_\Phi - \log H \\
F_{(5)} &= (1+*)\mathrm{d}(e^{2\Delta_{gf}}\mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3) \\
F_{(1)} &= -e^{-2c_\Phi}\partial_w(e^{-2c_\Phi}H + c_{f_5})w\mathrm{d}\phi
\end{aligned} \tag{4.12}$$

[2] Crucially, (4.10) arises from the Bianchi identity on $\mathrm{d}F_{(5)} = 0$. As we have seen before, these identities relate directly to the presence of sources and should be rewritten as $\mathrm{d}F_{(5)} = \Omega_{(6)}$ as we are looking for backgrounds with D3 sources. So strictly speaking, there should be a source density on the left hand side of (4.10), at least a δ-function. As we are looking for smeared D7 branes in backgrounds with localized D3s, we ignore this distinction and just keep in mind that when solving (4.10), we are looking for solutions that show singular behavior at $(v,w) = (0,0)$.

Comparing (4.9) and (4.12) shows the result of the dualitites to be a swap $-2\Delta_{gf} \leftrightarrow \log H$. Now note that while the Buscher rules for T-dualities in supergravity only apply for v^i to be an isometry of the background, i.e. for $\partial_{v^i} H = 0$, the substitution $-2\Delta_{gf} \leftrightarrow \log H$ is valid at the level of the BPS equations and equations of motion too. The simple reason is that the BPS equations are all trivially satisfied when written in terms of Δ_{gf} and H, the equation of motion for $F_{(1)}$ is always satisfied as well as $F_{(1)}$ depends only on $\mathrm{d}\phi$, so the only points of interest are the Bianchi identities for $F_{(5)}$ and $F_{(1)}$. These however do not need to be satisfied if we allow for smeared brane sources. Again we point out that we only included an explicit source term for the D7-branes, that have now been turned into D3s.

T-dualities along w^1, w^2 lead to

$$\begin{aligned}
\mathrm{d}s^2 &= e^{-\frac{\Delta_{gf}}{2}-\frac{c_\Phi}{2}+c_h}[H^{-1/4}(\mathrm{d}x_{1,3}^2 + e^{-2c_h}\mathrm{d}w_2^2) + H^{3/4}\mathrm{d}v_4^2] \\
\Phi &= 3\Delta_{gf} + c_\Phi - 2c_h - \frac{1}{2}\log H \\
F_{(7)} &= \mathrm{d}[(e^{-2c_\Phi}H^{-1} + c_{f_5})w\mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^3 \wedge \mathrm{d}w \wedge \mathrm{d}\phi] \\
F_{(1)} &= -\mathrm{d}(e^{-2\Delta_{gf}(w)})
\end{aligned} \tag{4.13}$$

For $\Delta_{gf} = 0$, (4.12) and (4.13) reduce to the standard flat D7 and D5 solutions. This is of course expected, as (4.7) describes a stack of D3-branes in flat space. Turning on Δ_{gf} adds five- and one-form flux to the T-dual backgrounds respectively; for (4.13) the one-form flux is exact, however, so there are only additional D7 sources if $\Delta_{gf}(w)$ is not differentiable at isolated points. We are dealing with a D7-D3 and a D5(-D7) system, respectively.

In the context of gauge/string duality, the T-dualities along the v^i should be of interest. After all, it exchanges the N_c color D3-branes with the N_f flavor D7-branes – at first glance, we have a duality $(N_c, N_f) \leftrightarrow (N_f, N_c)$. Of course the precise form of the duality depends on the brane distributions.

4.2.2 Simple, known solutions

For $\Delta_{gf} = 0, c_h = 0$, there is of course the standard D3-brane solution,

$$H_3 = 1 + \frac{r_3^4}{(v^2 + w^2)^2} \tag{4.14}$$

the laplacian of which has a δ-function singularity at $(v, w) = (0, 0)$ due to the presence of the D3-branes. The near horizon limit is given by

$$H_3 \mapsto \frac{r_3^4}{(v^2 + w^2)^2} \tag{4.15}$$

There are further solutions that depend on only one variable and have thus additional isometries in the background

$$\begin{aligned} H(v, w) &= 1 + \frac{r_5^2}{v^2} \\ H(v, w) &= 1 + r_7 \log w \end{aligned} \tag{4.16}$$

They are the harmonic functions in four and two dimensions respectively.[3] They are singular at $v = 0$ or $w = 0$. The standard interpretation here is to think of the D3-branes as having been smeared over (w, ϕ) or the v^i. I.e. the smeared branes are now codimensions four or codimension two objects. Performing two (four) T-dualities along the additional isometries leads to the standard D5 (D7) solutions. Remeber that (4.10) is linear, so any superposition of (4.14) and (4.16) is a solution as well.

4.2.3 Analytic solutions

Looking for new solutions of (4.10), we will make use of the fact that there are not cross derivative terms of the form $\partial_v \partial_w$. Hence the PDE is separable and we may look for solutions of the form

$$\begin{aligned} H(v, w) &= H_v^\times(v) \times H_w^\times(w) \\ H(v, w) &= H_v^+(v) + H_w^+(w) \end{aligned} \tag{4.17}$$

after which (4.10) takes the form

$$\begin{aligned} 0 &= H_w^\times [\frac{3}{v}(H_v^\times)' + (H_v^\times)''] + e^{\Delta_{gf} - c_h} H_v^\times [\frac{1}{w}(H_w^\times)' + (H_w^\times)''] \\ 0 &= [\frac{3}{v}(H_v^+)' + (H_v^+)''] + e^{\Delta_{gf} - c_h} [\frac{1}{w}(H_w^+)' + (H_w^+)''] \end{aligned} \tag{4.18}$$

The crucial point is that, independently of $\Delta_{gf}(w)$ and c_h, any solution of the ODEs

$$\begin{aligned} H_v''(v) &= -\frac{3}{v} H_v'(v) \\ H_w''(w) &= -\frac{1}{w} H_w'(w) \end{aligned} \tag{4.19}$$

[3]If one wonders why (4.10) is not symmetric under $v \leftrightarrow w$ for $\Delta_{gf} = c_h$, the explanation can be found here. v and w are the radial coordinate in spaces of different dimension.

gives a solution of type IIB supergravity. Of course, finding an analytic solution to (4.19) is quite straightforward. As a matter of fact, these are the harmonic functions of (4.16)

$$\begin{aligned} H_v &= \frac{c_{v1}}{v^2} + c_{v2} \\ H_w &= c_{w2} \log w + c_{w3} \end{aligned} \tag{4.20}$$

with $c_{v1}, c_{v2}, c_{w2}, c_{w3} \in \mathbb{R}$. And so we have two new families of analytic solutions

$$\begin{aligned} H^{\times}(v,w) &= (c_{w2} \log w + c_{w3})(\frac{c_{v1}}{v^2} + c_{v2}) \\ H^{+}(v,w) &= c_{w2} \log w + \frac{c_{v1}}{v^2} + c_{v2} + c_{w3} \end{aligned} \tag{4.21}$$

Of course these are just (4.16) and one might ask what is new. The point is that (4.21) hold together with (4.9) for arbitrary $\Delta_{gf}(w)$, and hence for arbitrary D7-brane distributions. The interpretation of these solutions is similar to that given at the end of section 4.2.2. The D3-branes are smeared over some of their transverse directions, but now also accomodate for any D7 distribution imposed by choice of $\Delta_{gf}(w)$.

We can generalize the H^+ solution slightly by demanding that the two terms in (4.18) do not vanish independently, but are each equal to a constant. I.e.

$$c_+ = e^{\Delta_{gf} - c_h}[\frac{1}{w}(H_w^+)' + (H_w^+)''] = -[\frac{3}{v}(H_v^+)' + (H_v^+)''] \tag{4.22}$$

These are solved by

$$\begin{aligned} H_v^+ &= \frac{c_{v1}}{v^2} + c_{v2} - \frac{c_+ v^2}{8} \\ H_w^+ &= c_{w2} + c_{w2} \log w + \int_0^w \frac{\int_0^{\hat{w}} c_+ e^{-\Delta_{gf}(\tilde{w}) + c_h} \tilde{w} \mathrm{d}\tilde{w}}{\hat{w}} \mathrm{d}\hat{w} \end{aligned} \tag{4.23}$$

This reduces to (4.21) in the case $c_+ = 0$. However, even if $c_+ \neq 0$, (4.23) reproduces (4.21) in the IR. Note that some care has to be taken when taking $c_+ \neq 0$, as our Ansatz demands for H to be positive definite.

It is interesting to note that (4.16) reappear as (4.21) independently of whether we add D7-sources or not. Of course, it would be much more interesting to find the equivalent of (4.14) in the presence of $\Delta_{gf} \neq 0$. We shall do so in section 4.2.5 numerically.

4.2.4 Brane distributions

At this point we will take a look at a few brane distributions. Note that

$$\begin{aligned}\Omega_{(2)} &= [(\partial_w^2 + \frac{1}{w}\partial_w)e^{-2\Delta_{gf}}]w\mathrm{d}w \wedge \mathrm{d}\phi \\ &= (\Box_w e^{-2\Delta_{gf}(w)})w\mathrm{d}w \wedge \mathrm{d}\phi\end{aligned} \tag{4.24}$$

From out ansatz it follows that D7-branes are always smeared along ϕ, so the simplest distribution is a δ-function one in the w direction,

$$\Omega_{(2)} = Q\delta(w - w_0)w\mathrm{d}w \wedge \mathrm{d}\phi \tag{4.25}$$

where Q is some normalization constant. We can integrate the resulting flux,

$$\int_{S^1} F_{(1)} = 2\pi Q w_0 \theta(w - w_0) \tag{4.26}$$

and it follows that

$$Q = \frac{N_f}{2\pi w_0} \tag{4.27}$$

Then

$$e^{-2\Delta_{gf}} = \frac{N_f}{2\pi w_0}[c_1 \log w + w_0\theta(w - w0)\log\frac{w}{w0} + c_{\mathrm{src}}] \tag{4.28}$$

Naturally this should be positive for all values of $w \in \mathbb{R}^+$, hence it seems appropriate to set $c_1 = 0$. Also, as both $e^{-\Delta_{gf}}$ and $e^{\Delta_{gf}}$ appear in the metric, $e^{-2\Delta_{gf}} \geq 0$ is a good assumption that is guaranteed by fixing $c_{\mathrm{src}} > 0$. Furthermore, our numerical studies in section 4.2.5 will show that varying c_{src} does influence the form of the solutions rather strongly. To avoid this, we will fix it to $c_{\mathrm{src}} = Q^{-1}$ so that the constant term in $e^{-2\Delta_{gf}}$ does not vary with N_f.

A similarly interesting case is given by

$$\begin{aligned}\Omega_{(2)} &= Q\theta(w - w_0)w\mathrm{d}w \wedge \mathrm{d}\phi \\ e^{-2\Delta_{gf}} &= Q[c_1 \log w + \frac{1}{4}\theta(w - w_0)(w^2 - w_0^2 - 2w_0^2 \log\frac{w}{w_0}) + c_{\mathrm{src}}]\end{aligned} \tag{4.29}$$

For the same reasons as above we fix $c_1 = 0$ and $c_{\mathrm{src}} = Q^{-1}$. Concerning the normalization, we have

$$F_{(1)} = \frac{Q}{2}(w^2 - w_0^2)\theta(w - w_0)\mathrm{d}\phi \tag{4.30}$$

leading to a radially dependent charge

$$N_f(w) = Q\pi(w^2 - w_0^2) \tag{4.31}$$

The fact that $N_f(w)$ behaves like a two-dimensional area is no accident. After all, we assume a homogeneous brane distribution in the (w, ϕ) plane for all $w \geq w_0$.

4.2.5 Numeric solutions

Let us now take a look at numeric solutions of (4.10). We are dealing with a deformation of the Laplace (Poisson) equation, that is, a homogeneous, elliptic, separable PDE of second order, and use the Fortran package Mudpack[4] to do so. Our aim is to perform a qualitative study of deformations of the original $AdS_5 \times S^5$ solution (4.14) that includes additional D7-branes. We fix the parameter $r_3 = 1$ and solve the equation in a rectangular domain in the (v, w) plane specified by

$$0.2 \leq v, w \leq 2.6 \tag{4.32}$$

on a 129×129 grid. Some experimentation shows that one obtains a good agreement with the analytic solutions in the absence of D7 branes when imposing the Neumann boundary conditions at $w = 0.2$ and $w = 2.6$ and Dirichlet ones at $v = 0.2$ and $v = 2.6$. I.e.

$$\begin{aligned} H &= \frac{1}{(v^2 + w^2)^2} \qquad \text{at } w = 0.2 \vee w = 2.6 \\ \partial_v H &= -\frac{4v}{(v^2 + w^2)^3} \quad \text{at } v = 0.2 \vee v = 2.6 \end{aligned} \tag{4.33}$$

However, the physical significance of the boundary conditions is not entirely clear and it might be appropriate to modify the boundary conditions when changing the source density Δ_{gf}.

Figure 4.1 shows the analytic solution $H_3 = \frac{1}{(v^2+w^2)^2}$. Our numeric solution for $e^{-\Delta_{gf}} = 1$ (not shown) agrees up to $\Delta H = \pm 0.0001$. We then proceed to include D7 branes via changing $e^{-2\Delta_{gf}}$. In all these cases we approximate Heaviside θ functions by

$$\begin{aligned} \theta(w) &= \frac{1}{2} + \frac{1}{2}\tanh(kw) \\ k &= 2.5 \end{aligned} \tag{4.34}$$

Larger values of k make for a sharper transition, in the case $k = 2.5$ we have $1 - \theta(0.5) \sim 0.0758$. Figure 4.2 shows the case $e^{-2\Delta_{gf}} = \theta(w-1)\log(w) + 1$

[4]Mudpack can be found at http://www.cisl.ucar.edu/css/software/mudpack/.

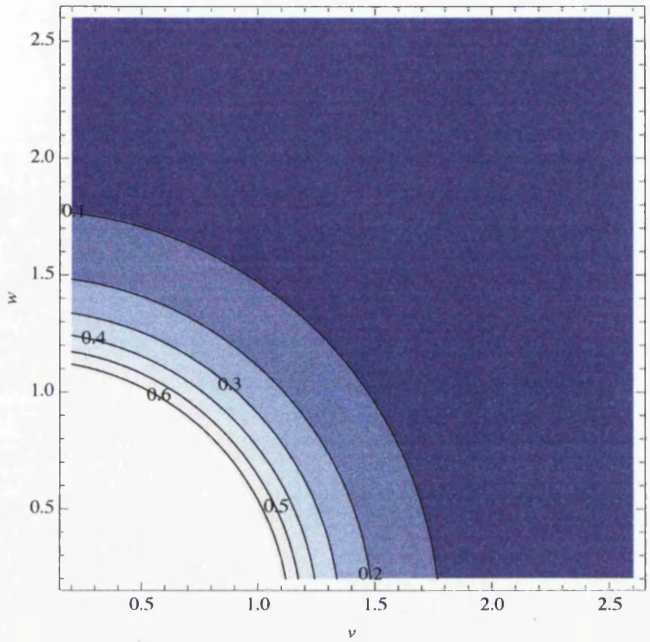

Figure 4.1: Plot of the analytic solution $H_3 = (v^2+w^2)^{-2}$. There is a singularity in H_3 at the origin charactereistic to the presence of D3 branes

while figure 4.3 uses $e^{-2\Delta_{gf}} = 10[\theta(w-1)\log(w)] + 1$. So in each case, there is a stack of D7 branes localized at $w = 1$, yet smeared along ϕ. The changes in the solutions are not drastic, but differ from H_3 by one or two orders of magnitude, so instead of plotting H for each case, we show the difference to the pure D3-brane solution of figure 4.1, $H - H_3$.

Things change considerably when we scale the source density by another factor of 50, i.e. we set $e^{-2\Delta_{gf}} = 500[\theta(w-1)\log(w)+]1$ (fig. 4.4). One can see quite clearly that the background is dominated by the D7 branes extending along the v^i while the boundary conditions, especially at $(0.2, 0.2)$ are still those of the D3 background.

Figure 4.5 shows a brane distribution along the lines of (4.29). That is, the number of flavors runs with w^2. Of course, here the UV should be dominated by increasing number of D7 branes and it might be appropriate to adjust the boundary conditions at $v = 2.6$ and $w = 2.6$. Based on the T-dual of the analytic D7 solution, we set them to

$$\begin{aligned} H &= \log w \quad \text{at } w = 2.6 \\ \partial_v H &= \frac{1}{w} \qquad \text{at } v = 2.6 \end{aligned} \tag{4.35}$$

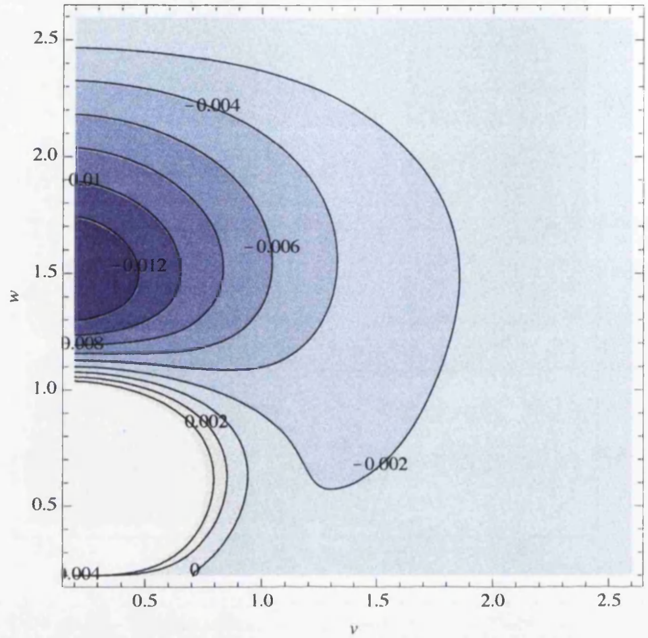

Figure 4.2: ΔH for $e^{-2\Delta_{gf}} = \theta(w-1)\log(w) + 1$.

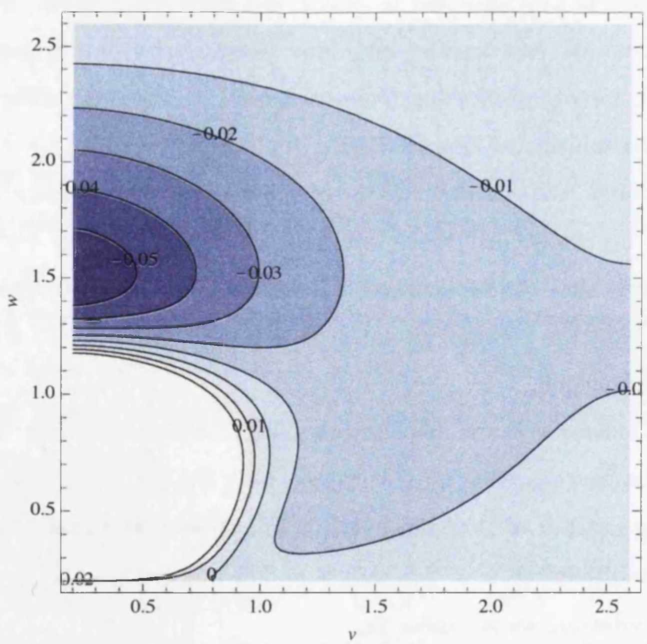

Figure 4.3: ΔH for $e^{-2\Delta_{gf}} = 10[\theta(w-1)\log(w)] + 1$.

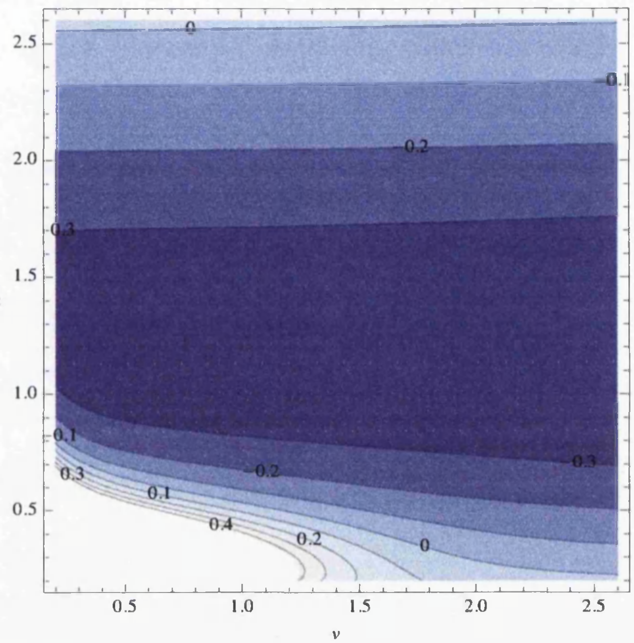

Figure 4.4: H for $e^{-2\Delta_{gf}} = 500[\theta(w-1)\log(w)] + 1$.

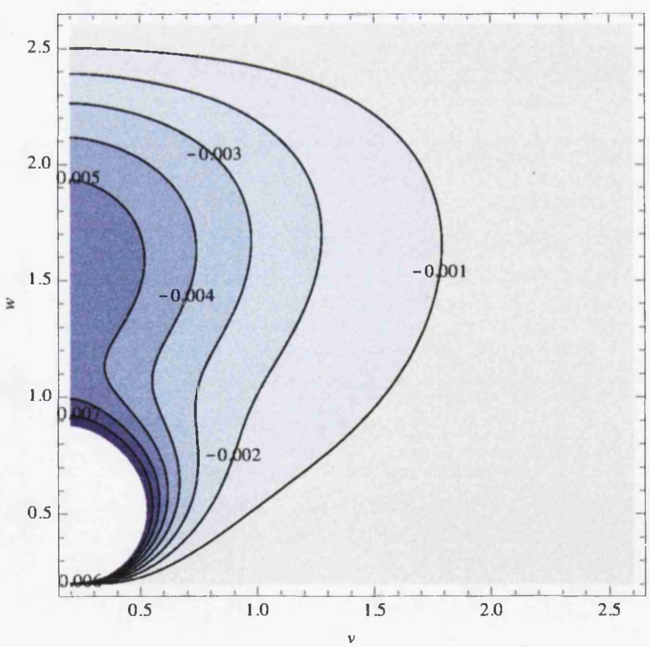

Figure 4.5: ΔH for $e^{-2\Delta_{gf}} = \frac{1}{4}\theta(w-1)[w^2 - 1 - 2w^2\log(w)] + 1$.

while those at $v = 0.2$ and $w = 0.2$ remain as in (4.33). The result is shown in 4.6. Note that having changed the boundary conditions, the solution is quite different to 4.1 and we plot H instead of ΔH.

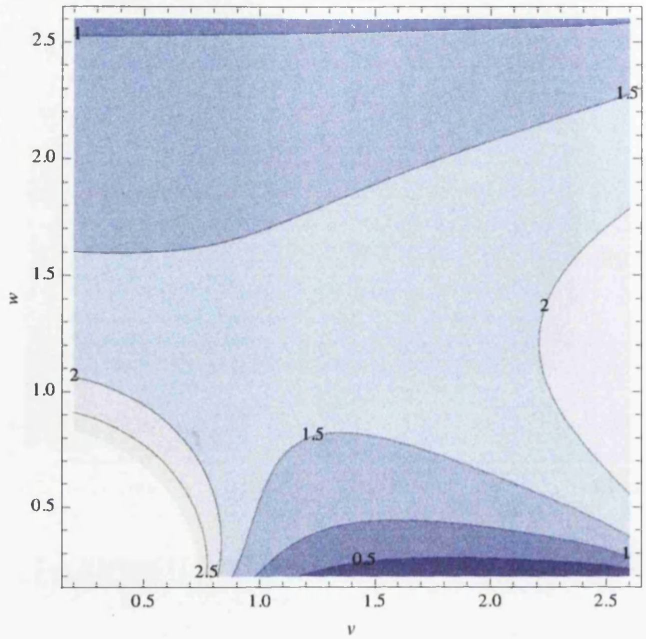

Figure 4.6: H for $e^{-2\Delta_{gf}}$ as in fig 4.5, however, at $v = 2.6$ and $w = 2.6$ we imposed the boundary conditions characteristic for D3 branes smeared along v^i – a system T-dual to D7 branes.

4.3 Discussion

We have now analyzed (1.3) from several perspectives. From the perspective of the p-brane action in chapters 2 and 3, we realized that color- and flavor-branes are actually on a very similar footing. In principle one should include source terms for both as was done in [83], so this observation is not new, as we mentioned before. When searcing for a new background, the use of source-terms is only necessary if the associated sources are to be smeared over an open subset of space-time, which is why the source-term is essential for the flavor-branes that are usually assumed to be smeared.

The impression that color and flavor-branes can – from the supergravity perspective – be taken to be on an equal footing was again confirmed by our observations in section 4.2.1, where we were able to exchange color and flavor-branes by performing four T-dualities in the directions transverse to the D3s. Curiously, we had included explicit source terms for the (smeared) flavor D7-branes while not doing so for their localized[5] cousins. One should also keep in mind that [81] obtained highly similar results working with the supergravity action alone – while including suitable δ-function sources, of course.

Considering our brief discussion of the equations of motion of the world-volume fields $X^\mu(\xi)$ in section 2.2, it is also appropriate to question whether it is generally possible to find a source-term for a given solution – especially in cases where supersymmetry is broken. As discussed in [58], the problem lies in the fact that for sources in theories of gravity, the energy of the source is not localized at the source but also stored in the self-energy of the surrounding gravitational field. Only in the presence of supersymmetry, where gravitational effects are canceled by those of a different field – the Maxwell-type p-form fields in this case – can one find a suitable source term. Again we point out [66] however, where the authors have constructed a finite-temperature background including additional flavor terms.

Naturally our comments and observations made here are only valid for the examples studied, and it would be interesting to study the issue of source terms for color-branes for more complex supergravity backgrounds dual to confining gauge theories, such as [8], [6] and [84]. From the point of gauge/string duality, the crucial point is there whether there are open string states in the spectrum, that should only appear in non-confining theories. In other words, one expects that for confining backgrounds it should not be possible to find source terms for the color branes, and it would be nice to verify this explicitly.

Finally, we found a series of new D3-D7 backgrounds with smeared D7-branes. Here, the various solutions shown have the interesting property that for any distribution of D7-branes encoded in Δ_{gf}, the D3s distribute themselves accordingly – a result one can attribute to the high amount of supersymmetry preserved by the backgrounds.

[5] As a matter of fact, the D3s were smeared over some of their transverse directions in the analytic solutions presented in 4.2.3, yet they were not smeared over an open subset of space-time.

Chapter 5

Kaluza-Klein monopole condensation

In the context of M-theory, the relations between type IIA string theory and eleven-dimensional supergravity are standard textbook material (see for example [58, 85, 86, 87]). The M2-brane gives rise to the D2 and the fundamental string, the M5 to the D4 and NS5 branes. The D0 and D6-branes on the other hand have a slightly different origin. Not being related to any brane-like object in eleven dimensions, they are results of the Kaluza-Klein (KK) reduction relating the two theories; the former being a particle-like, localized gravitational excitation on the KK-circle, the latter a peculiar fibration of said circle over the ten-dimensional base, known as a Kaluza-Klein monopole (a good review is given by [52]). In this chapter, we are concerned with a small gap in this formalism that becomes apparent when one tries to consider the M-theory lift of smeared D6-branes.

The problem can be quickly explained. The bosonic sector of eleven-dimensional supergravity contains only the graviton $\hat{g}_{MN}$ and a four-form field $\hat{F}_{(4)}$. Upon KK reduction, $\hat{F}_{(4)}$ gives rise to the Kalb-Ramond three-form field $H_{(3)}$ as well as the Ramond-Ramond four-form $F_{(4)}$. From $\hat{g}_{MN}$ one obtains the ten-dimensional metric $g_{\mu\nu}$, the dilaton Φ, and a one-form gauge potential $A_{(1)}$, with an associated field strength $F_{(2)} = \mathrm{d}A_{(1)}$. If we assume the KK-circle to be parameterized by z, the standard KK-ansatz relating the two geometries is[1]

$$\begin{aligned} \mathrm{d}s_{\mathrm{M}}^2 &= e^{-\frac{2}{3}\Phi}\mathrm{d}s_{\mathrm{IIA}}^2 + e^{\frac{4}{3}\Phi}(A_{(1)} + \mathrm{d}z)^2 \\ \hat{F}_{(4)} &= F_{(4)} + H_{(3)} \wedge \mathrm{d}z \end{aligned} \tag{5.1}$$

Given any solution to the equations of motion of type IIA supergravity, one can use (5.1) to lift to eleven dimensions and vice versa. However, as $A_{(1)}$ plays the role of a gauge potential, it is actually $F_{(2)} = \mathrm{d}A_{(1)}$ that contains the physically relevant degrees of freedom. Thus given a set $\{g_{\mu\nu}, \Phi, F_{(2)}, H_{(3)}, F_{(4)}\}$ one first has to find a gauge potential prior to lifting. Now assume that for some reason $\mathrm{d}F_{(2)} \neq 0$ on some subset of space-time that we will call Σ. As $F_{(2)}$ is no longer closed on Σ it cannot be expressed in terms of $A_{(1)}$ alone and we cannot use (5.1) to perform the lift. This is the apparent gap in the standard formalism we alluded to earlier.

The problem is not a purely formal one. D6-branes couple magnetically to $A_{(1)}$. As we have seen in the preceding chapters, the inclusion of D6 sources violates the Bianchi identity $\mathrm{d}F_{(2)} = 0$ at the position of the sources. While this is not a problem for localized sources – as a matter of fact it is the reason why the KK-monopole is a gravitational instanton – one encounters the problem at hand once one distributes the branes continuously and thus violates the Bianchi identity on an open subset of space-time.

