Singularities of the stochastic Burgers equation with vorticity.

Reasons, Scott

Singularities of the Stochastic Burgers Equation with Vorticity

Scott Reasons
masr@swan.ac.uk

Department of Mathematics
University of Wales Swansea
Singleton Park
Swansea SA2 8PP
United Kingdom

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Contents

Abstract

Within this thesis a model to initiate studies of rotational fluid flows (fluids with vorticity) is presented. We begin with a stochastic heat equation with a vector potential and relate this to a stochastic Burgers equation with vorticity. Consideration is given to level surfaces of the heat equation and classifying the corresponding shockwaves (caustics) of the Burgers equation. The theory is illustrated beautifully by both deterministic and stochastic examples.

In Chapter 1 the idea of a heat equation with a vector potential is introduced and its relation to a Burgers equation with vorticity discussed. The semi-classical solution of the heat equation is found by utalising Hamilton-Jacobi theory, Feynman-Kac formula and a variety of other stochastic analysis techniques.

In Chapter 2 we follow work of Truman and Zhao to use stochastic Hamilton-Jacobi theory to study stochastic heat equations and viscous Burgers equations with vorticity. We construct solutions of stochastic iterated Hamilton-Jacobi continuity equations to give semi-classical asymptotic expansions for the stochastic heat and Burgers equations. Finally, by using the logarithmic Hopf-Cole transformation and Nelson's stochastic mechanical processes a solution of the stochastic viscous Burgers equation with vorticity is given with the remainder term as a path integral.

In Chapter 3 we discuss and rework geometrical results of Davies, Truman and Zhao, that link caustics (shockwaves) of the Burgers equation and the corresponding level surfaces (wavefronts) of the heat equation. We introduce the notion of a reduced action function as a vehicle to study discontinuities across the caustics of the Burgers velocity field. Moreover, the notion of this function allows turbulence of an n dimensional problem to be studied using a one dimensional analysis.

Chapter 4 is dedicated to a deterministic harmonic oscillator example. This example is used to complement the analytical and geometrical results of previous chapters. Particular attention is taken with regards to methods of classifying the caustics as hot or cool.

In Chapter 5 we continue our theme by studying the consequences of adding a noise term to the underlying classical mechanics of our deterministic example. We consider touching points of pre curves to account for the turbulent times of the Burgers velocity field. The turbulent times are shown to be the zeros of a stochastic process whose properties are examined.

In Chapter 6 ideas for making the Burgers fluid incompressible is presented as motivations for possible future work.

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Chapter 1

Preliminaries on a Burgers Equation with Vorticity and Viscosity

Summary

This chapter is intended as a brief introduction to key ideas and concepts found later within the thesis. The idea of a heat equation with a vector potential is discussed and a semi-classical solution found using Hamilton-Jacobi theory. We show how this relates to a Burgers equation with vorticity.

1.1 Introduction

In what follows the element $\boldsymbol{a}$, which can be space, and time dependent, is referred to as the *vector potential*. The notion of a vector potential originates in the realms of electromagnetics where one often denotes a magnetic flux density $\boldsymbol{B}$ as $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{a}$. The idea of a vector potential is discussed concisely in Richard Feynman's second volume of his three volume work on physics [12]. Simon in [30] studies Schrödinger operators $H(\boldsymbol{a}, V)$ for electromagnetic forces

$$H(\boldsymbol{a}, V) = \frac{1}{2}(-i\boldsymbol{\nabla} - \boldsymbol{a})^2 + V,$$

where $\boldsymbol{a}$ and V are the vector and scalar potentials respectively. Simon points out that one can use the power of the Feynman-Kac formula to study e^{-tH} using Wiener integrals and proves in [31] that it can still be used when V is unbounded as it is for the harmonic oscillator potential. Here for simplicity we assume continuity and quadratic bounds on V, and that $\boldsymbol{a}$ is smooth in space and time variables and of compact support.

In this next section we use Simon's ideas of the Schrödinger operator for a particle moving in a magnetic field to construct a heat operator with a vector potential. The idea of studying a stochastic heat equation with a vector potential to obtain a stochastic Burgers fluid with vorticity is in no way my own. In addition to Simon, arriving at this work is largely thanks to previous studies by Herman Weyl in [39] and private communications with Aubrey Truman.

The following formal Proposition details the equivalence between the Stratonovich heat equation and the stochastic viscous Burgers equation. We will continuously lean upon this result throughout the thesis.

Proposition 1.1.1. *Let $\boldsymbol{a}(\boldsymbol{x},t) = \boldsymbol{a}_t(\boldsymbol{x}) \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}^1)$, $V(\boldsymbol{x}) \in C^2(\mathbb{R}^d)$ with quadratic bounds, $k(\boldsymbol{x},t) = k_t(\boldsymbol{x}) \in C^2(\mathbb{R}^d \times \mathbb{R}^1)$ and W_t be a one dimensional Wiener process on the probability space $\{\Omega, \mathcal{F}, P\}$. Then the Stratonovich heat equation for $u^\mu(\boldsymbol{x},t) = u_t(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^d$, $t > 0$*

$$du_t(\boldsymbol{x}) = \left(\frac{\mu^2}{2}\Delta u_t(\boldsymbol{x}) + (\boldsymbol{a}_t(\boldsymbol{x}) \cdot \boldsymbol{\nabla})u_t(\boldsymbol{x}) + \frac{1}{\mu^2}\Big(\frac{\boldsymbol{a}_t^2(\boldsymbol{x})}{2} + V(\boldsymbol{x})\Big)u_t(\boldsymbol{x})\right)dt + \varepsilon\frac{k_t(\boldsymbol{x})}{\mu^2}u_t \circ \partial W_t \qquad (1.1.1)$$

with initial condition $u^\mu(\boldsymbol{x},0) = \exp\left(-\frac{S_0(\boldsymbol{x})}{\mu^2}\right)$ is related to the stochastic viscous Burgers equation for velocity field $\boldsymbol{v}^\mu(\boldsymbol{x},t) = \boldsymbol{v}_t(\boldsymbol{x})$

$$d\boldsymbol{v}_t(\boldsymbol{x}) + \Big((\boldsymbol{v}_t(\boldsymbol{x}) \cdot \boldsymbol{\nabla})\boldsymbol{v}_t(\boldsymbol{x}) + \boldsymbol{v}_t(\boldsymbol{x}) \wedge \boldsymbol{curl}\ \boldsymbol{v}_t(\boldsymbol{x})\Big)dt = -d\boldsymbol{a}_t(\boldsymbol{x}) - \boldsymbol{\nabla}V(\boldsymbol{x})dt - \varepsilon\boldsymbol{\nabla}k_t(\boldsymbol{x})dW_t + 2^{-1}\mu^2\Delta\boldsymbol{v}_t(\boldsymbol{x})dt + 2^{-1}\mu^2\Delta\boldsymbol{a}_t(\boldsymbol{x})dt \qquad (1.1.2)$$

with initial velocity $\boldsymbol{v}^\mu(\boldsymbol{x},0) = \boldsymbol{\nabla}S_0(\boldsymbol{x}) - \boldsymbol{a}(\boldsymbol{x},0)$, for $S_0(\boldsymbol{x}) = S(\boldsymbol{x},0)$, by the logarithmic Hopf-Cole transformation $\boldsymbol{v}^\mu(\boldsymbol{x},t) = -\mu^2\boldsymbol{\nabla}_{\boldsymbol{x}}\log u^\mu(\boldsymbol{x},t) - \boldsymbol{a}(\boldsymbol{x},t)$.

Proof. Let u_t satisfy the Stratonovich heat equation (1.1.1) for almost all ω. Then $S_t(\boldsymbol{x}) = -\mu^2\log u_t(\boldsymbol{x})$ satisfies

$$\begin{aligned} dS_t(\boldsymbol{x}) &= -\mu^2\frac{du_t(\boldsymbol{x})}{u_t(\boldsymbol{x})} \\ &= -\frac{\mu^2}{u_t(\boldsymbol{x})}\left\{2^{-1}\mu^2\Delta u_t(\boldsymbol{x}) + (\boldsymbol{a}_t(\boldsymbol{x}) \cdot \boldsymbol{\nabla})u_t(\boldsymbol{x}) + \frac{1}{\mu^2}\left(\frac{\boldsymbol{a}^2(\boldsymbol{x})}{2} + V(\boldsymbol{x})\right)u_t(\boldsymbol{x})\right\}dt \\ &\quad - \varepsilon k_t(\boldsymbol{x}) \circ \partial W_t. \end{aligned}$$

But $u_t(\boldsymbol{x}) = \exp\left(-\frac{S_t(\boldsymbol{x})}{\mu^2}\right)$ gives

$$\begin{aligned} \Delta u_t(\boldsymbol{x}) &= \text{div}\left(-\frac{1}{\mu^2}\boldsymbol{\nabla}S_t(\boldsymbol{x})\exp\left(-\frac{S_t(\boldsymbol{x})}{\mu^2}\right)\right) \\ &= \left(-\frac{1}{\mu^2}\Delta S_t(\boldsymbol{x})\right)\exp\left(-\frac{S_t(\boldsymbol{x})}{\mu^2}\right) + \frac{1}{\mu^4}|\boldsymbol{\nabla}S_t(\boldsymbol{x})|^2\exp\left(-\frac{S_t(\boldsymbol{x})}{\mu^2}\right) \\ &= -\frac{1}{\mu^2}\Delta S_t(\boldsymbol{x})u_t(\boldsymbol{x}) + \frac{1}{\mu^4}|\boldsymbol{\nabla}S_t(\boldsymbol{x})|^2 u_t(\boldsymbol{x}). \end{aligned}$$

Therefore, we obtain $dS_t(\boldsymbol{x}) = -\frac{\mu^2}{u_t(\boldsymbol{x})}du_t(\boldsymbol{x})$, i.e.

$$\begin{aligned} dS_t(\boldsymbol{x}) = -\frac{\mu^2}{u_t(\boldsymbol{x})}\bigg\{&-\frac{1}{2}\Delta S_t(\boldsymbol{x})u_t(\boldsymbol{x}) + \frac{1}{2\mu^2}|\boldsymbol{\nabla}S_t(\boldsymbol{x})|^2 u_t(\boldsymbol{x}) - \Big(\boldsymbol{a}_t(\boldsymbol{x}) \cdot \frac{\boldsymbol{\nabla}S_t(\boldsymbol{x})}{\mu^2}\Big)u_t(\boldsymbol{x}) \\ &+ \frac{1}{\mu^2}\Big(\frac{\boldsymbol{a}_t{}^2(\boldsymbol{x})}{2} + V(\boldsymbol{x})\Big)u_t(\boldsymbol{x})\bigg\}dt - \varepsilon k_t(\boldsymbol{x}) \circ \partial W_t. \end{aligned}$$

That is to say $S_t(\boldsymbol{x})$ satisfies the viscous stochastic Hamilton-Jacobi equation for a vector potential

$$dS_t(\boldsymbol{x}) + 2^{-1}\left|\boldsymbol{\nabla} S_t(\boldsymbol{x}) - \boldsymbol{a}_t(\boldsymbol{x})\right|^2 dt + V(\boldsymbol{x})dt + \varepsilon k_t(\boldsymbol{x}) \circ \partial W_t = 2^{-1}\mu^2 \triangle S_t(\boldsymbol{x})dt.$$

However, since $k_t(\boldsymbol{x})$ is a deterministic function $S_t(\boldsymbol{x})$ also satisfies

$$dS_t(\boldsymbol{x}) + 2^{-1}\left|\boldsymbol{\nabla} S_t(\boldsymbol{x}) - \boldsymbol{a}_t(\boldsymbol{x})\right|^2 dt + V(\boldsymbol{x})dt + \varepsilon k_t(\boldsymbol{x}) dW_t = 2^{-1}\mu^2 \triangle S_t(\boldsymbol{x})dt.$$

If we set $\boldsymbol{v}_t(\boldsymbol{x}) = \boldsymbol{\nabla} S_t(\boldsymbol{x}) - \boldsymbol{a}_t(\boldsymbol{x})$, then

$$dS_t(\boldsymbol{x}) + 2^{-1}\left|\boldsymbol{v}_t(\boldsymbol{x})\right|^2 dt + V(\boldsymbol{x})dt + \varepsilon k_t(\boldsymbol{x}) dW_t = 2^{-1}\mu^2 \triangle S_t(\boldsymbol{x})dt,$$

taking the gradient and using the identity $\boldsymbol{\nabla}\left(\frac{\boldsymbol{v}_t{}^2}{2}\right) = (\boldsymbol{v}_t \cdot \boldsymbol{\nabla})\boldsymbol{v}_t + \boldsymbol{v}_t \wedge \mathbf{curl}\ \boldsymbol{v}_t$ we obtain

$$\begin{aligned} d(\boldsymbol{\nabla} S_t(\boldsymbol{x})) - d\boldsymbol{a}_t(\boldsymbol{x}) + d\boldsymbol{a}_t(\boldsymbol{x}) + ((\boldsymbol{v}_t(\boldsymbol{x}) \cdot \boldsymbol{\nabla})\boldsymbol{v}_t(\boldsymbol{x}) + \boldsymbol{v}_t(\boldsymbol{x}) \wedge \mathbf{curl}\ \boldsymbol{v}_t(\boldsymbol{x}))\, dt \\ = 2^{-1}\mu^2 \triangle(\boldsymbol{\nabla} S_t(\boldsymbol{x}) - \boldsymbol{a}_t(\boldsymbol{x}))dt + 2^{-1}\mu^2\triangle \boldsymbol{a}_t(\boldsymbol{x})dt - \boldsymbol{\nabla} V(\boldsymbol{x})dt - \varepsilon \boldsymbol{\nabla} k_t(\boldsymbol{x}) dW_t, \end{aligned}$$

therefore

$$\begin{aligned} d\boldsymbol{v}_t(\boldsymbol{x}) + \Big((\boldsymbol{v}_t(\boldsymbol{x}) \cdot \boldsymbol{\nabla})\boldsymbol{v}_t(\boldsymbol{x}) + \boldsymbol{v}_t(\boldsymbol{x}) \wedge \mathbf{curl}\ \boldsymbol{v}_t(\boldsymbol{x})\Big) dt \\ = -d\boldsymbol{a}_t(\boldsymbol{x}) - \boldsymbol{\nabla} V(\boldsymbol{x})dt - \varepsilon\boldsymbol{\nabla} k_t(\boldsymbol{x})dW_t + 2^{-1}\mu^2\triangle \boldsymbol{v}_t(\boldsymbol{x})dt + 2^{-1}\mu^2 \triangle \boldsymbol{a}_t(\boldsymbol{x})dt. \end{aligned}$$

□

Equation (1.1.2) can be thought of as the stochastic viscous Burgers equation with a viscosity term $2^{-1}\mu^2(\triangle \boldsymbol{v}_t + \triangle \boldsymbol{a}_t)dt$.

Having illustrated this important link via the Hamilton-Jacobi equation, let us proceed in the following section with a deterministic heat equation and an inviscid Burgers equation. We emphasise at this point the paramount importance throughout this work of the underlying Hamilton-Jacobi theory.

1.2 Heat equation with a vector potential

Here we present the main theorem of the chapter. A semi-classical solution for the heat equation, under a deterministic set-up, with an n-dimensional vector potential $\boldsymbol{a}(\boldsymbol{x}, t)$. We follow Simon by assuming $\boldsymbol{a}(\boldsymbol{x}, t) \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^1)$ throughout the thesis, V being continuous and bounded as explained above [31]. In practice the vector potential could belong to a bigger class of functions so long as the solution of the heat equation is unique. In this thesis we use Simon's result and therefore use the same class of functions.

Definition 1.2.1 (no caustic condition). There exists $T > 0$ such that the underlying classical mechanics is defined by a diffeomorphism $\Phi_t : \mathbb{R}^d \to \mathbb{R}^d$ for all $0 \le t \le T$. Under the no-caustic condition, for $0 \le t \le T$, $S(\boldsymbol{x}, t)$ is the classical $C^{1,2}$ solution of the Hamilton-Jacobi equation with a vector potential given by equation (1.2.8).

Remark 1.2.1. The precise definition of the Φ_t map is considered in Section (2.2) of Chapter 2.

Theorem 1.2.1. *Consider $u^\mu(\boldsymbol{x},t)$ a solution of the heat equation*

$$\begin{aligned}\frac{\partial}{\partial t}u^\mu(\boldsymbol{x},t) &= 2^{-1}\mu^2\triangle u^\mu(\boldsymbol{x},t) + \boldsymbol{a}(\boldsymbol{x},t)\cdot\boldsymbol{\nabla}u^\mu(\boldsymbol{x},t) + \mu^{-2}\left(\frac{\boldsymbol{a}(\boldsymbol{x},t)^2}{2} + V(\boldsymbol{x})\right)u^\mu(\boldsymbol{x},t)\\ &= -H(\boldsymbol{a},V)u^\mu(\boldsymbol{x},t) \qquad (1.2.1)\end{aligned}$$

with conditions on $\boldsymbol{a}(x,t)$ and $V(\boldsymbol{x})$ as described above and initial condition $u(\boldsymbol{x},0) = T_0(\boldsymbol{x})e^{-\frac{S_0(\boldsymbol{x})}{\mu^2}}$, here $T_0 : \mathbb{R}^d \to \mathbb{R}^1$ is positive and C^∞. Then, under a no-caustic condition we define a non explosive diffusion process

$$\begin{aligned} d\boldsymbol{X}_s^\mu &= \mu d\boldsymbol{B}_s - \left(\boldsymbol{\nabla}S_{t-s}(\boldsymbol{X}_s^\mu) - \boldsymbol{a}_{t-s}(\boldsymbol{X}_s^\mu)\right)ds,\\ \boldsymbol{X}_0^\mu &= \boldsymbol{x}, \qquad (1.2.2)\end{aligned}$$

where $\boldsymbol{B}_s$ is n-dimensional Brownian motion defined on $(\Omega,\mathcal{F},P)$ and $\boldsymbol{X}_s^\mu$ is assumed to be a unique solution of the stochastic differential equation (1.2.2). Here S satisfies the corresponding Hamilton-Jacobi equation (1.2.8) with $S(0,\boldsymbol{x}) = S_0(\boldsymbol{x})$. Following these assumptions the solution of the heat equation is given by the semi-classical representation

$$u^\mu(\boldsymbol{x},t) = e^{-\frac{S_t(\boldsymbol{x})}{\mu^2}}\mathbb{E}_P\left[T_0(\boldsymbol{X}_t^\mu)\exp\left(-\frac{1}{2}\int_0^t \triangle S_{t-s}(\boldsymbol{X}_s^\mu)ds\right)\right],$$

where $\mathbb{E}_P$ is the expectation with respect to the probability measure P.

Remark 1.2.2. In order to determine $\boldsymbol{a}$ uniquely we must impose a condition on its divergence, since $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{a} = \boldsymbol{\nabla}\times\tilde{\boldsymbol{a}}$ where $\tilde{\boldsymbol{a}} = \boldsymbol{a} + \nabla f$, irrespective of the form that the function f takes. Therefore, we must impose a condition on the divergence of $\boldsymbol{a}$. This condition is called the *gauge condition.* The natural choice being $\boldsymbol{\nabla}\cdot\boldsymbol{a} \equiv 0$.

To prove Theorem (1.2.1) we require a few elementary lemmas.

Lemma 1.2.2 (Feynman-Kac formula). *For $\mu \in \mathbb{R}^+$ and our usual conditions on $\boldsymbol{a}(\boldsymbol{x},t)$ and $V(\boldsymbol{x})$ define*

$$-H(\boldsymbol{a},V) = 2^{-1}\mu^2\triangle + \boldsymbol{a}(\boldsymbol{x},t)\cdot\boldsymbol{\nabla} + \mu^{-2}\left(\frac{\boldsymbol{a}(\boldsymbol{x},t)^2}{2} + V(\boldsymbol{x})\right) \qquad (1.2.3)$$

and

$$(P_T u)(\boldsymbol{x}) = \mathbb{E}_P\left[\exp\left(\mu^{-1}\int_0^T \boldsymbol{a}(T-s,\boldsymbol{B}_s^\mu)\cdot d\boldsymbol{B}_s + \mu^{-2}\int_0^T V(\boldsymbol{B}_s^\mu)ds\right)u(T-t,\boldsymbol{B}_t^\mu)\right], \qquad (1.2.4)$$

where $\boldsymbol{B}_t^\mu = \boldsymbol{x} + \mu\boldsymbol{B}_t$, then

$$\frac{d}{dt}(P_T u)(\boldsymbol{x}) = P_T(-Hu)(\boldsymbol{x}).$$

Proof. For fixed T let us compute the differential of

$$\begin{aligned} &Z_t u(\boldsymbol{B_t}^{\mu}, T-t) \\ &\equiv \exp\left(\mu^{-1}\int_0^t \boldsymbol{a}(\boldsymbol{B_s}^{\mu}, T-s)\cdot d\boldsymbol{B_s} + \mu^{-2}\int_0^t V(\boldsymbol{B_s}^{\mu})ds\right) u(\boldsymbol{B_t}^{\mu}, T-t) \\ &= \exp(Y_t)u(\boldsymbol{B_t}^{\mu}, T-t). \end{aligned}$$

By Itô's lemma we have

$$\begin{aligned} dZ_t &= Z_t\left(dY_t + 2^{-1}(dY_t)^2\right) \\ &= Z_t\Big(\mu^{-1}\boldsymbol{a}(\boldsymbol{B_t^{\mu}}, T-s)\cdot d\boldsymbol{B_t} + \mu^{-2}V(\boldsymbol{B_t^{\mu}})ds\Big) + 2^{-1}Z_t\Big(\mu^{-2}\boldsymbol{a}(\boldsymbol{B_t^{\mu}}, T-s)^2 ds\Big), \end{aligned}$$

and similarly

$$d\big(u(\boldsymbol{B_t^{\mu}}, T-t)\big) = -\frac{\partial u}{\partial(T-t)}(\boldsymbol{B_t^{\mu}}, T-t)dt + \mu^2 2^{-1}\triangle u(\boldsymbol{B_t^{\mu}}, T-t)dt + \mu\boldsymbol{\nabla}u(\boldsymbol{B_t^{\mu}}, T-t)\cdot d\boldsymbol{B_t}.$$

Suppressing the arguments,

$$\begin{aligned} d\big(Z_t u\big) &= Z_t du + u dZ_t + du dZ_t, \\ &= Z_t\Big(-\frac{\partial u}{\partial(T-t)}dt + \mu^2 2^{-1}\triangle u dt + \mu\boldsymbol{\nabla}u\cdot d\boldsymbol{B_t}\Big) \\ &+ u\Big(Z_t(\mu^{-1}\boldsymbol{a}\cdot d\boldsymbol{B_t} + \mu^{-2}Vdt) + 2^{-1}Z_t(\mu^{-2}\boldsymbol{a}^2 dt)\Big) + Z_t\boldsymbol{a}\cdot\boldsymbol{\nabla}u dt, \\ &= Z_t\Big[(\mu\boldsymbol{\nabla}u + \mu^{-1}\boldsymbol{a}u)\cdot d\boldsymbol{B_t} + \big(\mu^2 2^{-1}\triangle u + \boldsymbol{a}\cdot\boldsymbol{\nabla}u + \mu^{-2}Vu + \mu^{-2}2^{-1}\boldsymbol{a}^2 u\big)dt\Big]. \end{aligned}$$

Therefore, reinserting the $(\boldsymbol{B_t^{\mu}}, T-t)$ dependence, for $t > r$

$$\begin{aligned} &Z_t u(\boldsymbol{B_t^{\mu}}, T-t) - Z_r u(\boldsymbol{B_r^{\mu}}, T-r) \\ &= \int_r^t Z_s\left[\frac{\mu^2}{2}\triangle + \boldsymbol{a}(\boldsymbol{B_s^{\mu}}, T-s)\cdot\boldsymbol{\nabla} + \mu^{-2}\left(\frac{\boldsymbol{a}^2(\boldsymbol{B_s^{\mu}}, T-s)}{2} + V(\boldsymbol{B_s^{\mu}})\right)\right]u(\boldsymbol{B_s^{\mu}}, T-s)ds \\ &+ \int_r^t Z_s(\mu\boldsymbol{\nabla} + \mu^{-1}\boldsymbol{a}(\boldsymbol{B_s^{\mu}}, T-s))u(\boldsymbol{B_s^{\mu}}, T-s)\cdot d\boldsymbol{B_s}. \end{aligned}$$

Taking expectations yields

$$\mathbb{E}_P\Big[Z_t u(\boldsymbol{B_t^{\mu}}, T-t) - Z_r u(\boldsymbol{B_r^{\mu}}, T-r)\Big] = \int_u^r \mathbb{E}_P\Big[-Hu(\boldsymbol{B_s^{\mu}}, T-s)Z_s\Big]ds$$

Dividing by $(t-r)$ and letting $t \to r$ gives

$$\frac{d}{dt}(P_t u)(\boldsymbol{x}) = P_t(-Hu)(\boldsymbol{x}),$$

which yields the desired result. □

Lemma 1.2.3. *Set $\boldsymbol{X}(s) = \Phi_s \boldsymbol{x_0}$ for $0 \le s \le t$, where $\boldsymbol{x_0}(\boldsymbol{x}, t)$ is defined to be $\Phi_t^{-1}\boldsymbol{x}$. Here Φ_s is the diffeomorphism defined by*

$$\frac{d^2\Phi_s}{ds^2} = -\nabla V(\Phi_s) - \boldsymbol{curl}\ \boldsymbol{a}(\Phi_s, s) \wedge \frac{d\Phi_s}{ds} - \frac{\partial \boldsymbol{a}(\Phi_s, s)}{ds}, \tag{1.2.5}$$

initially $\Phi_0\boldsymbol{x_0} = \boldsymbol{x_0}$ and $\dot{\Phi}_0\boldsymbol{x_0} = \boldsymbol{\nabla} S_0(\boldsymbol{x_0}, 0) - \boldsymbol{a}(\boldsymbol{x_0}, 0)$. This Φ_t map corresponds to the Hamiltonian

$$H(\boldsymbol{q}, \boldsymbol{p}, t) = \frac{(\boldsymbol{p} - \boldsymbol{a}(\boldsymbol{q}, t))^2}{2} + V(\boldsymbol{q})$$

where $\boldsymbol{q} = (q_1, q_2, \dots, q_d)$ and $\boldsymbol{p} = (p_1, p_2, \dots, p_d)$ are canonical coordinates. Set

$$S_0(\boldsymbol{x}, t) = S_0(\boldsymbol{x_0}(\boldsymbol{x}, t)) + \int_0^t \mathcal{L}(\boldsymbol{X}(s), \dot{\boldsymbol{X}}(s), s) ds, \tag{1.2.6}$$

where $\mathcal{L}$ is defined as the Lagrangian

$$\mathcal{L}(\boldsymbol{X}(s), \dot{\boldsymbol{X}}(s), s) = 2^{-1}\dot{\boldsymbol{X}}^2(s) + \dot{\boldsymbol{X}}(s) \cdot \boldsymbol{a}(\boldsymbol{X}(s), s) - V(\boldsymbol{X}(s)), \tag{1.2.7}$$

here $\boldsymbol{X}(s) = \Phi_s\boldsymbol{x_0}$ satisfies the associated Euler-Lagrange equation with $\dot{\boldsymbol{X}}(0) = \boldsymbol{\nabla} S_0(\boldsymbol{x_0}, 0)$ $-\boldsymbol{a}(\boldsymbol{x_0}, 0)$, see Section (3.1) for more details on this. Then $S_0(\boldsymbol{x}, t)$ satisfies the Hamilton-Jacobi equation with a vector potential, namely

$$\frac{\partial S}{\partial t} + \frac{\left(\boldsymbol{\nabla} S(\boldsymbol{x}, t) - \boldsymbol{a}(\boldsymbol{x}, t)\right)^2}{2} + V(\boldsymbol{x}) = 0. \tag{1.2.8}$$

Proof. Differentiating (1.2.6) and using the chain rule gives

$$\frac{\partial S_0}{\partial \boldsymbol{x}}(\boldsymbol{x}, t) = \frac{\partial S_0}{\partial \boldsymbol{x_0}}(\boldsymbol{x_0}(\boldsymbol{x}, t))\frac{\partial \boldsymbol{x_0}}{\partial \boldsymbol{x}}(\boldsymbol{x}, t) + \int_0^t \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{X}}\frac{\partial \boldsymbol{X}}{\partial \boldsymbol{x}} + \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{X}}}\frac{\partial \dot{\boldsymbol{X}}}{\partial \boldsymbol{x}}\right) ds.$$

Abusing notation, since $\frac{\partial \dot{\boldsymbol{X}}}{\partial \boldsymbol{x}} = \frac{d}{ds}\frac{\partial \boldsymbol{X}}{\partial \boldsymbol{x}}$, integration by parts yields

$$\begin{aligned}
&\boldsymbol{\nabla}_{\boldsymbol{x}} S_0(\boldsymbol{x}, t) \\
&= \frac{\partial S_0}{\partial \boldsymbol{x}}(\boldsymbol{x}, t) \\
&= \frac{\partial S_0}{\partial \boldsymbol{x_0}}(\boldsymbol{x_0}(\boldsymbol{x}, t))\frac{\partial \boldsymbol{x_0}}{\partial \boldsymbol{x}}(\boldsymbol{x}, t) + \left[\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{X}}}(\boldsymbol{X}(s), \dot{\boldsymbol{X}}(s), s)\frac{\partial \boldsymbol{X}}{\partial \boldsymbol{x}}(s)\right]_{s=0}^{s=t} \\
&= \frac{\partial S_0}{\partial \boldsymbol{x_0}}(\boldsymbol{x_0}(\boldsymbol{x}, t))\frac{\partial \boldsymbol{x_0}}{\partial \boldsymbol{x}}(\boldsymbol{x}, t) + \dot{\boldsymbol{X}}(t) + \boldsymbol{a}(\boldsymbol{X}(t), t) - \left(\dot{\boldsymbol{X}}(0) + \boldsymbol{a}(\boldsymbol{X}(0), 0)\right)\frac{\partial \boldsymbol{x_0}}{\partial \boldsymbol{x}}(\boldsymbol{x}, t) \\
&= \dot{\boldsymbol{X}}(t) + \boldsymbol{a}(\boldsymbol{X}(t), t).
\end{aligned}$$

Moreover,

$$\begin{aligned}
\frac{\partial S}{\partial t}(\boldsymbol{x}, t) &= \frac{\partial S_0}{\partial \boldsymbol{x_0}}(\boldsymbol{x_0}(\boldsymbol{x}, t))\frac{\partial \boldsymbol{x_0}}{\partial t}(\boldsymbol{x}, t) + \left[\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{x}}}(\boldsymbol{x}(s), \dot{\boldsymbol{x}}(s))\frac{\partial \boldsymbol{x}}{\partial t}(s)\right]_{s=0}^{s=t} \\
&\quad + 2^{-1}\, \dot{\boldsymbol{x}}^2(t) + \boldsymbol{a}(\boldsymbol{x}(t), t).\dot{\boldsymbol{x}}(t) - V(\boldsymbol{x}(t)) \\
&= \left(\boldsymbol{a}(\boldsymbol{x}(t), t) + \dot{\boldsymbol{x}}(t)\right)\frac{\partial \boldsymbol{x}}{\partial t}(s = t) + \mathcal{L}(\boldsymbol{x}(t), \dot{\boldsymbol{x}}(t), t).
\end{aligned}$$

However,

$$\frac{\partial \boldsymbol{X}}{\partial t}(s=t) = \left(\frac{\partial \boldsymbol{X}}{\partial \boldsymbol{X_0}}\frac{\partial \boldsymbol{X_0}}{\partial t}\right)\Bigg|_{s=t}$$

and

$$\left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{x_0}}\frac{\partial \boldsymbol{x_0}}{\partial t}\right)\Bigg|_{s=t} + \dot{\boldsymbol{x}}(t) = \frac{\partial \boldsymbol{x}}{\partial t} = 0.$$

Consequently, we conclude

$$\frac{\partial S}{\partial t}(\boldsymbol{x},t) = -\frac{\dot{\boldsymbol{x}}^2(t)}{2} - V(\boldsymbol{x}),$$

and so

$$\frac{\partial S}{\partial t}(\boldsymbol{x},t) + \frac{\left(\boldsymbol{\nabla} S(\boldsymbol{x},t) - \boldsymbol{a}(\boldsymbol{x},t)\right)^2}{2} + V(\boldsymbol{x}) = 0$$

proving the result. □

Lemma 1.2.4. *Define a Hamiltonian function $H(\boldsymbol{q},\boldsymbol{p},t)$ by*

$$H(\boldsymbol{q},\boldsymbol{p},t) = \frac{(\boldsymbol{p} - \boldsymbol{a}(\boldsymbol{q},t))^2}{2} + V(\boldsymbol{q}) \tag{1.2.9}$$

and let $S(\boldsymbol{x},t)$ be the classical $C^{1,2}$ solution of the Hamilton-Jacobi equation (1.2.8) where $S(\boldsymbol{x},0) = S_0(\boldsymbol{x})$. Then $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x},t) = \boldsymbol{\nabla} S(\boldsymbol{x},t) - \boldsymbol{a}(\boldsymbol{x},t)$ is a classical solution of the Burgers equation

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\,\boldsymbol{v} + \boldsymbol{v}\wedge \boldsymbol{curl}\ \boldsymbol{v} = -\boldsymbol{\nabla} V - \frac{\partial \boldsymbol{a}}{\partial t}, \tag{1.2.10}$$

with $\boldsymbol{v}(\boldsymbol{x},0) = \boldsymbol{\nabla} S_0(\boldsymbol{x}) - \boldsymbol{a}(\boldsymbol{x},0)$.

