Stochastic differential delay equation with jumps and application to finance.

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Stochastic Differential Delay Equation with Jumps and Application to Finance

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Abstract

During the last decade Levy processes and other processes with jumps have become increasingly popular for modeling market fluctuations, both for risk management and option pricing purpose. In this thesis, we focus on investigating the approximate solutions for stochastic differential delay equations with jumps. The results include self-contained proof of moment bounds, the rate of convergence under a local Lipchitz condition, convergence in probability under nonlinear growth condition, and the minimal relative entropy martingale measure for a stochastic delay model.

DECLARATION

This work has not previously been accepted in substance for any degree and is not concurrently submitted in candidature for any degree.

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STATEMENT 1

This thesis is the result of my own independent investigation, except where otherwise stated. Other sources are acknowledged by footnotes giving explicit references. A bibliography is appended.

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Introduction

Mathematics develops fast in light of the needs of modern financial market and investment. There is a wide range of fascinating questions in understanding financial systems. Stochastic processes of importance in finance are developed in concert with the tools of stochastic calculus that are methods to solve problems of practical importance. With the development mathematical models, many financial problems are solved by mathematical tools, especially the stochastic differential equations (SDE). Lévy processes provide a convenient framework to model those empirical observations adequately both in the 'real'and in the 'risk-neutral'world, because their sample paths can have jumps and the generating distribution can be fat-tailed and skewed.

The importance of stochastic differential delay equations (SDDEs) derives from the fact that many of the phenomena witnessed around us do not have an effect immediately. However, there is seldom an explicit formula for the solution of an SDDEs, as a result, several numerical schemes have been developed to produce approximate solutions, for example E.Buckwar [3], [4], [5], and X.Mao [32], [34], [35], [36].

In chapter 2, we show that, given a stochastic differential equation with jumps corresponding to a bounded Lévy measure ν, the Euler-Maruyama approximations converge strongly to the exact solution under a global Lipchitz condition as well as under a local Lipchitz condition. An explicitly formula for the convergence rate under a local Lipchitz condition is given. We also investigate the convergence under nonlinear growth condition.

We consider the stochastic differential delay equation of the form

$$dX(t) = \alpha(X(t), X(\delta(t)))dt + \sigma(X(t), X(\delta(t)))dB(t) + \int_{\mathbb{R}^n} \gamma(X(t^-), X(\delta(t^-)), z)\tilde{N}(dt, dz).$$

In chapter 3, using the quadric form of a pure jump process [38], we present a self-contained proof of the $p-$th moment bounds. A proof that the Euler-Maruyama method as well as Stochastic Theta method strongly converge to the exact solution will be presented. We also obtain the rate of the strong convergence under a local Lipschitz condition and a linear growth condition, i.e., if the local Lipschitz constants for balls of radius R are supposed to grow not faster than $\log R$.

In chapter 4, consider the stochastic differential delay equation with jumps under more weaker conditions i.e., a local Lipchitz condition but non-linear growth. We prove that the solution is unique, and the approximations converge to the exact solution in probability.

In chapter 5, we apply such equation to finance. Consider a stock price process given by

$$dS(t) = \alpha S(t-\tau)S(t)dt + S(t^-)\int_{\mathbb{R}} \gamma(S(t-\tau)^-, z)\tilde{N}(dt, dz).$$

We then present the set of martingale measures under which the discounted process is a martingale. Using the Esscher transform and minimum relative entropy, we find a optimal martingale measure for this stochastic delay model.

Chapter 1

Preliminaries

In this chapter, we aim to introduce some notations used throughout this thesis and collect some preliminary results needed in further chapters. We quote the results and refer to Applebaum [1], Cont and Tankov [8], Ikeda [19], Klebaner [25], Øksendal [37] and Protter [40].

1.1 Definition of Lévy processes and related results

Suppose we are given a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with a filtration $(\mathcal{F})_{t\geq 0}$. By a filtration we mean a family of $\sigma-$algebras $(\mathcal{F})_{t\geq 0}$ that is increasing, i.e., $\mathcal{F}_s \subset \mathcal{F}_t$ if $s \leq t$.

Definition 1.1.1. (cf. Protter [40]) *A filtered complete space* $(\Omega, \mathcal{F}, \mathcal{P}, (\mathcal{F})_{t\geq 0})$ *is said to satisfy the usual hypotheses if*

1. $\mathcal{F}_0$ *contains all the* $\mathcal{P}$*-null sets of* $\mathcal{F}$*;*

2. $\mathcal{F}_t = \cap_{u>t}\mathcal{F}_u$*, for all* $0 \leq t \leq \infty$*, that is, the filtration* $(\mathcal{F})_{t\geq 0}$ *is right continuous.*

We always assume that the usual hypotheses hold.

Definition 1.1.2. (cf. Protter [40]) *Let $(\Omega, \mathcal{F}, \mathcal{P}, (\mathcal{F}_t)_{t\geq 0})$ he a filtered probability space. An $\mathcal{F}_t$ adapted process $X = X_t$ with $X_0 = 0$ a.s. is a Lévy process if*

(i) *X has independent increments; that is, $X_t - X_s$ is independent of $\mathcal{F}_s$, $0 \leq s \leq t \leq \infty$;*

(ii) *X has stationary increments; i.e., $X_t - X_s$ has the same distribution as X_{t-s}, $0 \leq s \leq t \leq \infty$;*

(iii) *X_t is continuous in probability; that is, $\lim_{t\to s} X_t = X_s$, where the limit is taken in probability.*

It is well-know that Brownian motion $B(t)$ and a Poisson process $N(t, z)$ have stationary and independent increments. Thus $B(t)$ and $N(t, z)$ are two Lévy processes.

Theorem 1.1.1. (cf. Protter [40]) *Let X_t be Lévy process. There exists a unique modification Y_t of X_t which is càdlàg and which is also a Lévy process.*

Theorem 1.1.2. *(**Lévy decomposition** Øksendal and Sulem [37]). Let X_t be a Lévy processes. Then X_t has the decomposition*

$$X_t = \alpha t + \beta B(t) + \int_{|z|<R} z\tilde{N}(t, dz) + \int_{|z|\geq R} zN(t, dz), \tag{1.1}$$

for some constants $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$, $R \in [0, \infty]$. Here

$$\tilde{N}(t, dz) = N(t, dz) - \nu(dz)t \tag{1.2}$$

is the compensated Poisson random measure of X_t, where $\nu(dz)$ is the Lévy measure of X_t, and $B(t)$ is an independent Brownian motion. Note the process

$$M_t := \tilde{N}(t, z) \text{ is a martingale.} \tag{1.3}$$

If $\alpha = 0$ and $R = \infty$, we call X_t a Lévy martingale.

Theorem 1.1.3. *(**The Lévy-Khintchine formula** Øksendal and Sulem [37]). Let X_t be a Lévy process with Lévy measure ν, such that $\int_{\mathbb{R}} \min(1, x^2)\nu(dx) < \infty$. Then*

$$\mathbb{E}\{e^{iuX_t}\} = e^{-t\psi(u)},$$

where

$$\psi(u) = \frac{\sigma^2}{2}u^2 - i\alpha u + \int_{|x \geq 1|}(1 - e^{iux})\nu(dx) + \int_{|x|<1}(1 - e^{iux} + iux)\nu(dx).$$

Moreover, given ν, σ^2, α the corresponding Lévy process is unique in distribution.

1.2 Martingales

Martingales play a central role in the modern theory of stochastic processes and stochastic calculus.

Definition 1.2.1. (cf. Cont and Tankov [8]) *A càdlàg process $(X_t)_{t\geq 0}$ is said to be a martingale if X_t is adapted to a filtration $(\mathcal{F})_{t\geq 0}$, $\mathbb{E}|X_t|$ is finite for any $t \geq 0$ and for all $s > t$,*

$$\mathbb{E}\left[X_s|\mathcal{F}_t\right] = X_t.$$

A special role in the theory of integration is played by square integrable martingales.

Definition 1.2.2. (cf. Klebaner [25]) *A process X_t is square integrable if $\sup_{t\geq 0} \mathbb{E}[X_t^2] < \infty$. If X_t is considered on a finite time interval $0 \leq t \leq T$, then it is square integrable if $\sup_{0\leq t\leq T} \mathbb{E}[X_t^2] < \infty$. If X_t is a martingale and is square integrable, then it is called a square integrable martingale.*

Theorem 1.2.1. *(**Martingale Convergence Theorem** Protter [40]) If X_t, is an integrable martingale, that is, if $\sup_{t\geq 0}\mathbb{E}|X_t| < \infty$, then there exists an almost sure limit*

$$\lim_{t\to\infty} X_t = Y$$

and Y is an integrable random variable.

Note that expectations $\mathbb{E}[X_t]$ may or may not convergence to the expectation of the limit $\mathbb{E}[Y]$.

We often have to deal with events happening at random times. A random time is nothing else than a positive random variable $T \geq 0$ which represents the time at which some event is going to take place.

Definition 1.2.3. (cf. Protter [40]) *A random variable $T : \Omega \to [0, \infty]$ is a stopping time if the event $\{T \leq t\} \in \mathcal{F}_t$, for every t, $0 \leq t \leq \infty$.*

The most interesting examples of stopping times are hitting times: given a non-anticipating càdlàg process X_t, the hitting time of an open set A is defined by the first time when X_t reaches A:

$$T_A = \inf\{t \geq 0,\ X_t \in A\}.$$

A key application of these concepts is in providing the following random time version of the martingale property.

Theorem 1.2.2. *(**Doob's Optional Stopping Theorem** Protter [40]) If X_t is a càdlàg martingale and S and T are bounded stopping times for which $S \leq T$ a.s., then X_S and X_T are both integrable, with*

$$\mathbb{E}[X_T|\mathcal{F}_S] = X_S \quad a.s..$$

Another useful generalisation of the martingale concept that we will use extensively is that of a local martingale.

Definition 1.2.4. (cf. Protter [40]) *An adapted, càdlàg process X_t is a local martingale if there exists a sequence of increasing stopping times, T_n, with $\lim_{n\to\infty} T_n = \infty$ a.s., such that $X_{t\wedge T_n} 1_{\{T_n>0\}}$ is a uniformly integrable martingale for each n. Such a sequence T_n of stopping times is called a fundamental sequence.*

Different martingales can be constructed from Lévy processes by the fact that they have independent increments. The following proposition shows some relations between Lévy processes and martingales.

Proposition 1.2.1. (cf. Cont and Tankov [8]) *Let $(X_t)_{t\geq 0}$ be a real-valued process with independent increments. Then*

1. $\left(\frac{e^{iuX_t}}{\mathbb{E}\left[e^{iuX_t}\right]}\right)_{t\geq 0}$ *is a martingale for all $u \in \mathbb{R}$.*

2. *If for some $u \in \mathbb{R}$, and all $t \geq 0$ it holds $\mathbb{E}\left[e^{iuX_t}\right] < \infty$ then $\left(\frac{e^{iuX_t}}{\mathbb{E}\left[e^{iuX_t}\right]}\right)_{t\geq 0}$ is a martingale.*

3. *If $\mathbb{E}[X_t] < \infty$ for all $t \geq 0$ then $M_t = X_t - \mathbb{E}[X_t]$ is a martingale, and also a process with independent increments.*

If X_t is a Levy process, for all of the processes introduced in this proposition to be martingales it suffices that the corresponding moments are finite for one value of t.

1.3 Semi-martingales and quadratic variation

Definition 1.3.1. (cf. Klebaner [25]) *A regular adapted càdlàg process is a semi-martingale if it can be represented as a sum of two processes: a local martingale M_t and a process of finite variation A_t,*

$$X_t = X_0 + M_t + A_t,$$

with $M_0 = A_0 = 0$.

The classical examples of semi-martingales are Poisson processes, Brownian motion, and more generally all Lévy processes.

We list some examples of semi-martingale.

1. Each adapted process with càdlàg paths of finite variation on compacts (of finite total variation) is a semi-martingale.

2. Each L^2-martingale with càdlàg paths is a semi-martingale.

3. Each càdlàg, locally square integrable, local martingale is a semi-martingale.

4. A local martingale with continuous paths is a semi-martingale.

5. A decomposable process is a semi-martingale.

The quadratic variation process of a semi-martingale, also known as the square bracket process, is a simple object that nevertheless plays a fundamental role.

Definition 1.3.2. (cf. Klebaner [25]) *Let* X, Y *be semi-martingales. The quadratic variation process of* X, *denoted* $[X,X] = ([X,X])_{t\geq 0}$ *is defined by*

$$[X,X]_t = \lim_{n\to 0} \sum_{i=0}^{n-1} (X(t_{i+1}^n) - X(t_i^n))^2,$$

where $\{t_i^n\}_{i=0}^n$ *is a partition of the interval* $[0,t]$ *and the limit is in probability when* $\delta_n = \max_i(t_{i+1}^n - t_i^n) \to 0$. *The quadratic covariation of* X, Y, *also called the square bracket process of* X, Y, *is defined by*

$$[X,Y]_t = \lim_{n\to 0} \sum_{i=0}^{n-1} (X(t_{i+1}^n) - X(t_i^n))(Y(t_{i+1}^n) - Y(t_i^n)).$$

It is clear that the operation $(X,Y) \to [X,Y]$ is bilinear and symmetric. We therefore have a polarization identity

$$[X,Y] = \frac{1}{2}([X+Y, X+Y] - [X,X] - [Y,Y]).$$

It is known that the quadratic variation of Brownian motion $B(t)$ is $[B,B](t) = t$ and of a Poisson process $N(t,z)$ is $[N,N](t) = N(t,z)$.

Proposition 1.3.1. (cf. Protter [40]) *Let X, Y be two locally square integrable local martingales. Then $[X,Y]$ is the unique adapted càdlàg process A with paths of finite variation on compacts satisfying the two properties:*

- *$XY - A$ is a local martingale;*
- $\Delta A = \Delta X \Delta Y$, $A_0 = X_0 Y_0$.

Proposition 1.3.2. (cf. Protter [40]) *Let X be a local martingale. Then X is a martingale with $\mathbb{E}[X_t^2] < \infty$, for all $t \geq 0$, if and only if $\mathbb{E}\{[X,X]_t\} < \infty$, for all $t \geq 0$. If $\mathbb{E}\{[X,X]_t\} < \infty$, then $\mathbb{E}[X_t^2] = \mathbb{E}\{[X,X]_t\}$.*

Proposition 1.3.3. (cf. Protter [40]) *Let X be a continuous local martingale. Then X and $[X,X]$ have almost surely the same intervals of constancy.*

Definition 1.3.3. (cf. Klebaner [25]) *Suppose that a filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ is given. A process X_t is called predictable (with respect to the filtration) if for each t, X_t is $\mathcal{F}_{t-1}$-measurable, that is, the value of the process X at time t is determined by the information up to and including time $t-1$.*

Definition 1.3.4. (cf. Klebaner [25]) *Let X_t be an adapted process of integrable or locally integrable variation. Its compensator A_t is the unique predictable process such that $M_t = X_t - A_t$ is a local martingale.*

Now, we arrive at another important definition of quadratic variation

Definition 1.3.5. (cf. Klebaner [25]) *The sharp bracket (or conditional quadratic variation)* $\langle X, X \rangle$ *process of a semi-martingale* X *is the compensator of* $[X, X](t)$*, that is, it is the unique predictable process that makes*

$$[X, X](t) - \langle X, X \rangle (t)$$

into a local martingale.

Let X_t be a Poisson process with intensity λ, the conditional quadratic variation $\langle X, X \rangle_t = \lambda t$. However, Let X_t be a Brownian motion the conditional quadratic variation $\langle X, X \rangle_t = [X, X]_t = t$.

Theorem 1.3.1. (cf. Klebaner [25]) *If* X_t *is a continuous semi-martingale with integrable quadratic variation, then* $\langle X, X \rangle_t = [X, X]_t$.

Theorem 1.3.2. (cf. Klebaner [25]) *Let* X_t *be a square integrable martingale. Then the sharp bracket process* $\langle X, X \rangle_t$ *is the unique predictable increasing process for which*

$$X_t^2 - \langle X, X \rangle_t$$

is a martingale.

1.4 Itô's formula

We call a stochastic process an Itô-Lévy process if it has the Lévy decomposition (see Theorem 1.1.2) and its differential version is given by

$$dX_t = \alpha dt + \beta dB(t) + \int_{|z|<R} z\tilde{N}(dt, dz) + \int_{|z|\geq R} zN(dt, dz). \tag{1.4}$$

Theorem 1.4.1. ***(The Itô formula*** *Øksendal and Sulem [37]). Suppose* $X(t) \in \mathbb{R}$ *is an Itô-Lévy process of the form*

$$dX(t) = \alpha(t, \omega)dt + \beta(t, \omega)dB(t) + \int_{\mathbb{R}} \gamma(t, z, \omega)\bar{N}(dt, dz),$$

where

$$\bar{N}(dt,dz) = \begin{cases} N(dt,dz) - \nu(dz)dt, & \text{if } |z| < R; \\ N(dt,dz), & \text{if } |z| \geq R. \end{cases}$$

for some $R \in [0,\infty]$. *Let* $f \in C^2(\mathbb{R}^2)$ *and define* $Y(t) = f(t, X(t))$. *Then* $Y(t)$ *is again an Itô-Lévy process and*

$$\begin{aligned} dY(t) =& \frac{\partial f}{\partial t}(t,X(t))dt + \frac{\partial f}{\partial x}(t,X(t))\left[\alpha(t,\omega)dt + \beta(t,\omega)dB(t)\right] \\ &+ \frac{1}{2}\beta^2(t,\omega)\frac{\partial^2 f}{\partial x^2}(t,X(t))dt \\ &+ \int_{|z|<R} \{f(t,X(t)+\gamma(t,z)) - f(t,X(t)) \\ &\quad - \frac{\partial f}{\partial x}(t,X(t))\gamma(t,z)\Big\}\,\nu(dz)dt \\ &+ \int_{\mathbb{R}} \{f(t,X(t)+\gamma(t,z)) - f(t,X(t))\}\,\bar{N}(dt,dz). \end{aligned}$$

Example 1.4.1. *(**The Itô isometry** Øksendal and Sulem [37]). Let* X_t *be a real-valued Itô-Lévy process with* $X_0 = 0$ *and* $\alpha = 0$. *Then*

$$\mathbb{E}\left[|X(t)|^2\right] = \mathbb{E}\left[\int_0^t |\beta|^2 dt + \int_0^t \int_{\mathbb{R}} |\gamma(t,z)|^2 \nu(dz)dt\right]$$

provided that the right hand side is finite.

1.5 Some useful inequalities

In this section, we present some useful inequalities.

Theorem 1.5.1. *(**Hölder's inequality** Applebaum [1]). Let* $p, q > 1$ *be such that* $1/p + 1/q = 1$. *Let* $f \in L^p(S)$ *and* $g \in L^p(S)$ *and define* $(f,g) : S \to \mathbb{R}$ *by* $(f,g)(x) = (f(x), g(x))$ *for all* $x \in S$. *Then* $(f,g) \in L^1(S)$ *and we have*

$$||(f,g)||_1 \leq ||f||_p ||g||_q.$$

Theorem 1.5.2. ***(Doob's martingale inequality*** *Protter [40]). If $(X_t)_{t\geq 0}$ is a martingale, then for any $p > 1$,*

$$\mathbb{E}\left[\sup_{0\leq s\leq t} |X_s|^p\right] \leq q^p \mathbb{E}\left[|X_t|^p\right],$$

where $1/p + 1/q = 1$.

Theorem 1.5.3. ***(Burkholder-Davis-Gundy inequality*** *Protter [40]). Let X_t be a martingale with càdlàg paths and let $p \geq 1$ be fixed, and T a finite stopping time. Then there exist constants c_p and C_p such that for any X_t*

$$c_p \mathbb{E}\left\{[X,X]_T^{p/2}\right\} \leq \mathbb{E}\left\{\sup_{0\leq t\leq T} |X_t|^p\right\} \leq C_p \mathbb{E}\left\{[X,X]_T^{p/2}\right\}.$$

The constants are universal: they do not depend on the choice of X_t.

Theorem 1.5.4. ***(Gronwall's inequality).*** *Let $[a,b] \subset \mathbb{R}$ be a closed interval and $\alpha,\ \beta :\ [a,b] \to \mathbb{R}$ be a non-negative with α bounded and β integrable. If there exists $C \geq 0$ such that, for all $t \in [a,b]$,*

$$\alpha(t) \leq C + \int_a^t \alpha(s)\beta(s)ds,$$

then we have

$$\alpha(t) \leq C \exp\left[\int_a^t \beta(s)ds\right]$$

for all $t \in [a,b]$.

Chapter 2

Approximate solutions of stochastic differential equations with jumps

2.1 Introduction

Consider the following stochastic differential equations (SDE) driven by a jump process and Brownian motion

$$dX(t) = \alpha(X(t))dt + \sigma(X(t))dB(t) + \int_{\mathbb{R}^n} \gamma(X(t^-), z)\tilde{N}(dt, dz) \qquad (2.1)$$

where $X(t^-)$ denotes $\lim_{s \to t^-} X(s)$, $\alpha : \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \to \mathbb{R}^{n \times d}$ and $\gamma : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n \times m}$. $B(t)$, $\tilde{N}(t, z)$ are standard Brownian motion and a compensated Poisson process respectively. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions. Let $| \cdot |$ denote the Euclidean norm for vectors or the trace norm for matrices. Further we require that the measure with respect to the compensated Poisson process is bounded. As a standing hypothesis we assume that

α, σ and γ are sufficiently smooth so that Eq. (2.1) has a unique solution.

For a given constant stepsize $\Delta t > 0$, let $t_n = n\Delta t$. We want to compute the discrete approximations $Y_n \approx X(t_n)$ by setting $Y_0 = X_0$ and forming

$$Y_{n+1} = Y_n + \alpha(Y_n)\Delta t + \sigma(Y_n)\Delta B_n + \int_{\mathbb{R}^n} \gamma(Y_n, z)\Delta \tilde{N}_n(t, dz). \qquad (2.2)$$

Here $\Delta B_n = B(t_{n+1}) - B(t_n)$ and $\Delta \tilde{N}_n = \tilde{N}(t_{n+1}, z) - \tilde{N}(t_n, z)$. Let

$$Y(t) = Y_n \quad \text{for} \quad t \in [t_n, t_{n+1})$$

and define a continuous time Euler-Maruyama approximation $\bar{Y}(t)$ by

$$\bar{Y}(t) = Y_0 + \int_0^t \alpha(Y(s^-))ds + \int_0^t \sigma(Y(s^-))dB(s) + \int_0^t \int_{\mathbb{R}^n} \gamma(Y(s^-), z)\tilde{N}(ds, dz). \qquad (2.3)$$

We notice that the Burkholder inequality can not give bounds of jumps straightforward, therefore an explicit method for $p-th$ moment bounds is shown in section 2. In section 3, we prove that the Euler-Maruyama method convergence strongly if the coefficients function are either global or local Lipschitz continuous. In section 4, we focus on the convergence rate. We show that the convergence rate is one half under local Lipschitz condition with a special coefficient (Theorem 2.4.1). A convergence in probability theorem is proved in section 5.

2.2 Moments bound property

In this section, we shall state a lemma which gives an estimate for the moments of the exact solution and the approximate solution. Generally speaking, in case $p = 2$ we can use the Itô isometry and the Burkholder-Davis-Gundy inequality to estimate the moments, however, a p-th ($p > 2$) moment bound is non-trivial, because we can not apply the Burkholder-Davis-Gundy inequality for jump terms to get the result straightforward.

Lemma 2.2.1. *Assume that the coefficients α, σ, γ satisfy the linear growth condition,*

(LG) *There exists a constant $h > 0$ such that*

$$||\sigma(x)||^2+|\alpha(x)|^2+\int_{\mathbb{R}}\sum_{k=1}^{m}|\gamma^{(k)}(x,z)|^2\nu_k(dz_k) \leq h(1+|x|^2) \quad \text{for all } x \in \mathbb{R}^n.$$

Then for any $p \geq 2$ there are constants K_1 and K_2 such that the exact solution and the approximate solution of (2.1) satisfy

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|\bar{Y}(t)|^p\right] \vee \mathbb{E}\left[\sup_{0\leq t\leq T}|X(t)|^p\right] < K_1 e^{TK_2},$$

where K_1 and K_2 are constants depending only on h, T, x_0 and K_p.

Remark 2.2.1. *We note that $\nu = (\nu_1, \cdots, \nu_n)$ is the Lévy measure, and moreover, if $N(dt,dz)$ is a standard Poisson process, then $\tilde{N}(dt,dz) = N(dt,dz) - \nu(dz)dt$. Each column $\gamma^{(k)}$ of the $n \times m$ matrix $\gamma = [\gamma_{ij}]$ depends on z only through the k^{th} coordinate z_k, i.e.*

$$\gamma^{(k)}(x,z) = \gamma^{(k)}(x,z_k); \quad z = (z_1, \cdots, z_m) \in \mathbb{R}^m.$$

Proof. Using the integral form of (2.1), we obtain

$$|X(t)|^p \leq 4^{p-1}\Bigg[|x_0|^p + \left|\int_0^t \alpha(X(s^-))ds\right|^p + \left|\int_0^t \sigma(X(s^-))dB(s)\right|^p + \left|\int_0^t\int_{\mathbb{R}^n}\gamma(X(s^-),z)\tilde{N}(ds,dz)\right|^p\Bigg].$$

Applying Hölder's inequality gives

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|X(t)|^p\right]$$

$$\leq 4^{p-1}\Big\{|X(0)|^p + T^{p-1}\mathbb{E}\int_0^T |\alpha(X(s^-))|^p ds$$
$$+\mathbb{E}\sup_{0\leq t\leq T}\Big[\Big|\int_0^t \sigma(X(s^-))dB(s)\Big|^p\Big]$$
$$+\mathbb{E}\sup_{0\leq t\leq T}\Big[\Big|\int_0^t\int_{\mathbb{R}^n}\gamma(X(s^-),z)\tilde{N}(ds,dz)\Big|^p\Big]\Big\}. \tag{2.4}$$

The continuous term is trivial, by Burkholder's inequality (cf. Protter [40]) and using the Hölder inequality we compute that

$$\mathbb{E}\sup_{0\leq t\leq T}\Big[\Big|\int_0^t \sigma(X(s^-))dB(s)\Big|^p\Big] \leq C_p\mathbb{E}\Big[\int_0^T |\sigma(X(s^-))|^2 ds\Big]^{\frac{p}{2}}$$
$$\leq C_p T^{\frac{p}{2}-1}\mathbb{E}\Big[\int_0^T |\sigma(X(s^-))|^p ds\Big], \tag{2.5}$$

where $C_p = \Big\{q^p\Big(\frac{p(p-1)}{2}\Big)\Big\}^{\frac{p}{2}}$, with $\frac{1}{p}+\frac{1}{q}=1$.

We need pay more attention to the jump term, since the Burkholder inequality can not give the nice result as we required. In the computations below, we take a recursive technique and the constant K_p depends only on p and varies from line to line. Let

$$d\eta(t) := \int_{\mathbb{R}^n}\gamma(X(s^-),z)\tilde{N}(ds,dz),$$

from the definition of the quadratic covariation (cf. Øksendal and Sulem [37]), using the integration by parts formula we compute

$$[\eta,\eta]_t = \eta_t^2 - 2\int \eta_{t^-}d\eta_t$$
$$= \int_0^t\int_{\mathbb{R}^n}\gamma^2(X(s^-),z)\nu(dz)dt + \int_0^t\int_{\mathbb{R}^n}\gamma^2(X(s^-),z)\tilde{N}(ds,dz).$$

Applying Burkholder-Davis-Gundy's (cf. Protter [40]) inequality we have

$$\mathbb{E}\sup_{0\leq t\leq T}\Big[\Big|\int_0^t\int_{\mathbb{R}^n}\gamma(X(s^-),z)\tilde{N}(ds,dz)\Big|^p\Big]$$
$$= \mathbb{E}\sup_{0\leq t\leq T}\Big|\int_0^t d\eta(s^-)\Big|^p$$

$$\leq C_p \mathbb{E}\left|\int_0^T d[\eta,\eta]_t\right|^{\frac{p}{2}},$$

and then we arrive at

$$\mathbb{E}\sup_{0\leq t\leq T}\left[\left|\int_0^t\int_{\mathbb{R}^n}\gamma(X(s^-),z)\tilde{N}(ds,dz)\right|^p\right]$$
$$\leq 2^{\frac{p}{2}-1}C_p\mathbb{E}\left\{\int_0^T\int_{\mathbb{R}^n}\left|\gamma(X(s^-),z)\right|^2\nu(dz)ds\right\}^{\frac{p}{2}}$$
$$+2^{\frac{p}{2}-1}C_p\mathbb{E}\left\{\int_0^T\int_{\mathbb{R}^n}\left|\gamma(X(s^-),z)\right|^2\tilde{N}(ds,dz)\right\}^{\frac{p}{2}}.$$

We apply the Burkholder-Davis-Gundy inequality again to the second term on the right hand side above to obtain

$$\mathbb{E}\left\{\int_0^T\int_{\mathbb{R}^n}\left|\gamma(X(s^-),z)\right|^2\tilde{N}(dt,dz)\right\}^{\frac{p}{2}}$$
$$\leq 2^{\frac{p}{4}-1}C_p\mathbb{E}\left\{\int_0^T\int_{\mathbb{R}^n}|\gamma(X(s^-),z)|^4\nu(dz)dt\right\}^{\frac{p}{4}}$$
$$+2^{\frac{p}{4}-1}C_p\mathbb{E}\left\{\int_0^T\int_{\mathbb{R}^n}|\gamma(X(s^-),z)|^4\tilde{N}(dt,dz)\right\}^{\frac{p}{4}}.$$

We continue recursively, which yields

$$\mathbb{E}\sup_{0\leq t\leq T}\left[\left|\int_0^t\int_{\mathbb{R}^n}\gamma(X(s^-),z)\right|^p\tilde{N}(ds,dz)\right]$$
$$\leq\sum_{i=1}^k 2^i 2^{p(1-\frac{2}{2^i})}(C_p)^i\mathbb{E}\left\{\int_0^T\int_{\mathbb{R}^n}|\gamma(X(s^-),z)|^{2^i}\nu(dz)ds\right\}^{\frac{p}{2^i}}$$
$$+2^{-k}2^{p(1-\frac{2}{2^k})}C_p^k\mathbb{E}\left\{\int_0^T\int_{\mathbb{R}^n}|\gamma(X(s^-),z)|^{2^i}\tilde{N}(dt,dz)\right\}^{\frac{p}{2^i}}. \tag{2.6}$$

We see that when $p=2^n$, for the second term above, choosing $n=i+1$, by the Itô isometry, we have

$$\mathbb{E}\left[\int_0^T\int_{\mathbb{R}^n}|\gamma(X(s^-),z)|^{2^i}\tilde{N}(dt,dz)\right]^{\frac{p}{2^i}}=\mathbb{E}\int_0^T\int_{\mathbb{R}^n}\left|\gamma(X(s^-),z)\right|^p\nu(dz)ds.$$

Moreover, all moments of order less than 2^n can be controlled by higher order moment, which follows form Hölder's inequality, i.e. $\mathbb{E}|XY| \le (\mathbb{E}|X|^p)^{1/p}(\mathbb{E}|Y|^q)^{1/q}$ for $p > 1$, $1/p + 1/q = 1$.