As an aside it is worthwhile to point out that the relation between D6-branes and the RR two-form is much the same as that between magnetic monopoles and the $F_{\mathrm{E\&M}}$ in standard electro-magnetism. The inclusion of magnetic sources restores the symmetry of the Maxwell equations. Schematically

$$\mathrm{d} * F_{\mathrm{E\&M}} = *j_{\mathrm{E}} \qquad \mathrm{d}F_{\mathrm{E\&M}} = *j_{\mathrm{M}} \tag{5.2}$$

Thus, the Bianchi identity is violated by the magnetic current j_{M}. In the context of quantum field theories one speaks of monopole condensation. (See e.g. [88])

In this chapter, we will not resolve the issue in full generality, but will focus on the inclusion of D6 sources in type IIA backgrounds of the form

$$\mathcal{M}_{10} = \mathbb{R}^{1,3} \times \mathcal{M}_6 \tag{5.3}$$

[1]Where the distinction is necessary, hats and tildes denote eleven-dimensional quantities. Capital letters describe eleven-dimensional indices. The M-theory circle will be parameterized by either z, ψ_+ or ψ.

without three of four-form flux, that preserve four supercharges. More precisely, we will be interested in the construction of string duals to $3+1$-dimensional $SU(N_c)$ gauge theories with $\mathcal{N}=1$ supersymmetry and N_f flavors using D6-branes.

The work presented here was originally born out of the interest to study the addition of flavor branes to type IIA backgrounds dual to $\mathcal{N}=1$, $SU(N_c)$ super Yang-Mills. Before flavoring, the geometry is that of N_c D6-branes wrapping a three-cycle in the deformed conifold.[2] In the limit $N_c g_{\rm YM}^2 \gg 1$, the backreaction of the color branes causes the system to undergo a geometric transition. The system is now best described in terms of the resolved conifold with the branes having been replaced by N_c units of two-form flux over a two-cycle. This was originally studied in [45, 31] and the geometric transition is based on the work of [43, 44]; an attempt at generalizing the duality to include finite-temperature duals was made in [89]. The resulting ten-dimensional background consists of metric, dilaton and RR two-form $(g_{\mu\nu}, \Phi, F_{(2)})$. Refering back to (5.1) one sees that it lifts to pure geometry in M-theory, as both $H_{(3)}$ and $F_{(4)}$ are set to zero. It is for this reason that it is particularly simple and interesting to study these geometries and dualities from the perspective of eleven-dimensional supergravity. Here, the equations of motion and supergravity variations simplify to

$$\hat{R}_{MN}=0 \qquad \delta_{\hat{\epsilon}}\hat{\psi}_M=\partial_M\hat{\epsilon}+\frac{1}{4}\hat{\omega}_{MAB}\hat{\Gamma}^{AB}\hat{\epsilon} \tag{5.4}$$

The eleven-dimensional geometry is of the form

$$\mathcal{M}_{11}=\mathbb{R}^{1,3}\times\mathcal{M}_7 \tag{5.5}$$

As the seven-dimensional manifold $\mathcal{M}_7$ preserves 1/8-SUSY and is Ricci flat, it is a manifold of G_2-holonomy. The concept of M-theory compactifications on such manifolds ([90]) is pretty much the same as that of the old heterotic string models on Calabi-Yau three-folds used in classic string phenomenology. Mathematically this is reflected by the presence of a three-form $\hat{\phi}_{G_2}$ that is

[2]To be precise, we will be dealing with conifolds deformed by the presence of branes or $F_{(2)}$ flux. They do carry $SU(3)$-structure but are not of $SU(3)$-holonomy. Therefore, they are not Calabi-Yau and strictly speaking we should not refer to them as (deformed/resolved) conifolds. For the lack of a better term however, we shall refer to the internal six-dimensional manifolds in this paper by that name though, as their topology is the same as that of their Calabi-Yau cousins.

closed and co-closed

$$\mathrm{d}\hat{\phi}_{G_2} = 0 \qquad \mathrm{d}(*_7\hat{\phi}_{G_2}) = 0 \tag{5.6}$$

where $*_7$ denotes the seven-dimensional Hodge dual on the internal space.

As we have discussed in considerable detail, the flavoring procedure is very straightforward from the point of view of type IIA string theory. For once we will refer to the smearing form as Ξ, as Ω will appear in the context of an $SU(3)$-structure later on. The brane action takes the form

$$S_{\text{Branes}} = -T_6 \int_{\mathcal{M}_{10}} (e^{-\Phi}\mathcal{K} - A_{(7)}) \wedge \Xi_{(3)} \tag{5.7}$$

and the presence of S_{Branes} in the modified action (1.3) gives source term contributions to the equations of motion. Most prominent among these is the appearance of a magnetic source term for the RR two-form,

$$\mathrm{d}F_{(2)} = -(2\kappa_{10}^2 T_6)\Xi_{(3)} \tag{5.8}$$

that violates the standard Bianchi identity. In type IIA one accomodates for this simply by adding a flavor contribution to the RR form,

$$F_{(2)} = \mathrm{d}A_{(1)} + (2\kappa_{10}^2 T_6) B_{(2)} \tag{5.9}$$

with $B_{(2)} \to 0$ as $N_f \to 0$. (Note that $B_{(2)}$ is not to be confused with the Kalb-Ramond two-form potential $H_{(3)} = \mathrm{d}B_{(2)}$ that will not appear in this chapter.) The smearing form $\Xi_{(3)}$ then satisfies

$$\mathrm{d} *_{10} \mathrm{d}(e^{-\Phi}\mathcal{K}) = -(2\kappa_{10}^2 T_6)\Xi_{(3)} \tag{5.10}$$

It is a priori not obvious how to accomodate the violation of the Bianchi identity (5.8) in M-theory. However, as the sources will not only modify the Bianchi identity, yet also the dilaton and Einstein equations, it is reasonable to expect that the eleven-dimensional geometry will not be Ricci flat. Instead, the Einstein equations should be supplemented by the presence of a source term,

$$\hat{R}_{MN} - \frac{1}{2}\hat{g}_{MN}\hat{R} = \hat{T}_{MN} \tag{5.11}$$

From the loss of Ricci flatness it follows that the manifold can no longer be of G_2-holonomy; as it preserves the same amount of supersymmetry however it is fair to expect it to carry a G_2-structure. Therefore, there is still a three-form $\hat{\phi}_{G_2}$ that now fails to be (co)closed. One can anticipate that the failure of the manifold to be of G_2-holonomy is parameterized by N_f and thus ultimately by the the difference between $F_{(2)}$ and $\mathrm{d}A_{(1)}$. As we will see, the precise relations are

$$\begin{aligned} \mathrm{d}\hat{\phi}_{G_2} &= -J \wedge (F_{(2)} - \mathrm{d}A_{(1)}) \\ \mathrm{d} *_7 \hat{\phi}_{G_2} &= e^{-\Phi/3}(*_6\Psi) \wedge (F_{(2)} - \mathrm{d}A_{(1)}) \end{aligned} \tag{5.12}$$

Now for a manifold carrying a G-structure, its failure to be of G-holonomy is measured by its intrinsic torsion.[3] Therefore, we expect the flavors in eleven dimensions to appear in the form of intrinsic torsion. A detailed study of the relation between the eleven and ten-dimensional supersymmetry variations will prompt us to consider eleven-dimensional backgrounds with torsion $\hat{\tau}$, where the torsion is related to $F - \mathrm{d}A = B$.

Finally we will see that an uplift of our ten-dimensional equations of motion is given by the relation

$$R^{(\tau)}_{MN} + \frac{1}{2} R^{(\tau)}_{KLRN} (*_7\hat{\phi})_M{}^{KLR} = 0 \tag{5.13}$$

which is the solution to our initial problem. $R^{(\tau)}$ is the eleven-dimensional Riemann (Ricci) tensor with torsion – we have discarded the use of hats to avoid an overly cluttered notation. As one can always rewrite the Riemann tensor as a combination of a torsion free Riemann tensor with additional terms depending on the torsion, it is possible to recast the above equation in the form of (5.11) with the energy-momentum tensor depending only on the torsion.

At first glance, equation (5.13) appears like a modification of M-theory and violates all intuition as eleven-dimensional supergravity is unique. However, one must not forget that we never assumed to solve the problem in its full generality. As a matter of fact, (5.13) has to be taken with several pinches of salt – which might not be a surprise, as the inclusion of source terms in theories of gravity is always a rather difficult business. First of all, (5.13) assumes the background to be of topology $\mathcal{M}_{11} = \mathbb{R}^{1,3} \times \mathcal{M}_7$, with the internal manifold carrying a G_2-structure. Furthermore this means that we are not dealing with maximal eleven-dimensional supergravity, but with a situation with reduced supersymmetry – 1/8 BPS – in which case the theory is no longer unique. Still,

[3] For intrinsic torsion in the context of string theory see [56].

as we will see, equation (5.13) manages what the standard KK-ansatz (5.1) does not. It gives the correct source-modified equations of motion in type IIA.

In section 5.1 we will begin with a review of the unflavored geometries in ten and eleven dimensions and then continue by studying the flavoring problem from the perspective of type IIA. Following this, we will turn to the issue of the M-theory lift in section 5.2. The chapter is ammended by appendices on brane embeddings, spinor conventions and KK reduction. For illustrative and motivational purposes we will be using a specific case of an M-theory G_2-holonomy manifold and its type IIA reduction in section 5.1. However, the results of section 5.2 on the M-theory lift of smeared D6-branes do not depend on this example or the type IIA reduction chosen. They only depend on the presence of a G_2-structure, four-dimensional Minkowski space and the absence of M-theory fluxes.

As in previous chapters, we will also present a new supergravity background dual to a flavored $\mathcal{N} = 1$, $SU(N_c)$ super Yang-Mills theory. For the specific ansatz of section 5.1, we derive a set of first-order equations – (5.45) and (5.49) – that have to be satisfied by smeared D6 sources in this geometry. For this we exhibit a one-parameter family of solutions in section 5.1.3. While the fluxes in this solution satisfy the flux quantization necessary for a string dual, the geometry is that of a cone over $S^2 \times S^3$ with a singularity at the origin. So we expect the interpretation of this solution as a suitable dual to be difficult. The presentation of the flavoring problem is supplemented by a discussion of D6-brane embeddings for the geometries at hand in appendix 5.A.

5.1 Flavored $\mathcal{N} = 1$ string duals from D6-branes

In this section, we will review the source-free string duals in their ten and eleven-dimensional formulations. Subsequently we will be turning to the issue of adding sources to the type IIA background. Let us once more emphasize that the particular choices of eleven-dimensional geometry (and its dimensional reduction) are of no direct consequence for our results concerning the M-theory lift of smeared D6-branes. The concrete geometry presented here is chosen due to its relevance to the flavoring problem in type IIA.

5.1.1 The eleven-dimensional dual without sources

Building on the work of Brandhuber [50] (see also [51, 91]) we consider the purely gravitational M-theory background given by the elfbein

$$\begin{aligned} &\tilde{e}^{\mu} = \mathrm{d}x^{\mu} && \tilde{e}^{\rho} = E(\rho)\mathrm{d}\rho \\ &\tilde{e}^{1,2} = A(\rho)\sigma_{1,2} && \tilde{e}^{3,4} = C(\rho)[\Sigma_{1,2} - f(\rho)\sigma_{1,2}] \\ &\tilde{e}^{5} = B(\rho)\sigma_3 && \tilde{e}^{6} = D(\rho)[\Sigma_3 - g(\rho)\sigma_3] \end{aligned} \tag{5.14}$$

σ_i, Σ_i are left-invariant Maurer-Cartan forms which we chose to be

$$\begin{aligned} &\sigma_1 = \cos\psi \mathrm{d}\theta + \sin\psi\sin\theta\mathrm{d}\phi && \Sigma_1 = \cos\tilde{\psi}\mathrm{d}\tilde{\theta} + \sin\tilde{\psi}\sin\tilde{\theta}\mathrm{d}\tilde{\phi} \\ &\sigma_2 = -\sin\psi\mathrm{d}\theta + \cos\psi\sin\theta\mathrm{d}\phi && \Sigma_2 = -\sin\tilde{\psi}\mathrm{d}\tilde{\theta} + \cos\tilde{\psi}\sin\tilde{\theta}\mathrm{d}\tilde{\phi} \\ &\sigma_3 = \mathrm{d}\psi + \cos\theta\mathrm{d}\phi && \Sigma_3 = \mathrm{d}\tilde{\psi} + \cos\tilde{\theta} \end{aligned} \tag{5.15}$$

The solutions we are interested in are 1/8-BPS; therefore one can impose the following constraints onto the SUSY spinor $\tilde{\epsilon}$:

$$\tilde{\Gamma}^{1234}\tilde{\epsilon} = \tilde{\epsilon} \quad \tilde{\Gamma}^{1356}\tilde{\epsilon} = -\tilde{\epsilon} \quad \tilde{\Gamma}^{\rho 126}\tilde{\epsilon} = -\tilde{\epsilon} \tag{5.16}$$

As a direct consequence we can calculate the following spinor bilinear, which turns out to be the G_2-structure

$$\begin{aligned} \tilde{\phi}_{G_2} &= (\bar{\tilde{\epsilon}}\tilde{\Gamma}_{A_0A_1A_2}\tilde{\epsilon})\tilde{e}^{A_0A_1A_2} \\ &= \tilde{e}^{\rho 13} + \tilde{e}^{\rho 24} + \tilde{e}^{\rho 56} + \tilde{e}^{146} + \tilde{e}^{345} - \tilde{e}^{125} - \tilde{e}^{236} \end{aligned} \tag{5.17}$$

In the absence of four-form flux the preservation of four supercharges is equivalent to the manifold being of G_2-holonomy. A necessary and sufficient condition is the closure and co-closure of the G_2-structure. By imposing $\mathrm{d}\tilde{\phi}_{G_2} = 0$ and $\mathrm{d}(*_7\tilde{\phi}_{G_2}) = 0$ we obtain the BPS equations

$$\begin{aligned} &A' = \frac{E[BD(g-f^2) + ACf(1-g)]}{2AB} && B' = \frac{ECf(1-g)}{A} \\ &D' = \frac{E[A^2(2C^2-D^2) + C^2D^2(f^2-g)]}{2A^2C^2} && C' = \frac{E[ABD - C^3f(1-g)]}{2ABC} \\ &f = \frac{BC}{2AD} && g = 1 - 2f^2 \end{aligned} \tag{5.18}$$

The same BPS system follows from demanding that $\delta_{\tilde{\epsilon}}\tilde{\psi}_M = 0$.

The best known solution to (5.18) is the Bryant-Salamon metric [92]. With

$$A^2 = B^2 = \frac{\rho^2}{12} \quad C^2 = D^2 = \frac{\rho^2}{9}(1 - \frac{\rho_0^3}{\rho^3}) \quad E^2 = (1 - \frac{\rho_0^3}{\rho^3})^{-1} \quad f = g = \frac{1}{2} \tag{5.19}$$

the metric takes the form

$$\mathrm{d}s^2 = \mathrm{d}x_{1,3}^2 + (1 - \frac{\rho_0^3}{\rho^3})^{-1}\mathrm{d}\rho^2 + \frac{\rho^2}{12}\sigma^2 + \frac{\rho^2}{9}(1 - \frac{\rho_0^3}{\rho^3})(\Sigma - \frac{1}{2}\sigma)^2 \qquad (5.20)$$

The seven-dimensional G_2 cone actually turns out to be the cotangent bundle T^*S^3. The geometry is that of a cone over $S^3 \times S^3$, with each sphere being parameterized by a set of Maurer-Cartan forms. At $\rho = \rho_0$, the minimum of the radial parameter, one of the spheres (Σ) collapses, while the other (σ) remains of finite size. M-theory dynamics on this type of manifold were discussed in [90]. Fluctuations in ρ_0 and the gauge potential A_3 can be combined into a complex parameter. However, as these fluctuations turn out to be non-normalizable, they do not parameterize a moduli space of vacua, yet rather a moduli space of theories.

There are three $U(1)$ isometries in (5.14) given by ∂_ϕ, $\partial_{\tilde{\phi}}$ and $\partial_\psi + \partial_{\tilde{\psi}}$ and there are therefore three different dimensional reductions to type IIA. In each case one obtains a conifold geometry with flux, with the conifold singularity being resolved by a deformation or resolution. I.e. there is a cone over $S^2 \times S^3$ and one of the spheres vanishes at at the minimal radius while the other remains of finite size. Furthermore, if we choose to reduce along an isometry embedded in the vanishing sphere, we need to recall that the vanishing of the M-theory circle indicates the presence of D6-branes. Thus the reduction along $\partial_{\tilde{\phi}}$ yields a deformed conifold with a D6-brane at $\rho = \rho_0$ extending along the Minkowski directions and wrapping the non-vanishing S^3. If one mods out the $U(1)$ by $\mathbb{Z}_{N_c}$ before reducing, the corresponding geometry is that of N_c branes. The other two reductions include non-singular $U(1)$'s, so we end up with resolved conifolds. As the M-theory circle is non-singular, there is no D6-brane. There is F_2 flux though on the finite-size two-sphere. The different geometries are related by a flop transition between the resolved conifolds and the conifold transition between the deformed and the resolved ones.

In the context of gauge/string duality, the deformed conifold corresponds to the weak 't Hooft coupling regime, while the resolved one is to be considered for large 't Hooft coupling. Thus the latter provides the appropriate supergravity dual. M-theory realizes the conifold dualities via the aforementioned moduli space of solutions. See [45, 31, 90].

Scherk-Schwarz gauge In what follows we will study the reduction along $\partial_\psi + \partial_{\tilde{\psi}}$. In the context of the flavoring problem of section 5.1.2 one expects the system to be best described by one of the resolved conifold geometries with additional flavor branes. Therefore, out of the three isometries discussed ∂_ϕ and $\partial_\psi + \partial_{\tilde{\psi}}$ are the obvious choices. We selected the latter as it leads to simpler equations in type IIA. The choice made here does affect the flavoring problem, yet not our results on the M-theory lift. As we are interested in the reduction of tangent-space quantities, we need to transform the elfbein to Scherk-Schwarz gauge

$$\hat{e}_M^A = \begin{pmatrix} e^{-\frac{1}{3}\Phi} e_\mu^a & e^{\frac{2}{3}\Phi} A_\mu \\ 0 & e^{\frac{2}{3}\Phi} \end{pmatrix}_{MA} \qquad \hat{E}_A^M = \begin{pmatrix} e^{\frac{1}{3}\Phi} E_a^\mu & -e^{\frac{1}{3}\Phi} A_a \\ 0 & e^{-\frac{2}{3}\Phi} \end{pmatrix}_{AM} \tag{5.21}$$

To obtain the gauge (5.21) from (5.14), we perform the following gauge transformation:

$$\Lambda = \Lambda^{(3)} \Lambda^{(2)} \Lambda^{(1)}$$

with the individual transformations $\Lambda^{(1)}, \Lambda^{(2)}, \Lambda^{(3)}$ being

$$\begin{aligned}
\Lambda^{(1)} &= \begin{pmatrix} \mathbb{I}_{9\times 9} & & \\ & \cos\alpha & -\sin\alpha \\ & \sin\alpha & \cos\alpha \end{pmatrix} \\
\Lambda^{(2)} &= \begin{pmatrix} \mathbb{I}_{5\times 5} & & & & & \\ & \cos\frac{\psi_+}{2} & -\sin\frac{\psi_+}{2} & & & \\ & \sin\frac{\psi_+}{2} & \cos\frac{\psi_+}{2} & & & \\ & & & \cos\frac{\psi_+}{2} & -\sin\frac{\psi_+}{2} & \\ & & & \sin\frac{\psi_+}{2} & \cos\frac{\psi_+}{2} & \\ & & & & & \mathbb{I}_{2\times 2} \end{pmatrix} \\
\Lambda^{(3)} &= \begin{pmatrix} \mathbb{I}_{6\times 6} & & & & \\ & \cos\alpha & 0 & \sin\alpha & \\ & 0 & 1 & 0 & \\ & -\sin\alpha & 0 & \cos\alpha & \\ & & & & \mathbb{I}_{2\times 2} \end{pmatrix}
\end{aligned} \tag{5.22}$$

and all other entries zero. Here we defined

$$\begin{aligned}
\cos\alpha(\rho) &= \frac{D(1-g)}{\sqrt{B^2 + (1-g)^2 D^2}} \qquad & \psi_+ &= \psi + \tilde{\psi} \\
\sin\alpha(\rho) &= \frac{B}{\sqrt{B^2 + (1-g)^2 D^2}} \qquad & \psi_- &= \psi - \tilde{\psi}
\end{aligned} \tag{5.23}$$

In principle one needs only $\Lambda^{(1)}$ and $\Lambda^{(2)}$ to obtain Scherk-Schwarz gauge; yet without $\Lambda^{(3)}$ the new projections satisfied by the SUSY spinor would be linear combinations of the old ones (5.16) with coefficients $\cos\alpha, \sin\alpha$. As it is, the form of the SUSY projections remains invariant under Λ. I.e.

$$\hat{\Gamma}^{1234}\hat{\epsilon} = \hat{\epsilon} \quad \hat{\Gamma}^{1356}\hat{\epsilon} = -\hat{\epsilon} \quad \hat{\Gamma}^{\rho 126}\hat{\epsilon} = -\hat{\epsilon} \tag{5.24}$$

Thus the G_2-structure (5.17) remains formally the same, with the vielbeins $\tilde{e}^A$ now replaced by $\hat{e}^A$. A disadvantage of the reducible gauge is that the new vielbein is rather complicated.

Dimensional reduction and type IIA string theory The resulting type IIA vielbein is given by

$$e^\mu = e^{\frac{1}{3}\Phi}\mathrm{d}x^\mu \tag{5.25a}$$

$$e^\rho = e^{\frac{1}{3}\Phi}E\mathrm{d}\rho \tag{5.25b}$$

$$e^1 = e^{\frac{1}{3}\Phi}A(\cos\frac{\psi_-}{2}\mathrm{d}\theta + \sin\theta\sin\frac{\psi_-}{2}\mathrm{d}\phi) \tag{5.25c}$$

$$\begin{aligned} e^2 &= e^{\frac{1}{3}\Phi}A\cos\alpha(\cos\frac{\psi_-}{2}\sin\theta\mathrm{d}\phi - \sin\frac{\psi_-}{2}\mathrm{d}\theta) \\ &\quad + e^{\frac{1}{3}\Phi}C\sin\alpha\left[\cos\frac{\psi_-}{2}(\sin\tilde{\theta}\mathrm{d}\tilde{\phi} - f\sin\theta\mathrm{d}\phi) + \sin\frac{\psi_-}{2}(\mathrm{d}\tilde{\theta} + f\mathrm{d}\theta)\right] \end{aligned} \tag{5.25d}$$

$$e^3 = e^{\frac{1}{3}\Phi}C\left[\cos\frac{\psi_-}{2}(\mathrm{d}\tilde{\theta} - f\mathrm{d}\theta) - \sin\frac{\psi_-}{2}(f\sin\theta\mathrm{d}\phi + \sin\tilde{\theta}\mathrm{d}\tilde{\phi})\right] \tag{5.25e}$$

$$\begin{aligned} e^4 &= -e^{\frac{1}{3}\Phi}A\sin\alpha(\cos\frac{\psi_-}{2}\sin\theta\mathrm{d}\phi - \sin\frac{\psi_-}{2}\mathrm{d}\theta) \\ &\quad + e^{\frac{1}{3}\Phi}C\cos\alpha\left[\cos\frac{\psi_-}{2}(\sin\tilde{\theta}\mathrm{d}\tilde{\phi} - f\sin\theta\mathrm{d}\phi) + \sin\frac{\psi_-}{2}(\mathrm{d}\tilde{\theta} + f\mathrm{d}\theta)\right] \end{aligned} \tag{5.25f}$$

$$e^5 = e^{\frac{1}{3}\Phi}D\sin\alpha(\cos\theta\mathrm{d}\phi - \cos\tilde{\theta}\mathrm{d}\tilde{\phi} + \mathrm{d}\psi_-) \tag{5.25g}$$

While the dilaton and gauge potential are

$$\begin{aligned} e^{\frac{2}{3}\Phi} &= \frac{B}{2\sin\alpha} = \frac{D(1-g)}{2\cos\alpha} \\ A_{(1)} &= \cos\theta\mathrm{d}\phi + \cos\tilde{\theta}\mathrm{d}\tilde{\phi} + \frac{B^2 - D^2(1-g^2)}{B^2 + (1-g)^2D^2}(\cos\theta\mathrm{d}\phi - \cos\tilde{\theta}\mathrm{d}\tilde{\phi} + \mathrm{d}\psi_-) \\ &= \cos\theta\mathrm{d}\phi + \cos\tilde{\theta}\mathrm{d}\tilde{\phi} + (\sin^2\alpha - \frac{1+g}{1-g}\cos^2\alpha)(\cos\theta\mathrm{d}\phi - \cos\tilde{\theta}\mathrm{d}\tilde{\phi} + \mathrm{d}\psi_-) \end{aligned} \tag{5.26}$$

Using $\hat{\Gamma}^{10} = \Gamma^{11}$, the reduction of the SUSY projections takes a more pleasing form:

$$\Gamma^{1234}\epsilon = \epsilon \quad \Gamma^{135}\Gamma^{11}\epsilon = -\epsilon \quad \Gamma^{\rho 12}\Gamma^{11}\epsilon = -\epsilon \tag{5.27}$$

This allows us to calculate the generalized calibration form for D6-branes in this background.

$$\mathcal{K} = (\bar{\epsilon}\Gamma_{a_0\dots a_6}\epsilon)e^{a_0\dots a_6} = e^{x^0x^1x^2x^3}\wedge(e^{125} - e^{345} - e^{\rho 24} - e^{\rho 13}) \tag{5.28}$$

Note, that the internal three-form part of this is, up to some overall dilaton factor, identical to that part of the G_2-structure (5.17) that is independent of

$\hat{e}^6$.

G-structures In terms of G-structures the situation in type IIA is the following. Because we preserve four supercharges, we expect space-time to carry an $SU(3)$-structure. As it was shown in [93], it can be directly derived from the G_2-structure of the KK-lift. Centerpiece of that reduction are the relations

$$\begin{aligned} J &= (\hat{\phi}_{G_2})_{ab6} e^{ab} \\ \Psi &= (\hat{\phi}_{G_2})_{abc} e^{abc} \end{aligned} \tag{5.29}$$

For the six-dimensional internal manifold, J defines an almost complex structure, with respect to which we can define from Ψ a $(3,0)$-form Ω as

$$\Omega = \Psi - i *_6 \Psi \tag{5.30}$$

These satisfy the equations

$$\begin{aligned} J \wedge \Omega &= 0 \\ J \wedge J \wedge J &= \frac{3i}{4} \Omega \wedge \bar{\Omega} \end{aligned} \tag{5.31}$$

In the case at hand we have

$$\begin{aligned} J &= e^{\rho 5} + e^{14} - e^{23} \\ \Psi &= e^{\rho 13} + e^{\rho 24} + e^{345} - e^{125} \end{aligned} \tag{5.32}$$

which gives

$$\Omega = \Psi - i *_6 \Psi = (e^\rho + ie^5) \wedge (e^1 + ie^4) \wedge (e^3 + ie^2) \tag{5.33}$$

Thinking about lifting from ten to eleven dimensions, we can invert equations (5.29) to express the eleven-dimensional G_2-structure in terms of the ten-dimensional quantities:

$$\begin{aligned} \hat{\phi}_{G_2} &= e^{-\Phi} \Psi + e^{-\frac{2}{3}\Phi} J \wedge \hat{e}^6 \\ *_7 \hat{\phi}_{G_2} &= -\frac{1}{2} e^{-4\Phi/3} J \wedge J + e^{-\Phi} (*_6 \Psi) \wedge \hat{e}^6 \end{aligned} \tag{5.34}$$

As previously stated, Ricci flatness, preservation of four supercharges and absence of four-form flux in eleven dimensions guarantee the G_2-holonomy of the internal manifold. This translates to the closure and co-closure of $\hat{\phi}_{G_2}$. As the fibration of the M-theory circle over the ten-dimensional base is non-trivial, one obtains non-vanishing two-form flux upon reduction to type IIA. Hence the internal six-dimensional manifold does not have $SU(3)$-holonomy due to its intrinsic torsion. This means that the forms J and Ω are not both closed. The relation they will obey can be derived from the closure and co-closure of $\hat{\phi}_{G_2}$ thanks to (5.34)

$$\begin{aligned} \mathrm{d}\hat{\phi}_{G_2} &= \mathrm{d}(e^{-\Phi}\Psi) + \mathrm{d}J \wedge (A_{(1)} + \mathrm{d}\psi_+) + J \wedge \mathrm{d}A_{(1)} = 0 \\ \mathrm{d} *_7 \hat{\phi}_{G_2} &= -\frac{1}{2}\mathrm{d}(e^{-4\Phi/3} J \wedge J) + \mathrm{d}(e^{-\Phi/3} *_6 \Psi) \wedge (A_{(1)} + \mathrm{d}\psi_+) \\ &\quad - e^{-\Phi/3}(*_6\Psi) \wedge \mathrm{d}A_{(1)} = 0 \end{aligned} \tag{5.35}$$

We know that none of the type IIA quantities depends on ψ_+. Hence, the contribution to the previous equations coming from $\mathrm{d}\psi_+$ must cancel by itself. It gives

$$\begin{aligned} 0 &= \mathrm{d}J \\ 0 &= \mathrm{d}(e^{-\Phi/3} *_6 \Psi) \\ 0 &= \mathrm{d}(e^{-\Phi}\Psi) + J \wedge \mathrm{d}A_{(1)} \\ 0 &= -\frac{1}{2}\mathrm{d}(e^{-4\Phi/3} J \wedge J) - e^{-\Phi/3}(*_6\Psi) \wedge \mathrm{d}A_{(1)} \end{aligned} \tag{5.36}$$

These equations can be rephrased (following [93] for example) as

$$\begin{aligned} \mathrm{d}J &= 0 \\ \mathrm{d}\Phi &= \frac{3}{4} e^{\Phi} \mathrm{d}A_{(1)} \lrcorner (*_6\Psi) \\ J \lrcorner \mathrm{d}A_{(1)} &= 0 \end{aligned} \tag{5.37}$$

where

$$G_{(p)} \lrcorner H_{(p+q)} = \frac{1}{p!} G^{\mu_1 \cdots \mu_p} H_{\mu_1 \ldots \mu_p \mu_{p+1} \ldots \mu_{p+q}} \mathrm{d}x^{\mu_{p+1}} \wedge \ldots \wedge \mathrm{d}x^{\mu_{p+q}} \tag{5.38}$$

We described in this section the construction of a type IIA background from the reduction of eleven-dimensional supergravity. We also derived the equations imposed on the structure by supersymmetry. Now we turn to the problem of adding backreacting flavors in this ten-dimensional context.