Proof. Hamilton's equations read,

$$\begin{aligned} \dot{q}_i &= p_i - a_i \\ \dot{p}_i &= -(p_j - a_j)\left(-\frac{\partial a_j}{\partial q_i}\right) - \frac{\partial V}{\partial q_i}. \end{aligned} \tag{1.2.11}$$

Differentiating Equation (1.2.11) and using Einstein's summation convention gives

$$\begin{aligned} \ddot{q}_i &= \dot{p}_i - \dot{a}_i \\ &= -(p_j - a_j)\left(-\frac{\partial a_j}{\partial q_i}\right) - \frac{\partial V}{\partial q_i} - \frac{\partial a_i}{\partial q_j}\dot{q}_j - \frac{\partial a_i}{\partial t}. \end{aligned}$$

For clarity set $i = 1$ to observe

$$\begin{aligned} \ddot{q}_1 &= \dot{q}_1\frac{\partial a_1}{\partial q_1} + \dot{q}_2\frac{\partial a_2}{\partial q_1} + \dot{q}_3\frac{\partial a_3}{\partial q_1} - \frac{\partial V}{\partial q_1} - \frac{\partial a_1}{\partial q_1}\dot{q}_1 - \frac{\partial a_1}{\partial q_2}\dot{q}_2 - \frac{\partial a_1}{\partial q_3}\dot{q}_3 - \frac{\partial a_1}{\partial t} \\ &= \dot{q}_2\frac{\partial a_2}{\partial q_1} + \dot{q}_3\frac{\partial a_3}{\partial q_1} - \frac{\partial a_1}{\partial q_2}\dot{q}_2 - \frac{\partial a_1}{\partial q_3}\dot{q}_3 - \frac{\partial V}{\partial q_1} - \frac{\partial a_1}{\partial t} \\ &= -\left\{\textbf{curl}\ \boldsymbol{a}\wedge\dot{\boldsymbol{q}}\right\}_1 - (\boldsymbol{\nabla} V)_1 - \frac{\partial a_1}{\partial t}. \end{aligned}$$

It becomes clear that this is nothing but

$$\ddot{\boldsymbol{q}} = -\textbf{curl } \boldsymbol{a} \wedge \dot{\boldsymbol{q}} - \boldsymbol{\nabla} V - \frac{\partial \boldsymbol{a}}{\partial t}.$$

Then for $S(\boldsymbol{x}, t)$ satisfying (1.2.8) we observe

$$\begin{aligned}
\boldsymbol{v} \wedge \textbf{curl } \boldsymbol{v} + (\boldsymbol{v}.\nabla)\boldsymbol{v} &= \varepsilon_{ijk} v_j (\textbf{curl } \boldsymbol{v})_k + v_j v_{i,j} \\
&= \varepsilon_{ijk}\varepsilon_{klm} v_j v_{m,l} + v_j v_{i,j} \\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) v_j v_{m,l} + v_j v_{i,j} \\
&= v_j v_{j,i} \\
&= \boldsymbol{\nabla}\left(\frac{1}{2}\boldsymbol{v}^2\right)
\end{aligned}$$

and so by taking the gradient of (1.2.8) and suppressing the $(\boldsymbol{x}, t)$ dependence we obtain

$$\frac{\partial}{\partial t}(\boldsymbol{\nabla} S) + (\boldsymbol{\nabla} S - \boldsymbol{a}) \wedge \textbf{curl } (\boldsymbol{\nabla} S - \boldsymbol{a}) + ((\boldsymbol{\nabla} S - \boldsymbol{a}) \cdot \boldsymbol{\nabla})(\boldsymbol{\nabla} S - \boldsymbol{a}) + \boldsymbol{\nabla} V = 0.$$

Setting $\boldsymbol{v} = \boldsymbol{\nabla} S - \boldsymbol{a}$ yields

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} + \boldsymbol{v} \wedge \textbf{curl } \boldsymbol{v} = -\boldsymbol{\nabla} V - \frac{\partial \boldsymbol{a}}{\partial t},$$

for $\boldsymbol{v}(\boldsymbol{x}, 0) = \boldsymbol{\nabla} S_0(\boldsymbol{x}) - \boldsymbol{a}(\boldsymbol{x}, 0)$. □

Remark 1.2.3. This can be thought of as an inviscid Burger's equation with non-zero vorticity $\boldsymbol{\xi}(\boldsymbol{x}, t) = \textbf{curl } \boldsymbol{v}(\boldsymbol{x}, t) = \textbf{curl } (\boldsymbol{\nabla} S(\boldsymbol{x}, t) - \boldsymbol{a}(\boldsymbol{x}, t)) = -\textbf{curl } \boldsymbol{a}(\boldsymbol{x}, t)$ prescribed in advance.

To prove the main theorem of this section we require a transformation of measures. For this we utilise Girsanov's theorem. In layman's terms this states that if we change the drift coefficient of a given Itô process (with a non-degenerate diffusion coefficient), then the law of the process will not change dramatically. Moreover, the law of the new process will be absolutely continuous with respect to the original process and we can compute explicitly the Radon-Nikodym derivative. We state Girsanov's theorem in terms of our set-up with the following lemma.

Lemma 1.2.5 (Girsanov). *Let $\boldsymbol{X}(t) = \boldsymbol{X}^{\boldsymbol{x}}(t) \in \mathbb{R}^n$ and $\boldsymbol{Y}(t) = \boldsymbol{Y}^{\boldsymbol{x}}(t) \in \mathbb{R}^n$ be an Itô diffusion and an Itô process, respectively, of the forms*

$$\begin{aligned}
d\boldsymbol{X}_t &= -(\boldsymbol{\nabla} S - \boldsymbol{a})dt + \mu d\boldsymbol{B}_t, \\
\boldsymbol{X}_0 &= \boldsymbol{x}
\end{aligned} \tag{1.2.12}$$

and

$$\begin{aligned}
d\boldsymbol{Y}_t &= \mu d\boldsymbol{B}_t, \\
\boldsymbol{Y}_0 &= \boldsymbol{x}
\end{aligned} \tag{1.2.13}$$

where $\boldsymbol{x} \in \mathbb{R}^n$, $t \leq T$, *for* T *a given finite constant,* $(\boldsymbol{\nabla} S - \boldsymbol{a}) : \mathbb{R}^n \to \mathbb{R}^n$ *is a bounded measurable function. Define a function* $\boldsymbol{h}(t, \omega) = \boldsymbol{h_0}(\boldsymbol{Y_t}(\omega))$ *for* $\boldsymbol{h_0} : \mathbb{R}^n \to \mathbb{R}^n$ *by*

$$\boldsymbol{h_0} = -\mu^{-1} (\boldsymbol{\nabla} S - \boldsymbol{a})$$

which satisfies the Novikov condition

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \boldsymbol{h}^2(s, \omega) ds\right)\right] < \infty.$$

Define the measure P_1 *on* $(\Omega, \mathcal{F})$ *by*

$$dP_1(\omega) = M_T(\omega) dP(\omega)$$

where

$$M_t = \exp\left[-\int_0^t \boldsymbol{h_0}(\boldsymbol{Y_s}) \cdot d\boldsymbol{B_s} - \frac{1}{2}\int_0^t \boldsymbol{h_0}^2(\boldsymbol{Y_s}) ds\right].$$

Then with $\widehat{\boldsymbol{B}(t)} = \widehat{\boldsymbol{B_t}}$ *defined by*

$$\widehat{\boldsymbol{B_t}} := \int_0^t \boldsymbol{h_0}(\boldsymbol{Y_s}) ds + \boldsymbol{B_t}$$

for $t \leq T$ *we have that*

$$d\boldsymbol{Y}(t) = (\boldsymbol{\nabla} S - \boldsymbol{a})\, dt + \mu d\widehat{\boldsymbol{B_t}} \tag{1.2.14}$$

and consequently

$$\mathbb{E}\left[f(\boldsymbol{X_t^x})\right] = \mathbb{E}\left[M_t f(\boldsymbol{B_t})\right]$$

for all $f \in C_0(\mathbb{R}^n)$ *and* $t \leq T$.

Remark 1.2.4. The Novikov condition is sufficient to guarantee that $\{M_t\}_{t \leq T}$ is a martingale. Moreover, since M_t is a martingale we have that

$$M_T dP = M_t dP \tag{1.2.15}$$

on $\mathcal{F}$ for $t \leq T$.

Remark 1.2.5. Equation (1.2.15) is equivalent to saying P_1 is absolutely continuous w.r.t. P with Radon-Nikodym derivative

$$\frac{dP_1}{dP} = M_T$$

on $\mathcal{F}$. One should observe that $M_T(\omega) > 0$ a.s., so P is also absolutely continuous w.r.t. P_1 therefore we have that the two measures P and P_1 are equivalent.

Proof. We have

$$\begin{aligned}M_t &= \exp\left[-\mu^{-1}\int_0^t (-\nabla S(\boldsymbol{X}_s, t-s) + \boldsymbol{a}(\boldsymbol{X}_s, t-s))\cdot d\boldsymbol{B}_s\right.\\ &\left.-\frac{1}{2\mu^2}\int_0^t |\nabla S(\boldsymbol{X}_s, t-s) - \boldsymbol{a}(\boldsymbol{X}_s, t-s)|^2\, ds\right]\\ &= \exp\left[\mu^{-1}\int_0^t (\nabla S(\boldsymbol{X}_s, t-s) - \boldsymbol{a}(\boldsymbol{X}_s, t-s))\cdot d\boldsymbol{B}_s\right.\\ &\left.-\frac{1}{2\mu^2}\int_0^t |\nabla S(\boldsymbol{X}_s, t-s) - \boldsymbol{a}(\boldsymbol{X}_s, t-s)|^2\, ds\right]\end{aligned} \tag{1.2.16}$$

and for fixed $T \leq \infty$

$$dP_1 = M_T dP$$

on $\mathcal{F}$. Then

$$\widehat{\boldsymbol{B}_t} = \mu^{-1}\int_0^t (\nabla S(\boldsymbol{X}_s, t-s) - \boldsymbol{a}(\boldsymbol{X}_s, t-s))\cdot d\boldsymbol{B}_s + \boldsymbol{B}_t$$

is a Brownian motion w.r.t. P_1 for $t \leq T$ and in terms of $\widehat{\boldsymbol{B}_t}$ the process $\boldsymbol{Y}_t$ has stochastic integral representation

$$d\boldsymbol{Y}_t = -\nabla S(\boldsymbol{X}_s, t-s)dt + \boldsymbol{a}(\boldsymbol{X}_s, t-s)dt + \mu d\widehat{\boldsymbol{B}_t}.$$

Now because $\boldsymbol{Y}_0 = \boldsymbol{x}$ the pair $(\boldsymbol{Y}_t, \widehat{\boldsymbol{B}_t})$ is then a weak solution of (1.2.12) for $t \leq T$. By weak uniqueness the P_1-law of $\boldsymbol{Y}_t = \mu\boldsymbol{B}_t$ coincides with the P-law of $\boldsymbol{X}^{\boldsymbol{x}}(t)$, so that

$$\begin{aligned}\mathbb{E}_P\left[f(\boldsymbol{X}_t^{\boldsymbol{x}})\right] &= \mathbb{E}_{P_1}\left[f(\boldsymbol{Y}_t)\right]\\ &= \mathbb{E}_P\left[M_T f(\boldsymbol{B}_t)\right]\end{aligned}$$

for all $f \in C_0^\infty(\mathbb{R}^n)$ and $t \leq T$. □

We are now in a position to prove the main result of this section

Proof of Theorem (1.2.1). Define an Itô diffusion and an Itô process, respectively, of the forms

$$\begin{aligned}d\boldsymbol{X}_t &= -(\nabla S - \boldsymbol{a})dt + \mu d\boldsymbol{B}_t,\\ \boldsymbol{X}_0 &= \boldsymbol{x},\end{aligned}$$

on $(\Omega, \mathcal{F}, P)$ and

$$\begin{aligned}d\boldsymbol{Y}_t &= \mu d\boldsymbol{B}_t,\\ \boldsymbol{Y}_0 &= \boldsymbol{x},\end{aligned}$$

on $(\Omega, \mathcal{F}, P_1)$. It then follows from lemma (1.2.2) that

$$\begin{aligned}\mathbb{E}_P[f(\boldsymbol{Y}_t)] &= \mathbb{E}_{P_1}[f(\boldsymbol{X}_t)]\\ &= \mathbb{E}_P\left[f(\boldsymbol{X}_t)\frac{dP_1}{dP}\right],\end{aligned}$$

where $\frac{dP_1}{dP}$ on the filtration $\mathcal{F}$ is given by Equation (1.2.16). Writing $u^\mu(t, \boldsymbol{x}) = u_t^\mu(\boldsymbol{x})$, recall from Lemma (1.2.2) that Equation (1.2.1) has a solution given by the Feynman-Kac formula

$$u_t^\mu(\boldsymbol{x}) = \mathbb{E}_P\left[T_0(\boldsymbol{B_t}^\mu)\exp\left(-\frac{S_0(\boldsymbol{B_t^\mu})}{\mu^2} + \mu^{-1}\int_0^t \boldsymbol{a}(\boldsymbol{B_s}^\mu, t-s)\cdot d\boldsymbol{B_s} + \mu^{-2}\int_0^t V(\boldsymbol{B_s}^\mu)ds\right)\right] \tag{1.2.17}$$

or in the equivalent notation

$$u_t^\mu(\boldsymbol{x}) = \mathbb{E}_P\left[T_0(\boldsymbol{Y_t})\exp\left(-\frac{S_0(\boldsymbol{Y_t})}{\mu^2} + \mu^{-2}\int_0^t \boldsymbol{a}(\boldsymbol{Y_s})\cdot d\boldsymbol{Y_s} + \mu^{-2}\int_0^t V(\boldsymbol{Y_s})ds\right)\right].$$

by Girsanov's theorem

$$\begin{aligned} u_t^\mu(\boldsymbol{x}) &= \mathbb{E}_{P_1}\left[T_0(\boldsymbol{X_t})\exp\left(-\frac{S_0(\boldsymbol{X_t})}{\mu^2} + \mu^{-2}\int_0^t \boldsymbol{a}_{t-s}(\boldsymbol{X_s})\cdot d\boldsymbol{X_s} + \mu^{-2}\int_0^t V(\boldsymbol{X_s})ds\right)\right] \\ &= \mathbb{E}_P\left[T_0(\boldsymbol{X_t})\exp\left(-\frac{S_0(\boldsymbol{X_t})}{\mu^2} + \mu^{-2}\int_0^t V(\boldsymbol{X_s})ds\right.\right. \\ &\quad \left.\left. + \,\mu^{-2}\int_0^t \boldsymbol{a}_{t-s}(\boldsymbol{X_s})\cdot(\mu d\boldsymbol{B_s} - \{\nabla S_{t-s}(\boldsymbol{X_s}) - \boldsymbol{a}_{t-s}(\boldsymbol{X_s})\}\,ds)\right)\cdot\frac{dP_1}{dP}\right]. \end{aligned}$$

This is nothing but

$$\begin{aligned} u_t^\mu(\boldsymbol{x}) = \mathbb{E}_P\Bigg[T_0(\boldsymbol{X_t})\exp\Bigg(&-\frac{S_0(\boldsymbol{X_t})}{\mu^2} + \mu^{-2}\int_0^t V(\boldsymbol{X_s})ds + \mu^{-1}\int_0^t \boldsymbol{a}_{t-s}(\boldsymbol{X_s})\cdot d\boldsymbol{B_s} \\ &- \mu^{-2}\int_0^t \boldsymbol{a}_{t-s}(\boldsymbol{X_s})\cdot(\nabla S_{t-s}(\boldsymbol{X_s}) - \boldsymbol{a}_{t-s}(\boldsymbol{X_s}))\,ds \\ &+ \mu^{-1}\int_0^t (\nabla S_{t-s}(\boldsymbol{X_s}) - \boldsymbol{a}_{t-s}(\boldsymbol{X_s}))\cdot d\boldsymbol{B_s} \\ &- \frac{1}{2\mu^2}\int_0^t |\nabla S_{t-s}(\boldsymbol{X_s}) - \boldsymbol{a}_{t-s}(\boldsymbol{X_s})|^2\,ds\Bigg)\Bigg] \end{aligned}$$

which reduces to

$$\begin{aligned} u_t^\mu(\boldsymbol{x}) = \mathbb{E}_P\Bigg[T_0(\boldsymbol{X_t})\exp\Bigg(&-\frac{S_0(\boldsymbol{X_t})}{\mu^2} + \mu^{-2}\int_0^t V(\boldsymbol{X_s})ds \\ &- \mu^{-2}\int_0^t \boldsymbol{a}_{t-s}(\boldsymbol{X_s}).\,(\nabla S_{t-s}(\boldsymbol{X_s}) - \boldsymbol{a}_{t-s}(\boldsymbol{X_s}))\,ds + \mu^{-1}\int_0^t \nabla S_{t-s}(\boldsymbol{X_s})\cdot d\boldsymbol{B_s} \\ &- \frac{1}{2\mu^2}\int_0^t |\nabla S_{t-s}(\boldsymbol{X_s}) - \boldsymbol{a}(\boldsymbol{X_s})|^2\,ds\Bigg)\Bigg]. \end{aligned} \tag{1.2.18}$$

We need to remove the stochastic exponential term, namely

$$\mu^{-1}\int_0^t \nabla S_{t-s}(\boldsymbol{X_s})\cdot d\boldsymbol{B_s}. \tag{1.2.19}$$

For this we invoke Itô's formula. Consider the stochastic differential equation $d\boldsymbol{X_s} = \boldsymbol{b}(\boldsymbol{X_s})ds + \sigma(\boldsymbol{X_s})d\boldsymbol{B_s}$, then

$$f(\boldsymbol{X_t},t) = f(\boldsymbol{X_0},0) + \int_0^t \left[\frac{\partial f}{\partial s}(\boldsymbol{X_s},s) + \boldsymbol{b}(\boldsymbol{X_s})\cdot\frac{\partial f}{\partial \boldsymbol{x}}(\boldsymbol{X_s},s) + 2^{-1}\sigma^2(\boldsymbol{X_s})\frac{\partial^2 f}{\partial \boldsymbol{x}^2}(\boldsymbol{X_s},s)\right] ds$$
$$+ \int_0^t \sigma(\boldsymbol{X_s})\frac{\partial f}{\partial \boldsymbol{x}}(\boldsymbol{X_s},s)\cdot d\boldsymbol{B_s}.$$

Consequently applying Itô's formula to $S_{t-s}(\boldsymbol{X_s})$ for the stochastic differential equation (1.2.2), yields

$$S(\boldsymbol{X_t},0) = S(\boldsymbol{x},t) + \int_0^t \left(\frac{\partial S_{t-s}}{\partial s}(\boldsymbol{X_s}) - \nabla S_{t-s}(\boldsymbol{X_s})\cdot\left(\nabla S_{t-s}(\boldsymbol{X_s}) - \boldsymbol{a}_{t-s}(\boldsymbol{X_s})\right)\right.$$
$$\left. + 2^{-1}\mu^2\Delta S_{t-s}(\boldsymbol{X_s})\right)ds + \mu\int_0^t \nabla S_{t-s}(\boldsymbol{X_s})\cdot d\boldsymbol{B_s}.$$

This implies that

$$\mu^{-1}\int_0^t \nabla S_{t-s}(\boldsymbol{X_s})\cdot d\boldsymbol{B_s} = \mu^{-2}S(\boldsymbol{X_t},0) - \mu^{-2}S(\boldsymbol{x},t) - \mu^{-2}\int_0^t \Big(\frac{\partial S_{t-s}}{\partial s}(\boldsymbol{X_s})$$
$$+ \nabla S_{t-s}(\boldsymbol{X_s})\cdot\left(\nabla S_{t-s}(\boldsymbol{X_s}) - \boldsymbol{a}_{t-s}(\boldsymbol{X_s})\right)\Big)ds$$
$$- 2^{-1}\int_0^t \Delta S_{t-s}(\boldsymbol{X_s})ds$$

Substituting into (1.2.18) gives

$$u_t^\mu(\boldsymbol{x}) = \mathbb{E}_P\Bigg[T_0(\boldsymbol{X_t})\exp\Bigg(-\frac{S_0(\boldsymbol{X_t})}{\mu^2} + \mu^{-2}\int_0^t V(\boldsymbol{X_s})ds$$
$$- \mu^{-2}\int_0^t \boldsymbol{a}_{t-s}(\boldsymbol{X_s})\cdot(\nabla S_{t-s}(\boldsymbol{X_s}) - \boldsymbol{a}_{t-s}(\boldsymbol{X_s}))\,ds + \mu^{-2}S(0,\boldsymbol{X_t}) - \mu^{-2}S(t,\boldsymbol{x})$$
$$- \mu^{-2}\int_0^t \frac{\partial S_{t-s}}{\partial s}(\boldsymbol{X_s})ds + \mu^{-2}\int_0^t \nabla S_{t-s}(\boldsymbol{X_s})\cdot\left(\nabla S_{t-s}(\boldsymbol{X_s}) - \boldsymbol{a}_{t-s}(\boldsymbol{X_s})\right)ds$$
$$- 2^{-1}\int_0^t \Delta S_{t-s}(\boldsymbol{X_s})ds - \frac{1}{2\mu^2}\int_0^t |\nabla S_{t-s}(\boldsymbol{X_s}) - \boldsymbol{a}_{t-s}(\boldsymbol{X_s})|^2\,ds\Bigg)\Bigg]$$

Collecting terms

$$u_t^\mu(\boldsymbol{x}) = e^{-\frac{S_t(\boldsymbol{x})}{\mu^2}}\mathbb{E}_P\Bigg[T_0(\boldsymbol{X_t})\exp\Bigg(-2^{-1}\int_0^t \Delta S_{t-s}(\boldsymbol{X_s})ds$$
$$- \mu^{-2}\int_0^t \left(\frac{\partial S_{t-s}}{\partial s}(\boldsymbol{X_s}) - 2^{-1}|\nabla S_{t-s}(\boldsymbol{X_s}) - \boldsymbol{a}_{t-s}(\boldsymbol{X_s})|^2 - V(\boldsymbol{X_s})\right)ds\Bigg)\Bigg].$$

Now since

$$\frac{\partial}{\partial(t-s)}S_{t-s}(\boldsymbol{X_s}) + 2^{-1}|\nabla S_{t-s}(\boldsymbol{X_s}) - \boldsymbol{a}_{t-s}(\boldsymbol{X_s})|^2 + V(\boldsymbol{X_s}) = 0$$

we obtain a semi-classical solution of the heat equation (1.2.1) in the form

$$u_t^\mu(\boldsymbol{x}) = e^{-\frac{S_t(\boldsymbol{x})}{\mu^2}} \mathbb{E}_P \left[T_0(\boldsymbol{X_t}) \exp\left(-\frac{1}{2} \int_0^t \triangle S_{t-s}(\boldsymbol{X_s}) ds \right) \right].$$

□

We now encounter the challenge of finding exact semi-classical expansions for solutions of viscous stochastic Burgers and heat equations up to arbitrarily high order powers of viscosity μ. We will handle this in the next chapter where we initiate our study by turning to Truman and Zhao [37].

Chapter 2

Semi-Classical Expansion of a Stochastic Viscous Burgers Equation with Vorticity

Summary

Here we mimic the workings of Truman and Zhao [37]. No startling originality is claimed for the arguments herein. Most of the contents of this chapter follow analogously from Truman and Zhao's study with little effort required to generalise their results to a Burgers fluid with vorticity. This part of the study centres on times before the caustics whereupon existing stochastic Hamilton-Jacobi theory is used to study stochastic heat and Burgers equations. We begin our study with a stochastic heat equation containing a vector potential as well as the usual scalar one. We present the solution to this equation and via the Hopf-Cole logarithmic transformation we give a solution to the stochastic viscous Burgers equation with vorticity.

2.1 Introduction

In this chapter we use the method of constructing iterated stochastic Hamilton-Jacobi continuity equations to obtain exact semi-classical expansions up to arbitrarily high order powers in the viscosity μ. The semi-classical asymptotic expansions for stochastic heat and Burgers equations are then presented and obtained using a Nelson stochastic mechanical process with drifts given in terms of the solution of the iterated stochastic Hamilton-Jacobi continuity equations. The explicit formula for the remainder term is given in terms of a path integral as in the Truman and Zhao paper [37].

Consider the stochastic viscous Burgers equation with vorticity

$$\frac{1}{2}\mu^2(\triangle v_t^\mu(x) + \triangle a_t(x))dt$$
$$= dv_t^\mu(x) + \left[\left(v_t^\mu(x)\cdot\nabla\right)v_t^\mu(x) + v_t^\mu(x)\wedge \text{curl } v_t^\mu(x) + \nabla V(x)\right]dt + da_t(x) + \varepsilon\nabla k_t(x)dW_t$$

here $v_t^\mu(x) = v^\mu(x,t) \in \mathbb{R}^d$ is the Burgers velocity field and μ^2 is the coefficient of viscosity. As before $a_t(x) = a(x,t)$, $V(x)$ and $k_t(x) = k(x,t)$ represent the vector, scalar and noise potential terms respectively, with conditions discussed shortly. $W_t = W(t)$ is a one-dimensional Wiener process on the probability triple $(\Omega, \mathcal{F}, P)$. Let $v^\mu(x,t) = \nabla S^\mu(x,t) - a(x,t)$ with an appropriate initial condition for $S^\mu(x,t)$. Then $S^\mu(x,t)$ satisfies the following stochastic Hamilton-Jacobi equation with a vector potential

$$dS^\mu(x,t) + \frac{|\nabla S^\mu(x,t) - a(x,t)|^2}{2}dt + V(x)dt + \varepsilon k dW(t) = \frac{\mu^2}{2}\triangle S^\mu(x,t)dt.$$

As we shall see S^μ admits an asymptotic expansion, so writing $S^\mu(x,t) \sim \sum_{j=0}^\infty \mu^{2j} S_j(x,t)$ and omitting the (x,t) dependence, it follows that

$$\sum_{j=0}^\infty \mu^{2j} dS_j + \frac{1}{2}\left|\sum_{j=0}^\infty \mu^{2j}\nabla S_j - a\right|^2 dt + Vdt + \varepsilon k dW \sim \frac{\mu^2}{2}\sum_{j=0}^\infty \mu^{2j}\triangle S_j dt.$$

Comparing coefficients of μ^{2j} for $j = 0, 1, 2, \ldots$, it can be shown that

$$\dot{S}_j + \frac{1}{2}\sum_{\substack{i_1,i_2\geq 0,\\ i_1+i_2=j}} \nabla S_{i_1}\cdot\nabla S_{i_2} - a\cdot\nabla S_j = \frac{1}{2}\triangle S_{j-1} \tag{2.1.1}$$

with $\frac{1}{2}\triangle S_{-1} = -\left(V + \frac{|a|^2}{2} + \varepsilon k \dot{W}_t\right)$ where $\dot{W}_t$ is the time derivative of a one dimensional Wiener process. These equations are the stochastic Hamilton-Jacobli continuity equations for a vector potential. In the following sections we shall place great importance on finding solutions to these equations which are given by the iterated continuity equations. The first m solutions of the iterated continuity equations give the first m terms in the asymptotics of the solution for the stochastic heat and stochastic viscous Burgers equations.

Remark 2.1.1. Truman and Zhao [37] give an excellent account of the history, motivation and work of predecessors. Consequently, in order to avoid duplicating their introductory section the reader is asked to consult the original text for such details.

Remark 2.1.2. Observe the cumbersome bold face notation for vectors has been omitted. We shall continue this theme for the remainder of the thesis.

2.2 The set-up

Assume our usual conditions on the vector and scalar potentials so that $V \in C^2(\mathbb{R}^d)$, $a \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}^1)$, $\text{div}(a) = 0$, $k \in C^2(\mathbb{R}^d \times \mathbb{R}^1)$ and $S(\cdot, 0) \in C^2(\mathbb{R}^d)$ and consider a stochastic classical mechanics defined by

$$\begin{aligned}
&d\dot{\Phi}_s = -\nabla V(\Phi_s)ds - \text{curl } a(\Phi_s, s) \wedge d\Phi_s - da(\Phi_s, s) - \varepsilon\nabla k(\Phi_s, s)dW_s,\\
&\Phi_0(x) = x,\\
&\dot{\Phi}_0(x) = \nabla S(x,0) - a(x,0).
\end{aligned} \tag{2.2.1}$$

For each x there exists a solution $\Phi_s(x)$, consequently $\Phi_s : \Omega \times \mathbb{R}^d \to \mathbb{R}^n$ is a random map for each s. In this set-up a no-caustic condition is assumed. This means that there exists a $T(\omega) \geq 0$ a.s. such that for $0 \leq t \leq T(\omega)$, $\Phi_t(\omega) : \mathbb{R}^d \to \mathbb{R}^d$ is a diffeomorphism for a.e. $\omega \in \Omega$. The proof of the existence of such a $T(\omega) \geq 0$ depends on whether each of the first and second order spatial derivatives are all bounded (Truman and Zhao [36]), if they are in fact bounded, then there exists a $T(\omega) > 0$ such that if $0 \leq s \leq T$, $\Phi_s(\omega) : \mathbb{R}^d \to \mathbb{R}^d$ is a diffeomorphism for a.e. $\omega \in \Omega$.

Define $\tilde{S}_0 : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ by the following non-anticipating Itô integral

$$\begin{aligned}\widetilde{S}_0(y,t) =& 2^{-1} \int_0^t \mid \dot{\Phi}_s(y) \mid^2 ds + S(y,0) + \int_0^t \dot{\Phi}_s(y) \cdot a(\Phi_s(y), s) ds \\ &- \int_0^t V(\Phi_s(y)) ds - \varepsilon \int_0^t k(\Phi_s(y), s) dW_s. \end{aligned} \tag{2.2.2}$$

For $0 \leq t \leq T(\omega)$, define $S_0(x,t)$ for a.e. $\omega \in \Omega$ to be

$$S_0(x,t) = \widetilde{S}_0(\Phi_t^{-1} x, t), \tag{2.2.3}$$

this is the infimum of the action. Here we assume $\dot{\Phi}_s$ is continuous in s, for all $y \in \mathbb{R}^d$, with probability one and $\Phi_s y$ is $\mathcal{F}_s$ measurable as usual.

2.3 Results

Theorem 2.3.1. *Assume that $V \in C^2(\mathbb{R}^d)$, $a \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}^1)$, $k \in C^2(\mathbb{R}^d \times \mathbb{R}^1)$, $S(\cdot, 0) \in C^2(\mathbb{R}^d)$, Φ_s defined by (2.2.1) satisfies a no-caustic condition for $0 \leq t \leq T(\omega)$ and S_0 defined by (2.2.3). Define the classical mechanical (diffeomorphism) flow by the classical Hamiltonian*

$$H(q,p,t) = 2^{-1}(p - a(q,t))^2 + V(q),$$

where $q = (q_1, q_2, \ldots, q_d)$, it follows that

$$\ddot{q}_i = -(curla \wedge \dot{q})_i - (\nabla V)_i - \frac{\partial a_i}{\partial t},$$

for $0 \leq s < T$, for some $T > 0$. In other words $\dot{q} = p - a$ and the classical flow map satisfies

$$\ddot{\Phi}_s \cdot = -B(\Phi_s \cdot, s) \wedge \dot{\Phi}_s \cdot - \nabla V(\Phi_s \cdot) - \frac{\partial a}{\partial s}(\Phi_s \cdot, s),$$

where $B = curl\, a(\Phi_s \cdot, s)$, in our case, with appropriate initial conditions defined by (2.2.1).

1. *Then for a.e. $\omega \in \Omega$ and $0 \leq t \leq T(\omega)$,*

$$\dot{\Phi}_t = \nabla S_0(\Phi_t, t) - a(\Phi_t, t), \tag{2.3.1}$$

and $S_0(x,t)$ satisfies the following stochastic Hamilton-Jacobi equation

$$dS_0(x,t) + 2^{-1} \left| \nabla S_0(x,t) - a(x,t) \right|^2 dt + V(x) dt + \varepsilon k(x,t) dW_t = 0. \tag{2.3.2}$$

2. Define $\phi(x,t) = \left|\det\left(\frac{\partial}{\partial x}\Phi_t^{-1}x\right)\right|$. *Then for a.e.* $\omega \in \Omega$, *any* $x \in \mathbb{R}^d$ *and* $0 \le t \le T(\omega)$, $\phi(x,t)$ *satisfies the following continuity equation*

$$\frac{\partial}{\partial t}\phi(x,t) + div\,\{\phi(x,t)(\nabla S_0(x,t) - a(x,t))\} = 0. \tag{2.3.3}$$

Suppose $T_0 : \mathbb{R}^d \to \mathbb{R}^d$ *is positive and* C^∞. *The function* $T_0(x)$ *is related to the initial condition of the stochastic heat equation* (2.3.43). *Define for the random map* $\Phi_s(\omega) : \mathbb{R}^d \mapsto \mathbb{R}^d$,

$$T_0(y,t) = T_0(y)$$

and

$$T_j(y,t) = \int_0^t \phi^{-\frac{1}{2}}(\cdot,s)\Delta T_{j-1}(\Phi_s^{-1}\cdot,s)\Big|_{\Phi_s(y)} ds \tag{2.3.4}$$

for $j = 1, 2, \ldots$, *and*

$$\psi_j(x,t) = T_j({\Phi_t}^{-1}x,t)\sqrt{\phi(x,t)}, \tag{2.3.5}$$

for $j = 0, 1, 2, \ldots$, *then we obtain the iterated continuity equations detailed in Lemma* (2.3.2).

Proof. We prove the theorem for $k = 0$ (for $k \ne 0$ see [38] and [36] for ideas). Let

$$H(q,p,t) = 2^{-1}(p - a(q,t))^2 + V(q). \tag{2.3.6}$$

Then Hamilton's equations read,

$$\dot{q}_i = p_i - a_i \tag{2.3.7}$$

$$\dot{p}_i = -(p_j - a_j)\left(-\frac{\partial a_j}{\partial q_i}\right) - \frac{\partial V}{\partial q_i}, \tag{2.3.8}$$

where $a = (a_1, a_2, \ldots, a_d)$ in Cartesian coordinates. Differentiating (2.3.7) gives

$$\begin{aligned}\ddot{q}_i &= \dot{p}_i - \dot{a}_i \\ &= -(p_j - a_j)\left(-\frac{\partial a_j}{\partial q_i}\right) - \frac{\partial V}{\partial q_i} - \frac{\partial a_i}{\partial q_j}\dot{q}_j - \frac{\partial a_i}{\partial t}\end{aligned}$$

Setting $i = 1$ helps to clarify what is happening,

$$\begin{aligned}\ddot{q}_1 &= \dot{q}_1\frac{\partial a_1}{\partial q_1} + \dot{q}_2\frac{\partial a_2}{\partial q_1} + \dot{q}_3\frac{\partial a_3}{\partial q_1} - \frac{\partial V}{\partial q_1} - \frac{\partial a_1}{\partial q_1}\dot{q}_1 - \frac{\partial a_1}{\partial q_2}\dot{q}_2 - \frac{\partial a_1}{\partial q_3}\dot{q}_3 - \frac{\partial a_1}{\partial t} \\ &= \dot{q}_2\frac{\partial a_2}{\partial q_1} + \dot{q}_3\frac{\partial a_3}{\partial q_1} - \frac{\partial a_1}{\partial q_2}\dot{q}_2 - \frac{\partial a_1}{\partial q_3}\dot{q}_3 - \frac{\partial V}{\partial q_1} - \frac{\partial a_1}{\partial t} \\ &= -\{\text{curl } a \wedge \dot{q}\}_1 - (\nabla V)_1 - \frac{\partial a_1}{\partial t},\end{aligned}$$

where curl a = curl $a(q_s, s)$. Hence we see that

$$\ddot{q}_i = -(\text{curl } a(q,t) \wedge \dot{q})_i - (\nabla V(q))_i - \frac{\partial a_i}{\partial t}(q,t),$$
$$= -(B(q,t) \wedge \dot{q})_i - (\nabla V(q))_i - \frac{\partial a_i}{\partial t}(q,t).$$

This nothing but

$$\ddot{\Phi}_s \cdot = -B(\Phi_s \cdot, s) \wedge \dot{\Phi}_s \cdot -\nabla V(\Phi_s \cdot) - \frac{\partial a}{\partial s}(\Phi_s \cdot, s).$$

For the moment let us note that defining the classical mechanical flow Φ_s as in (2.2.1) implies that $S(x,t)$ must satisfy the stochastic Hamilton-Jacobi equation (see Truman and Zhao [36]). In what follows we concentrate on proving the continuity equation (2.3.3).

We observe, from the first part of the proof that Φ_s corresponds to the classical Hamiltonian $H(q,p,t)$ as defined by (2.3.6). Let $S_0(x,t) = \widetilde{S}_0(\Phi_t^{-1}x, t)$ where $\widetilde{S}_0(y,t)$ is defined by (2.2.2).