On the other hand, for any bound measure ν, by Hölder inequality, we derive

$$\left|\int_{\mathbb{R}^n} \gamma(X(s^-), z)\nu(dz)\right|^p \le (\nu(\mathbb{R}^n))^{\frac{p}{p-1}} \int_{\mathbb{R}^n} |\gamma(X(s^-), z)|^p \nu(dz).$$

Now we compute the first term of (2.6)

$$\begin{aligned}
&\sum_{i=1}^{k} 2^i 2^{p(1-\frac{2}{2^i})}(C_p)^i \mathbb{E}\left\{\int_0^T \int_{\mathbb{R}^n} |\gamma(X(s^-), z)|^{2^i} \nu(dz)ds\right\}^{\frac{p}{2^i}} \\
&\le \sum_{i=1}^{k} 2^i 2^{p(1-\frac{2}{2^i})}(C_p)^i T^{\frac{p}{2^i}-1} \mathbb{E}\int_0^T \left\{\int_{\mathbb{R}^n} |\gamma(X(s^-), z)|^{2^i} \nu(dz)\right\}^{\frac{p}{2^i}} ds \\
&\le \sum_{i=1}^{k} 2^i 2^{p(1-\frac{2}{2^i})}(C_p)^i T^{\frac{p}{2^i}-1} (\nu(\mathbb{R}^n))^{\frac{\frac{p}{2^i}}{\frac{p}{2^i}-1}} \mathbb{E}\int_0^T \int_{\mathbb{R}^n} |\gamma(X(s^-), z)|^p \nu(dz)ds,
\end{aligned}$$

let

$$K_p = \sum_{i=1}^{k} 2^i 2^{p(1-\frac{2}{2^i})}(C_p)^i T^{\frac{p}{2^i}-1} (\nu(\mathbb{R}^n))^{\frac{\frac{p}{2^i}}{\frac{p}{2^i}-1}} + 2^{\frac{p}{2^i}-1} C_p,$$

and by the linear growth condition, we arrive at

$$\mathbb{E} \sup_{0\le t\le T} \left[|\int_0^t \int_{\mathbb{R}^n} \gamma(X(s^-), z)\tilde{N}(ds, dz)|^p \right] \le K_p h \mathbb{E}\int_0^T [1 + |X(s^-)|^p] ds. \tag{2.7}$$

Therefore, combining (2.4), (2.5) and (2.7), we have

$$\begin{aligned}
&\mathbb{E} \sup_{0\le t\le T} |X(t)|^p \\
&\le 4^{p-1}|x_0|^p + 4^{p-1} h(T^{p-1} + K_p + C_p T^{\frac{p}{2}-1})\mathbb{E}\int_0^T [1 + |X(s^-)|^p] ds \\
&\le 4^{p-1}\left[|x_0|^p + Th(T^{p-1} + K_p + C_p T^{\frac{p}{2}-1})\right] \\
&\quad + 4^{p-1} h(T^{p-1} + K_p + C_p T^{\frac{p}{2}-1}) \int_0^T \mathbb{E}\left\{\sup_{0\le u\le s} |X(u^-)|^p\right\} ds
\end{aligned}$$

$$:= K_1 + K_2 \int_0^T \mathbb{E} \sup_{0 \le s \le t} |X(s^-)|^p dt.$$

The well-known Gronwall inequality (see Higham [15]) gives the result as required. The same argument for $\bar{Y}$, we then complete the proof. □

2.3 Strong convergence under a Lipschitz condition

In this section, we shall discuss the strong convergence of the approximate solution to the exact solution under a Lipschitz condition

2.3.1 Convergence under a global Lipschitz condition

The following condition is called global Lipschitz condition

(GL) There exists a constant $L > 0$ such that

$$||\sigma(x) - \sigma(y)||^2 + |\alpha(x) - \alpha(y)|^2 + \sum_{k=1}^{m} \int_{\mathbb{R}} |\gamma^{(k)}(x, z_k) - \gamma^{(k)}(y, z_k)|^2 \nu_k(dz_k) \le L|x - y|^2 \tag{2.8}$$

for all $x, y \in \mathbb{R}^n$.

Theorem 2.3.1. *Under the global Lipschitz condition (GL) it holds*

$$\mathbb{E} \sup_{0 \le t \le T} |X(t) - \bar{Y}(t)|^2 \le 6LTC(\Delta t)(T + 8)e^{6LT(T+8)}$$

where C is a constant independent of Δt.

In order to prove this theorem, we need the following lemma.

Lemma 2.3.1. *For any $t \in [t_k, t_{k+1}]$,*

$$\mathbb{E} \sup_{t_k \le t \le t_{k+1}} |Y(s) - \bar{Y}(s)|^2 \le (3\Delta t + 24)h\Delta t(1 + K_1 e^{TK_2}) := C(\Delta t)$$

where h is a constant independent of Δt, and K_1, K_2 are defined as in Lemma 2.2.1.

Proof. Applying Doob's martingale inequality we obtain

$$\begin{aligned}
\mathbb{E} \sup_{t_k \le t \le t_{k+1}} |Y(s) - \bar{Y}(s)|^2 &= \mathbb{E} \sup_{t_k \le t \le t_{k+1}} |(t - t_k)\alpha(Y_k) + \sigma(Y_k)(B(t) - B(t_k)) \\
&\quad + \int_{\mathbb{R}^n} \gamma(Y_k, z)[\tilde{N}(t, dz) - \tilde{N}(t_k, dz)]|^2 \\
&\le 3(\Delta t)^2 \mathbb{E} \sup_{t_k \le t \le t_{k+1}} |\alpha(Y_k)|^2 + 12(\Delta t)\mathbb{E}|\sigma(Y_k)|^2 \\
&\quad + 3\mathbb{E} \sup_{t_k \le t \le t_{k+1}} \left| \int_{\mathbb{R}^n} \gamma(Y_k, z)[\tilde{N}(t, dz) - \tilde{N}(t_k, dz)] \right|^2 .
\end{aligned} \tag{2.9}$$

Note that $\tilde{N}(t, z)$ is a martingale, therefore we may apply Doob's martingale inequality and the Itô isometry to estimate the last term, which yields

$$\begin{aligned}
&\mathbb{E} \sup_{t_k \le t \le t_{k+1}} \left| \int_{\mathbb{R}^n} \gamma(Y_k, z)[\tilde{N}(t, dz) - \tilde{N}(t_k, dz)] \right|^2 \\
&\le 4\mathbb{E} \left| \int_{\mathbb{R}^n} \gamma(Y_k, z)[\tilde{N}(t, dz) - \tilde{N}(t_k, dz)] \right|^2 \\
&= 4\Delta t \mathbb{E} \int_{\mathbb{R}^n} |\gamma(Y_k, z)|^2 v(dz).
\end{aligned}$$

Substituting this estimate into (2.9), and using the linear growth condition we obtain

$$\mathbb{E} \sup_{t_k \le t \le t_{k+1}} |Y(s) - \bar{Y}(s)|^2 \le (3\Delta t + 24)h\Delta t \mathbb{E}(1 + |Y_k|^2),$$

and the lemma is proven. □

Proof of Theorem 2.3.1. Assume $Y(0) = X(0)$. From the identity

$$X(t) = X(0) + \int_0^t \alpha(X(s^-))ds + \int_0^t \sigma(X(s^-))dB(s) + \int_0^t \int_{\mathbb{R}^n} \gamma(X(s^-), z)\tilde{N}(dt, dz),$$

we have

$$
\begin{aligned}
&\mathbb{E} \sup_{0 \le t \le T} |X(t) - \bar{Y}(t)|^2 \\
&= \mathbb{E} \sup_{0 \le t \le T} \bigg| \int_0^t [\alpha(X(s^-)) - \alpha(Y(s^-))] ds \\
&\quad + \int_0^t [\sigma(X(s^-)) - \sigma(Y(s^-))] dB(s) \\
&\quad + \int_0^t \int_{\mathbb{R}^n} [\gamma(X(s^-), z) - \gamma(Y(s^-), z)] d\tilde{N}(dt, dz) \bigg|^2 \\
&\le 3T\mathbb{E} \sup_{0 \le t \le T} \int_0^t |\alpha(X(s^-)) - \alpha(Y(s^-))|^2 ds \\
&\quad + 3\mathbb{E} \sup_{0 \le t \le T} \left| \int_0^t [\sigma(X(s^-)) - \sigma(Y(s^-))] dB(s) \right|^2 \\
&\quad + 3\mathbb{E} \sup_{0 \le t \le T} \left| \int_0^t \int_{\mathbb{R}^n} [\gamma(X(s^-), z) - \gamma(Y(s^-), z)] \tilde{N}(dt, dz) \right|^2. \qquad (2.10)
\end{aligned}
$$

For the first term of (2.10), applying condition (2.8), we obtain

$$
\begin{aligned}
&\mathbb{E} \sup_{0 \le t \le T} \int_0^t |\alpha(X(s^-)) - \alpha(Y(s^-))|^2 ds \\
&= \mathbb{E} \sup_{0 \le t \le T} \int_0^t |\alpha(X(s^-)) - \alpha(\bar{Y}(s^-)) + \alpha(\bar{Y}(s^-)) - \alpha(Y(s^-))|^2 ds \\
&\le 2\mathbb{E} \sup_{0 \le t \le T} \int_0^t |\alpha(X(s^-)) - \alpha(\bar{Y}(s^-))|^2 ds \\
&\quad + 2\mathbb{E} \sup_{0 \le t \le T} \int_0^t |\alpha(\bar{Y}(s^-)) - \alpha(Y(s^-))|^2 ds \\
&\le 2L\mathbb{E} \sup_{0 \le t \le T} \int_0^t |X(t^-) - \bar{Y}(t^-)|^2 ds + 2LTC(\Delta t). \qquad (2.11)
\end{aligned}
$$

For the second term, Doob's inequality gives

$$
\begin{aligned}
&\mathbb{E} \sup_{0 \le t \le T} \left| \int_0^t [\sigma(X(s^-)) - \sigma(Y(s^-))] dB(s) \right|^2 \\
&\le 4\mathbb{E} \int_0^T |\sigma(X(s^-)) - \sigma(Y(s^-))|^2 ds \\
&\le 8\mathbb{E} \int_0^T |\sigma(X(s^-)) - \sigma(\bar{Y}(s^-))|^2 ds
\end{aligned}
$$

$$
\begin{aligned}
&+ 8\mathbb{E}\int_0^T |\sigma(\bar{Y}(s^-)) - \sigma(Y(s^-))|^2 ds \\
&\leq 8L\mathbb{E}\int_0^T |X(t^-) - \bar{Y}(t^-)|^2 ds + 8LTC(\Delta t) \\
&\leq 8L\mathbb{E}\sup_{0\leq t\leq T}\int_0^t |X(t^-) - \bar{Y}(t^-)|^2 ds + 8LTC(\Delta t).
\end{aligned} \tag{2.12}
$$

For the last term, we note that $\tilde{N}(t,z)$ is a martingale, then we apply Doob's martingale inequality and the Itô isometry (cf. Øksendal and Sulem [37])

$$
\begin{aligned}
&\mathbb{E}\sup_{0\leq t\leq T}\left|\int_0^t\int_{\mathbb{R}^n}[\gamma(X(s^-),z) - \gamma(Y(s^-),z)]\tilde{N}(ds,dz)\right|^2 \\
&\leq 4\mathbb{E}\left|\int_0^T\int_{\mathbb{R}^n}[\gamma(X(s^-),z) - \gamma(Y(s^-),z)]\tilde{N}(ds,dz)\right|^2 \\
&= 4\mathbb{E}\int_0^T\int_{\mathbb{R}^n}|\gamma(X(s^-),z) - \gamma(Y(s^-),z)|^2 v(dz)ds \\
&\leq 8\mathbb{E}\int_0^T\int_{\mathbb{R}^n}|\gamma(X(s^-),z) - \gamma(\bar{Y}(s^-),z)|^2 v(dz)ds \\
&\quad + 8\mathbb{E}\int_0^T\int_{\mathbb{R}^n}|\gamma(\bar{Y}(s^-),z) - \gamma(\bar{Y}(s^-),z)|^2 v(dz)ds \\
&\leq 8L\mathbb{E}\int_0^T |X(t^-) - \bar{Y}(t^-)|^2 ds + 8LTC(\Delta t) \\
&\leq 8L\mathbb{E}\sup_{0\leq t\leq T}\int_0^t |X(t^-) - \bar{Y}(t^-)|^2 ds + 8LTC(\Delta t).
\end{aligned} \tag{2.13}
$$

Together with (2.11), (2.12), it follows that

$$
\mathbb{E}\sup_{0\leq t\leq T}|X(t)-\bar{Y}(t)|^2 \leq 6L(T+8)\mathbb{E}\sup_{0\leq t\leq T}\int_0^t |X(t^-)-\bar{Y}(t^-)|^2 ds + 6LTC(\Delta t)(T+8),
$$

and the result follows from the Gronwall inequality (see Mao [33]). □

2.3.2 Convergence under a local Lipschitz and a linear Growth Condition

In Section 2.3.1, we have shown the strong convergence of the Euler method under a global Lipschitz condition. But in many situations the coefficients are only locally Lipschitz continuous. It is therefore useful to establish the strong convergence of the Euler method under a local Lipschitz condition. By a local Lipschitz condition we mean:

(LL) For each $R \in \mathbb{N}$, there exists a constant L_R such that

$$||\sigma(x)-\sigma(y)||^2+|\alpha(x)-\alpha(y)|^2+\sum_{k=1}^{m}\int_{\mathbb{R}^n}|\gamma(x,z_k)-\gamma(y,z_k)|^2\nu_k(dz_k) \leq L_R|x-y|^2, \tag{2.14}$$

for all $|x|,\ |y| \leq R$.

Theorem 2.3.2. *Under the local Lipschitz condition (LL) and the linear growth condition (LG), we define as before the continuous extension of the Euler scheme by*

$$\bar{Z}(t) = Z_0+\int_0^t \alpha(Z(s^-))ds+\int_0^t \sigma(Z(s^-))dB(s)+\int_0^t\int_{\mathbb{R}^n}\gamma(Z(s^-),z)\tilde{N}(ds,dz),$$

where $Z_0 = X_0$. Then it follows that

$$\lim_{\Delta t\to 0}\mathbb{E}\sup_{0\leq t\leq T}|\bar{Z}(t)-X(t)|^2 = 0$$

Proof. We define the stopping times

$$\tau_R = \inf\{t \geq 0 :\ |\bar{Z}(t)| \geq R\}, \quad \rho_R = \inf\{t \geq 0 :\ |\bar{X}(t)| \geq R\}$$

and write $\theta_R = \tau_R \wedge \rho_R$.

Recall Young's inequality: for $r^{-1}+q^{-1} = 1$ and all $a, b, \delta > 0$

$$ab \leq \frac{\delta}{r}a^r + \frac{1}{q\delta^{q/r}}b^q.$$

Thus, for any $\delta > 0$,

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|\bar{Z}(t)-X(t)|^2\right]$$

$$= \mathbb{E}\left[\sup_{0\le t\le T}|\bar{Z}(t)-X(t)|^2 I_{\{\tau_R>T,\rho_R>T\}}\right] + \mathbb{E}\left[\sup_{0\le t\le T}|\bar{Z}(t)-X(t)|^2 I_{\{\tau_R\le T \text{ or } \rho_R\le T\}}\right]$$
$$\le \mathbb{E}\left[\sup_{0\le t\le T}|\bar{Z}(t\wedge\theta_R)-X(t\wedge\theta_R)|^2 I_{\{\theta_R>T\}}\right] + \frac{2\delta}{p}\mathbb{E}\left[\sup_{0\le t\le T}|\bar{Z}(t)-X(t)|^p\right]$$
$$+ \frac{1-2/p}{p^{\delta^{2/(p-2)}}}\mathbb{P}(\tau_R \le T \text{ or } \rho_R\le T). \tag{2.15}$$

From Lemma 2.2.1, we deduce that

$$\mathbb{P}(\tau_R\le T) = \mathbb{E}\left[I_{\{\tau_R\le T\}}\frac{|X(\tau_R)|^p}{R^p}\right] \le \frac{1}{R^p}\mathbb{E}\left[\sup_{0\le t\le T}|X(t)|^p\right] \le \frac{K_1 e^{K_2}}{R^p}.$$

A similar result can be derived for ρ_R, so that

$$P(\tau_R\le T \text{ or } \rho_R\le T) \le \frac{2K_1e^{K_2}}{R^p}.$$

Also from the moment bounds see Lemma 2.2.1, we have

$$\mathbb{E}\left[\sup_{0\le t\le T}|\bar{Z}(t)-X(t)|^p\right] \le 2^{p-1}\left(\mathbb{E}\left[\sup_{0\ge t\le T}|\bar{Z}(t)|^p\right] + \mathbb{E}\left[\sup_{0\ge t\le T}|X(t)|^p\right]\right) \le 2^p K_1 e^{K_2}.$$

These bounds give

$$\mathbb{E}\sup_{0\le t\le T}|\bar{Z}(t)-X(t)|^2 \le \mathbb{E}\sup_{0\le t\le T}|\bar{Z}(t\wedge\theta_R)-X(t\wedge\theta_R)|^2$$
$$+\frac{2^{p+1}\delta K_1 e^{K_2}}{p} + \frac{2(p-2)K_1e^{K_2}}{p\delta^{2/(p-2)}R^p},$$

for any $\delta>0$, where K_1 and K_2 were defined in Lemma 2.2.1.

Further, we derive

$$\left|\bar{Z}(t\wedge\theta_R)-X(t\wedge\theta_R)\right|^2$$
$$= \Big|\int_0^{t\wedge\theta_R}\alpha(Z(s^-))-\alpha(X(s^-))ds + \int_0^{t\wedge\theta_R}\sigma(Z(s^-))-\sigma(X(s^-))dB(s)$$
$$+\int_0^{t\wedge\theta_R}\int_{\mathbb{R}^n}\gamma(Z(s^-),z)-\gamma(X(s^-),z)\tilde{N}(ds,dz)\Big|^2$$
$$\le 3\left[T\int_0^{t\wedge\theta_R}\left|\alpha(Z(s^-))-\alpha(X(s^-))\right|^2 ds + \left|\int_0^{t\wedge\theta_R}\sigma(Z(s^-))-\sigma(X(s^-))dB(s)\right|^2\right.$$
$$\left.+\left|\int_0^{t\wedge\theta_R}\int_{\mathbb{R}^n}\gamma(Z(s^-),z)-\gamma(X(s^-),z)\tilde{N}(ds,dz)\right|^2\right]$$

Applying Doob's martingale inequality and the Itô isometry under local Lipschitz condition, for any $t \leq T$ we obtain

$$\begin{aligned}
&\mathbb{E} \sup_{0 \leq t \leq T} |\bar{Z}(t \wedge \theta_R) - X(t \wedge \theta_R)|^2 \\
&\leq 3T\mathbb{E} \int_0^{T \wedge \theta_R} \left|\alpha(Z(s^-)) - \alpha(X(s^-))\right|^2 ds + 12\mathbb{E} \int_0^{T \wedge \theta_R} \left|\sigma(Z(s^-))\sigma(X(s^-))\right|^2 ds \\
&\quad + 12\mathbb{E} \int_0^{T \wedge \theta_R} \int_{\mathbb{R}^n} \left|\gamma(Z(s^-), z) - \gamma(X(s^-), z)\right|^2 ds \nu(dz) \\
&\leq 3L_R(T+8)\mathbb{E} \int_0^{T \wedge \theta_R} |Z(s^-) - X(s^-)|^2 ds \\
&\leq 6L_R(T+8) \Bigg[\mathbb{E} \int_0^{T \wedge \theta_R} |Z(s^-) - \bar{Z}(s^-)|^2 ds \\
&\qquad + 6L_R(T+8) \int_0^T \mathbb{E} \sup_{0 \leq r \leq s} |\bar{Z}(r \wedge \theta_R) - X(r \wedge \theta_R)|^2 ds\Bigg].
\end{aligned}$$

From Lemma 2.3.1 we have

$$\mathbb{E} \int_0^{T \wedge \theta_R} |Z(s) - \bar{Z}(s)|^2 ds \leq C(\Delta t),$$

where C is defined as before.

Then applying the continuous Gronwall inequality gives

$$\mathbb{E} \sup_{0 \leq t \leq T} |\bar{Z}(t \wedge \theta_R) - X(t \wedge \theta_R)|^2 \leq 6L_R(T+8)TC(\Delta t)e^{6L_R T(T+8)}.$$

Therefore,

$$\begin{aligned}
\mathbb{E} \sup_{0 \leq t \leq T} |\bar{Z}(t) - X(t)|^2 \leq & 6L_R(T+8)TC(\Delta t)e^{6L_R(T+8)} \\
& + \frac{2^{p+1}\delta K_1 e^{K_2}}{p} + \frac{2(p-2)K_1 e^{K_2}}{p^{\delta^2/(p-2)} R^p}.
\end{aligned}$$

Given any $\epsilon > 0$, we can choose δ such that

$$\frac{2^{p+1}\delta K_1 e^{K_2}}{p} < \frac{\epsilon}{3},$$

and then we choose R such that

$$\frac{2(p-2)K_1 e^{K_2}}{p^{\delta^2/(p-2)} R^p} < \frac{\epsilon}{3}.$$

Now for any sufficiently small Δt it follows that

$$6L_R(T+8)TC(\Delta t)e^{6L_RT(T+8)} < \frac{\epsilon}{3}$$

so that $\mathbb{E}\sup_{0\le t\le T}|\bar{Z}(t)-X(t)|^2 < \epsilon$, as required. □

2.4 Rate of the convergence under a local Lipschitz condition and a linear growth condition

So far we have investigated the behaviour of SDEs under a local Lipschitz condition, however, the rate of convergence with such condition can not be given explicitly. A joint work with N.Jacob and C.Yuan will be discussed in this section, which shows when the coefficient L_R satisfying some special condition, we still have the rate of convergence with a local Lipschitz.

Lemma 2.4.1. *Under a global Lipschitz condition, compare* (2.8), *it holds*

$$\mathbb{E}\sup_{0\le t\le T}|X(t)-\bar{Y}(t)|^4 \le 27L^2\xi(27\Delta t+128)h^2(\Delta t)^2(1+K_1e^{TK_2})^2Te^{27L^2\xi T}.$$

where ξ is a constant independent of Δt, more precisely,

$$\xi = T^3 + \left(\frac{512}{27}\right)^2 T + 2\left(\frac{512}{27}\right)^2 (T(\nu(\mathbb{R}^n))^2+1).$$

Proof. Set $e(t)=X(t)-\bar{Y}(t)$. We obtain first

$$\begin{aligned}
|e(t)|^4 &= |X(t)-\bar{Y}(t)|^4 \\
&\le 27\left|\int_0^t[\alpha(X(s^-))-\alpha(Y(s^-))]ds\right|^4 \\
&\quad + 27\left|\int_0^t[\sigma(X(s^-))-\sigma(Y(s^-))]dB(s)\right|^4
\end{aligned}$$

$$+ 27\left|\int_0^t \int_{\mathbb{R}^n} [\gamma(X(s^-), z) - \gamma(Y(s^-), z)] d\tilde{N}(dt, dz)\right|^4,$$

and we have

$$\begin{aligned}\mathbb{E} \sup_{0\leq t\leq T} |e(t)|^4 \leq 27\mathbb{E} \sup_{0\leq t\leq T} &\left|\int_0^t [\alpha(X(s^-)) - \alpha(Y(s^-))] ds\right|^4 \\ + 27\mathbb{E} \sup_{0\leq t\leq T} &\left|\int_0^t [\sigma(X(s^-)) - \sigma(Y(s^-))] dB(s)\right|^4 \\ + 27\mathbb{E} \sup_{0\leq t\leq T} &\left|\int_0^t \int_{\mathbb{R}^n} [\gamma(X(s^-), z) - \gamma(Y(s^-), z)] d\tilde{N}(dt, dz)\right|^4.\end{aligned}$$

By condition (2.8), and Hölder's inequality, we estimate the first term

$$\begin{aligned}&\mathbb{E} \sup_{0\leq t\leq T} \left|\int_0^t [\alpha(X(s^-)) - \alpha(Y(s^-))] ds\right|^4 \\ &\leq T^3\mathbb{E} \sup_{0\leq t\leq T} \int_0^t |\alpha(X(s^-)) - \alpha(Y(s^-))|^4 ds \\ &\leq T^3L^2\mathbb{E} \sup_{0\leq t\leq T} \int_0^t |X(s^-) - Y(s^-)|^4 ds. \qquad (2.16)\end{aligned}$$

Applying the Burkholder inequality for treating the second term we find

$$\begin{aligned}&\mathbb{E} \sup_{0\leq t\leq T} \left|\int_0^t [\sigma(X(s^-)) - \sigma(Y(s^-))] dB(s)\right|^4 \\ &\leq \left(\frac{512}{27}\right)^2 \mathbb{E}\left[\int_0^T |\sigma(X(s^-)) - \sigma(Y(s^-))|^2 ds\right]^2 \\ &\leq \left(\frac{512}{27}\right)^2 L^2\mathbb{E}\left[\int_0^T |X(s^-) - (Y(s^-)|^2 ds\right]^2 \\ &\leq \left(\frac{512}{27}\right)^2 L^2T\mathbb{E} \int_0^T |X(s^-) - (Y(s^-)|^4 ds. \qquad (2.17)\end{aligned}$$

Using Lemma 2.2.1, we find a bound of the third term

$$\begin{aligned}&\mathbb{E} \sup_{0\leq t\leq T} \left|\int_0^t \int_{\mathbb{R}^n} [\gamma(X(s^-), z) - \gamma(Y(s^-), z)] \tilde{N}(ds, dz)\right|^4 \\ &\leq 2\left(\frac{512}{27}\right)^2 \mathbb{E}\left|\int_0^T \int_{\mathbb{R}^n} [\gamma(X(s^-), z) - \gamma(Y(s^-), z)]^2 \nu(dz) ds\right|^2\end{aligned}$$

$$+ 2\left(\frac{512}{27}\right)^2 \mathbb{E}\left|\int_0^T \int_{\mathbb{R}^n} [\gamma(X(s^-), z) - \gamma(Y(s^-), z)]^2 \tilde{N}(dt, dz)\right|^2$$

$$\leq 2\left(\frac{512}{27}\right)^2 (T(\nu(\mathbb{R}^n))^2 + 1)\mathbb{E}\int_0^T \int_{\mathbb{R}^n} |\gamma(X(s^-), z) - \gamma(Y(s^-), z)|^4 \nu(dz)ds. \tag{2.18}$$

Combining (2.16) , (2.17) and (2.18) we have

$$\begin{aligned}
&\mathbb{E}\sup_{0\leq t\leq T} |X(t) - \bar{Y}(t)|^4 \\
&\leq 27T^3L^2\mathbb{E}\sup_{0\leq t\leq T}\int_0^t |X(s^-) - Y(s^-)|^4 ds \\
&\quad + 27\left(\frac{512}{27}\right)^2 L^2 T\mathbb{E}\int_0^T |X(s^-) - Y(s^-)|^4 ds \\
&\quad + 54\left(\frac{512}{27}\right)^2 (T(\nu(\mathbb{R}^n))^2 + 1)L^2\mathbb{E}\int_0^T |X(s^-) - Y(s^-)|^4 ds \\
&\leq 27L^2\left[T^3 + \left(\frac{512}{27}\right)^2 T + 2\left(\frac{512}{27}\right)^2 (T(\nu(\mathbb{R}^n))^2 + 1)\right] \\
&\qquad \times \mathbb{E}\sup_{0\leq t\leq T}\int_0^t [|X(s^-) - \bar{Y}(s^-)|^4 + |\bar{Y}(s^-) - Y(s^-)|^4] ds \\
&:= 27L^2\xi\mathbb{E}\sup_{0\leq t\leq T}\int_0^t [|X(s^-) - \bar{Y}(s^-)|^4 + |\bar{Y}(s^-) - Y(s^-)|^4] ds.
\end{aligned}$$

From Lemma 2.2.1, Lemma 2.3.1 and the linear growth condition, we obtain

$$\mathbb{E}|\bar{Y}(s) - Y(s)|^4 \leq (27\Delta t + 128)h^2(\Delta t)^2(1 + K_1 e^{TK_2})^2,$$

which substituted into the above, yields

$$\begin{aligned}
&\mathbb{E}\sup_{0\leq t\leq T} |X(t) - \bar{Y}(t)|^4 \\
&\leq 27L^2\xi\mathbb{E}\sup_{0\leq t\leq T}\int_0^t |X(s^-) - \bar{Y}(s^-)|^4 ds \\
&\quad + 27L^2\xi(27\Delta t + 128)h^2(\Delta t)^2(1 + K_1 e^{TK_2})^2 T.
\end{aligned}$$

Once again, the required assertion follows from the Gronwall inequality . □

The next theorem give the convergence rate under a local Lipschitz condition.