5.1.2 Smeared sources in type IIA supergravity

The source-modified first-order system We are now addressing the problem of flavoring the type IIA background obtained in the previous section using the methods of chapter 3. Hence we are looking for a solution of the equations of motion derived from

$$S = S_{IIA} - T_6 \int \left(e^{-\Phi}\mathcal{K} - A_{(7)}\right) \wedge \Xi_{(3)} \tag{5.39}$$

The sources in (5.39) modify the standard type IIA equations of motion and Bianchi identities to

$$\begin{aligned}
\mathrm{d}F_{(2)} &= -(2\kappa_{10}^2 T_6)\Xi_{(3)} \\
0 &= \mathrm{d} *_{10} F_{(2)} \\
0 &= \frac{1}{\sqrt{-g}}\partial_\kappa(\sqrt{-g}g^{\kappa\lambda}e^{-2\Phi}\partial_\lambda\Phi) - \frac{3}{8}F^2 - \frac{3}{4}e^{-\Phi}\Xi\lrcorner(*_{10}\mathcal{K}) \\
R_{\mu\nu} &= -2\nabla_\mu\partial_\nu\Phi + \frac{e^{2\Phi}}{2}(F_{\mu\kappa}F_\nu{}^\kappa - \frac{1}{4}g_{\mu\nu}F_{(2)}^2) \\
&\quad + \frac{e^\Phi}{4}\left((*_{10}\mathcal{K})_\mu{}^{\kappa\lambda}\Xi_{\nu\kappa\lambda} - g_{\mu\nu}\Xi\lrcorner(*_{10}\mathcal{K})\right)
\end{aligned} \tag{5.40}$$

Due to the standard integrability arguments ([38, 40]), it is sufficient to satisfy the Bianchi identities along with the first-order BPS equations. However, in section 5.2 we will show how to derive the second-order system directly from M-theory.

The metric ansatz is given by the vielbein (5.25) and the dilaton is assumed to depend only on the radial coordinate The calibration associated with κ-symmetric D6-branes is given by (5.28) which is

$$\mathcal{K} = e^{x^0x^1x^2x^3} \wedge \Psi \tag{5.41}$$

Supersymmetry requires the two-form flux to obey the generalized calibration condition

$$*_{10}\,\mathrm{d}(e^{-\Phi}\mathcal{K}) = F_{(2)} \tag{5.42}$$

This tells us that the most general ansatz for $F_{(2)}$ is

$$F_{(2)} = e^{-4\Phi/3}\left(F_{\rho 5}(\rho)e^{\rho 5} + F_{12}(\rho)e^{12} + F_{14}(\rho)e^{14} + F_{23}(\rho)e^{23} + F_{34}(\rho)e^{34}\right) \tag{5.43}$$

The conditions given by supersymmetry on this $SU(3)$ geometry with intrinsic torsion are still given by (see end of section 5.1.1)

$$\begin{aligned}
\mathrm{d}J &= 0 \\
\mathrm{d}\Phi &= \frac{3}{4}e^\Phi F_{(2)}\lrcorner(*_6\Psi) \\
J\lrcorner F_{(2)} &= 0
\end{aligned} \tag{5.44}$$

where we have now replaced $\mathrm{d}A_{(1)}$ by $F_{(2)}$, thus allowing for $\mathrm{d}F_{(2)} \neq 0$, as necessary for D6 sources. Together with the generalized calibration condition

(5.42), these equations give the first-order equations the system must satisfy:

$$\begin{aligned}
f &= \frac{A}{C\tan\alpha} \\
\alpha' &= \frac{E}{2}\left(\frac{2}{D\tan\alpha} - \frac{D}{C^2\tan\alpha} + \frac{D\cos\alpha\sin\alpha}{A^2} - 2F_{23}\right) \\
A' &= \frac{E}{2}\left(\frac{A}{D\tan^2\alpha} + \frac{D}{A} - \frac{AD}{C^2\tan^2\alpha} - \frac{2AF_{23}}{\tan\alpha} - AF_{34}\right) \\
C' &= \frac{E}{2}\left(-\frac{C}{D\tan^2\alpha} + \frac{D}{C} - CF_{34}\right) \\
D' &= \frac{E}{2}\left(-\frac{2D^2}{C^2} - \frac{D^2}{A^2} + \frac{DF_{23}}{\tan\alpha} + 2\right) \\
\Phi' &= \frac{3E}{2}\left(-\frac{D\cos^2\alpha}{2A^2} + \frac{D}{2C^2\tan^2\alpha} + \frac{F_{23}}{\tan\alpha} + F_{34}\right) \\
F_{\rho 5} &= \frac{D\cos\alpha\sin\alpha}{A^2} - \frac{D}{C^2\tan\alpha} \\
F_{12} &= \frac{D\cos^2\alpha}{A^2} - \frac{D}{C^2\tan^2\alpha} - \frac{2F_{23}}{\tan\alpha} - F_{34} \\
F_{14} &= \frac{D}{C^2\tan\alpha} - \frac{D\cos\alpha\sin\alpha}{A^2} + F_{23}
\end{aligned} \tag{5.45}$$

As mentioned before, the modified equations of motion relate the smearing form to the two-form flux.

$$\mathrm{d}F_{(2)} = -2\kappa_{10}^2 T_6 \Xi \tag{5.46}$$

This equation, combined with (5.43) and (5.45), tells us that the most general ansatz for Ξ is

$$\Xi = e^{-5\Phi/3}\left(\Xi_1(\rho)e^{\rho 34} + \Xi_2(\rho)(e^{\rho 23} + e^{\rho 14}) + \Xi_3(\rho)e^{\rho 12} + \Xi_4(\rho)(e^{135} + e^{245})\right) \tag{5.47}$$

with

$$\begin{aligned}
\Xi_3 &= -\Xi_1 - \frac{2\Xi_2}{\tan\alpha} \\
\Xi_4 &= \frac{F_{34}}{2\kappa_{10}^2 T_6 D\sin^2\alpha}
\end{aligned} \tag{5.48}$$

and the additional conditions

$$\begin{aligned}
F_{23}' =& E\left(-\frac{F_{34}}{D\tan\alpha} - \frac{DF_{23}\cos^2\alpha + DF_{34}\cos\alpha\sin\alpha}{A^2} - \frac{D^2\cos\alpha\sin\alpha}{A^2C^2} + \frac{2F_{23}^2}{\tan\alpha}\right. \\
&\left. + \frac{DF_{23}\cos(2\alpha) + DF_{34}\sin\alpha\cos\alpha}{C^2\sin^2\alpha} + \frac{D^2\cos\alpha}{C^4\sin\alpha} + 3F_{34}F_{23} - 2\kappa_{10}^2 T_6\Xi_2\right) \\
F_{34}' =& E\left(\frac{F_{34}}{D\tan^2\alpha} - \frac{DF_{34}}{2A^2} + \frac{DF_{34}\cos(2\alpha)}{2C^2\sin^2\alpha} + \frac{F_{34}F_{23}}{\tan\alpha} + 2F_{34}^2 - 2\kappa_{10}^2 T_6\Xi_1\right)
\end{aligned} \tag{5.49}$$

One can verify explicitly that any solution to equations (5.45) and (5.49) automatically verifies the source-modified equations of motion (5.40).

As we want to interpret the two-form flux $F_{(2)}$ as created by brane sources, we need the flux to be quantized, obeying $\int_{S^2} F_{(2)} = 2\pi N_c$. S^2 is a suitable two-cycle surrounding the branes in the transverse, three-dimensional space. This adds constraints on Ξ and $F_{(2)}$:

$$\begin{aligned}\Xi_1 &= \Xi_2 \tan\alpha \\ F_{23} &= \frac{-A^2 D + C^4 F_{34} \sin^2\alpha + C^2(2N_c e^{\frac{2}{3}\Phi} \sin\alpha \tan\alpha + D \sin^2\alpha + A^2 F_{34})}{(A^2C^2 + C^4 \sin^2\alpha)\tan\alpha}\end{aligned} \tag{5.50}$$

that are compatible with the equation (5.49).

5.1.3 Finding a solution

In this section, we present an analytic solution to the previous system of first-order equations. We will notice that this solution corresponds to the addition of sources in the singular conifold. First, we can directly solve one of the equations in (5.45):

$$D = e^{\frac{2}{3}\Phi} \frac{N_c C^2 \sin\alpha \tan\alpha}{A^2} \tag{5.51}$$

Let us now specialize to the case $\Xi_2 = 0$. We see that this reduces the freedom of the smearing form to

$$\Xi_{(3)} = \frac{e^{-5\Phi/3} F_{34}}{2\kappa_{10}^2 T_6 D \sin^2\alpha}(e^{135} + e^{245}) \tag{5.52}$$

The branes smeared with this particular form would correspond to branes extended in the radial direction ρ in a trivial way. For a discussion of κ-symmetric brane embeddings in this geometry, see appendix 5.A. This simplification enables us to solve the equation for the last unknown component of the two-form flux $F_{(2)}$:

$$F_{34} = e^{\frac{2}{3}\Phi} \frac{N_f \sin\alpha}{AC} \tag{5.53}$$

where N_f is a constant of integration related to the number of flavors in the dual field theory. We now suppose that the two-form flux is independent of the radial coordinate ρ, a property verified in other examples of string duals. This imposes that

$$A^2 = C^2 \sin^2\alpha \tag{5.54}$$

Finally, we assume f to be constant. A look at the original metric (5.25) tells us that f parameterizes the fibration between the two spheres – this becomes rather more obvious in (5.14). Thus if f is independent of ρ, the fibration does not change if we flow along the radial direction. Then we can solve the full BPS system analytically, and we find:

$$\begin{aligned}
D^2 &= e^{\frac{4}{3}\Phi}\frac{N_c^2}{f^2} \\
A^2 &= e^{\frac{4}{3}\Phi}\frac{4N_c^2(1-f^2)^2}{3f^2} \\
C^2 &= e^{\frac{4}{3}\Phi}\frac{4N_c^2(1-f^2)}{3f^2} \\
E^2 &= \frac{16N_c^2(1-f^2)^2}{f^2}[(e^{\frac{2}{3}\Phi})']^2 \\
\cos\alpha &= f \\
N_f &= \pm\frac{N_c(4f^2-1)}{3f}
\end{aligned} \tag{5.55}$$

where $0 < f < 1$. The two-form flux is

$$\begin{aligned}
F_{(2)} = &- N_c\Big(\sin\theta \mathrm{d}\theta\wedge\mathrm{d}\phi + \sin\tilde{\theta}\mathrm{d}\tilde{\theta}\wedge\mathrm{d}\tilde{\phi}\Big) \\
&+ N_f\sin\psi_-\Big(\mathrm{d}\theta\wedge\mathrm{d}\tilde{\theta} + \sin\theta\sin\tilde{\theta}\mathrm{d}\phi\wedge\mathrm{d}\tilde{\phi}\Big) \\
&+ N_f\cos\psi_-\Big(\sin\tilde{\theta}\mathrm{d}\theta\wedge\mathrm{d}\tilde{\phi} + \sin\theta\mathrm{d}\tilde{\theta}\wedge\mathrm{d}\phi\Big)
\end{aligned} \tag{5.56}$$

At this point we notice that we can write the metric explicitly as a cone upon redefinition of the radial coordinate. We take

$$r = \frac{4N_c(1-f^2)}{f}e^{2\Phi/3} \tag{5.57}$$

then $\mathrm{d}r^2 = E^2\mathrm{d}\rho^2$ and the metric is

$$\mathrm{d}s^2_{IIA} = e^{2\Phi/3}\Big(\mathrm{d}x^2_{1,3} + \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2_{int}\Big) \tag{5.58}$$

where

$$\begin{aligned}
\mathrm{d}\Omega^2_{int} =& \frac{1}{12}(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) + \frac{1}{12(1-f^2)}[(\omega_1 - f\mathrm{d}\theta)^2 + (\omega_2 - f\sin\theta\mathrm{d}\phi)^2] \\
&+ \frac{1}{16(1-f^2)}(\omega_3 - \cos\theta\mathrm{d}\phi)^2
\end{aligned} \tag{5.59}$$

We can first notice that taking the limit $N_f \to 0$ for this solution gives the singular conifold. It indeed corresponds to taking $f \longrightarrow \frac{1}{2}$, giving

$$\begin{aligned} \mathrm{d}s^2_{N_f\to 0} =& \frac{r}{6N_c}\Big(\mathrm{d}x^2_{1,3} + \mathrm{d}r^2 + \frac{r^2}{12}(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) \\ &+ \frac{r^2}{9}[(\omega_1 - \frac{1}{2}\mathrm{d}\theta)^2 + (\omega_2 - \frac{1}{2}\sin\theta\mathrm{d}\phi)^2] + \frac{r^2}{12}(\omega_3 - \cos\theta\mathrm{d}\phi)^2\Big) \end{aligned} \tag{5.60}$$

Secondly, we have quantization of the two-form color flux, which is necessary for the gauge/string duality.

The interpretion of the additional flavor terms to the flux is not clear. A look at the solution and appendix 5.A prompts us to suspect that the interpretation of the sources as being due to flavor branes is more straightforward if one reduces along ∂_ϕ. It should be interesting to consider the solution at hand in the context of conifold transitions though. Of course, this is just one solution of the BPS equations of this particular dimensional reduction. Other solutions might also present interesting properties. In either case, we now turn our attention back to the problem of the M-theory lift.

5.2 Back to M-theory

Having studied the flavoring problem of D6-branes in the background (5.25) in the previous section, we have sufficient intuition to turn back towards the more general case of smeared D6 sources in M-theory. The discussion here is fairly generic and requires only the presence of the various G-structures as well as the overall topology $\mathbb{R}^{1,3} \times \mathcal{M}$.

5.2.1 Lifting the SUSY variations

The G_2-structure

Our considerations in the introduction about the loss of Ricci flatness prompted us to consider the appearance of intrinsic torsion. So we will begin our attempt at finding a candidate M-theory lift with magnetic $A_{(1)}$ sources by studying the ten and eleven-dimensional G-structures. Originally we were dealing with a G_2-holonomy manifold in eleven dimensions. Then we reduced it to an $SU(3)$-structure in ten dimensions, following the equations (5.29). After this we flavored the theory, which changed the structure equations in ten dimensions (5.36) by replacing $\mathrm{d}A_{(1)}$ by $F_{(2)}$. However, after adding sources in type IIA supergravity, we have $F_{(2)} \neq \mathrm{d}A_{(1)}$. So, if we try to lift back to eleven dimensions, we start from

$$\begin{aligned}
0 &= \mathrm{d}J \\
0 &= \mathrm{d}(e^{-\Phi/3} *_6 \Psi) \\
0 &= \mathrm{d}(e^{-\Phi}\Psi) + J \wedge F_{(2)} \\
0 &= -\frac{1}{2}\mathrm{d}(e^{-4\Phi/3} J \wedge J) - e^{-\Phi/3}(*_6\Psi) \wedge F_{(2)}
\end{aligned} \tag{5.61}$$

When we then look at the G_2-structure we find, combining (5.35) with the above,

$$\begin{aligned}
\mathrm{d}\hat{\phi}_{G_2} &= -J \wedge (F_{(2)} - \mathrm{d}A_{(1)}) \\
\mathrm{d} *_7 \hat{\phi}_{G_2} &= e^{-\Phi/3}(*_6\Psi) \wedge (F_{(2)} - \mathrm{d}A_{(1)})
\end{aligned} \tag{5.62}$$

So sources in type IIA supergravity translate in eleven dimensions to the loss of G_2-holonomy and the appearance of torsion proportional to $F_{(2)} - \mathrm{d}A_{(1)} = B_{(2)}$.

The SUSY variations

The previous section gave a first confirmation of our suspicion that geometric torsion should allow us to accomodate for the sources in M-theory. This suggests that all geometric quantities such as covariant derivatives and curvature tensors should be replaced by their torsion-modified relatives. Simplest among these is the covariant derivative, which makes an explicit appearance in the eleven-dimensional supergravity variation $\delta_{\hat{\epsilon}}\hat{\psi}_M = D_M\hat{\epsilon}$, which yields the IIA supergravity variations upon KK-reduction. In appendix 5.B we therefore study how this equation and its Kaluza-Klein reduction change upon inclusion of a torsion tensor[4] $\hat{\tau}$

$$\delta_{\hat{\epsilon}}\hat{\psi}_M = \partial_M\hat{\epsilon} + \frac{1}{4}\hat{\omega}_{MAB}\hat{\Gamma}^{AB}\hat{\epsilon} + \frac{1}{4}\hat{\tau}_{MAB}\Gamma^{AB}\hat{\epsilon} \equiv D_M^{(\tau)}\epsilon \tag{5.63}$$

The result is given in (5.134) and we proceed by investigating what constraints we have to impose on $\hat{\tau}_{MAB}$ in order for the lower-dimensional variations to include magnetic sources.

[4] Of course, once we include the torsion and proceed from $D_M\hat{\epsilon}$ to $D_M^{(\tau)}\hat{\epsilon}$, it is not certain whether this defines a SUSY variation of a supergravity theory. What we do know for certain however – and will show in the following – is that the naive dimensional reduction of the usual eleven-dimensional SUSY variation does not yield the correct type IIA one and that (5.63) gives a first order differential on the spinor that does reduce to the correct equations. With this in mind, we write $\delta_{\hat{\epsilon}}\hat{\psi}_M = D_M^{(\tau)}\epsilon$.

Now from the form of the dilatino variation (Einstein frame),

$$\delta_\epsilon \lambda = \frac{3}{16}\frac{1}{\sqrt{2}} e^{\frac{3}{4}\Phi}(\mathrm{d}A_{bc} + 2e^{-\frac{3}{2}\Phi}\hat{\tau}_{zbc})\Gamma^{bc}\epsilon + \frac{\sqrt{2}}{4}(\partial_b\Phi + \frac{3}{2}e^{-\frac{3}{4}\Phi}\hat{\tau}_{zbz})\Gamma^b\Gamma^{11}\epsilon \tag{5.64}$$

it follows that we have to demand $\hat{\tau}_{zaz} = 0$ and $\hat{\tau}_{zbc} = T_6\kappa_{10}^2 \frac{e^{\frac{3}{2}\Phi}}{2} B_{bc}$, as (5.64) then takes the form

$$\delta_\epsilon \lambda = \frac{3}{16}\frac{1}{\sqrt{2}} e^{\frac{3}{4}\Phi} F_{bc}\Gamma^{bc}\epsilon + \frac{\sqrt{2}}{4}\partial_b\Phi\Gamma^b\Gamma^{11}\epsilon \tag{5.65}$$

with the two-form now no longer closed, $F = \mathrm{d}A + T_6\kappa_{10}^2 B$.

Substituting $\hat{\tau}_{zaz}$ and $\hat{\tau}_{zbc}$ into the gravitino varition of (5.134) we see that if we impose

$$\hat{\tau}_{zaz} = 0 \quad \hat{\tau}_{zbc} = \frac{e^{\frac{3}{2}\Phi}}{2} B_{bc} \quad \hat{\tau}_{\mu bc} = \frac{e^{\frac{3}{2}\Phi}}{2} A_\mu B_{bc} \quad \hat{\tau}_{\mu bz} = -\frac{e^{\frac{3}{4}\Phi}}{2} B_{\mu b} \tag{5.66}$$

the gravitino variation turns also to the desired form

$$\delta_\epsilon \psi_\mu = \partial_\mu \epsilon + e_\mu^a \frac{1}{4}\omega_{abc}\Gamma^{bc}\epsilon + \frac{1}{64} e^{\frac{3}{4}\Phi} e_\mu^a F_{cd}\left(\eta_{ab}\Gamma^{bcd} - 14\delta_a^c\Gamma^d\right)\Gamma^{11}\epsilon \tag{5.67}$$

Equations (5.65) and (5.67) are important results. If one performs a KK-reduction of the original supergravity variation without torsion, $\delta_{\hat{\epsilon}}\hat{\psi}_M = D_M\hat{\epsilon}$, one obtains supergravity variations including $\mathrm{d}A_{(1)}$, yet not $B_{(2)} = F_{(2)} - \mathrm{d}A_{(1)}$. By adding the torsion term to the eleven-dimensional supergravity variation, we are able to directly derive the ten-dimensional variations with $F_{(2)}$ instead of $\mathrm{d}A_{(1)}$. Looking back at (5.63) it is fair to say that the spin connection $\hat{\omega}_{MAB}$ contains $\mathrm{d}A_{(1)}$, while the torsion carries the $B_{(2)}$ term necessary to complete $F_{(2)}$. The right-hand side of (5.63) is constituted of two parts. The first two terms are the ones coming from the lift of the IIA part, and are exactly the terms present in eleven-dimensional supergravity. The last term, which is the only one involving the torsion, corresponds to the lift of the contribution of the sources to the ten-dimensional supergravity variations. Thus, it seems that, mimicking what happens in ten dimensions, we are in presence of the usual eleven-dimensional supergravity plus some sources.

Using the torsion-modified covariant derivative for spinors (5.63) we can also define such an operator $\nabla^{(\tau)}$ for tensors. The relevant connection coefficients Γ are

$$\begin{aligned} \Gamma^K{}_{LM} &= \left\{ {}^{K}_{LM} \right\} + K^K{}_{LM} \\ K_{AMB} &= \hat{\tau}_{MAB} \end{aligned} \tag{5.68}$$

where $\{ {}^{K}_{LM} \}$ is the Levi-Civita connection. With the help of $\nabla^{(\tau)}$, we can rewrite equations (5.62) as

$$\begin{aligned} \nabla^{(\tau)}_M \hat{\phi}_{G_2} &= 0 \\ \nabla^{(\tau)}_M (*_7 \hat{\phi}_{G_2}) &= 0 \end{aligned} \tag{5.69}$$

One should remember that the original BPS equations could be written geometrically as $\nabla_M \hat{\phi}_{G_2} = 0$ and $\nabla_M *_7 \hat{\phi}_{G_2} = 0$ yet that these ceased to be valid once we include the sources in ten dimensions – as we discussed in section 5.2.1. Equations (5.69) show however that these geometric BPS equations remain formally invariant once we include torsion.

5.2.2 The equations of motion

We shall finally turn to the search for equations of motion in M-theory that reduce to the source-modified second-order equations in type IIA as given in equation (5.40). To find these equations, we actually reverse the integrability argument that allowed us to consider the first instead of the second-order equations in sections 5.1.2 and 5.1.3.

To get an idea of what we are about to do, let us briefly digress to the simple case without any flavors or sources. The Bianchi identities are the usual ones, the equation of motion is simple Ricci flatness, $\hat{R}_{MN} = 0$, and the G_2-structure is closed and co-closed. Thus the latter satisfies $\nabla_M \hat{\phi}_{G_2} = 0$. Taking the commutator

$$\begin{aligned} 0 &= [\nabla_K, \nabla_L] \hat{\phi}_{G_2 MNP} \\ &= -\hat{R}^S{}_{MKL} \hat{\phi}_{G_2 SNP} - \hat{R}^S{}_{NKL} \hat{\phi}_{G_2 MSP} - \hat{R}^S{}_{PKL} \hat{\phi}_{G_2 MNS} \end{aligned} \tag{5.70}$$

Upon contraction of (5.70) with $\hat{\phi}_{G_2}$, we find[5]

$$0 = 2\hat{R}_{KL} + \hat{R}_{MNPL} (*_7 \hat{\phi}_{G_2})_K{}^{MNP} \tag{5.71}$$

In the absence of torsion, $\hat{R}_{MNPL} (*_7 \hat{\phi}_{G_2})_K{}^{MNP} = 0$, due to the well-known symmetries satisfied by the Riemann tensor,

$$\begin{aligned} \hat{R}_{K[LMN]} &= 0 \\ \hat{R}_{KLMN} = \hat{R}_{MNKL} &= -\hat{R}_{MNLK} \end{aligned} \tag{5.72}$$

[5] As one can verify by direct calculation using (5.17), the G_2-structure satisfies

$$\begin{aligned} \hat{\phi}_{G_2 lmn} \hat{\phi}_{G_2}{}^{kmn} &= 6\delta^k_l \\ \hat{\phi}_{G_2}{}^{klp} \hat{\phi}_{G_2 mnp} &= (*_7 \hat{\phi}_{G_2})_{mn}{}^{kl} + \delta^k_m \delta^l_n - \delta^k_n \delta^l_m \end{aligned}$$

k, l, m, n, p denote indices of the seven-dimensional internal manifold.

Therefore, our space-time is Ricci flat and the equations of motion are satisfied.

After this brief digression, we return to the original problem. Our aim is to find a suitable equation of motion in M-theory, that reduces to (5.40) upon dimensional reduction. For consistency this equation of motion needs to reduce to simple Ricci flatness in the limit where the type IIA source density Ξ – equivalently the torsion $\hat{\tau}$ in M-theory – vanishes. In opposite to our considerations in the previous paragraph, the G_2-structure does no longer satisfy $\nabla_M \hat{\phi}_{G_2} = 0$, but instead satisfies $\nabla^{(\tau)}_M \hat{\phi}_{G_2} = 0$. So we can once more consider the commutator of covariant derivatives. The identities of footnote 5 used to derive (5.71) still hold, yet (5.72) do not, and we arrive at the main result of this chapter, the M-theory lift of the source-modified equations of motion

$$0 = 2\hat{R}^{(\tau)}_{IKL} + \hat{R}^{(\tau)}_{MNPL}(*_7\hat{\phi}_{G_2})_K{}^{MNP} \tag{5.73}$$

where $\hat{R}^{(\tau)}$ is the Riemann (Ricci) tensor in the presence of torsion.

As we have pointed out before, the BPS equations in their geometric form – $\nabla^{(\tau)}_M \hat{\phi}_{G_2} = 0$ – are equivalent to those obtained from the SUSY spinor ϵ, $D^{(\tau)}_M \epsilon = 0$. Therefore we could have derived (5.73) also using (5.63). A commutator of covariant derivatives acting on the SUSY spinor yields

$$0 = \hat{R}^{(\tau)}_{CDML} \hat{\Gamma}^{CD} \hat{\epsilon} \tag{5.74}$$

We then contract with $\bar{\hat{\epsilon}} \hat{\Gamma}_K{}^M$ and make use of the identity

$$\begin{aligned} \Gamma^A\Gamma^B\Gamma^C\Gamma^D = {}& \Gamma^{ABCD} + \eta^{AB}\Gamma^{CD} - \eta^{CB}\Gamma^{DA} + \eta^{CD}\Gamma^{AB} + \eta^{DA}\Gamma^{BC} \\ & - \eta^{AC}\Gamma^{BC} - \eta^{BD}\Gamma^{AC} + \eta^{AB}\eta^{CD} - \eta^{AC}\eta^{BD} + \eta^{AD}\eta^{BC} \end{aligned} \tag{5.75}$$

it follows that

$$0 = 2(\bar{\hat{\epsilon}}\hat{\epsilon})\hat{R}^{(\tau)}_{KL} + (\bar{\hat{\epsilon}}\hat{\Gamma}_K{}^{MNP}\hat{\epsilon})\hat{R}_{MNPL} + \mathcal{O}\left(\bar{\hat{\epsilon}}\hat{\Gamma}^{AB}\hat{\epsilon}\right) \tag{5.76}$$

The assumptions made about the SUSY spinor $\hat{\epsilon}$ imply that there is a G_2 structure that can be expressed as

$$*_7\, \hat{\phi}_{G_2} = (\bar{\hat{\epsilon}}\Gamma_{ABCD}\hat{\epsilon})\hat{e}^{ABCD} \tag{5.77}$$

They also imply that all terms of the form $\bar{\hat{\epsilon}}\hat{\Gamma}^{AB}\hat{\epsilon}$ vanish. Hence (5.73) follows from (5.76).

The equations of motion (5.73) can be rewritten in a more typical and enlightning fashion using the Einstein tensor

$$\hat{R}_{KL} - \frac{1}{2}\hat{g}_{KL}\hat{R} = \hat{T}_{KL} \tag{5.78}$$

where $\hat{T}_{KL}$ is the energy-momentum tensor of the sources. It can be written in terms of the torsion as

$$\begin{aligned}
\hat{T}_{KL} = {} & \nabla_L K^M{}_{MK} - \nabla_M K^M{}_{LK} + K^M{}_{LP}K^P{}_{MK} - K^M{}_{MP}K^P{}_{LK} \\
& + \frac{1}{2}(\nabla_L K_{MPN} - \nabla_P K_{MLN} + K_{MLQ}K^Q{}_{PN} - K_{MPQ}K^Q{}_{LN})(*_7\hat{\phi}_{G_2})_K{}^{MNP} \\
& + \frac{1}{2}\hat{g}_{KL}(\nabla_M K^M{}_Q{}^Q - \nabla_Q K^M{}_M{}^Q + K^M{}_{MP}K^P{}_Q{}^Q - K^M{}_{QP}K^P{}_M{}^Q) \\
& + \frac{1}{2}\hat{g}_{KL}(\nabla_P K_{MQN} + K_{MPR}K^R{}_{QN})(*_7\hat{\phi}_{G_2})^{QMNP}
\end{aligned} \tag{5.79}$$

where K_{MNP} is the contorsion tensor (see (5.68)). From (5.78), we can see that the Einstein equation we are proposing contains two terms: on the left-hand side, one has the Einstein tensor one would get from varying the eleven-dimensional supergravity action with no four-form flux; on the right-hand side, one has an energy-momentum tensor that vanishes when the torsion is set to zero. When the torsion vanishes, so does $\hat{T}$ and one recovers the M-theory Einstein equation. Writing the equation in this form makes very clear the fact that the lift of type IIA supergravity with sources is eleven-dimensional supergravity supplemented by some sources. Unfortunately, we were not able to find an action that would be responsible for this energy-momentum tensor. To summarize, we claim that having sources in ten dimensions corresponds to having an energy-momentum tensor in eleven dimensions, of the form presented above.

To verify our claim, we will now perform the explicit dimensional reduction of (5.73), and show that we recover all the equations of motion of type IIA with sources. The calculations are – as so often in supergravity – straightforward yet tedious. We found [94] quite helpful, yet not essential. The reader not interested in mathematical details might want to skip ahead to the end of this section, where we summarize our findings. Notice that in the following, despite the fact that we dropped the superscript (τ) for simplicity of notation, all hatted Riemann and Ricci tensor are considered in the presence of torsion.