Consider

$$S_0(y,s) = S_0(x_0(y,s)) + \int_0^s \mathcal{L}(X(u), \dot{X}(u), u)du, \qquad (2.3.9)$$

for time s, such that $0 \leq s \leq T$ where the Lagrangian $\mathcal{L}$ is given by

$$\mathcal{L}(X(s), \dot{X}(s), s) = 2^{-1}\dot{X}^2(s) + \dot{X}(s).a(X(s)) - V(X(s)), \qquad (2.3.10)$$

with $y = \Phi_s x_0 = X(s)$ and $x_0 = x_0(x,s)$. Now differentiating (2.3.9) and using the chain rule gives

$$\frac{\partial S_0}{\partial y}(y,s) = \frac{\partial S_0}{\partial x_0}(x_0(y,s))\frac{\partial x_0}{\partial y}(y,s) + \int_0^s \left(\frac{\partial \mathcal{L}}{\partial X}\frac{\partial X}{\partial y} + \frac{\partial \mathcal{L}}{\partial \dot{X}}\frac{\partial \dot{X}}{\partial y}\right) du.$$

Since $\frac{\partial \dot{x}}{\partial y} = \frac{d}{ds}\frac{\partial X}{\partial y}$ integration by parts on the last term yields

$$\frac{\partial S_0}{\partial y}(y,s) = \frac{\partial S_0}{\partial x_0}(x_0(y,s))\frac{\partial x_0}{\partial y}(y,s) + \left[\frac{\partial \mathcal{L}}{\partial \dot{X}}(X(u), \dot{X}(u), u)\frac{\partial X}{\partial y}(u)\right]_{u=0}^{u=s}$$
$$= \frac{\partial \mathcal{L}}{\partial \dot{X}}(X(s), \dot{X}(s), s)\frac{\partial X}{\partial y}(s)$$
$$= \dot{X}(s) + a(X(s), s).$$

A simple rearrangement gives

$$\dot{X}(s) = \frac{\partial S_0}{\partial y}(y,s) - a(y,s). \qquad (2.3.11)$$

Recall, from the definition and the first part of the proof, that the diffeomorphism Φ_s satisfies $\dot{\Phi}_s q_0 = p - a(\Phi_s q, s)$ and $\ddot{\Phi}_s = -B \wedge \dot{\Phi}_s - \nabla V - \frac{\partial a}{\partial s}$, with $\Phi_0 x = x$ and $\dot{\Phi}_0(x) =$

$(\nabla S_0 - a)$. A restatement of equation (2.3.11) allows us to say for the diffeomorphism Φ_s, for $0 \leq s \leq T$, satisfies

$$\dot{\Phi}_s x_0 = \frac{\partial S_0}{\partial y}(\Phi_s x_0, s) - a(\Phi_s x_0, s). \tag{2.3.12}$$

Next, define the Jacobian determinant $J(x,t)$ as

$$J(x,t) = (\mathrm{Det} \nabla_x \Phi_t^{-1} x) = \mathrm{Det}\left(\frac{\partial x_0}{\partial x}\right) \tag{2.3.13}$$

and a vector field $j \in \mathbb{R}^4 = \Gamma \times \mathbb{R}$, where Γ is phase space, by

$$j(x,t) = T_0^2(x_0(x,t)) J(x,t) \left(\nabla S_0(x,t) - a(x,t), 1\right), \tag{2.3.14}$$

where $x_0(x,t) = \Phi_t^{-1} x$ and the final coordinate is the time coordinate t. Consider the surface Σ generated by the diffeomorphism (flow) Φ_s acting on a volume $\mathcal{V}$, see Figure (2.1).

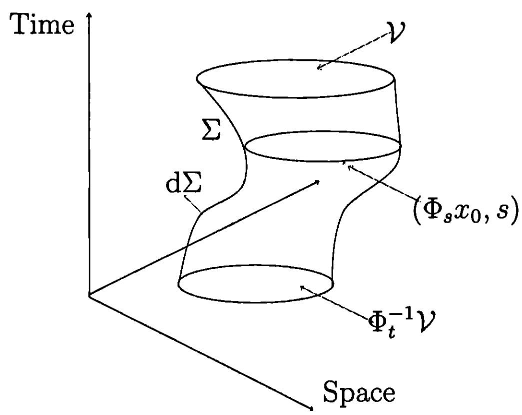


Figure 2.1: Diffeomorphism Φ_s on $\mathbb{R}^4$.

Then it follows from equations (2.3.12) and (2.3.14), that in the time-slice volume corresponding to time s, $j \perp d\Sigma$. Hence, the "surface integral" $\int\int\int j.d\Sigma$ consists of two terms, the first of which comes from $\mathcal{V}$ and the second from $\Phi_t^{-1}\mathcal{V}$ that is

$$\int_\Sigma = \int\int\int_{x\in\mathcal{V}} j \cdot \mathrm{d}\Sigma - \int\int\int_{\Phi_t^{-1}\mathcal{V}} j \cdot \mathrm{d}\Sigma.$$

By a change of variables together with the fact that $x_0(x,0) = x_0$ these are shown to cancel i.e.

$$\int\int\int_{x\in\mathcal{V}} T_0^2(x_0(x,t)) \frac{\partial x_0}{\partial x} dx - \int\int\int_{\Phi_t^{-1}\mathcal{V}} T_0^2(x_0) dx_0 = 0,$$

It follows that

$$\mathrm{div} j \equiv 0$$

and this reduces to the continuity equation (2.3.3).

□

Lemma 2.3.2 (Iterated Continuity Equations). *For a.e $\omega \in \Omega$, $\psi_j(x,t)$ defined by (2.3.5) satisfies the following iterated continuity equations for $0 \le t \le T(\omega)$:*

$$\frac{\partial \psi_j}{\partial t} + (\nabla \psi_j) \cdot (\nabla S_0(x,t) - a(x,t)) = -\frac{1}{2}\psi_j \Delta S_0(x,t) + \Delta(\psi_{j-1}) \tag{2.3.15}$$

for $j = 0, 1, 2, \ldots$, with the convention $\psi_{-1} \equiv 0$.

Proof. Differentiate the identity $\Phi_t(\Phi_t^{-1}x) = x$ with respect to x and t to obtain for $y = \Phi_t^{-1}x$

$$\left(\nabla_y \Phi_t(\Phi_t^{-1}x)\right) \nabla \Phi_t^{-1}x = I \tag{2.3.16}$$

and

$$\dot{\Phi}_t(\Phi_t^{-1}x) + \nabla_y \Phi_t(\Phi_t^{-1}x)(\dot{\Phi}_t^{-1}x) = 0. \tag{2.3.17}$$

Multiply both sides of (2.3.17) by $\nabla \Phi_t^{-1}x$ and using (2.3.16) we have

$$\begin{aligned} \nabla \Phi_t^{-1}x \left[\dot{\Phi}_t(\Phi_t^{-1}x) + \nabla_y \Phi_t(\Phi_t^{-1}x)(\dot{\Phi}_t^{-1}x)\right] &= 0, \\ \nabla \Phi_t^{-1}x \left[\dot{\Phi}_t(\Phi_t^{-1}x)\right] + \dot{\Phi}_t^{-1}x &= 0, \end{aligned} \tag{2.3.18}$$

in other words

$$\begin{aligned} \dot{\Phi}_t^{-1}x &= -(\nabla \Phi_t^{-1}x)\left[\dot{\Phi}_t(\Phi_t^{-1}x)\right] \\ &= -(\nabla \Phi_t^{-1}x)\left(\nabla S_0(x,t) - a(x,t)\right). \end{aligned} \tag{2.3.19}$$

Now using the definitions of ψ_j and T_j we have

$$\begin{aligned} &\frac{\partial}{\partial t}\psi_j(x,t) + (\nabla \psi_j) \cdot (\nabla S_0(x,t) - a(x,t)) \\ &= \nabla_y T_j(\Phi_t^{-1}x, t) \cdot (\dot{\Phi}_t^{-1}x)\phi_t^{\frac{1}{2}}(x) \\ &+ \frac{\partial}{\partial t}T_j(\Phi_t^{-1}x, t)\phi_t^{\frac{1}{2}}(x) + T_j(\Phi_t^{-1}x, t)\frac{\partial}{\partial t}\phi_t^{\frac{1}{2}}(x) \\ &+ \left(\nabla \Phi_t^{-1}x\right)^* \nabla_y T_j(\Phi_t^{-1}x, t)\phi_t^{\frac{1}{2}}(x) \cdot (\nabla S_0(x,t) - a(x,t)) \\ &+ T_j(\Phi_t^{-1}x, t)\nabla \phi_t^{\frac{1}{2}}(x) \cdot (\nabla S_0(x,t) - a(x,t)). \end{aligned}$$

Using equation (2.3.19) and the definition of ψ_{j-1} we find that

$$\begin{aligned}
&\frac{\partial}{\partial t}\psi_j(x,t) + (\nabla\psi_j)\cdot(\nabla S_0(x,t) - a(x,t))\\
&= \phi^{\frac{1}{2}}(x,t)\nabla_y T_j(\Phi_t^{-1}x,t)\cdot\left[\Phi_t^{-1}x + (\nabla\Phi_t^{-1}x)(\nabla S_0(x,t) - a(x,t))\right]\\
&+ T_j(\Phi_t^{-1}x,t)\left(\frac{\partial}{\partial t}\phi_t^{\frac{1}{2}}(x) + \nabla\phi_t^{\frac{1}{2}}(x)\cdot(\nabla S_0(x,t) - a)\right)\\
&+ \Delta\left(\phi_t^{\frac{1}{2}}(x)T_{j-1}(\Phi_t^{-1}(x,t))\right),\\
&= T_j(\Phi_t^{-1}x,t)\left(\frac{\partial}{\partial t}\phi_t^{\frac{1}{2}}(x) + \nabla\phi_t^{\frac{1}{2}}(x)\cdot(\nabla S_0(x,t) - a(x,t))\right)\\
&+ \Delta\left(\phi_t^{\frac{1}{2}}(x)T_{j-1}(\Phi_t^{-1}(x,t))\right),\\
&= T_j(\Phi_t^{-1}x,t)\left(\frac{\partial}{\partial t}\phi_t^{\frac{1}{2}}(x) + \nabla\phi_t^{\frac{1}{2}}(x)\cdot(\nabla S_0(x,t) - a(x,t))\right)\\
&+ \Delta\left(\psi_{j-1}(x,t)\right).
\end{aligned} \tag{2.3.20}$$

From the continuity equation (2.3.3) we immediately have

$$\begin{aligned}
\frac{\partial}{\partial t}\phi^{\frac{1}{2}} &= \frac{1}{2}\phi^{-\frac{1}{2}}\frac{\partial}{\partial t}\phi,\\
&= -\frac{1}{2}\phi^{-\frac{1}{2}}\mathrm{div}(\phi(\nabla S_0(x,t) - a(x,t))),\\
&= -\frac{1}{2}\phi^{-\frac{1}{2}}\left(\,\mathrm{div}(\phi\nabla S_0(x,t)) - \mathrm{div}(\phi a(x,t))\right),\\
&= -\frac{1}{2}\phi^{-\frac{1}{2}}\left(\phi\nabla(\nabla S_0(x,t)) + \nabla\phi\cdot\nabla S_0(x,t)\right)\\
&= -\frac{1}{2}\phi^{-\frac{1}{2}}\left(\phi\Delta S_0(x,t) + \nabla\phi\cdot\nabla S_0(x,t) - \nabla\phi\cdot a(x,t)\right).
\end{aligned}$$

Finally substituting this expression into (2.3.20) yields

$$\begin{aligned}
&\frac{\partial}{\partial t}\psi(x,t) + (\nabla\psi_j)\cdot(\nabla S_0(x,t) - a(x,t))\\
&= \psi_j\phi^{-\frac{1}{2}}(x)\Big(-\frac{1}{2}\phi^{-\frac{1}{2}}\left(\phi\Delta S_0(x,t) + \nabla\phi.\nabla S_0(x,t) - \nabla\phi\cdot a(x,t)\right)\\
&+ \nabla\phi_t^{\frac{1}{2}}(x)\cdot(\nabla S_0(x,t) - a(x,t))\Big) + \Delta\left(\psi_{j-1}(x,t)\right),\\
&= -\frac{1}{2}\psi_j\Delta S_0(x,t) + \Delta(\psi_{j-1})
\end{aligned}$$

as required. □

This result has the following important corollary

Corollary 2.3.3.

$$\frac{\partial}{\partial t}\left(\log\sum_{j=0}^{m}\frac{1}{2^j}\mu^{2j}\psi_j(x,t)\right)+\nabla\left(\log\sum_{j=0}^{m}\frac{1}{2^j}\mu^{2j}\psi_j(x,t)\right)\cdot(\nabla S_0(x,t)-a(x,t))$$
$$=-\frac{1}{2}\triangle S_0(x,t)+\frac{1}{2}\mu^2\frac{\triangle\left(\sum_{j=0}^{m-1}\frac{1}{2^j}\mu^{2j}\psi_j(x,t)\right)}{\left(\sum_{j=0}^{m}\frac{1}{2^j}\mu^{2j}\psi_j(x,t)\right)}. \qquad (2.3.21)$$

Proof. Consider the following linear combination of ψ_j

$$\frac{\partial}{\partial t}\left(\sum_{j=0}^{m}\frac{1}{2^j}\mu^{2j}\psi_j(x,t)\right)+\nabla\left(\sum_{j=0}^{m}\frac{1}{2^j}\mu^{2j}\psi_j(x,t)\right)\cdot(\nabla S_0(x,t)-a(x,t))$$
$$=-\frac{1}{2}\left(\sum_{j=0}^{m}\frac{1}{2^j}\mu^{2j}\psi_j(x,t)\right)\triangle S_0(x,t)+\frac{1}{2}\mu^2\triangle\left(\sum_{j=0}^{m-1}\frac{1}{2^j}\mu^{2j}\psi_j(x,t)\right),$$

on division by $\sum_{j=0}^{m}\frac{1}{2^j}\mu^{2j}\psi_j(x,t)$ the result follows. □

Let $v^\mu(x,t)=\nabla S^\mu(x,t)-a(x,t)$, $v\in C(\mathbb{R}^d\times\mathbb{R}^1)$, be a solution of the stochastic viscous Burgers equation with vorticity and appropriate initial conditions. Then $S^\mu(x,t)$ satisfies the stochastic Hamilton-Jacobi equation

$$dS^\mu(x,t)+\frac{1}{2}|\nabla S^\mu(x,t)-a(x,t)|^2dt+V(x)dt+\varepsilon k(x,t)dW_t=\frac{1}{2}\mu^2\triangle S^\mu(x,t)dt.$$

If we formally write $S^\mu=S^\mu(x,t)$, where

$$S^\mu\sim\sum_{j=0}^{\infty}\mu^{2j}S_j, \qquad (2.3.22)$$

then from the last equation, by equating coefficients of powers of μ^{2j} we arrive at the following theorem that gives the solutions of the iterated Hamilton-Jacobi continuity equations. Observe that the first equation will be the corresponding classical Hamilton-Jacobi equation for a vector potential.

Theorem 2.3.4 (Iterated Hamilton-Jacobi continuity equations). *Assume that $V\in C^2(\mathbb{R}^d)$, $a\in C_0^\infty(\mathbb{R}^d\times\mathbb{R}^1)$, $div(a)=0$, $k\in C^2(\mathbb{R}^d\times\mathbb{R}^1)$ and $S(\cdot,0)\in C^2(\mathbb{R}^d)$, Φ_s defined by* (2.2.1) *satisfies a no caustic condition for $0\le t\le T(\omega)$, then for a.e. $\omega\in\Omega$, the solutions of the following Hamilton-Jacobi continuity equations:*

$$\dot{S}_j+\frac{1}{2}\sum_{\substack{i_1,i_2\ge 0,\\ i_1+i_2=j}}\nabla S_{i_1}\cdot\nabla S_{i_2}-a\cdot\nabla S_j=\frac{1}{2}\triangle S_{j-1}, \qquad (2.3.23)$$

for $j\ge 0$, with the shorthand convention that $\frac{1}{2}\triangle S_{-1}=-\left(V(x)+\frac{|a(x,t)|^2}{2}+\varepsilon k(x,t)\dot{W}_t\right)$, for $0\le t\le T(\omega)$ are given by S_0 defined by (2.2.3), *$S_1(x,t)=-\log\psi_0(x,t)$ and for*

$j \geq 2$

$$S_j(x,t) = \frac{1}{2^{j-1}}\left(-\frac{\psi_{j-1}}{\psi_0} + \frac{\sum_{\substack{i_1,i_2\geq 1,\\ i_1+i_2=j-1}} \psi_{i_1}\psi_{i_2}}{2\psi_0^2} - \frac{\sum_{\substack{i_1,i_2,i_3\geq 1,\\ i_1+i_2+i_3=j-1}} \psi_{i_1}\psi_{i_2}\psi_{i_3}}{3\psi_0^3}\right.$$
$$\left. +\ldots + (-1)^{j-1}\frac{\psi_1^{j-1}}{(j-1)\psi_0^{j-1}}\right)(x,t). \tag{2.3.24}$$

Proof. For clarity suppress the space-time dependance of variables. Then for $j = 0$ the left hand side of (2.3.23) reads

$$\begin{aligned}
&\dot{S}_0 + \frac{1}{2}\nabla S_0 \cdot \nabla S_0 - a \cdot \nabla S_0 \\
&= \dot{S}_0 + \frac{1}{2}|\nabla S_0 + a|^2 + V + \varepsilon k \dot{W}_t - \left(\frac{|a|^2}{2} + V + \varepsilon k \dot{W}_t\right) \\
&= 0 - \left(\frac{|a|^2}{2} + V + \varepsilon k \dot{W}_t\right) \\
&= \frac{1}{2}\triangle S_{-1}
\end{aligned}$$

For $j = 1$ we begin by dividing both sides of the continuity equation (2.3.15) for ψ_0 by $-(\psi_0)$ to obtain

$$-\frac{1}{\psi_0}\frac{\partial}{\partial t}\psi_0 - \frac{1}{\psi_0}\nabla\psi_0 \cdot (\nabla S_0 - a) = \frac{1}{2}\Delta S_0 - \frac{\Delta\psi_{-1}}{\psi_0}.$$

This is nothing but

$$\frac{\partial}{\partial t}(-\log\psi_0) + \nabla(-\log\psi_0) \cdot (\nabla S_0 - a) = \frac{1}{2}\Delta S_0.$$

Recall that $S_1(x,t) = -\log\psi_0(x,t)$ so that

$$\frac{\partial}{\partial t}S_1 + \nabla S_1 \cdot \nabla S_0 - a \cdot \nabla S_1 = \frac{1}{2}\Delta S_0$$

as required. For $j = 2$ multiply the iterated continuity equation for ψ_1 by ψ_0 and the continuity equation for ψ_0 by ψ_1 to obtain

$$\psi_0\frac{\partial}{\partial t}\psi_1 + \psi_0\nabla\psi_1 \cdot (\nabla S_0 - a) = -\frac{1}{2}\psi_0\psi_1\Delta S_0 + \psi_0\Delta(\psi_0) \tag{2.3.25}$$

and

$$\psi_1\frac{\partial}{\partial t}\psi_0 + \psi_1\nabla\psi_0 \cdot (\nabla S_0 - a) = -\frac{1}{2}\psi_1\psi_0\Delta S_0. \tag{2.3.26}$$

Now if we subtract (2.3.26) from (2.3.25) and multiply through by $-\frac{1}{2\psi_0^2}$ we get

$$\frac{\partial}{\partial t}(-\frac{\psi_1}{2\psi_0}) + \nabla(-\frac{\psi_1}{2\psi_0}) \cdot (\nabla S_0 - a) = -\frac{\Delta(\psi_0)}{2\psi_0}. \tag{2.3.27}$$

By definition $S_1 := -\log \psi_0(x,t)$, which implies

$$\nabla S_1 = -\frac{\nabla \psi_0}{\psi_0}$$

and

$$\Delta S_1 = \left(\frac{\nabla \psi_0}{\psi_0}\right)^2 - \frac{\Delta \psi_0}{\psi_0}.$$

Consequently

$$\begin{aligned} \frac{1}{2}\Delta(S_1) + \frac{1}{2}(\nabla S_1)^2 &= \frac{1}{2}\left(\frac{\nabla \psi_0}{\psi_0}\right)^2 - \frac{\Delta \psi_0}{2\psi_0} - \frac{1}{2}\left(\frac{\nabla \psi_0}{\psi_0}\right)^2 \\ &= -\frac{\Delta \psi_0}{2\psi_0}, \end{aligned} \tag{2.3.28}$$

substituting (2.3.28) into (2.3.27) we obtain

$$\frac{\partial}{\partial t}\left(-\frac{\psi_1}{2\psi_0}\right) + \nabla\left(-\frac{\psi_1}{2\psi_0}\right) \cdot (\nabla S_0 - a) = \frac{1}{2}\Delta(S_1) - \frac{1}{2}(\nabla S_1)^2.$$

However, this is just

$$\frac{\partial}{\partial t} S_2 + \nabla S_2 \cdot (\nabla S_0 - a) = \frac{1}{2}\Delta S_1 - \frac{1}{2}(\nabla S_1)^2,$$

which rearranges to give the desired result

$$\frac{\partial}{\partial t} S_2 + \nabla S_2 \cdot \nabla S_0 + \frac{1}{2}\nabla S_1 \cdot \nabla S_1 - a \cdot \nabla S_2 = \frac{1}{2}\Delta S_1.$$

The method of proof for the case of an integer $j \geq 3$, although similar to the method previously undertaken, is quite involved and cumbersome. To aid the reader in following this part of the proof we list the systematic steps for obtaining the result.

1. Step 1: Having proved the result for $j = 0, 1, 2$, we concentrate, firstly, on the case where $i_1, i_2 \geq 1$ with $i_1 + i_2 = j - 1$. This ultimately leads to an alternate form to the expression
$$\frac{\partial}{\partial t} S_j + \nabla S_j \cdot (\nabla S_0 - a).$$

2. Step 2: We concentrate on simplifying the main body of the sum, namely
$$\sum_{\substack{i_1, i_2 \geq 2, \\ i_1 + i_2 = j}} \nabla S_{i_1} \cdot \nabla S_{i_2},$$
notice that this sum does not include the terms $i_1 = 1$ and $i_2 = 1$.

3. Step 3: We next find an answer to the remaining case needed to complete the desired limits of the summation term, i.e. when $i_1 = 1$ and $i_2 = j-1$ and the respective symmetric case $i_2 = 1$ and $i_1 = j-1$. For this we compute
$$\nabla S_1 \cdot \nabla S_{j-1}.$$
4. Step 4: We add the three quantities, together with $a \cdot \nabla S_j$, in their appropriate ratios, to obtain the desired result.

Step 1

To succeed in obtaining the first expression we begin by multiplying the continuity equation for ψ_0 by ψ_{j-1} and the iterated continuity equation for ψ_{j-1} by ψ_0 to obtain

$$\psi_{j-1}\frac{\partial}{\partial t}\psi_0 + \psi_{j-1}\nabla\psi_0 \cdot (\nabla S_0 - a) = -\frac{1}{2}\psi_{j-1}\psi_0\Delta S_0 \tag{2.3.29}$$

and

$$\psi_0\frac{\partial}{\partial t}\psi_{j-1} + \psi_0\nabla\psi_{j-1} \cdot (\nabla S_0 - a) = -\frac{1}{2}\psi_0\psi_{j-1}\Delta S_0 + \psi_0\Delta(\psi_{j-2}). \tag{2.3.30}$$

In a similar manner to before, taking (2.3.29) from (2.3.30) and multiplying through by $-\frac{1}{2^{j-1}\psi_0^2}$ gives

$$\frac{\partial}{\partial t}(-\frac{\psi_{j-1}}{2^{j-1}\psi_0}) + \nabla(-\frac{\psi_{j-1}}{2^{j-1}\psi_0}) \cdot (\nabla S_0 - a) = -\frac{\Delta(\psi_{j-2})}{2^{j-1}\psi_0}. \tag{2.3.31}$$

Now for any $i_1, i_2 \geq 1$ with $i_1 + i_2 = j-1$ multiply the iterated continuity equation for ψ_{i_1} by $\psi_0^2\psi_{i_2}$ and the iterated continuity equation for ψ_{i_2} by $\psi_0^2\psi_i$ we find

$$\psi_0^2\psi_{i_2}\frac{\partial}{\partial t}\psi_{i_1} + \psi_0^2\psi_{i_2}\nabla\psi_{i_1} \cdot (\nabla S_0 - a) = -\frac{1}{2}\psi_0^2\psi_{i_2}\psi_{i_1}\Delta S_0 + \psi_0^2\psi_{i_2}\nabla(\psi_{i_1-1}) \tag{2.3.32}$$

and

$$\psi_0^2\psi_{i_1}\frac{\partial}{\partial t}\psi_{i_2} + \psi_0^2\psi_{i_1}\nabla\psi_{i_2} \cdot (\nabla S_0 - a) = -\frac{1}{2}\psi_0^2\psi_{i_1}\psi_{i_2}\Delta S_0 + \psi_0^2\psi_{i_1}\nabla(\psi_{i_2-1}). \tag{2.3.33}$$

By adding (2.3.32) to (2.3.33) it turns out that

$$\begin{aligned}&\psi_0^2\frac{\partial}{\partial t}(\psi_{i_1}\psi_{i_2}) + \psi_0^2\nabla(\psi_{i_1}\psi_{i_2}) \cdot (\nabla S_0 - a)\\ &= -\frac{1}{2}\psi_0^2 \times 2(\psi_{i_1}\psi_{i_2})\Delta S_0 + \psi_0^2\left(\psi_{i_1}\nabla(\psi_{i_2-1}) + \psi_{i_2}\nabla(\psi_{i_1-1})\right).\end{aligned}$$

By symmetry of the indexes i_1 and i_2 we have that

$$\begin{aligned}&\psi_0^2\frac{\partial}{\partial t}\left(\sum_{\substack{i_1+i_2=j-1,\\ i_1,i_2\geq 1}}\psi_{i_1}\psi_{i_2}\right) + \psi_0^2\nabla\left(\sum_{\substack{i_1+i_2=j-1,\\ i_1,i_2\geq 1}}\psi_{i_1}\psi_{i_2}\right) \cdot (\nabla S_0 - a)\\ &= -\psi_0^2\left(\sum_{\substack{i_1+i_2=j-1,\\ i_1,i_2\geq 1}}\psi_{i_1}\psi_{i_2}\right)\Delta S_0 + 2\psi_0\left(\sum_{\substack{i_1+i_2=j-1,\\ i_1,i_2\geq 1}}\psi_{i_1}\Delta(\psi_{i_2-1})\right).\end{aligned} \tag{2.3.34}$$

However, if we now multiply the continuity equation for ψ_0 by

$$2\psi_0 \sum_{\substack{i_1+i_2=j-1,\\ i_1,i_2\geq 1}} \psi_{i_1}\psi_{i_2}$$

we have

$$\left(\sum_{\substack{i_1+i_2=j-1,\\ i_1,i_2\geq 1}} \psi_{i_1}\psi_{i_2}\right)\frac{\partial}{\partial t}\psi_0^2 + \left(\sum_{\substack{i_1+i_2=j-1,\\ i_1,i_2\geq 1}} \psi_{i_1}\psi_{i_2}\right)\psi_{j-1}\nabla\psi_0^2\cdot(\nabla S_0 - a)$$
$$= -\psi_0^2\left(\sum_{\substack{i_1+i_2=j-1,\\ i_1,i_2\geq 1}} \psi_{i_1}\psi_{i_2}\right)\Delta S_0. \tag{2.3.35}$$

It follows from (2.3.34) and (2.3.35) that

$$\frac{\partial}{\partial t}\left(\frac{\sum_{\substack{i_1+i_2=j-1,\\ i_1,i_2\geq 1}} \psi_{i_1}\psi_{i_2}}{2^{j-1}\times 2\psi_0^2}\right) + \nabla\left(\frac{\sum_{\substack{i_1+i_2=j-1,\\ i_1,i_2\geq 1}} \psi_{i_1}\psi_{i_2}}{2^{j-1}\times 2\psi_0^2}\right)\cdot(\nabla S_0 - a)$$
$$= \frac{\sum_{\substack{i_1+i_2=j-1,\\ i_1,i_2\geq 1}} \psi_{i_1}\Delta(\psi_{i_2-1})}{2^{j-1}\psi_0^2} \tag{2.3.36}$$

Using the same method we can prove that

$$\frac{\partial}{\partial t}\left(-\frac{\sum_{\substack{i_1+i_2+i_3=j-1,\\ i_1,i_2,i_3\geq 1}} \psi_{i_1}\psi_{i_2}\psi_{i_3}}{2^{j-1}\times 3\psi_0^3}\right) + \nabla\left(-\frac{\sum_{\substack{i_1+i_2+i_3=j-1,\\ i_1,i_2,i_3\geq 1}} \psi_{i_1}\psi_{i_2}\psi_{i_3}}{2^{j-1}\times 3\psi_0^3}\right)\cdot(\nabla S_0 - a)$$
$$= \frac{-\sum_{\substack{i_1+i_2+i_3=j-1,\\ i_1,i_2,i_3\geq 1}} \psi_{i_1}\psi_{i_2}\Delta(\psi_{i_3-1})}{2^{j-1}\psi_0^3}.$$

$$\vdots$$

$$\frac{\partial}{\partial t}\left((-1)^{j-1}\frac{\psi_1^{j-1}}{2^{j-1}\times(j-1)\psi_0^{j-1}}\right) + \nabla\left((-1)^{j-1}\frac{\psi_1^{j-1}}{2^{j-1}\times(j-1)\psi_0^{j-1}}\right)\cdot(\nabla S_0 - a)$$
$$= (-1)^{j-1}\frac{\psi_1^{j-2}\Delta(\psi_0)}{2^{j-1}\psi_0^{j-1}}. \tag{2.3.37}$$

Using equations (2.3.31), (2.3.36) and (2.3.37) this is nothing but

$$\frac{\partial}{\partial t}S_j + \nabla S_j\cdot(\nabla S_0 - a)$$
$$= \frac{1}{2^{j-1}}\left(-\frac{\Delta(\psi_{j-2})}{\psi_0} + \frac{\sum_{\substack{i_1+i_2=j-1,\\ i_1,i_2\geq 1}} \psi_{i_1}\Delta(\psi_{i_2-1})}{\psi_0^2} - \frac{\sum_{\substack{i_1+i_2+i_3=j-1,\\ i_1,i_2,i_3\geq 1}} \psi_{i_1}\psi_{i_2}\Delta(\psi_{i_3-1})}{\psi_0^3}\right.$$
$$\left. + \ldots + (-1)^{j-1}\frac{\psi_1^{j-2}\Delta\psi_0}{\psi_0^{j-1}}\right).$$

Which leads to

$$\frac{\partial}{\partial t}S_j + \nabla S_j \cdot (\nabla S_0 - a)$$
$$= \frac{1}{2^{j-1}}\left(-\frac{\Delta(\psi_{j-2})}{\psi_0} + \frac{\psi_{j-2}\Delta\psi_0}{\psi_0^2} + \frac{\sum_{\substack{i_1+i_2=j-2,\\ i_1,i_2\geq 1}} \psi_{i_1}\Delta(\psi_{i_2})}{\psi_0^2} \right.$$
$$- \frac{\sum_{\substack{i_1+i_2=j-2,\\ i_1,i_2\geq 1}} \psi_{i_1}\psi_{i_2}\Delta\psi_0}{\psi_0^3} - \frac{\sum_{\substack{i_1+i_2+i_3=j-2,\\ i_1,i_2,i_3\geq 1}} \psi_{i_1}\psi_{i_2}\Delta(\psi_{i_3})}{\psi_0^3}$$
$$\left. + \ldots + (-1)^{j-2}\frac{\psi_1^{j-3}\Delta\psi_1}{\psi_0^{j-2}} + (-1)^{j-1}\frac{\psi_1^{j-2}\Delta\psi_0}{\psi_0^{j-1}} \right). \tag{2.3.38}$$

<u>Step 2</u>

Differentiating S_{i_1} for $i_1 \geq 2$ with respect to the space variables x gives

$$\nabla S_{i_1}(x,t) = \frac{1}{2^{i_1-1}}\left(-\frac{\nabla\psi_{i_1-1}}{\psi_0} + \frac{\psi_{i_1-1}\nabla\psi_0}{\psi_0^2} + \frac{\sum_{\substack{j_1,j_2\geq 1,\\ j_1+j_2=i_1-1}} \nabla(\psi_{j_1})\psi_{j_2}}{\psi_0^2} \right.$$
$$- \frac{\sum_{\substack{j_1,j_2\geq 1,\\ j_1+j_2=i_1-1}} \psi_{j_1}\psi_{j_2}\nabla(\psi_0)}{\psi_0^3} - \frac{\sum_{\substack{j_1,j_2,j_3\geq 1,\\ j_1+j_2+j_3=i_1-1}} \nabla(\psi_{j_1})\psi_{j_2}\psi_{j_3}}{\psi_0^3}$$
$$+ \frac{\sum_{\substack{j_1,j_2,j_3\geq 1,\\ j_1+j_2+j_3=i_1-1}} \psi_{j_1}\psi_{j_2}\psi_{j_3}\nabla(\psi_0)}{\psi_0^4} + \ldots$$
$$\left. +(-1)^{i_1-1}\frac{\psi_1^{i_1-2}\nabla\psi_1}{\psi_0^{i_1-1}} - (-1)^{i_1-1}\frac{\psi_1^{i_1-1}\nabla\psi_0}{\psi_0^{i_1}} \right).$$

Similarly for $i_2 \geq 2$,

$$\nabla S_{i_2}(x,t) = \frac{1}{2^{i_2-1}}\left(-\frac{\nabla\psi_{i_2-1}}{\psi_0} + \frac{\psi_{i_2-1}\nabla\psi_0}{\psi_0^2} + \frac{\sum_{\substack{j_1,j_2\geq 1,\\ j_1+j_2=i_2-1}} \nabla(\psi_{j_1})\psi_{j_2}}{\psi_0^2} \right.$$
$$- \frac{\sum_{\substack{j_1,j_2\geq 1,\\ j_1+j_2=i_2-1}} \psi_{j_1}\psi_{j_2}\nabla(\psi_0)}{\psi_0^3}$$
$$- \frac{\sum_{\substack{j_1,j_2,j_3\geq 1,\\ j_1+j_2+j_3=i_2-1}} \nabla(\psi_{j_1})\psi_{j_2}\psi_{j_3}}{\psi_0^3}$$
$$+ \frac{\sum_{\substack{j_1,j_2,j_3\geq 1,\\ j_1+j_2+j_3=i_2-1}} \psi_{j_1}\psi_{j_2}\psi_{j_3}\nabla(\psi_0)}{\psi_0^4} + \ldots$$
$$\left. +(-1)^{i_2-1}\frac{\psi_1^{i_2-2}\nabla\psi_1}{\psi_0^{i_2-1}} - (-1)^{i_2-1}\frac{\psi_1^{i_2-1}\nabla\psi_0}{\psi_0^{i_2}} \right).$$

It follows that

$$\sum_{\substack{i_1,i_2\geq 2,\\ i_1+i_2=j}} \nabla S_{i_1}\cdot\nabla S_{i_2}$$
$$= \frac{1}{2^{j-2}}\sum_{\substack{i_1,i_2\geq 2,\\ i_1+i_2=j}}\Bigg(\frac{\nabla\psi_{i_1-1}\nabla\psi_{i_2-1}}{\psi_0^2} - \frac{(\nabla\psi_{i_1-1}\psi_{i_2-1}+\psi_{i_1-1}\nabla\psi_{i_2-1})\nabla\psi_0}{\psi_0^3}$$
$$+\frac{\psi_{i_1-1}\psi_{i_2-1}(\nabla\psi_0)^2}{\psi_0^4}$$
$$-\frac{\nabla\psi_{i_1-1}\sum_{\substack{j_1,j_2\geq 1,\\ j_1+j_2=i_2-1}}\nabla(\psi_{j_1})\psi_{j_2}+\nabla\psi_{i_2-1}\sum_{\substack{j_1,j_2\geq 1,\\ j_1+j_2=i_1-1}}\nabla(\psi_{j_1})\psi_{j_2}}{\psi_0^3}$$
$$+\ldots+(-1)^{i_1+i_2-2}\frac{\psi_1^{i_1+i_2-4}(\nabla\psi_1)^2}{\psi_0^{i_1+i_2-2}} - (-1)^{i_1+i_2-2}\frac{2\psi_1^{i_1+i_2-3}\nabla\psi_1\nabla\psi_0}{\psi_0^{i_1+i_2-1}}$$
$$+(-1)^{i_1+i_2-2}\frac{\psi_1^{i_1+i_2-2}(\nabla\psi_0)^2}{\psi_0^{i_1+i_2}}\Bigg). \tag{2.3.39}$$