Theorem 2.4.1. *Under a local Lipschitz condition (LL) and a linear growth condition (LG), if there is a constant* δ *such that* $(L_R)^2\xi T \leq \delta \log R$, *then the order of convergence is* $\frac{1}{2}$.

Proof. We consider a family of smooth functions $\varphi_R : \mathbb{R} \to \mathbb{R}$ satisfying $0 \leq \varphi_R \leq 1$ such that

$$\begin{cases} \varphi_R(x) = 1, & \text{for } |x| \leq R; \\ \varphi_R(x) = 0, & \text{for } |x| > R + 2. \end{cases}$$

Define

$$\alpha_R(x) = \varphi_R(x)\alpha(x), \ \sigma_R(x) = \varphi_R(x)\sigma(x)$$

and

$$\int_{\mathbb{R}^n} \gamma_R(x, z)\nu(dz) = \varphi_R(x) \int_{\mathbb{R}^n} \gamma(x, z)\nu(dz).$$

Let $\bar{Z}_R(t)$ be the Euler-Maruyama approximation to the following stochastic differential equation

$$dX_R(t) = \alpha_R(X_R(t))dt + \sigma_R(X_R(t))dB(t) + \int_{\mathbb{R}^n} \gamma_R(X_R(t), z)\tilde{N}(dt, dz)$$

with $\bar{Z}_R(0) = X_0$. By the Lemma 2.4.1 we obtain

$$\mathbb{E} \sup_{0 \leq t \leq T} |X_R(t) - \bar{Z}_R(t)|^4 \leq 27L_R^2\xi(27\Delta t + 128)h^2(\Delta t)^2(1 + K_1 e^{TK_2})^2 T e^{27L_R^2\xi T}.$$

Let

$$\hat{X}(t) = \sup_{0 \leq t \leq T} |X(t)| \ \textit{and} \ \hat{Z}(t) = \sup_{0 \leq t \leq T} |\bar{Z}(t)|.$$

On the one hand

$$|X(t) - \bar{Z}(t)|^2 = \sum_{R=1}^{\infty} |X(t) - \bar{Z}(t)|^2 I_{\{R-1 \leq \hat{X}(T) \vee \hat{Z}(T) \leq R\}}$$

$$= \sum_{R=1}^{\infty} |X_R(t) - \bar{Z}_R(t)|^2 I_{\{R-1 \le \hat{X}(T) \vee \hat{Z}(T) \le R\}},$$

where $\bar{Z}(t)$ is defined as in Theorem 2.3.2. Therefore

$$\begin{aligned}
&\mathbb{E} \sup_{0 \le t \le T} |X(t) - \bar{Z}(t)|^2 \\
&\le \sum_{R=1}^{\infty} (\mathbb{E}|X_R(t) - \bar{Z}_R(t)|^4)^{1/2} (\mathbb{E} I_{\{R-1 \le \hat{X}(T) \vee \hat{Z}(T) \le R\}})^{1/2} \\
&= \sum_{R=1}^{\infty} (\mathbb{E}|X_R(t) - \bar{Z}_R(t)|^4)^{1/2} \sqrt{\mathbb{P}(R-1 \le \hat{X}(T) \vee \hat{Z}(T) \le R)}
\end{aligned}$$

and from the condition $(L_R)^2 \xi T \le \delta \log R$, it follows that

$$\mathbb{E} \sup_{0 \le t \le T} |X_R(t) - \bar{Z}_R(t)|^4 \le 27(27\Delta t + 128) h^2 (\Delta t)^2 (1 + K_1 e^{TK_2})^2 T R^{54\delta}.$$

On the other hand, let $q \ge 2$, then

$$\mathbb{P}(R - 1 \le \hat{X}(T) \vee \hat{Z}(T)) \le \frac{\mathbb{E}|\hat{X}(T)|^q + \mathbb{E}|\hat{Z}(T)|^q}{R^q} \le 2\frac{K_1 e^{TK_2}}{R^q}.$$

Therefore,

$$\begin{aligned}
&\mathbb{E} \sup_{0 \le t \le T} |X(t) - \bar{Z}(t)|^2 \\
&\le \sum_{R=1}^{\infty} \sqrt{27(27\Delta t + 128) T h^2 (1 + K_1 e^{TK_2})^2} (\Delta t) R^{27\delta} 2 \frac{(K_1 e^{TK_2})^{1/2}}{R^{q/2}}.
\end{aligned}$$

If $q/2 > 27\delta$, the right hand side is convergent. $\square$

2.5 Convergence in probability

In this section, we concentrate on equation (2.1) satisfying only a local Lipschitz condition but no linear growth condition or the bounded pth moment property. The following theorem describes the convergence in probability, instead of L^2, of the EM solutions to the exact solution under some additional conditions in terms of Lyapunov-type functions.

Theorem 2.5.1. *Let the local Lipschitz condition hold. Assume that there exists a C^2 function $V : \mathbb{R}^n \to \mathbb{R}_+$ satisfying the following three conditions:*

(i) $\lim_{|x|\to\infty} V(x) = \infty$;

(ii) *for some $h > 0$,*

$$\mathcal{L}V(x) \leq h(1 + V(x)) \quad \forall x \in \mathbb{R}^n,$$

where

$$\mathcal{L}V(x) = \sum_{i=1}^{n} \alpha_i(x) \frac{\partial V}{\partial x_i}(x) + \frac{1}{2} \sum_{i,j=1}^{n} (\sigma\sigma^T)_{i,j}(x) \frac{\partial V}{\partial x_i \partial xj}(x)$$
$$+ \int_{\mathbb{R}} \sum_{k=1}^{m} \left\{ V(x + \gamma^{(k)}(x,z)) - V(x) - \nabla V(x) \cdot \gamma^{(k)}(x,z) \right\} \nu_k(dz_k).$$

(iii) *for each $R > 0$ there exists a positive constant K_R such that for all $x, y \in \mathbb{R}^n$ with $|x| \vee |y| \leq R$,*

$$|V(x) - V(y)| \vee |V_x(x) - V_x(y)| \vee |V_{xx}(x) - V_{xx}(y)| \leq K_R |x - y|,$$

where $V_x(x) = \left(\frac{\partial V(x)}{\partial x_1}, \ldots, \frac{\partial V(x)}{\partial x_n} \right)$, $\quad V_{xx}(x) = \left(\frac{\partial^2 V(x)}{\partial x_i \partial x_j} \right)_{n \times n}$.

Then

$$\lim_{\Delta t \to 0} \left(\sup_{0 \leq t \leq T} |X(t) - \bar{Y}(t)|^2 \right) = 0 \quad \textit{in probability.} \tag{2.19}$$

Proof. For sufficiently large R, define stopping times

$$\theta = \inf\{t \in [0,T] : \ |X(t)| \geq R\},$$

and

$$\rho = \inf\{t \in [0,T] : \ |\bar{Y}(t)| \geq R\}.$$

Applying the multi-dimensional Itô's formula (cf. Øksendal and Sulem [37]) and applying condition (ii) to $V(X(t))$, we obtain

$V(X(t \wedge \theta))$

$$
\begin{aligned}
&= V(X(0)) + \int_0^{t\wedge\theta} \sum_{i=1}^{n} \frac{\partial V}{\partial X_i(s^-)} \alpha_i(X(s^-))ds \\
&+ \int_0^{t\wedge\theta} \sum_{i=1}^{n} \frac{\partial V}{\partial X_i(s^-)} \sigma_i(X(s^-))dB(s) + \frac{1}{2}\int_0^{t\wedge\theta} \sum_{i,j=1}^{n} (\sigma\sigma^T)_{i,j} \frac{\partial V^2}{\partial X_i(s^-)\partial X_j(s^-)} ds \\
&+ \int_0^{t\wedge\theta} \sum_{k=1}^{m} \int_{\mathbb{R}} \left\{ V\left[X(s^-) + \gamma^{(k)}(X(s^-), z_k)\right] - V(X(s^-)) \right. \\
&\qquad \left. - \nabla V(X(s^-))\gamma^{(k)}(X(s^-), z_k) \right\} \nu_k(dz_k)ds \\
&+ \int_0^{t\wedge\theta} \sum_{k=1}^{m} \int_{\mathbb{R}} \left\{ V\left[X(s^-) + \gamma^{(k)}(X(s^-), z_k)\right] - V(X(s^-)) \right\} \tilde{N}_k(ds, dz_k) \\
&= V(X(0)) + \int_0^{t\wedge\theta} \mathcal{L}V(X(s^-))ds + \int_0^{t\wedge\theta} \sum_{i=1}^{n} \frac{\partial V}{\partial X_i(s^-)} \sigma_i(X(s^-))dB(s) \\
&+ \int_0^{t\wedge\theta} \sum_{k=1}^{m} \int_{\mathbb{R}} \left\{ V\left[X(s^-) + \gamma^{(k)}(X(s^-), z_k)\right] - V(X(s^-)) \right\} \tilde{N}_k(ds, dz_k)
\end{aligned}
$$

where $\gamma^{(k)}$ is the $k-$th column of the matrix (γ_{ik}).

We note that the third and the sixth term are martingales with zero mean, taking expectation of $V(X(t \wedge \theta))$, and by condition (ii) we arrive at

$$
\begin{aligned}
\mathbb{E}\left[V(X(t\wedge\theta))\right] &= V(X(0)) + \mathbb{E}\int_0^{t\wedge\theta} \mathcal{L}V(X(s^-))ds \\
&\leq V(X(0)) + h\mathbb{E}\int_0^{t\wedge\theta} (1 + V(X(s^-))ds \\
&\leq V(X(0)) + hT + h\mathbb{E}\int_0^{t} V(X(s\wedge\theta))ds.
\end{aligned}
$$

Using the Gronwall inequality we have

$$
\mathbb{E}\left[V(X(t\wedge\theta))\right] \leq (V(X(0)) + hT)e^{hT}. \tag{2.20}
$$

Let

$$
v_R = \inf\{V(x) : |x| \geq R\}.
$$

By condition (i), $v_R \to \infty$ as $R \to \infty$. Note that $|X(\theta)| = R$ whenever $\theta < T$, and therefore we derive from (2.20) that

$$
v_R\mathbb{P}(\theta < T) \leq \mathbb{E}V(X(\theta)I_{\{\theta<T\}}) \leq V(X(0)) + hT)e^{hT},
$$

that is

$$\mathbb{P}(\theta < T) \leq \frac{V(X(0)) + hT)e^{hT}}{v_R}. \tag{2.21}$$

Once again applying the multi-dimensional Itô formula to $V(\bar{Y}(t \wedge \rho))$ yields

$$\begin{aligned}
& V(\bar{Y}(t \wedge \rho)) \\
&= V(Y(0)) + \int_0^{t\wedge\rho} \sum_{i=1}^{n} \frac{\partial V(\bar{Y}(s^-))}{\partial \bar{Y}_i(s^-)} \alpha_i(Y(s^-))ds \\
&+ \int_0^{t\wedge\rho} \sum_{i=1}^{n} \frac{\partial V(\bar{Y}(s^-))}{\partial \bar{Y}_i(s^-)} \sigma_i(Y(s^-))dB(s) + \frac{1}{2}\int_0^{t\wedge\rho} \sum_{i,j=1}^{n} (\sigma\sigma^T)_{i,j} \frac{\partial V^2(\bar{Y}(s^-))}{\partial \bar{Y}_i(s^-))\partial \bar{Y}_j(s^-)} ds \\
&+ \int_0^{t\wedge\rho} \sum_{k=1}^{m} \int_{\mathbb{R}} \{V\left[\bar{Y}(s^-)) + \gamma^{(k)}(Y(s^-), z_k)\right] - V(\bar{Y}(s^-)) \\
&\qquad - \nabla V(\bar{Y}(s^-))\gamma^{(k)}(Y(s^-), z_k)\} \nu_k(dz_k)ds \\
&+ \int_0^{t\wedge\rho} \sum_{k=1}^{m} \int_{\mathbb{R}} \{V\left[\bar{Y}(s^-) + \gamma^{(k)}(Y(s^-), z_k)\right] - V(\bar{Y}(s^-))\} \tilde{N}_k(ds, dz_k) \\
&= V(Y(0)) + \int_0^{t\wedge\rho} \mathcal{L}V(Y(s^-))ds + \int_0^{t\wedge\rho} \sum_{i=1}^{n} \frac{\partial V(\bar{Y}(s^-))}{\partial \bar{Y}_i(s^-)} \sigma_i(Y(s^-))dB(s) \\
&+ \frac{1}{2}\int_0^{t\wedge\rho} \sum_{i,j=1}^{n} (\sigma\sigma^T)_{i,j} \left[\frac{\partial V^2(\bar{Y}(s^-))}{\partial \bar{Y}_i(s^-)\partial \bar{Y}_j(s^-)} - \frac{\partial V^2(Y(s^-))}{\partial Y_i(s^-)\partial Y_j(s^-)}\right] ds \\
&+ \int_0^{t\wedge\rho} \sum_{k=1}^{m} \int_{\mathbb{R}} \{\left[V(\bar{Y}(s^-) + \gamma^{(k)}(Y(s^-), z_k)) - V(Y(s^-) + \gamma^{(k)}(Y(s^-), z_k))\right] \\
&\qquad\qquad - \left[V(\bar{Y}(s^-)) - V(Y(s^-))\right] \\
&\qquad - \left[\nabla V(\bar{Y}(s^-)) - \nabla V(Y(s^-))\right] \gamma^{(k)}(Y(s^-), z_k)\} \nu_k(dz_k)ds \\
&+ \int_0^{t\wedge\rho} \sum_{k=1}^{m} \int_{\mathbb{R}} \{\left[V(\bar{Y}(s^-) + \gamma^{(k)}(Y(s^-), z_k)) - V(Y(s^-) + \gamma^{(k)}(Y(s^-), z_k))\right] \\
&\qquad - \left[V(\bar{Y}(s^-)) - V(Y(s^-))\right]\} \tilde{N}_k(ds, dz_k).
\end{aligned}$$

From the definition of the stopping times, we note that there must exist a constant $C(R)$ such that

$$\alpha(Y(s)) \vee \sigma(Y(s)) \vee \sum_{k=1}^{m} \int_{\mathbb{R}} \gamma^{(k)}(Y(s^-), z_k)\nu_k(dz_k) \leq C(R)$$

when $|Y(s)| \leq R$, and in the computation below, the constant $C(R)$ varies line by line. Taking expectation of $V(\bar{Y}(s^-))$ and by condition (iii), we have

$$\begin{aligned}\mathbb{E}(V(\bar{Y}(t\wedge\rho))) \leq V(Y(0)) &+ h\mathbb{E}\int_0^{t\wedge\rho}(1+V(Y(s^-))ds \\ &+ \frac{1}{2}K_R\mathbb{E}\int_0^{t\wedge\rho}\sum_{i,j=1}^{n}(\sigma\sigma^T)_{i,j}|\bar{Y}(s^-)-Y(s^-)|ds \\ &+ K_R\mathbb{E}\int_0^{t\wedge\rho}|\bar{Y}(s^-)-Y(s^-)|\sum_{k=1}^{m}\int_{\mathbb{R}}\gamma^{(k)}(Y(s^-),z_k)\nu_k(dz_k)ds.\end{aligned}$$

In view of Lemma 2.3.1, we can show that

$$\mathbb{E}|\bar{Y}(t) - Y(t)| \leq C(R)\Delta t,$$

so that

$$\mathbb{E}(V(\bar{Y}(t\wedge\rho))) \leq V(Y(0)) + hT + h\mathbb{E}\int_0^t V(\bar{Y}(s\wedge\rho))ds + C(R)\Delta t.$$

By the same computation leading to (2.21), we obtain

$$\mathbb{P}(\rho < T) \leq [V(Y(0)) + hT + C(R)\Delta t]\,\frac{e^{hT}}{v_R}. \tag{2.22}$$

Now, let$\tau = \rho \wedge \theta$. Recall Theorem 3.3, then we have

$$\mathbb{E}\left[\sup_{0\leq t\leq\tau\wedge T}|X(t)-\bar{Y}(t)|^2\right] \leq C(R)(\Delta t).$$

Assume $\epsilon,\ \alpha \in (0,1)$, set

$$\bar{\Omega} = \{\omega : \sup_{0\leq t\leq T}|X(t)-Y(t)|^2 \geq \alpha\}.$$

We compute

$$\alpha\mathbb{P}(\bar{\Omega}\cap\{\tau\geq T\}) = \alpha\mathbb{E}\left[I_{\tau\geq T}I_{\bar{\Omega}}\right]$$

$$
\begin{aligned}
&\le \mathbb{E}\left[I_{\tau\ge T}\sup_{0\le t\le \tau\wedge T}|X(t)-\bar{Y}(t)|^2\right]\\
&\le \mathbb{E}\left[\sup_{0\le t\le \tau\wedge T}|X(t)-\bar{Y}(t)|^2\right]\\
&\le C(R)(\Delta t),
\end{aligned}
$$

together with (2.21) and (2.22), this yields

$$
\begin{aligned}
\mathbb{P}(\bar{\Omega}) &\le \mathbb{P}(\bar{\Omega}\cap\{\tau\ge T\}) + \mathbb{P}(\tau<T)\\
&\le \mathbb{P}(\bar{\Omega}\cap\{\tau\ge T\}) + \mathbb{P}(\theta<T) + \mathbb{P}(\rho<T)\\
&\le \frac{C(R)}{\alpha}(\Delta t) + \frac{(V(X(0))+hT)e^{hT}}{v_R}\\
&\qquad + [V(Y(0))+hT+C(R)\Delta t]\,\frac{e^{hT}}{v_R}.
\end{aligned}
$$

Recalling that $v_R \to \infty$ as $R\to\infty$, we can choose R sufficiently large for obtaining

$$
\frac{e^{hT}}{v_R}2[V(Y(0))+hT] < \frac{\epsilon}{2},
$$

and then choose Δt sufficiently small to get

$$
\frac{C(R)}{\alpha}(\Delta t) + \frac{e^{hT}}{v_R}C(R)(\Delta t) < \frac{\epsilon}{2},
$$

hence we arrive at

$$
\mathbb{P}(\bar{\Omega}) = \mathbb{P}\left(\sup_{0\le t\le T}|X(t)-Y(t)|^2 \ge \alpha\right) < \epsilon.
$$

□

Chapter 3

Approximate solutions of stochastic differential delay equations with jumps

3.1 Introduction

The importance of stochastic differential delay equations (SDDEs) derives from the fact that many of phenomena do not have an effect immediately. However, there are seldom explicit formula for solutions of SDDEs, and several numerical schemes have been developed to produce approximate solutions, for example [1], [2], [3], [4], [32], [34], [35]. In this chapter, we investigate numerical schemes for SDDEs with jumps.

Throughout this chapter, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathcal{F}_0$ contains all P-null sets). Let $B(t)$ be $m-$dimensional Brownian motion and $N(dt, dz)$ be a Poisson measure and denote the compensated Poisson measure by

$$\tilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt.$$

Let $|\cdot|$ denote the Euclidean norm as well as the matrix trace norm. Let $\tau > 0$ and $C([-\tau, 0]; \mathbb{R}^d)$ denote the family of continuous function ϕ from $[-\tau, 0]$ to $\mathbb{R}^d$ with the norm $||\phi|| = \sup_{-\tau \leq \theta \leq 0} |\phi(\theta)|$. Denote by $C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^d)$ the family of all bounded, $\mathcal{F}_0-$measurable, $C([-\tau, 0]; \mathbb{R}^d)-$valued random variables.

Let τ and T be positive constants. Consider the $d-$dimensional stochastic differential delay equation with jumps

$$\begin{aligned} dX(t) = \alpha(X(t), X(\delta(t)))dt + \sigma(X(t), X(\delta(t)))dB(t) \\ + \int_{\mathbb{R}^n} \gamma(X(t^-), X(\delta(t^-)), z)\tilde{N}(dt, dz) \end{aligned} \tag{3.1}$$

on $t \in [0, T]$ with initial data

$$\{X(t) : \ -\tau \leq t \leq 0\} = \{\zeta(t) : \ -\tau \leq t \leq 0\} \in C^b_{\mathcal{F}_0}([-\tau, 0]),$$

and where $X(t^-) = \lim_{s \to t} X(s)$, $\alpha : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$, $\sigma : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}^{d \times m}$, $\gamma : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}^{d \times n}$. We note that each column $\gamma^{(k)}$ of the $d \times n$ matrix $\gamma = [\gamma_{ij}]$ depends on z only through the k^{th} coordinate z_k, i.e.

$$\gamma^{(k)}(x, z) = \gamma^{(k)}(x, z_k); \quad z = (z_1, \cdots, z_n) \in \mathbb{R}^n.$$

As the standing hypotheses we assume that α, σ and γ are sufficiently smooth so that (3.1) has a unique solution (see [39]). Moreover, we always make the following assumptions:

(A1) The Lipschitz continuous function $\delta : [0, \infty] \to \mathbb{R}$ stands for the time delay and satisfies

$$-\tau \leq \delta(t) \leq t \text{ and } |\delta(t) - \delta(s)| \leq \rho|t - s|, \ \forall t, s \geq 0$$

for some positive constat ρ.

(A2) The coefficients α, σ, γ are sufficiently smooth in order that Eq. (3.1) has unique solution on $[-\tau, T]$.

(A3) (Hölder continuity of the initial data) There exist constants $K_0 > 0$ such that for all $-\tau \leq s < t \leq 0$,

$$\mathbb{E}|\zeta(t) - \zeta(s)| \leq K_0|t - s|.$$

(A4) The measures $\nu = (\nu_1, \cdots, \nu_n)$ are bounded Lévy measures.

The rest of the chapter is arranged as follows, Section 2 prepares two auxiliary lemmas, namely the moment bounds property. In Section 3, we state the strong convergence result for the Euler-Maruyama method and the stochastic theta method respectively. The convergence rate under a local Lipschitz condition is discussed in Section 4.

3.2 The Euler-Maruyama method

We define the Euler-Maruyama (EM) numerical solution. Let the time step-size $\Delta \in (0, 1)$ be a fraction of τ, that is $\Delta = \frac{\tau}{N}$ for some sufficiently large integer N. The discrete EM approximate solution is defined by

$$\begin{aligned} Y((k+1)\Delta) =& Y(k\Delta) + \alpha(Y(k\Delta), Y(I_\Delta[\delta(k\Delta)]\Delta))\Delta \\ &+ \sigma(Y(k\Delta), Y(I_\Delta[\delta(k\Delta)]\Delta))\Delta B_k \end{aligned}$$

$$+\int_{\mathbb{R}^n}\gamma(Y(k\Delta),Y(I_\Delta[\delta(k\Delta)]\Delta,z))\Delta\tilde{N}_k(dz)$$

with $Y(0)=\zeta(0)$ on $-\tau\le t\le 0$. Here $k=0,1,2,...,$ and $I_\Delta[\delta(k\Delta)]$ denotes the integer part of $\delta(k\Delta)/\Delta$, $\Delta B_k=B((k+1)\Delta)-B(k\Delta)$, and $\Delta\tilde{N}_k(dz)=\tilde{N}((k+1)\Delta,dz)-\tilde{N}(k\Delta,dz)$. We note that

$$-\tau\le I_\Delta[\delta(k\Delta)]\Delta\quad\text{for every } k\ge 0.$$

In fact

$$-N=-\frac{\tau}{\Delta}\le\frac{\delta(k\Delta)}{\Delta}\le k,$$

so

$$-N\le I_\Delta[\delta(k\Delta)]\le k.$$

To define the continuous extension, we need introduce two step processes

$$z_1(t)=\sum_{k=0}^{\infty}1_{[k\Delta,(k+1)\Delta]}(t)Y(k\Delta),$$

$$z_2(t)=\sum_{k=0}^{\infty}1_{[k\Delta,(k+1)\Delta]}(t)Y(I_\Delta[\delta(k\Delta)]\Delta).$$

The continuous EM numerical solution is defined by

$$\bar{Y}(t)=\begin{cases}\zeta(t), & -\tau\le t\le 0;\\[2ex] \zeta(0)+\int_0^t\alpha(z_1(s^-),z_2(s^-))ds+\int_0^t\sigma(z_1(s^-),z_2(s^-))dB(s) & \\ \quad+\int_0^t\int_{\mathbb{R}^n}\gamma(z_1(s^-),z_2(s^-),z)\tilde{N}(ds,dz), & 0\le t\le T.\end{cases}$$

Note that $Y(k\Delta)=\bar{Y}(k\Delta)$ for every $k\ge 0$.

3.2.1 Auxiliary lemmas

In this section, we will give estimates for moments of the exact solution of Eq.(3.1) as well as the EM numerical solution.

Lemma 3.2.1. *Assume that α, σ, γ satisfy the linear growth condition:*

(LG) *There exists a constant $h > 0$ such that*

$$|\sigma(x,y)|^2 + |\alpha(x,y)|^2 + \int_{\mathbb{R}} \sum_{k=1}^{m} |\gamma_k(x,y,z)|^2 \nu_k(dz_k)$$
$$\leq h(1 + |x|^2 + |y|^2) \quad \textit{for all } x,\ y \in \mathbb{R}^d.$$

Then there is a constant K_1, which dependents only on T, h, ζ but is independent of Δ, such that the exact solution and the EM numerical solution to the SDDE (3.1) *satisfy*

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |X(t)|^2\right] \vee \mathbb{E}\left[\sup_{0\leq t\leq T} |\bar{Y}(t)|^2\right] \leq K_1. \tag{3.2}$$

The proof of this lemma is similar to that for SDEs by using Burkholder-Davis-Gundy's inequality, and we omit here. The reader is referred to [9] and [16]. However, since the quadratic variation of the jump term is different from the one of a continuous martingale (for example, Brownian motion), the estimation of the $p-$moment ($p > 2$) is different from that of SDEs driven by Brownian motion. Although there are some results in the literature (cf. [11]), in order to be self-contained we prove

Lemma 3.2.2. *Under the linear growth condition (LG), for any $p > 2$, there is a positive constant K_p which depends only on p, ν, T, h, but is independent of Δ such that*

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |X(t)|^p\right] \vee \mathbb{E}\left[\sup_{0\leq t\leq T} |\bar{Y}(t)|^p\right] \leq K_p. \tag{3.3}$$

Proof. Using the continuous form of the EM approximation and Hölder's inequality, we compute straightforward

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |\bar{Y}(t)|^p\right]$$

$$
\begin{aligned}
&\leq 4^{p-1}\left[\mathbb{E}|\zeta(0)|^p + \mathbb{E}\sup_{0\leq t\leq T}\left|\int_0^t \alpha(z_1(s^-), z_2(s^-))ds\right|^p\right. \\
&\quad + \mathbb{E}\sup_{0\leq t\leq T}\left|\int_0^t \sigma(z_1(s^-), z_2(s^-))dB(s)\right|^p \\
&\quad \left. + \mathbb{E}\sup_{0\leq t\leq T}\left|\int_0^t\int_{\mathbb{R}^n} \gamma(z_1(s^-), z_2(s^-), z)\tilde{N}(ds, dz)\right|^p\right] \\
&\leq 4^{p-1}\left[\mathbb{E}\sup_{-\tau\leq s\leq 0}|\zeta(s)|^p + T^{p-1}\mathbb{E}\int_0^T |\alpha(z_1(s^-), z_2(s^-))|^p ds\right. \\
&\quad + \mathbb{E}\sup_{0\leq t\leq T}\left|\int_0^t \sigma(z_1(s^-), z_2(s^-))dB(s)\right|^p \\
&\quad \left. + \mathbb{E}\sup_{0\leq t\leq T}\left|\int_0^t\int_{\mathbb{R}^n} \gamma(z_1(s^-), z_2(s^-), z)\tilde{N}(ds, dz)\right|^p\right]. \qquad (3.4)
\end{aligned}
$$

To the term including Brownian motion, we apply Burkholder-Davis-Gundy's inequality to get

$$
\begin{aligned}
\mathbb{E}\sup_{0\leq t\leq T}\left|\int_0^t \sigma(z_1(s^-), z_2(s^-))dB(s)\right|^p &\leq C_p\mathbb{E}\left[\int_0^T |\alpha(z_1(s^-), z_2(s^-))|^2 ds\right]^{p/2} \\
&\leq C_p T^{\frac{p}{2}-1}\mathbb{E}\left[\int_0^T |\alpha(z_1(s^-), z_2(s^-))|^p ds\right].
\end{aligned}
\qquad (3.5)
$$

For the last term in (3.4), we need to use the quadratic variation of jump processes. In the following computations the constant $K_p = K(p,\ \nu,\ T,\ C_p)$ depends on $p,\ \nu,\ T,\ C_p$ and it may vary line by line. Let

$$
d\eta(s) := \int_{\mathbb{R}^n} \gamma(z_1(s), z_2(s), z)\tilde{N}(ds, dz)
$$

and compute the quadratic variation

$$
\begin{aligned}
[\eta, \eta]_t &= \int_0^t\int_{\mathbb{R}^n}\sum_{i=1}^d\sum_{j=1}^n \gamma_{ij}^2(z_1(s^-), z_2(s^-), z_j)\nu(dz_j)dt \\
&\quad + \int_0^t\int_{\mathbb{R}^n}\sum_{i=1}^d\sum_{j=1}^n \gamma_{ij}^2(z_1(s^-), z_2(s^-), z_j)\tilde{N}(ds, dz_j),
\end{aligned}
$$

to find

$$d[\eta,\eta]_t = \int_{\mathbb{R}^n} |\gamma(z_1(t^-),z_2(t^-),z)|^2\nu(dz)dt + \int_{\mathbb{R}^n} |\gamma(z_1(t^-),z_2(t^-),z)|^2\tilde{N}(dt,dz).$$