Let us start with the zz-component of (5.71). We find

$$\begin{aligned}\hat{R}_{zz} &= -\frac{2}{3}e^{4\Phi}\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}e^{-2\Phi}\partial^\mu\Phi) + \frac{1}{4}e^{4\Phi}F^2 \\ (*_7\hat{\phi}_{G_2})_z{}^{SPK}\hat{R}_{SPKz} &= e^{3\Phi}(*_6\Psi)\lrcorner \mathrm{d}B\end{aligned} \tag{5.80}$$

from which it follows that

$$\begin{aligned}0 &= 2\hat{R}_{zz} + (*_7\hat{\phi}_{G_2})_z{}^{SPK}\hat{R}_{SPKz} \\ &= -\frac{4}{3}e^{4\Phi}\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}e^{-2\Phi}\partial^\mu\Phi) + \frac{1}{2}e^{4\Phi}F^2 + e^{3\Phi}(*_6\Psi)\lrcorner \mathrm{d}B \\ &= \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}e^{-2\Phi}\partial^\mu\Phi) - \frac{3}{8}F^2 - \frac{3}{4}e^{-\Phi}(*_{10}\mathcal{K})\lrcorner\Xi\end{aligned} \tag{5.81}$$

Here we used that $*_6\Psi = -*_{10}\mathcal{K}$ and $\mathrm{d}B = \mathrm{d}F = -\Xi$. And we notice that we find the source-modified ten-dimensional equation of motion for the dilaton as in (5.40).

Now we investigate the μz-component of (5.71). We find

$$\begin{aligned}\hat{R}_{\mu z} &= -\frac{1}{2}e^{2\Phi}\nabla^\nu F_{\nu\mu} + A_\mu\hat{R}_{zz} \\ (*_7\hat{\phi}_{G_2})_\mu{}^{SPK}\hat{R}_{SPKz} &= -\frac{1}{6}e_{a\mu}(*_7\hat{\phi}_{G_2})^{abcd}(\mathrm{d}B)_{bcd} + A_\mu(e^{3\Phi}(*_6\Psi)\lrcorner \mathrm{d}B)\end{aligned} \tag{5.82}$$

Now we have

$$\begin{aligned}\frac{1}{6}e_{a\mu}(*_7\hat{\phi}_{G_2})^{abcd}(\mathrm{d}B)_{bcd} &= \frac{1}{6}(*_6J)^{\mu bcd}(\mathrm{d}B)_{bcd} \\ &= \frac{1}{12}\sqrt{-g_{(6)}}\epsilon^{\alpha\beta\mu\nu\rho\sigma}J_{\alpha\beta}(\mathrm{d}B)_{\nu\rho\sigma} \\ &= \frac{1}{12}\frac{1}{6!}*_6(\mathrm{d}x^\mu \wedge J \wedge \mathrm{d}B) \\ &= \frac{1}{12}\frac{1}{6!}*_6[\mathrm{d}x^\mu \wedge \mathrm{d}(J\wedge B)] \\ &= 0\end{aligned} \tag{5.83}$$

because supersymmetry tells us that $\mathrm{d}J = 0$ and $\mathrm{d}(J\wedge B) = \mathrm{d}(\mathrm{d}\hat{\phi}_{G_2}) = 0$. Thus

$$\begin{aligned}0 &= 2\hat{R}_{\mu z} + (*_7\hat{\phi}_{G_2})_\mu{}^{SPK}\hat{R}_{SPKz} \\ &= -e^{2\Phi}\nabla^\nu F_{\nu\mu} + 2A_\mu\hat{R}_{zz} - \frac{1}{6}e_{a\mu}(*_7\hat{\phi}_{G_2})^{abcd}(\mathrm{d}B)_{bcd} + A_\mu(e^{3\Phi}(*_6\Psi)\lrcorner \mathrm{d}B) \\ &= -e^{2\Phi}\nabla^\nu F_{\nu\mu} + A_\mu[2\hat{R}_{zz} + e^{3\Phi}(*_6\Psi)\lrcorner \mathrm{d}B]\end{aligned} \tag{5.84}$$

The term in square brackets is equal to the zz-component of (5.71) and the remaining part corresponds to the Maxwell equation for $F_{(2)}$.

The $z\nu$-component[6] of (5.71) gives

$$\begin{aligned}\hat{R}_{z\nu} &= -\frac{1}{2}e^{2\Phi}\nabla^\rho F_{\rho\nu} + A_\nu \hat{R}_{zz} + \frac{2}{3}e^{2\Phi}(\mathrm{d}A - F)_{\nu\rho}\partial^\rho\Phi \\ (*_7\hat{\phi}_{G_2})_z{}^{SPK}\hat{R}_{SPK\nu} &= e^{3\Phi}B_{\nu\beta}[F\lrcorner(*_6\Psi)]^\beta + A_\nu(e^{3\Phi}(*_6\Psi)\lrcorner\mathrm{d}B) \\ &= \frac{4}{3}e^{2\Phi}B_{\nu\beta}\partial^\beta\Phi + A_\nu(e^{3\Phi}(*_6\Psi)\lrcorner\mathrm{d}B)\end{aligned} \tag{5.85}$$

with $\mathrm{d}\Phi = \frac{3}{4}e^\Phi F\lrcorner(*_6\Psi)$ due to sypersymmetry. Putting things together

$$\begin{aligned}0 &= 2\hat{R}_{z\nu} + (*_7\hat{\phi}_{G_2})_z{}^{SPK}\hat{R}_{SPK\nu} \\ &= -e^{2\Phi}\nabla^\rho F_{\rho\nu} + A_\nu[2\hat{R}_{zz} + e^{3\Phi}(*_6\Psi)\lrcorner\mathrm{d}B]\end{aligned} \tag{5.86}$$

This agrees with the μz-component. Let us finally look at the $\mu\nu$-component of (5.71). We have

$$\begin{aligned}\hat{R}_{\mu\nu} = R_{\mu\nu} + 2\nabla_\mu\partial_\nu\Phi - \frac{e^{2\Phi}}{2}(F_{\mu\rho}(\mathrm{d}A)_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}F^2) - \frac{1}{2}A_\nu\nabla^\rho F_{\rho\mu} + A_\mu\hat{R}_{z\nu} \\ - \frac{e^{-2\Phi}}{2}g_{\mu\nu}\hat{R}_{zz}\end{aligned} \tag{5.87}$$

and

$$\begin{aligned}(*_7\hat{\phi}_{G_2})_\mu{}^{SPK}\hat{R}_{SPK\nu} = A_\mu[(*_7\hat{\phi}_{G_2})_z{}^{SPK}\hat{R}_{SPK\nu}] + \frac{4}{3}e^\Phi(*_6\Psi)_\mu{}^{cd}B_{\nu d}\partial_c\Phi \\ - \frac{1}{6}A_\nu e^{2\Phi}e_{a\mu}(*_7\hat{\phi}_{G_2})^{abcd}(\mathrm{d}B)_{bcd} - e^\Phi(*_6\Psi)_\mu{}^{cd}\nabla_d B_{\nu c} \\ + \frac{1}{2}e^{2\Phi}e_{a\mu}(*_7\hat{\phi}_{G_2})^{abcd}B_{\nu d}F_{cb} - \frac{1}{2}e^\Phi(*_6\Psi)_\mu{}^{cd}\nabla_\nu B_{cd}\end{aligned} \tag{5.88}$$

Let us first notice that

$$(*_6\Psi)_\mu{}^{cd}(\nabla_d B_{\nu c} + \frac{1}{2}\nabla_\nu B_{cd}) = \frac{1}{2}(*_6\Psi)_\mu{}^{cd}(\mathrm{d}B)_{\nu cd} \tag{5.89}$$

Then from previous computation we know that

$$e_{a\mu}(*_7\hat{\phi}_{G_2})^{abcd}(\mathrm{d}B)_{bcd} = 0 \tag{5.90}$$

In the following we will also use the identities

$$\begin{aligned}(*_6\Psi)^f{}_{ab}(*_6\Psi)_{cd}{}^f &= \eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc} + J_{ac}J_{db} + J_{ad}J_{bc} \\ \frac{1}{2}(J\wedge J)_{abcd} &= J_{ab}J_{cd} + J_{ac}J_{db} + J_{ad}J_{bc}\end{aligned} \tag{5.91}$$

[6] One might suspect this to be identical to the μz-component. Due to the presence of torsion however, the Ricci tensor is no longer symmetric and one has to check this independently. Interestingly, the Kaluza-Klein reductions of μz and $z\nu$ are already different in the torsion-free case. Here the two differ by $F - \mathrm{d}A$ however, which vanishes in source and torsion-free geometries.

and once again

$$\partial_a\Phi = \frac{3}{4}e^{\Phi}(F\lrcorner(*_6\Psi))_a = \frac{3}{8}e^{\Phi}F^{bc}(*_6\Psi)_{bca} \tag{5.92}$$

So

$$\begin{aligned}(*_7\hat{\phi}_{G_2})^{abcd}F_{cb} &= -\frac{1}{2}(J\wedge J)^{abcd}F_{cb} \\ &= F_{cb}(J^{ab}J^{cd} + J^{ac}J^{db} + J^{ad}J^{bc}) \\ &= 2J^{ab}F_{bc}J^{cd}\end{aligned} \tag{5.93}$$

because supersymmetry dictates that $F\lrcorner J = 0$. And

$$\begin{aligned}(*_6\Psi)_\mu{}^{cd}\partial_c\Phi &= -\frac{3}{8}e^{\Phi}F^{fg}(*_6\Psi)_\mu{}^{dc}(*_6\Psi)_{fgc} \\ &= -\frac{3}{4}e^{\Phi}(F_\mu{}^d + J_\mu{}^f F_{fg}J^{gd})\end{aligned} \tag{5.94}$$

Putting everything together, we get

$$\begin{aligned}(*_7\hat{\phi}_{G_2})_\mu{}^{SPK}\hat{R}_{SPK\nu} &= A_\mu[(*_7\hat{\phi}_{G_2})_z{}^{SPK}\hat{R}_{SPK\nu}] + e^{2\Phi}e_{a\mu}B_{\nu d}J^{ab}F_{bc}J^{cd} \\ &\quad - \frac{1}{2}e^{\Phi}(*_6\Psi)_\mu{}^{cd}(\mathrm{d}B)_{\nu cd} - e^{2\Phi}B_{\nu d}(F_\mu{}^d + J_\mu{}^f F_{fg}J^{gd}) \\ &= A_\mu[(*_7\hat{\phi}_{G_2})_z{}^{SPK}\hat{R}_{SPK\nu}] - \frac{1}{2}e^{\Phi}(*_6\Psi)_\mu{}^{cd}(\mathrm{d}B)_{\nu cd} \\ &\quad - e^{2\Phi}F_\mu{}^d B_{\nu d}\end{aligned} \tag{5.95}$$

In total

$$\begin{aligned}0 &= 2\hat{R}_{\mu\nu} + (*_7\hat{\phi}_{G_2})_\mu{}^{SPK}\hat{R}_{SPK\nu} \\ &= 2R_{\mu\nu} + 4\nabla_\mu\partial_\nu\Phi - e^{2\Phi}(F_{\mu\rho}(\mathrm{d}A)_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}F^2) - A_\nu\nabla^\rho F_{\rho\mu} - e^{2\Phi}F_\mu{}^d B_{\nu d} \\ &\quad - e^{-2\Phi}g_{\mu\nu}\hat{R}_{zz} + A_\mu[(*_7\hat{\phi}_{G_2})_z{}^{SPK}\hat{R}_{SPK\nu}] - \frac{1}{2}e^{\Phi}(*_6\Psi)_\mu{}^{cd}(\mathrm{d}B)_{\nu cd} + A_\mu 2\hat{R}_{z\nu} \\ &= 2R_{\mu\nu} + 4\nabla_\mu\partial_\nu\Phi - e^{2\Phi}(F_{\mu\rho}(\mathrm{d}A+B)_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}F^2) - A_\nu\nabla^\rho F_{\rho\mu} \\ &\quad + A_\mu[2\hat{R}_{z\nu} + (*_7\hat{\phi}_{G_2})_z{}^{SPK}\hat{R}_{SPK\nu}] - e^{-2\Phi}g_{\mu\nu}[\hat{R}_{zz} + \frac{1}{2}e^{3\Phi}(*_6\Psi)\lrcorner\mathrm{d}B] \\ &\quad - \frac{1}{2}e^{\Phi}(*_{10}\mathcal{K})_\mu{}^{\rho\sigma}\Xi_{\nu\rho\sigma} + \frac{1}{2}e^{-2\Phi}g_{\mu\nu}e^{3\Phi}(*_6\Psi)\lrcorner\mathrm{d}B\end{aligned} \tag{5.96}$$

which gives

$$\begin{aligned}0 = 2R_{\mu\nu} &+ 4\nabla_\mu\partial_\nu\Phi - e^{2\Phi}(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}F^2) \\ &- \frac{1}{2}e^{\Phi}\left((*_{10}\mathcal{K})_\mu{}^{\rho\sigma}\Xi_{\nu\rho\sigma} - g_{\mu\nu}(*_{10}\mathcal{K})\lrcorner\Xi\right) \\ &- A_\nu\nabla^\rho F_{\rho\mu} + A_\mu[2\hat{R}_{z\nu} + (*_7\hat{\phi}_{G_2})_z{}^{SPK}\hat{R}_{SPK\nu}] \\ &- \frac{e^{-2\Phi}}{2}g_{\mu\nu}[2\hat{R}_{zz} + e^{3\Phi}(*_{10}\mathcal{K})\lrcorner\Xi]\end{aligned} \tag{5.97}$$

where we recognize the first two lines of this equation as being the Einstein equation of type IIA supergravity with sources and the rest vanishes thanks to other components of (5.71). This completes the reduction of eleven-dimensional Einstein equations to the type IIA supergravity equations of motion.

To summarize, in this section we showed that the equation of motion of eleven-dimensional supergravity with torsion (5.73), which is given to us by integrability, reduces to the source-modified type IIA supergravity equations of motion (5.40). It thus shows that adding torsion to eleven-dimensional supergravity reduces to adding smeared D6 sources in type IIA supergravity.

5.3 Discussion

We have studied two related issues: the addition of D6-branes as smeared sources to a type IIA background, and the lifting of such a system to eleven-dimensional supergravity. We considered these in the context of 1/8 BPS solutions of the form $\mathbb{R}^{1,3} \times \mathcal{M}$, a fact represented by the presence of a G_2 or $SU(3)$-structure.

Concerning the problem of the M-theory lift, we saw that ordinary eleven dimensional-supergravity cannot accommodate for the presence of the additional sources and argued that a possible solution might lie in the inclusion of geometric torsion. While our argument was founded on the observed loss of Ricci flatness in the higher-dimensional theory, we were able to show by explicit calculation that the supersymmetry variations take the required form upon addition of torsion. Moreover, the torsion must take the form (5.66), related to the distribution $\Xi_{(3)}$ of the sources in the reduced theory. Subsequently we derived a set of second order equations that could be the equations of motion of some eleven-dimensional supergravity with torsion, and proved that they reduce to the type IIA equations of motion with smeared D6-branes. As we pointed out, this work is not in contradiction with the uniqueness of supergravity in eleven dimensions, because we are only considering a theory that preserves four supercharges. We did not of course show that there is a well defined theory in eleven dimensions that is supersymmetric and has the field content of both eleven dimensional supergravity as well as of the additional torsion. One should not forget however, that we are not studying the uplift of S_{IIA}, which is well known, but of

$$S = S_{\text{IIA}} + S_{\text{D6-sources}} \tag{5.98}$$

The problem was first addressed in [46] whose authors found a seven-dimensional gauged sigma model action that reduces to the DBI-term of the D6 brane. They were unable to find a suitable uplift of the Wess-Zumino term however. While this chapter does not solve the problem in the sense of [46], it does succeed in lifting the ten-dimensional equations of motion to pure eleven-dimensional geometry. The question whether the results are just an accidental rewriting of type IIA dynamics in higher-dimensional notation or do actually point to a higher dimensional supersymmetric theory that includes torsion remains open.

While there is a long history of the uses of torsion in the context of string theory, the torsion used in papers such as [95] and [56] is related to the presence of fluxes, not of sources. Therefore the addition of further torsion is a rather unorthodox concept. So it is necessary to wonder if we would not have been able to solve the problem at hand with simpler methods. As mentioned before, our argument was based on the loss of Ricci flatness in eleven dimensions. One might guess that it is possible to use the four-form in M-theory, $\hat{F}_{(4)}$, to obtain a suitable energy-momentum tensor to supplement the Einstein equations. This however leads to four and three-form flux in type IIA, in contradiction with our results of section 5.1. Another possibility would be to use the KK-monopole action of [46]. There the authors constructed a gauged sigma model action (5.99) that is the dimensional uplift of the DBI term of a D6-brane. Using this, one could try to lift the action (1.3) to M-theory. Yet considered in connection with the standard Kaluza-Klein mechanism, (1.3) is an action in terms of $\mathrm{d}A_{(1)}$, not $F_{(2)}$. So even if one were able to lift the brane contribution to (1.3), the supergravity part would still be lacking the source contribution. Still, it might be interesting to try to match the sigma model action [46] with the inclusion of torsion.

Note that the considerations made in section 5.2 make hardly any use of string or M-theory. The setup is merely that of a $U(1)$ Kaluza-Klein theory in d and $d+1$ dimensions with monopole condensation in the lower dimensional theory. Hence the results of this chapter may be re-expressed as follows: a monople condensate in a d-dimensional Kaluza-Klein theory might be described as torsion in $d+1$ dimensions.

The other problem studied here was the construction of a gravity dual to

$\mathcal{N} = 1$, $SU(N_c)$ super Yang-Mills with flavors. We addressed this in section 5.1. Here we found a system of first-order BPS equations that describes the addition of D6 sources to the type IIA background (5.25). At the end of section 5.1, we presented a family of exact solutions.

5.A D6-brane embeddings

We will now discuss D6-brane embeddings in the three type IIA reductions of the Bryant-Salamon metric (5.20). In principle one would have to study each of the three reductions independently, but as we will show now it is actually possible to search for these embeddings directly in M-theory. Strictly speaking we will do nothing but rewriting the calibration condition of type IIA string theory in eleven-dimensional notation. However this turns out to be quite useful, as the M-theory expressions are more compact and less convoluted than their lower-dimensional counterparts.

The starting point is the gauged sigma model action of [46]. Here, the authors constructed an action that is the eleven-dimensional uplift of the DBI action of a D6-brane. In other words, it can be thought of as the world-volume action of a Kaluza-Klein monopole. Let the M-theory circle be described by the Killing vector $K = \partial_z$. Then

$$\begin{aligned} S_{\mathrm{KK7}} &= -T_{\mathrm{KK7}} \int \mathrm{d}^7\xi K^2 \sqrt{-\det \partial_i X^M \partial_j X^N \Pi_{MN}} \\ \Pi_{MN} &= g_{MN} - K^{-1} K_M K_N \end{aligned} \tag{5.99}$$

The action is that of a gauged sigma model. $\hat{\Pi}^M{}_N$ projects eleven to ten-dimensional vectors. One verifies by explicit calculation that (5.99) reduces to the DBI action of a D6-brane.

We want to use Π_{MN} to describe calibrated cycles of D6-branes in type IIA using M-theory notation. Recall that a D6-brane embedding $X^\mu(\xi^i)$ is supersymmetric if it satisfies the calibration condition

$$X^* \mathcal{K} = \mathrm{d}^7\xi \sqrt{-g_{\mathrm{ind.}}} \tag{5.100}$$

Here $(g_{\mathrm{ind.}})_{ij} = \partial_i X^\mu \partial_j X^\nu g_{\mu\nu}$ is the induced metric and $\mathcal{K}$ the calibration form (5.28). Defining $(\hat{e}^{\mathrm{IIA}})^A_M = \Pi_{MN} \hat{E}^{NA}$ we have, using (5.21), $(\hat{e}^{\mathrm{IIA}})^a = e^{-\frac{\Phi}{3}} e^a$. We can now define the M-theory lift of the type IIA calibration form as

$$\phi_{\mathrm{KK7}} = (\hat{e}^{\mathrm{IIA}})^{x^0 x^1 x^2 x^3} \wedge [(\hat{e}^{\mathrm{IIA}})^{125} - (\hat{e}^{\mathrm{IIA}})^{345} - (\hat{e}^{\mathrm{IIA}})^{\rho 24} - (\hat{e}^{\mathrm{IIA}})^{\rho 13}] \tag{5.101}$$

In an abuse of notation, we have labeled this the calibration form of a KK-monopole. Also $\sqrt{-\Pi} = e^{-\frac{7}{3}\Phi}\sqrt{-g_{\text{ind.}}}$, and we arrive at a lifted form of the calibration condition (5.100),

$$X^*\phi_{\text{KK7}} = \sqrt{-\Pi} \tag{5.102}$$

We will now use (5.102) to study D6-brane embeddings. Recall that there are three $U(1)$ isometries, with three different dimensional reductions

$$\begin{array}{ll} \partial_{\psi+\tilde{\psi}} \subset \sigma \times \Sigma & \text{Resolved conifold} \\ \partial_{\phi} \subset \sigma & \text{Resolved conifold} \\ \partial_{\tilde{\phi}} \subset \Sigma & \text{Deformed conifold} \end{array} \tag{5.103}$$

Color embeddings Color embeddings are those which wrap only a compact cycle. In the case at hand they do not extend along the radial direction at all. If we specify to the deformed conifold, that is, we choose the isometry $K = \partial_{\tilde{\phi}}$, we find an embedding parameterized by[7]

$$\begin{array}{cccccccc} x^\mu & \rho & \theta & \phi & \psi & \tilde{\theta} & \tilde{\phi} & \tilde{\psi} \\ \hline - & \rho_0 & \circ & \circ & \circ & . & K & . \end{array} \tag{5.104}$$

The embedding exists only at $\rho = \rho_0$ as

$$\begin{aligned} & X^*\phi_{\text{KK7}} = -\frac{2\rho^3 + \rho_0^3}{72\sqrt{3}}\sin\theta \quad \sqrt{-\Pi} = \frac{-4\rho^3 + \rho_0^3}{72\sqrt{3}}\sin\theta \\ & X^*\phi_{\text{KK7}} \overset{\rho\to\rho_0}{=} \sqrt{-\Pi} \end{aligned} \tag{5.105}$$

So we recover the color brane embedding of the string dual we started with. Note that this cycle is calibrated in M-theory though. I.e. it is a minimum volume cycle of the eleven-dimensional geometry.

For the resolved conifold associated with $K = \partial_\phi$ one might try an embedding as

$$\begin{array}{cccccccc} x^\mu & \rho & \theta & \phi & \psi & \tilde{\theta} & \tilde{\phi} & \tilde{\psi} \\ \hline - & \rho_0 & . & K & . & \circ & \circ & \circ \end{array} \tag{5.106}$$

However, the cycle in question vanishes at $\rho = \rho_0$, as one would expect.

Massless flavor embeddings Massless flavor branes extend fully along the radial direction ρ. Therefore they only need to wrap a two-cycle in the internal

[7] The notation for these embedding diagrams is as follows: a $-$ signals a non-compact direction along which the brane extends, a $\circ$ a wrapped compact one. K denotes the M-theory circle associated with the Killing vector K, . finally stands for localized directions.

geometry and one can make the following guess

$$\begin{array}{cccccccc} x^\mu & \rho & \theta & \phi & \psi & \tilde{\theta} & \tilde{\phi} & \tilde{\psi} \\ \hline - & - & . & . & \circ & . & . & \circ \end{array} \tag{5.107}$$

Note that this embedding works for both the ∂_ϕ and the $\partial_{\tilde{\phi}}$ isometries.

For the deformed conifold, i.e. reduction along $\partial_{\tilde{\phi}}$, we obtain the relation

$$\begin{aligned} X^*\phi_{\text{KK7}} &= \frac{\rho^2}{6\sqrt{3}}\sin^2\tilde{\theta} \\ \sqrt{-\Pi} &= \frac{\rho^2}{6\sqrt{3}}\sin\tilde{\theta} \end{aligned} \tag{5.108}$$

demanding $\tilde{\theta} = \frac{\pi}{2}$. The resolved conifold associated with ∂_ϕ gives

$$\begin{aligned} X^*\phi_{\text{KK7}} &= \frac{\rho^2}{6\sqrt{3}}\sin^2\theta \\ \sqrt{-\Pi} &= \frac{\rho^2}{6\sqrt{3}}\sin\theta \end{aligned} \tag{5.109}$$

demanding $\theta = \frac{\pi}{2}$; whereas for the $\partial_\psi + \partial_{\tilde{\psi}}$ reduction both $X^*\phi$ and $\sqrt{-\Pi}$ vanish. Interestingly, in M-theory the cycle $(x^\mu, \rho, \psi, \tilde{\psi})$ is calibrated in the traditional sense; that is, it is a minimal volume cycle.

Massive flavor embeddings Naturally one would like to relax the constraints on θ and $\tilde{\theta}$ respectively from the above paragraph. A good guess to do so lies in assuming a relation between ρ and θ (or $\tilde{\theta}$).

In the case of the $\partial_{\tilde{\phi}}$ reduction, we assume

$$\begin{array}{cccccccc} x^\mu & \rho & \theta & \phi & \psi & \tilde{\theta} & \tilde{\phi} & \tilde{\psi} \\ \hline - & \rho(\tilde{\theta}) & . & . & \circ & \circ & . & \circ \end{array} \tag{5.110}$$

Then

$$\begin{aligned} X^*\phi_{\text{KK7}} &= \frac{(\rho^3 - \rho_0^3)\cos\tilde{\theta} + 3\rho^2\rho'\sin\tilde{\theta}}{18\sqrt{3}}\sin\tilde{\theta} \\ \sqrt{-\Pi} &= \frac{\sqrt{(\rho^3-\rho_0^3)^2 + 9\rho^4(\rho')^2}}{18\sqrt{3}}\sin\tilde{\theta} \end{aligned} \tag{5.111}$$

Demanding the two expressions to agree, it follows that

$$\begin{aligned} \rho'(\tilde{\theta}) &= \frac{\rho^3 - \rho_0^3}{3\rho^2}\tan\tilde{\theta} \\ \rho(\tilde{\theta}) &= \left(\rho_0^3 + e^{3C_1}\sec\tilde{\theta}\right)^{1/3} \end{aligned} \tag{5.112}$$

with C_1 being a constant of integration, associated with the mass of the flavors, as we will show now. $\sec\tilde{\theta} \in [1, \infty)$, so the brane reaches down to $(\rho_0^3 + e^{3C_1})^{1/3}$.

Thus the massless limit is given by $C_1 \to -\infty$. In order to compare this embedding with the massless one of the previous paragraph, we have to solve the embedding equation for $\tilde{\theta}$ before taking this limit – as we expect the brane to be localized in $\tilde{\theta}$, so the mapping $\tilde{\theta} \mapsto \rho$ is ill defined. The result is

$$\tilde{\theta} = \arccos \frac{e^{3C_1}}{\rho^3 - \rho_0^3} \tag{5.113}$$

So in the limit $C_1 \to -\infty$, the brane sits once more at $\tilde{\theta} = \frac{\pi}{2}$, which is also the position of the brane for $\rho \gg \rho_0$.

For the ∂_ϕ reduction, one needs to swap θ for $\tilde{\theta}$. Then,

$$\begin{array}{cccccccc} x^\mu & \rho & \theta & \phi & \psi & \tilde{\theta} & \tilde{\phi} & \tilde{\psi} \\ \hline - & \rho(\theta) & \circ & . & \circ & . & . & \circ \end{array} \tag{5.114}$$

The calibration condition is given by

$$\begin{aligned} X^*\phi_{\mathrm{KK7}} &= \frac{(8\rho^6 - 7\rho^3\rho_0^3 - \rho_0^6)\cos\theta + 6\rho^2(4\rho^3 - \rho_0^3)\rho'\sin\theta}{36\sqrt{3}(4\rho^3 - \rho_0^3)}\sin\theta \\ \sqrt{-\Pi} &= \frac{\sqrt{4\rho^6 - 6\rho^3\rho_0^3 + \rho_0^6 + 36\rho^4(\rho')^2}}{36\sqrt{3}}\sin\theta \end{aligned} \tag{5.115}$$

leading to a differential equation for ρ that is considerably harder to solve than the previous one. One can study it numerically, obtaining results similar to those of the previous embedding. As to analytic results, setting $\rho_0 \to 0$, leads to simplifications allowing for

$$\rho(\theta) = C_1(\sec\theta)^{1/3} \tag{5.116}$$

which is identical to (5.113) in the same limit.

5.B Kaluza-Klein reduction of supergravity variations with torsion

We review the dimensional reduction of the SUSY variations – with an additional torsion term – from eleven to ten-dimensional supergravity. Conceptually we follow [58], our conventions are slightly different though. We assume a spacetime with topology $\mathcal{M}_{10} \times S^1$ and label the eleventh coordinate as z. Naturally all fields will be independent of z. Further assuming the eleven-dimensional background to be purely gravitational, we only need to consider the variation of the gravitino,

$$\delta_{\hat{\epsilon}}\hat{\psi}_M = \partial_M\hat{\epsilon} + \frac{1}{4}\hat{\omega}_{MAB}\Gamma^{AB}\hat{\epsilon} + \frac{1}{4}\hat{\tau}_{MAB}\hat{\Gamma}^{AB}\hat{\epsilon} \tag{5.117}$$

which we have modified by the presence of the torsion term $\hat{\tau}$. As in section 5.1.1 we take the vielbein to be in Scherk-Schwarz gauge (5.21).

We shall perform the reduction of (5.117) step by step and our first aim shall be the reduction of the spin connection

$$\hat{\omega}_{ABC} = \frac{1}{2}\left(\hat{\Omega}_{CAB} - \hat{\Omega}_{BAC} - \hat{\Omega}_{ABC}\right) \tag{5.118}$$

with the objects of anholomorphicity defined as

$$\hat{\Omega}_{ABC} = \left(\partial_M \hat{e}_N^K - \partial_N \hat{e}_M^K\right) \hat{\eta}_{KA} \hat{E}_B^N \hat{E}_C^M \tag{5.119}$$

Then

$$\begin{aligned} \hat{\omega}_{zbc} &= +\frac{e^{\frac{4}{3}\Phi}}{2}(\mathrm{d}A)_{bc} \quad \hat{\omega}_{abc} = \frac{e^{\frac{1}{3}\Phi}}{3}(\eta_{ab}\partial_c\Phi - \eta_{ac}\partial_b\Phi) + e^{\frac{1}{3}\Phi}\omega_{abc} \\ \hat{\omega}_{abz} &= -\frac{e^{\frac{4}{3}\Phi}}{2}(\mathrm{d}A)_{ab} \quad \hat{\omega}_{zaz} = \frac{2}{3}e^{\frac{1}{3}\Phi}\partial_a\Phi \end{aligned} \tag{5.120}$$

Note that we use $\mathrm{d}A_{\mu\nu}$ instead of $F_{\mu\nu}$ as we are anticipating the inclusion of sources such that F is no longer exact.