Step 3

Note that it immediately follows from the definition of S_1 that $\nabla S_1 = -\frac{\nabla\psi_0}{\psi_0}$. Consequently this means

$$\nabla S_1\cdot\nabla S_{j-1} - = \frac{1}{2^{j-2}}\Bigg(-\frac{\nabla\psi_{j-2}\nabla\psi_0}{\psi_0^2}+\frac{\psi_{j-2}(\nabla\psi_0)^2}{\psi_0^3}$$
$$+\frac{\sum_{\substack{j_1,j_2\geq 1,\\ j_1+j_2=j-2}}\nabla(\psi_{j_1})\psi_{j_2}\nabla\psi_0}{\psi_0^3} - \frac{\sum_{\substack{j_1,j_2\geq 1,\\ j_1+j_2=j-2}}\psi_{j_1}\psi_{j_2}(\nabla\psi_0)^2}{\psi_0^4}$$
$$+\ldots+(-1)^{j-2}\frac{\psi_1^{j-3}\nabla\psi_1\nabla\psi_0}{\psi_0^{j-1}} - (-1)^{j-2}\frac{\psi_1^{j-2}(\nabla\psi_0)^2}{\psi_0^{j}}\Bigg). \tag{2.3.40}$$

Step 4

Finally using (2.3.38), (2.3.39) and (2.3.40)

$$
\begin{aligned}
&\dot{S}_j + \frac{1}{2} \sum_{\substack{i_1,i_2\geq 0,\\ i_1+i_2=j}} \nabla S_{i_1}\cdot\nabla S_{i_2} - a\cdot\nabla S_j \\
&= \frac{1}{2^{j-1}}\left[-\frac{\triangle(\psi_{j-2})}{\psi_0} + \frac{\psi_{j-2}\triangle\psi_0}{\psi_0^2} + 2\left(\frac{\nabla\psi_{j-2}\nabla\psi_0}{\psi_0^2} - \frac{\psi_{j-2}(\nabla\psi_0)^2}{\psi_0^3}\right)\right. \\
&+ \frac{\sum_{\substack{i_1,i_2\geq 1,\\ i_1+i_2=j-2}} \psi_{i_1}\triangle(\psi_{i_2})}{\psi_0^2} - \frac{\sum_{\substack{i_1,i_2\geq 1,\\ i_1+i_2=j-2}} \psi_{i_1}\psi_{i_2}\triangle(\psi_0)}{\psi_0^3} \\
&+ \frac{\sum_{\substack{i_1,i_2\geq 1,\\ i_1+i_2=j-2}} \nabla\psi_{i_1}\nabla\psi_{i_2}}{\psi_0^2} \\
&- \frac{4\times\sum_{\substack{i_1,i_2\geq 1,\\ i_1+i_2=j-2}} (\nabla\psi_{i_1})\psi i_2\nabla\psi_0}{\psi_0^3} + \frac{3\times\sum_{\substack{i_1,i_2\geq 1,\\ i_1+i_2=j-2}} \psi_{i_1}\psi i_2(\nabla\psi_0)^2}{\psi_0^4} \\
&+ \ldots + (-1)^{j-2}\left(\frac{\psi_1^{j-3}\triangle\psi_1}{\psi_0^{j-2}} - \frac{\psi_1^{j-2}\triangle\psi_0}{\psi_0^{j-1}} + \frac{(j-3)\psi_1^{j-4}(\nabla\psi_1)^2}{\psi_0^{j-2}}\right. \\
&\left.\left. \frac{(j-1)\psi_1^{j-2}(\nabla\psi_0)^2}{\psi_0^{j}} - \frac{2(j-2)\psi_1^{j-3}\nabla\psi_1\nabla\psi_0}{\psi_0^{j-1}}\right)\right]
\end{aligned}
$$

A simple straightforward computation shows the right hand side of the above expression equals $\frac{1}{2}\triangle S_{j-1}$. Therefore, for $j \geq 0$,

$$
\dot{S}_j + \frac{1}{2} \sum_{\substack{i_1,i_2\geq 0,\\ i_1+i_2=j}} \nabla S_{i_1}\cdot\nabla S_{i_2} - a\cdot\nabla S_j = \frac{1}{2}\triangle S_{j-1}
$$

□

Corollary 2.3.5. *Assume that $V \in C^2(\mathbb{R}^d)$, $a \in C_0^\infty(\mathbb{R}^d\times\mathbb{R}^1)$, $div(a) = 0$, $k \in C^2(\mathbb{R}^d\times\mathbb{R}^1)$ and $S(\cdot,0) \in C^2(\mathbb{R}^d)$, Φ_s defined by (2.2.1) satisfies a no caustic condition for $0 \leq t \leq T(\omega)$, then for a.e. $\omega \in \Omega$,*

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\sum_{j=0}^{m}\mu^{2j}S_j\right) + \frac{1}{2}\sum_{j=0}^{m}\mu^{2j}\left(\sum_{\substack{i_1,i_2\geq 0,\\ i_1+i_2=j}}\nabla S_{i_1}\cdot\nabla S_{i_2}\right) - a\cdot\nabla\left(\sum_{j=0}^{m}\mu^{2j}S_j\right) + \frac{|a|^2}{2} \\
&+ V + \varepsilon k\dot{W}_t = \frac{1}{2}\mu^2\triangle\left(\sum_{j=0}^{m-1}\mu^{2j}S_j\right) \qquad (2.3.41)
\end{aligned}
$$

Proof. This follows directly from Equation (2.3.23). □

In what proceeds we follow Truman and Zhao's notation in [37] by introducing a stochastic process x_s^μ defined on a probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}})$ by the following stochastic differential equation

$$dx_s^\mu = \mu dB_s - (\nabla S_0(x_s^\mu, t-s) - a(x_s^\mu, t-s))\, ds + \mu^2 \nabla \log \sum_{j=0}^{m} \frac{1}{2^j} \mu^{2j} \psi_j(x_s^\mu, t-s) ds,$$
$$x_0^\mu = x \tag{2.3.42}$$

where B_s is a standard Brownian motion on $\mathbb{R}^d$ on the probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}})$.

Turning to earlier work of Truman and Zhao ([38], [36] and [35]) we observe that x_s^μ is also a process on $(\Omega, \mathcal{F}, \mathbb{P})$. In the endeavour to make the stochastic integral $\int_0^t k(t-s, x_s) dW_{t-s}$ well defined in the Itô sense we denote $W_s^* = W_{t-s}$ and $\mathcal{F}_s^*$ is the enlargement of the filtration $\{\mathcal{F}_s^*\}$, where

$$\mathcal{F}_s^0 = \sigma(W_s^* : r \leq s).$$

Then $W_{t-s} = W_s^*$ is $\mathcal{F}_s^*$ measurable and $\mathcal{F}_{s_1}^* \subset \mathcal{F}_{s_2}^*$ if $s_1 \leq s_2$.

We will use the process x_s^μ, if it is $\mathcal{F}_s^*$ measurable and non-explosive to construct a solution to the following stochastic heat equation of Stratonovich type:

$$\partial u_t^\mu(x) = \left[\frac{1}{2}\mu^2 \triangle u_t^\mu(x) + a(x,t) \cdot \nabla u_t^\mu(x) + \mu^{-2} \left(\frac{a^2(x,t)}{2} + V(x)\right) u_t^\mu(x)\right] dt$$
$$+ \varepsilon \frac{k(x,t)}{\mu^2} u_t^\mu(x) \circ \partial W_t,$$
$$u_0^\mu(x) = T_0(x) e^{-\frac{S(x,0)}{\mu^2}}. \tag{2.3.43}$$

Theorem 2.3.6. *Assume that* $V \in C^2(\mathbb{R}^d)$, $a \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}^1)$, $k \in C^2(\mathbb{R}^d \times \mathbb{R}^1)$ *and* $S(\cdot, 0) \in C^2(\mathbb{R}^d)$, Φ_s *defined by* (2.2.1) *satisfies a no caustic condition for* $0 \leq t \leq T(\omega)$, $\psi_j(x,t)$ *are defined by* (2.3.5) *and the stochastic process defined by* (2.3.42) *is* $\mathcal{F}_s^*$ *measurable and non-explosive. Then for a.e.* $\omega \in \Omega$, $0 \leq t \leq T(\omega)$, *the solution of the heat equation* (2.3.43) *is given by*

$$u_t^\mu(x) = \exp\left(-\frac{S_0(x,t)}{\mu^2}\right) \sum_{j=0}^{m} \frac{1}{2^j} \mu^{2j} \psi_j(x,t)$$
$$\times \widehat{\mathbb{E}} \left\{ \exp\left(\frac{1}{2^{m+1}} \mu^{2(m+1)} \int_0^t \frac{\triangle \left(\psi_m(x_s^\mu, t-s)\right)}{\sum_{j=0}^m \frac{1}{2^j} \mu^{2j} \psi_j(x_s^\mu, t-s)}\right)\right\}$$

Proof. The Radon-Nikodym derivative with respect to a new probability measure $\widehat{\mathbb{P}_1}$ is given by

$$\frac{d\widehat{\mathbb{P}}_1}{d\widehat{\mathbb{P}}} = \mathcal{M}_t^\mu$$
$$= \exp\left\{-\mu^{-1}\int_0^t \Bigg(- (\nabla S_0(x_s^\mu, t-s) - a(x_s^\mu, t-s)) \right.$$
$$+ \mu^2 \nabla \log \sum_{j=0}^m \frac{1}{2^j}\mu^{2j}\psi_j(x_s^\mu, t-s)\Bigg)\cdot dB_s$$
$$\left. -\frac{1}{2\mu^2}\int_0^t \left| -(\nabla S_0(x_s^\mu, t-s) - a(x_s^\mu, t-s)) + \mu^2\nabla\log\sum_{j=0}^m \frac{1}{2^j}\mu^{2j}\psi_j(x_s^\mu, t-s)\right|^2 ds\right\}.$$

Using $\mathbb{E}_{\widehat{\mathbb{P}}_1}$ to denote the expectation with respect to the measure $\widehat{\mathbb{P}_1}$ and simply $\mathbb{E}$ for the measure $\widehat{\mathbb{P}}$. If we define $B_t^\mu = x + \mu B_t$ the Feynman-Kac formula gives

$$u_t^\mu(x) = \mathbb{E}\left[T_0(B_t^\mu)\exp\left\{-\frac{S_0(B_t^\mu)}{\mu^2} + \mu^{-1}\int_0^t a(B_s^\mu, t-s)\cdot dB_s^\mu + \mu^{-2}\int_0^t V(B_s^\mu)ds \right.\right.$$
$$\left.\left. + \mu^{-2}\varepsilon\int_0^t k(B_s^\mu, t-s)dW_{t-s}\right\}\right].$$

For the new probability measure $\widehat{\mathbb{P}_1}$ and the stochastic process x_s^μ, using Girsanov's theorem, we have

$$u_t^\mu(x) = \mathbb{E}_{\widehat{\mathbb{P}}_1}\left[T_0(x_s^\mu)\exp\left\{-\frac{S_0(x_s^\mu)}{\mu^2} + \mu^{-2}\int_0^t a(x_s^\mu, t-s)\cdot dx_s^\mu + \mu^{-2}\int_0^t V(x_s^\mu)ds \right.\right.$$
$$\left.\left. + \mu^{-2}\varepsilon\int_0^t k(x_s^\mu, t-s)dW_{t-s}\right\}\right]$$

and consequently

$$u_t^\mu(x) = \mathbb{E}\left[T_0(x_s^\mu)\exp\left\{-\frac{S_0(x_s^\mu)}{\mu^2} + \mu^{-2}\int_0^t a(x_s^\mu, t-s)\right.\right.$$
$$\times\left(\mu dB_s - (\nabla S_0(x_s^\mu, t-s) - a(x_s^\mu, t-s))ds + \mu^2\nabla\log\sum_{j=0}^m \frac{1}{2^j}\mu^{2j}\psi_j(x_s^\mu, t-s)ds\right)$$
$$\left.\left. + \mu^{-2}\int_0^t V(x_s^\mu)ds + \mu^{-2}\varepsilon\int_0^t k(x_s^\mu, t-s)dW_{t-s}\right\}.\mathcal{M}_t^\mu\right]. \tag{2.3.44}$$

We now concentrate on the exponential stochastic terms, namely

$$\mu^{-1}\int_0^t a(x_s^\mu, t-s)\cdot dB_s$$
$$-\mu^{-1}\int_0^t\left(-(\nabla S_0(x_s^\mu, t-s) - a(x_s^\mu, t-s)) + \mu^2\nabla\log\sum_{j=0}^m \frac{1}{2^j}\mu^{2j}\psi_j(x_s^\mu, t-s)\right)\cdot dB_s$$
$$= \mu^{-1}\left(\nabla S_0(x_s^\mu, t-s) - \mu^2\nabla\log\sum_{j=0}^m\frac{1}{2^j}\mu^{2j}\psi_j(x_s^\mu, t-s)\right)\cdot dB_s.$$

In an analogous manner to Truman and Zhao in [37] and similar to Chapter 1, we invoke Itô's formula for

$$S_0(x_s^\mu, t-s) - \mu^2 \nabla \log \sum_{j=0}^{m} \frac{1}{2^j} \mu^{2j} \psi_j(x_s^\mu, t-s)$$

which yields

$$\begin{aligned}
&S_0(x_t, 0) - \mu^2 \log T_0(x_t) = \\
&S_0(x,t) - \mu^2 \log \sum_{j=0}^{m} \frac{1}{2^j} \mu^{2j} \psi_j(x_s^\mu, t-s) \\
&+ \int_0^t \left(d_s S_0(x_s^\mu, t-s) - \mu^2 \frac{\partial}{\partial s} \log \sum_{j=0}^{m} \frac{1}{2^j} \mu^{2j} \psi_j(x_s^\mu, t-s) \right) ds \\
&\int_0^t \left(-(\nabla S_0(x_s^\mu, t-s) - a(x_s^\mu, t-s)) + \mu^2 \nabla \log \sum_{j=0}^{m} \frac{1}{2^j} \mu^{2j} \psi_j(x_s^\mu, t-s) \right) \\
&\times \left(\nabla S_0(x_s^\mu, t-s) - \mu^2 \nabla \log \sum_{j=0}^{m} \frac{1}{2^j} \mu^{2j} \psi_j(x_s^\mu, t-s) \right) ds \\
&+ \frac{\mu^2}{2} \int_0^t \left(\triangle S_0(x_s^\mu, t-s) - \mu^2 \triangle \log \sum_{j=0}^{m} \frac{1}{2^j} \mu^{2j} \psi_j(x_s^\mu, t-s) \right) ds \\
&+ \mu \int_0^t \left(\nabla S_0(x_s^\mu, t-s) - \mu^2 \nabla \log \sum_{j=0}^{m} \frac{1}{2^j} \mu^{2j} \psi_j(x_s^\mu, t-s) \right) \cdot dB_s.
\end{aligned}$$

This immediately implies

$$\begin{aligned}
&\mu^{-1} \int_0^t \left(\nabla S_0(x_s^\mu, t-s) - \mu^2 \nabla \log \sum_{j=0}^{m} \frac{1}{2^j} \mu^{2j} \psi_j(x_s^\mu, t-s) \right) \cdot dB_s \\
&= \frac{1}{\mu^2} \left(S_0(x_t, 0) - \mu^2 \log T_0(x_t) - S_0(x,t) + \mu^2 \log \sum_{j=0}^{m} \frac{1}{2^j} \mu^{2j} \psi_j(x_s^\mu, t-s) \right) \\
&- \frac{1}{\mu^2} \int_0^t \left(d_s S_0(x_s^\mu, t-s) - \mu^2 \frac{\partial}{\partial s} \log \sum_{j=0}^{m} \frac{1}{2^j} \mu^{2j} \psi_j(x_s^\mu, t-s) \right) ds \\
&+ \frac{1}{\mu^2} \int_0^t \left(\nabla S_0(x_s^\mu, t-s) - \mu^2 \nabla \log \sum_{j=0}^{m} \frac{1}{2^j} \mu^{2j} \psi_j(x_s^\mu, t-s) \right) \\
&\times \left((\nabla S_0(x_s^\mu, t-s) - a(x_s^\mu, t-s)) + \mu^2 \nabla \log \sum_{j=0}^{m} \frac{1}{2^j} \mu^{2j} \psi_j(x_s^\mu, t-s) \right) ds \\
&- \frac{1}{2} \int_0^t \left(\triangle S_0(x_s^\mu, t-s) - \mu^2 \triangle \log \sum_{j=0}^{m} \frac{1}{2^j} \mu^{2j} \psi_j(x_s^\mu, t-s) \right) ds. \qquad (2.3.45)
\end{aligned}$$

Observe the identity

$$\triangle \log \left(\sum_{j=0}^{m} \frac{1}{2^j} \mu^{2j} \psi_j(x_s^\mu, t-s) \right) = \frac{\triangle \left(\sum_{j=0}^{m} \frac{1}{2^j} \mu^{2j} \psi_j(x_s^\mu, t-s) \right)}{\sum_{j=0}^{m} \frac{1}{2^j} \mu^{2j} \psi_j(x_s^\mu, t-s)} - \left| \nabla \log \sum_{j=0}^{m} \frac{1}{2^j} \mu^{2j} \psi_j(x_s^\mu, t-s) \right|^2 . \tag{2.3.46}$$

Now consider, for the purposes of cancellation, the following expression

$$\begin{aligned}
&\mathcal{M}_t^\mu . \exp\left\{ \mu^{-1} \int_0^t a(x_s^\mu, t-s) \cdot dB_s \right\} \\
&= \exp\left\{ \mu^{-1} \int_0^t \left(\nabla S_0(x_s^\mu, t-s) - \mu^2 \nabla \log \sum_{j=0}^{m} \frac{1}{2^j} \mu^{2j} \psi_j(x_s^\mu, t-s) \right) \cdot dB_s \right. \\
&\left. -\frac{1}{2\mu^2} \int_0^t \left| -(\nabla S_0(x_s^\mu, t-s) - a(x_s^\mu, t-s)) + \mu^2 \nabla \log \sum_{j=0}^{m} \frac{1}{2^j} \mu^{2j} \psi_j(x_s^\mu, t-s) \right|^2 ds \right\}.
\end{aligned}$$

Using equation (2.3.45) and (2.3.46) the above turns out to be

$$\begin{aligned}
&\mathcal{M}_t^\mu . \exp\left\{ \mu^{-1} \int_0^t a(x_s^\mu, t-s) \cdot dB_s \right\} \\
&= \exp\left\{ \frac{1}{\mu^2} \left(S_0(x_t, 0) - \mu^2 \log T_0(x_t) - S_0(x, t) + \mu^2 \log \sum_{j=0}^{m} \frac{1}{2^j} \mu^{2j} \psi_j(x_s^\mu, t-s) \right) \right. \\
&+ \frac{1}{\mu^2} \int_0^t \left(-d_s S_0(x_s^\mu, t-s) - \frac{1}{2} \left| \nabla S(x_s^\mu, t-s) - a(x_s^\mu, t-s) \right|^2 ds \right) \\
&+ \frac{1}{\mu^2} \int_0^t \nabla S(x_s^\mu, t-s) \cdot (\nabla S(x_s^\mu, t-s) - a(x_s^\mu, t-s)) \, ds \\
&+ \frac{\mu^2}{2} \int_0^t \left| \nabla \log \sum_{j=0}^{m} \frac{1}{2^j} \mu^{2j} \psi_j(x_s^\mu, t-s) \right|^2 ds \\
&+ \int_0^t \left[\frac{\partial}{\partial s} \log \sum_{j=0}^{m} \frac{1}{2^j} \mu^{2j} \psi_j(x_s^\mu, t-s) - \frac{1}{2} \triangle S_0(x_s^\mu, t-s) \right. \\
&- \nabla S_0(x_s^\mu, t-s) . \log \sum_{j=0}^{m} \frac{1}{2^j} \mu^{2j} \psi_j(x_s^\mu, t-s) \qquad (2.3.47) \\
&\left. \left. + \frac{\mu^2}{2} \frac{\triangle \left(\sum_{j=0}^{m} \frac{1}{2^j} \mu^{2j} \psi_j(x_s^\mu, t-s) \right)}{\sum_{j=0}^{m} \frac{1}{2^j} \mu^{2j} \psi_j(x_s^\mu, t-s)} - \frac{\mu^2}{2} \left| \nabla \log \sum_{j=0}^{m} \frac{1}{2^j} \mu^{2j} \psi_j(x_s^\mu, t-s) \right|^2 \right] ds \right\}.
\end{aligned}$$

If we transfer our attention to (2.3.44) we find that we may rewrite this equation as

$$
\begin{aligned}
u_t^\mu(x) = \mathbb{E}\Bigg[T_0(x_t^\mu)\exp\Bigg\{&-\frac{S_0(x_t^\mu)}{\mu^2} - \frac{1}{\mu^2}\int_0^t a(x_s^\mu, t-s) \\
&\times\left((\nabla S_0(x_s^\mu, t-s) - a(x_s^\mu, t-s))ds - \mu^2\nabla\log\sum_{j=0}^m \frac{1}{2^j}\mu^{2j}\psi_j(x_s^\mu, t-s)ds\right) \\
&+\frac{1}{\mu^2}\int_0^t V(x_s^\mu)ds + \frac{\varepsilon}{\mu^2}\int_0^t k(x_s^\mu, t-s)dW_{t-s}\Bigg\} \\
&\times\mathcal{M}_t^\mu . \exp\left\{\mu^{-1}\int_0^t a(x_s^\mu, t-s)\cdot dB_s\right\}\Bigg].
\end{aligned}
$$

Substitution using (2.3.47) yields

$$
\begin{aligned}
u_t^\mu(x) = \mathbb{E}\Bigg[T_0(x_t)\exp\Bigg\{&-\frac{S_0(x_t^\mu)}{\mu^2} + \int_0^t a(x_s^\mu, t-s)\cdot\left(\nabla\log\sum_{j=0}^m \frac{1}{2^j}\mu^{2j}\psi_j(x_s^\mu, t-s)\right)ds \\
&-\frac{1}{\mu^2}\int_0^t a(x_s^\mu, t-s)\cdot(\nabla S_0(x_s^\mu, t-s) - a(x_s^\mu, t-s))\,ds \\
&+\frac{1}{\mu^2}\int_0^t V(x_s^\mu)ds + \frac{\varepsilon}{\mu^2}\int_0^t k(x_s^\mu, t-s)dW_{t-s} \\
&+\frac{1}{\mu^2}\left(S_0(x_t, 0) - \mu^2\log T_0(x_t) - S_0(x,t) + \mu^2\log\sum_{j=0}^m \frac{1}{2^j}\mu^{2j}\psi_j(x_s^\mu, t-s)\right) \\
&+\frac{1}{\mu^2}\int_0^t\left(-d_sS_0(x_s^\mu, t-s) - \frac{1}{2}\left|\nabla S(x_s^\mu, t-s) - a(x_s^\mu, t-s)\right|^2 ds\right) \\
&+\frac{1}{\mu^2}\int_0^t \nabla S(x_s^\mu, t-s)\cdot(\nabla S(x_s^\mu, t-s) - a(x_s^\mu, t-s))\,ds \\
&+\frac{\mu^2}{2}\int_0^t\left|\nabla\log\sum_{j=0}^m \frac{1}{2^j}\mu^{2j}\psi_j(x_s^\mu, t-s)\right|^2 ds \\
&+\int_0^t\left(\frac{\partial}{\partial s}\log\sum_{j=0}^m \frac{1}{2^j}\mu^{2j}\psi_j(x_s^\mu, t-s) - \frac{1}{2}\triangle S_0(x_s^\mu, t-s)\right. \\
&-\nabla S_0(x_s^\mu, t-s).\log\sum_{j=0}^m \frac{1}{2^j}\mu^{2j}\psi_j(x_s^\mu, t-s) \\
&\left.+\frac{\mu^2}{2}\frac{\triangle\left(\sum_{j=0}^m \frac{1}{2^j}\mu^{2j}\psi_j(x_s^\mu, t-s)\right)}{\sum_{j=0}^m \frac{1}{2^j}\mu^{2j}\psi_j(x_s^\mu, t-s)} - \frac{\mu^2}{2}\left|\nabla\log\sum_{j=0}^m \frac{1}{2^j}\mu^{2j}\psi_j(x_s^\mu, t-s)\right|^2\right)ds\Bigg\}\Bigg].
\end{aligned}
$$

Cancellation and collecting like terms gives

$$
\begin{aligned}
u_t^\mu(x) = \mathbb{E}\Bigg[T_0(x_t)\exp\Bigg\{&\frac{-S_0(x,t)}{\mu^2} \\
&+\frac{1}{\mu^2}\int_0^t\Big(-d_sS_0(x_s^\mu,t-s)-\frac{1}{2}\left|\nabla S(x_s^\mu,t-s)-a(x_s^\mu,t-s)\right|^2 \\
&+V(x_s^\mu)+\varepsilon k(x_s^\mu,t-s)\dot{W}_{t-s}\Big)\,ds \\
&+\int_0^t\Bigg(\frac{\partial}{\partial s}\log\sum_{j=0}^m\frac{1}{2^j}\mu^{2j}\psi_j(x_s^\mu,t-s) \\
&-\nabla\log\sum_{j=0}^m\frac{1}{2^j}\mu^{2j}\psi_j(x_s^\mu,t-s)\cdot\left(\nabla S(x_s^\mu,t-s)-a(x_s^\mu,t-s)\right) \\
&-\frac{1}{2}\triangle S_0(x_s^\mu,t-s)+\frac{\mu^2}{2}\frac{\triangle\left(\sum_{j=0}^m\frac{1}{2^j}\mu^{2j}\psi_j(x_s^\mu,t-s)\right)}{\sum_{j=0}^m\frac{1}{2^j}\mu^{2j}\psi_j(x_s^\mu,t-s)}\Bigg)\,ds \\
&-\log T_0(x_t)+\log\sum_{j=0}^m\frac{1}{2^j}\mu^{2j}\psi_j(x_s^\mu,t-s)\Bigg\}\Bigg]
\end{aligned}
$$

Finally, by using the stochastic Hamilton-Jacobi equation (2.3.2) and the continuity equation (2.3.21) this reduces to

$$
\begin{aligned}
u_t^\mu(x) &= \exp\left(-\frac{S_0(x,t)}{\mu^2}\right)\sum_{j=0}^m\frac{1}{2^j}\mu^{2j}\psi_j(x,t) \\
&\times\mathbb{E}\left[\exp\left(\frac{1}{2^{m+1}}\mu^{2(m+1)}\int_0^t\frac{\triangle\left(\psi_m(x_s^\mu,t-s)\right)}{\sum_{j=0}^m\frac{1}{2^j}\mu^{2j}\psi_j(x_s^\mu,t-s)}\right)\right]
\end{aligned}
$$

Observe this result to be consistent with the zero vorticity case, i.e. when $a(x,t)\equiv 0$, published in Truman and Zhao's paper [37]. The only difference being that the x_s^μ process satisfies a stochastic differntial equation with different drifts. □

We have succeeded in obtaining the solution of the stochastic heat equation (2.3.43). The remainder term is nothing but a functional integral. However, problems with the remainder term can occur if we endeavor to perform the logarithmic Hopf-Cole transformation. As stated by Truman and Zhao, under this transformation the remainder term is not explicit and it turns out to be given by a series whose convergence seems questionable. To circumvent this problem and arrive at a solution to the stochastic Burgers equation with vorticity we consider an alternative, slightly different stochastic process, namely the Nelson stochastic process defined by

$$
\begin{aligned}
dy_s^\mu &= \mu dB_s-\left(\sum_{j=0}^m\mu^{2j}\nabla S_j(y_s^\mu,t-s)-a(y_s^\mu,t-s)\right)ds, \\
y_0^\mu &= x. \qquad (2.3.48)
\end{aligned}
$$

As before, B_s is a standard Brownian motion on $\mathbb{R}^d$ on the probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}})$. Under these definitions we have the following theorem

Theorem 2.3.7. *Assume that $V \in C^2(\mathbb{R}^d)$, $a \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}^1)$, $div\,(a) = 0$, $k \in C^2(\mathbb{R}^d \times \mathbb{R}^1)$ and $S(\cdot, 0) \in C^2(\mathbb{R}^d)$, Φ_s defined by (2.2.1) satisfies a no caustic condition for $0 \leq t \leq T(\omega)$, $S_j(x,t)$ are defined by (2.3.24). Suppose the stochastic process (2.3.48) is $\mathcal{F}_s^*$ measurable and non-explosive. Then the solution of the heat equation (2.3.43) is given by*

$$u_t^\mu(x) = \exp\left(-\frac{1}{\mu^2}\sum_{j=0}^{m}\mu^{2j}S_j(x,t)\right) \times \widehat{\mathbb{E}}_P\left[\exp\left(-\frac{1}{2}\mu^{2m}\int_0^t \triangle S_m(y_s^\mu, t-s)ds\right.\right.$$
$$\left.\left.+\frac{1}{2}\int_0^t\sum_{j=m+1}^{2m}\mu^{2(j-1)}\sum_{0\leq i_1,i_2\leq m, i_1+i_2=j}\nabla S_{i_1}\cdot\nabla S_{i_2}\right)\right]. \tag{2.3.49}$$

Proof. Let y_s^μ be defined by (2.3.48). Define a new probability measure $\widehat{\mathbb{P}_1}$, for each $\omega \in \Omega$, by

$$\frac{d\widehat{\mathbb{P}_1}}{d\widehat{\mathbb{P}}} = \mathcal{M}_t^\mu$$
$$= \exp\left\{-\frac{1}{\mu}\int_0^t - \left(\sum_{j=0}^{m}\mu^{2j}\nabla S_j(y_s^\mu, t-s) - a(y_s^\mu, t-s)\right)\cdot dB_s\right.$$
$$\left.-\frac{1}{2\mu^2}\int_0^t\left|-\left(\sum_{j=0}^{m}\mu^{2j}\nabla S_j(y_s^\mu, t-s) - a(y_s^\mu, t-s)\right)\right|^2 ds\right\}.$$

Then for each $\omega \in \Omega$, y_s^μ is a Brownian motion on $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}_1})$ with variance μ^2. Then y_s^μ must be "isometric" to $B_t^\mu = x + \mu B_t$ on $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}})$. Isometric in the sense that the $\widehat{\mathbb{P}_1}$ law of y_s^μ is identical to the $\widehat{\mathbb{P}}$ law of B_t^μ. In a similar manner to the previous proof, using the Feynman-Kac formula and Girsanov's theorem yields

$$u_t^\mu(x) = \mathbb{E}\left[T_0(y_t^\mu)\exp\left\{-\frac{S_0(y_t^\mu)}{\mu^2} + \frac{1}{\mu^2}\int_0^t a(y_s^\mu, t-s)\cdot dy_s^\mu + \frac{1}{\mu^2}\int_0^t V(y_s^\mu)ds\right.\right.$$
$$\left.\left.+\frac{\varepsilon}{\mu^2}\int_0^t k(y_s^\mu, t-s)dW_{t-s}\right\}\mathcal{M}_t^\mu\right].$$

By the definition of dy_s^μ this becomes

$$u_t^\mu(x) = \mathbb{E}\left[T_0(y_t^\mu)\exp\left\{-\frac{S_0(y_t^\mu)}{\mu^2}\right.\right.$$
$$+\frac{1}{\mu^2}\int_0^t a(y_s^\mu, t-s)\cdot\left(\mu dB_s - \left(\sum_{j=0}^{m}\mu^{2j}\nabla S_j(y_s^\mu, t-s) - a(y_s^\mu, t-s)\right)ds\right)$$
$$\left.\left.+\frac{1}{\mu^2}\int_0^t V(y_s^\mu) + \frac{\varepsilon}{\mu^2}\int_0^t k(y_s^\mu, t-s)dW_{t-s}\right\}.\mathcal{M}_t^\mu\right]. \tag{2.3.50}$$

We concentrate on the removal of the exponential stochastic terms. Applying Itô's formula to $\sum_{j=0}^{m} \mu^{2j} S_j(y_s^\mu, t-s)$ yields

$$
\begin{aligned}
&S_0(y_t, 0) - \mu^2 \log T_0(y_t) \\
&= \sum_{j=0}^{m} \mu^{2j} S_j(x,t) + \int_0^t \sum_{j=0}^{m} \mu^{2j} d_s S_j(y_s^\mu, t-s) ds \\
&+ \int_0^t - \left(\sum_{j=0}^{m} \mu^{2j} \nabla S_j(y_s^\mu, t-s) - a(y_s^\mu, t-s) \right) \cdot \left(\sum_{j=0}^{m} \mu^{2j} \nabla S_j(y_s^\mu, t-s) \right) ds \\
&+ \mu \int_0^t \left(\sum_{j=0}^{m} \mu^{2j} \nabla S_j(y_s^\mu, t-s) \right) \cdot dB_s \\
&+ \frac{\mu^2}{2} \int_0^t \left(\sum_{j=0}^{m} \mu^{2j} \triangle S_j(y_s^\mu, t-s) \right) ds
\end{aligned}
$$

which immediately implies

$$
\begin{aligned}
&\frac{1}{\mu} \int_0^t \left(\sum_{j=0}^{m} \mu^{2j} \nabla S_j(y_s^\mu, t-s) \right) \cdot dB_s \\
&= \frac{1}{\mu^2} \left(S_0(y_t, 0) - \mu^2 \log T_0(y_t) - \sum_{j=0}^{m} \mu^{2j} S_j(x,t) \right) \\
&+ \frac{1}{\mu^2} \int_0^t \left(- \sum_{j=0}^{m} \mu^{2j} d_s S_j(y_s^\mu, t-s) + \left| \sum_{j=0}^{m} \mu^{2j} \nabla S_j(y_s^\mu, t-s) \right|^2 \right. \\
&\left. - a(y_s^\mu, t-s) \cdot \sum_{j=0}^{m} \mu^{2j} \nabla S_j(y_s^\mu, t-s) \right) ds \\
&- \frac{1}{2} \int_0^t \left(\sum_{j=0}^{m} \mu^{2j} \triangle S_j(y_s^\mu, t-s) \right) ds. \qquad (2.3.51)
\end{aligned}
$$

Substituting (2.3.51) into (2.3.50) and collecting like terms yields

$$
\begin{aligned}
u_t^\mu(x) = \mathbb{E} \Bigg[T_0(y_t^\mu) \exp \Bigg\{ &\frac{1}{2\mu^2} \int_0^t \left| \sum_{j=0}^{m} \mu^{2j} \nabla S_j(y_s^\mu, t-s) - a(y_s^\mu, t-s) \right|^2 ds \\
&+ \mu^{-2} \int_0^t V(y_s^\mu) ds + \mu^{-2} \varepsilon \int_0^t k(y_s^\mu, t-s) dW_{t-s} \\
&+ \frac{1}{\mu^2} \left(-\mu^2 \log T_0(y_t) - \sum_{j=0}^{m} \mu^{2j} S_j(x,t) \right) + \frac{1}{\mu^2} \int_0^t \left(- \sum_{j=0}^{m} \mu^{2j} d_s S_j(y_s^\mu, t-s) \right) ds \\
&- \frac{1}{2} \int_0^t \sum_{j=0}^{m} \mu^{2j} \triangle S_j(y_s^\mu, t-s) ds \Bigg\} \Bigg].
\end{aligned}
$$