Applying the Burkholder-Davis-Gundy inequality (see [11]) on page 223), gives

$$\begin{aligned}
&\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t\int_{\mathbb{R}^n}\gamma(z_1(s^-),z_2(s^-),z)\tilde{N}(ds,dz)\right|^p \\
&= \mathbb{E}\sup_{0\le t\le T}\left|\int_0^t d\eta(s^-)\right|^p \le K_p\mathbb{E}\left|\int_0^T d\,[\eta,\eta]_{t^-}\right|^{p/2},
\end{aligned}$$

and we obtain

$$\begin{aligned}
&\mathbb{E}\sup_{0\le t\le T}\left[\left|\int_0^t\int_{\mathbb{R}^n}\gamma(z_1(s^-),z_2(s^-),z)\tilde{N}(ds,dz)\right|^p\right] \\
&\le K_p\mathbb{E}\left[\int_0^T\int_{\mathbb{R}^n}|\gamma(z_1(s^-),z_2(s^-),z)|^2\nu(dz)ds\right]^{\frac{p}{2}} \\
&\quad + K_p\mathbb{E}\left[\int_0^T\int_{\mathbb{R}^n}|\gamma(z_1(s^-),z_2(s^-),z)|^2\tilde{N}(ds,dz)\right]^{\frac{p}{2}}.
\end{aligned}$$

Applying the Burkholder-Davis-Gundy inequality again to the second term on the right hand side above we obtain

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T\int_{\mathbb{R}^n}|\gamma(z_1(s^-),z_2(s^-),z)|^2\tilde{N}(dt,dz)\right]^{\frac{p}{2}} \\
&\le K_p\mathbb{E}\left[\int_0^T\int_{\mathbb{R}^n}|\gamma(z_1(s^-),z_2(s^-),z)|^4\nu(dz)dt\right]^{\frac{p}{4}} \\
&\quad + K_p\mathbb{E}\left[\int_0^T\int_{\mathbb{R}^n}|\gamma(z_1(s^-),z_2(s^-),z)|^4\tilde{N}(dt,dz)\right]^{\frac{p}{4}}.
\end{aligned}$$

By a recursive argument we arrive at

$$\begin{aligned}
&\mathbb{E}\sup_{0\le t\le T}\left[\left|\int_0^t\int_{\mathbb{R}^n}\gamma(z_1(s^-),z_2(s^-),z)\tilde{N}(ds,dz)\right|^p\right] \\
&\le K_p\sum_{i=1}^k\mathbb{E}\left[\int_0^T\int_{\mathbb{R}^n}|\gamma(z_1(s^-),z_2(s^-),z)|^{2^i}\nu(dz)ds\right]^{\frac{p}{2^i}}
\end{aligned}$$

$$+ K_p\mathbb{E}\left[\int_0^T\int_{\mathbb{R}^n}|\gamma(z_1(s^-),z_2(s^-),z)|^{2^k}\tilde{N}(dt,dz)\right]^{\frac{p}{2^k}}. \qquad (3.6)$$

Since the measure ν is bounded, by Hölder's inequality, we estimate the first term in (3.6)

$$\begin{aligned}
&K_p\sum_{i=1}^k\mathbb{E}\left[\int_0^T\int_{\mathbb{R}^n}|\gamma(z_1(s^-),z_2(s^-),z)|^{2^i}\nu(dz)ds\right]^{\frac{p}{2^i}}\\
&\leq K_p\sum_{i=1}^k T^{\frac{p}{2^i}-1}\mathbb{E}\int_0^T\left[\int_{\mathbb{R}^n}|\gamma(z_1(s^-),z_2(s^-),z)|^{2^i}\nu(dz)\right]^{\frac{p}{2^i}}ds\\
&\leq K_p\sum_{i=1}^k T^{\frac{p}{2^i}-1}\nu(\mathbb{R}^n)^{\frac{p}{2^i}-1}\mathbb{E}\int_0^T\int_{\mathbb{R}^n}|\gamma(z_1(s^-),z_2(s^-),z)|^p\nu(dz)ds\\
&\leq K_p\mathbb{E}\int_0^T\int_{\mathbb{R}^n}|\gamma(z_1(s^-),z_2(s^-),z)|^p\nu(dz)ds.
\end{aligned}$$

For the second term of (3.6), choosing $p=2^n$ and $k=n-1$, by the Itô isometry, we have

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T\int_{\mathbb{R}^n}|\gamma(z_1(s^-),z_2(s^-),z)|^{2^k}\tilde{N}(dt,dz)\right]^{\frac{p}{2^k}}\\
&=\mathbb{E}\left[\int_0^T\int_{\mathbb{R}^n}|\gamma(z_1(s^-),z_2(s^-),z)|^p\nu(dz)ds\right].
\end{aligned}$$

Moreover, any $p-th$ moment less than 2^n can be dominated by the 2^nth moment, which follows from Hölder's inequality, and by the linear growth condition (LG), we arrive at

$$\begin{aligned}
&\mathbb{E}\sup_{0\leq t\leq T}\left|\int_0^t\int_{\mathbb{R}^n}\gamma(z_1(s^-),z_2(s^-),z)\tilde{N}(ds,dz)\right|^{2^n}\\
&\leq K_p\mathbb{E}\int_0^T\int_{\mathbb{R}^n}|\gamma(z_1(s^-),z_2(s^-),z)|^p\nu(dz)ds\\
&\leq K_ph\mathbb{E}\int_0^T(1+|z_1(s^-)|^p+|z_2(s^-)|^p)ds. \qquad (3.7)
\end{aligned}$$

Combing with (3.5) and (3.4), we obtain

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|\bar{Y}(t)|^p\right]$$

$$
\begin{aligned}
&\leq 4^{p-1}\mathbb{E} \sup_{-\tau\leq s\leq 0} |\zeta(s)|^p + 4^{p-1}h(K_p + C_pT^{p/2-1} + T^{p-1}) \\
&\qquad \times \mathbb{E}\int_0^T (1 + |z_1(s^-)|^p + |z_2(s^-)|^p)ds \\
&\leq 4^{p-1}\mathbb{E} \sup_{-\tau\leq s\leq 0} |\zeta(s)|^p + 4^{p-1}hT(K_p + C_pT^{p/2-1} + T^{p-1}) \\
&\qquad + 2^{2p-1}h(K_p + C_pT^{p/2-1} + T^{p-1}) \int_0^T \mathbb{E} \sup_{0\leq s\leq t} |\bar{Y}(s)|^p dt.
\end{aligned}
$$

The result follows now from Gronwall's inequality

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq t\leq T} |\bar{Y}(t)|^p\right] \\
&\leq \left[4^{p-1}\mathbb{E} \sup_{-\tau\leq s\leq 0} |\zeta(s)|^p + 4^{p-1}hT(K_p + C_pT^{p/2-1} + T^{p-1})\right] \\
&\qquad \times e^{2^{2p-1}Th(K_p + C_pT^{p/2-1} + T^{p-1})} \\
&:= K_p
\end{aligned}
$$

□

3.2.2 Strong convergence of the EM method

Convergence under a global Lipschitz condition

In this section we show strong convergence of the EM approximate solution to the exact solution under the following global Lipschitz condition:

(GL) There exists a constant $L > 0$ such that

$$
\begin{aligned}
&|\sigma(x,y) - \sigma(\bar{x},\bar{y})|^2 + |\alpha(x,y) - \alpha(\bar{x},\bar{y})|^2 \\
&+ \sum_{k=1}^{n} \int_{\mathbb{R}} |\gamma^{(k)}(x,y,z_k) - \gamma^{(k)}(\bar{x},\bar{y},z_k)|^2 \nu_k(dz_k) \leq L(|x-\bar{x}|^2 + |y-\bar{y}|^2)
\end{aligned} \tag{3.8}
$$

for all $x, y, \bar{x}, \bar{y} \in \mathbb{R}^n$.

To prove the main result we need the following lemmas.

Lemma 3.2.3. *Under the conditions of Lemma 3.2.1*

$$\mathbb{E}|\bar{Y}(t) - z_1(t)|^2 \le K_2\Delta, \quad \textit{for all } t \in [0,T], \tag{3.9}$$

where K_2 is a constant independent of Δ.

Proof. For any $t \in [0,T]$ choose a k such that $t \in [k\Delta, (k+1)\Delta)$. Then

$$\begin{aligned}\bar{Y}(t) - z_1(t) &= \bar{Y}(t) - Y(k\Delta) = \bar{Y}(t) - \bar{Y}(k\Delta)\\ &= \int_{k\Delta}^t \alpha(z_1(s^-), z_2(s^-))ds\\ &\quad + \int_{k\Delta}^t \sigma(z_1(s^-), z_2(s^-))dB(s)\\ &\quad + \int_{k\Delta}^t \int_{\mathbb{R}^n} \sigma(z_1(s^-), z_2(s^-), z)\tilde{N}(ds, dz).\end{aligned}$$

Thus by the linear growth condition (GL) and Lemma 3.2.1,

$$\begin{aligned}\mathbb{E}|\bar{Y}(t) - z_1(t)|^2 &\le (3\Delta + 24)h \int_{k\Delta}^t (1 + |z_1(s^-)|^2 + |z_{2^-}(s)|^2)ds\\ &\le 27h(1 + 2K_1)\Delta,\end{aligned}$$

which is the desired assertion with $K_2 = 27h(1 + 2K_1)$. □

Lemma 3.2.4. *Under Assumption A1 and A3, if for Δ, it holds $(\rho+1)\Delta \le 1$, then*

$$\mathbb{E}|\bar{Y}(\delta(t)) - z_2(t)|^2 \le K_3\Delta, \quad \forall t \in [0,T], \tag{3.10}$$

where K_3 is a constant independent of Δ.

We follows the proof of Mao [16].

Proof. For any $t \in [0,T]$, choose a k such that $t \in [k\Delta, (k+1)\Delta]$. Then

$$\bar{Y}(\delta(t)) - z_2(t) = \bar{Y}(\delta(t)) - \bar{Y}(I_\Delta[\delta(k\Delta)]\Delta),$$

and note that

$$\delta(k\Delta) - \Delta \leq I_\Delta[\delta(k\Delta)]\Delta \leq \delta(k\Delta).$$

As in Mao [16], we also consider the following possible cases:

case 1. If $\delta(t) \geq I_\Delta[\delta(k\Delta)]\Delta \geq 0$, then

$$\delta(t) - I_\Delta[\delta(k\Delta)]\Delta \leq \delta(t) - \delta(k\Delta) + \Delta \leq (\rho+1)\Delta.$$

Therefore, by the Itô isometry and the linear growth condition, we have

$$\begin{aligned}
&\mathbb{E}|\bar{Y}(\delta(t)) - z_2(t)|^2 \\
&\leq 3\mathbb{E}\left|\int_{I_\Delta[\delta(k\Delta)]\Delta}^{\delta(t)} \alpha(z_1(s^-), z_2(s^-))ds\right|^2 + 3\mathbb{E}\left|\int_{I_\Delta[\delta(k\Delta)]\Delta}^{\delta(t)} \sigma(z_1(s^-), z_2(s^-))dB(s)\right|^2 \\
&+ 3\mathbb{E}\left|\int_{I_\Delta[\delta(k\Delta)]\Delta}^{\delta(t)} \int_{\mathbb{R}^n} \gamma(z_1(s^-), z_2(s^-), z)\tilde{N}(ds, dz)\right|^2 \\
&\leq 3[(\rho+1)\Delta + 2]h \int_{I_\Delta[\delta(k\Delta)]\Delta}^{\delta(t)} (1 + \mathbb{E}|z_1(s^-)|^2 + \mathbb{E}|z_2(s^-)|^2)ds \\
&\leq K_3\Delta.
\end{aligned}$$

case 2. If $0 \leq \delta(t) \leq I_\Delta[\delta(k\Delta)]\Delta$, then

$$I_\Delta[\delta(k\Delta)]\Delta - \delta(t) \leq \delta(k\Delta) - \delta(t) \leq \rho\Delta$$

and again by the Itô isometry and the linear growth condition

$$\begin{aligned}
&\mathbb{E}|\bar{Y}(\delta(t)) - z_2(t)|^2 \\
&\leq 3[\rho\Delta + 2]h \int_{I_\Delta[\delta(k\Delta)]\Delta}^{\delta(t)} (1 + \mathbb{E}|z_1(s^-)|^2 + \mathbb{E}|z_2(s^-)|^2)ds \\
&\leq K_3\Delta.
\end{aligned}$$

case 3. If $0 \geq \delta(t) \geq I_\Delta[\delta(k\Delta)]\Delta$ or $0 \geq I_\Delta[\delta(k\Delta)]\Delta \geq \delta(t)$ then by Assumptions A1 and A3, we obtain

$$|\delta(t) - I_\Delta[\delta(k\Delta)]\Delta| \leq (\rho+1)\Delta,$$

hence

$$\begin{aligned}\mathbb{E}|\bar{Y}(\delta(t)) - z_2(t)|^2 &= \mathbb{E}|\zeta(\delta(t)) - \zeta(I_\Delta[\delta(k\Delta)]\Delta)|^2 \\ &\le K_0|\delta(t) - I_\Delta[\delta(k\Delta)]\Delta|^\beta \\ &\le K_0(\rho+1)^\beta\Delta^\beta \\ &\le K_3\Delta.\end{aligned}$$

case 4. If $\delta(t) \ge 0 \ge I_\Delta[\delta(k\Delta)]\Delta$ then

$$-I_\Delta[\delta(k\Delta)]\Delta \le (\rho+1)\Delta, \text{ and } \delta(t) \le (\rho+1)\Delta,$$

thus,

$$\begin{aligned}\mathbb{E}|\bar{Y}(\delta(t)) - z_2(t)|^2 &= \mathbb{E}|\bar{Y}(\delta(t)) - \zeta(I_\Delta[\delta(k\Delta)]\Delta)|^2 \\ &\le 2\mathbb{E}|\bar{Y}(\delta(t)) - \zeta(0)|^2 + 2\mathbb{E}|\zeta(0) - \zeta(I_\Delta[\delta(k\Delta)]\Delta)|^2 \\ &\le K_3\Delta\end{aligned}$$

case 5. If $I_\Delta[\delta(k\Delta)]\Delta \ge 0 \ge \delta(t)$ then

$$-\delta(t) \le \rho\Delta, \text{ and } I_\Delta[\delta(k\Delta)]\Delta \le \rho\Delta.$$

and therefore, $\mathbb{E}|\bar{Y}(\delta(t)) - z_2(t)|^2 \le K_3\Delta$.

Combing these different cases, we get

$$\mathbb{E}|\bar{Y}(\delta(t)) - z_2(t)|^2 \le K_3\Delta$$

for all $t \in [0, T]$ as required. □

Let us now state our main convergence result under a global Lipschitz condition.

Theorem 3.2.1. *Under the global Lipschitz condition, we have*

$$\lim_{\Delta\to 0} \mathbb{E}\left[\sup_{0\le t\le T} |X(t) - \bar{Y}(t)|^2\right] = 0. \tag{3.11}$$

Proof. Although the proof is similar to that of the proof of SDDEs without jump, for further purposes (see section 5) we shall give the details of the proof. Using the continuous extension of the numerical solution, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T}|X(t)-\bar{Y}(t)|^2\right] \\
&\le 3\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t[\alpha(X(s^-),X(\delta(s^-)))-\alpha(z_1(s^-),z_2(s^-))]ds\right|^2 \\
&\quad+3\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t[\sigma(X(s^-),X(\delta(s^-)))-\sigma(z_1(s^-),z_2(s^-))]dB(s)\right|^2 \\
&\quad+3\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t\int_{\mathbb{R}^n}[\gamma(X(s^-),X(\delta(s^-)),z)-\gamma(z_1(s^-),z_2(s^-),z)]\tilde{N}(ds,dz)\right|^2.
\end{aligned} \tag{3.12}$$

By Hölder's inequality and (GL), we estimate the first term,

$$\begin{aligned}
&3\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t[\alpha(X(s^-),X(\delta(s^-)))-\alpha(z_1(s^-),z_2(s^-))]ds\right|^2 \\
&\le 3T\mathbb{E}\sup_{0\le t\le T}\int_0^t|\alpha(X(s^-),X(\delta(s^-)))-\alpha(z_1(s^-),z_2(s^-))|^2ds \\
&\le 3TL\mathbb{E}\sup_{0\le t\le T}\int_0^t[|X(s^-)-z_1(s^-)|^2+|X(\delta(s^-))-z_2(s)|^2]ds.
\end{aligned} \tag{3.13}$$

For the second term, Doob's martingale inequality gives

$$\begin{aligned}
&3\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t[\sigma(X(s^-),X(\delta(s^-)))-\sigma(z_1(s^-),z_2(s^-))]dB(s)\right|^2 \\
&\le 12\mathbb{E}\int_0^T\left|\sigma(X(s^-),X(\delta(s^-)))-\sigma(z_1(s^-),z_2(s^-))\right|^2ds \\
&\le 12L\mathbb{E}\int_0^T[|X(s^-)-z_1(s^-)|^2+|X(\delta(s^-))-z_2(s^-)|^2]ds.
\end{aligned} \tag{3.14}$$

Applying Itô's isometry and the Doob martingale inequality for the third term, yields

$$3\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t\int_{\mathbb{R}^n}[\gamma(X(s^-),X(\delta(s^-)),z)-\gamma(z_1(s^-),z_2(s^-),z)]\tilde{N}(ds,dz)\right|^2$$

$$\leq 12\mathbb{E}\left|\int_0^T\int_{\mathbb{R}^n}[\gamma(X(s^-),X(\delta(s^-)),z)-\gamma(z_1(s^-),z_2(s^-),z)]\tilde{N}(ds,dz)\right|^2$$
$$\leq 12\mathbb{E}\int_0^T\int_{\mathbb{R}^n}|\gamma(X(s^-),X(\delta(s^-)),z)-\gamma(z_1(s^-),z_2(s^-),z)|^2\nu(dz)ds$$
$$\leq 12L\mathbb{E}\int_0^T[|X(s^-)-z_1(s^-)|^2+|X(\delta(s^-))-z_2(s^-)|^2]ds. \tag{3.15}$$

Using (3.14) and (3.13) in (3.12), and by the Lemma 3.2.3 and Lemma 3.2.4, we arrive at

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|X(t)-\bar{Y}(t)|^2\right]$$
$$\leq 3L(T+8)\mathbb{E}\int_0^T[|X(s^-)-z_1(s^-)|^2+|X(\delta(s^-))-z_2(s^-)|^2]ds$$
$$\leq 6L(T+8)\mathbb{E}\int_0^T\left[|X(s^-)-\bar{Y}(s^-)|^2+|X(\delta(s^-))-\bar{Y}(\delta(s^-))|^2\right]ds$$
$$+6L(T+8)\mathbb{E}\int_0^T[|\bar{Y}(s^-)-z_1(s)|^2+|\bar{Y}(\delta(s^-))-z_2(s^-)|^2]ds$$
$$\leq 12L(T+8)\mathbb{E}\int_0^T(X(s^-)-\bar{Y}(s^-))ds+6L(T+8)(K_2\Delta+K_3\Delta)T.$$
$$\leq 12L(T+8)\int_0^T\mathbb{E}\sup_{0\leq s\leq t}(X(s^-)-\bar{Y}(s^-))dt$$
$$+6L(T+8)(K_2\Delta+K_3\Delta)T.$$

By the Gronwall inequality we obtain

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|X(t)-\bar{Y}(t)|^2\right]\leq 6L(T+8)(K_2+K_3)Te^{12LT(T+8)}\Delta \tag{3.16}$$

and the required assertion follows. □

Remark 3.2.1. *Under a global Lipschitz condition, the conclusion of this lemma not only tells us the strong convergence, but also tells us the rate of the convergence by* (3.16).

Convergence under a local Lipschitz and a linear growth condition

In many situation, the coefficients α, σ and γ are only locally Lipschitz continuous. In this section, we shall discuss the strong convergence of the EM scheme for the stochastic differential delay equation with jumps under a local Lipschitz condition.

(LL) For each $R \in \mathbb{N}$ there exists a constant L_R such that

$$|\sigma(x,y) - \sigma(\bar{x},\bar{y})|^2 + |\alpha(x,y) - \alpha(\bar{x},\bar{y})|^2 \\ + \int_{\mathbb{R}^n} |\gamma(x,y,z) - \gamma(\bar{x},\bar{y},z)|^2 \nu(dz) \le L_R(|x-\bar{x}|^2 + |y-\bar{y}|^2), \tag{3.17}$$

for all $x,\ y,\ \bar{x},\ \bar{y} \in \mathbb{R}^n$, and $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \le R$.

Theorem 3.2.2. *Under the local Lipschitz condition (LL) and the linear growth condition (LG), the EM approximate solution converges to the exact solution of the SDDE with jumps* (5.1), *in sense that*

$$\lim_{\Delta \to 0} \mathbb{E}\left[\sup_{0\le t\le T} |X(t) - \bar{Y}(t)|^2\right] = 0. \tag{3.18}$$

Proof. The techniques of the proof have been developed in [9] where the author showed the strong convergence of the EM scheme for the SDE without jumps under a local Lipschitz condition. We highlight here, that the proof tells us how to control the error of the convergence. Define the stopping times

$$\tau_R = \inf\{t \ge 0 : \ |X(t)| \ge R\}, \quad \rho_R = \inf\{t \ge 0 : \ |\bar{Y}(t)| \ge R\}$$

and write $\theta_R = \tau_R \wedge \rho_R$.

Recall Young's inequality: for $r^{-1} + q^{-1} = 1$ and all $a, b, \delta > 0$

$$ab \le \frac{\delta}{r} a^r + \frac{1}{q\delta^{q/r}} b^q.$$

Thus, for any $\delta > 0$,

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T}|X(t)-\bar{Y}(s)|^2\right]\\
&=\mathbb{E}\left[\sup_{0\le t\le T}|X(t)-\bar{Y}(t)|^2 I_{\{\tau_R>T,\rho_R>T\}}\right]+\mathbb{E}\left[\sup_{0\le t\le T}|X(t)-\bar{Y}(t)|^2 I_{\{\tau_R\le T \text{ or } \rho_R\le T\}}\right]\\
&\le \mathbb{E}\left[\sup_{0\le t\le T}|X(t\wedge\theta_R)-\bar{Y}(t\wedge\theta_R)|^2 I_{\{\theta_R>T\}}\right]+\frac{2\delta}{p}\mathbb{E}\left[\sup_{0\le t\le T}|X(t)-\bar{Y}(t)|^p\right]\\
&\quad+\frac{1-2/p}{\delta^{2/(p-2)}}\mathbb{P}(\tau_R\le T \text{ or } \rho_R\le T).
\end{aligned}
\tag{3.19}
$$

From Lemma 3.2.2, we deduce that

$$
\mathbb{P}(\tau_R\le T)=\mathbb{E}\left[I_{\{\tau_R\le T\}}\frac{|X(\tau_R)|^p}{R^p}\right]\le\frac{1}{R^p}\mathbb{E}\left[\sup_{0\le t\le T}|X(t)|^p\right]\le\frac{K_p}{R^p}.
$$

A similar result can be derived for ρ_R, so that

$$
P(\tau_R\le T \text{ or } \rho_R\le T)\le\frac{K_p}{R^p}.
$$

These bounds give

$$
\begin{aligned}
\mathbb{E}\sup_{0\le t\le T}|X(t)-\bar{Y}(t)|^2\le{}&\mathbb{E}\sup_{0\le t\le T}|X(t\wedge\theta_R)-\bar{Y}(t\wedge\theta_R)|^2\\
&+\frac{2^{p+1}\delta K_p}{p}+\frac{2(p-2)K_p}{p\delta^{2/(p-2)}R^p}.
\end{aligned}
$$

In a very similar way as in the proof of Theorem 2.4, we can obtain that

$$
\mathbb{E}\left[\sup_{0\le t\le T}|X(t\wedge\theta_R)-\bar{Y}(t\wedge\theta_R)|^2\right]\le C_R\Delta,
\tag{3.20}
$$

where C_R is a constant and independents of Δ.

Given any $\epsilon > 0$, we can choose δ such that

$$
\frac{2^{p+1}\delta K_p}{p}<\frac{\epsilon}{3},
$$

and then we choose R such that

$$
\frac{2(p-2)K_p}{p^{\delta^{2/(p-2)}}R^p}<\frac{\epsilon}{3}.
$$

Now for any sufficiently small Δ it following that

$$C_R\Delta < \frac{\epsilon}{3},$$

combing the bounds above and with (3.19), we finally get

$$\mathbb{E}\sup_{0\le t\le T}|X(t)-\bar{Y}(t)|^2 < \epsilon.$$

□

Remark 3.2.2. *Under a local Lipschitz condition and a linear growth condition, so far we can not get the rate of the strong convergence.*

3.3 A stochastic theta method

In this section, we shall develop the stochastic theta method (STM) for (3.1) (see [6]). For a sufficiently small stepsize $\Delta = \frac{\tau}{N} > 0$ for some large integer N and a particular choice of $\theta \in [0,1]$, the theta method is defined by

$$\begin{aligned} Z((k+1)\Delta) =& Z(k\Delta) + (1-\theta)\alpha(Z(k\Delta), Z(I_\Delta[\delta(k\Delta)]\Delta))\Delta \\ &+ \theta\alpha(Z((k+1)\Delta), Z(I_\Delta[\delta((k+1)\Delta)]\Delta))\Delta \\ &+ \sigma(Z(k\Delta), Z(I_\Delta[\delta(k\Delta)]\Delta))\Delta B_k \\ &+ \int_{\mathbb{R}^n}\gamma(Z(k\Delta), Z(I_\Delta[\delta(k\Delta)]\Delta, z))\Delta\tilde{N}_k(dz) \end{aligned} \tag{3.21}$$

with initial data

$$\begin{aligned} \{Z(t):\ -\tau\le t\le 0\} &= \{X(t):\ -\tau\le t\le 0\} \\ &= \{\zeta(t):\ -\tau\le t\le 0\} \in C^b_{\mathcal{F}_0}([-\tau,0]), \end{aligned}$$

where ΔB_k and $\Delta\tilde{N}_k(dz)$ denote the increments of the Brownian motion and the compensated Poisson process, respectively. As before $I_\Delta[\delta((k+1)\Delta)]$

denotes the integer part of $\delta((k+1)\Delta)/\Delta$. For convenience, we will extend the discrete numerical solution to continuous time. In addition we also need introduce four step processes

$$\begin{aligned}
z_1(t) &= \sum_{k=0}^{\infty} 1_{[k\Delta,(k+1)\Delta]}(t) Z(k\Delta),\\
z_2(t) &= \sum_{k=0}^{\infty} 1_{[k\Delta,(k+1)\Delta]}(t) Z(I_\Delta[\delta(k\Delta)]\Delta),\\
z_3(t) &= \sum_{k=0}^{\infty} 1_{[k\Delta,(k+1)\Delta]}(t) Z((k+1)\Delta),\\
z_4(t) &= \sum_{k=0}^{\infty} 1_{[k\Delta,(k+1)\Delta]}(t) Z(I_\Delta[\delta((k+1)\Delta)]\Delta).
\end{aligned}$$

The continuous theta approximate solution is defined as

$$\bar{Z}(t) = \begin{cases} \zeta(t), & -\tau \le t \le 0;\\[2ex] \zeta(0) + \int_0^t (1-\theta)\alpha(z_1(s^-), z_2(s^-))ds \\ \quad + \int_0^t \theta\alpha(z_3(s^-), z_4(s^-))ds \\ \quad + \int_0^t \sigma(z_1(s^-), z_2(s^-))dB(s) \\ \quad + \int_0^t \int_{\mathbb{R}^n} \gamma(z_1(s^-), z_2(s^-), z)\tilde{N}(ds, dz), & 0 \le t \le T. \end{cases}$$

Clearly, $z_1(k\Delta) = z_3((k-1)\Delta) = \bar{Z}(k\Delta)$.

Note that when $\theta = 0$ the numerical solution becomes the EM approximation, however, when $\theta \neq 0$, the θ method is defined by an implicit equation. In order to guarantee (3.21) can be solved uniquely, in this section we require the Assumption **(A1)** replaced by

(A*1) There exists a positive constant ε such that

$$-\tau \le \delta(t) \le t - \varepsilon,$$

and

$$|\delta(t)-\delta(s)| \le \rho|t-s|, \quad \text{for all } t,s \ge 0 \text{ for some positive constat } \rho.$$

By this assumption, $\delta(0) \le -\varepsilon$ and $\delta(k\Delta)/\Delta \le k - \varepsilon/\Delta$, therefore there exists a sufficient small stepsize Δ^* such that

$$\delta(\Delta^*) \le -\varepsilon/2 \text{ and } \frac{\delta(k\Delta^*)}{\Delta} \le k-1. \tag{3.22}$$

The next lemma will show that the numerical solution is well defined.

Lemma 3.3.1. *Under a global Lipschitz condition (GL) with Lipschitz constant L, choose the stepsize Δ sufficiently small such that $\Delta < \max\{\frac{1}{\theta\sqrt{L}}, \Delta^*\}$ then (3.21) can be solved uniquely for $Z((k+1)\Delta)$ with probability 1.*

Proof.