Turning to the gravitino, one could make an ansatz

$$\hat{\psi}_M = \left(e^{m\Phi}\psi_\mu, e^{n\Phi}\lambda\right) \tag{5.121}$$

and

$$\hat{\epsilon} = e^{l\Phi}\epsilon \tag{5.122}$$

with $l, m, n \in \mathbb{C}$. Yet, as we will see, we will need to consider linear combinations such as $\hat{\psi}_\mu = e^{m\Phi}\psi_\mu + e^{n\Phi}\Gamma_\mu\lambda + e^{p\Phi}\Gamma_\mu\Gamma^{11}\lambda$.

We begin with the covariant derivative of the SUSY spinor, looking first at the vector components:

$$\begin{aligned} e^{-l\Phi}\hat{D}_\mu\hat{\epsilon} = {} & (l\partial_\mu\Phi\epsilon + \partial_\mu\epsilon) + e_\mu^a\left[\frac{1}{4}\omega_{abc} + \frac{1}{12}(\eta_{ab}\partial_c\Phi - \eta_{ac}\partial_b\Phi)\right]\Gamma^{bc}\epsilon \\ & - \frac{1}{4}e^\Phi e_\mu^a \mathrm{d}A_{ab}\Gamma^b\Gamma^{11}\epsilon + \left(\frac{1}{8}e^{2\Phi}A_\mu \mathrm{d}A_{bc}\Gamma^{bc} + \frac{1}{3}e^\Phi A_\mu\partial_b\Phi\Gamma^b\Gamma^{11}\right)\epsilon \\ & + \frac{1}{4}\hat{\tau}_{\mu bc}\Gamma^{bc}\epsilon + \frac{1}{2}\hat{\tau}_{\mu bz}\Gamma^b\Gamma^{11}\epsilon \end{aligned} \tag{5.123}$$

The scalar component satisfies

$$e^{-l\Phi}\hat{D}_z\hat{\epsilon} = \frac{e^{2\Phi}}{8}\mathrm{d}A_{bc}\Gamma^{bc}\epsilon + \frac{e^\Phi}{3}\partial_b\Phi\Gamma^b\Gamma^{11}\epsilon + \frac{1}{4}\hat{\tau}_{zbc}\Gamma^{bc}\epsilon + \frac{1}{2}\hat{\tau}_{zbz}\Gamma^b\Gamma^{11}\epsilon \tag{5.124}$$

Equations (5.123) and (5.124) hold in string frame. To convert to Einstein frame we need to recall that the gamma matrices are defined in tangent space, from which it follows that only the curved space gamma matrices are affected by Weyl transformations. For a generic Weyl transformation, we have

$$
\begin{aligned}
&e^a_\mu \mapsto e^{\delta\Phi} e^a_\mu && \Omega_{abc} \mapsto e^{-\delta\Phi}\Omega_{abc} + e^{-\delta\Phi}\delta(\eta_{ab}\partial_c\Phi - \eta_{ac}\partial_b\Phi) \\
&E^\mu_a \mapsto e^{-\delta\Phi} E^\mu_a && \omega_{abc} \mapsto e^{-\delta\Phi}\omega_{abc} - \delta e^{-\delta\Phi}(\eta_{ab}\partial_c\Phi - \eta_{ac}\partial_b\Phi) \\
&\partial_a \mapsto e^{-\delta\Phi}\partial_a && \mathrm{d}A_{a_1\ldots a_p} \mapsto e^{-p\delta\Phi}\mathrm{d}A_{a_1\ldots a_p} \\
&\Gamma^a \mapsto \Gamma^a && \eta_{ab} \mapsto \eta_{ab} \\
&\hat{\tau}_{\mu bc} \mapsto \hat{\tau}_{\mu bc}
\end{aligned}
\tag{5.125}
$$

So that with $(e^S)^a_\mu = e^{\frac{1}{4}\Phi}(e^E)^a_\mu$, $\delta = \frac{1}{4}$,

$$
\begin{aligned}
e^{-l\Phi}\hat{D}_\mu\hat{\epsilon} = {}& (l\partial_\mu\Phi\epsilon + \partial_\mu\epsilon) + e^a_\mu\left[\frac{1}{4}\omega_{abc} + \frac{1}{48}(\eta_{ab}\partial_c\Phi - \eta_{ac}\partial_b\Phi)\right]\Gamma^{bc}\epsilon \\
& - \frac{1}{4}e^{\frac{3}{4}\Phi}e^a_\mu \mathrm{d}A_{ab}\Gamma^b\Gamma^{11}\epsilon + \left(\frac{1}{8}e^{\frac{3}{2}\Phi}A_\mu \mathrm{d}A_{bc}\Gamma^{bc} + \frac{1}{3}e^{\frac{3}{4}\Phi}A_\mu\partial_b\Phi\Gamma^b\Gamma^{11}\right)\epsilon \\
& + \frac{1}{4}\hat{\tau}_{\mu bc}\Gamma^{bc}\epsilon + \frac{1}{2}\hat{\tau}_{\mu bz}\Gamma^b\Gamma^{11}\epsilon \\
e^{-l\Phi}\hat{D}_z\hat{\epsilon} = {}& \frac{e^{\frac{3}{2}\Phi}}{8}\mathrm{d}A_{bc}\Gamma^{bc}\epsilon + \frac{e^{\frac{3}{4}\Phi}}{3}\partial_b\Phi\Gamma^b\Gamma^{11}\epsilon + \frac{1}{4}\hat{\tau}_{zbc}\Gamma^{bc}\epsilon + \frac{1}{2}\hat{\tau}_{zbz}\Gamma^b\Gamma^{11}\epsilon
\end{aligned}
\tag{5.126}
$$

One needs to compare (5.123) and (5.124) or (5.126) repectively to the SUSY variations of the ansatz (5.121)

$$
\begin{aligned}
\hat{D}_\mu\hat{\epsilon} &= \delta_{\hat{\epsilon}}\hat{\psi}_\mu = e^{m\Phi}\left(m\delta_{\hat{\epsilon}}\Phi\psi_\mu + \delta_{\hat{\epsilon}}\psi_\mu\right) = e^{m\Phi}\delta_{\hat{\epsilon}}\psi_\mu \\
\hat{D}_z\hat{\epsilon} &= \delta_{\hat{\epsilon}}\hat{\psi}_z = e^{n\Phi}\left(n\delta_{\hat{\epsilon}}\Phi\lambda + \delta_{\hat{\epsilon}}\lambda\right) = e^{n\Phi}\delta_{\hat{\epsilon}}\lambda
\end{aligned}
\tag{5.127}
$$

The last equalities follow from the fact that we assume the spinor fields to vanish. However the resulting variations will explicitly depend on the gauge-potential A. We therefore replace the original ansatz (5.121) with

$$
\begin{aligned}
\psi_\mu &= \hat{\psi}_\mu - x_2 e^a_\mu\eta_{ab}\Gamma^b\Gamma^{11}\hat{\psi}_z - x_3 A_\mu\hat{\psi}_z \\
\lambda &= x_1\hat{\psi}_z \\
\hat{\epsilon} &= e^{l\Phi}\epsilon
\end{aligned}
\tag{5.128}
$$

which amounts to a field redefinition in ten dimensions. If one was to work properly, one had to peform the dimensional reduction of the action as well in order to make sure that the fermion terms have the proper normalizations. The

SUSY variations of (5.128) are

$$
\begin{aligned}
\delta_\epsilon \psi_\mu &= \delta_\epsilon \hat{\psi}_\mu - x_2 e^a_\mu \eta_{ab} \Gamma^b \Gamma^{11} \delta_\epsilon \hat{\psi}_z - x_3 A_\mu \delta_\epsilon \hat{\psi}_z \\
&= e^{-l\Phi} \hat{D}_\mu \hat{\epsilon} - x_2 e^a_\mu \eta_{ab} \Gamma^b \Gamma^{11} e^{-l\Phi} \hat{D}_z \hat{\epsilon} - x_3 A_\mu e^{-l\Phi} \hat{D}_z \hat{\epsilon} \\
\delta_\epsilon \lambda &= x_1 e^{-l\Phi} \hat{D}_z \hat{\epsilon}
\end{aligned}
\tag{5.129}
$$

Note that the variations of the bosonic fields all vanish, as we have set the fermions explicitly to zero. Our aim is to compare (5.129) with the IIA Einstein frame SUSY variations as taken from [96]

$$
\delta\lambda = \frac{\sqrt{2}}{4} \partial_\mu \Phi \Gamma^\mu \Gamma^{11} \epsilon + \frac{3}{16} \frac{1}{\sqrt{2}} e^{\frac{3}{4}\Phi} \mathrm{d}A_{\mu_1\mu_2} \Gamma^{\mu_1\mu_2} \epsilon \tag{5.130a}
$$

$$
\delta\psi_\mu = D_\mu \epsilon + \frac{1}{64} e^{\frac{3}{4}\Phi} \mathrm{d}A_{\mu_1\mu_2} \left(\Gamma_\mu{}^{\mu_1\mu_2} - 14 \delta^{\mu_1}_\mu \Gamma^{\mu_2} \right) \Gamma^{11} \epsilon \tag{5.130b}
$$

Before evaluating (5.129), we calculate[8]

$$
\begin{aligned}
& x_2 e^a_\mu \eta_{ab} \Gamma^b \Gamma^{11} e^{-l\Phi} \hat{D}_z \hat{\epsilon} \\
&= x_2 \frac{1}{8} e^{\frac{3}{2}\Phi} e^a_\mu \eta_{ab} \mathrm{d}A_{cd} (\Gamma^{bcd} + 2\eta^{bc}\Gamma^d) \Gamma^{11} \epsilon \\
&- x_2 \frac{1}{3} e^{\frac{3}{4}\Phi} e^a_\mu \partial_a \Phi \epsilon - x_2 \frac{1}{6} e^{\frac{3}{4}\Phi} e^a_\mu (\eta_{ab} \partial_c \Phi - \eta_{ac} \partial_b \Phi) \Gamma^{bc} \epsilon \\
&- \frac{1}{2} x_2 e^a_\mu \eta_{ab} \hat{\tau}_{zaz} \epsilon - \frac{1}{2} x_2 e^a_\mu \eta_{ab} \hat{\tau}_{zcz} \Gamma^{bc} \epsilon + \frac{1}{4} x_2 e^a_\mu \hat{\tau}_{zcd} (\eta_{ab} \Gamma^{bcd} + 2\delta^c_a \Gamma^d) \Gamma^{11} \epsilon
\end{aligned}
\tag{5.131}
$$

[8] The following is used here:

$$
\begin{aligned}
\Gamma^a \Gamma^b &= \Gamma^{ab} + \eta^{ab} \\
\Gamma^a \Gamma^b \Gamma^c &= \Gamma^{abc} + \eta^{ab} \Gamma^c - \eta^{ca} \Gamma^b + \eta^{bc} \Gamma^a
\end{aligned}
$$

Putting things together, we use equations (5.126) and (5.129)

$$\begin{aligned}
\delta_\epsilon\psi_\mu &= (l\partial_\mu\Phi\epsilon + \partial_\mu\epsilon) + e^a_\mu\left[\frac{1}{4}\omega_{abc} + \frac{1}{48}(\eta_{ab}\partial_c\Phi - \eta_{ac}\partial_b\Phi)\right]\Gamma^{bc}\epsilon \\
&- \frac{1}{4}e^{\frac{3}{4}\Phi}e^a_\mu \mathrm{d}A_{ab}\Gamma^b\Gamma^{11}\epsilon \\
&- x_2\frac{1}{8}e^{\frac{3}{2}\Phi}e_\mu\eta_{ab}\mathrm{d}A_{cd}(\Gamma^{bcd} + 2\eta^{bc}\Gamma^d)\Gamma^{11}\epsilon \\
&+ x_2\frac{1}{3}e^{\frac{3}{4}\Phi}e^a_\mu\partial_a\Phi\epsilon + x_2\frac{1}{6}e^{\frac{3}{4}\Phi}e^a_\mu(\eta_{ab}\partial_c\Phi - \eta_{ac}\partial_b\Phi)\Gamma^{bc}\epsilon \\
&+ \left(\frac{1}{8}e^{\frac{3}{2}\Phi}A_\mu \mathrm{d}A_{bc}\Gamma^{bc} + \frac{1}{3}e^{\frac{3}{4}\Phi}A_\mu\partial_b\Phi\Gamma^b\Gamma^{11}\right)\epsilon \\
&- x_3\left(\frac{e^{\frac{3}{2}\Phi}}{8}e^a_\mu A_a \mathrm{d}A_{bc}\Gamma^{bc} + \frac{e^{\frac{3}{4}\Phi}}{3}e^a_\mu A_a\partial_b\Phi\Gamma^b\Gamma^{11}\right)\epsilon \\
&+ \frac{1}{4}\hat{\tau}_{\mu bc}\Gamma^{bc}\epsilon + \frac{1}{2}\hat{\tau}_{\mu bz}\Gamma^b\Gamma^{11}\epsilon \\
&+ \frac{1}{2}x_2 e^a_\mu\eta_{ab}\hat{\tau}_{zaz}\epsilon + \frac{1}{2}x_2 e^a_\mu\eta_{ab}\hat{\tau}_{zcz}\Gamma^{bc}\epsilon - \frac{1}{4}x_2 e^a_\mu\hat{\tau}_{zcd}(\eta_{ab}\Gamma^{bcd} + 2\delta^c_a\Gamma^d)\Gamma^{11}\epsilon \\
&- x_3 A_\mu(\frac{1}{4}\hat{\tau}_{zbc}\Gamma^{bc} + \frac{1}{2}\hat{\tau}_{zbz}\Gamma^b\Gamma^{11})\epsilon \\
\delta_\epsilon\lambda &= x_1\frac{e^{\frac{3}{2}\Phi}}{8}\mathrm{d}A_{bc}\Gamma^{bc}\epsilon + x_1\frac{e^{\frac{3}{4}\Phi}}{3}\partial_b\Phi\Gamma^b\Gamma^{11}\epsilon + x_1(\frac{1}{4}\hat{\tau}_{zbc}\Gamma^{bc} + \frac{1}{2}\hat{\tau}_{zbz}\Gamma^b\Gamma^{11})\epsilon
\end{aligned} \tag{5.132}$$

Investigating this and comparing with (5.130) one sets $l = \frac{1}{24}$ and

$$\begin{aligned}
x_1 &= \frac{3\sqrt{2}}{4}e^{-\frac{3}{4}\Phi} \\
x_2 &= -\frac{1}{8}e^{-\frac{3}{4}\Phi} \\
x_3 &= 1
\end{aligned} \tag{5.133}$$

to obtain the standard type IIA SUSY variations garnished with some additional torsion terms:

$$\begin{aligned}
\delta_\epsilon\psi_\mu &= \partial_\mu\epsilon + e^a_\mu\frac{1}{4}\omega_{abc}\Gamma^{bc}\epsilon + \frac{1}{64}e^{\frac{3}{4}\Phi}e^a_\mu \mathrm{d}A_{cd}\left(\eta_{ab}\Gamma^{bcd} - 14\delta^c_a\Gamma^d\right)\Gamma^{11}\epsilon \\
&+ \frac{1}{4}\hat{\tau}_{\mu bc}\Gamma^{bc}\epsilon + \frac{1}{2}\hat{\tau}_{\mu bz}\Gamma^b\Gamma^{11}\epsilon \\
&- \frac{1}{16}e^{-\frac{3}{4}\Phi}e^a_\mu\eta_{ab}\hat{\tau}_{zaz}\epsilon - \frac{1}{16}e^{-\frac{3}{4}\Phi}e^a_\mu\eta_{ab}\hat{\tau}_{zcz}\Gamma^{bc}\epsilon \\
&+ \frac{1}{32}e^{-\frac{3}{4}\Phi}e^a_\mu\hat{\tau}_{zcd}(\eta_{ab}\Gamma^{bcd} + 2\delta^c_a\Gamma^d)\Gamma^{11}\epsilon \\
&- A_\mu(\frac{1}{4}\hat{\tau}_{zbc}\Gamma^{bc} + \frac{1}{2}\hat{\tau}_{zbz}\Gamma^b\Gamma^{11})\epsilon \\
\delta_\epsilon\lambda &= \frac{3}{16}\frac{1}{\sqrt{2}}e^{\frac{3}{4}\Phi}(\mathrm{d}A_{bc} + 2e^{-\frac{3}{2}\Phi}\hat{\tau}_{zbc})\Gamma^{bc}\epsilon + \frac{\sqrt{2}}{4}(\partial_b\Phi + \frac{3}{2}e^{-\frac{3}{4}\Phi}\hat{\tau}_{zbz})\Gamma^b\Gamma^{11}\epsilon
\end{aligned} \tag{5.134}$$

Chapter 6

The duality at finite temperature

In the final chapter of this thesis, we shall now turn to the study of the quark-gluon plasma (QGP) as produced in relativistic heavy ion collisions [97, 98, 99, 100]. The QGP is also relevant to early universe cosmology. Among the items that have been studied using a gravity dual are the plasma's shear viscosity [47], photoproduction [101], jet-quenching [102], and drag force [103].[1]

A large portion of the research conducted in this area centers on $\mathcal{N} = 4$ super Yang-Mills and AdS/CFT in its best understood form, D3-branes in type IIB theory. Apart from the fact that this is the most tractable of gravity duals, one reason for choosing $\mathcal{N} = 4$ is that albeit having properties very different from those of QCD at $T = 0$, the two theories start to appear more and more similiar as soon as there is finite temperature. Despite these successes however a complete study of QGP physics based on string theory demands for an investigation of the $T \neq 0$ behavior of other gravity duals showing a stronger resemblance to QCD even at zero temperature. Conversely (qualitative) comparison with experimental data is also an excelent test for new proposed dualities. Some work in this direction was undertaken in [107, 108, 109, 110, 111]

In this chapter, which is based on [89], we will investigate the possibility of constructing a supergravity background dual to an $\mathcal{N} = 1$ QGP based on D6-branes wrapping an S^3 in the deformed conifold. In other words, we will be trying to generalize the the backgrounds studied in chapter 5 to describe gauge

[1] A review on the uses of gauge/string duality and QGP physics is [104]. The general properties of the plasma in general and RHIC physics are summarized in [105] and [106].

theories at finite temperature, but in the absence of fundamental matter.

If one wants to use this gravity dual to study the QGP, one needs to add a black hole to the supergravity background. As the theory is purely gravitational when lifting to eleven dimensions, the equations of motion take the simplest form possible here,

$$R_{\mu\nu} = 0, \tag{6.1}$$

making this the best place to perform the search for a black hole solution. As we find in section 6.3, if one wants to keep the ansatz for the new metric as simple as possible by making the substitutions

$$\mathrm{d}t^2 \to f(\rho)\mathrm{d}t^2 \qquad\qquad \mathrm{d}\rho^2 \to \frac{\mathrm{d}\rho^2}{f(\rho)}, \tag{6.2}$$

there is a non-trivial solution if and only if one makes the geometry of the G_2 manifold singular. The unique solution is then $f = 1 - \rho_h^5/\rho^5$, where the singularity at $\rho = 0$ is hidden by the horizon $\rho_h > 0$. When studying the thermodynamics of this new solution, we will see that the black hole behaves in many ways as the ordinary Schwarzschild solutions in four and eleven dimensions. I.e. the temperature is proportional to the inverse of the horizon, $T = \frac{5}{4\pi\rho_h}$, and the specific heat is negative. As the horizon of the black hole covers the six-dimensional base of the internal G_2 cone, the entropy behaves as $S \propto \rho_h^6$, leading to the surprising relation $S \propto T^{-6}$. While our subsequent calculation of the quark-antiquark potential and the shear-viscosity show the expected results, that is confinement and a shear-viscosity to entropy ratio of $\eta/s = 1/4\pi$, the discussion of parton energy loss leads to a puzzling pathological property of the solution. The energy loss as calculated from the jet-quenching parameter and the damping coefficient of the drag force are both vanishing.

Sections 6.1 and 6.2 are dedicated to an extensive review of the string theory and its gauge dual at zero temperature. Here our discussion starts with eleven-dimensional supergravity and then proceeds via type IIA to the four-dimensional super Yang-Mills theory. As we will make extensive use of the machinery of Wilson lines, we shall give a brief introduction to this subject before calculating the quark-antiquark potential, paying special attention to the boundary conditions imposed on worldsheets used to calculate Wilson lines. After these preliminaries we finally turn to the subject of finite temperature. The discussion mimicks that of the $T = 0$ case in that we will start from the eleven-dimensional gravity dual (section 6.3) and then progress via type IIA to the gauge-theory (section 6.4). Here we study the quark-antiquark potential, the shear-viscosity, and parton energy loss as it is parametrized by the jet-quenching factor $\hat{q}$ and the drag-force. There is an appendix with basics on the bundle structure of S^3 6.A.

6.1 The supergravity dual at zero temperature

Starting point is again the Bryant-Salamon metric (5.20) that we encountered already in chapter 5. In this chapter we will of course be working towards the possibility of a dual to pure super Yang-Mills at finite temperature without fundamental matter rather than the flavoring problem. Recall that depending on the energy scale one needs to work either in eleven-dimensional M- or ten-dimensional type IIA string theory.

6.1.1 M-theory on the G_2 holonomy manifold

We change conventions and notations slightly compared to chapter 5 and write the metric as

$$\mathrm{d}s_M^2 = \mathrm{d}x_{1,3}^2 + \frac{\mathrm{d}\rho^2}{1 - a^3/\rho^3} + \frac{\rho^2}{12}\tilde{w}^{a2} + \frac{\rho^2}{9}\left(1 - \frac{a^3}{\rho^3}\right)\left(w^a - \frac{1}{2}\tilde{w}^a\right)^2 . \tag{6.3}$$

with Maurer-Cartan forms

$$\begin{aligned} w^1 &= \cos\phi \mathrm{d}\theta + \sin\theta \sin\phi \mathrm{d}\psi \\ w^2 &= \sin\phi \mathrm{d}\theta - \sin\theta \cos\phi \mathrm{d}\psi \\ w^3 &= \mathrm{d}\phi + cos\theta \mathrm{d}\psi , \end{aligned} \tag{6.4}$$

and

$$\theta \in [0, \pi] \quad \phi \in [0, 2\pi] \quad \psi \in [0, 4\pi] . \tag{6.5}$$

We have also relabeled the resolution parameter $\rho_0 \equiv a$.

Let us recall a few further facts from our discussion in section 5.1.1. The gauge-theory we are interested in is living on the world volume of N D6-branes wrapping a calibrated three-cycle in the deformed conifold. The UV completion of this theory is given by M-theory on (6.3). Here there are two S^3, parametrized by $w, \tilde{w}$. In opposite to S^3, $\tilde{S}^3$ has a finite radius a as we take $\rho \to a$, resolving the singularity at $\rho = a$. One could also have picked the other sphere, S^3, to do this, and the flop transition of [45] provides the duality between these two cases.

6.1.2 Wrapped D6-branes in type IIA string theory

There are three $U(1)$ isometries along which we can reduce to type IIA when flowing towards the IR. This leads to an effective description in terms of type IIA string theory on a space with topology

$$\mathbb{R}^{1,3} \times \mathbb{R}_+ \times S^3 \times S^2. \tag{6.6}$$

If we choose an S^1 in the singular three-sphere, $S^1 \subset S^3$, the resulting geometry is a singular S^2 and a non-singular S^3 known as the deformed conifold. See fig. 6.1(a). The converse case, the resolved conifold, is depicted in fig. 6.1(c). See [112] for a discussion of the conifold. As depicted in fig. 6.1(b), there is a singularity at which both spheres have a vanishing radius. From a mathematician's point of view one deals with this singularity by giving one of the spheres a finite radius, leading to the deformed and the resolved conifold. Physics allows for the following interpretation of this[2] [45, 113, 85]: If one considers the singularity as the $a \to 0$ limit of the deformed conifold, there is a logarithmic singularity in the metric. This may be interpreted as the result of having integrated out a field whose mass is dependent on a, $m = m(a)$. When approaching the singularity,

$$m(a) \to 0 \qquad \text{as} \qquad a \to 0. \tag{6.7}$$

Therefore the physical interpretation of the singularity lies in the fact that one has attempted to integrate out a massless field. As we will see in section 6.3.1 however, the finite temperature theory makes use of another method of dealing with the singularity. The theory will be defined on the singular conifold with the singularity hidden behind a black hole's event horizon.

The string theory equivalent of the flop transition is the conifold transition[44]. It relates the two geometries via a large N duality. For small 't Hooft coupling

$$\lambda = N g_{\mathrm{YM}}^2 = N g_s \ll 1, \tag{6.8}$$

[2]The interpretation of the singularity in terms of having integrated out a massless field appears in [113] for the generic case of string theory on a deformed conifold. We did not explicitly verify that it holds in our case. See also [85].

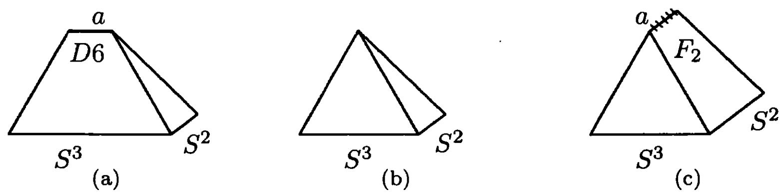

Figure 6.1: The deformed 6.1(a), singular 6.1(b), and resolved 6.1(c) conifold. In the type IIA theory discussed in section 6.1.2, there are N D6-branes wrapping the non-vanishing S^3 in 6.1(a), while in in the dual geometry 6.1(c) the branes have disappeared and been replaced by a two-form flux F_2.

one considers a stack of N D6-branes wrapping the non-singular S^3 in the deformed conifold. Taking the 't Hooft coupling large on the other hand one cannot neglect the branes' backreaction and does therefore pass to the resolved conifold. Here the branes have disappeared and been replaced by N units of two-form flux through the now blown up S^2.

Being interested in a strongly coupled quark-gluon plasma, we choose to reduce along the non-singular $S^1 \subset \tilde{S}^3$. Before doing so, we have to identify the S^1 fibre along which we want to reduce. A generic three-sphere may be written as

$$S^3 \equiv \{(z_0, z_1) \in \mathbb{C}^2 | |z_0|^2 + |z_1|^2 = 1\}. \tag{6.9}$$

The coordinates $z_{0,1}$ are related to those of (6.4) by

$$\begin{aligned} z_0 &= \cos\frac{\tilde{\theta}}{2} e^{\imath\frac{\tilde{\psi}+\tilde{\phi}}{2}} \\ z_1 &= \imath \sin\frac{\tilde{\theta}}{2} e^{\imath\frac{\tilde{\psi}-\tilde{\phi}}{2}}. \end{aligned} \tag{6.10}$$

(6.155) tells us, that the projection $S^3 \xrightarrow{\pi} S^2$ acts on this as

$$\begin{aligned} &-\imath \cot\frac{\tilde{\theta}}{2} e^{\imath 2\tilde{\phi}} && \tilde{\theta} \neq 0 \\ &\imath \tan\frac{\tilde{\theta}}{2} e^{-\imath 2\tilde{\phi}} && \tilde{\theta} \neq \pi, \end{aligned} \tag{6.11}$$

depending on the coordinate patch. One sees immediately that the fibre coordinate is $\tilde{\psi}$, as it does not survive the projection.

Before actually reducing we mod out by

$$\mathbb{Z}_N \subset S^1 \subset \tilde{S}^3. \tag{6.12}$$

This means a change in the periodicity of $\tilde{\psi}$,

$$\begin{aligned}\tilde{\psi} \in [0, 2\pi] \to \tilde{\psi} \in [0, 2\pi/N]\,, \\ \mathrm{d}\tilde{\psi} \to \frac{\mathrm{d}\tilde{\psi}}{N}.\end{aligned} \tag{6.13}$$

As we will see soon, N gives the F_2 flux through $\tilde{S}^2$ and therefore the number of D6-branes present in the dual type IIA geometry.

In order to perform the reduction, we could simply expand the metric. However, there is a smarter way to go about this. Defining

$$n^a \equiv \tilde{w}^a \left(\partial_{\tilde{\psi}}\right) = \left(\sin\tilde{\theta}\sin\tilde{\phi}, -\sin\tilde{\theta}\cos\tilde{\phi}, \cos\tilde{\theta}\right), \tag{6.14}$$

we may rewrite the metric (6.3) in terms of a new set of differential forms $\hat{w}^a$ independent of $\mathrm{d}\tilde{\psi}$,

$$\tilde{w}^a = \hat{w}^a + n^a \frac{\mathrm{d}\tilde{\psi}}{N}. \tag{6.15}$$

With $\beta \equiv 1 - a^3/\rho^3$ we obtain

$$\begin{aligned}\mathrm{d}s^2_{\mathrm{M}} = {} & \mathrm{d}x^2_{1,3} + \frac{\mathrm{d}\rho^2}{\beta} + \frac{\rho^2(3+\beta)}{36}\hat{w}^2 + \frac{\rho^2}{9}\beta w^2 - \frac{\rho^2}{9}\beta w.\hat{w} \\ & + \underbrace{\frac{\rho^2(3+\beta)}{36N^2a^2}}_{e^{\frac{4\Phi}{3}}} a^2 \mathrm{d}\tilde{\psi}^2 + \underbrace{\left(\frac{2\rho^2}{12Na}n.\hat{w} + \frac{\rho^2}{18Na}\beta n.\hat{w} - \frac{\rho^2}{9Na}\beta n.w\right)}_{2e^{\frac{4\Phi}{3}}A_{(1)}} a\mathrm{d}\tilde{\psi}\end{aligned} \tag{6.16}$$

We included several factors of a to make sure that everything has the correct dimensions. Dimensional reduction along an S^1 yields apart from the new metric $g_{\mu\nu}$ a one-form potential and the dilaton.

$$e^{\frac{4\Phi}{3}} = \frac{\rho^2(3+\beta)}{36N^2a^2} \tag{6.17}$$

$$A_{(1)} = Na\left(\hat{w}.n - \frac{2\beta}{3+\beta}w.n\right) \tag{6.18}$$

$$\begin{aligned}\mathrm{d}s^2_{IIA} = e^{\frac{2}{3}\Phi}\Bigg(& \mathrm{d}x^2_{1,3} + \frac{\mathrm{d}\rho^2}{\beta} + \frac{\rho^2(3+\beta)}{36}\hat{w}^2 + \frac{\rho^2}{9}\beta w^2 \\ & -\frac{\rho^2}{9}\beta w.\hat{w} - e^{\frac{4}{3}\Phi}A_{(1)}A_{(1)}\Bigg).\end{aligned} \tag{6.19}$$

We will also need the ten-dimensional Ricci scalar. In the string frame it reads

$$R = -9aN\frac{832\rho^9 - 240a^3\rho^6 + 63a^6\rho^3 - 7a^9}{2\sqrt{4 - \frac{a^3}{\rho^3}}\rho^6\left(4\rho^3 - a^3\right)^2}. \tag{6.20}$$

R is not singular at $\rho = a$. As a matter of fact,

$$R|_{\rho=a} = -108\sqrt{3}\frac{N}{a^2} \tag{6.21}$$

which gives us an explicit expression for the conifold singularity in the limit $a \to 0$.