If we observe that

$$\left|\sum_{j=0}^{m}\mu^{2j}\nabla S_j - a(y_s^\mu, t-s)\right|^2$$
$$= \sum_{j=0}^{m}\mu^{2j}\sum_{\substack{i_1,i_2\geq 0,\\ i_1+i_2=j}}\nabla S_{i_1}\cdot\nabla S_{i_2} + \sum_{j=m+1}^{2m}\mu^{2j}\sum_{\substack{0\leq i_1,i_2\leq m,\\ i_1+i_2=j}}\nabla S_{i_1}\cdot\nabla S_{i_2}$$
$$- 2a(y_s^\mu, t-s)\cdot\sum_{j=0}^{m}\mu^{2j}\nabla S_j + |a(y_s^\mu, t-s)|^2,$$

then one obtains

$$u_t^\mu(x) = \mathbb{E}\left[T_0(y_t^\mu)\exp\left\{\frac{1}{2\mu^2}\int_0^t\left[\sum_{j=0}^{m}\mu^{2j}\sum_{\substack{i_1,i_2\geq 0,\\ i_1+i_2=j}}\nabla S_{i_1}\cdot\nabla S_{i_2}\right.\right.\right.$$
$$\left. + \sum_{j=m+1}^{2m}\mu^{2j}\sum_{\substack{0\leq i_1,i_2\leq m,\\ i_1+i_2=j}}\nabla S_{i_1}\cdot\nabla S_{i_2} - 2a(y_s^\mu, t-s)\cdot\sum_{j=0}^{m}\mu^{2j}\nabla S_j + |a(y_s^\mu, t-s)|^2\right]ds$$
$$+ \frac{1}{\mu^2}\int_0^t V(y_s^\mu)ds + \frac{\varepsilon}{\mu^2}\int_0^t k(y_s^\mu, t-s)dW_{t-s}$$
$$+ \frac{1}{\mu^2}\left(-\mu^2\log T_0(y_t) - \sum_{j=0}^{m}\mu^{2j}S_j(x,t)\right) + \frac{1}{\mu^2}\int_0^t\left(-\sum_{j=0}^{m}\mu^{2j}d_sS_j(y_s^\mu, t-s)\right)ds$$
$$\left.\left. - \frac{1}{2}\int_0^t\sum_{j=0}^{m}\mu^{2j}\triangle S_j(y_s^\mu, t-s)ds\right\}\right].$$

Which can be rearranged to give

$$u_t^\mu(x) = \mathbb{E}\left[T_0(y_t^\mu)\exp\left\{\frac{1}{\mu^2}\int_0^t -\sum_{j=0}^{m}\mu^{2j}d_sS_j(y_s^\mu, t-s)\right.\right.$$
$$+ \Big(\frac{1}{2}\sum_{j=0}^{m}\mu^{2j}\sum_{\substack{i_1,i_2\geq 0,\\ i_1+i_2=j}}\nabla S_{i_1}\cdot\nabla S_{i_2} - a(y_s^\mu, t-s)\cdot\sum_{j=0}^{m}\mu^{2j}\nabla S_j$$
$$+ V(y_s^\mu) + \varepsilon k(y_s^\mu, t-s)\dot{W}_{t-s}\Big)ds$$
$$- \frac{1}{2}\int_0^t\left(\sum_{j=0}^{m}\mu^{2j}\triangle S_j(y_s^\mu, t-s)\right)ds + \frac{1}{\mu^2}\left(-\mu^2\log T_0(y_t) - \sum_{j=0}^{m}\mu^{2j}S_j(x,t)\right)$$
$$\left.\left. + \frac{1}{2\mu^2}\int_0^t\left(\sum_{j=m+1}^{2m}\mu^{2j}\sum_{\substack{0\leq i_1,i_2\leq m,\\ i_1+i_2=j}}\nabla S_{i_1}\cdot\nabla S_{i_2} + |a(y_s^\mu, t-s)|^2\right)ds\right\}\right].$$

By using (2.3.41)

$$u_t^\mu(x) = \mathbb{E}\left[T_0(y_t^\mu)\exp\left\{\int_0^t \frac{1}{2}\triangle\sum_{j=0}^{m-1}\mu^{2j}S_j(x,t) - \frac{|a(x,t)|^2}{2\mu^2}\right) ds\right.$$
$$-\frac{1}{2}\int_0^t\left(\sum_{j=0}^{m}\mu^{2j}\triangle S_j(y_s^\mu, t-s)ds + \frac{1}{\mu^2}\left(-\mu^2\log T_0(y_t) - \sum_{j=0}^{m}\mu^{2j}S_j(x,t)\right)\right.$$
$$\left.\left. + \frac{1}{2\mu^2}\left(\int_0^t\sum_{j=m+1}^{2m}\mu^{2j}\sum_{\substack{0\le i_1,i_2\le m,\\ i_1+i_2=j}}\nabla S_{i_1}\cdot\nabla S_{i_2} + a^2(y_s^\mu, t-s)\right)ds\right\}\right]$$

which turns out be

$$u_t^\mu(x) = \exp\left(-\frac{1}{\mu^2}\sum_{j=0}^{m}\mu^{2j}S_j(x,t)\right)\mathbb{E}\left[\exp\left(-\frac{1}{2}\mu^{2m}\int_0^t\triangle S_m(y_s^\mu, t-s)ds\right.\right.$$
$$\left.\left.+\frac{1}{2}\sum_{j=m+1}^{2m}\mu^{2(j-1)}\sum_{\substack{0\le i_1,i_2\le m,\\ i_1+i_2=j}}\int_0^t\nabla S_{i_1}\cdot\nabla S_{i_2}(y_s^\mu, t-s)ds\right)\right]$$

□

In equation (2.3.43) consider the special case when $T_0 = 1$. The Hopf-Cole logarithmic transformation $v^\mu(x,t) = -\mu^2\nabla\log u_t^\mu(x) - a(x,t)$ then gives the solution of the following viscous stochastic Burgers equation with vorticity:

$$\begin{aligned}&\frac{1}{2}\mu^2\triangle v_t^\mu(x)dt + \frac{1}{2}\mu^2\triangle a(x,t)dt\\ &= dv_t^\mu(x) + \left[(v_t^\mu(x).\nabla)\,v_t^\mu(x) + v_t^\mu(x)\wedge\text{curl }v_t^\mu(x) + \nabla V(x)\right]dt + da(x,t) + \varepsilon\nabla k(x,t)dW_t\end{aligned} \tag{2.3.52}$$

with the initial condition

$$v_0^\mu(x) = \nabla S_0(x) - a(x,0) = v_0(x).$$

If we denote $v_0(x,t) = \nabla S_0(x,t) - a(x,t)$ and $v_j(x,t) = \nabla S_j$ then, in a similar manner to the iterated Hamilton-Jacobi continuity equations, $v_j(x,t)$ must satisfy the following iterated Burgers equation for $j \ge 0$, $j \ne 1$

$$\frac{\partial}{\partial t}v_j + \sum_{\substack{i_1,i_2\ge 0,\\ i_1+i_2=j}}\left[(\nabla\cdot v_{i_1})v_{i_2} + v_{i_1}\wedge\text{curl}v_{i_2}\right] = \frac{1}{2}\triangle v_{j-1}$$

and

$$\frac{\partial}{\partial t}v_j + \sum_{\substack{i_1,i_2\ge 0,\\ i_1+i_2=j}}\left[(\nabla\cdot v_{i_1})v_{i_2} + v_{i_1}\wedge\text{curl}v_{i_2}\right] = \frac{1}{2}\triangle v_{j-1} + \frac{1}{2}\triangle a, \quad \text{if} \quad j = 1,$$

with the convention that $\frac{1}{2}\triangle v_{-1} = -\frac{\partial a}{\partial t}(x,t) - \nabla V(x) - \varepsilon \nabla k(x,t)\dot{W}_t$. The initial condition for these iterated equations being $v_0(x,0) = \nabla S_0(x) - a(x,0)$, $v_j(x,0) = 0$ for $j = 1, 2, \cdots$.
A straight forward application of the Hopf-Cole transformation to equation (2.3.49) provides us with the following theorem:

Theorem 2.3.8. *Assume that $V \in C^2(\mathbb{R}^d)$, $a \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}^1)$, div $(a) = 0$, $k \in C^2(\mathbb{R}^d \times \mathbb{R}^1)$ and $S(\cdot,0) \in C^2(\mathbb{R}^d)$, Φ_s defined by* (2.2.1) *satisfies a no caustic condition for $0 \leq t \leq T(\omega)$ and $S_j(x,t)$ are defined by* (2.3.24). *Suppose that the stochastic process y_s defined by* (2.3.48) *is $\mathcal{F}_s^*$ measurable and non-explosive. Then the solution of the viscous stochastic Burgers equation* (2.3.52) *is given by*

$$v_t^\mu(x) = \sum_{j=0}^{m} \mu^{2j} v_j(x,t) - \mu^2 \nabla \log \mathbb{E}\left\{ \exp\left(-\frac{\mu^{2m}}{2} \int_0^t \nabla \cdot v_m(y_s^\mu, t-s)ds \right.\right.$$

$$\left.\left. + \frac{1}{2} \sum_{j=m+1}^{2m} \mu^{2(j-1)} \sum_{\substack{0 \leq i_1, i_2 \leq m, \\ i_1 + i_2 = j}} \int_0^t v_{i_1} \cdot v_{i_2}(y_s^\mu, t-s)ds \right) \right\},$$

where $v_0(x,t) = \nabla S_0(x,t) - a(x,t)$ for $j = 0$ and $v_j(x,t) = \nabla S_j(x,t)$ for $j \geq 1$.

This last result provides a solution to the viscous stochastic Burgers equation up to an arbitrarily high order in the viscosity μ^2. The Burgers equation (2.3.52) describes a fluid whose particles pass through each other subject to a body force $-\nabla V(x) - \frac{\partial a(x,t)}{\partial t} - \varepsilon \nabla k(x,t)\dot{W}_t$ and vorticity $\xi = \text{curl } v$. As Truman and Zhao remark, it is immediate from the above formula that iterated Burgers equations give the higher order terms of the asymptotics which arise beyond the inviscid limit. The explicit formula for the remainder term is given by the logarithmic derivative of a path integral.

Chapter 3

Extending Existing Geometrical Results

Summary

In this chapter we follow earlier results of Davies, Truman and Zhao (DTZ) in [8] and try to extend their findings for rotational fluids, i.e. fluids with vorticity. We discuss their results which link caustics (shockwaves) for Burgers equation and wavefronts of the corresponding heat equation. Furthermore, we use these results to account for the turbulence for the Burgers velocity field.

3.1 Introduction

Recall from the previous chapter the stochastic viscous Burgers equation with vorticity, for the velocity field $v_t^\mu(x) = v^\mu(x,t)$, $x \in \mathbb{R}^d$, $t > 0$

$$\begin{aligned}&\frac{\mu^2}{2}(\triangle v_t^\mu(x) + \triangle a_t(x))dt \\ &= dv_t^\mu(x) + [(v_t^\mu(x)\cdot\nabla)\,v_t^\mu(x) + v_t^\mu(x)\wedge \operatorname{curl}\, v_t^\mu(x) + \nabla V(x)]\,dt + da_t(x) + \varepsilon\nabla k_t(x)dW_t\end{aligned}$$

where the initial velocity is $v^\mu(x,0) = \nabla S_0(x) - a(x,0)$, μ^2 is the coefficient of viscosity and W_t is a one dimensional Wiener process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. As before $a_t(x) = a(x,t)$, $V(x)$ and $k_t(x) = k(x,t)$ represent the vector, scalar and noise potential terms respectively. The Stratonovich heat equation corresponding to this Burgers equation was shown to be

$$\begin{aligned}\partial u_t^\mu(x) &= \left[\frac{1}{2}\mu^2\triangle u_t^\mu(x) + a(x,t)\cdot\nabla u_t^\mu(x) + \mu^{-2}\left(\frac{a^2(x,t)}{2} + V(x)\right)u_t^\mu(x)\right]dt \\ &\quad + \varepsilon\frac{k(x,t)}{\mu^2}u_t^\mu(x)\circ\partial W_t, \\ u_0^\mu(x) &= T_0(x)e^{-\frac{S(x,0)}{\mu^2}}. \end{aligned} \tag{3.1.1}$$

where $u_t^\mu(x) = u^\mu(x,t)$ is the temperature and $u^\mu(x,0) = T_0(x)e^{-\frac{S_0(x)}{\mu^2}}$. Function T_0 is a convergence factor related to the initial Burgers fluid density. The connection between these two equations is the logarithmic Hopf-Cole transformation

$$v^\mu(x,t) = -\mu^2 \nabla \log u_t^\mu(x) - a(x,t).$$

In this chapter we study the discontinuities of the Burgers velocity in the inviscid limit, i.e. discontinuities in the velocity field

$$v^0(x,t) = \lim_{\mu \to 0} v^\mu(x,t).$$

Following DTZ's notation, define

$$\mathcal{S}(x,t) := \inf_{X(0)} \left[A(X(0),x,t) + S_0(X(0)) \right],$$

where, under the vorticity set up,

$$A(X(0),x,t) = \inf_{\substack{X(0) \\ X(t)=x}} A[X]$$

$$A[X] = \int_0^t \left(\frac{1}{2}\dot{X}(s)^2 - V(X(s)) \right) ds + \int_0^t a(X(s),s) \cdot dX(s) - \varepsilon \int_0^t k(X(s),s) dW_s. \tag{3.1.2}$$

Here $A[X(s)]$ denotes the stochastic action of the stochastic mechanical path $X(s) \in \mathbb{R}^d$. Here we assume $\dot{X}_s$ is continuous in s with probability one and is $\mathcal{F}_s$ measurable as usual. From results of Freidlin et al in [14] and [15], as $\mu \searrow 0$ one can expect

$$-\mu^2 \ln u^\mu(x,t) \to \mathcal{S}(x,t). \tag{3.1.3}$$

where $\mathcal{S}(x,t)$ satisfies the stochastic Hamilton-Jacobi equation for a vector potential

$$dS(x,t) + \frac{|\nabla S(x,t) - a(x,t)|^2}{2} dt + V(x)dt + \varepsilon k(x,t) dW_t = 0, \qquad S_0(x) = S(x,0).$$

Definition 3.1.1. Define $\mathcal{A}[X] := A(X(0),x,t) + S_0(X(0))$.

Remark 3.1.1. Minimising $\mathcal{A}[X]$ over $X(0)$ gives $\mathcal{S}(x,t)$ which satisfies the above Hamilton-Jacobi equation and hence is a Hamilton characteristic function. See following paragraphs for more information on this.

Definition 3.1.2. Define the stochastic wavefront by $\{x : \mathcal{S}(x,t) = 0\}$.

As in the case of just a scalar potential, we can still expect from (3.1.3), that as $\mu \to 0$, u^μ can switch continuously from being exponentially large to exponentially small as we cross the wavefront. However, u^μ and v^μ can also switch discontinuously, as will be explained shortly. The conclusion of $\mathcal{S}(x,t)$ being Hamilton's characteristic function for a stochastic mechanical path can be seen with the following argument:

Assume that $\dot{X}(s)$ is a continuous process with the integration by parts property (ibp^2). This means for all deterministic $u \in C^1(0,t)$

$$\int_0^t \dot{u}(s)\dot{X}(s)ds = u(t)\dot{X}(t) - u(0)\dot{X}(0) - \int_0^t u(s)d\dot{X}(s), \tag{3.1.4}$$

this last term being a stochastic integral. If we fix $u(t) = 0$ and set $\mathcal{A}[X] = A[X] + S_0(X(0))$ then we obtain for $S_0 \in C^1$, $V \in C^1$, $k \in C^{1,0}$

$$\begin{aligned} -\frac{d}{d\eta}\bigg|_{\eta=0} \mathcal{A}[X+\eta u] = -\frac{d}{d\eta}\bigg|_{\eta=0} \bigg\{ & \int_0^t \left(\frac{1}{2}[\dot{X}+\eta\dot{u}]^2(s) - V((X+\eta u)(s))\right) ds \\ & + \int_0^t a((X+\eta u)(s), s) \cdot d\big(X+\eta u\big)(s) \\ & -\varepsilon \int_0^t k((X+\eta u)(s), s)dW_s + S_0((X+\eta u)(0))\bigg\}. \end{aligned} \tag{3.1.5}$$

The term that causes most difficulty is

$$\frac{d}{d\eta}\bigg|_{\eta=0} a((X+\eta u)(s), s) \cdot d\big(X+\eta u\big)(s).$$

However, dropping the s dependence, one should realise that the above quantity is nothing but

$$\begin{aligned} & \lim_{\eta \searrow 0} \left\{ \frac{a(X+\eta u) \cdot \big(dX + \eta\dot{u}\big) - a(X)\cdot dX}{\eta} \right\} \\ & = \lim_{\eta \searrow 0} \left\{ \frac{\eta(u_j\nabla_j)a_k dX_k + a_k\eta\dot{u}_k}{\eta} \right\} \\ & = (u_j\nabla_j)a_k dX_k + a_k\dot{u}_k \end{aligned}$$

and so

$$\begin{aligned} & \int_0^t \frac{d}{d\eta}\bigg|_{\eta=0} a((X+\eta u)(s), s) \cdot d\big(\,X+\eta u\big)(s)ds \\ & = \int_0^t (u_j\nabla_j a_k)dX_k(s) + \int_0^t a_k\dot{u}_k(s)ds. \end{aligned}$$

Applying the ibp^2 property on the last term above yields

$$\int_0^t u_j\nabla_j a_k dX_k - \int_0^t \left(\frac{\partial}{\partial s}a_k u_k - \dot{X}_l\nabla_l a_k u_k\right) ds - a_k(X(0))u_k(0).$$

Due to our gauge condition, $\nabla \cdot a \equiv 0$, this is nothing more than

$$\int_0^t \left(-\text{curl } a(X(s), s) \wedge \dot{X}(s) - \frac{\partial a}{\partial s}(X(s), s)\right) u(s)ds - a(X(0))u(0).$$

Once again, using the ibp[2] property and setting $X(s) = X_s$ Equation (3.1.5) reduces to

$$
\begin{aligned}
&-\frac{d}{d\eta}\bigg|_{\eta=0} \mathcal{A}[X+\eta u] \\
&= \int_0^t \left\{ d\dot{X}_s + \nabla V(X_s)ds + \text{curl } a(X_s,s) \wedge dX_s + da(X_s,s) + \varepsilon \nabla k(X_s,s)dW_s \right\} u(s) \\
&+ u(0)\left(\dot{X}_0 - \nabla S_0(X_0) + a(X_0,0)\right).
\end{aligned}
$$

Hence, the necessary conditions for the extremiser are, for $s \in [0,t]$,

$$
\begin{aligned}
&d\dot{X}_s + \text{curl } a(X_s,s) \wedge dX_s + \nabla V(X_s)ds + da(X_s,s) + \varepsilon \nabla k(X_s,s)dW_s = 0, \\
&\dot{X}_0 = \nabla S_0(X_0) - a(X_0,0).
\end{aligned} \tag{3.1.6}
$$

As observed by DTZ, if one demands that, for a fixed time t and position x, $X(t) = x$, then the $X(s)$ process satisfying Equation (3.1.6) is not necessarily unique. Consequently, the shockwaves for the Burgers velocity v occur from precaustics (in (x_0, t) variables) where infinitely many of these classical mechanical paths originating at x_0, and its neighborhood, focus on a set of zero volume centred at $X(t) = x$. This scenario can be stated succinctly with the following condition: for paths originating at x_0 and focusing on a point x at time t,

$$
\text{Det}\left(\frac{\partial X(t)}{\partial x_0}\right) = 0 \qquad \text{(Precaustic)}.
$$

Define the random map $\Phi_s : \mathbb{R}^d \to \mathbb{R}^d$ corresponding to the classical mechanical flow by

$$
\begin{aligned}
&d_s\dot{\Phi}_s = -\nabla V(\Phi_s)ds - \text{curl } a(\Phi_s,s) \wedge d\Phi_s - da(\Phi_s,s) - \varepsilon \nabla k(\Phi_s,s)dW_s, \\
&\Phi_0(x) = x, \\
&\dot{\Phi}_0(x) = \nabla S_0(x_0) - a(x_0,0)
\end{aligned}
$$

Under this set up $X(s) = \Phi_s\Phi_t^{-1}x$. The image of the precaustic under the Φ_t map is denoted by the term *caustic*

$$
\text{Det}\left(\frac{\partial X(t)}{\partial x_0}\right)\bigg|_{x_0=\Phi_t^{-1}x} = 0 \qquad \text{(Caustic)}.
$$

It is expected that the non-uniqueness of $x_0(x,t) = \Phi_t^{-1}x$ will be associated with the occurrence of discontinuities in $v^0(x,t)$ and $u^0(x,t)$.

Remark 3.1.2.
Up to the caustic time $T(\omega)$, $x_0(x,t)$ is unique and therefore

$$
v^0(x,t) = \dot{\Phi}_t\Phi_t^{-1}x \tag{3.1.7}
$$

is a C^1 solution of the classical inviscid Burgers equation with vorticity. After the caustic time, for polynomial S_0, $x_0(x,t)$ will have finite multiplicity so long as the minimising $x_0(x,t)$ is unique. In which case Equation (3.1.7) may still be assumed true if we only consider the part of the level surface of Hamilton's characteristic function $S(x,t)$ which corresponds to the minimising $x_0(x,t)$.

Following DTZ's analysis in [7] and [8] we assume $\mathcal{A}(x_0, x, t) \in C^2$ in space for $t > 0$ so that the classical mechanical flow is determined by the equation

$$\nabla_{x_0}\mathcal{A}(x_0, x, t) = 0. \tag{3.1.8}$$

The level surface conditions being

$$\mathcal{A}(x_0, x, t) = c \qquad \text{and} \qquad \nabla_{x_0}\mathcal{A}(x_0, x, t) = 0. \tag{3.1.9}$$

The prelevel surface is obtained by eliminating x and the level surface obtained by eliminating x_0 from the two above equations. If x is a non-degenerate critical point and $x_0(x,t)$ has a finite multiplicity $n \in \mathbb{N}$ then we can write

$$\Phi_t^{-1}\{x\} = \{x_0^1(x,t), x_0^2(x,t), \ldots, x_0^n(x,t)\}.$$

From our work in the previous chapter we saw

$$u^\mu(x,t) \sim \sum \theta_i \exp\left\{-\frac{S_0^i(x,t)}{\mu^2}\right\}, \tag{3.1.10}$$

where

$$S_0^i(x,t) = S_0(x_0^i(x,t)) + A(x_0^i(x,t), x, t) \tag{3.1.11}$$

for $i = 1, 2, \ldots, n$. The detailed structure of θ_i can be found in the penultimate result of the preceding chapter, but in short, is an asymptotic series in μ^2. From Theorem (2.3.8) of Chapter 2 we can also deduce that as $\mu \sim 0$ the leading term in v^μ is

$$v^\mu(x,t) \sim \nabla S_0(x,t) - a(x,t) + \mathrm{O}(\mu^2). \tag{3.1.12}$$

Here, $S_0(x,t)$ is the solution of the Hamilton-Jacobi equation which minimises the action $\mathcal{A}$. Since

$$\mathcal{S}(x,t) = \min_{i=1,2,\ldots,n} S_0^i(x,t),$$

we define the zero level surface of Hamilton's characteristic function by

$$H_t^0 = \{x : S_0^i(x,t) = 0, \text{for some i}\}.$$

Notice that the definition of H_t^0 will include the wavefront. From Equations (3.1.10) and (3.1.12) we observe that the dominant contribution to the 'blow-up' of $u^0(x,t)$ and $v^0(x,t)$ comes from the minimising $\tilde{x}_0(x,t)$.

In forthcoming sections work is presented consistent with thoughts of Freidlin et al in [14] and [15] and DTZ in [7] and [8] in so much that $u^0(x,t)$ can switch discontinuously from being exponentially large to exponentially small due to the possible disappearance of the minimising $S_0^i(x,t)$.

3.2 Extending DTZ's results for a stochastic Burgers fluid with vorticity

Recall the stochastic action $A(x_0, x, t)$ given by Equation (3.1.2) as

$$A(X(0), x, t) = \int_0^t \left(\frac{1}{2}\dot{X}(s)^2 - V(X_s)\right) ds + \int_0^t a(X_s, s) \cdot dX_s - \varepsilon \int_0^t k(X_s, s) dW_s.$$

where $X_s = X(s, x_0, p_0)$ must satisfy the second order stochastic differential equation

$$\begin{aligned} &d\dot{X}_s + \nabla V(X_s)ds + \operatorname{curl} a(X_s, s) \wedge dX_s + da(X_s, s) + \varepsilon \nabla k(X_s, s) dW_s = 0, \\ &\dot{X}(0) = p_0 \end{aligned} \tag{3.2.1}$$

for $s \in [0, t]$ with $X(0) = x_0$. We remark that, as usual, X_s is assumed to be a unique $\mathcal{F}_s$ measurable process. x_0, $p_0 \in \mathbb{R}^d$ and p_0 is as yet an unspecified function of x_0.

Previously we used the ibp^2 (see Equation (3.1.4)) on the continuous process $\dot{X}(s)$ and deterministic $u \in C^1(0, t)$ to prove necessary conditions for the extremiser. In fact the ibp^2 holds for random u and continuous process $\dot{X}(s) = \dot{X}(s, x_0, p_0)$ if the stochastic integral is of Itô form and, $du_s d\dot{X}_s$, the corresponding Itô correction is zero. Following DTZ we study the important case when

$$u_s = \frac{\partial X_s}{\partial x_0^\alpha},$$

for $s \in [0, t]$ and $\alpha = 1, 2, ..., d$. Kunita in [20] proves that if ∇V and ∇k Lipschitz and all second derivatives with respect to space variables of V and k bounded then $\partial X_s / \partial x_0^\alpha$ satisfies

$$\frac{d}{ds}\left(\frac{\partial X_s}{\partial x_0^\alpha}\right) = \frac{\partial \dot{X}_s}{\partial x_0^\alpha} \qquad \alpha = 1, 2, ..., d. \tag{3.2.2}$$

Essentially, this means the Itô correction is zero and so for $u_s = \partial X_s / \partial x_0^\alpha$ the ibp^2 property can be legitimately used. We use this idea for:-

Lemma 3.2.1. *Assume our usual conditions on a and V together with $S_0, V \in C^2$ and $k(x, t) \in C^{2,0}$, ∇V, ∇k are Lipschitz, with Hessians $\nabla^2 V$, $\nabla^2 k$ and all second derivatives with respect to space variables of V and k are bounded. If $\dot{X}_s$ satisfies Equation* (3.2.2) *and p_0 is possibly x_0 dependent, then*

$$\frac{\partial A}{\partial x_0^\alpha}(x_0, p_0, t) = \left(\dot{X}(t) + a(X(t), t)\right) \frac{\partial X(t)}{\partial x_0^\alpha} - \left(\dot{X}_\alpha(0) + a(X(0))\right), \quad for \quad \alpha = 1, 2, ..., d. \tag{3.2.3}$$

almost surely.

Proof. From Equation (3.1.2) the stochastic action may be written as

$$A(x_0, p_0, t) = \int_0^t \left(\frac{1}{2}\dot{X}_s^2 + a(X_s, s) \cdot \dot{X}_s - V(X_s)\right) ds - \varepsilon \int_0^t k(X_s, s) dW_s.$$

for X_s satisfying Equation (3.2.1). Consider the term

$$\int_0^t \frac{\partial}{\partial x_0^\alpha}\Big(a(X(s),s)\cdot \dot{X}(s)\Big)\,ds$$
$$= \int_0^t \left\{ -\frac{\partial X(s)}{\partial x_0^\alpha}\text{curl } a(X(s),s)\wedge \dot{X}(s) + a(X(s),s)\frac{d}{ds}\left(\frac{\partial X(s)}{\partial x_0^\alpha}\right)\right\} ds$$

The ibp^2 property gives

$$\int_0^t \frac{\partial}{\partial x_0^\alpha}\Big(a(X(s),s)\cdot \dot{X}(s)\Big)\,ds$$
$$= \int_0^t -\frac{\partial X(s)}{\partial x_0^\alpha}\text{curl } a(X(s),s)\wedge \dot{X}(s)ds$$
$$+\left\{ a(X(t),t)\frac{\partial X(t)}{\partial x_0^\alpha} - a(X(0))\frac{\partial X(0)}{\partial x_0^\alpha} - \int_0^t \frac{\partial a(X(s),s)}{\partial s}\frac{\partial X(s)}{\partial x_0^\alpha}ds\right\}.$$

This would mean that

$$\frac{\partial A}{\partial x_0^\alpha}(x_0,p_0,t)$$
$$= \int_0^t \left\{ \dot{X}(s).\frac{\partial \dot{X}(s)}{\partial x_0^\alpha} - \nabla V(X(s)).\frac{\partial X(s)}{\partial x_0^\alpha}\right.$$
$$\left. - \Big(\text{curl } a(X(s),s)\wedge \dot{X}(s)\Big)\frac{\partial X(s)}{\partial x_0^\alpha} - \frac{\partial a(X(s),s)}{\partial s}\frac{\partial X(s)}{\partial x_0^\alpha}\right\} ds$$
$$-\varepsilon\int_0^t \nabla k(X(s),s)\frac{\partial X(s)}{\partial x_0^\alpha}dW_s$$
$$+ a(X(t),t)\frac{\partial X(t)}{\partial x_0^\alpha} - a(X(0))\frac{\partial X(0)}{\partial x_0^\alpha}$$

Once again, using Kunita's identity (3.2.2) and ibp^2 formula on the first term in the integrand yields

$$\frac{\partial A}{\partial x_0^\alpha}(x_0,p_0,t)$$
$$= -\int_0^t \frac{\partial X(s)}{\partial x_0^\alpha}\Big[d\dot{X}(s) + \nabla V(X(s))ds + \Big(\text{curl } a(X(s),s)\wedge \dot{X}(s)\Big)ds$$
$$+\frac{\partial a(X(s),s)}{\partial s}ds + \varepsilon\nabla k(X(s),s)dW_s\Big] + \left[\dot{X}(s).\frac{\partial X(s)}{\partial x_0^\alpha}\right]_0^t$$
$$+ a(X(t),t)\frac{\partial X(t)}{\partial x_0^\alpha} - a(X(0))\frac{\partial X(0)}{\partial x_0^\alpha}.$$

then a.s.

$$\begin{aligned}\frac{\partial A}{\partial x_0^\alpha}(x_0,p_0,t) &= \dot{X}(t)\frac{\partial X(t)}{\partial x_0^\alpha} - \dot{X}(0)\frac{\partial X(0)}{\partial x_0^\alpha} + a(X(t),t)\frac{\partial X(t)}{\partial x_0^\alpha} - a(X(0))\frac{\partial X(0)}{\partial x_0^\alpha} \\ &= \left\{\dot{X}(t) + a(X(t),t)\right\}\frac{\partial X(t)}{\partial x_0^\alpha} - \left\{\dot{X}(0) + a(X(0))\right\}\frac{\partial X(0)}{\partial x_0^\alpha} \\ &= \left\{\dot{X}(t) + a(X(t),t)\right\}\frac{\partial X(t)}{\partial x_0^\alpha} - \left\{\dot{X}(0) + a(X(0))\right\}.\end{aligned}$$

□

Remark 3.2.1. Observe for fixed $X(t)$ we obtain a.s.

$$\frac{\partial A}{\partial x_0^\alpha}(x_0,p_0,t) = -\left(\dot{X}(0) + a(X(0))\right)_\alpha$$

for $\alpha = 1, 2, \ldots, d$, representing coordinate position.

Following DTZ's notation we set

$$X(s,x_0,x) = X(s,x_0,p_0)\big|_{p_0=p(x_0,x,t)} ,$$

where $p_0 = p(x_0,x,t)$ is the (random) minimiser, which we assume unique, of $A(x_0,p_0,t)$ with $X(t,x_0,p_0) = x$. Likewise, set

$$A(x_0,x,t) = A(x_0,p_0,t)\big|_{p_0=p(x_0,x,t)} ,$$

so that

Theorem 3.2.2. *Defining $A(x_0,x,t)$ as above*

$$\left.\frac{\partial}{\partial x_0^\alpha}\right|_{\text{fixed }(x,t)} A(x_0,x,t) = -\left(\dot{X}(0) + a(X(0))\right)_\alpha , \qquad \alpha = 1,2,\ldots d,$$

and so

$$\left.\frac{\partial}{\partial x_0^\alpha}\right|_{\text{fixed }(x,t)} [A(x_0,x,t) + S_0(x_0)] = 0, \qquad \alpha = 1,2,...,d,$$

will define the stochastic mechanical flow map Φ_t with $x = \Phi_t x_0$.

Proof. Follows from Lemma (3.2.1) and above remark. □

We now introduce the stochastic action corresponding to the initial momentum $\nabla S_0(x_0) - a(x_0,0)$ by

$$\mathcal{A}(x_0,x,t) = A(x_0,x,t) + S_0(x_0) .$$

3.3 Geometrical results for the level surface

Here we closely follow DTZ's account of *Level Surface Geometry* in [8]. In this section we assume that

$$\text{Det}\left[\frac{\partial^2 \mathcal{A}}{\partial x_0 \partial x}(x_0, x, t)\right] \neq 0 \,, \qquad x_0, x \in \mathbb{R}^d \tag{3.3.1}$$

It is possible to weaken this last assumption, however, by assuming this condition the proofs are much simpler. Moreover, this assumption is sufficient for us to extend DTZ's results to incorporate studies of vorticity.

For this section let us remind ourselves that the prelevel surface of Hamilton's characteristic function, denoted by $\Phi_t^{-1} H_t$ is found on elimination of x from

$$\mathcal{A}(x_0, x, t) = c \quad \text{and} \quad \frac{\partial \mathcal{A}}{\partial x_0^\alpha}(x_0, x, t) = 0, \qquad \alpha = 1, 2, \ldots, d.$$

Likewise, the level surface, H_t, is determined by the elimination of x_0 from the above two equations. The precaustic, which shall be noted by $\Phi_t^{-1} C_t$, is obtained on elimination of x from

$$\text{Det}\left(\frac{\partial^2 \mathcal{A}}{\partial x_0^\alpha}(x_0, x, t)\right) = 0 \quad \text{and} \quad \frac{\partial \mathcal{A}}{\partial x_0^\alpha}(x_0, x, t) = 0 \quad \alpha = 1, 2, \ldots, d$$

and the caustic C_t being found on elimination of x_0.

Lemma 3.3.1. *The classical flow map $x = \Phi_t(x_0)$ is a differentiable map from $\Phi_t^{-1} H_t$ to H_t with Fréchet derivative*

$$D\Phi_t(x_0) = \left(-\frac{\partial^2 \mathcal{A}}{\partial x \partial x_0}(x_0, x, t)\right)^{-1} \left(\frac{\partial^2 \mathcal{A}}{\partial x_0^2}(x_0, x, t)\right) \,,$$

if $\mathcal{A}$ is C^3 in space variables with probability one.