For any $k \ge 0$, we assume that there exist two solutions of (3.21),

$$\begin{aligned} Z_1((k+1)\Delta) =& Z(k\Delta) + (1-\theta)\alpha(Z(k\Delta), Z(I_\Delta[\delta(k\Delta)]\Delta))\Delta \\ &+ \theta\alpha(Z_1((k+1)\Delta), Z_1(I_\Delta[\delta((k+1)\Delta)]\Delta))\Delta \\ &+ \sigma(Z(k\Delta), Z(I_\Delta[\delta(k\Delta)]\Delta))\Delta B_k \\ &+ \int_{\mathbb{R}^n} \gamma(Z(k\Delta), Z(I_\Delta[\delta(k\Delta)]\Delta, z))\Delta\tilde{N}_k(dz) \end{aligned}$$

and

$$\begin{aligned} Z_2((k+1)\Delta) =& Z(k\Delta) + (1-\theta)\alpha(Z(k\Delta), Z(I_\Delta[\delta(k\Delta)]\Delta))\Delta \\ &+ \theta\alpha(Z_2((k+1)\Delta), Z_2(I_\Delta[\delta((k+1)\Delta)]\Delta))\Delta \\ &+ \sigma(Z(k\Delta), Z(I_\Delta[\delta(k\Delta)]\Delta))\Delta B_k \\ &+ \int_{\mathbb{R}^n} \gamma(Z(k\Delta), Z(I_\Delta[\delta(k\Delta)]\Delta, z))\Delta\tilde{N}_k(dz). \end{aligned}$$

Under the globe Lipschitz condition, we have

$$|Z_1((k+1)\Delta) - Z_2((k+1)\Delta)|$$

$$
\begin{aligned}
&= |\theta\alpha(Z_1((k+1)\Delta), Z_1(I_\Delta[\delta((k+1)\Delta)]\Delta) \\
&\quad - \theta\alpha(Z_2((k+1)\Delta), Z_2(I_\Delta[\delta((k+1)\Delta)]\Delta)| \\
&\leq \theta\Delta\sqrt{L}\,(|Z_1((k+1)\Delta) - Z_2((k+1)\Delta)| + |Z_1(I_\Delta[\delta((k+1)\Delta)]\Delta) \\
&\quad - Z_2(I_\Delta[\delta((k+1)\Delta)]\Delta)|)\,.
\end{aligned}
$$

Let $k = 0$, using the condition $\Delta < \Delta^*$, we have $I_\Delta[\delta(\Delta)] \leq 0$, this implies,

$$
Z_1(I_\Delta[\delta(\Delta)]\Delta) = Z_2(I_\Delta[\delta(\Delta)]\Delta) = \zeta(I_\Delta[\delta(\Delta)]\Delta),
$$

therefore $|Z_1(\Delta) - Z_2(\Delta)| \leq \theta\Delta\sqrt{L}|Z_1(\Delta) - Z_2(\Delta)|$, thus, when $\theta\Delta\sqrt{L} < 1$, $Z_1(\Delta) = Z_2(\Delta)$. Let $k = 1$, by (3.22), we observe that $Z_1(I_\Delta[\delta(2\Delta)]\Delta)$ and $Z_2(I_\Delta[\delta(2\Delta)]\Delta)$ are determined in the interval $[-\tau, \Delta)$, so that

$$
Z_1(I_\Delta[\delta(2\Delta)]\Delta) = Z_2(I_\Delta[\delta(2\Delta)]\Delta),
$$

thus, provided $\theta\Delta\sqrt{L} < 1$, $Z_1(2\Delta) = Z_2(2\Delta)$. Using the same arguments for $k = 2, 3\cdots$, we arrive at, $Z_1((k+1)\Delta) = Z_2((k+1)\Delta)$, when $\theta\sqrt{L}\Delta < 1$ and $\Delta < \Delta^*$, that is (3.21) has unique solution, i.e., the numerical solution of theta method is well defined. □

To state the strong convergence result, we need the following lemmas.

Lemma 3.3.2. *Under the linear growth condition (LG), for a sufficiently small* Δ

$$
\mathbb{E}|Z(k\Delta)|^2 \leq C_1,
$$

where C_1 *is a constant independent of* Δ.

Proof. By the definition of the step process, we have

$$
\begin{aligned}
Z((k+1)\Delta) =&\zeta(0) + \int_0^{(k+1)\Delta} (1-\theta)\alpha(z_1(s^-), z_2(s^-))ds \\
&+ \int_0^{(k+1)\Delta} \theta\alpha(z_3(s^-), z_4(s^-))ds
\end{aligned}
$$

$$+\int_0^{(k+1)\Delta}\sigma(z_1(s^-),z_2(s^-))dB(s)$$
$$+\int_0^{(k+1)\Delta}\int_{\mathbb{R}^n}\gamma(z_1(s^-),z_2(s^-),z)\tilde{N}(ds,dz),$$

thus, for $(k+1)\Delta\le T$

$$\begin{aligned}\mathbb{E}|Z((k+1)\Delta)|^2 \le & 4|\zeta(0)|^2+4\mathbb{E}\left|\int_0^{(k+1)\Delta}\left[(1-\theta)\alpha(z_1(s^-),z_2(s^-))\right.\right.\\ & \left.\left.+\theta\alpha(z_3(s^-),z_4(s^-))\right]ds\right|^2\\ & +4\mathbb{E}\left|\int_0^{(k+1)\Delta}\sigma(z_1(s^-),z_2(s^-))dB(s)\right|^2\\ & +4\mathbb{E}\left|\int_0^{(k+1)\Delta}\int_{\mathbb{R}^n}\gamma(z_1(s^-),z_2(s^-),z)\tilde{N}(ds,dz)\right|^2.\end{aligned}$$

By the Hölder inequality and the linear growth bound, we obtain

$$\begin{aligned}&\mathbb{E}\left|\int_0^{(k+1)\Delta}\left[(1-\theta)\alpha(z_1(s^-),z_2(s^-))+\theta\alpha(z_3(s^-),z_4(s^-))\right]ds\right|^2\\ &\le T\mathbb{E}\int_0^{(k+1)\Delta}\left|(1-\theta)\alpha(z_1(s^-),z_2(s^-))+\theta\alpha(z_3(s^-),z_4(s^-))\right|^2ds\\ &\le 2T\mathbb{E}\int_0^{(k+1)\Delta}\left[\left|\alpha(z_1(s^-),z_2(s^-))\right|^2+\left|\alpha(z_3(s^-),z_4(s^-))\right|^2\right]ds\\ &\le 2Th\mathbb{E}\int_0^{(k+1)\Delta}\left[2+|z_1(s^-)|^2+|z_2(s^-)|^2+|z_3(s^-)|^2+|z_4(s^-)|^2\right]ds\\ &\le 4T^2h+8Th\Delta\sum_{i=0}^{k}\mathbb{E}|Z(i\Delta)|^2+4Th\Delta\mathbb{E}|Z((k+1)\Delta)|^2. \qquad (3.23)\end{aligned}$$

For the second term, the Itô isometry gives

$$\begin{aligned}&\mathbb{E}\left|\int_0^{(k+1)\Delta}\sigma(z_1(s^-),z_2(s^-))dB(s)\right|^2\\ &=\mathbb{E}\int_0^{(k+1)\Delta}|\sigma(z_1(s^-),z_2(s^-))|^2ds\\ &\le \mathbb{E}h\int_0^{(k+1)\Delta}(1+|z_1(s^-)|^2+|z_2(s^-)|^2)ds\end{aligned}$$

$$\leq hT + 2h\Delta \sum_{i=0}^{k} \mathbb{E}|Z(i\Delta)|^2, \tag{3.24}$$

and by the same argument for the third term, we have

$$\mathbb{E}\left|\int_0^{(k+1)\Delta}\int_{\mathbb{R}^n}\gamma(z_1(s^-),z_2(s^-),z)\tilde{N}(ds,dz)\right|^2 \leq hT + 2h\Delta\sum_{i=0}^{k}\mathbb{E}|Z(i\Delta)|^2. \tag{3.25}$$

Combining (3.23), (3.24) and (3.25), we arrive at

$$\mathbb{E}|Z((k+1)\Delta)|^2 \leq 4|\zeta(0)|^2 + 16T^2h + 8Th + 8h\Delta(4T+2)\sum_{i=0}^{k}\mathbb{E}|Z(i\Delta)|^2$$
$$+ 16Th\Delta\mathbb{E}|Z((k+1)\Delta)|^2.$$

Choosing Δ sufficiently small, for $1 - 16Th\Delta \geq \frac{1}{2}$, we obtain,

$$\mathbb{E}|Z((k+1)\Delta)|^2 \leq 8|\zeta(0)|^2 + 32T^2h + 16Th + 16h\Delta(4T+2)\sum_{i=0}^{k}\mathbb{E}|Z(i\Delta)|^2.$$

The result now follows as an application of the discrete Gronwall inequality (see [15]). □

The next lemma shows that the second moment of the continuous approximation is bounded in a strong sense.

Lemma 3.3.3. *Under the linear growth condition (LG), for a sufficiently small Δ we have*

$$\mathbb{E}\sup_{0\leq t\leq T}|Z(t)|^2 \leq C_2,$$

where C_2 is a constant independent of Δ.

Proof. From the continuous extension and by the Cauchy-Schwarz inequality, we obtain

$$\mathbb{E}\sup_{0\leq t\leq T}|\bar{Z}(t)|^2 \leq 4\mathbb{E}|\zeta(0)|^2 + 4\mathbb{E}\sup_{0\leq t\leq T}\left|\int_0^t\left[(1-\theta)\alpha(z_1(s^-),z_2(s^-))\right.\right.$$

$$
\begin{aligned}
&+\theta\alpha(z_3(s^-), z_4(s^-))\big]\, ds\Big|^2 \\
&+ 4\mathbb{E} \sup_{0\le t\le T} \left| \int_0^t \sigma(z_1(s^-), z_2(s^-)) dB(s) \right|^2 \\
&+ 4\mathbb{E} \sup_{0\le t\le T} \left| \int_0^t \int_{\mathbb{R}^n} \gamma(z_1(s^-), z_2(s^-), z) \tilde{N}(ds, dz) \right|^2 \\
\le\ & 4\mathbb{E}|\zeta(0)|^2 + 8T\mathbb{E} \sup_{0\le t\le T} \int_0^t \left[|\alpha(z_1(s^-), z_2(s^-))|^2 \right. \\
&\left. + |\alpha(z_3(s^-), z_4(s^-))|^2 \right] ds \\
&+ 4\mathbb{E} \sup_{0\le t\le T} \left| \int_0^t \sigma(z_1(s^-), z_2(s^-)) dB(s) \right|^2 \\
&+ 4\mathbb{E} \sup_{0\le t\le T} \left| \int_0^t \int_{\mathbb{R}^n} \gamma(z_1(s^-), z_2(s^-), z) \tilde{N}(ds, dz) \right|^2 .
\end{aligned}
$$

Using the Itô isometry and Doob's martingale inequality, we have

$$
\begin{aligned}
&\mathbb{E} \sup_{0\le t\le T} |\bar{Z}(t)|^2 \\
\le\ & 4\mathbb{E}|\zeta(0)|^2 + 8Th\mathbb{E} \sup_{0\le t\le T} \int_0^t \left[1 + |z_1(s^-)|^2 + |z_2(s^-)|^2 \right. \\
&\qquad \left. + |z_3(s^-)|^2 + |z_4(s^-)|^2 \right] ds \\
&+ 16\mathbb{E} \int_0^T |\sigma(z_1(s^-), z_2(s^-))|^2 ds \\
&+ 16\mathbb{E} \left| \int_0^T \int_{\mathbb{R}^n} \gamma(z_1(s^-), z_2(s^-), z) \tilde{N}(ds, dz) \right|^2 \\
\le\ & 4\mathbb{E}|\zeta(0)|^2 + 8Th(T+4) + 8h(T+4) \int_0^T [|z_1(s^-)|^2 + |z_2(s^-)|^2] ds \\
&+ 8Th \int_0^T [|z_3(s^-)|^2 + |z_4(s^-)|^2] ds.
\end{aligned}
$$

Since some $z_3(t)$ and $z_4(t)$ can be extended beyond T, over the interval $[0, T+1]$,we apply Lemma 3.3.2 to get the result. □

In the following, we will show that the continuous approximation convergence to the step process in a strong sense.

Lemma 3.3.4. *Under the linear growth condition (LG), for a sufficiently small* Δ

$$\mathbb{E}\sup_{0\le t\le T}|\bar{Z}(t)-z_1(t)|^2 \le C_3\Delta$$

and

$$\mathbb{E}\sup_{0\le t\le T}|\bar{Z}(t)-z_3(t)|^2 \le C_4\Delta,$$

where C_3 *and* C_4 *are constants independent of* Δ.

Proof. Consider $t\in[k\Delta,(k+1)\Delta]$, we have

$$\begin{aligned}\bar{Z}(t)-z_1(t) &= \int_{k\Delta}^{t}[(1-\theta)\alpha(z_1(s^-),z_2(s^-))+\theta\alpha(z_3(s^-),z_4(s^-))]ds\\ &\quad+\int_{k\Delta}^{t}\sigma(z_1(s^-),z_2(s^-))dB(s)\\ &\quad+\int_{k\Delta}^{t}\int_{\mathbb{R}^n}\gamma(z_1(s^-),z_2(s^-),z)\tilde{N}(ds,dz).\end{aligned}$$

Thus,

$$\begin{aligned}&\mathbb{E}\sup_{0\le t\le T}|\bar{Z}(t)-z_1(t)|^2\\ &\le 3\mathbb{E}\max_{k=0,1,\cdot,T/\Delta-1}\sup_{s\in[k\Delta,(k+1)\Delta]}\left\{3\left|\int_{k\Delta}^{s}[(1-\theta)\alpha(z_1(u^-),z_2(u^-))\right.\right.\\ &\qquad\left.+\theta\alpha(z_3(u^-),z_4(u^-))]du\right|^2\\ &\quad+3\left|\int_{k\Delta}^{s}\sigma(z_1(u^-),z_2(u^-))dB(u)\right|^2\\ &\quad\left.+3\left|\int_{k\Delta}^{s}\int_{\mathbb{R}^n}\gamma(z_1(u^-),z_2(u^-),z)\tilde{N}(du,dz)\right|^2\right\}.\end{aligned}$$

Applying Doob's martingale inequality and the Itô isometry, yields

$$\begin{aligned}&\mathbb{E}\sup_{0\le t\le T}|\bar{Z}(t)-z_1(t)|^2\\ &\le \max_{k=0,1,\cdot,T/\Delta-1}\left\{6\Delta h(\Delta+4)+6h(\Delta+4)\mathbb{E}\int_{k\Delta}^{(k+1)\Delta}[|z_1(s^-)|^2+|z_2(s^-)|^2]ds\right.\end{aligned}$$

$$+6\Delta h\mathbb{E}\int_{k\Delta}^{(k+1)\Delta}[|z_3(s^-)|^2+|z_4(s^-)|^2]ds\Bigg\}.$$

Now the result follows from Lemma 3.3.2, and the similar analysis gives

$$\mathbb{E}\sup_{0\le t\le T}|\bar{Z}(t)-z_3(t)|^2\le C_4\Delta.$$

□

Lemma 3.3.5. *Under the linear growth condition (LG), for a sufficiently small* Δ

$$\mathbb{E}|\bar{Z}(\delta(t))-z_2(t)|^2\le C_5\Delta$$

and

$$\mathbb{E}|\bar{Z}(\delta(t))-z_4(t)|^2\le C_6\Delta,$$

where C_5 *and* C_6 *are constants independent of* Δ.

We omit the proof here, because it is very similar to the proof of Lemma 3.2.4.

Now we state the strong convergence result.

Theorem 3.3.1. *Under a global Lipschitz condition, for a sufficiently small* Δ,

$$\mathbb{E}\left[\sup_{0\le t\le T}|\bar{Z}(t)-X(t)|^2\right]\le C_7\Delta.$$

where C_7 *is a constant independent of* Δ.

Proof. By construction,

$$\begin{aligned}&\mathbb{E}\left[\sup_{0\le t\le T}|\bar{Z}(t)-X(t)|^2\right]\\ &\le 4\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t(1-\theta)[\alpha(z_1(s^-),z_2(s^-))-\alpha(X(s^-),X(\delta(s^-)))]ds\right|^2\\ &\quad+4\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t\theta[\alpha(z_3(s^-),z_4(s^-))-\alpha(X(s^-),X(\delta(s^-)))]\right|^2\end{aligned}$$

$$+ 4\mathbb{E} \sup_{0\le t\le T} \left| \int_0^t [\sigma(z_1(s^-), z_2(s^-)) - \sigma(X(s^-), X(\delta(s^-)))] dB(s) \right|^2$$

$$+ 4\mathbb{E} \sup_{0\le t\le T} \left| \int_0^t \int_{\mathbb{R}^n} [\gamma(z_1(s^-), z_2(s^-), z) - \gamma(X(s^-), X(\delta(s^-)), z)] \tilde{N}(ds, dz) \right|^2$$

$$\le 4T\mathbb{E} \sup_{0\le t\le T} \int_0^t |\alpha(z_1(s^-), z_2(s^-)) - \alpha(X(s^-), X(\delta(s^-)))|^2 ds$$

$$+ 4T\mathbb{E} \sup_{0\le t\le T} \int_0^t |\alpha(z_3(s^-), z_4(s^-)) - \alpha(X(s^-), X(\delta(s^-)))|^2 ds$$

$$+ 4\mathbb{E} \sup_{0\le t\le T} \left| \int_0^t [\sigma(z_1(s^-), z_2(s^-)) - \sigma(X(s^-), X(\delta(s^-)))] dB(s) \right|^2$$

$$+ 4\mathbb{E} \sup_{0\le t\le T} \left| \int_0^t \int_{\mathbb{R}^n} [\gamma(z_1(s^-), z_2(s^-), z) - \gamma(X(s^-), X(\delta(s^-)), z)] \tilde{N}(ds, dz) \right|^2 .$$

By Doob's martingale inequality and the global Lipschitz condition, we derive for the third term,

$$4\mathbb{E} \sup_{0\le t\le T} \left| \int_0^t [\sigma(z_1(s^-), z_2(s^-)) - \sigma(X(s^-), X(\delta(s^-)))] dB(s) \right|^2$$

$$\le 16\mathbb{E} \int_0^T [|\sigma(z_1(s^-), z_2(s^-)) - \sigma(X(s^-), X(\delta(s^-)))|^2] ds$$

$$\le 16L\mathbb{E} \int_0^T [|z_1(s^-) - X(s^-)|^2 + |z_2(s^-) - X(\delta(s^-))|^2] ds,$$

and for the last term, we apply the Itô isometry to obtain

$$4\mathbb{E} \sup_{0\le t\le T} \left| \int_0^t \int_{\mathbb{R}^n} [\gamma(z_1(s^-), z_2(s^-), z) - \gamma(X(s^-), X(\delta(s^-)), z)] \tilde{N}(ds, dz) \right|^2$$

$$\le 16\mathbb{E} \left| \int_0^T \int_{\mathbb{R}^n} [\gamma(z_1(s^-), z_2(s^-), z) - \gamma(X(s^-), X(\delta(s^-)), z)] \tilde{N}(ds, dz) \right|^2$$

$$= 16\mathbb{E} \int_0^T \int_{\mathbb{R}^n} \left| \gamma(z_1(s^-), z_2(s^-), z) - \gamma(X(s^-), X(\delta(s^-)), z) \right|^2 \nu(dz) ds$$

$$\le 16L\mathbb{E} \int_0^T [|z_1(s^-) - X(s^-)|^2 + |z_2(s^-) - X(\delta(s^-))|^2] ds.$$

Now by Lemma 3.3.4 and Lemma 3.3.5, we arrive at

$$\mathbb{E} \left[\sup_{0\le t\le T} |\bar{Z}(t) - X(t)|^2 \right]$$

$$
\begin{aligned}
&\le (32L + 4TL)\mathbb{E}\int_0^T [|z_1(s^-) - X(s^-)|^2 + |z_2(s^-) - X(\delta(s^-))|^2]ds \\
&\quad + 4TL\mathbb{E}\int_0^T [|z_3(s^-) - X(s^-)|^2 + |z_4(s^-) - X(\delta(s^-))|^2]ds \\
&\le (64L + 16TL)\mathbb{E}\int_0^T [|\bar{Z}(s^-) - X(s^-)|^2 + |\bar{Z}(\delta(s^-)) - X(\delta(s^-))|^2]ds \\
&\quad + (64L + 8TL)\mathbb{E}\int_0^T [|\bar{Z}(s^-) - z_1(s^-)|^2 + |\bar{Z}(\delta(s^-)) - z_2(s^-)|^2]ds \\
&\quad + 8TL\mathbb{E}\int_0^T [|\bar{Z}(s^-) - z_3(s^-)|^2 + |\bar{Z}(\delta(s^-)) - z_4(s^-)|^2]ds \\
&\le (128L + 32TL)\mathbb{E}\int_0^T |\bar{Z}(s^-) - X(s^-)|^2 ds \\
&\quad + (64L + 8TL)T(C_3\Delta + C_5\Delta) + 8T^2L(C_4\Delta + C_6\Delta).
\end{aligned}
$$

The Gronwall inequality gives the result as required. □

3.4 Rate of convergence under a local Lipschitz condition and a linear growth condition

3.4.1 The Euler-Maruyama method

In this section, we will discuss the rate of convergence for the EM method under a local Lipschitz condition with the Lipschitz coefficient L_R satisfying

$$L_R^2 \lambda T \le \varrho \log R,$$

where

$$\lambda = \left[T^3 + \left(\frac{512}{27}\right)^2 (4T + 3)\right].$$

Lemma 3.4.1. *Under a global Lipschitz condition (GL),*

$$\mathbb{E}\sup_{0\le t\le T}|X(t)-\bar{Y}(t)|^4 \le 432L^2\lambda\left[81h^2(1+2K_1^2+K_5)\Delta^2\right]Te^{864L^2\lambda T}.$$

Proof. Set $e(t)=X(t)-\bar{Y}(t)$, we obtain first

$$\begin{aligned}
&\mathbb{E}\sup_{0\le t\le T}|e(t)|\\
&\le 27\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t[\alpha(X(s^-),X(\delta(s^-)))-\alpha(z_1(s^-),z_2(s^-))]ds\right|^4\\
&+27\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t[\sigma(X(s^-),X(\delta(s^-)))-\sigma(z_1(s^-),z_2(s^-))]dB(s)\right|^4\\
&+27\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t\int_{\mathbb{R}^n}[\gamma(X(s^-),X(\delta(s^-)),z)-\gamma(z_1(s^-),z_2(s^-),z)]\tilde{N}(ds,dz)\right|^4.
\end{aligned}$$

By Hölder's inequality, and the global Lipschitz condition (GL), we estimate the first term,

$$\begin{aligned}
&27\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t[\alpha(X(s^-),X(\delta(s^-)))-\alpha(z_1(s^-),z_2(s^-))]ds\right|^4\\
&\le 27T^3\mathbb{E}\sup_{0\le t\le T}\int_0^t|\alpha(X(s^-),X(\delta(s^-)))-\alpha(z_1(s^-),z_2(s^-))|^4ds\\
&\le 54T^3L^2\mathbb{E}\sup_{0\le t\le T}\int_0^t\left[|X(s^-)-z_1(s^-)|^4+|X(\delta(s^-))-z_2(s^-)|^4\right]ds. \quad (3.26)
\end{aligned}$$

Next we apply the Burkholder inequality to the second term, obtaining

$$\begin{aligned}
&27\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t[\sigma(X(s^-),X(\delta(s^-)))-\sigma(z_1(s^-),z_2(s^-))]dB(s)\right|^4\\
&\le 27\left(\frac{512}{27}\right)^2\mathbb{E}\left\{\int_0^T|\sigma(X(s^-),X(\delta(s^-)))-\sigma(z_1(s^-),z_2(s^-))|^2ds\right\}^2\\
&\le 27\left(\frac{512}{27}\right)^2T\mathbb{E}\int_0^T|\sigma(X(s^-),X(\delta(s^-)))-\sigma(z_1(s^-),z_2(s^-))|^4ds\\
&\le 54\left(\frac{512}{27}\right)^2TL^2\mathbb{E}\int_0^T\left[|X(s^-)-z_1(s^-)|^4+|X(\delta(s^-))-z_2(s^-)|^4\right]ds.
\end{aligned} \quad (3.27)$$

Recall the technique we used in Lemma 2.2.1, and using the Itô isometry, we drive for the third term

$$\begin{aligned}
&\mathbb{E} \sup_{0\le t\le T} \left| \int_0^t \int_{\mathbb{R}^n} [\gamma(X(s^-), X(\delta(s^-)), z) - \gamma(z_1(s^-), z_2(s^-), z)] \tilde{N}(ds, dz) \right|^4 \\
&\le 54 \left(\frac{512}{27}\right)^2 \mathbb{E} \Big\{ \int_0^T \int_{\mathbb{R}^n} |\gamma(X(s^-), X(\delta(s^-)), z) \\
&\qquad - \gamma(z_1(s^-), z_2(s^-), z)|^2 \nu(dz) ds \Big\}^2 \\
&\quad + 54 \left(\frac{512}{27}\right)^2 \mathbb{E} \Big\{ \int_0^T \int_{\mathbb{R}^n} |\gamma(X(s^-), X(\delta(s^-)), z) \\
&\qquad - \gamma(z_1(s^-), z_2(s^-), z)|^2 \tilde{N}(ds, dz) \Big\}^2 \\
&\le 54 \left(\frac{512}{27}\right)^2 L^2 \mathbb{E} \left\{ \int_0^T [|X(s^-) - z_1(s^-)|^2 + |X(\delta(s^-)) - z_2(s^-)|^2] ds \right\}^2 \\
&\quad + 54 \left(\frac{512}{27}\right)^2 \mathbb{E} \int_0^T \int_{\mathbb{R}^n} |\gamma(X(s^-), X(\delta(s^-)), z) \\
&\qquad - \gamma(z_1(s^-), z_2(s^-), z)|^4 \nu(dz) ds \\
&\le 162 \left(\frac{512}{27}\right)^2 L^2 (T+1) \mathbb{E} \int_0^T [|X(s^-) - z_1(s^-)|^4 + |X(\delta(s^-)) - z_2(s^-)|^4] ds.
\end{aligned} \tag{3.28}$$

Combing (3.26), (3.27) and (3.28), we arrive at

$$\begin{aligned}
&\mathbb{E} \sup_{0\le t\le T} |X(t) - \bar{Y}(t)|^4 \\
&\le 54 T^3 L^2 \mathbb{E} \sup_{0\le t\le T} \int_0^t \left[|X(s^-) - z_1(s^-)|^4 + |X(\delta(s^-)) - z_2(s^-)|^4 \right] ds \\
&\quad + 54 \left(\frac{512}{27}\right)^2 T L^2 \mathbb{E} \int_0^T \left[|X(s^-) - z_1(s^-)|^4 + |X(\delta(s^-)) - z_2(s^-)|^4 \right] ds \\
&\quad + 162 \left(\frac{512}{27}\right)^2 L^2 (T+1) \mathbb{E} \int_0^T [|X(s^-) - z_1(s^-)|^4 + |X(\delta(s^-)) - z_2(s^-)|^4] ds \\
&= 54 L^2 \left[T^3 + \left(\frac{512}{27}\right)^2 (4T+3) \right] \mathbb{E} \int_0^T (X(s^-) - z_1(s^-))^4 ds \\
&\quad + 54 L^2 \left[T^3 + \left(\frac{512}{27}\right)^2 (4T+3) \right] \mathbb{E} \int_0^T |X(\delta(s^-)) - z_2(s^-)|^4 ds
\end{aligned}$$

$$\le 432L^2\lambda\mathbb{E}\int_0^T \left[|X(s^-) - \bar{Y}(s^-)|^4 + |\bar{Y}(s^-) - z_1(s^-)|^4\right] ds$$

$$+ 432L^2\lambda\mathbb{E}\int_0^T \left[|X(\delta(s^-)) - \bar{Y}(\delta(s^-))|^4 + |\bar{Y}(\delta(s^-)) - z_2(s^-)|^4\right] ds$$

$$= 864L^2\lambda\mathbb{E}\int_0^T |X(s^-) - \bar{Y}(s^-)|^4 ds$$

$$+ 432L^2\lambda\mathbb{E}\int_0^T \left[|\bar{Y}(s^-) - z_1(s^-)|^4 + |\bar{Y}(\delta(s^-)) - z_2(s^-)|^4\right] ds, \tag{3.29}$$

where $\lambda = \left[T^3 + \left(\frac{512}{27}\right)^2 (4T+3)\right]$. By the same computation as in Lemma 3.2.3 and Lemma 3.2.4, we have

$$\mathbb{E}|\bar{Y}(t) - z_1(t)|^4 \le 81h^2(1 + 2K_1^2)\Delta^2,$$

and

$$\mathbb{E}|\bar{Y}(\delta(t)) - z_2(t)|^4 \le K_5\Delta^2,$$

where K_1 is defined as in Lemma 2.1, K_5 is a constant independent of Δ. Now we can rewrite (3.29) as

$$\mathbb{E}\sup_{0\le t\le T}|X(t) - \bar{Y}(t)|^4$$

$$\le 864L^2\lambda\mathbb{E}\int_0^T \sup_{0\le s\le t}|X(s^-) - \bar{Y}(s^-)|^4 dt + 432L^2\lambda\left[81h^2(1 + 2K_1^2 + K_5)\Delta^2\right]T,$$

and the assertion follows from the Grownwall inequality. □

The next theorem gives the convergence rate under a local Lipschitz condition.

Theorem 3.4.1. *Under a local Lipichitz condition (LL) and the linear growth condition (LG), if there exists a constant ϱ such that $L_R^2\lambda T \le \varrho \log R$, where*

$$\lambda = \left[T^3 + \left(\frac{512}{27}\right)^2 (4T+3)\right],$$

the order of convergence of the EM approximation is one half.

Proof. For each $R \geq 1$, define the function

$$\alpha_R(x,y) = \begin{cases} \alpha(x,y), & \text{if } |x| \vee |y| < R; \\ \alpha(Rx/|x|, Ry/|y|), & \text{if } |x| \wedge |y| \geq R, \end{cases}$$

$\sigma_R(x,y)$ and $\gamma_R(x,y,z)$ similarly. Let $\bar{Y}_R(t)$ be the Euler-Maruyama approximation to the following stochastic differential equation

$$\begin{aligned} dX_R(t) &= \alpha_R(X_R(t), X_R(\delta(t)))dt + \sigma_R(X_R(t), X_R(\delta(t)))dB(t) \\ &+ \int_{\mathbb{R}^n} \gamma_R(X_R(t^-), X_R(\delta(t^-)), z)\tilde{N}(dt, dz) \end{aligned} \tag{3.30}$$

with $\bar{Y}_R(0) = X_0$. By the Lemma 3.4.1 we obtain

$$\mathbb{E} \sup_{0 \leq t \leq T} |X_R(t) - \bar{Y}_R(t)|^4 \leq 432 L_R^2 \lambda \left[81 h^2 (1 + 2K_1^2 + K_5)\Delta^2\right] T e^{864 L_R^2 \lambda T}.$$

Let

$$\hat{X}(t) = \sup_{0 \leq t \leq T} |X(t)| \text{ and } \hat{Y}(t) = \sup_{0 \leq t \leq T} |\bar{Y}(t)|.$$

Define the stopping time

$$\rho_R = T \wedge \inf\{t \in [0,T] : |X_R(t)| \vee ||\bar{Y}_R(t) \geq R\}.$$

Clearly, $|X_R(t)| \vee |X_R(\delta(t))| \leq R$ for $0 \leq t < \rho_R$, hence

$$\alpha_R(X_R(t), X_R(\delta(t))) = \alpha_{R+1}(X_R(t), X_R(\delta(t))),$$

$$\sigma_R(X_R(t), X_R(\delta(t))) = \sigma_{R+1}(X_R(t), X_R(\delta(t))),$$

and

$$\int_{\mathbb{R}^n} \gamma_R(X_R(t), X_R(\delta(t)), z)\nu(dz) = \int_{\mathbb{R}^n} \gamma_{R+1}(X_R(t), X_R(\delta(t)), z)\nu(dz)$$

on $\tau \leq t \leq \rho_R$. Therefore,

$$X_R(t) = X_{R+1}(t) \text{ and } \bar{Y}_R(t) = \bar{Y}_{R+1}(t) \text{ if } 0 \leq t \leq \rho_R.$$

This implies that ρ_R is increasing in R. Let $\rho = \lim_{R\to\infty} \rho_R$. The property above also enables us to define $X(t)$ for $t \in [-\tau, \rho)$ as follows

$$X(t) = X_R(t) \text{ if } -\tau \le t \le \rho_R.$$

It is clear that $X(t)$ is the unique solution to equation (5.1) for $t \in [\tau, \rho)$.