We claimed that in the above geometry there are N units of Ramond-Ramond flux through the two-sphere. To check this we simply calculate the $F_{(2)}$ flux through the S^2 parametrized by $\tilde{\theta}$ and $\tilde{\phi}$.

$$\int_{S^2} *F_{(8)} = \int_{S^2} ** F_{(2)} = -\int_{S^2} dA_{(1)} = 4\pi N a \tag{6.22}$$

Now the conifold transition relates the above to a stack of N D6-branes on the deformed conifold. One may obtain this dual geometry from eleven-dimensional supergravity by reducing along the singular three-sphere. Indications towards the presence of the branes are the resulting one-form potential, which couples magnetically to the branes, and the behavior of the Ricci scalar near the singularity. See [31, 51].

6.2 The gauge theory at zero temperature

We shall now turn to the discussion of the dual gauge theory at $T = 0$. With the exception of the Yang-Mills coupling in section 6.2.1 and the $q\bar{q}$-potential in section 6.2.5 this section contains mostly review material. The relation between the supergravity backgrounds, the gauge theory, and gauged supergravity was exhibited in [31]. For a review on this issue see [114].

6.2.1 The coupling constant of the gauge theory

In the following we elaborate on the developments in [115, 77]. To find the super Yang-Mills theory's coupling constant g_{YM}, we place a D6-probe brane at constant ρ, extending along x^μ and wrapping the resolved conifold's S^3. Recall that we may think of our original stack of D6-branes as wrapping $\tilde{S}^3$ in the deformed conifold. We also fix the brane's position in the S^2 to be $\tilde{\theta} = \tilde{\phi} = 0$. The general idea is to identify the gauge field living on the probe brane with that of the dual super Yang-Mills theory. Thus we may extract information about the dual theory from the probe's DBI action. Using world-volume coordinates

ξ^a and labeling the brane-tension T_6, we expand the DBI action in powers of α'

$$\begin{aligned} S_{DBI} &= -T_6 \int d^7\xi e^{-\Phi} \sqrt{-\det X^*[g] + 2\pi\alpha' F} + T_6 \int \sum_n C_{(n)} \wedge e^{2\pi F} \\ &= -T_6 \int d^7\xi e^{-\Phi} \sqrt{-\det X^*[g]} \left(1 + (\alpha'\pi)^2 F^2\right) + \mathcal{O}(\alpha')^3 + \dots \end{aligned} \tag{6.23}$$

X^* denotes the pullback onto the brane. For the embedding we have chosen, the induced metric $X^*[g]$ is

$$\mathrm{d}s_6^2 = e^{\frac{2}{3}\Phi} \left(\mathrm{d}x_{1,3}^2 + \frac{\rho^2}{9}\beta w^2 - e^{\frac{4}{3}\Phi} N^2 a^2 \left(\frac{2\beta}{3+\beta}\right)^2 \left(w^3\right)^2 \right) \tag{6.24}$$

Now notice that after Kaluza-Klein decomposition the massless modes of $F_{\mu\nu}$ are functions of the x^μ alone, while all the other terms in (6.23) do not depend on the flat part of the world-volume. Therefore that part of (6.23) containing F^2 may be written as

$$-\left(T_6(\pi\alpha')^2 \int \mathrm{d}\theta \mathrm{d}\phi \mathrm{d}\psi e^{-\Phi} \sqrt{-\det X^*[g]}\right) \int d^4x F^2. \tag{6.25}$$

Comparing the Yang-Mills action

$$S_{YM} = -\frac{1}{4g_{YM}^2} \int d^4x F^2 + \frac{\theta_{YM}}{32\pi^2} \int d^4x F\tilde{F}, \tag{6.26}$$

and using the explicit expression for the D-brane tension

$$T_p = \frac{1}{(2\pi)^p \alpha'^{\frac{p+1}{2}}} \tag{6.27}$$

we obtain

$$g_{\mathrm{YM}} = 18(12)^{\frac{1}{4}} \frac{Na\pi\sqrt{\alpha'^{9/2}\rho}}{((4\rho^3 - a^3)(\rho^3 - a^3)^3)^{1/4}}. \tag{6.28}$$

Note that the coupling is dimensionless, as it should be the case for a four-dimensional Yang-Mills theory. We have plotted g_{YM} in figure 6.2. The AdS/CFT dictionary tells us that we may relate the radial coordinate ρ to the energy scale. To obtain a precise relation one may consider chiral symmetry breaking and the vev of the gluino condensate $\langle\lambda\lambda\rangle$ [115]. Yet for our purposes it is sufficient to think of $\rho \to \infty$ as the UV regime of the gauge theory and $\rho \to a$ as the IR.[3] Then (6.28) clearly shows asymptotic freedom.

[3] As we mentioned earlier, the UV completion is given by M-theory, while in the infrared the relevant degrees of freedom are best described by the gauge theory. See section 6.2.3 and [4].

6.2.2 Field Content

We shall take a look at the massless excitations. Prior to wrapping, the theory living on the world volume of N D6-branes is a super Yang-Mills theory with 16 supercharges, as the branes are half-BPS. Upon wrapping, the global symmetries break as

$$SO(1,6) \times SO_R(3) \rightarrow SO(1,3) \times SO(3) \times SO_R(3). \tag{6.29}$$

From dimensional analysis it follows that the Kaluza-Klein modes become relevant at energy scales of order

$$\Lambda_{\mathrm{KK}} \sim \frac{\alpha'^{3/2}}{\mathrm{Vol}\, S^3} = \frac{\alpha'^{3/2}}{2\pi^2 a^3}. \tag{6.30}$$

Ignoring all massive modes, the bosonic sector includes now the gauge potential and three massless scalars transforming as a $\mathbf{3}$ under the R-symmetry. The representation for the fermions changes under (6.29) from $(\mathbf{8},\mathbf{2})$ to $(\mathbf{4},\mathbf{2},\mathbf{2})$.

This is not the complete picture however. Consider the behavior of the gravitino under SUSY transformations,

$$\delta_\epsilon \Psi_\mu|_{\Psi=0} = \nabla_\mu \epsilon = \left(\partial_\mu + \frac{1}{2}\omega_\mu\right)\epsilon, \tag{6.31}$$

with ω being the spin connection. For the theory to be supersymmetric we need a covariantly constant spinor satisfying $\nabla_\mu \epsilon = 0$. As the spin structure on S^3 does not allow for such a spinor to exist, supersymmetry is completely broken upon wrapping. Raising the status of the R-symmetry to that of a gauge symmetry, we may modify (6.31) to

$$\nabla_\mu \epsilon = \left(\partial_\mu + \frac{1}{2}\omega_\mu + A_\mu^{(R)}\right)\epsilon. \tag{6.32}$$

Fixing $A_\mu^{(R)} = 2\omega_\mu$ resolves the issue. This topological twist was first introduced by Witten in [116]. While it changes the behavior of the 6+1 dimensional theory significantly, the consequence for the 3 + 1 dimensional one we are interested in consists in keeping only those fields that transform as a singlet under the diagonal

$$SO(3) \times SO_R(3) \rightarrow SO_D(3). \tag{6.33}$$

The gauge potential is not affected by the whole construction, whereas all of the scalars disappear from the spectrum. The representation of the fermions decomposes as

$$(\mathbf{4},\mathbf{2},\mathbf{2}) \to (\mathbf{4},\mathbf{1}) \oplus (\mathbf{4},\mathbf{3}), \tag{6.34}$$

because $\mathbf{2}\times\mathbf{2} = \mathbf{1}\oplus\mathbf{3}$. So recalling that the branes are half-BPS we are left with $\frac{1}{2}\times\frac{1}{4}\times 32 = 4$ supercharges, confirming the previous calculation based on the holonomy of the eleven-dimensional background. Thus the massless spectrum is given by pure $\mathcal{N}=1$ super Yang-Mills.

6.2.3 The gauge/gravity correspondence

Knowing the energy scale of the KK-modes (6.30) and the behavior of the Ricci scalar (6.20), the Yang-Mills coupling constant (6.28), and the dilaton (6.17) enables us to address the issue at which energy scales the system is best described by super Yang-Mills, type IIA, or M-theory. As in the previous section we do not know the precise relation between the radial coordinate ρ and the energy scale μ in question, and are therefore only able to make qualitative statements identifying the large-ρ regime as the UV and vice versa. Figure 6.2 shows the behavior of all three relevant quantities.

We see that in the IR the relevant degrees of freedom are best described in type IIA theory. While it might seem that the UV completion is given by both M-theory and super Yang-Mills one should not forget that figure 6.2 shows the four-dimensional gauge coupling. At sufficiently high energies the $\mathcal{N}=1$ theory will begin to fully explore the compact dimensions; the gauge theory becomes $6+1$ dimensional. Purely gravitational M-theory gives the only UV-completion.

If we want to use this overall setup to study zero-temperature, non-perturbative gauge dynamics, it follows from (6.30) that we want the resolution parameter a to satisfy $a < \sqrt{\alpha'}$. However we also need

$$\alpha' R \ll 1 \qquad \lambda = g_{\mathrm{YM}}^2 N > 1 \qquad e^{\Phi} \ll 1. \tag{6.35}$$

For $\rho \to a$, these quantities behave as

$$\begin{aligned}
\lambda &\sim \frac{N^3 a^{3/2} \alpha'^{3/2}}{(\rho^3 - a^3)^{3/2}} \\
-\alpha' R &\leq \sqrt{3}\, 108 \frac{N\alpha'}{a^2} \\
e^{\Phi} &= \frac{1}{\sqrt{2}N^{3/2}}\left(\frac{1}{2\, 3^{3/4}} + \frac{3^{5/4}}{8a}(\rho - a)\right) + \mathcal{O}(\rho - a)^2
\end{aligned} \tag{6.36}$$

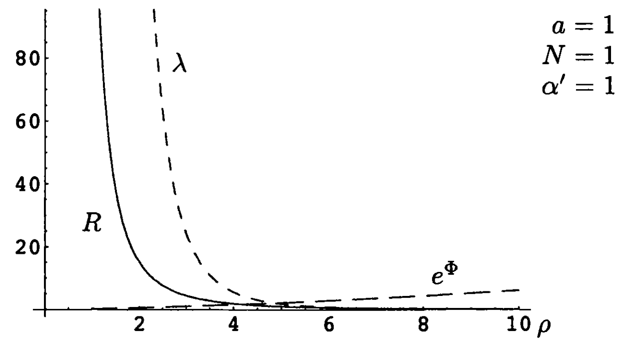

Figure 6.2: Ricci scalar, 't Hooft coupling, and dilaton in terms of ρ. One sees clearly that the IR phsics is captured by type IIA string theory while the UV completion is given by M-theory on the G_2 holonomy manifold. Note that despite appearances R is not singular at $\rho = a$. The 't Hooft coupling however is.

Comparing this with figure 6.2 we conclude that there is a limit for N, a, α' in which the supergravity approximation captures non-perturbative gauge dynamics. However the massive Kaluza-Klein modes do not fully decouple and thus spoil the behavior of pure $\mathcal{N} = 1$ super Yang-Mills. If one were able to perform computations beyond the supergravity limit one could easily avoid this issue.

6.2.4 Wilson loops and minimal surfaces

The AdS/CFT-correspondence is a powerful tool for the study of Wilson lines [117], [118], and [119]. In the next section (6.2.5) we shall use it to study the $q\bar{q}$-potential at $T = 0$. Further applications will be the finite-temperature $q\bar{q}$-potential and the jet-quenching factor in sections 6.4.2 and 6.4.4 respectively, while the the method used to compute the drag-force in section 6.4.4 takes a similiar approach.

For a generic gauge theory a Wilson loop is defined as[4]

$$W(\mathcal{C}) = \mathcal{P}e^{\imath \oint_{\mathcal{C}} \mathrm{d}A}. \tag{6.38}$$

$\mathcal{P}$ denotes path ordering and $\mathcal{C}$ the contour of integration.

[4]The expression presented here is not entirely generic. E.g. for $d = 4, \mathcal{N} = 4$ super Yang Mills whose gravity dual is defined on $AdS_5 \times S^5$, one needs also to consider scalar fields Φ^I. The index I may be considered as a representation index of $SO(6)$. The Wilson line is given by

$$W^A(\mathcal{C}) = \mathcal{P}e^{\imath \oint_{\mathcal{C}} \mathrm{d}s(\dot{x}^\mu A_\mu + |\dot{x}| n^I \Phi^I)}. \tag{6.37}$$

However, as (6.38) is entirely sufficient in the context presented here, we shall not elaborate on the issue.

To see how to calculate the expectation value $\langle W(\mathcal{C})\rangle$ for a generic contour $\mathcal{C}$ using the AdS/CFT-correspondence, consider the following. If we do not close the loop $\mathcal{C}$, but instead consider a line, (6.38) is a non-local operator transforming at it's endpoints under the fundamental- and anti-fundamental representation respectively. The gauge theory and its gravity dual as discussed above are free of any fundamental degrees of freedom. In order to introduce these we start with a stack of $N+1$ D6-branes and place one of them at a large yet finite radius ρ_Λ. The gauge symmetry is broken as

$$SU(N+1) \to SU(N) \times U(1). \tag{6.39}$$

We have Higgsed the theory. From the point of view of the gauge theory we therefore expect the appearance of massive W-bosons, which we will treat as highly massive probe quarks. In the string theory these bosons are realized by strings stretching between the stack of branes and the separated one transforming in the (anti-)fundamental representation of the two new gauge groups. The new $U(1)$ gauge field may be ignored as it's living on the brane which is at a large separation from the stack of D6s.[5] When taking the decoupling limit the N branes at $\rho = 0$ are replaced by the background geometry while the single brane at ρ_Λ may be treated as a probe. As the branes are replaced by their geometry, the correct way for the W-bosons to interact with the gauge theory is not by ending on the branes but by interacting with the background. Therefore one evaluates $\langle W(\mathcal{C})\rangle$ by embedding the contour $\mathcal{C}$ into the probe brane and using it as a boundary condition for the worldsheets of open-strings exploring the bulk. See figure 6.3(a). The AdS/CFT-dictionary tells us then to calculate the expectation value of the Wilson loop for the adjoint representation by minimizing the Nambu-Goto action for the corresponding world-sheets,

$$\langle W^A(\mathcal{C})\rangle = \lim e^{-\mathcal{S}_{NG}}. \tag{6.40}$$

$\mathcal{S}_{NG}$ is the Nambu-Goto action

$$\begin{aligned} \mathcal{S}_{\mathrm{NG}} &= \frac{1}{2\pi\alpha'} \int \mathrm{d}\tau \mathrm{d}\sigma \sqrt{-\det \partial_\alpha X^\mu \partial_\beta X_\mu} \\ &= \frac{1}{2\pi\alpha'} \int \mathrm{d}\tau \mathrm{d}\sigma \sqrt{-\dot{X}^2 X'^2 + (\dot{X}.X')^2} \end{aligned} \tag{6.41}$$

[5] An alternative approach would be to take the flavor brane to wrap the S^2 and to extend along ρ from ρ_Λ to ∞. In this case one argues that the gauge-theory living on the probe is non-dynamical as seen from the four-dimensional theory as the probe wraps a non-compact dimension.

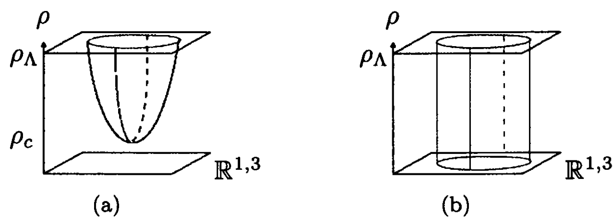

Figure 6.3: A Wilson 6.3(a) loop in the gauge theory is evaluated by using the loop as the boundary condition of a worldsheet ending on a probe brane. The worldsheet reaches a minimum at $\rho = \rho_c \geq 0$. The action is renormalised by that of strings stretching straight from the loop to the bottom of the space, sometimes given by the horizon of a black hole 6.3(b). As was argued in [120], one also needs to consider strings stretching from the probe away from the horizon.

While one usually takes the limit $\rho_\Lambda \to \infty$, one may also keep ρ_Λ finite and consider it as the energy the gauge theory is defined at. Note that the prescription given in (6.40) requires some sort of renormalization, usually given by the mass of the W bosons. This again is calculated from the action of a string stretching directly from the contour on the D7-brane to the bottom of the space as depicted in fig. 6.3(b). Note that this configuration is not physical, as it is not possible to define suitable boundary conditions at $\rho = a$. This will change in section 6.4.2, where we shall be considering the finite-temperature theory. Finite temperature is achieved by the presence of a black hole who's horizon gives suitable boundary conditions for the worldsheet in fig. 6.3(b) to be considered physical.

Boundary conditions There is a crucial aspect of (6.40) that appears to be frequently overlooked.[6] If we force the string to end on the contour $\mathcal{C}$, the resulting boundary conditions in at least some of the directions tangential to the brane are not von Neumann, but Dirichlet. One needs to ask for the object that restricts the string to lie on the contour.

As it is the easiest to understand this in terms of specific examples we shall delay explicit calculations to sections 6.2.5, 6.4.2, and 6.4.4. The technical aspects for all of these will be the same however, which is why we shall discuss them now.

Consider the Nambu-Goto action (6.41). It has a symmetry under transla-

[6]See however [120].

tions

$$X^\mu \to X^\mu + Y^\mu \qquad Y^\mu = \text{const.}, \tag{6.42}$$

which we know from ordinary classical mechanics to be related to energy-momentum conservation in space-time. Specialising to infinitesimal transformations, we can calculate the conserved current with the Noether prescription. As an intermediate result we obtain

$$2\pi\alpha' j^\alpha_\mu = \frac{\partial \mathcal{L}}{\partial \partial_\alpha X^\mu} = g_{\mu\nu} \begin{cases} \frac{-\dot{X}^\nu X'^2 + X'^\nu \dot{X}.X'}{\sqrt{-\dot{X}^2 X'^2 + (\dot{X}.X')^2}} & \alpha = \tau \\ \frac{-\dot{X}^2 X'^\nu + \dot{X}^\nu \dot{X}.X'}{\sqrt{-\dot{X}^2 X'^2 + (\dot{X}.X')^2}} & \alpha = \sigma \end{cases} \tag{6.43}$$

$j^\tau_{\bar{\mu}}$ gives the energy ($\bar{\mu} = 0$) or $\bar{\mu} = \bar{m}$-momentum density on the string. $j^\sigma_{\bar{\mu}}$ on the other hand denotes the flux of energy or momentum along the string. Thus we can calculate the total energy and momentum to be

$$E = \int \mathrm{d}\sigma j^\tau_0 \tag{6.44}$$

$$P_{\bar{m}} = \int \mathrm{d}\sigma j^\tau_{\bar{m}}. \tag{6.45}$$

The fluxes are related to an open string's boundary conditions. A string satisfying von Neumann boundary conditions does not allow for momentum to flow off the string, requiring

$$j^\sigma_{\bar{\mu}}\big|_{\text{boundary}} = 0. \tag{6.46}$$

The solution of the issue of defining Dirichlet boundary conditions in directions tangent to a brane will be turning on $U(1)$ gauge fields on the brane whose interaction with the string endpoints will exactly cancel the energy-momentum flow defined by these equations. The authors of [120] pointed out that as long as one keeps the position of the probe brane ρ_Λ finite, it is more sensible to think of a constant force of the $U(1)$ field on the string's endpoints rather than of a constant separation L separating them.

6.2.5 The quark-antiquark potential & confinement

Our first application of the concepts introduced in section 6.2.4 shall be the $q\bar{q}$-potential in the zero-temperature gauge theory. We follow [121]. Conceptually one studies this by placing two infinitively heavy and therefore static probe-quarks at a fixed separation L into the gauge theory. For such a configuration,

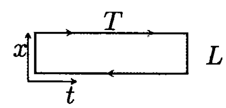

Figure 6.4: The rectangular Wilson loop used in section 6.2.5 as seen in the (t, x)-plane.

the action is independent of the time-like extension of the loop and therefore behaves as $S = ET$, with E the energy of the system.

Now if the gauge theory is confining, the energy is proportional to L from which it follows that

$$E(L) \propto L \quad \Rightarrow \quad S \propto LT. \tag{6.47}$$

LT is the area surrounded by such a Wilson loop, so that for a confining theory we expect the action for the quark loop to satisfy an area law.[7] In the following we shall study the $q\bar{q}$-potential of our gauge dual and whether it exhibits confinement.

The profile In this section we will use the static Wilson loop shown in fig. 6.4. Fixing $x \equiv x^2$, we may parametrize the loop and the corresponding worldsheet as

$$x^0 = \tau \qquad\qquad x = \sigma \qquad\qquad \rho = \rho(\sigma) \tag{6.48}$$

where $\tau \in [0, T]$ and $\sigma \in \left[-\frac{L}{2}, \frac{L}{2}\right]$. Also we will need to impose the boundary conditions

$$\rho\left(\sigma = \pm L/2\right) = \rho_\Lambda. \tag{6.49}$$

Note that the parametrization (6.48) does not define a complete Wilson loop but two Wilson lines separated by a distance L. Assuming $T \gg L$ however we may neglect the contribution from the pieces needed to close of the loop. Upon plugging (6.48) into the Nambu-Goto action (6.41) one notices immediately that

[7]Technically (6.47) shows only that confinement leads to an area law. We are reversing the argument simply claiming that the converse is also true, i.e. that confinement occurs iff the action satisfies an area law. The relation between confinement and an area law for the Wilson loop was first discussed in [122].

the integration over τ is trivial giving an overall factor of T,

$$S_{\text{NG}} = \frac{T}{2\pi\alpha'} \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}\sigma \underbrace{\sqrt{g_{tt}\left(g_{xx} + \rho'^2 g_{\rho\rho}\right)}}_{\mathcal{L}}. \tag{6.50}$$

The idea is to treat this formally as a system from classical mechanics with Lagrangian $\mathcal{L}(\sigma)$. With σ playing the role one would usually associate with time t and identifying $\rho(\sigma)$ as the system's time coordinate, one calculates the canonical momentum π and performs a Legendre transformation

$$\begin{aligned} \pi &= \frac{\partial \mathcal{L}}{\partial \rho'} \\ \mathcal{H} &= \rho'\pi - \mathcal{L} = \frac{-g_{xx}g_{tt}}{\sqrt{g_{tt}\left(g_{xx} + \rho'^2 g_{\rho\rho}\right)}}. \end{aligned} \tag{6.51}$$

From $\frac{\partial \mathcal{H}}{\partial \sigma} = 0$ it follows with Hamilton's equations that $\frac{d\mathcal{H}}{d\sigma} = 0$. Hence there is a conserved quantity

$$\mathcal{H} \equiv \kappa \in \mathbb{R}. \tag{6.52}$$

It might seem surprising that we emphasize that κ is real. However we will encounter examples where this is not the case. As

$$-g_{tt}g_{xx} = e^{\frac{4}{3}\Phi} \in \left[(12N^2)^{-1}, \infty\right) \xrightarrow{N\to\infty} [0,\infty), \tag{6.53}$$

there exists $\rho_c \geq a$ s.t. $\kappa^2 = -g_{tt}g_{xx}\big|_{\rho=\rho_c}$. One sees immediately that $\rho'\big|_{\rho_c} = 0$, which means that ρ_c denotes the lowest point reached by the string. $\kappa = 0$ holds if and only if the string reaches the bottom of the space.

Solving (6.54) for ρ' yields a first order equation for the profile

$$\rho'^2 = \frac{g_{xx}}{g_{\rho\rho}} \left(\frac{-g_{tt}g_{xx} - \kappa^2}{\kappa^2} \right). \tag{6.54}$$

Note that $g_{tt} \leq 0$. We assume the system to be symmetric about $\sigma = 0$, which leads to the constraint $\rho'(0) = 0$. A look at the profile tells us that this is satisfied for

$$\rho = \rho_c \geq a. \tag{6.55}$$

See fig. 6.3(a). Note that ρ' is real as long as $\rho \geq \rho_c$.

Boundary conditions We briefly turn to the issue of the string's boundary conditions at the probe brane. Following the discussion in section 6.2.4 we are interested in the momentum flux at the endpoints of the string. Therefore we evaluate j^σ_μ as in (6.43) for the metric and profile in question and find

$$j^\sigma_\mu = \frac{1}{2\pi\alpha'}\frac{\kappa}{g_{xx}}\left(\delta^x_\mu g_{xx} + \delta^\rho_\mu g_{\rho\rho}\rho'\right). \tag{6.56}$$

The crucial observation is $j^\sigma_x \propto \kappa$. That is as long as the string does not reach the bottom of the space (i.e. $\rho_c = a$), there is momentum in the x-direction flowing through the string, violating von Neumann boundary conditions (6.46). We may easily fix this by turning on a $U(1)$ gauge-field in the world-volume of the brane. Note that $\kappa \in \mathbb{R}$ tells us that one may choose the direction of momentum flow. This makes sense, as, the problem is symmetric and there is no reason a priori why the momentum should flow in a specified direction. We may interpret this as our freedom to choose which of the two heavy W-bosons represents the quark and which represents the anti-quark. In other words while we set of with a mathematical model which was symmetric under a $q \leftrightarrow \bar{q}$ exchange, the appearance of the $U(1)$ gauge field breaks this discrete symmetry.

j^σ_ρ is also non-vanishing. Yet as ρ denotes a direction transverse to the probe, this is in accordance with the Dirichlet boundary conditions in that direction.

Separation of the quarks ρ_c is not a parameter but depends on the separation of the quarks. Regard

$$L = 2\int_0^{\frac{L}{2}} \mathrm{d}x = 2\int_{\rho_c}^{\rho_\Lambda} \mathrm{d}\rho\rho'^{-1}. \tag{6.57}$$

One obtains a relation $L(\rho_c)$ which may be inverted to eliminate ρ_c. Albeit the integrand's singularity for $\rho \to \rho_c$, the integral is finite for fixed values of ρ_c and ρ_Λ. For large ρ however the integrand behaves roughly as

$$\rho'^{-1} \overset{\rho\to\infty}{\approx} \frac{1}{\rho}, \tag{6.58}$$

such that one does not obtain a finite value for L when taking $\rho_\Lambda \to \infty$. This is in contrast to asymptotically AdS_5 backgrounds, and might be related to the lack of a conformal boundary.

Renormalization As outlined in section 6.2.4 one renormalizes the action by evaluating the Nambu-Goto action for the worldsheet

$$\tau = x^0 \qquad \sigma = \rho \qquad x \in \left\{-\frac{L}{2}, \frac{L}{2}\right\} \qquad \tau \in \{0, T\}. \tag{6.59}$$

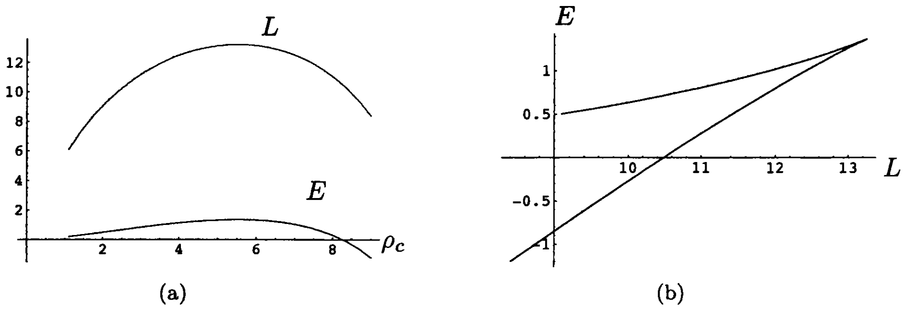

Figure 6.5: Separation (L) and potential energy (E) of the $q\bar{q}$ system for $a = 1$, $\rho_\Lambda = 10$, and $N = 5$. ρ_c denotes the lowest point in the bulk reached by the string. The $E(L)$ plot shows that for most values of L there are 2 energy levels corresponding to a large and a small value of ρ_c. Minimizing its energy the system will choose the lower branch corresonding to larger values of ρ_c.

As with (6.48) this does not define a complete loop, but two separate lines. Again we may ignore this issue as long as we assume that $T \gg L$. Physically the overall procedure corresonds to subtracting the energy of two independent, static quarks. Proceeding as before, the counterterm is given by

$$S_{\mathrm{R}} = \frac{T}{\pi\alpha'} \int_a^{\rho_\Lambda} \mathrm{d}\rho \sqrt{-g_{tt} g_{\rho\rho}}. \tag{6.60}$$

One should emphasize again that, while being an admissible solution of the equations of motion, the solution used for renormalization here is not physical as there are no suitable boundary conditions to be defined at $\rho = a$. One should simply think of this as a method to calculate the mass of the W-bosons.