Proof. Assume $x = \Phi_t(x_0)$, $x_0 \in \Phi_t^{-1} H_t$, $x \in H_t$, so that

$$\mathcal{A}(x_0, x, t) = c \quad \text{and} \quad \frac{\partial \mathcal{A}}{\partial x_0^\alpha}(x_0, x, t) = 0 \quad \alpha = 1, 2, \ldots, d.$$

Consider a neighbouring point $(x_0 + \delta x_0, x + \delta x, t)$ on $\Phi_t^{-1} H_t$ and H_t respectively. From the above this immediately implies

$$\mathcal{A}(x_0 + \delta x_0, x + \delta x, t) = c \quad \text{and} \quad \frac{\partial \mathcal{A}}{\partial x_0^\alpha}(x_0 + \delta x_0, x + \delta x, t) = 0 \quad \alpha = 1, 2, \ldots, d.$$

Working with the left most equation it follows that

$$\underbrace{\mathcal{A}(x_0, x, t)}_{=c} + \delta x_0 . \underbrace{\frac{\partial \mathcal{A}}{\partial x_0}(x_0, x, t)}_{=0} + \delta x . \frac{\partial \mathcal{A}}{\partial x}(x_0, x, t) + O(\delta^2) = c$$

So that, correct to first order, we obtain

$$\delta x . \frac{\partial \mathcal{A}}{\partial x}(x_0, x, t) = 0.$$

Likewise, the right most equation yields

$$\frac{\partial \mathcal{A}}{\partial x_0^\alpha}(x_0, x, t) + \delta x_0 . \frac{\partial^2 \mathcal{A}}{\partial x_0^2}(x_0, x, t) + \delta x . \frac{\partial^2 \mathcal{A}}{\partial x \partial x_0}(x_0, x, t) + O(\delta^2) = 0,$$

hence

$$\left[\frac{\partial^2 \mathcal{A}}{\partial x_0^2}(x_0, x, t)\right] \delta x_0 + \left[\frac{\partial^2 \mathcal{A}}{\partial x \partial x_0}(x_0, x, t)\right] \delta x = 0.$$

This Allows us to write

$$\delta x = \left[-\frac{\partial^2 \mathcal{A}}{\partial x \partial x_0}(x_0, x, t)\right]^{-1} \left[\frac{\partial^2 \mathcal{A}}{\partial x_0^2}(x_0, x, t)\right] \delta x_0.$$

Noting that

$$\begin{aligned} x + \delta x &= \Phi_t(x_0) + D\Phi_t(x_0)\delta x_0 \\ \delta x &= D\Phi_t(x_0)\delta x_0 \end{aligned}$$

the result follows. □

This allows us to prove the following proposition in d dimensions

Proposition 3.3.2. *Consider the random prelevel surface obtained by eliminating x between the equations*

$$\mathcal{A}(x_0, x, t) = c \quad \textit{and} \quad \frac{\partial \mathcal{A}}{\partial x_0^\alpha}(x_0, x, t) = 0, \quad \alpha = 1, 2 \ldots, d.$$

Then, for a.e. $\omega \in \Omega$, the normal to the prelevel surface at x_0 is to within a scalar multiplier given by

$$n(x_0) = -\left(\frac{\partial^2 \mathcal{A}}{\partial x_0^2}\right)\left(\frac{\partial^2 \mathcal{A}}{\partial x_0 \partial x}\right)^{-1} \dot{X}(t, x_0, p_0)\ , \tag{3.3.2}$$

where $p_0 = p_0(x_0, x, t) = \nabla S_0(x_0) - a(x_0, 0)$.

Proof. Evidently, $n(x_0) = \nabla_{x_0}\mathcal{A}(x_0, x, t)$ where $x = \Phi_t(x_0)$, the map Φ_t being defined by

$$\frac{\partial \mathcal{A}}{\partial x_0^\alpha}(x_0, x, t) = 0 \quad \alpha = 1, 2, \ldots, d.$$

Hence only the partial derivatives with respect to x contribute to $n(x_0)$ giving

$$n = (D\Phi_t(x_0))^T \frac{\partial \mathcal{A}}{\partial x^\alpha}(x_0, x, t)\bigg|_{x=\Phi_t(x_0)},$$

^{T} denoting the transpose. However, following the method of Lemma (3.2.1) it is not difficult to show that

$$\frac{\partial \mathcal{A}}{\partial x^\alpha}(x_0, p_0, t) = \dot{X}_\alpha(t, x_0, p_0)$$

almost surely for $\alpha = 1, 2, \ldots, d$, proving the proposition. □

Corollary 3.3.3. *In three dimensions at any point $x_0 \in \Phi_t^{-1} C_t \cap \Phi_t^{-1} H_t$ (i.e. where the prelevel surface meets the precaustic at a point x_0) where $n(x_0) \neq 0$ and*

$$\operatorname{Ker}\left(\frac{\partial^2 \mathcal{A}}{\partial x_0^2}(x_0, \Phi_t(x_0), t)\right) =< e_0 >,$$

e_0 being the zero eigenvector. Then the tangent plane to the prelevel surface is spanned by e_0 and $(n(x_0) \wedge e_0)$.

Proof. By symmetry of $\left(\frac{\partial^2 \mathcal{A}}{\partial x_0^2}\right)$

$$\begin{aligned} e_0.n &= e_0.\left(-\left(\frac{\partial^2 \mathcal{A}}{\partial x_0^2}\right)\left(\frac{\partial^2 \mathcal{A}}{\partial x_0 \partial x}\right)^{-1}\right)\dot{X}(t, x_0, p_0) \\ &= -\left(\frac{\partial^2 \mathcal{A}}{\partial x_0^2}\right) e_0.\left(\frac{\partial^2 \mathcal{A}}{\partial x_0 \partial x}\right)^{-1} \dot{X}(t, x_0, p_0) \\ &= 0 . \end{aligned}$$

This is implies e_0 must be in the tangent plane T_{x_0}. By definition $e_0 \wedge n$ is perpendicular to n and e_0. Hence, $e_0^{\perp} = n \wedge e_0 \in T_{x_0}$. □

Remark 3.3.1. Observe from Equation (3.3.2) that if $|\dot{X}(t, x_0, p_0)| = 0$, i.e. if the speed is identically zero, then $n(x_0) = 0$. Consequently, x_0 is a singular point of the prelevel surface, either a node with two distinct directions for the tangent plane or a cusped singularity. Because, $\frac{\partial A}{\partial x} = \dot{X}$ necessarily $x = \Phi_t(x_0)$ is a singular point of the level surface even if $x_0 \notin \Phi_t^{-1} C_t$. If both $x_0 \notin \Phi_t^{-1} C_t$ and $\dot{X} = 0$ then the singularity may only be a node because the tangent space is fully two dimensional at this point.

Corollary 3.3.4. *In two dimensions let the prelevel surface meet the precaustic at a point x_0 where $n(x_0) \neq 0$ and* Ker $\left(\frac{\partial^2 \mathcal{A}}{\partial x_0^2}(x_0, \Phi_t(x_0), t)\right) =< e_0 >$, *$e_0$ being the zero eigenvector. Then the tangent plane to the prelevel surface is spanned by e_0 .*

Proof. Simple consequence of the above proposition. □

Proposition 3.3.5. *Assume that in two dimensions a point $x_0 \in \Phi_t^{-1} H_t$ where $n(x_0) \neq 0$ so that $\Phi_t^{-1} H_t$ does not have a generalised cusp at x_0. Then H_t will have a cusp at $x = \Phi_t(x_0)$ if $\Phi_t(x_0) \in C_t$, the caustic surface. Moreover, if $x = \Phi_t(x_0) \in \Phi_t\{\Phi_t^{-1} C_t \cap \Phi_t^{-1} H_t\}$, then H_t will have a generalised cusp at x.*

Proof. When $n(x_0) \neq 0$ the direction of the tangent to the prelevel surface, $\Phi_t^{-1} H_t$, is well defined at x_0. Let $x_0 = x_0(\gamma)$ be a parametrisation of the two dimensional prelevel surface with $\gamma \in N(\gamma_0, \delta)$ in a neighborhood of $x_0 = x_0(\gamma)$. It follows that the tangent to the level surface is

$$\begin{aligned} \left.\frac{dx(\gamma)}{d\gamma}\right|_{\gamma=\gamma_0} &= D\Phi_t(x_0) \left.\frac{dx_0(\gamma)}{d\gamma}\right|_{\gamma=\gamma_0} \\ &= \left(-\frac{\partial^2 \mathcal{A}}{\partial x \partial x_0}\right)^{-1} \underbrace{\left(\frac{\partial^2 \mathcal{A}}{\partial x_0^2}\right)}_{\text{Det}=0 \text{ if } x \in C_t} \underbrace{\left.\frac{dx_0(\gamma)}{d\gamma}\right|_{\gamma=\gamma_0}}_{T_{x_0}} \\ &= 0, \end{aligned}$$

because T_{x_0} is spanned by e_0, hence we conclude x_0 must lie on the intersection of the precurves, i.e. $x_0 \in \Phi_t^{-1}H_t \cap \Phi_t^{-1}C_t$. Therefore, $\Phi_t(x_0) = x$ is a cusp on H_t if $x \in C_t$. If $\frac{dx_0(\gamma_0)}{d\gamma} = 0$ then necessarily Det $\left(\frac{\partial^2 \mathcal{A}}{\partial x_0^2}\right) = 0$ if x_0 is not a generalised cusp. □

If the intrinsic parameterisation of the curve is $x_0 = x_0(\gamma)$, $\gamma \in N(\gamma_0, \delta)$ in a neighbourhood of $x_0 = x_0(\gamma_0)$ then it follows from DTZ's above proof that if $\frac{dx_0(\gamma_0)}{d\gamma} = 0$ then $\frac{dx(\gamma_0)}{d\gamma} = 0$. Consequently, generalised cusps map to generalised cusps regardless of whether or not $x_0 \in \Phi_t^{-1}C_t$.

Definition 3.3.1. Define the cusped part of the level surface as

$$\mathrm{Cusp}(H_t) = \{x \in H_t : x \in \Phi_t\left(\Phi_t^{-1}C_t \cap \Phi_t^{-1}H_t\right), x = \Phi_t(x_0), n(x_0) \neq 0\}.$$

Proposition 3.3.6. *Let $x \in Cusp(H_t)$ then in three dimensions , T_x, the tangent space to the level surface at x is one dimensional at most.*

Proof. We have seen, via the preceding results, that if $x = \Phi_t(x_0)$, $x_0 \in \Phi_t^{-1}C_t \cap \Phi_t^{-1}H_t$ with $n(x_0) \neq 0$. Consequently, T_{x_0} is a well defined two dimensional tangent plane to the prelevel surface at x_0 where T_{x_0} is spanned by e_0 and $(n \wedge e_0)$. Yet the derivative of the flow map is given by

$$D\Phi_t(x_0) = \left(-\frac{\partial^2 \mathcal{A}}{\partial x \partial x_0}\right)^{-1} \left(\frac{\partial^2 \mathcal{A}}{\partial x_0^2}\right)$$

so $D\Phi_t(x_0).e_0 = 0$ and therefore $T_x = < D\Phi_t(x_0)(n \wedge e_0) >$ can be at most one dimensional. □

Remark 3.3.2. A similar result holds in higher dimensions.

The next result explains why we can expect to see generalised cusps on planar cross sections of level surfaces in three dimensions. δx is to be considered as second order of small quantities. The reader is asked to consult [8].

Theorem 3.3.7. *Any point x on the level surface H_t, $x = \Phi_t(x_0)$ with $x_0 = \Phi_t^{-1}x$ on the prelevel surface, can only be a generalised cusp of a curve on H_t if x_0 is a generalised cusp of the precurve on the prelevel surface or if $x_0 \in \Phi_t^{-1}C_t$ the precaustic.*

Proof. See [8]. □

3.4 Consequences for the Burgers fluid

Consider the zero limiting viscosity solution $v^0(x,t)$ of the deterministic free Burgers equation with vorticity.

Theorem 3.4.1. *Let $\tau(p)$ be the first time such that there exists minimisers $y_1 \neq y_2$, $y_1, y_2 \in \mathbb{R}^d$ such that $\Phi_\tau(y_1) = \Phi_\tau(y_2) = p$ and $\dot{\Phi}_\tau(y_1) \neq \dot{\Phi}_\tau(y_2)$, i.e τ is a caustic time at a point p. Then $v^0(x,t)$ is discontinuous at (τ, p).*

Proof. See [8]. □

In two dimensions this theorem provides an important application to the Burgers fluid. We illustrate For smooth S_0, we can divide one side of the caustic into *hot* and *cool* parts (we will define this concept shortly). Consider the level surface $S_0^i(x,t)$ defined by Equation (3.1.11). Now consider points α on the caustic where the caustic intersects a level surface $S_0^i(x,t) = c$. Recall that $\{x : S_0^i(x,t) = c\}$ will have a cusp at α if $\nabla S_0^i(\alpha,t) = 0$. If this level surface cusped at α corresponds to the minimising $\widetilde{x_0}(\alpha,t)$ then $\nabla S_0^i(x,t) \longrightarrow \nabla S_0^i(\alpha,t) = 0$, as $x \to \alpha$ from the cusped side of the caustic. The occurrence of a cusp can be explained by the fact that the minimising surface on the cusped side of the caustic cannot be continued as we cross the caustic. Necessarily, $u^0(x,t)$ will be exponentially discontinuous as we cross these parts of the caustic. At such points on the caustic two of the minimising $x_0(x,t)$'s coalesce at the cusp and then disappear. This is something we will expound on in the next section. We conclude this part of the study with a result for the random Burgers velocity field for any potential $V(x)$ and non-zero $k(x,t)$. We remind ourselves that all results in the previous section hold for any finite ε, the strength of white noise, so long as as Det $\left(\frac{\partial A}{\partial x_0 \partial x}\right) \neq 0$.

Recall our random map $\Phi_s(\omega) : \mathbb{R}^d \to \mathbb{R}^d$ satisfies

$$d\dot{\Phi}_s = -\nabla V(\Phi_s)ds - \text{curl}\; a(\Phi_s, s) \wedge d\Phi_s - da(\Phi_s, s) - \varepsilon \nabla k(\Phi_s, s) dW_s,$$

where $\Phi_0 = I$ and $\dot{\Phi}_0 = \nabla S_0 - a$. Applying the global inverse function theorem from [35] yields

Proposition 3.4.2. *With the usual conditions on a and V and if V, k and S_0 are smooth with bounded second order partial derivatives, there exists $T(\omega)$ such that $\Phi_s(\omega) : \mathbb{R}^d \to \mathbb{R}^d$ is a random diffeomorphism.*

Remark 3.4.1. For large a, $T(\omega) = \inf_{s>0} \{\det(\nabla_{x_0} \Phi_s(x_0)) = a\}$ is assumed to be a stopping time.

In light of this we conclude that we have been able to use DTZ's analysis in [8] to describe how after the caustic time $T(\omega)$ the stochastic action can be used to portray how the level surface meets the caustics of the Burgers fluid with vorticity in cusps. In higher dimensions, as observed by DTZ, we saw how a zero speed condition (see Remark (3.3.1)) on the precaustic and prelevel surfaces lead to turbulent behavior of the Burgers fluid at corresponding image points on the caustic. Under this condition the deterministic Burgers fluid will behave in a similar manner to its stochastic counterpart. Turbulent behaviour is characterised by the geometry of the minimising level surface of Hamilton's characterisitic function changing infinitely rapidly at points where it meets the caustic at cusps. The next section uses a sophisticated, yet simple, argument to comprehensively investigate jump discontinuities and turbulence in the inviscid limiting Burgers velocity field.

3.5 The reduced action functional

In this section we illustrate how work of Reynolds, Truman and Williams [RTW] in [26] has shown that the construction of an elementary action functional, in one space variable, can be used as a vehicle to divide caustics into hot and cool parts. This segregation allows detailed analysis in to the jump discontinuities across cool parts of the caustic in the inviscid limiting Burgers velocity field. Across hot parts of the caustic the field is simply continuous.

We show how the ideas in the preceding section can be used to investigate the comparisons between the possible *intermittence* of the stochastic turbulence and the corresponding deterministic turbulence. We will show that turbulent behavior can in fact be associated with the number of cusped curves on the level surface. Theorem (3.3.7) and Proposition (3.3.6) indicate that the times t at which the precurves *touch* give rise to turbulent behavior. Towards the end of this section we show, in the deterministic case, that the times t at which the number of cusped curves change will be given by the zeros (commonly isolated) of a deterministic function ζ. In the analogous stochastic case the zeros come from a stochastic process ζ^ϵ which form a perfect set. It will be shown that at these times the number of cusped curves change with infinite frequency due to the infinitely rapid oscillations of the stochastic process ζ^ϵ. Indeed, when this stochastic process ζ^ϵ is *recurrent* we regard this turbulent behavior as intermittent since the fluctuations vary in a random way.

Below we introduce the notion of global reducibility and the reduced one dimensional action function as first used by RTW in [26]. Under mild restrictions, this function allows us to reduce an d-dimensional problem into a study of a single dimension. It allows progress in areas that would previously have been almost impossible. Particularly for fluids that incorporate vorticity since studying a rotational fluid adds another level of complexity to the problem. These difficulties will be made explicit in the following chapter. We begin by encapsulating earlier findings of [26].

Definition 3.5.1. The classical flow map Φ_t is *globally reducible* if

$$x = \Phi_t x_0 \Longrightarrow x_0^r = x_0^r(x, x_0^1, x_0^2, \ldots x_0^{r-1}, t)$$

where $x_0 = (x_0^1, x_0^2, \ldots x_0^d)$, $x = (x^1, x^2, \ldots x^d)$ and $r = d, d-1, d-2, \ldots 2$.

For the above Φ_t map, in d-dimensions, we require twice differentiable functions $x_0^d, x_0^{d-1}, \ldots,$ so that

$$\begin{aligned}
x_0^d &= x_0^d(x, x_0^1, x_0^2, \ldots x_0^{d-1}, t) \\
&\Longleftrightarrow \frac{\partial \mathcal{A}}{\partial x_0^d}(x_0, x, t) = 0 \\
x_0^{d-1} &= x_0^{d-1}(x, x_0^1, x_0^2, \ldots x_0^{d-2}, t) \\
&\Longleftrightarrow \frac{\partial \mathcal{A}}{\partial x_0^{d-1}}(x_0^1, x_0^2, \ldots x_0^{d-1}, x_0^d(), x, t) = 0 \\
&\vdots \\
&\vdots \\
x_0^2 &= x_0^2(x, x_0^1, t) \\
&\Longleftrightarrow \frac{\partial \mathcal{A}}{\partial x_0^2}(x_0^1, x_0^2, x_0^3(x, x_0^1, x_0^2, t), x_0^4(), \ldots, x_0^d(), x, t) = 0,
\end{aligned} \tag{3.5.1}$$

where $x_0^d() = x_0^d(x, x_0^1, x_0^2, \ldots, x_0^{d-1}, t)$. Since no root is repeated the second derivatives of $\mathcal{A}$ will not vanish. With this ordering of the coordinates and corresponding decomposition of Φ_t the non-uniqueness in d-dimensions can be reduced to a study of the x_0^1 coordinate.

Proposition 3.5.1. *Assuming the Φ_t map is globally reducible we define the one dimensional reduced action functional as*

$$f(x_0^1, x, t) = \mathcal{A}(x_0^1, x_0^2(x, x_0^1, t), x_0^3(x, x_0^1, x_0^2(x, x_0^1, t), t), \ldots, x, t)$$

Then

1. $\frac{\partial f}{\partial x_0^1}(x_0^1, x, t) = 0$ *and* *Equations* (3.5.1) $\Longleftrightarrow x = \Phi_t x_0$
2. *Equations* (3.5.1) *and* $\frac{\partial f}{\partial x_0^1}(x_0^1, x, t) = \frac{\partial^2 f}{(\partial x_0^1)^2}(x_0^1, x, t) = 0 \Longleftrightarrow x = \Phi_t x_0$ *is such that the number of solutions* x_0 *of this equation changes.*

Proof. See RTW in [26]. □

Lemma 3.5.2.

$$\left|\mathrm{Det}\left(\frac{\partial^2 \mathcal{A}}{\partial x_0^2}(x_0, x, t)\right)\Big|_{x=\Phi_t x_0}\right|$$
$$= \prod_{i=0}^{d-1}\left|\frac{\partial^2 \mathcal{A}}{(\partial x_0^{d-i})^2}\left(x_0^1, x_0^2(x, x_0^1, t), x_0^3(x, x_0^1, x_0^2(x, x_0^1, t), t), \ldots, x, t\right)\right|$$

the last term is $f''(x_0^1, x, t)$ and the first $(d-1)$ terms are non-zero as above.

Proof. RTW in [26] indicate that the above result is obtained by applying the principle of stationary phase to

$$I = \int_{\mathbb{R}^d} G(x_0) \exp\left(-\frac{i}{\mu^2}\mathcal{A}(x_0, x, t)\right) dx_0.$$

See [11] for details of this method. □

By stationary phase if $\frac{\partial^2 f}{(\partial x_0^1)^2}(x_0^1, x, t) \neq 0$ then $\frac{\partial f}{\partial x_0^1}(x_0^1, x, t) = 0$ must have exactly n roots $x_0^1 = \beta_1^1(x, t), \beta_2^1(x, t), \ldots, \beta_n^1(x, t)$. However, if x is varied in such a way that $\frac{\partial^2 f}{(\partial x_0^1)^2}(x_0^1, x, t) = 0$ two of the above critical points say β_{i-1} and β_i, $i \leq n$, which had previously formed a local maximum and local minimum, now coalesce to form a point of inflexion. Algebraically, we interpret this as a repeated root in $f'(x_0^1)$.

To illustrate this concept consider $D_n \frac{\partial^2 f}{(\partial x_0^1)^2}(x_0^1, x, t) \neq 0$, where D_n is the directional derivative. The series of figures portray how the graph of $f_{(x,t)}(x_0^1)$ deforms as we vary x moving in the direction of n and crossing a cool part of the caustic.

Cusped side of the caustic

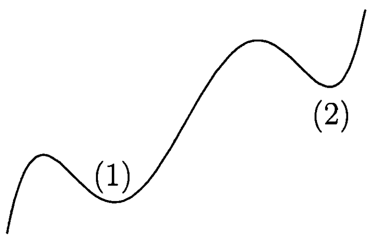


Figure 3.1: Graph of $f_{(x,t)}(x_0^1)$ as we move in the direction n where $D_n \frac{\partial^2 f}{(\partial x_0^1)^2}(x_0^1, x, t) > 0$.

On cool part of caustic

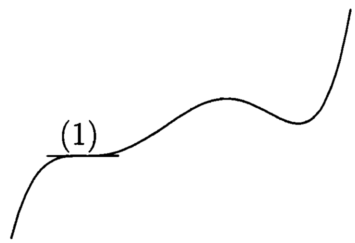


Figure 3.2: Here two consecutive critical points say β_{i-1}^1 and β_i^1 have coalesced to form the repeated root $\beta_r^1(x,t) = \beta_{i-1}^1(x,t) = \beta_i^1(x,t)$.

Beyond caustic

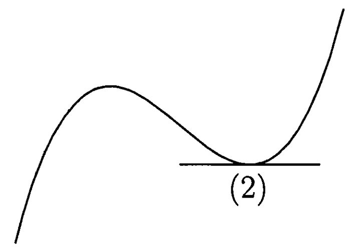


Figure 3.3: The point of inflexion at (1) has disappeared

Remark 3.5.1.

1. In the series of figures the progression from figure (3.2) to figure (3.3) illustrates the root that disappears is in fact the minimising one.

2. On the cool part of the caustic the global minimising root corresponds to a point of inflexion. This inflexion point cannot be continued beyond the caustic and hence the minimising root must jump form (1) to (2).

3. It is precisely this concept that allows us to analyse the jump discontinuities in the v^0 Burgers solution and the exponential jump discontinuities in the u^0 solution of the heat equation.

This leads us to a formal definition of the hot and cool classification of the caustic.

Definition 3.5.2. Let $\lambda \mapsto x_t(\lambda)$ be a parameterisation of the caustic C_t for $\lambda \in \mathbb{R}$. Suppose the derivative of the reduced one dimensional action functional $f'_{(x_t(\lambda),t)}(x_0^1)$ has roots $\beta_1^1, \beta_2^1, \ldots, \beta_{d-1}^1, \beta_d^1$. Proposition (3.5.1) guarantees one of these roots, label this β_r^1, will be repeated. Consider a fixed point $\lambda = \tilde{\lambda}$. One side of C_t, the caustic, will be cool at $x_t(\tilde{\lambda})$ if

$$f_{(x_t(\tilde{\lambda}),t)}(\beta_r^1) \leq \min_{\substack{i=1,2,\ldots,d \\ i \neq r}} f_{(x_t(\tilde{\lambda}),t)}(\beta_i^1). \tag{3.5.2}$$

The boundary of the cool part is given by

$$f_{(x_t(\tilde{\lambda}),t)}(\beta_r^1) = \min_{\substack{i=1,2,\ldots,d \\ i \neq r}} f_{(x_t(\tilde{\lambda}),t)}(\beta_i^1).$$

C_t is considered to be hot if Equation (3.5.2) is not satisfied at $x_t(\tilde{\lambda})$.

Corollary 3.5.3. *The $v^0(x,t)$ solution of Burgers equation with vorticity is smooth across hot parts of C_t but switches discontinuously across cool parts.*

Proof. As we cross C_t the point of inflexion disappears. Consequently, If we cross a cool part of C_t the minimising x_0^1 must switch discontinuously to another one. If we cross a hot part of the caustic the disappearance of the inflexion has no effect on $v^0(x,t)$ since it has no effect on the minimiser. □

In short, the reduced action functional $f(x_0^1) = f(x_0^1, x, t)$ provides a complete one dimensional method for analysing an d-dimensional problem. Moreover, its usefulness in determining jump discontinuities in the inviscid limiting Burgers velocity is paramount.

3.6 Intermittence of stochastic turbulence

The Zeta process is a stochastic process ζ^ϵ whose zeros are the turbulent times t at which the prelevel surface touches the precaustic. The idea holds in any number of dimensions so long as the map Φ_t is globally reducible. For the sake of clarity we work here in two dimensions.

Definition 3.6.1. *Turbulent times* are those times when the prelevel surface and precaustic touch.

Remark 3.6.1. It follows immediately from Proposition (3.3.5) and the above definition that these are the times at which the number of cusped meeting points on the level surface with the caustic changes.

Recall from Proposition (3.5.1) that $f'_{(x,t)}(x_0^1)$ has a repeated root if, and only if, x is on the caustic. In two dimensions, if the precaustic is parametrised by $\lambda \mapsto (\lambda, x_0^2(\lambda))$, $\lambda \in \mathbb{R}$, so that $\Phi_t((\lambda, x_0^2(\lambda))$ is a parameterisation of the caustic, the repeated root must be $x_0^1 = \lambda$. The number of meeting points of the precurves is therefore given by

$$\#\left\{\lambda \in \mathbb{R} : f_{(x_t(\lambda),t)}(\lambda) = c\right\} ,$$

In order to find the turbulent times we wish to know when this value changes. Consequently, the turbulent times must satisfy

$$f_{(x_t(\lambda),t)}(\lambda) = c \quad \text{and} \quad f'_{(x_t(\lambda),t)}(\lambda) = 0.$$

This leads to the following results on the so called *Zeta Process.*

Definition 3.6.2. Assume the stochastic classical mechanical flow map Φ_t is globally reducible and that the precaustic is parametrised by $\lambda \mapsto (\lambda, x_0^2(\lambda))$, $\lambda \in \mathbb{R}$. Then the Zeta Process is defined by

$$\zeta_c(t) := f_{(x_t(\lambda_0),t)}(\lambda_0) - c,$$

where $f_{(x,t)}(x_0^1)$ is the reduced action evaluated at points $x = x_t(\lambda_0) = \Phi_t(\lambda_0, x_0^2(\lambda_0))$ on the caustic and $\lambda = \lambda_0$ satisfies

$$\dot{X}_t(\lambda). \frac{dx_t}{d\lambda}(\lambda) = 0,$$

where $\dot{X}(\lambda) = \dot{\Phi}_t(\lambda, x_0^2(\lambda)), \Phi_t((\lambda_0, x_0^2(\lambda_0)) \in \text{Cool}(C_t)$.

Remark 3.6.2. Defining $\zeta_c(t)$ in this way indicates that the ζ_0 processes are simply the stochastic action evaluated at cusps on the caustics and their inverse images.

Proposition 3.6.1. *The turbulent times of the Burgers fluid are the random times* $t(\omega)$ *which are the zeros of the* $\zeta^c(t)$ *stochastic process. Moreover, the turbulence occurs at* $\Phi_t((\lambda_0, x_0^2(\lambda_0)) \in Cool(C_t)$.

Proof. Let $\lambda \mapsto (\lambda, x_0^2(\lambda))$, $\lambda \in \mathbb{R}$ be a parameterisation of the precaustic $\Phi_t^{-1}C_t$. Then the number of cusps on the level surface $\mathcal{S} = c$ is

$$\#\left\{\lambda \in \mathbb{R} : f_{(x_t(\lambda),t)}(\lambda) = c\right\} .$$

Differentiating this expression with respect to λ yields $\lambda = \lambda_0$ satisfying

$$\dot{X}_t(\lambda).\frac{dx_t}{d\lambda}(\lambda) = 0,$$

and $\zeta_c(t) = 0$. □

RTW remark that the above suggests three kinds of turbulence:-

1. Cusped, where there is a cusp on the caustic,
2. Zero speed, where the Burgers fluid velocity is zero,
3. Orthogonal, where the Burgers fluid velocity is orthogonal to the caustic.

Remark 3.6.3. At the turbulent times $t(\omega)$ the number of cusps on a cool part of the caustic change. At such times the geometry of the level surface of Hamilton's characteristic function changes infinitely rapidly giving rise to the turbulent behaviour.

3.7 Analytical results of DTZ for small noise

We conclude this chapter by extending some small ε analytical results of DTZ in [7] and [8] to the Burgers fluid with vorticity. Consider, for $v = v(x,t)$,

$$dv + (v \cdot \nabla)\, vdt + (v \wedge \text{curl } v)dt = -\nabla V(x)dt - da(x,t) - \varepsilon \nabla k(x,t)dW_t$$

with the corresponding stochastic classical mechanics

$$\begin{aligned} d\dot{X}^\varepsilon(x_0,s) &= -\nabla V(X^\varepsilon(x_0,s))ds - \text{curl } a(X^\varepsilon(x_0,s)) \wedge dX^\varepsilon(x_0,s) \\ &\quad - da(X^\varepsilon(x_0,s)) - \varepsilon \nabla k(X^\varepsilon(x_0,s))dW_s, \end{aligned} \qquad (3.7.1)$$

with $X^\varepsilon(x_0,0) = x_0$ and $\dot{X}^\varepsilon(x_0,0) = \nabla S_0(x_0) - a(x,0)$, $0 < s < t$. Let $X^0(x,s) = \Phi_s^0 x_0$ satisfy the deterministic ($\varepsilon = 0$) version of Equation (3.7.1). Let $\mathcal{G}$ be represented by

$$\mathcal{G}_{ij} = \left\{X_i^0(u), X_j^0(s)\right\}\theta(s-u),$$

which is the product of the Poisson bracket $\{\}$ and the Heaviside function θ.

Lemma 3.7.1. *$\mathcal{G}$ satisfies the matrix Jacobi equation*

$$\left(\frac{d^2}{ds^2} + V''(X^0(x_0,s))\right)\mathcal{G}(x_0,s,u) = 0,$$

with boundary condition

$$\mathcal{G}(x_0,s_+,s) = 0, \quad \frac{d\mathcal{G}}{ds}(x_0,s,u)\bigg|_{u=s_-} = I.$$

Let

$$\tilde{X}^\varepsilon(x_0,s) = \Phi_s^0 x_0 - \varepsilon\int_0^s \mathcal{G}(x_0,s,u)\nabla k(\Phi_u^0 x_0,u)dW_u,$$

for $s \in [0,t]$. This is the first term in the perturbation expansion for X^ε.

Proof. Consider our underlying classical Hamiltonian

$$H(q,p,t) = \frac{1}{2}(p-a(q,t))^2 + V(q)$$

and formally write its small noise stochastic analogue as

$$H(q,p,t) = \frac{1}{2}(p-a(q,t))^2 + V(q) + \varepsilon k(q,t)\dot{W}_t.$$

Write the classical mechanical path as

$$\ddot{X}^0(s) = f(X^0(s),\dot{X}^0(s),s) \tag{3.7.2}$$

which we assume to have a well behaved solution $X^0(s)(q_0,p_0)$, $(q_0,p_0)\in\mathbb{R}^6$ with initial conditions $\dot{X}^0(0) = q_0$, $\dot{X}^0(0) = p_0 - a(q_0)$. Recall from Equation (3.1.6) that we are interested in solutions of

$$\ddot{X}(s) = f(X(s),\dot{X}(s),s) - \varepsilon\nabla k(X(s),s)\dot{W}_s.$$

To solve this equation write

$$X(s) = X_0(s) + \varepsilon X_1(s) + \mathrm{O}(\varepsilon^2)$$

and try to find $X_1(s)$. If we assume f is smooth then, dropping the s dependence,

$$\ddot{X}_0 + \varepsilon\ddot{X}_1 + \mathrm{O}(\varepsilon^2) = f(X_0+\varepsilon X_1, \dot{X}_0+\varepsilon\dot{X}_1, s) - \varepsilon\nabla k\dot{W}_s + O(\varepsilon^2)$$

where $\nabla k = \nabla k(X_0(s),s)$. Equating coefficients of powers of ε we obtain

$$\ddot{X}_0 = f(X_0(s),\dot{X}_0(s),s) \tag{3.7.3}$$

and

$$\left(D^2 - \frac{\partial f}{\partial\dot{X}}D - \frac{\partial f}{\partial\dot{X}}\right)X_1 = -\varepsilon\nabla k\dot{W}_s \tag{3.7.4}$$

where D is the derivative operator. We proceed by trying to find the matrix $L = (L_{i,j})$ where

$$L_{ij} = \frac{d^2}{ds^2}\delta_i j - \frac{\partial f^i}{\partial \dot{X}_0^j}\frac{d}{ds} - \frac{\partial f^i}{\partial X_0^j}, \qquad i,j = 1,2,3,$$

with $\frac{\partial f^i}{\partial \dot{X}_0^j} = \frac{\partial f^i}{\partial \dot{X}_0^j}(X_0(s), \dot{X}_0(s), s)$ etc. so that

$$\sum_{j=1}^{3} L_{ij}X_1^j = -\varepsilon \nabla_i k(X_0(s), s)\dot{W}_s \qquad i = 1,2,3.$$

Observe that for any parameter α, partially differentiating Equation (3.7.3) yields

$$\frac{d^2}{ds^2}\left(\frac{\partial X_0^j}{\partial \alpha^0}\right) = \frac{\partial f^i}{\partial X^j}\frac{\partial X_0^j}{\partial \alpha^0} + \frac{\partial f^i}{\partial \dot{X}_0^j}\frac{d}{ds}\left(\frac{\partial X_0^j}{\partial \alpha^0}\right), \qquad i,j = 1,2,3.$$

Using the matrix notation we can say

$$\sum_{j=1}^{3} L_{ij}\frac{\partial X_0^j}{\partial \alpha} = 0 \qquad \text{for} \quad i = 1,2,3,$$

since $\{X_0^i(s), X_0^j(t)\}_{q_0,p_0}$ is a linear combination of such derivatives. Define

$$\mathcal{G}_{ij}(s,t) = \{X_0^i(s), X_0^j(t)\}\theta(t-s)$$

to be the product of the Poisson bracket $\{\}$ and heaviside function θ which satisfies

$$\lim_{s\nearrow t}\mathcal{G}_{ij}(s,t) = \{X_0^i(t), X_0^j(t)\} = \{q_i(t), q_j(t)\}_{q_0,p_0} = 0.$$

Then $\mathcal{G}_{ij}(s,t)$ must satisfy

$$\begin{aligned}
\lim_{s\nearrow t}\frac{d}{ds}\mathcal{G}_{ij}(s,t) &= \lim_{s\nearrow t}\left\{\dot{X}_0^i(s), X_0^j(t)\right\} \\
&= \{p_i(t) - a_i(q(t),t), q_j(t)\}_{q_0,p_0} \\
&= \{p_i(t), q_j(t)\} \\
&= -\delta_{ij} \qquad i,j = 1,2,3.
\end{aligned}$$

So $\mathcal{G}$ must be the matrix solution of

$$L(s)\mathcal{G}(s,t) = 0, \qquad s < t,$$

with

$$\lim_{s\nearrow t}\mathcal{G}(s,t) = 0 \quad \text{and} \quad \lim_{s\nearrow t}\frac{d}{ds}\mathcal{G}(s,t) = I.$$

Hence, $\mathcal{G}$ is the matrix green function for our set-up. It follows from a simple calculation that

$$X_1(s) = -\varepsilon\int_0^s \mathcal{G}(u,s)du\nabla k(X_0(u),u)\dot{W}_u$$

that is to say that X_1 is the vector

$$X_1(s) = -\varepsilon \int_0^s \mathcal{G}(u,s)\nabla k(X_0(u),u)dW_u.$$

All that remains is to find the differential operator when L is in matrix form where $f = -\nabla V - \left(\text{curl } a \wedge \dot{X}\right) - \frac{\partial a}{\partial t}$. So

$$\begin{aligned}
\frac{\partial f^i}{\partial X^j} &= -\nabla_i\nabla_j V - \nabla_j\left(\text{curl}a \wedge \dot{X}\right)_i - \nabla_j\frac{\partial a^i}{\partial t} \\
&= -\nabla_i\nabla_j V - \nabla_j\left(\varepsilon_{ilm}\varepsilon_{lnp}\nabla_n a_p \dot{X}_m\right) - \nabla_j\frac{\partial a^i}{\partial t} \\
&= -\nabla_i\nabla_j V + \nabla_j\left(\varepsilon_{iml}\varepsilon_{lnp}\nabla_n a_p \dot{X}_m\right) - \nabla_j\frac{\partial a^i}{\partial t} \\
&= -\nabla_i\nabla_j V + \left(\delta_{in}\delta_{mp} - \delta_{ip}\delta_{mn}\right)\nabla_j\nabla_n a_p \dot{X}_m - \nabla_j\frac{\partial a^i}{\partial t} \\
&= -\nabla_i\nabla_j V + \left(\nabla_j\nabla_i a_m\right)\dot{X}_m - \left(\dot{X}_n\nabla_n\right)\nabla_j a_i - \nabla_j\frac{\partial a^i}{\partial t} \quad \text{for} \quad i,j = 1,2,3.
\end{aligned}$$

Likewise,

$$\begin{aligned}
\frac{\partial f^i}{\partial \dot{X}^j} &= -\varepsilon_{ilm}\varepsilon_{lpq}\nabla_p a_q \delta_{mj} \\
&= \varepsilon_{ijl}\varepsilon_{lpq}\nabla_p a_q \\
&= \nabla_i a_j - \nabla_j a_i \quad \text{for} \quad i,j = 1,2,3.
\end{aligned}$$

□

Remark 3.7.1. For details of convergence in the above proof see [9].