On the other hand, for $t \in [0, T]$, we compute

$$\begin{aligned}\mathbb{E}[X(t\wedge\rho)] &= \mathbb{E}[\zeta(0)] + \mathbb{E}\int_0^{t\wedge\rho} \alpha(X(s^-), X(\delta(s^-)))ds \\ &\le \mathbb{E}[\zeta(0)] + 2hT + 2\mathbb{E}\int_0^{t\wedge\rho} X(s^-)ds,\end{aligned}$$

and by the Gronwall inequality, we get

$$\mathbb{E}[X(t\wedge\rho)] \le [\mathbb{E}[\zeta(0)] + 2hT]e^{2hT}.$$

Note that $|X(\rho)| \ge R$, whenever $\rho < T$, and therefore we derive

$$R\mathbb{P}(\rho < T) \le \mathbb{E}[X(t\wedge\rho)I_{\{\rho<T\}}] \le [\mathbb{E}[\zeta(0)] + 2hT]e^{2hT}$$

that is

$$\mathbb{P}(\rho < T) \le \frac{[\mathbb{E}[\zeta(0)] + 2hT]e^{2hT}}{R}.$$

Letting $R \to \infty$, we obtain $\mathbb{P}(\rho_R < T) = 0$, this implies $\lim_{R\to\infty} \rho_R = T$ a.s. Let $\rho_0 = 0$ we compute, for $t \in [0, T)$

$$\begin{aligned}|X(t) - \bar{Y}(t)|^2 &= \sum_{R=1}^{\infty} |X(t) - \bar{Y}(t)|^2 I_{\{R-1\le \hat{X}(T)\vee\hat{Y}(T)\le R\}} \\ &= \sum_{R=1}^{\infty} |X_R(t) - \bar{Y}_R(t)|^2 I_{\{R-1\le \hat{X}(T)\vee\hat{Y}(T)\le R\}}.\end{aligned}$$

Therefore

$$\mathbb{E} \sup_{0\le t\le T} |X(t) - \bar{Y}(t)|^2$$

$$\leq \sum_{R=1}^{\infty}(\mathbb{E}|X_R(t)-\bar{Y}_R(t)|^4)^{1/2}(\mathbb{E}I_{\{R-1\leq \hat{X}(T)\vee\hat{Y}(T)\leq R\}})^{1/2}$$
$$= \sum_{R=1}^{\infty}(\mathbb{E}|X_R(t)-\bar{Y}_R(t)|^4)^{1/2}\sqrt{\mathbb{P}(R-1\leq \hat{X}(T)\vee\hat{Y}(T)\leq R)}$$

and from the condition $(L_R)^2\lambda T \leq \varrho \log R$, it follows that

$$\mathbb{E}\sup_{0\leq t\leq T}|X_R(t)-\bar{Y}_R(t)|^4 \leq 432\left[81h^2(1+2K_1^2+K_5)\Delta^2\right]TR^{1728\varrho}.$$

On the other hand, if let $q \geq 2$ then

$$\mathbb{P}(R-1\leq \hat{X}(T)\vee\hat{Y}(T)) \leq \frac{\mathbb{E}|\hat{X}(T)|^q+\mathbb{E}|\hat{Y}(T)|^q}{R^q} \leq 2\frac{K_p}{R^q},$$

and therefore,

$$\mathbb{E}\sup_{0\leq t\leq T}|X(t)-\bar{Y}(t)|^2 \leq \sum_{R=1}^{\infty}\sqrt{432\left[81h^2(1+2K_1^2+K_5)\right]T}\Delta R^{862\varrho}\frac{(2K_p)^{1/2}}{R^{q/2}}.$$

Let q be sufficiently large for $q/2 > 862\varrho$, we see that the right hand side is convergent, whence we get the rate of convergence is one half. □

3.4.2 The stochastic theta method

In this section, we will discuss the convergence rate of STM, when the coefficients function $\alpha(x,y)$ satisfying a global Lipschitz condition, and $\sigma(x,y)$ as well as $\int_{\mathbb{R}^n}\gamma(x,y,z)\nu(dz)$ satisfying a local Lipschitz condition i.e.

(A5).

$$|\alpha(x,y)-\alpha(\bar{x},\bar{y})|^2 \leq L(|x-\bar{x}|^2+|y-\bar{y}|^2)$$

and

$$|\sigma(x,y)-\sigma(\bar{x},\bar{y})|^2 + \sum_{k=1}^{n}\int_{\mathbb{R}}|\gamma^{(k)}(x,y,z_k)-\gamma^{(k)}(\bar{x},\bar{y},z_k)|^2\nu_k(dz_k)$$
$$\leq L_R(|x-\bar{x}|^2+|y-\bar{y}|^2).$$

Since $\alpha(x,y)$ satisfies the global Lipschitz condition, by Lemma 3.3.1, if $\Delta < \min\{\frac{1}{\theta\sqrt{L}}, \Delta^*\}$, the numerical solution of theta method is well defined.

Lemma 3.4.2. *Under a global Lipschitz condition (GL), we have*

$$\mathbb{E}\sup_{0\le t\le T}|X(t)-\bar{Z}(t)|^4$$
$$\le 512L^2\left[4T^3+\left(\frac{512}{27}\right)^2(5T+3)\right]T\Delta^2C^+e^{1024L^2\left[4T^3+\left(\frac{512}{27}\right)^2(5T+3)\right]T}$$

Where

$$C^+ = \max\{(C_3+C_5),(C_4+C_6)\},$$

and C_3, C_4, C_5, C_6 were defined in the Section 3.

Proof. For any $t\in[0,T]$, we obtain

$$\begin{aligned}
&\mathbb{E}\sup_{0\le t\le T}|X(t)-\bar{Z}(t)|^4\\
&\le 64\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t(1-\theta)[\alpha(z_1(s^-),z_2(s^-))-\alpha(X(s^-),X(\delta(s^-)))]ds\right|^4\\
&+64\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t\theta[\alpha(z_3(s^-),z_4(s^-))-\alpha(X(s^-),X(\delta(s^-)))]ds\right|^4\\
&+64\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t[\sigma(z_1(s^-),z_2(s^-))-\sigma(X(s^-),X(\delta(s^-)))]dB(s)\right|^4\\
&+64\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t\int_{\mathbb{R}^n}[\gamma(z_1(s^-),z_2(s^-),z)-\gamma(X(s^-),X(\delta(s^-))),z)]\tilde{N}(ds,dz)\right|^4\\
&\le 128T^3L^2\mathbb{E}\sup_{0\le t\le T}\int_0^t[|X(s^-)-z_1(s^-)|^4+|X(\delta(s^-))-z_2(s^-)|^4]ds\\
&+128T^3L^2\mathbb{E}\sup_{0\le t\le T}\int_0^t[|X(s^-)-z_3(s^-)|^4+|X(\delta(s^-))-z_4(s^-)|^4]ds\\
&+64\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t[\sigma(z_1(s^-),z_2(s^-))-\sigma(X(s^-),X(\delta(s^-)))]dB(s)\right|^4\\
&+64\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t\int_{\mathbb{R}^n}[\gamma(z_1(s^-),z_2(s^-),z)-\gamma(X(s^-),X(\delta(s^-))),z)]\tilde{N}(ds,dz)\right|^4.
\end{aligned}$$

For the third and the fourth term, the same computation as in Lemma 3.4.1, gives

$$64\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t[\sigma(z_1(s^-),z_2(s^-))-\sigma(X(s^-),X(\delta(s^-)))]dB(s)\right|^4$$
$$\le 128\left(\frac{512}{27}\right)^2 TL^2\int_0^T[|X(s^-)-z_1(s^-)|^4+|X(\delta(s^-))-z_2(s^-)|^4]ds$$

and

$$64\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t\int_{\mathbb{R}^n}[\gamma(z_1(s^-),z_2(s^-),z)-\gamma(X(s^-),X(\delta(s^-))),z)]\tilde{N}(ds,dz)\right|^4$$
$$\le 192\left(\frac{512}{27}\right)^2 L^2(T+1)\int_0^T[|X(s^-)-z_1(s^-)|^4+|X(\delta(s^-))-z_2(s^-)|^4]ds.$$

From Lemma 3.3 then we arrive at

$$\mathbb{E}\sup_{0\le t\le T}|X(t)-\bar{Z}(t)|^4$$
$$\le 64L^2(2T^3+\left(\frac{512}{27}\right)^2(5T+3))\mathbb{E}\int_0^T[|X(s^-)-z_1(s^-)|^4+|X(\delta(s^-))-z_2(s^-)|^4]ds$$
$$+512\mathbb{E}T^3L^2\int_0^T[|\bar{Z}(s^-)-z_3(s^-)|^4+|\bar{Z}(s^-)-z_4(s^-)|^4]ds$$
$$+1024\mathbb{E}T^3L^2\int_0^T|X(s^-)-\bar{Z}(s^-)|^4ds$$
$$\le 1024L^2\left[4T^3+\left(\frac{512}{27}\right)^2(5T+3)\right]\mathbb{E}\sup_{0\le t\le T}\int_0^t|X(s^-)-\bar{Z}(s^-)|^4ds$$
$$+512L^2T\Delta^2\left[4T^3+\left(\frac{512}{27}\right)^2(5T+3)\right]C^+,$$

where $C^+=\max\{(C_3+C_5),(C_4+C_6)\}$. By the Grownwall inequality, the result follows. □

Now we can proof

Theorem 3.4.2. *Under the condition (**A5**) and a linear growth condition*

(LG), for ϱ such that

$$L_R^2\left(\frac{512}{27}\right)^2(5T+3)\le \max(\varrho\log R-4T^3L^2,0) \tag{3.31}$$

we have as order of convergence of theta method one half.

Proof. Using the methods and ideas of the proof of Theorem 3.4.1 and denoting by $\bar{Z}(t)$ the theta method approximation to (3.30), we have the estimation

$$\begin{aligned}
&\mathbb{E}\sup_{0\le t\le T}|X(t)-\bar{Z}(t)|^2\\
&\le \sum_{R=1}^{\infty}(\mathbb{E}|X_R(t)-\bar{Z}_R(t)|^4)^{1/2}(\mathbb{E}I_{\{R-1\le \hat{X}(T)\vee\hat{Z}(T)\le R\}})^{1/2}\\
&=\sum_{R=1}^{\infty}(\mathbb{E}|X_R(t)-\bar{Z}_R(t)|^4)^{1/2}\sqrt{\mathbb{P}(R-1\le \hat{X}(T)\vee\hat{Z}(T)\le R)}.
\end{aligned}$$

By the Lemma 3.4.2, and the condition (**A5**), we estimate

$$\begin{aligned}
&\mathbb{E}|X_R(t)-\bar{Z}_R(t)|^4\\
&\le\left[2048C^+T^3L^2\Delta^2+512C^+L_R^2\Delta^2\left(\frac{512}{27}\right)^2(5T+3)\right]T\\
&\quad\times\exp\left\{\left[4096T^3L^2+1024L_R^2\left(\frac{512}{27}\right)^2(5T+3)\right]T\right\}
\end{aligned}$$

and from the condition (3.31), it follows that

$$\mathbb{E}\sup_{0\le t\le T}|X_R(t)-\bar{Z}_R(t)|^4\le 512\left[C^+\Delta^2\right]TR^{1024\varrho}.$$

On the other hand, for $p>2$, and any $t\in[0,T]$

$$\begin{aligned}
&\mathbb{E}|\bar{Z}(t)|^p\\
&\le 4^{p-1}\varsigma(0)+4^{p-1}\mathbb{E}\left|\int_0^t[(1-\theta)\alpha(z_1(s^-),z_2(s^-))+\theta\alpha(z_3(s^-),z_4(s^-))]ds\right|^p\\
&\quad+4^{p-1}\mathbb{E}\left|\int_0^t\sigma(z_1(s^-),z_2(s^-))dB(s)\right|^p
\end{aligned}$$

$$+ 4^{p-1}\mathbb{E}\left|\int_0^t \int_{\mathbb{R}^n} \gamma(z_1(s^-), z_2(s^-), z)\tilde{N}(ds, dz)\right|^p.$$

By the Hölder inequality and the linear growth condition, we estimate the second term,

$$\begin{aligned}
&\mathbb{E}\left|\int_0^t [(1-\theta)\alpha(z_1(s^-), z_2(s^-)) + \theta\alpha(z_3(s^-), z_4(s^-))]ds\right|^p \\
&\leq T^{p-1}\mathbb{E}\int_0^t |(1-\theta)\alpha(z_1(s^-), z_2(s^-)) + \theta\alpha(z_3(s^-), z_4(s^-))|^p ds \\
&\leq 2^{p-1}T^{p-1}\mathbb{E}\int_0^t [|\alpha(z_1(s^-), z_2(s^-))|^p + |\alpha(z_3(s^-), z_4(s^-))|^p] ds \\
&\leq 2^{p-1}T^{p-1}\mathbb{E}\int_0^t [|z_1(s^-)|^p + |z_2(s^-)|^p + |z_3(s^-)|^p + |z_4(s^-)|^p] ds + 2^p T^p h.
\end{aligned} \tag{3.32}$$

For the third and the fourth term we can take the result in the proof of Lemma 3.2.2, and together with (3.32), we obtain

$$\mathbb{E}|\bar{Z}(t)|^p \leq K_{p1}.$$

Let $q \geq 2$

$$\mathbb{P}(R - 1 \leq \hat{X}(T) \vee \hat{Z}(T)) \leq \frac{\mathbb{E}|\hat{X}(T)|^q + \mathbb{E}|\hat{Z}(T)|^q}{R^q} \leq 2\frac{K_p'}{R^q},$$

where $K_p' = \max\{K_p, K_{p1}\}$. It follows that

$$\mathbb{E}\sup_{0\leq t\leq T} |X(t) - \bar{Z}(t)|^2 \leq \sum_{R=1}^{\infty} \sqrt{512C^+T}\Delta R^{1024\varrho}\frac{(2K_p')^{1/2}}{R^{q/2}}.$$

For $q/2 > 1024\varrho$, we see that the right hand side is convergent, whence we get the rate of convergence is one half.

□

Chapter 4

Stochastic differential delay equations with jumps, under nonlinear growth condition

4.1 Introduction

The classical existence and uniqueness result for solutions of a stochastic differential delay equations (SDDEs) require the coefficients function satisfy a local Lipschitz condition and a linear growth condition. However, there are many SDDEs which do not satisfy the linear growth condition, for example, the geometric Lévy process:

$$dX(t) = X(t)\left[(a - X^2(t))dt + bX(t)dB(t) + \int_{\mathbb{R}} z\tilde{N}(dt, dz)\right].$$

Since it preserves positivity i.e. an initial value $X(0) \geq 0$ implies $X(t) \geq 0$, it is often used as a model for stock prices, such as the exponential-Lévy model (cf. [7]).

In this chapter, we investigate the existence and uniqueness of SDDEs with jumps which coefficients which do not satisfy the linear growth condition, we also study their numerical solutions.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathcal{F}_0$ contains all P-null sets). Let $B(t)$ be $m-$dimensional Brownian motion and $N(t,z)$ be a Poisson measure and denote the compensated Poisson measure

$$\tilde{N}(dt,dz) = N(dt,dz) - \nu(dz)dt.$$

Let $|\cdot|$ denote the Euclidean norm as well as the matrix trace norm. Let $\tau > 0$ and $C([-\tau,0];\mathbb{R}^d)$ denote the family of continuous function ϕ from $[-\tau,0]$ to $\mathbb{R}^d$ with the norm $||\phi|| = \sup_{-\tau\leq\theta\leq 0}|\phi(\theta)|$. Denote by $C^b_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^d)$ the family of all bounded, $\mathcal{F}_0-$measurable, $C([-\tau,0];\mathbb{R}^d)-$valued random variables.

Consider the $d-$dimensional stochastic differential delay equation with jumps

$$\begin{aligned} dX(t) &= \alpha(X(t), X(t-\tau))dt + \sigma(X(t), X(t-\tau))dB(t) \\ &\qquad + \int_{\mathbb{R}^n} \gamma(X(t^-), X((t-\tau)^-), z)\tilde{N}(dt,dz) \end{aligned} \tag{4.1}$$

for $t \in [0,T]$ with initial data

$$\{X(t) : -\tau \leq t \leq 0\} = \{\zeta(t) : -\tau \leq t \leq 0\} \in C^b_{\mathcal{F}_0}([-\tau,0]),$$

and where $X(t^-) = \lim_{s\to t} X(s)$, $\alpha : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$, $\gamma : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d\times n}$. We note that each column $\gamma^{(k)}$ of the $d \times n$ matrix $\gamma = [\gamma_{ij}]$ depends on z only through the k^{th} coordinate z_k, i.e.

$$\gamma^{(k)}(x,z) = \gamma^{(k)}(x,z_k); \quad z = (z_1, \cdots, z_n) \in \mathbb{R}^n.$$

Given $V \in C^{2,1}(\mathbb{R}^d \times [-\tau, T); \mathbb{R}_+)$, we define the operator $\mathcal{L}V$ by

$$\mathcal{L}V(x,y) = V_x \alpha(x,y) + \frac{1}{2}\text{trace}[\sigma^T(x,y)V_{xx}\sigma(x,y)] + \int_{\mathbb{R}^n} \sum_{k=1}^{n} \left\{ V(x + \gamma^{(k)}(x,y,z)) - V(x) - V_x(\gamma^{(k)}(x,y,z)) \right\} \nu_k(dz_k).$$

where

$$V_x(x) = \left(\frac{\partial V(x)}{\partial x_1}, \ldots, \frac{\partial V(x)}{\partial x_n} \right), \quad V_{xx}(x) = \left(\frac{\partial^2 V(x)}{\partial x_i \partial x_j} \right)_{d \times d}.$$

Assumption 1. (LL) For each $R = 1, 2, \cdots$, there exists a constant L_R such that

$$|\sigma(x,y) - \sigma(\bar{x},\bar{y})|^2 + |\alpha(x,y) - \alpha(\bar{x},\bar{y})|^2 + \int_{\mathbb{R}^n} |\gamma(x,y,z) - \gamma(\bar{x},\bar{y},z)|^2 \nu(dz) \leq L_R(|x-\bar{x}|^2 + |y-\bar{y}|^2), \tag{4.2}$$

for all $x,\ y,\ \bar{x},\ \bar{y} \in \mathbb{R}^n$, and $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \leq R$.

Assumption 2. There are two functions $V \in C^2(\mathbb{R}^d \times [-\tau, T); \mathbb{R}_+)$ and $U \in C(\mathbb{R}^d \times [-\tau, T); \mathbb{R}_+)$ as well as two positive constants λ_1 and λ_2, such that

$$\lim_{|x| \to \infty} V(x) = \infty \tag{4.3}$$

and

$$\mathcal{L}V(x,y) \leq \lambda_1[1 + V(x) + V(y) + U(y)] - \lambda_2 U(x).$$

Assumption 3. The jumps are bounded for all moment i.e. $\int_{\mathbb{R}^n} |\gamma(x,z)|^p \nu(dz) = C$, where p and C are positive constants.

The rest of the chapter is arranged as follows, in section 2, we extend Mao's work (cf. [36]) to SDDEs with jumps. In section 3, we present the Euler-Maruyama scheme and Stochastic Theta Method approximation convergence to the exact solution in probability.

4.2 Existence and uniqueness

Theorem 4.2.1. *Let Assumption 1 and 2 hold. Then for any given initial data* $\{\zeta(t) : \ -\tau \le t \le 0\} \subset C^b_{\mathcal{F}_0}([-\tau, 0])$, *there is a unique global solution* $X(t)$ *to equation* (4.1) *on* $t \in [-\tau, T]$. *Moreover, the solution has the properties that*

$$\mathbb{E}V(X(t)) < \infty \text{ and } \mathbb{E}\int_0^t U(X(s))ds < \infty$$

for any $t \in [0, T]$.

Proof. By Assumption 1, for any given initial data $\zeta(t)$ there is a unique maximal local solution $X(t)$ on $t \in [-\tau, \tau_e)$, where τ_e is the explosion time. For each integer R, we define the stopping time

$$\tau_R = \inf\{t \in [0, \tau_e) : |X(t^-)| \ge R\},$$

and we define $\inf \emptyset = \infty$, $\emptyset$ denotes the empty set. Clearly τ_R is increasing as $R \to \infty$. Set $\tau_\infty = \lim_{R\to\infty} \tau_R$. Note that $\tau_\infty \le \tau_e$ a.s. To complete the proof, we need to show $\mathbb{P}\{\tau_\infty < T\} = 0$ a.s. For any $R > 0$ we derive from the Itô formula on $t_1 \in [0, \tau]$,

$$\begin{aligned}
&V(X(t_1 \wedge \tau_R)) \\
&= V(\zeta(0)) + \int_0^{t_1\wedge\tau_R} V_x(X(s^-))\alpha(X(s^-), X((s-\tau)^-))ds \\
&+ \int_0^{t_1\wedge\tau_R} V_x(X(s^-))\sigma(X(s^-), X((s-\tau)^-))dB(s) \\
&+ \frac{1}{2}\int_0^{t\wedge\theta} \text{trace}[\sigma^T(X(s^-), X((s-\tau)^-))V_{xx}(X(s^-))\sigma(X(s^-), X((s-\tau)^-))]ds \\
&+ \int_0^{t_1\wedge\tau_R}\int_{\mathbb{R}^n} \{V(X(s^-)) + \gamma(X(s^-), X((s-\tau)^-), z)) - V(X(s^-)) \\
&\qquad -V_x(X(s^-)\gamma(X(s^-), X((s-\tau)^-), z))\}\,\nu(dz)ds \\
&+ \int_0^{t_1\wedge\tau_R}\int_{\mathbb{R}^n} \{V(X(s^-) + \gamma(X(s^-), X((s-\tau)^-), z)) - V(X(s^-))\}\,\tilde{N}(dz, ds) \\
&+ \int_0^{t_1\wedge\tau_R}
\end{aligned}$$

$$= V(\zeta(0)) + \int_0^{t_1\wedge\tau_R} \mathcal{L}V(X(s^-), X((s-\tau)^-))ds$$
$$+ \int_0^{t_1\wedge\tau_R} V_x(X(s^-))\sigma(X(s^-), X((s-\tau)^-))dB(s)$$
$$+ \int_0^{t_1\wedge\tau_R} \int_{\mathbb{R}^n} \{V(X(s^-) + \gamma(X(s^-), X((s-\tau)^-), z)) - V(X(s^-))\} \tilde{N}(dz, ds),$$

note the last two terms are martingales, we take the expectation and by the Assumption 2 to get

$$\begin{aligned}
&\mathbb{E}V(X(t_1\wedge\tau_R)) \\
&= \mathbb{E}V(\zeta(0)) + \mathbb{E}\int_0^{t_1\wedge\tau_R} \mathcal{L}V(X(s^-)), X((s-\tau)^-))ds \\
&\le V(\zeta(0)) - \lambda_2\mathbb{E}\int_0^{t_1\wedge\tau_R} U(X(s^-))ds \\
&\qquad + \lambda_1\mathbb{E}\int_0^{t_1\wedge\tau_R} (1 + V(X(s^-)) + V(X((s-\tau)^-)) + U(X((s-\tau)^-)))ds \\
&\le c_1 + \lambda_1\mathbb{E}\int_0^{t_1\wedge\tau_R} V(X(s^-))ds - \lambda_2\mathbb{E}\int_0^{t_1\wedge\tau_R} U(X(s^-))ds \qquad (4.4)
\end{aligned}$$

where

$$\begin{aligned}
c_1 &= V(\zeta(0)) + \lambda_1\mathbb{E}\int_0^{\tau} (1 + V(X((s-\tau)^-)) + U(X((s-\tau)^-)))ds \\
&= V(\zeta(0)) + \lambda_1\mathbb{E}\int_{-\tau}^{0} (1 + V(\zeta(s^-)) + U(\zeta(s^-)))ds \\
&< \infty.
\end{aligned}$$

Since λ_2 is a positive constant, and the function $U : \mathbb{R}^n \to \mathbb{R}_+$, so we obtain

$$\begin{aligned}
\mathbb{E}V(X(t_1\wedge\tau_R)) &\le c_1 + \lambda_1\mathbb{E}\int_0^{t_1\wedge\tau_R} V(X(s^-))ds \\
&\le c_1 + \lambda_1\mathbb{E}\int_0^{t_1} V(X(s\wedge\tau_R)^-)ds,
\end{aligned}$$

for any $t_1 \in [0, \tau]$, and by the Gronwall inequality

$$\mathbb{E}V(X(t_1\wedge\tau_R)) \le c_1 e^{\lambda_1\tau}. \qquad (4.5)$$

In particular, for any $R > 0$

$$\mathbb{E}V(X(\tau \wedge \tau_R)) \leq c_1 e^{\lambda_1 \tau}.$$

Define

$$\mu_R = \inf_{|x| \geq R} V(x) \quad \text{for all } R > 0.$$

It then follows that

$$\mu_R \mathbb{P}(\tau_R < \tau) \leq \mathbb{E}V(X(\tau \wedge \tau_R)) \leq c_1 e^{\lambda_1 \tau}.$$

By the condition (4.3), $\lim_{R\to\infty} \mu_R = \infty$. Letting $R \to \infty$, we arrive at

$$\mathbb{P}(\tau_R < \tau) = 0.$$

Letting $R \to \infty$ in (4.5) yields

$$\mathbb{E}V(X(t_1)) \leq c_1 e^{\lambda_1 \tau} \text{ for } 0 \leq t_1 \leq \tau.$$

Moreover, setting $t_1 = \tau$, we observe that

$$\lambda_2 \mathbb{E} \int_0^{\tau \wedge \tau_R} U(X(s^-))ds \leq c_1 + \lambda_1 \mathbb{E} \int_0^{\tau \wedge \tau_R} V(X(s^-))ds.$$

Letting $R \to \infty$, we have

$$\mathbb{E} \int_0^{\tau} U(X(s^-))ds \leq \frac{1}{\lambda_2}(c_1 + \tau \lambda_1 c_1 e^{\lambda_1 \tau}) < \infty.$$

Now we proceed to prove $\tau_\infty > 2\tau$ a.s. For any $R > 0$ and $t_2 \in [0, 2\tau]$, nothing changed but τ_1 substituted by τ_2 in (4.4) and we get

$$\mathbb{E}V(X(t_2 \wedge \tau_R)) \leq c_2 + \lambda_1 \mathbb{E} \int_0^{t_2 \wedge \tau_R} V(X(s^-))ds - \lambda_2 \mathbb{E} \int_0^{t_2 \wedge \tau_R} U(X(s^-))ds, \tag{4.6}$$

where

$$c_2 = V(\zeta(0)) + \lambda_1 \mathbb{E} \int_0^{2\tau} (1 + V(X((s-\tau)^-)) + U(X((s-\tau)^-)))ds$$

$$= \lambda_1 \mathbb{E} \int_0^{\tau} (1 + V(X(s^-)) + U(X(s^-)))ds$$
$$+ \lambda_1 \mathbb{E} \int_{-\tau}^{0} (1 + V(\zeta(s^-)) + U(\zeta(s^-)))ds$$
$$< \infty.$$

Therefore, we have

$$\mathbb{E}V(X(t_2 \wedge \tau_R)) \leq c_2 e^{2\lambda_1 \tau}, \quad 0 \leq t_1 \leq 2\tau. \tag{4.7}$$

In particular,

$$\mathbb{E}V(X(2\tau \wedge \tau_R)) \leq c_2 e^{2\lambda_1 \tau}, \quad \text{for any } R \geq R_0.$$

This implies

$$\mu_R \mathbb{P}(\tau_R \leq 2\tau) \leq c_2 e^{2\lambda_1 \tau}.$$

Letting $R \to \infty$, we then obtain that $\mathbb{P}(\tau_R \leq 2\tau) = 0$.