Evaluation Using $TE = S_{\mathrm{NG}} - S_{\mathrm{R}}$ and (6.41), (6.57), and (6.60) one obtains for the energy

$$\begin{aligned} E(\rho_c, \rho_\Lambda) = {} & \sqrt{-g_{tt}(\rho_c) g_{xx}(\rho_c)} L(\rho_c, \rho_\Lambda) \\ & + 2 \int_{\rho_c}^{\rho_\Lambda} \sqrt{\frac{g_{\rho\rho}}{g_{xx}}} \left(\sqrt{-g_{tt} g_{xx} + g_{tt}(\rho_c) g_{xx}(\rho_c)} - \sqrt{-g_{tt} g_{xx}} \right) \mathrm{d}\rho \\ & - 2 \int_a^{\rho_c} \sqrt{-g_{tt} g_{xx}} \mathrm{d}\rho. \end{aligned} \tag{6.61}$$

Numerical results are shown in figure 6.5 and show clearly that

$$E(L) \propto L, \tag{6.62}$$

for $L \lessapprox 13$. In order to properly exhibit confinement we would need to discuss the potential for $L > 13$ in order to show that the proportionality holds for all values of L.

As a matter of fact the behavior of L around $L \approx 13$ stems from the fact that we did not take the $\rho_\Lambda \to \infty$ limit. That is, the separation between the branes is still finite and so is the mass of the probe quarks. Indeed, when running the same numerics for larger values of ρ_Λ, one ends up with similiar plots yet valid for larger values of L, which we take as a indication that the proportionality $E \propto L$ holds for any L. In order to properly establish confinement however, we shall use a different method. According to a theorem[8] by Kinar, Schreiber, and Sonnenschein [121], a sufficient condition for confinement is given by the following: Consider the function

$$f^2(\rho) \equiv -g_{00}g_{xx}\big|_\rho \,. \tag{6.63}$$

Then the dual gauge theory is confining if f has a minimum at some ρ_{min} and $f(\rho_{\text{min}}) \neq 0$. The metric (6.19) satisfies this and we conclude the discussion of the zero temperature theory by noting that the field theory is a confining.

6.3 The supergravity theory at finite temperature

Having completed our review of the zero-temperature theory, we shall discuss the finite-temperature case. Proceding in the same way as before, we begin with eleven-dimensional supergravity.

6.3.1 The eleven-dimensional black hole

Studying the quark gluon plasma means studying finite temperature physics. As for the gauge theory, finite-temperature field theory is – in the Matsubara formalism – defined on Euclidean space-time compactified to $S^1 \times \mathbb{R}^3$. The previously time-like direction x_E^0 is now periodic with period $\beta = T^{-1}$. In the supergravity dual, we do also need to add an event-horizon to the background, turning the previously extremal p-brane solutions into non-extremal black branes [124]. One should picture this departure from extremality as adding energy to the background while keeping all charges constant. As the extremal solutions satisfy a BPS bound, adding temperature corresponds to using non-BPS branes.

[8] For a proof of the relevant theorem see [123].

In order to do so, we modify the eleven-dimensional metric (6.3) to

$$\mathrm{d}s_M^2 = -f(\rho)\mathrm{d}t^2 + \mathrm{d}\mathbf{x}^2 + \frac{\mathrm{d}\rho^2}{f(\rho)\left(1 - \frac{a^3}{\rho^3}\right)} + \dots . \tag{6.64}$$

Note that we are using Minkowski-signature here, albeit the previous comments about the Matsubara formalism. The reason is that the procedure we use for finding the black brane solution does not depend on the signature and that we will be mostly using the Minkowski-space solution later on, because Euclidean time does not allow the study of dynamical quantities. However, in order to study genuinely themodynamical issues such as temperature, entropy, or specific heat, as we will do in section 6.3.2, we need to compactifiy to periodic, Euclidean time.

Enforcing the equation of motion $R_{\mu\nu} = 0$ on the above gives a system of differential equations for $f(\rho)$. While there will certainly be the trivial solution $f(\rho) = 1$, we are looking for a nontrivial one exhibiting a horizon structure $f(\rho_h) = 0$.

Calculating the Ricci tensor for the above ansatz one sees quickly that there is a non-trivial solution if and only if one takes $a \to 0$. While one might object that we are not allowed to take this limit as the zero temperature requires $a > 0$ to resolve the conifold singularity, one should not forget that the singularity will be hidden by the black hole's horizon. The unique solution is

$$f(\rho) = 1 - \frac{\rho_h^5}{\rho^5}. \tag{6.65}$$

with

$$\rho \in [\rho_h, \infty) . \tag{6.66}$$

The new metric is given by

$$\mathrm{d}s_M^2 = -f(\rho)\mathrm{d}t^2 + \mathrm{d}\mathbf{x}^2 + \frac{\mathrm{d}\rho^2}{f(\rho)} + \frac{\rho^2}{9}\left(\tilde{w}^{a2} + w^{a2} - \tilde{w}^a w^a\right) . \tag{6.67}$$

Most of our discussion will only require knowledge of the precise form of the $t, \mathbf{x}, \rho$ directions. When using Euclidean signature, we shall denote the metric by $\hat{g}_{\mu\nu}$.

6.3.2 Thermodynamics

We will now turn to a discussion of some of the thermodynamical properties of the solution (6.67).

Temperature Consider the (t_E, ρ) plane in the finite-temperature formalism. It has topology $S^1 \times \mathbb{R}_{>0}$ with $\rho \in [\rho_h, \infty)$ and $t_E \in [0, \beta]$. One proceeds by demanding that there be no conical singularity at the origin. Mathematically this may be expressed by considering the ratio of circumference and radius of a small circle around the origin and solving for

$$2\pi \stackrel{!}{=} \lim_{\rho \to \rho_h} \frac{\text{circumference}}{\text{radius}}. \tag{6.68}$$

Using the standard expression for arclength, we obtain

$$\text{circ.} = \int_0^\beta \mathrm{d}t_E \sqrt{\hat{g}_{tt}} \approx \beta \rho \partial_\rho \sqrt{\hat{g}_{tt}(\rho)} \tag{6.69}$$

$$\text{rad.} = \int_0^\rho \mathrm{d}\rho' \sqrt{\hat{g}_{\rho\rho}} \approx \rho \sqrt{\hat{g}_{\rho\rho}}. \tag{6.70}$$

Plugging these into (6.68) yields

$$\begin{aligned} 2\pi &\stackrel{!}{=} \beta \lim_{\rho \to \rho_h} \frac{\beta \partial_\rho \sqrt{\hat{g}_{tt}}}{\sqrt{\hat{g}_{\rho\rho}}} \\ \Rightarrow T &= \lim_{\rho \to \rho_h} \frac{\partial_\rho \hat{g}_{tt}}{4\pi \sqrt{\hat{g}_{tt} \hat{g}_{\rho\rho}}} = \frac{5}{4\pi \rho_h}. \end{aligned} \tag{6.71}$$

One should pay attention to the slightly unusual dependence of the temperature on the position on the horizon. For the $AdS_5 \times S^5$ black hole for example, the relation is $T \propto \rho_h$. We will return to this issue in section 6.3.3.

Evaluation of the partition function To study further thermodynamic properties of the solution (6.67), we need to evaluate the partition function $\mathcal{Z} = e^{-\mathcal{S}_E}$. As the eleven dimensional theory is purely gravitational, this boils down to calculating the action

$$\mathcal{S} = \frac{1}{16\pi} \int_{\mathcal{M}} d^{\mathrm{d}}x \sqrt{\hat{g}} R + \frac{1}{8\pi} \int_{\partial \mathcal{M}} d^{d-1}x K \sqrt{\hat{h}} \tag{6.72}$$

for Euclidean space-time. Where $\mathcal{M}$ is a volume of spacetime defined by $\rho < \rho_\Lambda$. As in the absence of any further fields the equations of motion simplify to $R_{\mu\nu} = 0$, the Einstein-Hilbert term vanishes leaving us with the Gibbons-Hawking term.

The metric induced on $\partial\mathcal{M}$ is denoted by h. K is the extrinsic curvature defined by

$$K_{ab} \equiv \partial_a x^\mu \partial_b x^\nu \nabla_\mu n_\nu. \tag{6.73}$$

The coordinates x^μ are that of the eleven-dimensional background, while the x^a parametrize the boundary of the region of integration $\partial\mathcal{M}$. Due to our choice of volume $\mathcal{M}$ we may pick the x^a such that

$$\partial_a x^\mu = \left\{ \begin{array}{ll} \delta^\mu_a & \mu \neq \rho \\ 0 & \mu = \rho \end{array} \right. \tag{6.74}$$

n is a unit normal to $\partial\mathcal{M}$. We choose $n = \sqrt{g^{\rho\rho}}\partial_\rho$. Now (6.73) simplifies considerably.

$$K_{ab} = \partial_a n_b - \Gamma_{\lambda ab} n^\lambda = -\Gamma_{\rho ab}\sqrt{g^{\rho\rho}} = \frac{1}{2}\sqrt{g^{\rho\rho}}\partial_\rho g_{ab} \tag{6.75}$$

Similarly $h_{ab} = \partial_a x^\mu \partial_b x^\nu g_{\mu\nu}$ and thus

$$\sqrt{h} = \frac{\rho^6\sqrt{f}\sin\theta\sin\tilde{\theta}}{648} \tag{6.76}$$

Also

$$K = h^{ab}K_{ab} = \frac{\sqrt{g^{\rho\rho}}}{2} g^{ab}\partial_\rho g_{ab} = \frac{\sqrt{f}}{2}\left(f^{-1}f' + \frac{12}{\rho}\right). \tag{6.77}$$

Applying this to the action (6.72) one realizes that the integration is trivial as the radial variable is not integrated over. Then

$$\begin{aligned} \mathcal{S} &= \overbrace{\left(\frac{1}{10368\pi\sqrt{3}}\int \mathrm{dx}\mathrm{d}\theta\mathrm{d}\phi\mathrm{d}\psi\mathrm{d}\tilde{\theta}\mathrm{d}\tilde{\phi}\mathrm{d}\tilde{\psi}\sin\theta\sin\tilde{\theta}\right)}^{\mathcal{A}} \int_0^\beta \mathrm{d}x^0 f\rho^6\left(f^{-1}f' + \frac{12}{\rho}\right)\Bigg|_{\rho=\rho_\Lambda} \\ &= \left\{ \begin{array}{ll} \mathcal{A}\beta\left(12\rho_\Lambda^5 - 7\rho_h^5\right) & T>0 \\ \mathcal{A}\beta 12\rho_\Lambda^5 & T=0 \end{array} \right. \end{aligned} \tag{6.78}$$

Note that $\mathcal{A} = \frac{2\pi^3 \mathrm{Vol}\mathbb{R}^3}{81\sqrt{3}}$.

Renormalisation If we take the cutoff to infinity, $\rho_\Lambda \to \infty$, the result of (6.78) is divergent and does need to be renormalized. The easiest way to do so is by subtracting the action of some reference space-time. As we are only considering the Gibbons-Hawking term, the natural candidate is the zero-temperature solution as defined on the singular conifold, whose action may be obtained directly from (6.78) by setting $f \to 1$. We call this reference action $S_{T=0}$. We could have also calculated this reference action by starting from the non-singular zero-temperature metric (6.3), evaluating the on-shell action and taking the limit $a \to 0$ before $\rho_\Lambda \to \infty$.

Again we need to compactify the Euclidean t_E direction on an S^1. Yet in opposite to the black hole solution (6.67) it is not obvious what the periodicity of the circle should be. Therefore consider a particle, whose energy is equal to the thermal energy T, in the finite-temperature solution propagating at a radius of ρ_Λ. To an observer at spatial infinity, its thermal energy will appear redshifted to

$$E_\infty^T = \sqrt{-g^{tt}(\rho_\Lambda)p_0p_0} = \frac{T}{\sqrt{\hat{g}_{tt}(\rho_\Lambda)}} = \frac{T}{\sqrt{f(\rho_\Lambda)}}. \tag{6.79}$$

In the zero temperature solution on the other hand, $\hat{g}_{tt} = 1$, and there is no redshift. Comparing energies in the two solutions by means of hypothetical observers at $\rho = \infty$, the energies correspond as

$$E_{\rho_\Lambda}^{T=0} = \frac{E_{\rho_\Lambda}^T}{\sqrt{f(\rho_\Lambda)}}, \tag{6.80}$$

which leads us to

$$\beta_{T=0} = \beta_T\sqrt{1 - \frac{\rho_h^5}{\rho_\Lambda^5}}. \tag{6.81}$$

We shall use this result to evaluate and compare (6.78) for the zero- and finite-temperature backgrounds with t_E periodic and periodicity $\beta_{T=0}, \beta_T$, yielding

$$\mathcal{S}_{T=0} = 12\mathcal{A}\beta\rho_\Lambda^5\sqrt{1 - \frac{\rho_h^5}{\rho_\Lambda^5}} \tag{6.82}$$

$$\mathcal{S}_{T>0} = -7\mathcal{A}\beta\rho_h^5 + 12\beta\mathcal{A}\rho_\Lambda^5. \tag{6.83}$$

Taking the cutoff ρ_Λ to infinity, evaluating $\mathcal{A}$ explicitly, and dividing by the volume of $\mathbb{R}^3$, the final, renormalized result for the action density is

$$\mathcal{S}_E = \lim_{\rho_\Lambda \to \infty} \mathcal{S}_{T>0} - \mathcal{S}_{T=0} = -\frac{8\pi^4\rho_h^6}{405\sqrt{3}}. \tag{6.84}$$

The fact that this seems to be negative should not disturb us. In the contrary, as it implies $s_{T=0} > s_{T>0}$, the finite temperature solution will be the leading order contribution in a saddle point approximation to the path integral. If this was not the case, we were not allowed to study finite temperature effects using the solution (6.67). Naturally, when computing further quantities, we will use the absolute value of (6.84).

One should wonder about the N dependence of (6.84). After all our aim is to study the physics of the QGP, which is in a deconfined phase of QCD.

So the entropy should reflect the N^2 color-degrees of freedom. On the other hand, (6.84) cannot contain any factor N, as the UV completion does not know about the number of colors. One may try to resolve this issue by substituting the 't Hooft coupling λ for ρ_h. We will compute λ in section 6.4.1, yet for our discussion here it is sufficient to know that when expressed in terms of N, λ, and energy-scale ρ, ρ_h has a

$$\rho_h^5 \sim \frac{N^5}{\lambda} \tag{6.85}$$

dependence, leading to a $N^6\lambda^{-6/5}$ dependence for the entropy. While this is not fully satisfactory - after all, one would expect N^2, it shows the correct qualitative behavior.

Mass, Entropy, Specific heat Using the renormalized Euclidean action (6.84) and some standard relations of thermodynamics one can calculate a variety of properties of the background. Mass, entropy-density and specific heat are given by

$$\mathcal{Z} = e^{-\mathcal{S}_E} \tag{6.86}$$

$$M = \langle E \rangle = \frac{\partial \mathcal{S}_E}{\partial \beta} = \frac{5}{4\pi}\frac{\partial \mathcal{S}_E}{\partial \rho_h} \tag{6.87}$$

$$S = \beta\langle E \rangle - \mathcal{S}_E \tag{6.88}$$

$$C = T\frac{\partial S}{\partial T}. \tag{6.89}$$

Therefore

$$M = \frac{4\pi^3\rho_h^5}{27\sqrt{3}} \tag{6.90}$$

$$S = \frac{8\pi^4\rho_h^6}{81\sqrt{3}} \tag{6.91}$$

$$C = -\frac{16\pi^4\rho_h^6}{27\sqrt{3}}. \tag{6.92}$$

Equations (6.90) show a rather surprising thermodynamic behavior – especially as we are trying to identify it with that of a four-dimensional gauge theory. First of all, the specific heat C is negative, probably denoting an instability of the solution. More importantly, the entropy behaves as $S \propto T^{-6}$, which is rather puzzling. As a first check of the above results, one can compare (6.90) to the Bekenstein-Hawking entropy, which in our conventions takes the form $S_{\rm BH} = \frac{A}{4}$, with A being the area of the black hole horizon. A direct calculation gives $S_{\rm BH} = \frac{8\pi^4}{81\sqrt{3}}\rho_h^6$, which agrees with the previous result. One should also note that the first law of thermodynamics, $\mathrm{d}M = T\mathrm{d}S$, is satisfied by the solution, as can be verified explicitly.

So while the thermodynamical properties of the system appear sensible from the point of view of eleven-dimensional supergravity, it is difficult to interpret them as those of a four-dimensional gauge theory. We will try to find a partial explanation for this behavior in the next section.

6.3.3 Comparison with the Schwarzschild solution

In comparison with the AdS-black hole [124] properties of the finite-temperature G_2 holonomy solution (6.67) might seem a bit surprising. However, there is a very well understood solution of the four-dimensional Einstein equations with similiar characteristics, the Schwarzschild black hole. So let us recall the properties of its generalization, the four-dimensional Reissner-Nordstrom solution.

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_2^2$$

$$r_\pm = M \pm \sqrt{M^2 - Q^2} \quad T = \frac{1}{4\pi}\left(\frac{2M}{r_+^2} - \frac{2Q^2}{r_+^3}\right) \quad F = \frac{Q}{r^2}\mathrm{d}t \wedge \mathrm{d}r$$

M is the mass, Q the charge, T the temperature, and $r_\pm$ are the inner and outer horizons. The Schwarzschild solution is obtained in the $Q \to 0$ limit. As one may see from the equations, there is a BPS constraint on the mass $M \geq Q$.

As long as we keep $Q > 0$, the temperature vanishes in the extremal limit $M \to Q$. This changes in the Schwarzschild case $Q = 0$. Here the temperature is singular when taking the mass to zero. Mathematically this is expressed by the absence of the $+\frac{Q^2}{r^2}$ term in the Schwarzschild metric. As there is no such term in the eleven-dimensional metric (6.67) and as both the Schwarzschild and the Reissner-Nordstrom solution have negative specific heat[9], one may speculate that the singular behavior of the temperature of the gravity dual in question may be related to the dual being of Schwarzschild- rather than Reissner-Nordstrom type.

[9] For Schwarzschild one sees this by realizing that any increase in M leads to a decrease in T. So whenever we increase the energy, keeping the charge constant, the temperature decreases. For Reissner-Nordstrom the situation is slightly more complicated. While T vanishes with M for sufficiently small M the behavior reduces to that of Schwarzschild in the large M limit. It follows that Reissner-Nordstrom black holes of small masses have positive specific heat, while those of large mass have negative specific heat.

We may pursue the comparison with the Schwarzschild solution even further. Our zero-temperature background has the topology $\mathbb{R}^{1,3} \times \mathbb{R} \times \mathcal{M}$, with $\mathcal{M}$ being the G_2-holonomy manifold. If we were simply to replace $\mathcal{M}$ by an S^6, we were dealing with ordinary Minkowski space in eleven dimensions. Now searching for a black hole of with the Ansatz

$$\mathrm{d}s^2 = -f(\rho)\mathrm{d}t^2 + \mathrm{d}\mathbf{x}^2 + \frac{\mathrm{d}\rho}{f(\rho)} + \rho^2 \mathrm{d}\Omega_6^2 \tag{6.93}$$

we find the identical solution to the equations of motion, $R_{\mu\nu} = 0$, given by (6.65). Performing the same calculations on this eleven-dimensional Schwarzschild black hole that we did before, we see, that the Bekenstein-Hawking entropy behave as $S_{\mathrm{BH}} \propto \rho_h^6$, whereas the temperature will satisfy $T = \frac{5}{4\pi\rho_h}$, showing thermodynamic behavior identical to that of our solution (6.67). Thus it appears as if the rather undesirable behavior of the entropy $S \propto T^{-6}$ might be related to the fact that the string dual may be traced back to pure gravity in eleven dimensions. In analogy with the four-dimensional case one might expect the thermodynamics of our solution to improve once the black hole is charged under some gauge field. Generalizing the ansatz (6.64) to include the three-form potential of eleven-dimensional supergravity however will make the task of finding a solution considerably more difficult.

6.3.4 Dimensional Reduction

In the same way that we went from M-theory to type IIA at zero temperature in section 6.1.2, one may perform dimensional reduction for the finite-temperature background.

$$e^{\frac{4\Phi}{3}} = \frac{\rho^2}{9N^2\rho_h^2} \tag{6.94}$$

$$A_{(1)} = N\rho_h \left(n.\hat{w} - \frac{1}{2} n.w \right) \tag{6.95}$$

$$\mathrm{d}s^2_{IIA} = e^{\frac{2}{3}\Phi} \left[-f(\rho)\mathrm{d}t^2 + \mathrm{d}\mathbf{x}^2 + \frac{\mathrm{d}\rho^2}{f(\rho)} + \frac{\rho^2}{9}\left(w^2 + \hat{w}^2 - w.\hat{w}\right) - e^{\frac{4}{3}\Phi} A_{(1)} A_{(1)} \right] \tag{6.96}$$

The Ricci scalar in the string frame is

$$R = \frac{9N\rho_h}{\rho^8}\left(-13\rho^5 + 3\rho_h^5\right). \tag{6.97}$$

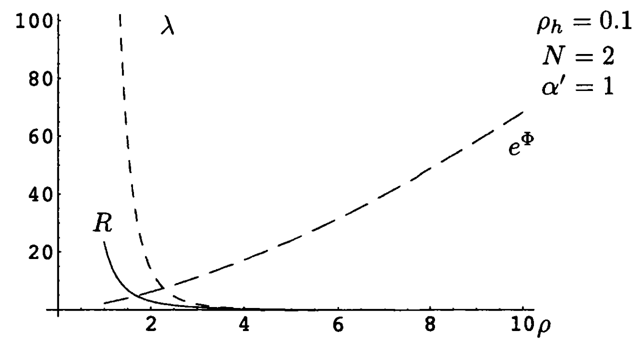

Figure 6.6: As previously done for the zero temperature gauge theory in figure 6.2, we discuss the curvature and couplings of the finite temperature solution. Again there is clearly a regime in the IR where non-perturbative gauge-dynamics are captured by type IIA string theory. As in the zero-temperature case though, g_{YM} is singular at $\rho = \rho_h$.

6.4 The field theory at finite temperature

6.4.1 Properties of the Dual Field Theory

Turning on a temperature does naturally break the supersymmetry, so that we are dealing with the same modes as in the zero-temperature case, except that there is no supersymmetry. Now however the mass of the Kaluza-Klein modes is given by the size of the wrapped S^3 in the far IR, that is by the location of the horizon. We may use (6.71) to relate it to the temperature as

$$\Lambda_{KK} = \frac{\alpha'}{2\pi^2\rho_h^3} = \frac{1}{2}\left(\frac{4}{5}\right)^3 \alpha' T^3 \tag{6.98}$$

In all other aspects the discussion of the theory's field content is identical to that performed in section 6.2.2.

The same holds true for the derivation of the Yang-Mills coupling constant from the DBI action (6.2.1). The induced metric is

$$\begin{aligned} \mathrm{d}s_6^2 = e^{\frac{2}{3}\Phi}\Big[&-f\mathrm{d}t^2 + \mathrm{d}\mathbf{x}^2 + \frac{\rho^2}{9}\mathrm{d}\tilde{\theta}^2 + \frac{\rho^2}{12}\mathrm{d}\tilde{\phi}^2 + \frac{\rho^2}{9}\left(1 - \frac{1}{4}\cos^2\tilde{\theta}\right)\mathrm{d}\tilde{\psi}^2 \\ &+2\frac{\rho^2}{12}\cos\tilde{\theta}\mathrm{d}\tilde{\phi}\mathrm{d}\tilde{\psi}\Big], \end{aligned} \tag{6.99}$$

leading to

$$g_{\mathrm{YM}} = \frac{3^{13/4} N\pi\alpha'^{3/4}\rho_h}{\rho^{5/4}\left(\rho^5 - \rho_h^5\right)^{1/4}} \tag{6.100}$$

Having already calculated the dilaton (6.94) and the Ricci scalar (6.97), we are again able to discuss the decoupling limit. To get a qualitative understanding we have plotted the relevant quantities in fig. 6.6.

$$-\alpha' R \leq \frac{90 N \alpha'}{\rho_h^2} \tag{6.101}$$

$$\lambda = \frac{324\sqrt{3} N^3 \pi^2 \alpha^{3/2} \rho_h^2}{\rho^{5/2}\sqrt{\rho^5 - \rho_h^5}} \tag{6.102}$$

$$e^{\Phi} = \left(\frac{\rho}{3N\rho_h}\right)^{\frac{3}{2}} \tag{6.103}$$

Again the supergravity description is valid in the large N, small α' limit while it is not possible to ignore the KK-modes (ρ_h small) at the same time.

6.4.2 Quark-Antiquark Potential

We perform a numerical analysis of the quark-antiquark potential. The results presented here were derived in exactly the same way as in section 6.2.5 with the finite temperature metric (6.96) replacing the zero temperature background (6.19).

The results are depicted in fig. 6.7. At first sight it appears as if there are again two solutions with the minimum energy one showing a direct proportionality $E \propto L$ and thus confinement. If this were the complete story the physical system dual to our finite-temperature background were certainly not a deconfined QGP.

Now recall from our discussion of the Wilson loop's renormalization in sections 6.2.4 and 6.2.5 that for the zero temperature solution the configuration of two strings stretching from the probe brane to the bottom of the space ($\rho = a$) was not physical as it is not possible to define suitable boundary conditions for the worldsheet. In other words, there is nothing at the bottom of the space for the open strings to end on. This is different for the finite temperature case though, where it is possible for a string to end (or fall through) a black hole's horizon, as long as suitable boundary conditions are satisfied; i.e. there may be no excitations leaving the black hole. Therefore renormalization in the finite temperature theory is not interpreted as merely subtracting the mass of the two W-bosons. Instead one actually considers two competing, physical solutions. That of two quarks connected by a string and that of two independent quarks. The system chooses the minimum energy configuration and therefore we may interpret the point in fig. 6.7(b) at $L \approx 21$ where $E(L) = 0$ as the transition

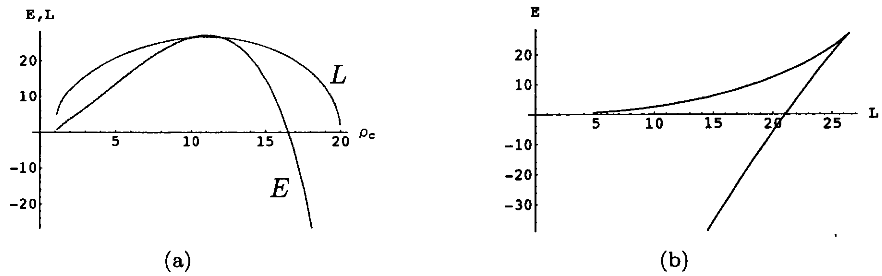

Figure 6.7: The quark-antiquark potential at finite temperature. Compare the zero temperature case shown in figure 6.5.

between the two solutions. For $L > 21$ we have two quarks propagating independently,[10] while for $L < 21$ the two quarks interact via a string. Therefore we claim that the finite temperature theory is not confining, as expected for the QGP.

As to the issue of the world-sheet's boundary conditions, the discussion is identical to that of the zero temperature case in section 6.2.5. The x-momentum flux along the string is proportional to a constant of integration κ with $\kappa = 0$ if and only if the string stretches all the way to the horizon. Again one fixes the failure of the boundary conditions to be properly von Neumann by turning on a $U(1)$ gauge field in the probe brane.

6.4.3 Shear Viscosity

One of the first properties of the $\mathcal{N} = 4$ QGP calculated from the dual $AdS_5 \times S^5$ geometry was the plasma's shear viscosity η.[11] The original ansatz of [47] uses the *Kubo relations* which stem from the formalism of finite-temperature field theory. These relate the shear viscosity to the energy-momentum tensor as

$$\eta = \lim_{\omega \to 0} \frac{1}{2\omega} \int \mathrm{d}t\mathrm{d}\mathbf{x} e^{\imath\omega t} \langle [T_{xy}(t, \mathbf{x}), T_{xy}(0, 0)] \rangle. \tag{6.104}$$

While one may simply use the gauge/gravity correspondence to directly calculate the above correlator, the authors of [48] were able to identify hydrodynamic behavior in the gravity dual by studying metric perturbations in the background. Thus they obtained an explicit expression for the shear viscosity in terms of the

[10] The quarks are not fully independent. The two worldsheets interact via graviton exchange in the bulk spacetime.

[11] Brief reviews of relativistic hydrodynamics and their relevance to relativistic heavy ion collisions may be found in [105, 106].

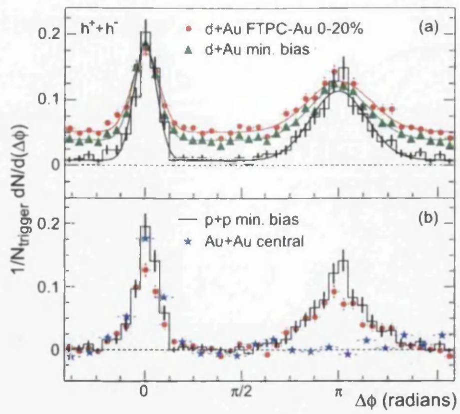

Figure 6.8: Experimental evidence for jet quenching in heavy ion collisions (Source: [126]).

entropy density. Defining $g = \det g_{\mu\nu}$,

$$\frac{\eta}{s} = T \left. \frac{\sqrt{-g}}{\sqrt{-g_{00} g_{\rho\rho}}} \right|_{\rho_h} \int_{\rho_h}^{\infty} \mathrm{d}\rho \frac{-g_{00} g_{\rho\rho}}{g_{xx}\sqrt{-g}}. \tag{6.105}$$

Evaluating the above for the type IIA or 11-dimensional background (6.96), (6.67) yields

$$\frac{\eta}{s} = \frac{1}{4\pi}. \tag{6.106}$$

The above result confirms a general theorem [49], [125] according to which the ratio $\eta/s = 1/4\pi$ is of the same value for a fairly large class of gravity duals.

6.4.4 Energy Loss of a Heavy Quark

Our final object of study shall be the radiative energy loss of a heavy quark traversing the plasma. Prior to exhibiting how this may be modeled in terms of the AdS/CFT correspondence and the G_2 holonomy manifold we shall take a brief excursion into experimental data obtained at the relativistic heavy ion collider in order to see why radiative energy loss is a problem of interest.