Theorem 3.7.2. *Given that the vector potential $a \in C_0^\infty$ together with some mild conditions on continuity and boundedness of V and k and their derivatives, there exists a constant $M > 0$ such that for any $\delta > 0$ and sufficiently small $\varepsilon > 0$*

$$\mathbb{P}\left\{\frac{1}{\varepsilon^{\frac{3}{2}}}\sup_{x_0}\left|X^\varepsilon(x_0,s) - \tilde{X}^\varepsilon(x_0,s)\right| > \delta \quad \textit{for some} \quad \delta \in [0,t]\right\} < \frac{M\varepsilon^2}{\delta^4}$$

and

$$\mathbb{P}\left\{\frac{1}{\varepsilon^{\frac{3}{2}}}\sup_{x_0}\left|\nabla X^\varepsilon(x_0,s) - \nabla\tilde{X}^\varepsilon(x_0,s)\right| > \delta \quad \textit{for some} \quad \delta \in [0,t]\right\} < \frac{M\varepsilon}{\delta^2}.$$

In particular,

$$X^\varepsilon(x_0,s) - \tilde{X}^\varepsilon(x_0,s) = O(\varepsilon^{\frac{3}{2}})$$

and

$$\nabla X^\varepsilon(x_0,s) - \nabla\tilde{X}^\varepsilon(x_0,s) = O(\varepsilon^{\frac{3}{2}})$$

as $\varepsilon \searrow 0$ in probability.

Proof. See [9] for details. □

With no consideration for a vector potential, DTZ indicate that it is not difficult to show that the precaustic surface of the stochastic mechanics converges to the precaustic surface of classical mechanics as $\varepsilon \searrow 0$ in probability. Caustic surfaces are stable in probability. We would expect this to be true with a vector potential as well as the following theorem which is pertinent to the stability of level surfaces of the Hamilton-Jacobi function.

Theorem 3.7.3. *Let Φ_s be the minimiser of*

$$\frac{1}{2}\int_0^t \left|\dot{\Phi}_s x_0\right|^2 ds + S_0(\Phi_t x_0) + \int_0^t a(\Phi_s x_0).\dot{\Phi}_s x_0 ds - \int_0^t V(\Phi_s x_0) ds.$$

such that $\Phi_t x_0 = x$, with corresponding minimum $\mathcal{S}^0(x,t)$ and let Φ_s^ε be the minimiser of

$$\frac{1}{2}\int_0^t \left|\dot{\Phi}_s^\varepsilon x_0\right|^2 ds + S_0(\Phi_t^\varepsilon x_0) + \int_0^t a(\Phi_t^\varepsilon x_0).\dot{\Phi}_s^\varepsilon x_0 ds - \int_0^t V(\Phi_s^\varepsilon x_0) ds - \varepsilon \int_0^t k(\Phi_s^\varepsilon x_0) dW_s,$$

satisfying $\Phi_t^\varepsilon x_0 = x$ for almost all $\omega \in \Omega$ with corresponding minimum $\mathcal{S}^\varepsilon(x,t)$. Then we have for almost all $\omega \in \Omega$

$$\mathcal{S}^0(x,t) - \varepsilon \int_0^t k(\Phi_s^\varepsilon x_0) dW_s \leq \mathcal{S}^\varepsilon(x,t) \leq \mathcal{S}^0(x,t) - \varepsilon \int_0^t k(\Phi_s x_0) dW_s.$$

In particular, as $\varepsilon \searrow 0$, $\mathcal{S}^\varepsilon(x,t) \to \mathcal{S}^0(x,t)$ a.s.

Proof. See [9]. Suffice to say the set of ω's for which the above inequalities fail depends on ε. □

Following this result DTZ remark:

Remark 3.7.2. If one assumes there exists a unique x_0 for fixed t and x such that $\Phi_t x_0 = x$, then the first approximation is

$$\mathcal{S}^\varepsilon(x,t) = \mathcal{S}^0(x,t) - \varepsilon \int_0^t k(\Phi_s x_0) dW_s + O(\varepsilon^2),$$

where $\mathcal{S}^0(x,t)$ is Hamilton's characteristic function for the path $X^0(x_0,s)$. Similar results hold for $x_0^i(x,t)$ and corresponding $\mathcal{S}^i$.

3.8 Zeta Process for small noise in 2 dimensions

We conclude this chapter with an elegant result of DTZ concerning the ζ process for small noise in 2 dimensions. We are interested in the consequences of adding a small noise potential term $\varepsilon k(x,s)\dot{W}_s$ to our underlying deterministic classical mechanical system. The next proposition indicates what one can expect of the stochastic turbulence at the displaced deterministic cusp $x_t(\lambda_0)$. We use the notation Φ_t^0 to represent the globally reducible deterministic flow map where $X_s^0 = \Phi_s^0 x_0$. It is important to realise that $X_0^0 = x_0^r(x,t)$ and $x = X_t^0 = x_t(\lambda_0)$ with x_0^r the repeated root vector. When $\dot{X}_t^0 = 0$ a straightforward result exists for the small noise ζ process.

Proposition 3.8.1. *Let ζ_c^0 be the stochastic turbulence at the cusp on the deterministic caustic $x_t(\lambda_0)$. Assuming $\dot{X}_t^0 = 0$, formally correct to first order in ε, the stochastic turbulence processes ζ are given by*

$$\zeta_c(t) = \zeta_c^0(t) - \varepsilon \int_0^t k_s(\Phi_s^0(x_0^r(x_t(\lambda_0), t)))dW_s, \qquad c \in \mathbb{R},$$

where $x_0^r(x_t(\lambda_0), t)$ is the repeated root vector evaluated at $x_t(\lambda_0)$. ζ_c^0 is the reduced classical action at $x_t(\lambda_0)$ and $x_0^r(x_t(\lambda_0), t)$.

Remark 3.8.1. $\zeta_c^0(t)$ is a deterministic function which is simply the reduced classical action evaluated for the repeated root at the cusp on the caustic, i.e. $\zeta_c^0 = f^0_{(x_t(\lambda),t)}(x_0^r)$ using our usual notation.

Remark 3.8.2. A simple extension of work by Reynolds in [25] give numerous examples of turbulence when $\dot{X}_t^0 = 0$ in the free case.

In Chapter 5, by means of an example, we explicitly illustrate the usefulness of the ζ process and how the recurrent nature gives rise to the intermittence of the stochastic turbulence.

Chapter 4

Deterministic Burgers Fluid in a Rotating Bucket under a Harmonic Oscillator Potential

Summary

In this chapter we use a detailed deterministic example to complement geometrical and analytical results of previous chapters. For this illustration we use a deterministic Burgers fluid in a rotating bucket under a harmonic oscillator potential.

4.1 The set-up

In this chapter our equation of interest is the deterministic inviscid Burgers equation with vorticity, namely

$$\frac{\partial v}{\partial t} + (v.\nabla)\, v + v \wedge \text{curl} v = -\nabla V(x),$$

where $v = v(x,t)$, $x \in \mathbb{R}^3$ and $v(x,0) = \nabla S_0(x) - a(x,0)$. The domain of this equation is the whole of $\mathbb{R}^3$. However, the fluid ceases to rotate at infinity since the vector potential is C_0^∞. Working in three dimensions, consider the Hamiltonian

$$H(q,p) = 2^{-1}(p - a(q))^2 + V(q),$$

where $q = (q_1, q_2, q_3)$ and $p = (p_1, p_2, p_3)$ are canonical cartesian coordinates. Hamilton's equations read

$$\dot{q} = p - a(q), \tag{4.1.1}$$

$$\dot{p} = -(\text{curl } a(q) \wedge \dot{q}) - \nabla V(q), \tag{4.1.2}$$

where $a = (a_1, a_2, a_3)$ and $V = (V_1, V_2, V_3)$ represents the vector and scalar potentials respectively. Differentiating Equation (4.1.1) gives

$$\begin{aligned} \ddot{q} &= \dot{p} - \dot{a}(q) \\ &= -(\text{curl } a(q) \wedge \dot{q}) - \nabla V(q) - \frac{\partial a(q)}{\partial t}. \end{aligned} \tag{4.1.3}$$

Consider a uniform field Ω, which is everywhere parallel to the q_3 axis, acting upon the fluid particles, i.e. $\Omega = (0, 0, \Omega)$. The vector potential $a(q)$ for such a field is defined to be

$$\begin{aligned} a(q) &:= -\frac{1}{2} q \wedge \Omega \\ &= \left(-\frac{\Omega q_2}{2}, \frac{\Omega q_1}{2}, 0 \right). \end{aligned}$$

Observe that $a(q)$ is time independent and

$$\text{curl } a = \left\| \begin{matrix} \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ -\frac{\Omega q_2}{2} & \frac{\Omega q_1}{2} & 0 \end{matrix} \right\| = (0, 0, \Omega) = \Omega.$$

Equation (4.1.3) now simplifies to

$$\ddot{q} = -\Omega \wedge \dot{q} - \nabla V(q). \tag{4.1.4}$$

Set V as the harmonic oscillator potential $V = \frac{1}{2} q \omega^2 q^T$, where ω^2 is a real symmetric positive definite 3×3 matrix with

$$(\omega)_{ij} = \begin{cases} \omega & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

In short, $V \equiv \frac{\omega^2}{2} |q|^2$, then Equation (4.1.4) becomes

$$\ddot{q} = -\Omega \wedge \dot{q} - \omega^2 q. \tag{4.1.5}$$

Changing notation to 3-dimensional Cartesian coordinates so that $q = (x, y, z)$ the above equation reduces to

$$\begin{aligned} (\ddot{x}, \ddot{y}, \ddot{z}) + \left\| \begin{matrix} 0 & 0 & \Omega \\ \dot{x} & \dot{y} & \dot{z} \end{matrix} \right\| + \omega^2 (x, y, z) = 0, \\ \Leftrightarrow (\ddot{x}, \ddot{y}, \ddot{z}) + (-\Omega \dot{y}, \Omega \dot{x}, 0) + \omega^2 (x, y, z) = 0. \end{aligned}$$

Coordinate wise, this result yields three equations. The last being the differential equation $\ddot{z} = -\omega^2 z$ which describes simple harmonic motion along the z-axis with solution

$$z(t) = z(0) \cos(\omega t) + \dot{z}(0) \sin(\omega t).$$

The remaining two equations govern motion in the (x, y) plane

$$\begin{aligned} \ddot{x} - \Omega \dot{y} + \omega^2 x = 0, \\ \ddot{y} + \Omega \dot{x} + \omega^2 y = 0. \end{aligned}$$

To solve these we multiply the second equation by i and add it to the first to obtain

$$(\ddot{x} + i\ddot{y}) - \Omega \dot{y} + i\Omega \dot{x} + \omega^2 (x + iy) = 0$$

or

$$(\ddot{x} + i\ddot{y}) + i\Omega(\dot{x} + i\dot{y}) + \omega^2(x + iy) = 0.$$

Setting $\varsigma(s) = x(s) + iy(s)$ this simply reduces to

$$\ddot{\varsigma} + i\Omega\dot{\varsigma} + \omega^2\varsigma = 0. \tag{4.1.6}$$

Consider the identity

$$g(D)e^{\alpha x} f(x) = e^{\alpha x} g(D + \alpha) f(x)$$

where D represents the derivative operator with respect to x, and $g(D)$ is some polynomial function of D.

Remark 4.1.1.

To see this observe that

$$\begin{aligned}\frac{d}{dx}(e^{\alpha x} f(x)) &= \alpha e^{\alpha x} f(x) + e^{\alpha x} D f(x),\\ &= e^{\alpha x}(D + \alpha) f(x),\end{aligned}$$

and

$$\frac{d^2}{dx^2}(e^{\alpha x} f(x)) = e^{\alpha x}(D + \alpha)^2 f(x).$$

We require the result when $\alpha = \frac{i\Omega}{2}$, so

$$\begin{aligned}\frac{d^2}{ds^2} e^{\frac{i\Omega}{2}s}\varsigma &= e^{\frac{i\Omega}{2}s}\left(\frac{d}{ds} + \frac{i\Omega}{2}\right)^2 \varsigma,\\ &= e^{\frac{i\Omega}{2}s}\left(\frac{d^2}{ds^2} + i\Omega\frac{d}{ds} - \frac{\Omega^2}{4}\right)\varsigma.\end{aligned}$$

and therefore Equation (4.1.6) becomes

$$\left(\frac{d^2}{ds^2} + \frac{\Omega^2}{4}\right) e^{\frac{i\Omega}{2}s}\varsigma = -\omega^2 e^{\frac{i\Omega}{2}s}\varsigma.$$

This gives

$$\frac{d^2}{ds^2}\left(e^{\frac{i\Omega}{2}s}\varsigma\right) = -\left(\omega^2 + \frac{\Omega^2}{4}\right)\left(e^{\frac{i\Omega}{2}s}\varsigma\right). \tag{4.1.7}$$

We recognise this format as that of a differential equation for simple harmonic motion. Observing that $\ddot{r}(t) = -\omega^2 r(t)$ has a solution $r(t) = r(0)\cos(\omega t) + \frac{\dot{r}(0)}{\omega}\sin(\omega t)$ we conclude that

$$\begin{aligned}e^{\frac{i\Omega}{2}s}\varsigma(s) = \left[e^{\frac{i\Omega}{2}s}\right]_{s=0}&\varsigma(0)\cos\left(s\sqrt{\omega^2 + \frac{\Omega^2}{4}}\right)\\ &+ \frac{1}{\sqrt{\omega^2 + \frac{\Omega^2}{4}}}\left[\frac{d}{ds}\left(e^{\frac{i\Omega}{2}s}\varsigma(s)\right)\right]_{s=0}\sin\left(s\sqrt{\omega^2 + \frac{\Omega^2}{4}}\right),\end{aligned}$$

$$e^{\frac{i\Omega}{2}s}\varsigma(s) = \varsigma(0)\cos\left(\frac{s}{2}\sqrt{4\omega^2+\Omega^2}\right) + \frac{2\left(\frac{i\Omega}{2}\varsigma(0)+\dot{\varsigma}(0)\right)}{\sqrt{4\omega^2+\Omega^2}}\sin\left(\frac{s}{2}\sqrt{4\omega^2+\Omega^2}\right),$$

$$\varsigma(s) = e^{-\frac{i\Omega}{2}s}\left[\varsigma(0)\cos\left(\frac{s}{2}\sqrt{4\omega^2+\Omega^2}\right) + \frac{(i\Omega\varsigma(0)+2\dot{\varsigma}(0))}{\sqrt{4\omega^2+\Omega^2}}\sin\left(\frac{s}{2}\sqrt{4\omega^2+\Omega^2}\right)\right],$$

$$\varsigma(s) = e^{-\frac{i\Omega}{2}s}\left[\varsigma(0)\cos\left(\frac{s\varpi}{2}\right) + \frac{(i\Omega\varsigma(0)+2\dot{\varsigma}(0))}{\varpi}\sin\left(\frac{s\varpi}{2}\right)\right], \qquad (4.1.8)$$

on setting $\sqrt{4\omega^2+\Omega^2} = \varpi$. Of course in order to make sense of the above equation we have assumed that $4\omega^2+\Omega^2 \geq 0$. If this was not the case then the trigonometric functions would simply be replaced by corresponding hyperbolic ones. We also need to address what is a reasonable initial assumption about the motion of the fluid particles in the (x, y) plane?
We assume that at time $t = 0$ the particles originate from

$$\varsigma(0) = x_0 + iy_0,$$

with time derivative

$$\dot{\varsigma}(0) = \dot{x}_0 + i\dot{y}_0.$$

Setting $a(x, 0) = a(x_0)$, recall from work in previous chapters that the initial momentum of such a system was defined to be $\nabla S_0(x_0, y_0, z_0) - a(x_0, y_0, z_0)$. Therefore, we must demand that

$$\dot{x}(0) = \nabla_{x_0}S_0(x_0, y_0) - a_{x_0}(x_0, y_0),$$
$$\dot{y}(0) = \nabla_{y_0}S_0(x_0, y_0) - a_{y_0}(x_0, y_0),$$

where ∇_{x_0} and ∇_{x_0} represent spatial derivatives with respect to x_0 and y_0. a_{x_0} and a_{y_0} are the first and second components of the vector potential respectively. Let us take the initial function $S_0(x_0, y_0, z_0)$ to be the generic cusp condition, i.e. $S_0(x_0, y_0, z_0) = \frac{x_0^2 y_0}{2}$. It follows that

$$\dot{x}(0) = x_0 y_0 + \frac{\Omega y_0}{2},$$
$$\dot{y}(0) = \frac{x_0^2}{2} - \frac{\Omega x_0}{2}.$$

This immediately implies

$$\dot{\varsigma}(0) = x_0 y_0 + \frac{\Omega y_0}{2} + i\left(\frac{x_0^2}{2} - \frac{\Omega x_0}{2}\right).$$

To simplify equation (4.1.8) we insert these initial conditions to obtain, for $s = t$,

$$\varsigma(t) = e^{-\frac{i\Omega}{2}t}\left[(x_0+iy_0)\cos\left(\frac{\varpi t}{2}\right) + \frac{(2x_0y_0+ix_0^2)}{\varpi}\sin\left(\frac{\varpi t}{2}\right)\right]. \qquad (4.1.9)$$

Remark 4.1.2. Recall for that $r, w \in \mathbb{C}$, $|rw| = |r|.|w|$ and $\arg(rw) = \arg(r) + \arg(w)$, which implies

$$|\varsigma(t)| = \left[\left(x_0 \cos\left(\frac{t\varpi}{2}\right) + \frac{2x_0y_0}{\varpi}\sin\left(\frac{t\varpi}{2}\right)\right)^2 + \left(y_0 \cos\left(\frac{t\varpi}{2}\right) + \frac{x_0^2}{\varpi}\sin\left(\frac{t\varpi}{2}\right)\right)^2\right]^{\frac{1}{2}}.$$

This expression gives us information on the distance of the fluid particles from the z-axis as time increases. Furthermore,

$$\begin{aligned}\arg\varsigma(t) &= \frac{\Omega t}{2} + \tan^{-1}\left(\frac{x_0\cos\left(\frac{t\varpi}{2}\right) + \frac{2x_0y_0}{\varpi}\sin\left(\frac{t\varpi}{2}\right)}{y_0\cos\left(\frac{t\varpi}{2}\right) + \frac{x_0^2}{\varpi}\sin\left(\frac{t\varpi}{2}\right)}\right), \\ &= \frac{\Omega t}{2} + \tan^{-1}\left(\frac{\varpi x_0\cos\left(\frac{t\varpi}{2}\right) + 2x_0y_0\sin\left(\frac{t\varpi}{2}\right)}{\varpi y_0\cos\left(\frac{t\varpi}{2}\right) + x_0^2\sin\left(\frac{t\varpi}{2}\right)}\right).\end{aligned}$$

Observe that the first term provides linear growth in time whilst the second gives an oscillatory behavior.

To obtain the caustic equation consider

$$\text{Det}\left(\frac{\partial X(t)}{\partial x_0}\right) = 0$$

for the classical mechanical flow path $X(s) \in \mathbb{R}^3$ where

$$X(s) := (\text{Re}\ \varsigma(s), \text{Im}\ \varsigma(s), z(s))$$

with boundary conditions $X(t) = (x, y, z)(t)$, $X(0) = (x_0, y_0, z_0)$. By eliminating the (x, y, z) variables we obtain the precaustic, likewise eliminating the (x_0, y_0, z_0) variables yields the caustic. Separation of Equation (4.1.9) into real and imaginary parts yields

$$\begin{aligned}x(t) &= \text{Re}\{\varsigma(t)\} \\ &= x_0\cos\left(\frac{t\varpi}{2}\right)\cos\left(\frac{t\Omega}{2}\right) + \frac{2x_0y_0\cos\left(\frac{t\Omega}{2}\right)\sin\left(\frac{t\varpi}{2}\right)}{\varpi} + y_0\cos\left(\frac{t\varpi}{2}\right)\sin\left(\frac{t\Omega}{2}\right) \\ &+ \frac{{x_0}^2\sin\left(\frac{t\varpi}{2}\right)\sin\left(\frac{t\Omega}{2}\right)}{\varpi}\end{aligned} \tag{4.1.10}$$

and

$$\begin{aligned}y(t) &= \text{Im}\{\varsigma(t)\} \\ &= y_0\cos\left(\frac{t\varpi}{2}\right)\cos\left(\frac{t\Omega}{2}\right) + \frac{{x_0}^2\cos\left(\frac{t\Omega}{2}\right)\sin\left(\frac{t\varpi}{2}\right)}{\varpi} - x_0\cos\left(\frac{t\varpi}{2}\right)\sin\left(\frac{t\Omega}{2}\right) \\ &- \frac{2x_0y_0\sin\left(\frac{t\varpi}{2}\right)\sin\left(\frac{t\Omega}{2}\right)}{\varpi},\end{aligned} \tag{4.1.11}$$

in addition to $z(t) = z_0\cos(\omega t)$. Therefore

$$\text{Det}\left(\frac{\partial X(t)}{\partial x_0}\right) = \text{Det}\begin{pmatrix} \frac{\partial}{\partial x_0}Re\{\varsigma(t)\} & \frac{\partial}{\partial y_0}Re\{\varsigma(t)\} & \frac{\partial}{\partial z_0}Re\{\varsigma(t)\} \\ \frac{\partial}{\partial x_0}Im\{\varsigma(t)\} & \frac{\partial}{\partial y_0}Im\{\varsigma(t)\} & \frac{\partial}{\partial z_0}Im\{\varsigma(t)\} \\ \frac{\partial}{\partial x_0}z(t) & \frac{\partial}{\partial y_0}z(t) & \frac{\partial}{\partial z_0}z(t)\end{pmatrix} = 0 \tag{4.1.12}$$

where

$$\frac{\partial}{\partial x_0} Re\{\varsigma(t)\} = \cos\left(\frac{\varpi t}{2}\right)\cos\left(\frac{\Omega t}{2}\right) + \frac{2y_0\cos\left(\frac{\Omega t}{2}\right)\sin\left(\frac{\varpi t}{2}\right)}{\varpi} + \frac{2x_0\sin\left(\frac{\varpi t}{2}\right)\sin\left(\frac{\Omega t}{2}\right)}{\varpi},$$

$$\frac{\partial}{\partial y_0} Re\{\varsigma(t)\} = \frac{2x_0\cos\left(\frac{\Omega t}{2}\right)\sin\left(\frac{\varpi t}{2}\right)}{\varpi} + \cos\left(\frac{\varpi t}{2}\right)\sin\left(\frac{\Omega t}{2}\right),$$

$$\frac{\partial}{\partial z_0} Re\{\varsigma(t)\} = 0,$$

$$\frac{\partial}{\partial x_0} Im\{\varsigma(t)\} = \frac{2x_0\cos\left(\frac{\Omega t}{2}\right)\sin\left(\frac{\varpi t}{2}\right)}{\varpi} - \cos\left(\frac{\varpi t}{2}\right)\sin\left(\frac{\Omega t}{2}\right) - \frac{2y_0\sin\left(\frac{\varpi t}{2}\right)\sin\left(\frac{\Omega t}{2}\right)}{\varpi},$$

$$\frac{\partial}{\partial y_0} Im\{\varsigma(t)\} = \cos\left(\frac{\varpi t}{2}\right)\cos\left(\frac{\Omega t}{2}\right) - \frac{2x_0\sin\left(\frac{\varpi t}{2}\right)\sin\left(\frac{\Omega t}{2}\right)}{\varpi},$$

$$\frac{\partial}{\partial x_0} z(t) = 0,$$

$$\frac{\partial}{\partial y_0} z(t) = 0,$$

$$\frac{\partial}{\partial z_0} z(t) = \cos(\omega t).$$

Elimination of x and y from Equations (4.1.10), (4.1.11) and (4.1.12) yields the precaustic equation

$$y_0(x_0) = -\frac{\varpi}{2}\cot\left(\frac{t\varpi}{2}\right) + \frac{2}{\varpi}\tan\left(\frac{t\varpi}{2}\right){x_0}^2. \tag{4.1.13}$$

Remark 4.1.3. Observe this equation to be a quadratic function of x_0. The vertical orientation of the precaustic is determined by sgn $\left[\frac{2}{\varpi}\tan\left(\frac{t\varpi}{2}\right)\right]$. This will be all important in later sections of the chapter.

Remark 4.1.4. For the remainder of the chapter we shall neglect the uninteresting simple harmonic motion of the fluid along the z axis.

Suppressing our considerations to two dimensions the caustic equation is best represented in terms of two parametric equations in x_0

$$\begin{aligned} x(x_0, t) = &- \frac{\varpi}{2}\cos\left(\frac{\varpi t}{2}\right)\cot\left(\frac{\varpi t}{2}\right)\sin\left(\frac{\Omega t}{2}\right) \\ &+ \frac{3}{\varpi}\sin\left(\frac{\varpi t}{2}\right)\sin\left(\frac{\Omega t}{2}\right)x_0^2 \\ &+ \frac{4}{\varpi^2}\cos\left(\frac{\Omega t}{2}\right)\sin\left(\frac{\varpi t}{2}\right)\tan\left(\frac{\varpi t}{2}\right)x_0^3 \end{aligned} \tag{4.1.14}$$

and

$$\begin{aligned} y(x_0,t) = &-\frac{\varpi}{2}\cos\left(\frac{\varpi t}{2}\right)\cos\left(\frac{\Omega t}{2}\right)\cot\left(\frac{\varpi t}{2}\right) \\ &+\frac{3}{\varpi}\cos\left(\frac{\Omega t}{2}\right)\sin\left(\frac{\varpi t}{2}\right)x_0^2 \\ &-\frac{4}{\varpi^2}\sin\left(\frac{\varpi t}{2}\right)\sin\left(\frac{\Omega t}{2}\right)\tan\left(\frac{\varpi t}{2}\right)x_0^3. \end{aligned} \tag{4.1.15}$$

The cusped singularities of the caustic are confirmed in Figure (4.1).

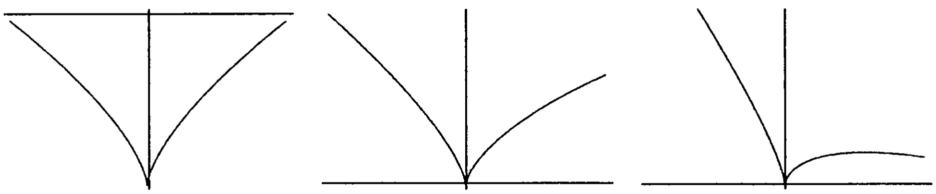


Figure 4.1: Left to right shows a typical time evolution of a cusped caustic.

Previously we had mentioned that our two dimensional classical mechanical path was given by $X(s) = (\text{Re}\,\varsigma(s), \text{Im}\,\varsigma(s))$. However, to calculate the classical action of our path $X(s)$ when it starts at a point x_0 and reaches a point x in a time t we first need to find the initial momentum vector $p(0) = p_0(x_0, y_0, x, y, t) = (p_{x_0}, p_{y_0})$ satisfying these boundary conditions. Consider

$$\begin{aligned} \varsigma(t) &= x(t) + iy(t), \\ \dot{\varsigma}(t) &= \dot{x}(t) + i\dot{y}(t). \end{aligned}$$

Hamilton's equation (4.1.1) gives us

$$\dot{x}(0) = p_{x_0} - a_{x_0}(x_0, y_0) \qquad \text{and} \qquad \dot{y}(0) = p_{y_0} - a_{y_0}(x_0, y_0)$$

and so

$$\begin{aligned} \dot{\varsigma}(0) &= \dot{x}(0) + i\dot{y}(0) \\ &= p_{x_0} - a_{x_0}(x_0, y_0) + i(p_{y_0} - a_{y_0}(x_0, y_0)). \end{aligned}$$

Substituting into the equation of motion (4.1.8) for $\varsigma(t)$, we obtain

$$\begin{aligned} \varsigma(t) &= \varsigma(x_0, y_0, p_0(x_0, y_0, x, y, t), x, y, t) \\ &= e^{-\frac{i\Omega t}{2}}\left[(x_0 + iy_0)\cos\left(\frac{\varpi t}{2}\right) + \frac{i\Omega}{2\varpi}(x_0 + iy_0)\sin\left(\frac{\varpi t}{2}\right)\right. \\ &\left.+\frac{2\sin\left(\frac{\varpi t}{2}\right)}{\varpi}\left(p_{x_0} + \frac{y_0\Omega}{2} + i\left(p_{y_0} - \frac{x_0\Omega}{2}\right)\right)\right], \end{aligned} \tag{4.1.16}$$

where $\varpi = \sqrt{4\omega^2 + \Omega^2}$. Observing the real and imaginary parts of Equation (4.1.16) to be equal to $x(t)$ and $y(t)$ respectively, the initial momentum vector $p_0 = (p_{x_0}, p_{y_0})$ of the classical mechanical path originating from (x_0, y_0) and arriving at (x, y) in a time t can be found explicitly. It turns out that

$$p_{x_0} = \frac{\varpi}{2}\operatorname{cosec}\left(\frac{\varpi t}{2}\right)\left(x\cos\left(\frac{\Omega t}{2}\right) - y\sin\left(\frac{\Omega t}{2}\right) - x_0\cos\left(\frac{\varpi t}{2}\right)\right)$$

and

$$p_{y_0} = \frac{\varpi}{2}\operatorname{cosec}\left(\frac{\varpi t}{2}\right)\left(y\cos\left(\frac{\Omega t}{2}\right) + x\sin\left(\frac{\Omega t}{2}\right) - y_0\cos\left(\frac{t\varpi}{2}\right)\right).$$

Substitution into Equation (4.1.16) and taking the real and imaginary parts yields the desired classical mechanical path $X(x_0, y_0, p_0(x_0, x, t), x, y, s) = (X(s), Y(s))$, where

$$\begin{aligned} X(s) &= \cos\left(\frac{\varpi s}{2}\right)\left(x_0\cos\left(\frac{\Omega s}{2}\right) + y_0\sin\left(\frac{\Omega s}{2}\right)\right) \\ &\quad - \cot\left(\frac{\varpi t}{2}\right)\sin\left(\frac{\varpi s}{2}\right)\left(x_0\cos\left(\frac{\Omega s}{2}\right) + y_0\sin\left(\frac{\Omega s}{2}\right)\right) \\ &\quad + \operatorname{cosec}\left(\frac{\varpi t}{2}\right)\sin\left(\frac{\varpi s}{2}\right)\left(x\cos\left(\frac{\Omega(s-t)}{2}\right) + y\sin\left(\frac{\Omega(s-t)}{2}\right)\right) \end{aligned}$$

and

$$\begin{aligned} Y(s) &= \cos\left(\frac{\varpi s}{2}\right)\left(y_0\cos\left(\frac{\Omega s}{2}\right) - x_0\sin\left(\frac{\Omega s}{2}\right)\right) \\ &\quad + \sin\left(\frac{\varpi s}{2}\right)\cot\left(\frac{\varpi t}{2}\right)\left(x_0\sin\left(\frac{\Omega s}{2}\right) - y_0\cos\left(\frac{\Omega s}{2}\right)\right) \\ &\quad + \sin\left(\frac{\varpi s}{2}\right)\operatorname{cosec}\left(\frac{\varpi t}{2}\right)\left(y\cos\left(\frac{\Omega(s-t)}{2}\right) - x\sin\left(\frac{\Omega(s-t)}{2}\right)\right). \end{aligned}$$

We use this path to calculate the action for a classical mechanical path starting at a point (x_0, y_0) and finishing at a point (x, y) in a time t. The action $A(x_0, y_0, x, y, t)$ of such a path was defined in Equation (3.1.2) as

$$A(x_0, y_0, x, y, t) = \int_0^t \left(\frac{1}{2}\dot{X}^2(s) + a(X(s)).\dot{X}(s) - V(X(s))\right) ds.$$

Via lengthy computation $\mathcal{A}(x_0, y_0, x, y, t) := A(x_0, y_0, x, y, t) + S_0(x_0, y_0)$ is given by

$$\begin{aligned} &\mathcal{A}(x_0, y_0, x, y, t) \\ &= \frac{x_0^2 y_0}{2} + \frac{\varpi}{4}\operatorname{cosec}\left(\frac{t\varpi}{2}\right) \times \\ &\left(\cos\left(\frac{t\varpi}{2}\right)\left(x^2 + {x_0}^2 + y^2 + {y_0}^2\right) - 2\left(xx_0 + yy_0\right)\cos\left(\frac{\Omega t}{2}\right) + 2\left(x_0 y - x y_0\right)\sin\left(\frac{\Omega t}{2}\right)\right) \end{aligned} \tag{4.1.17}$$

where $S_0(x_0, y_0) = \frac{x_0^2 y_0}{2}$. Using the equation $\mathcal{A} = c$, together with the relationship $\nabla_{x_0}\mathcal{A} = 0$ and $\nabla_{y_0}\mathcal{A} = 0$, it follows that the prelevel surface equation is given by

$$y_0(x_0) = \frac{2{x_0}^2\varpi - 6{x_0}^2\varpi\cos(\varpi t) \pm \sqrt{\mathcal{D}(x_0)}}{2\left(4{x_0}^2\sin(\varpi t) - \varpi^2\sin(\varpi t)\right)}$$

where

$$\begin{aligned}\mathcal{D}(x_0) :=& \left(2{x_0}^2\varpi - 6{x_0}^2\varpi\cos(\varpi t)\right)^2 \\ &- 4\left(\varpi^2\sin(\varpi t) - 4{x_0}^2\sin(\varpi t)\right)\left(8c\varpi - {x_0}^4\sin(\varpi t) + {x_0}^2\varpi^2\sin(\varpi t)\right).\end{aligned}$$

The level surface equations are

$$\begin{aligned}x(x_0,t) =& \frac{x_0\varpi}{2\varpi^2\left(\varpi^2 - 4{x_0}^2\right)} \\ &\times\left(2x_0\operatorname{cosec}\left(\frac{\varpi t}{2}\right)\sin\left(\frac{\Omega t}{2}\right)\left(\left(2{x_0}^2+\varpi^2\right)\cos\left(\varpi t\right) - 2{x_0}^2\right)\right. \\ &\left.- \varpi\sec\left(\frac{\varpi t}{2}\right)\cos\left(\frac{\Omega t}{2}\right)\left(\varpi^2 - 6{x_0}^2 + \left(2{x_0}^2+\varpi^2\right)\cos\left(\varpi t\right)\right)\right) \\ &\pm\frac{\sin(\frac{\varpi t}{2})^2\tan(\frac{\varpi t}{2})\sqrt{\mathcal{R}\left(2x_0\sin\left(\frac{\varpi t}{2}\right)\cos\left(\frac{\Omega t}{2}\right) + \varpi\cos\left(\frac{\varpi t}{2}\right)\sin\left(\frac{\Omega t}{2}\right)\right)^2}}{\varpi\left(\varpi^2 - 4{x_0}^2\right)},\end{aligned}$$

$$\begin{aligned}y(x_0,t) =& \frac{x_0\varpi}{2\varpi^2\left(\varpi^2 - 4{x_0}^2\right)} \\ &\times\left(2x_0\operatorname{cosec}(\frac{\varpi t}{2})\cos(\frac{\Omega t}{2})\left(\left(2{x_0}^2+\varpi^2\right)\cos\left(\varpi t\right) - 2{x_0}^2\right)\right. \\ &\left.- \varpi\sec\left(\frac{\varpi t}{2}\right)\sin\left(\frac{\Omega t}{2}\right)\left(\varpi^2 - 6{x_0}^2 + \left(2{x_0}^2+\varpi^2\right)\cos(\varpi t)\right)\right) \\ &\pm\frac{\sin(\frac{\varpi t}{2})^2\tan(\frac{\varpi t}{2})\sqrt{\mathcal{R}\left(\varpi\cos\left(\frac{\varpi t}{2}\right)\sin\left(\frac{\Omega t}{2}\right) - 2x_0\sin\left(\frac{\varpi t}{2}\right)\cos\left(\frac{\Omega t}{2}\right)\right)^2}}{\varpi\left(\varpi^2 - 4{x_0}^2\right)},\end{aligned}$$

where

$$\begin{aligned}\mathcal{R} = \frac{\operatorname{cosec}^8(\frac{\varpi t}{2})}{8}&\left({x_0}^2\left(2{x_0}^2+\varpi^2\right)^2\cos(2\varpi t) - {x_0}^2\left(4{x_0}^4 - 16{x_0}^2\varpi^2 + \varpi^4\right)\right. \\ &\left.- 12{x_0}^4\varpi^2\cos(\varpi t) - 16c\varpi\left(\varpi^2 - 4{x_0}^2\right)\sin(\varpi t)\right)\end{aligned}$$

and $\varpi = \sqrt{4\omega^2 + \Omega^2}$.