Moreover, by letting $R \to \infty$ and setting $t_2 = 2\tau$ in (4.6), we observe that

$$\mathbb{E} \int_0^{2\tau} U(X(s^-))ds \leq \frac{1}{\lambda_2} \left[c_2 + \lambda_1 \mathbb{E} \int_0^{2\tau} V(X(s^-))ds \right]$$
$$\leq \frac{1}{\lambda_2}(c_2 + 2\lambda_1 \tau c_2 e^{2\lambda_1 \tau})$$
$$< \infty.$$

Repeating this procedure, we can show, for any integer $1 \leq i \leq \lfloor \frac{T}{\tau} \rfloor + 1$, that $\tau_\infty \geq i\tau$ a.s. and

$$\mathbb{E}V(X(t)) \leq c_i e^{i\lambda_1 \tau} \; 0 \leq t \leq i\tau,$$

as well as

$$\mathbb{E} \int_0^{i\tau} U(X(s^-))ds \leq \frac{1}{\lambda_2}(c_i + i\lambda_1 \tau c_i e^{i\lambda_1 \tau})$$

where

$$c_i = V(\zeta(0)) + \lambda_1 \mathbb{E} \int_0^{i-1\tau} (1 + V(X(s^-)) + U(X(s^-)))ds < \infty.$$

Therefore, we have $\mathbb{P}(\tau_\infty < T) = 0$ as claimed. □

Example 4.2.1. Consider a one-dimensional SDDE with jumps

$$dX(t) = X(t)\left[aX(t-\tau) - X^2(t)dt + bX(t-\tau)dB(t) + \int_{\mathbb{R}} z\tilde{N}(dz, dt)\right], \tag{4.8}$$

where $B(t)$ is a one dimensional Brownian motion and $\tilde{N}(dz, dt)$ is a Poisson measure. Both a, and b are constant. We now let $V(x) = x^2$, Then the corresponding operator $\mathcal{L}V(x, y)$ has the form

$$\begin{aligned}\mathcal{L}V(x,y) &= 2x^2(ay - x^2) + x^2b^2y^2 + x^2\int_{\mathbb{R}} z^2\nu(dz) \\ &\leq 2ax^2y - 2x^4 + b^2x^2y^2 + Cx^2.\end{aligned}$$

By the elementary inequality,

$$uv \leq \frac{(u^2 + v^2)}{2},$$

we derive

$$b^2x^2y^2 \leq \frac{(x^4 + b^4y^4)}{2}. \tag{4.9}$$

Recalling the Young inequality

$$u^\beta v^{1-\beta} \leq \beta u + (1-\beta)v, \text{ for any } u, v \geq 0,\ \beta \in [0, 1],$$

we compute, for some $\epsilon > 0$

$$x^2y = \left(\epsilon x^4\right)^{1/2}\left(\frac{y^2}{\epsilon}\right)^{1/2} \leq \frac{\epsilon}{2}x^4 + \frac{1}{2\epsilon}y^2.$$

Choosing $\epsilon = \frac{1}{a}$, we have

$$2ax^2y \leq x^4 + a^2y^2. \tag{4.10}$$

Combining (4.9) and (4.10), we arrive at

$$\mathcal{L}V(x,y) \le \lambda_1(x^2+y^2+y^4) - \frac{1}{2}x^4,$$

where $\lambda_1 = \max\{C, a^2, \frac{b^2}{4}\}$. Therefore, we choose $U(x) = x^4$ and $\lambda_2 = \frac{1}{2}$ to fulfill Assumption 2. Moreover, by Theorem 4.2.1 we conclude for any initial data $\zeta(t)$, $-\tau \le t \le 0$, that there is a unique global solution $X(t)$ on $t \in [-\tau, T]$.

Remark 4.2.1. By Theorem 4.2.1,

$$\mathbb{E}V(X(t)) < \infty \text{ and } \mathbb{E}\int_0^t U(X(s))ds < \infty,$$

if we let $V(x,y) = |x|^p$ and $U(x,y) = |x|^q$ for any $p,\ q \ge 2$ we get the moment bounded property.

4.3 Convergence in probability

In this section, we will introduce the Euler-Maruyama scheme and Stochastic Theta Method for SDDEs with jumps and prove convergence to the exact solution in probability under some additional conditions in terms of Lyapunov-type functions. We replace Assumption 2 by :

Assumption 4 The C^2 function $V : \mathbb{R}^n \to \mathbb{R}_+$ satisfying

$$\lim_{|x|\to\infty} V(x) = \infty$$

and

$$\mathcal{L}V(x,y) \le h[1 + V(x) + V(y)].$$

Assumption 5 for each $R > 0$ there exists a positive constant K_R such that for all $x, y \in \mathbb{R}^n$ with $|x| \vee |y| \le R$,

$$|V(x) - V(y)| \vee |V_x(x) - V_x(y)| \vee |V_{xx}(x) - V_{xx}(y)| \le K_R|x-y|.$$

4.3.1 Euler-Maruyama method

We now define the Euler-Maruyama approximation. Let the time stepsize $\Delta \in (0,1)$ be a fraction of τ, that is $\Delta = \frac{\tau}{N}$ for some sufficiently large integer N. The discrete EM approximate solution is defined by

$$\begin{aligned} Y((k+1)\Delta) =& Y(k\Delta) + \alpha(Y(k\Delta), Y(I_\Delta[k\Delta - \tau]\Delta))\Delta \\ &+ \sigma(Y(k\Delta), Y(I_\Delta[k\Delta - \tau]\Delta))\Delta B_k \\ &+ \int_{\mathbb{R}^n} \gamma(Y(k\Delta), Y(I_\Delta[k\Delta - \tau]\Delta, z))\Delta \tilde{N}_k(dz) \end{aligned}$$

with $Y(0) = \zeta(0)$ on $-\tau \leq t \leq 0$. Here $k = 0, 1, 2, \ldots$, and $I_\Delta[k\Delta - \tau]$ denotes the integer part of $k\Delta - \tau/\Delta$, and $\Delta B_k = B((k+1)\Delta) - B(k\Delta)$ and $\Delta \tilde{N}_k(dz) = \tilde{N}((k+1)\Delta, dz) - \tilde{N}(k\Delta, dz)$.

To define the continuous extension, we need to introduce two step processes

$$z_1(t) = \sum_{k=0}^{\infty} 1_{[k\Delta,(k+1)\Delta]}(t) Y(k\Delta),$$

$$z_2(t) = \sum_{k=0}^{\infty} 1_{[k\Delta,(k+1)\Delta]}(t) Y(I_\Delta[k\Delta - \tau]\Delta).$$

The continuous EM numerical solution is defined by

$$\bar{Y}(t) = \begin{cases} \zeta(t), & -\tau \leq t \leq 0; \\ \zeta(0) + \int_0^t \alpha(z_1(s^-), z_2(s^-))ds \\ \quad + \int_0^t \sigma(z_1(s^-), z_2(s^-))dB(s) \\ \quad + \int_0^t \int_{\mathbb{R}^n} \gamma(z_1(s^-), z_2(s^-), z)\tilde{N}(ds, dz), & 0 \leq t \leq T. \end{cases}$$

Note that $Y(k\Delta) = \bar{Y}(k\Delta)$ for every $k \geq 0$.

Theorem 4.3.1. *Let Assumptions 1, 3, 4, 5 hold. Then*

$$\lim_{\Delta\to 0}\left(\sup_{0\le t\le T}|X(t)-\bar{Y}(t)|^2\right)=0 \quad \textit{in probability.} \tag{4.11}$$

Proof. We divide the whole proof into three steps.

Step 1. For sufficiently large R, define the stopping time

$$\theta_1=\inf\{t\in[0,T]:|X(t^-)|\ge R\}.$$

Applying the generalized Itô formula, yields

$$\begin{aligned}
&V(X(t\wedge\theta_1))\\
&=V(\zeta(0))+\int_0^{t\wedge\theta_1}V_x(X(s^-))\alpha(X(s^-),X((s-\tau)^-))ds\\
&\quad+\int_0^{t\wedge\theta_1}V_x(X(s^-))\sigma(X(s^-),X((s-\tau)^-))dB(s)\\
&\quad+\frac{1}{2}\int_0^{t\wedge\theta_1}\text{trace}[\sigma^T(X(s^-),X((s-\tau)^-))V_{xx}(X(s^-))\\
&\qquad\qquad\times\sigma(X(s^-),X((s-\tau)^-))]ds\\
&\quad+\int_0^{t\wedge\theta_1}\int_{\mathbb{R}^n}\{V(X(s^-))+\gamma(X(s^-),X((s-\tau)^-),z))-V(X(s^-))\\
&\qquad\qquad-V_x(X(s^-)\gamma(X(s^-),X((s-\tau)^-),z))\}\,\nu(dz)ds\\
&\quad+\int_0^{t\wedge\theta_1}\int_{\mathbb{R}^n}\{V(X(s^-)+\gamma(X(s^-),X((s-\tau)^-),z))\\
&\qquad\qquad-V(X(s^-))\}\,\tilde{N}(dz,ds)\\
&=V(\zeta(0))+\int_0^{t\wedge\theta_1}\mathcal{L}V(X(s^-),X((s-\tau)^-))ds\\
&\quad+\int_0^{t\wedge\theta_1}V_x(X(s^-))\sigma(X(s^-),X((s-\tau)^-))dB(s)\\
&\quad+\int_0^{t\wedge\theta_1}\int_{\mathbb{R}^n}\{V(X(s^-)+\gamma(X(s^-),X((s-\tau)^-),z))-V(X(s))\}\,\tilde{N}(dz,ds),
\end{aligned}$$

Note that the last two terms are martingales, taking the expectation of $V(X(t))$, and by the Assumption 4 we arrive at

$$\mathbb{E}(V(X(t\wedge\theta_1)))$$

$$= V(\zeta(0)) + \mathbb{E}\int_0^{t\wedge\theta_1} \mathcal{L}V(X(s^-), X((s-\tau)^-))ds$$
$$\leq V(\zeta(0)) + h\mathbb{E}\int_0^{t\wedge\theta_1} [1 + V(X(s^-)) + V(X((s-\tau)^-))]ds$$
$$\leq V(\zeta(0)) + hT + \sup_{-\tau\leq s\leq 0} \mathbb{E}V(\zeta(s)) + 2h\mathbb{E}\int_0^t V(X(s\wedge\theta_1))ds,$$

by the Gronwall inequality we have

$$\mathbb{E}(V(X(t\wedge\theta_1))) \leq (V(\zeta(0)) + hT + \sup_{-\tau\leq s\leq 0} \mathbb{E}V(\zeta(s)))e^{2hT}. \tag{4.12}$$

Let

$$v_R = \inf\{V(x) : |x| \geq R\}.$$

By condition (i), $v_R \to \infty$ as $R \to \infty$. Note that $|X(\theta_1)| = R$ whenever $\theta_1 < T$, and therefore we derive from (4.12) that

$$v_R\mathbb{P}(\theta_1 < T) \leq \mathbb{E}V(X(\theta_1)I_{\{\theta_1<T\}}) \leq (V(\zeta(0)) + hT + \sup_{-\tau\leq s\leq 0} \mathbb{E}V(\zeta(s)))e^{2hT},$$

that is

$$\mathbb{P}(\theta_1 < T) \leq \frac{(V(\zeta(0)) + hT + \sup_{-\tau\leq s\leq 0} \mathbb{E}V(\zeta(s)))e^{2hT}}{v_R}. \tag{4.13}$$

Step 2. For sufficiently large R define the stopping time

$$\rho_1 = \inf\{t \in [0,T] : |\bar{Y}(t^-)| \geq R\}.$$

Once again, we apply the Itô formula to $V(\bar{Y}(t))$, and taking the expectation we obtain

$$\mathbb{E}[V(\bar{Y}(t\wedge\rho_1))]$$
$$= V(\zeta(0)) + \mathbb{E}\int_0^{t\wedge\rho_1} V_x(\bar{Y}(s^-))\alpha(z_1(s^-), z_2(s^-))ds$$
$$+ \frac{1}{2}\mathbb{E}\int_0^{t\wedge\rho_1} \text{trace}[\sigma^T(z_1(s^-), z_2(s^-))V_{xx}(\bar{Y}(s^-))\sigma(z_1(s^-), z_2(s^-))]ds$$

$$
\begin{aligned}
&+\mathbb{E}\int_0^{t\wedge\rho_1}\int_{\mathbb{R}^n}\big\{V(\bar{Y}(s^-)+\gamma(z_1(s^-),z_2(s^-),z))-V(\bar{Y}(s^-))\\
&\qquad -V_x(\bar{Y}(s^-))\gamma(z_1(s^-),z_2(s^-),z)\big\}\,\nu(dz)ds\\
&=V(\zeta(0))+\mathbb{E}\int_0^{t\wedge\rho_1}\mathcal{L}V(z_1(s^-),z_2(s^-))ds\\
&+\mathbb{E}\int_0^{t\wedge\rho_1}\left[V_x(\bar{Y}(s^-))-V_x(z_1(s^-))\right]\alpha(z_1(s^-),z_2(s^-))ds\\
&+\frac{1}{2}\mathbb{E}\int_0^{t\wedge\rho_1}\text{trace}\left[\sigma^T(z_1(s^-),z_2(s^-))\left(V_{xx}(\bar{Y}(s^-))-V_{xx}(z_1(s^-))\right)\right.\\
&\quad\left.\times\sigma(z_1(s^-),z_2(s^-))\right]ds\\
&+\mathbb{E}\int_0^{t\wedge\rho_1}\int_{\mathbb{R}^n}\big\{V(\bar{Y}(s^-)+\gamma(z_1(s^-),z_2(s^-),z))\\
&\quad -V(z_1(s^-)+\gamma(z_1(s^-),z_2(s^-),z))\\
&\quad +V(z_1(s^-))-V(\bar{Y}(s^-))\\
&\quad -\left[V_x(\bar{Y}(s^-))-V_x(z_1(s^-))\right]\gamma(z_1(s^-),z_2(s^-),z)\big\}\,\nu(dz)ds\\
&\le V(\zeta(0))+hT+2h\mathbb{E}\int_0^{t\wedge\rho_1}V(\bar{Y}(s^-))ds\\
&+h\mathbb{E}\int_0^{t\wedge\rho_1}|V(\bar{Y}(s^-))-V(z_1(s^-))|ds\\
&+h\mathbb{E}\int_0^{t\wedge\rho_1}|V(\bar{Y}(s^-))-V(z_2(s^-))|ds\\
&+C_R\mathbb{E}\int_0^{t\wedge\rho_1}|\bar{Y}(s^-)-z_1(s^-)|ds,\\
&\le V(\zeta(0))+hT+2h\mathbb{E}\int_0^{t\wedge\rho_1}V(\bar{Y}(s^-))ds+hK_R\int_0^{t\wedge\rho_1}|\bar{Y}(s^-)-z_1(s^-)|ds\\
&+hK_R\int_0^{t\wedge\rho_1}|\bar{Y}(s^-)-z_2(s^-)|ds+C_R\mathbb{E}\int_0^{t\wedge\rho_1}|\bar{Y}(s^-)-z_1(s^-)|ds,
\end{aligned}
$$

where C_R is a constant independent of Δ, and in the computation below C_R varies line by line. In the same way as we proved in the previous Lemma 3.2.3 we can show that

$$
\mathbb{E}|\bar{Y}(s)-z_1(s)|^2\le C(R)\Delta
$$

and

$$\mathbb{E}|\bar{Y}(s) - z_2(s)|^2 \leq C(R)\Delta,$$

so that

$$\mathbb{E}V(\bar{Y}(s)) \leq V(\zeta(0)) + hT + C_R\Delta + 2h\mathbb{E}\int_0^{t\wedge\rho_1} V(\bar{Y}(s^-))ds.$$

By the same computation leading to (4.13), we obtain

$$\mathbb{P}(\rho_1 < T) \leq [V(\zeta(0)) + hT + C_R\Delta]\frac{e^{2hT}}{v_R}. \tag{4.14}$$

Step 3. Let $\tau_1 = \rho_1 \wedge \theta_1$. Recall Theorem 3.2.1, then we have

$$\mathbb{E}\left[\sup_{0\leq t\leq \tau_1\wedge T} |X(t) - \bar{Y}(t)|^2\right] \leq C(R)\Delta.$$

Assume $\epsilon,\ \alpha \in (0,1)$, set

$$\bar{\Omega} = \{\omega : \sup_{0\leq t\leq T} |X(t) - \bar{Y}(t)|^2 \geq \alpha\}.$$

We compute

$$\begin{aligned}
\alpha\mathbb{P}(\bar{\Omega}\cap\{\tau_1 \geq T\}) &= \alpha\mathbb{E}\left[I_{\tau_1\geq T} I_{\bar{\Omega}}\right] \\
&\leq \mathbb{E}\left[I_{\tau_1\geq T}\sup_{0\leq t\leq \tau_1\wedge T} |X(t) - \bar{Y}(t)|^2\right] \\
&\leq \mathbb{E}\left[\sup_{0\leq t\leq \tau_1\wedge T} |X(t) - \bar{Y}(t)|^2\right] \\
&\leq C(R)\Delta,
\end{aligned}$$

together with (4.13) and (4.14), this yields

$$\begin{aligned}
\mathbb{P}(\bar{\Omega}) &\leq \mathbb{P}(\bar{\Omega}\cap\{\tau_1 \geq T\}) + \mathbb{P}(\tau_1 < T) \\
&\leq \mathbb{P}(\bar{\Omega}\cap\{\tau_1 \geq T\}) + \mathbb{P}(\theta_1 < T) + \mathbb{P}(\rho_1 < T) \\
&\leq \frac{C(R)}{\alpha}\Delta + [V(\zeta(0)) + hT + C_R\Delta]\frac{e^{2hT}}{v_R}
\end{aligned}$$

$$+\frac{(V(\zeta(0))+hT+\sup_{-\tau\le s\le 0}\mathbb{E}V(\zeta(s)))e^{2hT}}{v_R}.$$

Recalling that $v_R \to \infty$ as $R \to \infty$, we can choose R sufficiently large for obtaining

$$[V(\zeta(0))+hT]\frac{e^{2hT}}{v_R}+\frac{(V(\zeta(0))+hT+\sup_{-\tau\le s\le 0}\mathbb{E}V(\zeta(s)))e^{2hT}}{v_R}<\frac{\epsilon}{2},$$

and then choose Δ sufficiently small to get

$$\frac{C(R)}{\alpha}\Delta+C_R\Delta\frac{e^{2hT}}{v_R}<\frac{\epsilon}{2}$$

hence we aim at

$$\mathbb{P}(\bar{\Omega})=\mathbb{P}\left(\sup_{0\le t\le T}|X(t)-\bar{Y}(t)|^2\ge\alpha\right)<\epsilon.$$

□

4.3.2 The stochastic theta method

Given a sufficiently small stepsize $\Delta=\frac{\tau}{N}>0$ for some large integer N and a particular choice of $\theta\in[0,1]$, the theta method is defined by

$$\begin{aligned}Z((k+1)\Delta)=&Z(k\Delta)+(1-\theta)\alpha(Z(k\Delta),Z(I_\Delta[k\Delta-\tau]\Delta))\Delta\\&+\theta\alpha(Z((k+1)\Delta),Z(I_\Delta[(k+1)\Delta-\tau]\Delta))\Delta\\&+\sigma(Z(k\Delta),Z(I_\Delta[k\Delta-\tau]\Delta))\Delta B_k\\&+\int_{\mathbb{R}^n}\gamma(Z(k\Delta),Z(I_\Delta[k\Delta-\tau]\Delta,z))\Delta\tilde{N}_k(dz)\end{aligned}\tag{4.15}$$

with initial data

$$\begin{aligned}\{Z(t):\ -\tau\le t\le 0\}&=\{X(t):\ -\tau\le t\le 0\}\\&=\{\zeta(t):\ -\tau\le t\le 0\}\in C^b_{\mathcal{F}_0}([-\tau,0]),\end{aligned}$$

where ΔB_k and $\Delta\tilde{N}_k(dz)$ denote the increments of the Brownian motion and the compensated Poisson process respectively. $I_\Delta[(k+1)\Delta-\tau]$ denotes the integer part of $(k+1)\Delta-\tau/\delta$.

For convenience, we will extend the discrete numerical solution to continuous time. In addition, we also need introduce four step process

$$
\begin{aligned}
z_1(t) &= \sum_{k=0}^{\infty} 1_{[k\Delta,(k+1)\Delta]}(t) Z(k\Delta), \\
z_2(t) &= \sum_{k=0}^{\infty} 1_{[k\Delta,(k+1)\Delta]}(t) Z(I_\Delta[k\Delta-\tau]\Delta), \\
z_3(t) &= \sum_{k=0}^{\infty} 1_{[k\Delta,(k+1)\Delta]}(t) Z((k+1)\Delta), \\
z_4(t) &= \sum_{k=0}^{\infty} 1_{[k\Delta,(k+1)\Delta]}(t) Z(I_\Delta[(k+1)\Delta-\tau]\Delta).
\end{aligned}
$$

The continuous theta approximate solute is defined by

$$
\bar{Z}(t) = \begin{cases}
\varsigma(t), & -\tau \le t \le 0; \\[2em]
\varsigma(0) + \int_0^t (1-\theta)\alpha(z_1(s^-), z_2(s^-))ds & \\
\quad + \int_0^t \theta\alpha(z_3(s^-), z_4(s^-))ds & \\
\quad + \int_0^t \sigma(z_1(s^-), z_2(s^-))dB(s) & \\
\quad + \int_0^t \int_{\mathbb{R}^n} \gamma(z_1(s^-), z_2(s^-), z)\tilde{N}(ds, dz), & 0 \le t \le T.
\end{cases}
$$

Clearly, $z_1(k\Delta) = z_3((k-1)\Delta) = \bar{Z}(k\Delta)$.

Clearly, when $\theta = 0$ the numerical solution becomes EM approximation, however, when $\theta \neq 0$, the θ method is defined by an implicit equation. In order to make sure the Stochastic Theta Method is well defined, we require the coefficient function $\alpha(x, y)$ satisfying a global Lipschitz condition, i.e. there exists a constant $L > 0$ such that

Assumption 6

$$|\alpha(x,y)-\alpha(\bar{x},\bar{y})|^2 \leq L(|x-\bar{x}|^2+|y-\bar{y}|^2), \tag{4.16}$$

and $\sigma(x,y)$ as well as $\int_{\mathbb{R}^n}\gamma(x,y,z)\nu(dz)$ still satisfying the condition in Assumptions 1, i.e.

$$|\sigma(x,y)-\sigma(\bar{x},\bar{y})|^2+\int_{\mathbb{R}^n}|\gamma(x,y,z)-\gamma(\bar{x},\bar{y},z)|^2\nu(dz)$$
$$\leq L_R(|x-\bar{x}|^2+|y-\bar{y}|^2),$$

for all $x,\ y,\ \bar{x},\ \bar{y}\in\mathbb{R}^n$, and $|x|\vee|y|\vee|\bar{x}|\vee|\bar{y}|\leq R$.

Then for $\Delta<\min\{\frac{1}{\theta\sqrt{L}},\tau\}$, the numerical solution of theta method is well defined. (see [23])

Theorem 4.3.2. *Let Assumption 3, 4, 5, 6 hold, then we have*

$$\lim_{\Delta\to 0}\left(\sup_{0\leq t\leq T}|X(t)-\bar{Z}(t)|^2\right)=0 \quad \textit{in probability.} \tag{4.17}$$

Proof. Again we divide the whole proof into three steps.

Step 1. For sufficiently large R, define the stopping time

$$\theta_2=\inf\{t\in[0,T]:|X(t)|\geq R\}.$$

By the same computation as in the proof of Theorem 4.3.1, we have

$$\mathbb{P}(\theta_2<T)\leq\frac{(V(\zeta(0))+hT+\sup_{-\tau\leq s\leq 0}\mathbb{E}V(\zeta(s)))e^{2hT}}{v_R}. \tag{4.18}$$

Step 2. For sufficiently large R define the stopping time

$$\rho_2=\inf\{t\in[0,T]:|\bar{Z}(t)|\geq R\}.$$

Once again, we apply the Itô formula to $V(\bar{Z}(t))$, and taking the expectation to obtain

$$\mathbb{E}[V(\bar{Z}(t\wedge\rho_2))]$$

$$
\begin{aligned}
&= V(\zeta(0)) + \mathbb{E}\int_0^{t\wedge\rho_2} V_x(\bar{Z}(s^-))[(1-\theta)\alpha(z_1(s^-), z_2(s^-)) \\
&\qquad + \theta\alpha(z_3(s^-), z_4(s^-))]ds \\
&\quad + \frac{1}{2}\mathbb{E}\int_0^{t\wedge\rho_2} \text{trace}[\sigma^T(z_1(s^-), z_2(s^-))V_{xx}(\bar{Z}(s^-))\sigma(z_1(s^-), z_2(s^-))]ds \\
&\quad + \mathbb{E}\int_0^{t\wedge\rho_2}\int_{\mathbb{R}^n} \left\{V(\bar{Z}(s^-) + \gamma(z_1(s^-), z_2(s^-), z)) - V(\bar{Z}(s^-))\right. \\
&\qquad \left. - V_x(\bar{Z}(s^-))\gamma(z_1(s^-), z_2(s^-), z)\right\}\nu(dz)ds \\
&= V(\zeta(0)) + \mathbb{E}\int_0^{t\wedge\rho_2} \mathcal{L}V(z_1(s^-), z_2(s^-))ds \\
&\quad + \mathbb{E}\int_0^{t\wedge\rho_2} V_x(\bar{Z}(s^-))\theta[\alpha(z_3(s^-), z_4(s^-)) - \alpha(z_1(s^-), z_2(s^-))]ds \\
&\quad + \mathbb{E}\int_0^{t\wedge\rho_2} \left[V_x(\bar{Z}(s^-)) - V_x(z_1(s^-))\right]\alpha(z_1(s^-), z_2(s^-))ds \\
&\quad + \frac{1}{2}\mathbb{E}\int_0^{t\wedge\rho_2} \text{trace}\left[\sigma^T(z_1(s^-), z_2(s^-))\left(V_{xx}(\bar{Z}(s^-)) - V_{xx}(z_1(s^-))\right)\right. \\
&\qquad \left.\sigma(z_1(s^-), z_2(s^-))\right] ds \\
&\quad + \mathbb{E}\int_0^{t\wedge\rho_2}\int_{\mathbb{R}^n} \left\{V(\bar{Z}(s^-) + \gamma(z_1(s^-), z_2(s^-), z))\right. \\
&\qquad - V(z_1(s^-) + \gamma(z_1(s^-), z_2(s^-), z)) \\
&\qquad + V(z_1(s^-)) - V(\bar{Z}(s^-)) \\
&\qquad \left. - \left[V_x(\bar{Z}(s^-)) - V_x(z_1(s^-))\right]\gamma(z_1(s^-), z_2(s^-), z)\right\}\nu(dz)ds. \qquad (4.19)
\end{aligned}
$$

Let us now have a look the behavior of the third term of (4.19). We know that at each gridpoint $z_1(k\Delta) = z_3((k-1)\Delta)$, and for each $\tau = N\Delta$, $z_2(k\Delta) = z_4((k-1))\Delta$. Since some $z_3(t), z_4(t)$ may extend beyond T, we have to work in the interval $[0, T+1]$, by the Lipschitz condition

$$
\begin{aligned}
&\mathbb{E}\int_0^{t\wedge\rho_2} V_x(\bar{Z}(s^-))\theta[\alpha(z_3(s^-), z_4(s^-)) - \alpha(z_1(s^-), z_2(s^-))]ds \\
&\le \mathbb{E}\int_0^{t\wedge\rho_2} V_x(\bar{Z}(s^-))\theta\sqrt{L}(|z_3(s^-) - z_1(s^-)| + |z_4(s^-) - z_2(s^-)|)ds \\
&= 0.
\end{aligned}
$$

Therefore

$$\begin{aligned}
&\mathbb{E}[V(\bar{Z}(t\wedge\rho_2))]\\
&\le V(\zeta(0)) + hT + 2h\mathbb{E}\int_0^{t\wedge\rho_2} V(\bar{Z}(s^-))ds\\
&\quad + h\mathbb{E}\int_0^{t\wedge\rho_2} |V(\bar{Z}(s^-)) - V(z_1(s^-))|ds\\
&\quad + h\mathbb{E}\int_0^{t\wedge\rho_2} |V(\bar{Z}(s^-)) - V(z_2(s^-))|ds\\
&\quad + C_R\mathbb{E}\int_0^{t\wedge\rho_2} |V(\bar{Z}(s^-)) - V(z_1(s^-))|ds\\
&\le V(\zeta(0)) + hT + 2h\mathbb{E}\int_0^{t\wedge\rho_2} V(\bar{Z}(s^-))ds + hK_R\mathbb{E}\int_0^{t\wedge\rho_2} |\bar{Z}(s^-) - z_1(s^-)|ds\\
&\quad + hK_R\mathbb{E}\int_0^{t\wedge\rho_2} |\bar{Z}(s^-) - z_2(s^-)|ds\\
&\quad + C_R\mathbb{E}\int_0^{t\wedge\rho_2} |V(\bar{Z}(s^-)) - V(z_1(s^-))|ds
\end{aligned}$$

where C_R is a constant independent of Δ, and in the computation below C_R varies line by line. In the same way as was proved Lemma 3.2.3 in [23] we can show that

$$\mathbb{E}|\bar{Z}(s) - z_1(s)|^2 \le C(R)\Delta$$

and

$$\mathbb{E}|\bar{Z}(s) - z_2(s)|^2 \le C(R)\Delta,$$

so that

$$\mathbb{E}V(\bar{Z}(s)) \le V(\zeta(0)) + hT + C_R\Delta + 2h\mathbb{E}\int_0^{t\wedge\rho_2} V(\bar{Z}(s^-))ds.$$

By the same computation leading to (4.13), we obtain

$$\mathbb{P}(\rho_2 < T) \le [V(\zeta(0)) + hT + C_R\Delta]\frac{e^{2hT}}{v_R}. \tag{4.20}$$

Step 3. Let $\tau_2 = \rho_2 \wedge \theta_2$. Recalling Theorem 3.3.1 in [23], we have

$$\mathbb{E}\left[\sup_{0\le t\le \tau_2\wedge T} |X(t) - \bar{Z}(t)|^2\right] \le C(R)\Delta.$$

Assume $\epsilon,\ \alpha \in (0,1)$, set

$$\bar{\Omega} = \{\omega : \sup_{0\le t\le T} |X(t) - \bar{Z}(t)|^2 \ge \alpha\}.$$

We compute

$$\begin{aligned}\alpha\mathbb{P}(\bar{\Omega} \cap \{\tau_2 \ge T\}) &= \alpha\mathbb{E}\left[I_{\tau_2\ge T} I_{\bar{\Omega}}\right] \\ &\le \mathbb{E}\left[I_{\tau_2\ge T} \sup_{0\le t\le \tau_2\wedge T} |X(t) - \bar{Z}(t)|^2\right] \\ &\le \mathbb{E}\left[\sup_{0\le t\le \tau_2\wedge T} |X(t) - \bar{Z}(t)|^2\right] \\ &\le C(R)\Delta,\end{aligned}$$

together with (4.18) and (4.20), this yields

$$\begin{aligned}\mathbb{P}(\bar{\Omega}) &\le \mathbb{P}(\bar{\Omega} \cap \{\tau_2 \ge T\}) + \mathbb{P}(\tau_2 < T) \\ &\le \mathbb{P}(\bar{\Omega} \cap \{\tau_2 \ge T\}) + \mathbb{P}(\theta_2 < T) + \mathbb{P}(\rho_2 < T) \\ &\le \frac{C(R)}{\alpha}\Delta + [V(\zeta(0)) + hT + C_R\Delta]\frac{e^{2hT}}{v_R} \\ &\quad + \frac{(V(\zeta(0)) + hT + \sup_{-\tau\le s\le 0}\mathbb{E}V(\zeta(s)))e^{2hT}}{v_R}.\end{aligned}$$

Recalling that $v_R \to \infty$ as $R \to \infty$, we can choose R sufficiently large for obtaining

$$[V(\zeta(0)) + hT]\frac{e^{2hT}}{v_R} + \frac{(V(\zeta(0)) + hT + \sup_{-\tau\le s\le 0}\mathbb{E}V(\zeta(s)))e^{2hT}}{v_R} < \frac{\epsilon}{2},$$

and then choose Δ sufficiently small to get

$$\frac{C(R)}{\alpha}\Delta + C_R\Delta\frac{e^{2hT}}{v_R} < \frac{\epsilon}{2}$$

hence we find

$$\mathbb{P}(\bar{\Omega}) = \mathbb{P}\left(\sup_{0\le t\le T} |X(t) - \bar{Z}(t)|^2 \ge \alpha\right) < \epsilon.$$

□

Chapter 5

Application to finance

During the last decade, Lévy processes (especially the exponential Lévy processes) have become increasingly popular for modeling market fluctuations, both for risk management and option pricing purposes. However, the Lévy market is incomplete, that is contingent claims cannot in general be hedged by a suitable portfolio. Many different approaches to this problem have been proposed in recent years. In this chapter, we focus on seeking a minimum relative entropy martingale measure for the stochastic jump-diffusion delay models.