Experimental Background The relativistic heavy ion collider performs central Au+Au collisions at about 200GeV. After the collision the system quickly reaches a local thermal equilibrium at a temperature of about 170MeV and is assumed to be a quark-gluon plasma.[12] Naturally the plasma is not the only

[12] For a review of relativistic heavy ion collisions see [105].

result of the collision. Instead there is also a number of partons whith energies of up to $\mathcal{O}(1\text{GeV})$. One might expect that these should be created in two- or three-jet events. Specializing to back-to-back scattering, figure 6.8(b) shows the yield of such partons in terms of their angular distribution in the reaction plane. The concept is to wait for a trigger particle with transverse momentum $4 < p_{T,\text{Trig.}} < 6\text{GeV}/c$ and then search for further particles with $2\text{GeV}/c < p_{T,\text{Trig.}}$. With the trigger particle at $\Delta\Phi = 0$ one sees clearly a suppression of such back-to-back events in the Au+Au heavy ion collisions in comparision to ordinary p+p scattering. The reason for this suppression lies in the fact that, as sketched in figure 6.9, one of the partons needs to traverse the plasma. In doing so it interacts with the plasma leading to an overall energy loss. The answer to our initial question should be clear from this: As this phenomenon is specific to heavy ion collisions, it may be directly attributed to the presence of the plasma and is therefore an experimental indicator to the QGP being created in the course of the experiment.

When applying the AdS/CFT-correspondence to describe parton energy loss, there are two fundamentally different approaches. One, referred to in the literature as the jet quenching calculation [102],[109] models the problem in terms of ordinary particle physics and uses the correspondence exclusively for purposes of computation. The concept of the drag force on the other hand is intrinsically stringy as the quark is depicted as a string hanging from a probe brane into the bulk geometry [103, 127]. There is a further difference between the two approaches. While the former relies on the energy of the quark being highly relativistic, the latter is not only free of this assumption but is moreover frequently used to make statements about the non-relativistic limit.

Jet Quenching In the jet quenching picture, the energy loss of the high energy quark is captured by the jet quenching parameter $\hat{q}$ which again is defined in terms of the expectation value of a Wilson loop:

$$\langle W(\mathcal{C})\rangle = e^{-\frac{1}{4}\hat{q}L^-L^2} \tag{6.107}$$

Here $\mathcal{C}$ is a light-like Wilson loop in the $x^2, x^- = \frac{x^0-x^3}{\sqrt{2}}$ plane. The extension along the light-cone is L^- while that along x^2 is L. One assumes $L^- \gg L$. One should note that albeit the loop being defined in Minkowski space, the

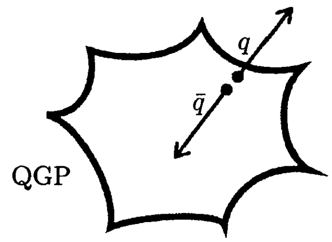

Figure 6.9: Jet Quenching in Relativistic Heavy Ion Collisions is due to radiative energy loss of a parton - here the antiquark $\bar{q}$ - traversing the plasma.

exponential on the right hand side of (6.107) is a real quantity. This is in contrast to (6.40), which is defined in Euclidean space.

The derivation of (6.107) is purely based on particle theory and rather non-trivial. We shall only briefly describe how $\hat{q}$ captures the phenomenon of radiative energy loss and why one may use a Wilson loop to calculate it. The interested reader is referred to the literature [128], [129] for details on how (6.107) arises.

To answer the first of these questions, note that parton energy loss is directly proportional to the jet quenching factor,

$$\Delta E \propto \hat{q} L^{-2}. \tag{6.108}$$

As to the question of why this may be calculated using a Wilson loop, consider the following: Due to the quarks' high energy, we may think of it as actually moving along the light-cone. Interaction with the gluons of the plasma leads to color rotations. One may think of *in*- and *out*-states related by a Wilson line along the light-cone

$$|\Psi_{\text{out}}\rangle = Tr\mathcal{P}e^{i\int_0^{L^-} \mathrm{d}x_- A^-}|\Psi_{\text{in}}\rangle. \tag{6.109}$$

Expectation values involve the hermitian conjugate of this, leading to a Wilson line in the opposite direction. As $L^- \gg L$, one may join the two lines giving us the loop $\mathcal{C}$.

Taking a closer look at (6.107), a crucial observation is that we are dealing with the exponenential of a real quantity albeit using Minkowskian signature. This is directly related to the occurence of the light-like Wilson loop. Although it is technically possible to obtain a result for the jet-quenching factor using such a loop, as was done in [109], we will see that one needs to consider such a light-like loop as the limiting case of space- or time-like ones extending either down- or upwards from the flavor brane they are attached to. Note that while the original paper [128] considered only a space-like string stretching from the flavor brane towards the horizon and approaching the light-like limit from below, $v < 1$, it was argued in [130, 120, 131] that all four cases need to be investigated. As the technicalities follow analogous steps in all four cases, we will only exhibit a detailed calculation for the space-like down string followed by some remarks about the three remaining configurations.

The space-like down-string We consider the quark-antiquark pair as moving with constant speed $v = \tanh \eta$. Eventually we will take the limit $v \to 1$. At first we will assume the string to stretch from the flavor brane at $\rho_\Lambda = \Lambda \rho_0$ towards the horizon at ρ_h. We are interested in the limit $\Lambda \to \infty$, the case of infinitely heavy quarks. Moving to a coordinate frame in which the pair lies at rest leads to a new metric given by

$$\begin{aligned} g'_{00} &= \frac{\rho}{3N\rho_h}\left[-f\cosh^2\eta + \sinh^2\eta\right] && (6.110)\\ &= -\frac{\rho}{3N\rho_h}\left[1 - \left(\frac{\rho_h}{\rho}\right)^5\cosh^2\eta\right] && (6.111)\\ g'_{x^3x^3} &= \frac{\rho}{3N\rho_h}\left[\cosh^2\eta - f\sinh^2\eta\right] && (6.112)\\ g'_{0x^3} &= \frac{\rho}{3N\rho_h}\left[-f\cosh\eta\sinh\eta + \cosh\eta\sinh\eta\right] && (6.113)\end{aligned}$$

with the other components as before in (6.96). As x^3 will not appear in our calculations, we shall ignore the primes from now on and define $x \equiv x^2$. In these coordinates the profile is that of a static quark-antiquark pair and therefore the same as in (6.48) in section 6.2.5. Note that if the elongation along $x^0 = t$ in this reference frame is $\mathcal{T}$, then it is $L^- = \mathcal{T}\cosh\eta$ in the laboratory frame.

The Nambu-Goto action is

$$S_{\rm NG} = \frac{\mathcal{T}}{\pi\alpha'}\int_0^{\frac{L}{2}} d\sigma\sqrt{-g_{00}\left(g_{xx} + \rho'^2 g_{\rho\rho}\right)}. \qquad (6.114)$$

Ignoring the overall normalisation,

$$\begin{aligned}\mathcal{L} &= \sqrt{-g_{00}\left(g_{xx} + \rho'^2 g_{\rho\rho}\right)} \\ &= \sqrt{\frac{\rho}{3N\rho_0}\left[1 - \left(\frac{\rho_0}{\rho}\right)^5 \cosh^2\eta\right]\left(g_{xx} + \rho'^2 g_{\rho\rho}\right)}.\end{aligned} \tag{6.115}$$

While the second term is positive definite, the first term however might change sign, depending on the values of η and Λ, with the Λ dependence arising as $\rho \in \{\rho_0, \Lambda\rho_0\}$. We see that as long as

$$\cosh^2\eta > \Lambda^5, \tag{6.116}$$

the Lagrangian $\mathcal{L}$ is imaginary. This is what guarantees the exponent in (6.107) to be real, as required. Therefore the limits $\eta \to \infty$ and $\Lambda \to \infty$ do not commute.

The Hamiltonian is

$$\mathcal{H} = \frac{g_{00} g_{xx}}{\mathcal{L}} \equiv \kappa \qquad \kappa \in \imath\mathbb{R} \tag{6.117}$$

In the problem in question κ is purely imaginary, as the Lagrangian is imaginary. The profile is given by

$$\begin{aligned}\rho'^2 &= \frac{g_{xx}}{g_{\rho\rho}}\left(\frac{-g_{00}g_{xx} - \kappa^2}{\kappa^2}\right) \\ &= \frac{f}{\kappa^2}\left\{\frac{\rho^2}{9N^2\rho_h^2}\left[1 - \left(\frac{\rho_h}{\rho}\right)^5\cosh^2\eta\right] - \kappa^2\right\}\end{aligned} \tag{6.118}$$

with $\kappa^2 \leq 0$. For this to be real and positive, one needs to impose constraints on κ.

$$\cosh^2\eta - \Lambda^5 - 9N^2\Lambda^3\left|\kappa\right|^2 \geq 0 \tag{6.119}$$

So from now on we shall assume $|\kappa| \ll 1$.

Evaluating the length and the action We choose new coordinates,

$$\rho = \rho_h y \qquad\qquad L = \rho_h l. \tag{6.120}$$

Then

$$\begin{aligned}l &= \frac{2}{\rho_h}\int_0^{\frac{L}{2}} \mathrm{d}x = \frac{2}{\rho_h}\int_{\rho_h}^{\Lambda\rho_h} \mathrm{d}\rho\rho'^{-1} \\ &= 2\left|\kappa\right|\int_1^{\Lambda}\mathrm{d}y\left(\frac{y^8}{y^5 - 1}\right)^{\frac{1}{2}}\left(\frac{9N^2}{\cosh^2\eta - y^5 - \left|\kappa\right|^2 9N^2 y^3}\right)^{\frac{1}{2}}\end{aligned} \tag{6.121}$$

We are interested in the small l behavior, which is equivalent to assuming κ to be small. Expanding the integrand gives

$$
\begin{aligned}
&= 6N\left|\kappa\right| \int_1^\Lambda \mathrm{d}y \frac{y^4}{\sqrt{y^5-1}} \left(\frac{1}{\sqrt{\cosh^2\eta - ny^5}} + \frac{\left|\kappa\right|^2 9N^2 y^3}{2\left(\cosh^2\eta - y^5\right)^{3/2}} + \mathcal{O}\left(\left|\kappa\right|^4\right)\right) \\
&= 6N\left|\kappa\right| \underbrace{\int_1^\Lambda \mathrm{d}y \frac{y^4}{\sqrt{y^5-1}} \frac{1}{\sqrt{\cosh^2\eta - y^5}}}_{A} + \mathcal{O}\left(\left|\kappa\right|^3\right) \\
&= \frac{6N}{\cosh\eta}\left|\kappa\right| \underbrace{\int_1^\Lambda \mathrm{d}y \frac{y^4}{\sqrt{y^5-1}}}_{B} + \mathcal{O}\left(\left|\kappa\right|^3, \frac{1}{\cosh\eta}\right)
\end{aligned}
\tag{6.122}
$$

In the last equation we assumed that $\frac{\Lambda^5}{\cosh^2\eta}$ is sufficiently small in order to develop the expression in $\cosh^{-1}\eta$. In the $\Lambda \to \infty$ limit, the integral B is certainly divergent, which might raise the question wheter l may truly be considered to be small. Closer examination however shows that for large y,

$$
B \sim \Lambda^{\frac{5}{2}} \cosh^{-1}\eta. \tag{6.123}
$$

As $\cosh^2\eta \geq \Lambda^5$, our assumption about l is justified.

Similarly to the lenght we may treat the action,

$$
\begin{aligned}
S_{\mathrm{NG}} &= \frac{\imath T}{\pi\alpha'} \int_{\rho_h}^{\Lambda\rho_h} \mathrm{d}\rho \rho'^{-1} \sqrt{\frac{g_{00}^2 g_{xx}^2}{\left|\kappa\right|^2}} \\
&= \frac{\imath T \rho_0}{3\pi\alpha' N} \int_1^\Lambda \mathrm{d}y \sqrt{\frac{y^2\left(\cosh^2\eta - y^5\right)^2}{y^5-1}} \frac{1}{\sqrt{\cosh^2\eta - y^5 - \left|\kappa\right|^2 y^3 9N^2}} \\
&= \frac{\imath T \rho_0}{3\pi\alpha' N} \int_1^\Lambda \mathrm{d}y \sqrt{\frac{y^2\left(\cosh^2\eta - y^5\right)}{y^5-1}} \\
&+ \frac{3\imath T N \rho_0 \left|\kappa\right|^2}{2\pi\alpha'} \underbrace{\int_1^\Lambda \mathrm{d}y \frac{y^4}{\sqrt{y^5-1}} \frac{1}{\sqrt{\cosh^2\eta - y^5}}}_{A} + \mathcal{O}\left(\left|\kappa\right|^4\right) \\
&\equiv S^{(0)} + \left|\kappa\right|^2 S^{(1)} + \mathcal{O}\left(\left|\kappa\right|^4\right)
\end{aligned}
\tag{6.124}
$$

If one again only looks into the leading order behavior for $\cosh^{-1}\eta$, the $\mathcal{O}\left(\left|\kappa\right|^2\right)$ term is

$$
S^{(1)} = \frac{3\imath T N \rho_0 \left|\kappa\right|^2}{2\pi\alpha' \cosh\eta} \underbrace{\int_1^\Lambda \mathrm{d}y \frac{y^4}{\sqrt{y^5-1}}}_{B} = \imath \frac{T L^2 L^-}{30\alpha' B} \tag{6.125}
$$

Note the reappearance of the integrals A, B. Renormalizing the above action as described in section 6.2.4 yields a counterterm that exactly cancels $S^{(0)}$. So to first order in $|\kappa|^2$ we may work with $S^{(1)}$.

The remaining configurations & jet quenching From equation (6.118) it follows that one may also consider a world-sheet ending on the flavor brane yet stretching away from the horizon s.t. $\rho \geq \Lambda \rho_h$. Using the same approximations as for the down-string of the previous paragraph, one arrives at an expression identical to (6.125) except for the integration bounds. Once more, taking $\eta \to \infty$ before $\Lambda \to \infty$, the relevant integral B diverges.

For the string with $v > 1$ one boosts to a faster than light frame. Technically this amounts to substituting $\cosh \eta \mapsto \frac{1}{\imath \sinh \zeta}$ and $\sinh \eta \mapsto \frac{1}{\imath \tanh \zeta}$ and eventually taking the limit $\zeta \to 0$. Keeping track of all the factors of $\imath$ appearing in the calculations, one arrives at (6.125) for the down-string, thus recovering the $v < 1$ result exactly. In this case, there is no up-string solution.

No matter which of the three configurations we use, we can write down the expression for the Wilson loop and extract the Jet-Quenching parameter

$$\langle W(\mathcal{C}) \rangle = e^{\imath (S(\mathcal{C}) - S_0)} = e^{-\frac{TL^2 L^-}{30\alpha' B}} \stackrel{!}{=} e^{-\frac{1}{4\sqrt{2}} \hat{q} L^2 L^-} + \mathcal{O}\left(\frac{1}{N^2}\right) \tag{6.126}$$

In each case the integral B is divergent, and so the jet-quenching factor vanishes.

$$\hat{q} = 0 \tag{6.127}$$

On the non-commutativity of the limits taken As we have seen above and as was noted first in [128] the limits $\eta \to \infty$ and $\Lambda \to \infty$ do not commute. In the same paper, Liu, Rajagopal, and Wiedemann gave a very nice discussion of this issue, which we shall summarize here.

Mathematics From a purely formal point of view, the first indication for noncommutativity is that one needs the Lagrangian to be imaginary in order for the expectation value to be real. This leads to

$$\frac{\Lambda^5}{\cosh^2 \eta} < 1. \tag{6.128}$$

Now regard (6.122). In going from the second line to the third, we need to assume

$$\frac{y^5}{\cosh^2\eta} \ll 1 \quad \Rightarrow \quad \frac{\Lambda^5}{\cosh^2\eta} \ll 1. \tag{6.129}$$

While this is a pretty strong assumption, it is certainly satisfied if one takes the $\eta \to \infty$ limit first. This corresponds with the ansatz taken in [109] where the authors work with a light-like worldsheet in the first place.

Physics As to physics, one need to consider that different types of Wilson loops may be used to study different physical problems. On the one hand, we have jet-quenching, related to a Wilson loop which is again related to the exponential of a real quantity. This is the regime $\cosh\eta \gg \Lambda$. On the other there is the behavior of the (possibly moving) $q\bar{q}$ pair, where the Wilson loop is related to the exponential of an imaginary quantity. Here we have $\cosh\eta \ll \Lambda$. Between these two regions there is a discontinuity at $\cosh\eta \sim \Lambda$.

The authors of [128] go on to point out that if $\cosh\eta \gg 1$ but $\cosh\eta < \Lambda$, the screening length L_{max} is given by

$$L_{\text{max}} = \frac{0.743}{\pi\sqrt{\cosh\eta}T}. \tag{6.130}$$

Also, there is a size δ associated with every external quark, given by

$$\delta \sim \frac{\sqrt{\lambda}}{M} \sim \frac{1}{\Lambda T}. \tag{6.131}$$

$M = M(\Lambda)$ is the mass of the quark. So at the singularity, the screening length is similiar to the size of the quark

$$\delta \sim L_{\text{max}}. \tag{6.132}$$

Now if

$$1 \ll \cosh\eta \ll \Lambda \qquad \text{then} \qquad \delta \ll L_{\text{max}} \tag{6.133}$$

which confirms that the string represents a quarkonium meson. If we trust the above formulas to be true in the limit $\cosh\eta \gg \Lambda$, albeit not having assumed this when defining L_{max}, we realize that because of

$$\delta \gg L_{\text{max}} \tag{6.134}$$

the quark is bigger than its screening length, meaning that there are no $q\bar{q}$ bound states. So there are two different regimes with different physics, depending on

$$\cosh^2 \eta \lessgtr \Lambda^5. \tag{6.135}$$

If we want to examine certain physics, we have to make a choice on how to take the limit.

Drag Force While the jet-quenching method described above only uses the gauge/gravity correspondence to calculate the expectation value of a wilson line, the concept of the drag force, which was introduced in [103, 127], is fully based on the existence of a holographic dual. The main idea is that if one is able to describe a massive quark-antiquark pair as an open string whose both ends are attached to a probe brane at large radius, one might be able to think of a single quark as a single string stretching from the probe to the horizon. Again one uses the Nambu-Goto action in order to study the string's dynamics.

Generically the movement of the quark trough the plasma is governed by

$$\dot{p} = -\mu p + f, \tag{6.136}$$

where p is the quarks momentum, μ a damping coefficient, and f a possible external force. There are two situations of interest here. $f = 0$ and $\dot{p} = 0$.

In the first case, it follows that $\frac{\dot{p}}{p} = -\mu$ and therefore

$$p(t) = e^{-\mu t} p(0). \tag{6.137}$$

One may extract μ numerically from a quasi normal mode analysis of a string stretching between the probe and the boundary.

We shall however not perform the numerical analysis and instead only focus on the second case. A quark moving at a constant speed through the plasma satisfies $\dot{p} = 0$. Yet as the plasma is continuously draining the quark's energy, there has to be an external force f constantly repleneshing the quark's energy and momentum.

Again we place the probe brane at $\rho = \Lambda\rho_0$. To study a single open string hanging down to the horizon, we assume a profile of the form

$$\tau = t \qquad\qquad \sigma = \rho \qquad\qquad x = x(\tau, \sigma) \tag{6.138}$$

where in opposite to (6.48) we allow x to depend on the time. The Nambu-Goto action (6.41) yields the following equations of motion

$$0 = -g_{\rho\rho} g_{xx} \partial_\tau \frac{\dot{x}}{\sqrt{-g}} + \partial_\rho \frac{-g_{tt} g_{xx} x'}{\sqrt{-g}} \tag{6.139}$$

where we defined

$$\begin{aligned} g &= g_{tt} g_{\rho\rho} + g_{tt} g_{xx} x'^2 + g_{\rho\rho} g_{xx} \dot{x}^2, \\ &= -\frac{\rho^2}{9N^2\rho_0^2} - \frac{\rho^2}{9N^2\rho_0^2} f(\rho) x'^2 + \frac{\rho^2}{9N^2\rho_0^2} \frac{1}{f(\rho)} \dot{x}^2, \end{aligned} \tag{6.140}$$

We shall now examine the properties of a specific time-dependent solution. As we will see one may extract information about the string and the quark it describes without fully solving the equations of motion.

Assume $\partial_t x = v$, a constant. Then the equations (6.140) and (6.139) simplify to

$$\begin{aligned} g &= g_{tt} g_{\rho\rho} + g_{tt} g_{xx} x'^2 + g_{\rho\rho} g_{xx} v^2 \\ &= -\frac{\rho^2}{9N^2\rho_0^2} - \frac{\rho^2}{9N^2\rho_0^2} f(\rho) x'^2 + \frac{\rho^2}{9N^2\rho_0^2} \frac{1}{f(\rho)} v^2 \end{aligned} \tag{6.141}$$

and

$$0 = \partial_\rho \frac{-g_{tt} g_{xx} x'}{\sqrt{-g}} \tag{6.142}$$

as $\partial_\tau g = 0$. This can be integrated once and solved for x' to give

$$x'^2 = -\frac{C^2 g_{\rho\rho} \left(g_{tt} + v^2 g_{xx}\right)}{g_{tt} g_{xx} \left(g_{tt} g_{xx} + C^2\right)}, \tag{6.143}$$

where C is a constant of integration.

Plugging this back into (6.44), (6.45) yields

$$\frac{dE}{dt} = \pi_t^\sigma = -\frac{Cv}{2\pi\alpha'} \tag{6.144}$$

$$\frac{dP}{dt} = -\pi_x^\sigma = -\frac{C}{2\pi\alpha'} \tag{6.145}$$

We want the string to reach the horizon. To see whether this is possible, we need to check if the solution is well defined in the region $\rho_0 \leq \rho \leq \Lambda\rho_0$. As usual one needs to require $\sqrt{-g}, x'^2 \geq 0$. From

$$\sqrt{-g} = -g_{tt} g_{xx} x' C^{-1} \tag{6.146}$$

it follows that $\sqrt{-g}$ is real if that is the case for x'^2. A look at (6.143) tells us that we cannot avoid its numerator to change the sign as long as $v \neq 0$. Hence one needs to make sure that both the numerator and the denominator change sign at the same radial position $\rho_\pm$. This amounts to solving

$$g_{tt} + g_{xx}v^2\big|_{\rho=\rho_\pm} = 0 = g_{xx}g_{tt} + C^2\big|_{\rho=\rho_\pm} \tag{6.147}$$

for C. The former equation leads to $\rho_+ = \rho_h(1-v^2)^{1/5}$, from which it follows that

$$C = \frac{v}{3N_c\,(1-v^2)^{1/6}}. \tag{6.148}$$

Hence the energy and momentum loss are

$$\frac{dP}{dt} = -\frac{v}{6\pi N\alpha'\,(1-v^2)^{1/5}} \tag{6.149}$$

$$\frac{dE}{dt} = -\frac{v^2}{6\pi N\alpha'\,(1-v^2)^{1/5}} \tag{6.150}$$

Going back to (6.136), setting $\dot{p} = 0$, taking (6.149) for $-f$, and making use of the relativistic relation $p = \frac{mv}{\sqrt{1-v^2}}$, leads to

$$\mu m = \frac{\left(1-v^2\right)^{3/10}}{6\pi N\alpha'} \tag{6.151}$$

This result has some interesting properties. As long as we consider α' to be finite, the strict $N \to \infty$ limit leads to a vanishing μm. So in this case there is no radiative energy loss. This agrees nicely with the vanishing of the jet-quenching factor $\hat{q}$ studied in section 6.4.4. Furthermore (6.151) even extends that result to quarks of any non-vanishing mass.[13] If we only take N to be large however, equation (6.151) seems rather awkward, as the damping decreases the faster the probe moves.

Also one should not forget that we need α' to be small in order to use the supergravity approximation. More precisely, as was studied in section 6.4.1, the 't Hooft coupling behaves as $\lambda \sim N^5 \frac{\alpha'^2}{\rho^5(\rho^5-\rho_h^5)}$. Thus

$$\mu m \sim \frac{N^{3/2}}{\sqrt{\lambda\,(\rho^{10}-\rho^5\rho_h^5)}}. \tag{6.152}$$

So making a definite statement about the fate of the damping coefficient μ requires a more rigorous study of the relation between the gauge- and the string theory's couplings and energy scales.

[13] Note however that to take the limit $m \to 0$ one needs to bring the probe brane arbitrarily close to the horizon. One should assume that something should happen in this case, i.e. the brane might fall into the horizon.

6.5 Discussion

We have constructed a new solution (6.67) to the equations of motion of eleven-dimensional supergravity. As our discussion of its thermodynamical properties in section 6.3.2 shows there is reason to doubt that it is dual to a four-dimensional gauge theory at finite temperature, leaving us with the question what the field-theory dual of the background in question is. Our comparison with the four- and eleven-dimensional Schwarzschild black holes shows however that the surprising thermodynamical features are to be expected from a solution that is purely gravitational in eleven dimensions. Therefore one might expect to find a better supergravity dual upon generalizing the ansatz (6.64) such that the black hole is charged under the three-form gauge field of eleven-dimensional supergravity.

Despite these problems we were able to exhibit some of the expected features of a gauge-dual at $T > 0$, such as deconfinement and the universal ratio of shear-viscosity and entropy density. Further pathologies of our background are the negative specific heat and the vanishing parton energy loss.

As to the issue of the specific heat one should call to mind the work done by Gubser and Mitra [132, 133, 134], indicating that in fairly general settings a thermodynamic instability is leading to a dynamical one.

One might also consider the following: While our derivation of the shear-viscosity to entropy ratio uses the concept of the stretched horizon introduced by Kovtun, Son, and Starinets [48], one expects to obtain the same universal result from the more standard calculation based on the evaluation of the Kubo-relations. Now as the derivation of photon and dilepton production in the dual plasma [101] is quite similiar ot that of the shear-viscosity one might conjecture these quantities to behave better then the energy loss that was was discussed in this chapter.

6.A The bundle structure of S^3

We examine the bundle structure of S^3, following the classic book by Nakahara [135]. The 3-sphere can be defined as

$$S^3 \equiv \{(z_0, z_1) \in \mathbb{C}^2 | |z_0|^2 + |z_1|^2 = 1\} \qquad (6.153)$$

In the language of [135] this is our total space. Being a manifold, we can equip it with an open covering

$$\begin{aligned} U_0 &\equiv \{(z_0, z_1) \in S^3 | |z_0|^2 \leq \frac{1}{2}\} \\ U_1 &\equiv \{(z_0, z_1) \in S^3 | |z_1|^2 \leq \frac{1}{2}\} \\ U_0 \cap U_1 &= \{(z_0, z_1) | |z_0| = \frac{1}{\sqrt{2}} = |z_1|\} \end{aligned} \tag{6.154}$$

We claim that the base space is S^2 and the fibre $S^1 \simeq U(1)$. To show this, let us first define the projection.

$$\begin{aligned} &\pi : S^3 \to S^2 \simeq \mathbb{CP}^1 \\ &(z_0, z_1) \mapsto [(z_0, z_1)] = \{\lambda(z_0, z_1) | \lambda \in \mathbb{C} \setminus \{0\}\} \end{aligned} \tag{6.155}$$

Now on $U_{0,1}$, we know that $z_{1,0} \neq 0$ and can thus choose $\lambda = z_{1,0}^{-1}$. That means we have the following coordinates on $V_{0,1} \equiv \pi(U_{0,1})$:

$$\zeta_{0,1} \equiv \frac{z_{0,1}}{z_{1,0}} \quad |\zeta_{0,1}| \leq 1. \tag{6.156}$$

There is an overlap between the two coordinate patches

$$V_0 \cap V_1 = \{|\zeta_0| = 1 = |\zeta_1|\} \tag{6.157}$$

on which the coordinates are related as $\zeta_0 = \zeta_1^{-1}$. Our base space has thus the topology of two discs glued together along their boundaries and is therefore a two-sphere.

To confirm that the fibre is indeed $U(1)$, we need to examine π^{-1}. Choose $\zeta \in S^2$. We shall assume w.l.o.g. $\zeta \in V_0$. We can somewhat lift ζ to $\mathbb{CP}^1$ by writing

$$\zeta = (\zeta, 1) \simeq \lambda(\zeta, 1) \quad \lambda \in \mathbb{C} \setminus \{0\} \tag{6.158}$$

We are now looking for points in S^3 which are projected onto this element of $\mathbb{CP}^1$. This is summarised by the equation

$$\kappa(z_0, z_1) = \lambda(\zeta, 1) \tag{6.159}$$

The $\mathbb{C}$-number κ is redundant, leading us to

$$(z_0, z_1) = (\lambda\zeta, \lambda) \Rightarrow |\lambda|^2|\zeta|^2 + |\lambda|^2 = 1 \tag{6.160}$$

While this uniquely determines the modulus of λ, its complex phase remains fully arbitrary. We can summarize this as

$$\pi^{-1}(\zeta) \simeq U(1). \tag{6.161}$$

If we assume the structural group to be $U(1)$, it is obvious that there is a well defined left action on the fibre.

To define the local trivilisations, we shall use the open covering V_i of S^2 that we defined previously. Thanks to our work in the previous paragraphs, it is no work at all to write an explicit expression.

$$\begin{aligned} \Phi_0 : V_0 \times U(1) &\to \pi^{-1}(V_0) = U_0 \\ (\zeta, \phi) &\mapsto \left(re^{\imath\phi}\zeta, re^{\imath\phi}\right) \end{aligned} \tag{6.162}$$

with

$$r = |\lambda| = \sqrt{\frac{1}{1+|\zeta|^2}} \tag{6.163}$$

One can check that

$$\pi\left(re^{\imath\phi}\zeta, re^{\imath\phi}\right) = \lambda\left(re^{\imath\phi}\zeta, re^{\imath\phi}\right) = (\zeta, 1) = \zeta. \tag{6.164}$$

A virtually identical definition holds for V_1.

$$\begin{aligned} \Phi_1 : V_1 \times U(1) &\to \pi^{-1}(V_1) = U_1 \\ (\zeta, \phi) &\mapsto \left(re^{\imath\phi}, re^{\imath\phi}\zeta\right) \end{aligned} \tag{6.165}$$

Finally, we check the transition functions. Assume $\zeta \in V_0 \cap V_1$; it follows that $\zeta = e^{\imath\theta}$.

$$\begin{aligned} t_{01,\zeta}(\phi) &= \Phi_1^{-1}\left(re^{\imath\phi}\zeta, re^{\imath\phi}\right) \\ &= (\zeta^{-1}, \phi+\theta) \in V_1 \times U(1) \end{aligned} \tag{6.166}$$

This shows that the transition function is a simple shift in the fibre and thus certainly a diffeomorphism. Note that in going to the last line, we had to acknowledge that when going from V_0 to V_1 coordinates, we have to invert the element.

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