5.1 Stochastic jump delay models for the stock price

We let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathcal{F}_0$ contains all P-null sets). Let $\tilde{N}(t,z)$ be a compensated Poisson process with bounded intensity $\lambda = \nu(\mathbb{R})$ where ν is a bounded Lévy measure. Let $\tau > 0$ and $C([-\tau, 0]; \mathbb{R})$ denote the family of continuous function ϕ from $[-\tau, 0]$ to $\mathbb{R}$

with the norm $||\phi|| = \sup_{-\tau\leq\theta\leq 0} |\phi(\theta)|$. Denote by $C^b_{\mathcal{F}_0}([-\tau,0];\mathbb{R})$ the family of all bounded, $\mathcal{F}_0-$measurable, $C([-\tau,0];\mathbb{R})-$valued random variables.

We consider the stock price process $S(t)$ satisfying the following stochastic differential delay equation with jumps on $t \in [0,T]$:

$$dS(t) = \alpha S(t-\tau)S(t)dt + S(t)\int_{\mathbb{R}} \gamma(S(t-\tau)^-, z)\tilde{N}(dt,dz) \qquad (5.1)$$

and with initial data

$$\{S(t): \ -\tau \leq t \leq 0\} = \{\zeta(t): \ -\tau \leq t \leq 0\} \in C^b_{\mathcal{F}_0}([-\tau,0]).$$

Where α is a constant and $\gamma : \mathbb{R} \to \mathbb{R}$ is continuous functions.

As defined in the real world, we would like to show that positive initial data leads to positive solutions.

Theorem 5.1.1. *The SDDE* (5.1) *has a pathwise unique solution $S(t)$ for a given $\mathcal{F}_0-$measurable initial process $\zeta(t)$. Furthermore, if $\zeta(0) > 0$ a.s. and for any $x \in \mathbb{R}$ $\gamma(x,z) > -1$ then $S(t) > 0$ for all $t \in [0,T]$ a.s.*

Proof. For any $t \in [0,\tau]$, with the initial value $S(0) = \zeta(0)$, we have

$$dS(t) = S(t)\left[\alpha\zeta(t-\tau)dt + \int_{\mathbb{R}} \gamma(\zeta(t-\tau)^-, z)\tilde{N}(dt,dz)\right] \qquad (5.2)$$

Set

$$\hat{S}(t) = \frac{S(t)}{\zeta(0)}$$

and define

$$\bar{Y}(t) = \int_0^t \alpha\zeta(s-\tau)^- ds + \int_0^t \int_{\mathbb{R}} \gamma(\zeta(s-\tau)^-, z)\tilde{N}(ds,dz),$$

then

$$d\hat{S}(t) = \hat{S}(t)d\bar{Y}(t)$$

with initial value $\hat{S}(0) = 1$. The condition $\gamma(\zeta(t-\tau)^-, z) > -1$ implies $\Delta\bar{Y}(t) > -1$. Apply Doléans-Dade exponential formula (see [6]), we have

$$\hat{S}(t) = \exp\{\bar{Y}_{t^-}\} \prod_{0\le s\le t} (1 + \Delta\bar{Y}_s).$$

So the solution of (5.2) is given by

$$S(t) = \zeta(0)\exp\{\bar{Y}_{t^-}\} \prod_{0\le s\le t} (1 + \Delta\bar{Y}_s).$$

Since $\Delta\bar{Y}(t) > -1$, it is obviously that $S(t) > 0$ for all $t \in [0,\tau]$ almost surely, when $\zeta(0) > 0$ a.s.. By a similar argument, we may obtain $S(t) > 0$ for all $t \in [\tau, 2\tau]$ a.s.. Therefore $S(t) > 0$ for all $t \in [0,T]$ a.s..

Note $\gamma(x,z) > -1$, for all $t \in [0,\tau]$, define

$$\begin{aligned} Y(t) =& \ln\zeta(0) + \int_0^t \alpha\zeta(s-\tau)^- ds + \int_0^t \int_{\mathbb{R}} \ln\left[1 + \gamma(\zeta(s-\tau)^-, z)\right] \tilde{N}(dz, ds) \\ &+ \int_0^t \int_{\mathbb{R}} \left\{\ln\left[1 + \gamma(\zeta(s-\tau)^-, z)\right] - \gamma(\zeta(s-\tau)^-, z)\right\} \nu(dz)ds. \end{aligned}$$

Let $X(t) = e^{Y(t)}$, and the Itô formula gives

$$\begin{aligned} dX(t) =& e^{Y(t)}\left\{\alpha\zeta(t-\tau)^- + \int_{\mathbb{R}} \left\{\ln\left[1+\gamma(\zeta(t-\tau)^-, z)\right] - \gamma(\zeta(t-\tau)^-, z)\right\}\nu(dz)\right\} dt \\ &+ \int_{\mathbb{R}} e^{Y(t) - \ln\left[1+\gamma(\zeta(t-\tau)^-, z)\right]} - e^{Y(t)}\tilde{N}(dz, dt) \\ &+ \int_{\mathbb{R}} \left\{e^{Y(t) - \ln\left[1+\gamma(\zeta(t-\tau)^-, z)\right]}\right. \\ &\qquad \left. - e^{Y(t)} - e^{Y(t)}\ln\left[1 + \gamma(\zeta(t-\tau)^-, z)\right]\right\}\nu(dz)dt \\ =& X(t)\left\{\alpha\zeta(t-\tau)dt + \int_{\mathbb{R}} \gamma(\zeta(t-\tau)^-, z)\tilde{N}(dt, dz)\right\}. \end{aligned}$$

Since the solution to Eq.(5.1) is unique, $S(t) = X(t) = e^{Y(t)}$, and this gives the solution

$$S(t) = \zeta(0)\exp\left\{\int_0^t \alpha\zeta(s-\tau)^- ds + \int_0^t \int_{\mathbb{R}} \ln\left[1 + \gamma(\zeta(s-\tau)^-, z)\right] \tilde{N}(dz, ds)\right.$$

$$+\int_0^t \int_{\mathbb{R}} \{\ln[1+\gamma(\zeta(s-\tau)^-,z)] - \gamma(\zeta(s-\tau)^-,z)\}\, \nu(dz)ds \Big\}$$

This clearly implies $S(t) > 0$ for all $t \in [0,\tau]$ a.s.. Repeating this procedure above on $[\tau, 2\tau]$, therefore, we have the solution

$$S(t) = \zeta(0)\exp\Big\{\int_0^t \alpha S(s-\tau)^- ds + \int_0^t \int_{\mathbb{R}} \ln[1+\gamma(S(s-\tau)^-,z)]\, \tilde{N}(dz,ds)$$
$$+\int_0^t \int_{\mathbb{R}} \{\ln[1+\gamma(S(s-\tau)^-,z)] - \gamma(S(s-\tau)^-,z)\}\, \nu(dz)ds \Big\}$$

Therefore $S(t) > 0$ for all $t \in [0,T]$ a.s. when $\gamma(S(s-\tau)^-,z) > -1$ and $\zeta(0) > 0$.

□

5.2 Equivalent martingales

The fundamental asset pricing theorem showed us "there is no free lunch with vanishing risk", in a mathematical word

Proposition 5.2.1. *The market model defined by $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ and asset prices $S(t)$ is arbitrage-free if and only if there exists a probability measure $Q \sim P$ such that the discounted assets $\tilde{S}(t)$ are martingales with respect to Q.*

We consider a world with just one risky asset with price process $S(t)$ satisfying the SDDE (5.1) and a risk-free saving account paying constant interest rate r, we set the bond $B(t) = e^{rt}$. Consider an option, written on a stock, with maturity at some future time $T > t$ and exercise price K. Assume also that there are no transaction costs and that the underlying stock pays no dividends.

We denote the discounted stock price process by

$$\tilde{S}(t) := \frac{S(t)}{B(t)} = e^{-rt}S(t), \quad t \in (0,T].$$

Then by Itô's formula, we obtain

$$\begin{aligned} d\tilde{S}(t) &= e^{-rt}dS(t) + S(t)(-re^{-rt})dt \\ &= \tilde{S}(t)\left[(\alpha S(t-\tau) - r)dt + \int_{\mathbb{R}} \gamma(S(t-\tau), z)\tilde{N}(dz, dt)\right]. \end{aligned}$$

We seek an equivalent martingale measure Q of the process $\tilde{S}(t)$.

Theorem 5.2.1. *Given the discounted price $\tilde{S}(t)$ of the form*

$$d\tilde{S}(t) = \tilde{S}(t)\left[(\alpha S(t-\tau) - r)dt + \int_{\mathbb{R}} \gamma(S(t-\tau), z)\tilde{N}(dz, dt)\right].$$

Assume that there exists a Borel measurable function $q(x,z)$, such that

$$q(x,z) < 1$$

and

$$\int_{\mathbb{R}} \gamma(x,z)q(x,z)\nu(dz) = \alpha x - r. \tag{5.3}$$

Define the process

$$\begin{aligned} Z(t) = \exp\Big\{ &\int_0^t \int_{\mathbb{R}} \ln\left(1 - q(S(s-\tau)^-, z)\right) \tilde{N}(ds, dz) \\ &+ \int_0^t \int_{\mathbb{R}} \left[\ln\left(1 - q(S(s-\tau)^-, z)\right) + q(S(s-\tau)^-, z)\right] \nu(dz)ds \Big\} \end{aligned}$$

Define a measure Q on $\mathcal{F}_T$ by

$$dQ = Z(T)dP,$$

if

$$\mathbb{E}_P[Z(T)] = 1,$$

then Q is an equivalent local martingale measure for $\tilde{S}(t)$.

Proof. Set $\mathcal{F}_t := \mathcal{F}_0$ for any $t \leq 0$. Since $q(x,z)$ is measurable with respect to the $\sigma-$algebra $\mathcal{F}_0$. Then for any $t \in [0,T]$, Set

$$Y(t) = \int_0^t \int_{\mathbb{R}} \ln\left(1 - q(S(s-\tau)^-, z)\right) \tilde{N}(ds, dz)$$

$$+\int_0^t \int_{\mathbb{R}} \left[\ln\left(1-q(S(s-\tau)^-,z)\right)+q(S(s-\tau)^-,z)\right]\nu(dz)ds$$

by Itô's formula, we obtain

$$\begin{aligned} e^{Y(t)} &= 1+\int_0^t \Big[e^{Y(s)+\ln\left(1-q(S(s-\tau)^-,z)\right)} \\ &\quad -e^{Y(s)}-e^{Y(s)}\ln\left(1-q(S(s-\tau)^-,z)\right)\Big]\nu(dz)ds \\ &\quad +\int_0^t \int_{\mathbb{R}} \left[e^{Y(s)+\ln\left(1-q(S(s-\tau)^-,z)\right)}-e^{Y(s)}\right]\tilde{N}(dz,ds) \\ &\quad +\int_0^t e^{Y(s)}\int_{\mathbb{R}} \left[\ln\left(1-q(S(s-\tau)^-,z)\right)+q(S(s-\tau)^-,z)\right]\nu(dz)ds \\ &= 1+\int_0^t \int_{\mathbb{R}} \left[e^{Y(s)+\ln\left(1-q(S(s-\tau)^-,z)\right)}-e^{Y(s)}\right]\tilde{N}(dz,ds). \end{aligned}$$

We note that $\tilde{N}(dt,dz)$ is a martingale measure, therefore

$$\mathbb{E}_P\left\{1+\int_0^t \int_{\mathbb{R}} \left[e^{Y(t)+\ln\left(1-q(S(s-\tau)^-,z)\right)}-e^{Y(t)}\right]\tilde{N}(dz,dt)|\mathcal{F}_0\right\}=1.$$

Define measure

$$\tilde{N}^Q(dt,dz)=\tilde{N}(dt,dz)+q(S(t-\tau),z)\nu(dz)dt,$$

by Girsanov's theorem ([5])

$$\int_0^t \int_{\mathbb{R}} \tilde{N}^Q(dt,dz)=\int_0^t \int_{\mathbb{R}} \tilde{N}(dt,dz)+\int_0^t \int_{\mathbb{R}} q(S(t-\tau),z)\nu(dz)dt$$

is a $Q-$martingale.

We then derive for the discounted price

$$\begin{aligned} \tilde{S}(t) &= \zeta(0)\exp\left\{\int_0^t (\alpha S(s-\tau)^- - r)ds+\int_0^t \int_{\mathbb{R}} \ln\left[1+\gamma(S(s-\tau)^-,z)\right]\tilde{N}(dz,ds)\right. \\ &\quad \left.+\int_0^t \int_{\mathbb{R}} \left\{\ln\left[1+\gamma(S(s-\tau)^-,z)\right]-\gamma(S(s-\tau)^-,z)\right\}\nu(dz)ds\right\} \\ &= \zeta(0)\exp\left\{\int_0^t \int_{\mathbb{R}} \ln\left[1+\gamma(S(s-\tau)^-,z)\right]\tilde{N}^Q(dz,ds)\right. \\ &\quad -\int_0^t \int_{\mathbb{R}} \left\{\ln\left[1+\gamma(S(s-\tau)^-)\right]-\gamma(S(s-\tau)^-,z)\right\} \end{aligned}$$

$$\times(1-q(S(t-\tau),z))\nu(dz)ds$$
$$+\int_0^t\left[(\alpha S(s-\tau)^- - r - \int_{\mathbb{R}} q(S(t-\tau),z))\gamma(S(s-\tau)^-)\nu(dz)\right]ds\bigg\}.$$

by the condition (5.3), $\tilde{S}(t)$ is a martingale with respect to the measure Q.

$\square$

Consider a portfolio in this market is a predictable process

$$\phi(t)=(\phi_0(t),\phi_1(t))\in\mathbb{R}^2$$

such that

$$\int_0^T[\phi_0^2(t)+\phi_1^2(t)]dt<\infty \text{ a.s.}$$

The value process is a stochastic process and defined by

$$V(t)=\phi_0(t)B(t)+\phi_1(t)S(t), \quad 0\le t\le T. \tag{5.4}$$

Assume that $(\phi_0(t),\phi_1(t))$ is self-financing, then $V(t)$ is given by

$$V(t)=V(0)+\int_0^t\phi_0(s)dB(s)+\int_0^t\phi_1(s)dS(s).$$

From (5.4), we obtain

$$\phi_0=e^{-rt}(V(t)-\phi_1(t)S(t)),$$

and

$$dV(t)=rV(t)dt+\phi_1S(t)\left[(\alpha S(t-\tau)-r)dt+\int_{\mathbb{R}}\gamma(S(t-\tau),z)\tilde{N}(dz,dt)\right].$$

This implies

$$\begin{aligned}
&d(e^{-rt}V(t))\\
&=e^{-rt}dV(t)-re^{-rt}V(t)dt\\
&=\phi_1e^{-rt}S(t)\left[(\alpha S(t-\tau)-r)dt+\int_{\mathbb{R}}\gamma(S(t-\tau),z)\tilde{N}(dz,dt)\right]
\end{aligned}$$

$$= \phi_1 d\tilde{S}(t).$$

That is

$$e^{-rt}V(t) = V(0) + \int_0^T \phi_1 d\tilde{S}(t).$$

Assume there exists a measure Q absolute continuous to P, such that the discount asset processes $\tilde{S}(t)$ are martingales with respect to Q. When $V(0) = 0$, we see that

$$\mathbb{E}_Q \left[e^{-rt}V(t)\right] = \mathbb{E}_Q \left[\int_0^T \phi_1 d\tilde{S}(t)\right] = 0,$$

this shows that there is no arbitrage in this market, i.e. Proposition 5.2.1 holds.

5.3 Minimum relative entropy martingale measure

So far we have discussed the equivalent martingale measure and its relation with absence of arbitrage, which is both a reasonable property to assume in a real market and a generic property of many stochastic models. However, for the option pricing model we described in the previous section, there exist many equivalent measures under which the discounted price process is martingale, in other words, such a market is incomplete. In a complete market there is only one arbitrage-free way to value an option. In real markets, as well as in jump models described markets, contingent claims cannot in general be hedged by a suitable portfolio. This forced us to select an appropriate martingale measure from among the uncountably many such measures with which to price a contingent claim. During the last decade, many investigations and approaches to this problem, such as [4]. Nevertheless, there is no definitive way of pricing contingent claims in incomplete markets which is preferable to the other possible methods in all situations. In this section, we focus on the minimal relative entropy martingale measure for the stochastic jump delay models.

Definition 5.3.1. *For a fixed measure P, the relative entropy $I_P(Q)$ of any measure Q with respect to P is defined to be*

$$I_P(Q) = \mathbb{E}^Q\left[\ln\frac{dQ}{dP}\right] = \mathbb{E}^P\left[\frac{dQ}{dP}\ln\frac{dQ}{dP}\right].$$

By Theorem 5.2.1, for any $s \in [0,T]$ the relative entropy in terms of the $Q-$martingale $\tilde{N}^Q$ is therefore

$$\begin{aligned}
I_P(Q) &= \mathbb{E}^Q\left[\ln\frac{dQ}{dP}\right]\\
&= \mathbb{E}^Q\bigg[\int_0^t\int_{\mathbb{R}}\ln\left(1-q(S(s-\tau)^-,z)\right)\tilde{N}(ds,dz)\\
&\quad+\int_0^t\int_{\mathbb{R}}\left[\ln\left(1-q(S(s-\tau)^-,z)\right)+q(S(s-\tau)^-,z)\right]\nu(dz)ds\bigg]\\
&= \mathbb{E}^Q\bigg[\int_0^t\int_{\mathbb{R}}\ln\left(1-q(S(s-\tau)^-,z)\right)\tilde{N}^Q(ds,dz)\\
&\quad-\int_0^t\int_{\mathbb{R}}\ln\left(1-q(S(s-\tau)^-,z)\right)q(S(s-\tau)^-,z)\nu(dz)ds\\
&\quad+\int_0^t\int_{\mathbb{R}}\left[\ln\left(1-q(S(s-\tau)^-,z)\right)+q(S(s-\tau)^-,z)\right]\nu(dz)ds\bigg]\\
&= \mathbb{E}^Q\left[\int_0^t\int_{\mathbb{R}}[(1-q(S(s-\tau)^-,z))\ln(1-q(S(s-\tau)^-,z))+q(S(s-\tau)^-,z)]\nu(dz)ds\right].
\end{aligned}$$

The problem of finding the equivalent martingale measure of minimum relative entropy can clearly be reduced to that for a fixed $s\in[0,T]$, $q(S(s-\tau)^-,z)$ satisfying (5.3), and minimizing

$$\mathbb{E}^Q\left[\int_{\mathbb{R}}[(1-q(S(s-\tau)^-,z))\ln(1-q(S(s-\tau)^-,z))+q(S(s-\tau)^-,z)]\nu(dz)\right].$$

Note that $I_P(Q)\geq 0$ for any Q. If Q is not absolutely continuous with respect to P, $I_P(Q)$ is infinite. Since the problem can be reduce further to that of minimizing

$$\int_{\mathbb{R}}[(1-q(S(s-\tau)^-,z))\ln(1-q(S(s-\tau)^-,z))+q(S(s-\tau)^-,z)]\nu(dz) \quad (5.5)$$

In terms of the variation of $q(S(s-\tau)^-,z)$ leads to different measure, one can choose the optimal $q(S(s-\tau),z)$, and denote by Q^* the corresponding measure. The optional value of (5.5) is therefore deterministic and for any other choice of $q(S(s-\tau),z)$ with associated measure $\tilde{Q}$, we have

$$\tilde{Q}\left[\int_{\mathbb{R}}[(1-q(S(s-\tau)^-,z))\ln(1-q(S(s-\tau)^-,z))+q(S(s-\tau)^-,z)]\nu(dz)\right]$$
$$\geq I^* = Q^*[I^*].$$

Letting λ be a Lagrange multiplier associated with the constraint (5.3), then we arrive at

$$\begin{aligned}
&L(\lambda, q(S(s-\tau),z))\\
&= \int_{\mathbb{R}}[(1-q(S(s-\tau)^-,z))\ln(1-q(S(s-\tau)^-,z))+q(S(s-\tau)^-,z)]\nu(dz)\\
&+\int_{\mathbb{R}}\lambda\gamma(S(t-\tau)^-,z)q(S(s-\tau)^-,z)\nu(dz).
\end{aligned}$$

For all u and F, we require

$$\frac{d}{du}L(\lambda, q(S(s-\tau),z)+uF)\bigg|_{u=0} = 0, \quad (5.6)$$

we calculate (5.6) as follows

$$\begin{aligned}
&\frac{d}{du}L(\lambda, q(S(s-\tau),z)+uF)\bigg|_{u=0}\\
&= \frac{d}{du}\int_{\mathbb{R}}\{(1-q(S(s-\tau)^-,z)-uF)\ln(1-q(S(s-\tau)^-,z)-uF)\\
&\quad +q(S(s-\tau),z)+uF+\lambda\gamma(S(t-\tau)^-,z)(q(S(s-\tau)^-,z)+uF)\}\,\nu(dz)\bigg|_{u=0}\\
&= \int_{\mathbb{R}}\frac{d}{du}\{\ln(1-q(S(s-\tau)^-,z)-uF)
\end{aligned}$$

$$
\begin{aligned}
&-q(S(s-\tau)^-,z)\ln(1-q(S(s-\tau)^-,z)-uF) \\
&-uF\ln(1-q(S(s-\tau)^-,z)-uF)+q(S(s-\tau)^-,z)+uF \\
&+\lambda\gamma(S(t-\tau)^-,z)q(S(s-\tau)^-,z)+\lambda\gamma(S(t-\tau)^-,z)uF\Big\}\,\nu(dz)\Big|_{u=0} \\
=&\int_{\mathbb{R}} F(\ln(1-q(S(s-\tau)^-,z))+\lambda\gamma(S(t-\tau)^-,z))\nu(dz) \\
=&\,0,
\end{aligned}
$$

therefore,

$$\ln(1-q(S(s-\tau)^-,z))+\lambda\gamma(S(t-\tau)^-,z)=0$$

and we have

$$q(S(s-\tau)^-,z))=1-e^{-\lambda\gamma(S(t-\tau)^-,z)}.$$

Therefore, writing the discounted price $\tilde{S}(t)$ in terms of Q^*-martingale, we have

$$
\begin{aligned}
\tilde{S}(t)=&\,\zeta(0)\exp\Big\{\int_0^t[\alpha S(s-\tau)^- - r]ds \\
&+\int_0^t\int_{\mathbb{R}}\{\ln\left[1+\gamma(S(s-\tau)^-,z)\right]-\gamma(S(s-\tau)^-,z)\}\,\nu(dz)ds \\
&+\int_0^t\int_{\mathbb{R}}\ln\left[1+\gamma(S(s-\tau)^-,z)\right]\tilde{N}(dz,ds)\Big\} \\
=&\,\zeta(0)\exp\Big\{\int_0^t[\alpha S(s-\tau)^- - r]ds \\
&+\int_0^t\int_{\mathbb{R}}\{\ln\left[1+\gamma(S(s-\tau)^-,z)\right]-\gamma(S(s-\tau)^-,z)\}\,\nu(dz)ds \\
&+\int_0^t\int_{\mathbb{R}}\ln\left[1+\gamma(S(s-\tau)^-,z)\right]\tilde{N}^{Q^*}(dz,ds) \\
&-\int_0^t e^{\lambda\gamma(S(s-\tau)^-,z)}\ln\left[1+\gamma(S(s-\tau)^-,z)\right]\nu(dz)ds\Big\} \\
=&\,\zeta(0)\exp\Big\{+\int_0^t\int_{\mathbb{R}}\ln\left[1+\gamma(S(s-\tau)^-,z)\right]\tilde{N}^{Q^*}(dz,ds) \\
&+\int_0^t\int_R\{\ln\left[1+\gamma(S(s-\tau)^-,z)\right]-\gamma(S(s-\tau)^-,z)\}
\end{aligned}
$$

$$
\begin{aligned}
&\times e^{\lambda\gamma(S(s-\tau)^-,z)}\nu(dz)ds \\
&-\int_0^t\int_R \left\{\ln\left[1+\gamma(S(s-\tau)^-,z)\right]-\gamma(S(s-\tau)^-,z)\right\} \\
&\qquad \times e^{\lambda\gamma(S(s-\tau)^-,z)}\nu(dz)ds \\
&+\int_0^t\int_{\mathbb{R}} \left\{\ln\left[1+\gamma(S(s-\tau)^-,z)\right]-\gamma(S(s-\tau)^-,z)\right\}\nu(dz)ds \\
&-\int_0^t\int_{\mathbb{R}} (1-e^{\lambda\gamma(S(s-\tau)^-,z)})\ln\left[1+\gamma(S(s-\tau)^-,z)\right]\nu(dz)ds \\
&+\left.\int_0^t [\alpha S(s-\tau)^- - r]ds\right\} \\
&= \zeta(0)\exp\left\{+\int_0^t\int_{\mathbb{R}} \ln\left[1+\gamma(S(s-\tau)^-,z)\right]\tilde{N}^{Q^*}(dz,ds)\right. \\
&+\int_0^t\int_R \left\{\ln\left[1+\gamma(S(s-\tau)^-,z)\right]-\gamma(S(s-\tau)^-,z)\right\} \\
&\qquad \times e^{\lambda\gamma(S(s-\tau)^-,z)}\nu(dz)ds \\
&+\int_0^t\left[-\int_{\mathbb{R}}(1-e^{\lambda\gamma(S(s-\tau)^-,z)})\gamma(S(s-\tau)^-,z)\nu(dz)\right. \\
&\qquad \left.\left.+(\alpha S(s-\tau)^- - r)\right]ds\right\},
\end{aligned}
$$

by the condition (5.3), we see that

$$
\begin{aligned}
\tilde{S}(t) = &= \zeta(0)\exp\left\{+\int_0^t\int_{\mathbb{R}} \ln\left[1+\gamma(S(s-\tau)^-,z)\right]\tilde{N}^{Q^*}(dz,ds)\right. \\
&+\int_0^t\int_R \left\{\ln\left[1+\gamma(S(s-\tau)^-,z)\right]-\gamma(S(s-\tau)^-,z)\right\} \\
&\qquad \left.\times e^{\lambda\gamma(S(s-\tau)^-,z)}\nu(dz)ds\right\}
\end{aligned}
$$

is a martingale with respect to Q^*.

We then have the following theorem

Theorem 5.3.1. *Given the jump delay model of the form*

$$
dS(t) = \alpha S(t-\tau)S(t)dt + S(t)\int_{\mathbb{R}}\gamma(S(t-\tau)^-,z)\tilde{N}(dt,dz)
$$

with bounded moments. The minimum relative entropy martingale measure is constructed via a Lagrange multiplier λ*, such as*

$$q(x,z) = 1 - e^{-\lambda\gamma(x,z)}$$

for any $t \in [0, T]$*, where* $q(x,z)$ *is defined in Theorem 5.2.1.*

Remark 5.3.1. Open question: there is a lack of technique to control models such as:

$$\begin{aligned} dS(t) =& \alpha S(t-\tau)S(t)dt + \sigma(S(t-\tau))S(t)dW(t) \\ &+ \int_{\mathbb{R}} \gamma(S(t-\tau)^-, z)S(t^-)\tilde{N}(dt, dz). \end{aligned}$$

Since coefficients involves the initial process, which is not independent on $W(t)$ and $\tilde{N}(t,z)$. It is difficult to seek a equivalent martingale measure.

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Index of notation

$\mathbb{R}^n$	n-dimensional Euclidean space.
$\mathbb{E}$	the expectation.
$\mathbb{E}_P$	the expectation with respect to the measure P.
$N(dz, dt)$	Poisson measure.
$\tilde{N}(dz, dt)$	compensated Poisson measure.
ν	bounded Lévy measure
$\Delta X(t)$	the jump of $X(t)$ defined by $\Delta X(t) = X(t) - X(t^-)$.
$X(t^-)$	the left limit of X at time t.
$\delta(t)$	variable time delay.
$\lvert \cdot \rvert$	the Euclidean norm for vectors or the trace norm for matrices.
$C^b_{\mathcal{F}_0}$	the family of all bounded, $\mathcal{F}_0$-measurable functions.
$[X, X]_t$	quadratic variation of X_t.
càdlàg	right continuous with left limits.
SDE	stochastic differential equation.
SDDE	stochastic differential delay equation.
EM	Euler-Maruyama approximation.