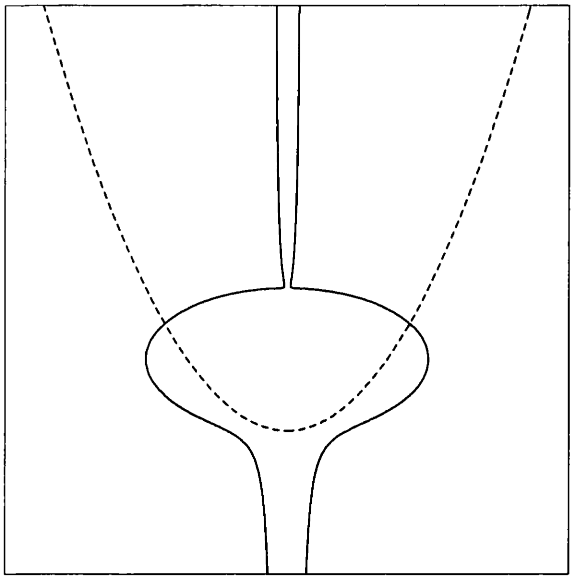


Figure 4.2: Zero prelevel surface and precaustic parabola (dashed line) for unit time, $\Omega = 0.2$, and $\omega = 0.01$, so that $\varpi = \sqrt{4\omega^2 + \Omega} = \sqrt{0.0404}$.

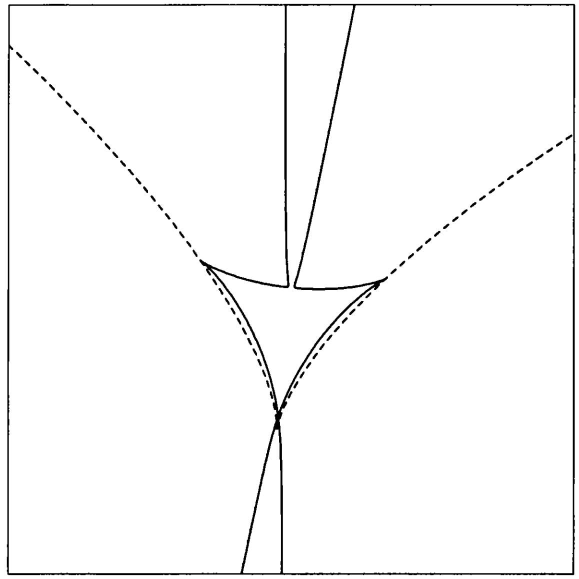


Figure 4.3: Zero level surface and cusped caustic (dashed line) for unit time, $\Omega = 0.2$, and $\omega = 0.01$, so that $\varpi = \sqrt{4\omega^2 + \Omega} = \sqrt{0.0404}$.

4.2 Finding the cusped points of the caustic

Definition 4.2.1. A curve $x = x(\lambda)$, $\lambda \in N(\lambda_0, \delta)$ has a generalised cusp at $\lambda = \lambda_0$, λ being arc-length, if

$$\left.\frac{dx}{d\lambda}\right|_{\lambda=\lambda_0} = 0.$$

Parameterising the caustic equations (4.1.14) and (4.1.15) by λ yields

$$x(\lambda, t) = -\frac{\varpi}{2}\cos\left(\frac{\varpi t}{2}\right)\cot\left(\frac{\varpi t}{2}\right)\sin\left(\frac{\Omega t}{2}\right) + \frac{3}{\varpi}\sin\left(\frac{\varpi t}{2}\right)\sin\left(\frac{\Omega t}{2}\right)\lambda^2 + \frac{4}{\varpi^2}\cos\left(\frac{\Omega t}{2}\right)\sin\left(\frac{t\varpi}{2}\right)\tan\left(\frac{\varpi t}{2}\right)\lambda^3 \quad (4.2.1)$$

and

$$y(\lambda, t) = -\frac{\varpi}{2}\cos\left(\frac{\varpi t}{2}\right)\cos\left(\frac{\Omega t}{2}\right)\cot\left(\frac{\varpi t}{2}\right) + \frac{3}{\varpi}\cos\left(\frac{\Omega t}{2}\right)\sin\left(\frac{\varpi t}{2}\right)\lambda^2 - \frac{4}{\varpi^2}\sin\left(\frac{\varpi t}{2}\right)\sin\left(\frac{\Omega t}{2}\right)\tan\left(\frac{\varpi t}{2}\right)\lambda^3 \quad (4.2.2)$$

By definition, if λ is arc length, the condition for cusps on the caustic in two dimensions is simply $\frac{dx}{d\lambda}(\lambda) = \frac{dy}{d\lambda}(\lambda) = 0$.

$$\frac{dx}{d\lambda}(\lambda) = \frac{6\lambda\sin(\frac{s\varpi}{2})\left(\varpi\sin(\frac{s\Omega}{2}) + 2\lambda\cos(\frac{s\Omega}{2})\tan(\frac{s\varpi}{2})\right)}{\varpi^2} = 0$$

and

$$\frac{dy}{d\lambda}(\lambda) = \frac{6\lambda\sin(\frac{s\varpi}{2})\left(\varpi\cos(\frac{s\Omega}{2}) - 2\lambda\sin(\frac{s\Omega}{2})\tan(\frac{s\varpi}{2})\right)}{\varpi^2} = 0$$

Clearly, both equations are simultaneously satisfied if $\lambda = 0$ or $t = \frac{2k\pi}{\varpi}$, for $k \in \mathbb{N}$. When $\lambda = 0$ the cusped point of the caustic is given by Equations (4.2.1) and (4.2.2) as

$$\begin{aligned}(x(0,t), y(0,t)) = \Bigg(&-\frac{\varpi}{2}\cos\left(\frac{\varpi t}{2}\right)\cot\left(\frac{\varpi t}{2}\right)\sin\left(\frac{\Omega t}{2}\right), \\ &-\frac{\varpi}{2}\cos\left(\frac{\varpi t}{2}\right)\cos\left(\frac{\Omega t}{2}\right)\cot\left(\frac{\varpi t}{2}\right)\Bigg). \qquad (4.2.3)\end{aligned}$$

Corollary 4.2.1. *As the time $t \to \infty$ the coordinate position of this cusp oscillates infinitely often in the interval $[-\infty, \infty]$. Consequently, there exists infinitely many zeros of $(x(0,t), y(0,t))$ as $t \to \infty$.*

Proof. Observe that whenever $\sin(\frac{t\varpi}{2}) = 0$ we have division by zero and consequent blow-up. This inevitably happens for those times $t = \frac{2k\pi}{\varpi}$, where $k \in \mathbb{N}$ and $\varpi = \sqrt{4\omega^2 + \Omega^2}$. For fixed values of Ω and ω, as $t \to \infty$ the first and second components of this two dimensional cusp oscillate infinitely often in the interval $(-\infty, \infty)$. Hence, by continuity and the intermediate value theorem there must exist infinitely many zeros. □

4.3 Using the reduced action functional to determine the cool parts of the caustic

Recall the following definition

Definition 4.3.1. The classical flow map Φ_t is globally reducible if

$$x = \Phi_t x_0 \Longleftrightarrow x_0^r = x_0^r(x, x_0^1, x_0^2, \ldots x_0^{r-1}, t)$$

where $x_0 = (x_0^1, x_0^2, \ldots x_0^d)$, $x = (x^1, x^2, \ldots x^d)$ and $r = d, d-1, d-2, \ldots 2$.

Definition 4.3.2. Assume the two-dimensional Φ_t map is globally reducible. Then the reduced action functional is given by

$$f_{(x,t)}(x_0^1) = \mathcal{A}(x_0^1, x_0^2(x, x_0^1, t), x, t),$$

where $x = (x^1, x^2)$, and $x_0 = (x_0^1, x_0^2)$. Moreover, if $x \in C_t$ and $\lambda \mapsto x_t(\lambda)$ is a parameterisation of C_t for $\lambda \in \mathbb{R}$ then the reduced action functional on the caustic is denoted by

$$f_{(x_t(\lambda),t)}(x_0) = \mathcal{A}(x_0^1, x_0^2(x(\lambda), x_0^1, t), x(\lambda), t).$$

Lemma 4.3.1. *The 2-dimensional Φ_t map, defined by Equations (4.1.10) and (4.1.11), is globally reducible.*

Proof. It follows from Equation (4.1.11)

$$y(x_0, y_0, x, t) = \frac{\sin\left(\frac{\varpi t}{2}\right)}{\varpi}\left(\cos(\frac{\Omega t}{2})\left(x_0{}^2 + y_0\varpi\cot(\frac{\varpi t}{2})\right)\right.$$
$$\left. - x_0\sin(\frac{\Omega t}{2})\left(2y_0 + \varpi\cot(\frac{\varpi t}{2})\right)\right)$$

which can be solved explicitly for $y_0 = y_0(x_0, x, y, t)$ as

$$y_0(x_0, x, y, t) = \frac{y\varpi - x_0{}^2\cos(\frac{\Omega t}{2})\sin(\frac{\varpi t}{2}) + x_0\varpi\cos(\frac{\varpi t}{2})\sin(\frac{\Omega t}{2})}{\varpi\cos(\frac{\varpi t}{2})\cos(\frac{\Omega t}{2}) - 2x_0\sin(\frac{\varpi t}{2})\sin(\frac{\Omega t}{2})}. \tag{4.3.1}$$

□

Proposition 4.3.2. *Let $\lambda \mapsto x_t(\lambda)$ be the 2-dimensional parameterisation of the caustic C_t for $\lambda \in \mathbb{R}$. Setting $\sqrt{4\omega^2 + \Omega^2} = \varpi$, the reduced one dimensional action function on the caustic is*

$$f_{(x_t(\lambda),t)}(x_0) = \frac{p_5x_0^5 + p_4x_0^4 + p_3x_0^3 + p_2x_0^2 + p_1x_0 + p_0}{(q_1x_0 + q_0)^2} \tag{4.3.2}$$

where

$$\begin{aligned}
p_5 &= p_5(t) \\
&= 8\varpi^3\sin(\frac{t\varpi}{2})^2\sin(t\Omega), \\
p_4 &= p_4(t) \\
&= -2\varpi^4\cos(\frac{t\Omega}{2})^2\sin(t\varpi), \\
p_3 &= p_3(\lambda, t) \\
&= -32\lambda^2\varpi^2\sin(\frac{t\varpi}{2})^2\sin(\frac{t\Omega}{2})\left(3\varpi\cos(\frac{t\Omega}{2}) + 2\lambda\sin(\frac{t\Omega}{2})\tan(\frac{t\varpi}{2})\right) \\
p_2 &= p_2(\lambda, t) \\
&= \frac{1}{2}\Bigg[\sin(t\varpi)\Big(-192\lambda^6 + 36\lambda^4\varpi^2 + 24\lambda^2\varpi^4 - \varpi^6 + \cos(t\Omega)\left(192\lambda^6 - 36\lambda^4\varpi^2 + \varpi^6\right)\Big) \\
&+ \sin(2t\varpi)\sin(\frac{t\Omega}{2})^2\left(64\lambda^6 - 36\lambda^4\varpi^2 - 12\lambda^2\varpi^4 - \varpi^6\right) \\
&- 64\lambda^3\varpi^3\sin(t\Omega)\Big(\cos(t\varpi) - 1\Big) - 256\lambda^6\tan(\frac{t\varpi}{2})\Big(-1 + \cos(t\Omega)\Big)\Bigg]
\end{aligned}$$

$$
\begin{aligned}
p_1 &= p_1(\lambda, t)\\
&= \frac{\varpi \sec(\frac{t\varpi}{2})}{16}\Big[64\lambda^3\varpi\left(6\lambda^2-\varpi^2\right)\sin\left(\frac{3t\varpi}{2}\right) - 64\lambda^3\varpi\left(18\lambda^2+\varpi^2\right)\sin\left(\frac{t\varpi}{2}\right)\\
&+ 128\lambda^6\sin\Big(t(\frac{\varpi}{2}-\Omega)\Big) + 576\lambda^5\varpi\sin\Big(t(\frac{\varpi}{2}-\Omega)\Big) - 72\lambda^4\varpi^2\sin\Big(t(\frac{\varpi}{2}-\Omega)\Big)\\
&- 32\lambda^3\varpi^3\sin\Big(t(\frac{\varpi}{2}-\Omega)\Big) + 24\lambda^2\varpi^4\sin\Big(t(\frac{\varpi}{2}-\Omega)\Big) - 10\varpi^6\sin\Big(t(\frac{\varpi}{2}-\Omega)\Big)\\
&- 192\lambda^6\sin\Big(t(\frac{3\varpi}{2}-\Omega)\Big) - 192\lambda^5\varpi\sin\Big(t(\frac{3\varpi}{2}-\Omega)\Big) + 108\lambda^4\varpi^2\sin\Big(t(\frac{3\varpi}{2}-\Omega)\Big)\\
&- 32\lambda^3\varpi^3\sin\Big(t(\frac{3\varpi}{2}-\Omega)\Big) - 12\lambda^2\varpi^4\sin\Big(t(\frac{3\varpi}{2}-\Omega)\Big) - 5\varpi^6\sin\Big(t(\frac{3\varpi}{2}-\Omega)\Big)\\
&+ 64\lambda^6\sin\Big(t(\frac{5\varpi}{2}-\Omega)\Big) - 36\lambda^4\varpi^2\sin\Big(t(\frac{5\varpi}{2}-\Omega)\Big) - 12\lambda^2\varpi^4\sin\Big(t(\frac{5\varpi}{2}-\Omega)\Big)\\
&- \varpi^6\sin\Big(t(\frac{5\varpi}{2}-\Omega)\Big) + 192\lambda^6\sin\Big(t(\frac{3\varpi}{2}+\Omega)\Big) - 192\lambda^5\varpi\sin\Big(t(\frac{3\varpi}{2}+\Omega)\Big)\\
&- 108\lambda^4\varpi^2\sin\Big(t(\frac{3\varpi}{2}+\Omega)\Big) - 32\lambda^3\varpi^3\sin\Big(t(\frac{3\varpi}{2}+\Omega)\Big) + 12\lambda^2\varpi^4\sin\Big(t(\frac{3\varpi}{2}+\Omega)\Big)\\
&+ 5\varpi^6\sin\Big(t(\frac{3\varpi}{2}+\Omega)\Big) - 64\lambda^6\sin\Big(t(\frac{5\varpi}{2}+\Omega)\Big) + 36\lambda^4\varpi^2\sin\Big(t(\frac{5\varpi}{2}+\Omega)\Big)\\
&+ 12\lambda^2\varpi^4\sin\Big(t(\frac{5\varpi}{2}+\Omega)\Big) + \varpi^6\sin\Big(t(\frac{5\varpi}{2}+\Omega)\Big)\\
&+ 2\Big(-64\lambda^6+288\lambda^5\varpi+36\lambda^4\varpi^2-16\lambda^3\varpi^3-12\lambda^2\varpi^4+5\varpi^6\Big)\sin\Big(t\Big(\frac{\varpi}{2}+\Omega\Big)\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
p_0 &= p_0(\lambda, t)\\
&= \frac{\varpi^2\operatorname{cosec}(t\varpi)}{64}\Big[1792\lambda^6 - 144\lambda^4\varpi^2 + 48\lambda^2\varpi^4 - 20\varpi^6\\
&+ \cos(t\varpi)\Big(-2176\lambda^6 + 72\lambda^4\varpi^2 + 24\lambda^2\varpi^4 - 30\varpi^6\Big)\\
&+ 4\cos(2t\varpi)\Big(64\lambda^6 + 36\lambda^4\varpi^2 - 12\lambda^2\varpi^4 - 3\varpi^6\Big)\\
&+ 128\lambda^6\cos(3t\varpi) - 72\lambda^4\varpi^2\cos(3t\varpi) - 24\lambda^2\varpi^4\cos(3t\varpi) - 2\varpi^6\cos(3t\varpi)\\
&+ 960\lambda^6\cos\left(t\left(\varpi-\Omega\right)\right) + 36\lambda^4\varpi^2\cos\left(t\left(\varpi-\Omega\right)\right) + 12\lambda^2\varpi^4\cos\left(t\left(\varpi-\Omega\right)\right)\\
&-15\varpi^6\cos\left(t\left(\varpi-\Omega\right)\right) - 384\lambda^6\cos(t\,(2\varpi-\Omega)) + 72\lambda^4\varpi^2\cos(t\,(2\varpi-\Omega))\\
&-24\lambda^2\varpi^4\cos(t\,(2\varpi-\Omega)) - 6\varpi^6\cos(t\,(2\varpi-\Omega)) + 64\lambda^6\cos\left(t\left(3\varpi-\Omega\right)\right)\\
&-36\lambda^4\varpi^2\cos\left(t\left(3\varpi-\Omega\right)\right) - 12\lambda^2\varpi^4\cos\left(t\left(3\varpi-\Omega\right)\right) - \varpi^6\cos\left(t\left(3\varpi-\Omega\right)\right)\\
&-1280\lambda^6\cos(t\Omega) - 144\lambda^4\varpi^2\cos(t\Omega) + 48\lambda^2\varpi^4\cos(t\Omega) - 20\varpi^6\cos(t\Omega)\\
&+960\lambda^6\cos\left(t\left(\varpi+\Omega\right)\right) + 36\lambda^4\varpi^2\cos\left(t\left(\varpi+\Omega\right)\right) + 12\lambda^2\varpi^4\cos\left(t\left(\varpi+\Omega\right)\right)\\
&-15\varpi^6\cos\left(t\left(\varpi+\Omega\right)\right) - 384\lambda^6\cos\left(t\left(2\varpi+\Omega\right)\right) + 72\lambda^4\varpi^2\cos\left(t\left(2\varpi+\Omega\right)\right)\\
&-24\lambda^2\varpi^4\cos\left(t\left(2\varpi+\Omega\right)\right) - 6\varpi^6\cos\left(t\left(2\varpi+\Omega\right)\right)\\
&+ \cos\left(t\left(3\varpi+\Omega\right)\right)\Big(64\lambda^6 - 36\lambda^4\varpi^2 - 12\lambda^2\varpi^4 - \varpi^6\Big)\Big],
\end{aligned}
$$

$$q_1 = q_1(t) = -32\varpi^3 \sin\left(\frac{t\varpi}{2}\right) \sin\left(\frac{t\Omega}{2}\right),$$
$$q_0 = q_0(t) = 16\varpi^4 \cos\left(\frac{t\varpi}{2}\right) \cos\left(\frac{t\Omega}{2}\right).$$

and $x = (x, y)$, $x_0 = (x_0, y_0)$.

Proof. Use the fact that the classical mechanical flow map is globally reducible to substitute Equation (4.3.1) and the parameterised caustic Equations (4.2.1) and (4.2.2) into the stochastic action $\mathcal{A}(x_0, x, t)$ given by Equation (4.1.17). □

Remark 4.3.1. See Section (4.5) for detailed graphs of this function.

Corollary 4.3.3. *On the caustic, the reduced action function* $f_{(x_t(\lambda),t)}(x_0)$ *has a singularity independent of the position on the caustic. Furthermore, this singularity occurs at* $x_0 = x_0^b$ *where*

$$x_0^b = \frac{\varpi \cot\left(\frac{\Omega t}{2}\right) \cot\left(\frac{t\varpi}{2}\right)}{2}. \tag{4.3.3}$$

We shall refer to this value as the break point of $f_{(x_t(\lambda),t)}(x_0)$.

Proof. On the caustic the denominator of the reduced one dimensional action function $f_{(x(\lambda,t))}(x_0)$ given by Equation (4.3.2) takes the form

$$16\varpi^3\left(\varpi \cos\left(\frac{\varpi t}{2}\right) \cos\left(\frac{\Omega t}{2}\right) - 2x_0 \sin\left(\frac{\varpi t}{2}\right) \sin\left(\frac{\Omega t}{2}\right)\right)^2,$$

consequently, for non-trivial values of t, ϖ and Ω, $f_{(x(\lambda,t))}(x_0)$ is undefined at

$$x_0 = x_0^b = \frac{\varpi \cot\left(\frac{\Omega t}{2}\right) \cot\left(\frac{t\varpi}{2}\right)}{2}.$$

□

Remark 4.3.2. Since the denominator of $f_{(x_t(\lambda),t)}(x_0)$ is independent of λ, the break point x_0^b is independent of the position on the caustic .

Proposition 4.3.4. *On the caustic there exist four stationary points of* $f_{(x_t(\lambda),t)}(x_0)$, *one of which is repeated. Label these*

$$x_0 = x_0^r = \lambda,$$
$$x_0 = x_{0,1} = -2\lambda,$$
$$x_0 = x_{0,2} = \nu - \kappa \left|\xi(\lambda)\right| \sqrt{\eta(\lambda)}$$
$$x_0 = x_{0,3} = \nu + \kappa \left|\xi(\lambda)\right| \sqrt{\eta(\lambda)},$$

where x_0^r is the repeated root of $f'_{(x_t(\lambda),t)}(x_0)$ and

$$\nu = \frac{\varpi \cot(\frac{\varpi t}{2}) \cot(\frac{\Omega t}{2})}{2},$$
$$\kappa = \frac{\operatorname{cosec}(\frac{\varpi t}{2})^2 \operatorname{cosec}(\frac{\Omega t}{2})^2 \sec(\frac{\varpi t}{2}) \sec(\frac{\Omega t}{2})}{8\sqrt{3}\varpi},$$
$$\xi(\lambda) = 4\lambda (\cos(\Omega t) - 1) \sin\left(\frac{\varpi t}{2}\right)^2 + \varpi \sin(\varpi t) \sin(\Omega t),$$
$$\eta(\lambda) = \varpi \left(\varpi \cos\left(\frac{\varpi t}{2}\right)^2 \cos\left(\frac{\Omega t}{2}\right)^2 + \lambda \sin(\varpi t) \sin(\Omega t) \right). \tag{4.3.4}$$

Proof. Differentiating Equation (4.3.2) with respect to x_0 yields a quintic numerator in x_0 which factorises as

$$\tan\left(\frac{t\varpi}{2}\right) (x_0 - \lambda)^2 (x_0 + 2\lambda) \left(x_0 - \nu + \kappa |\xi(\lambda)| \sqrt{\eta(\lambda)}\right) \left(x_0 - \nu - \kappa |\xi(\lambda)| \sqrt{\eta(\lambda)}\right),$$

where κ, ν, $\xi(\lambda)$ and $\eta(\lambda)$ are detailed above. □

Remark 4.3.3. Observe that κ and ν are independent of λ.

4.4 Finding hot and cool parts of the caustic

Recall from Theorem (3.5.1) and Definition (3.5.2) that one side of C_t will be *cool* at a point $x_t(\tilde{\lambda})$ if, and only if,

$$f_{(x_t(\tilde{\lambda}),t)}(x_0^r) \leq \min_{i=1,2,\ldots,n} f_{(x_t(\tilde{\lambda}),t)}(x_{0,i}), \tag{4.4.1}$$

where, in our example, $x_0 = x_{0,i}$ are the solutions of $f'_{(x_t(\tilde{\lambda}),t)}(x_0) = 0$ for $i = 1, 2, 3$.

Remark 4.4.1. Recall that all four stationary points x_0^r, $x_{0,1}$, $x_{0,2}$ and $x_{0,3}$ have been found explicitly, see Proposition (4.3.4). However, there exists no trivial way of determining, as we move along the parameterised caustic, if $f_{(x_t(\lambda),t)}(x_0^r)$ is the minimum over all extremum values.

In the case of a free Burgers fluid with zero vorticity Reynolds in [25] is able classify the cool parts of the caustic, more or less, trivially. Even with a harmonic oscillator potential his study turns out to be little more complicated. Here we address how, and why, Reynold's analysis, and indeed the simplified approach of using the reduced one dimensional action function, breaks down for the vorticity study.

Studying a Burgers fluid with zero vorticity corresponds to setting the vector potential $a(x) \equiv 0$. However, we defined this as $a(x) := -\frac{1}{2}x \wedge \Omega$. Consequently, setting $\Omega = (0, 0, \Omega) = 0$ in Equation (4.3.2) reduces our analysis to zero vorticity. In this case the reduced action function $f_{(x_t(\lambda),t)}(x_0)$ takes the form

$$\begin{aligned} & f_{(x_t(\lambda),t)}(x_0) \\ & = -\left(\frac{\tan(\omega t)}{8\omega}\right) x_0^4 + \left(\frac{3\lambda^2 \tan(\omega t)}{4\omega}\right) x_0^2 - \left(\frac{\lambda^3 \tan(\omega t)}{\omega}\right) x_0 \\ & + \frac{1}{16\omega^3}\left(2\lambda^4\left(4\lambda^2 - 9\omega^2\right)\tan(\omega t) - 8\omega^6\cot(\omega t) - 4\lambda^6\sin(2\omega t) + \left(3\lambda^2\omega + 2\omega^3\right)^2\sin(2\omega t)\right). \end{aligned} \tag{4.4.2}$$

This representation is considerably shorter than the corresponding case for vorticity. Furthermore, simplifying the above by considering a *free* Burgers fluid with zero vorticity turns out to be more or less trivial

$$f_{(x_t(\lambda),t)}(x_0) = \frac{t}{8}\left(-{x_0}^4 + 6{x_0}^2\lambda^2 - 8x_0\lambda^3 + 4t^2\lambda^6\right).$$

Observe that in both instances the reduced action function is no longer a polynomial quotient in x_0. In fact, for a Burgers fluid with zero vorticity, determining the cool parts of the caustic involves simply solving cubics. Unfortunately, with vorticity, the reduced action functional, in both the free case and harmonic oscillator case, is a polynomial quotient with a quintic numerator and quadratic denominator. Consequently, there is no reason why one should expect to find its stationary points, and hence the nature of the caustic, trivially.

With zero vorticity and a harmonic oscillator potential, the first derivative of the reduced action functional (4.4.2) is simply

$$f'_{x(\lambda,t)}(x_0) = -\frac{1}{2\omega}\tan(\omega t)(x_0 - \lambda)^2(x_0 + 2\lambda),$$

yielding two stationary points

$$x_0 = x_0^r = \lambda \qquad x_0 = x_{0,1} = -2\lambda$$

where $x_0 = x_0^r$ is repeated.

Remark 4.4.2. Observe these two stationary points occur in an identical manner for the vorticity case.

Examining the extremum values at each of these stationary points yield

$$\begin{aligned} & f_{(x_t(\lambda),t)}(x_0^r) \\ & = \frac{4\omega^6\sin(2\omega t) - 8\omega^6\cot(\omega t)}{16\omega^3} + \frac{3\omega\sin(2\omega t)}{4}\lambda^2 + \frac{(9\omega^2\sin(2\omega t) - 24\omega^2\tan(\omega t))}{16\omega^3}\lambda^4 \\ & + \frac{(8\tan(\omega t) - 4\sin(2\omega t))}{16\omega^3}\lambda^6 \end{aligned} \tag{4.4.3}$$

and

$$\begin{aligned} & f_{(x_t(\lambda),t)}(x_{0,1}) \\ & = \frac{4\omega^6\sin(2\omega t) - 8\omega^6\cot(\omega t)}{16\omega^3} + \frac{3\omega\sin(2\omega t)}{4}\lambda^2 + \frac{(9\omega^2\sin(2\omega t) + 30\omega^2\tan(\omega t))}{16\omega^3}\lambda^4 \\ & + \frac{(8\tan(\omega t) - 4\sin(2\omega t))}{16\omega^3}\lambda^6 \end{aligned} \tag{4.4.4}$$

As expected, the structure of these two extremum values are very similar. Only the coefficient of λ^4 differs. The problem of ascertaining cool parts of the caustic thus reduces to determining when

$$f_{(x_t(\lambda),t)}(x_0^r) \leq f_{(x_t(\lambda),t)}(x_{0,1}).$$

A simple comparison of Equations (4.4.3) and (4.4.4) indicates

$$f_{(x_t(\lambda),t)}(x_0^r) \leq f_{(x_t(\lambda),t)}(x_{0,1}) \Longleftrightarrow \tan(\omega t) \geq 0.$$

In other words the whole of the caustic is cool when $\tan(\omega t) \geq 0$.

Remark 4.4.3. For a cusp we demand $\lambda = 0$ (see Equation (4.2.3)) . Therefore, by virtue of the above analysis $f_{(x_t(\lambda),t)}(x_{0,1}) = f_{(x_t(\lambda),t)}(x_0^r)$ at the cusp. Under our definition the cusp is defined as a cool part of the caustic.

We endeavour to make progress for a Burgers fluid with vorticity and a harmonic oscillator potential with the following proposition.

Proposition 4.4.1. *Recall that $x_0 = x_0^r = \lambda$ and $x_0 = x_{0,1} = -2\lambda$, are two from four of the roots of $f'_{(x_t(\lambda),t)}(x_0)$ with $x_0 = x_0^r$ repeated. The orientation of the precaustic is all important in determining the hot and cool parts of the caustic. If the precaustic has its usual upright orientation then $f_{(x_t(\lambda),t)}(x_0^r) \leq f_{(x_t(\lambda),t)}(x_{0,1})$ for all points along the caustic.*

Proof. Via lengthy computation, it turns out by using Equation (4.3.2) the difference of the extremum values at $x_0 = x_0^r$ and $x_0 = x_{0,1}$ is

$$\begin{aligned} f_{(x_t(\lambda),t)}(x_{0,1}) - f_{(x_t(\lambda),t)}(x_0^r) &= f_{(x_t(\lambda),t)}(-2\lambda) - f_{(x_t(\lambda),t)}(\lambda) \\ &= \frac{27\lambda^4 \tan(\frac{\varpi t}{2})}{4\varpi}. \end{aligned}$$

Thus

$$f_{(x_t(\lambda),t)}(x_{0,1}) \geq f_{(x_t(\lambda),t)}(x_0^r) \qquad \Leftrightarrow \qquad \tan\left(\frac{\varpi t}{2}\right) \geq 0.$$

Recall the precaustic equation from (4.1.13)

$$y_0(x_0) = -\frac{\varpi}{2}\cot\left(\frac{\varpi t}{2}\right) + \frac{2}{\varpi}\tan\left(\frac{\varpi t}{2}\right) {x_0}^2$$

where $\varpi = \sqrt{4\omega^2 + \Omega}$. Assuming $\varpi > 0$, observe the vertical orientation of the precaustic to be determined by

$$\operatorname{sgn}\left[\tan\left(\frac{\varpi t}{2}\right)\right].$$

In conclusion, if the precaustic has its usual upright orientation then $f_{(x_t(\lambda),t)}(x_{0,1}) \geq f_{(x_t(\lambda),t)}(x_0^r)$ at all points along the caustic. □

The above result is not sufficient to guarantee a cool part of the caustic since we must examine what happens to the remaining two stationary points $x_0 = x_{0,2}$ and $x_0 = x_{0,3}$. However, in light of this Proposition we obtain the following results:-

Corollary 4.4.2. *If* $\tan\left(\frac{\varpi t}{2}\right) < 0$ *the caustic is hot.*

Proof. For $\tan\left(\frac{\varpi t}{2}\right) < 0$, by the above proposition, $f_{(x_t(\lambda),t)}(x_{0,1}) < f_{(x_t(\lambda),t)}(x_0^r)$. □

Lemma 4.4.3. *The critical values* $x_0 = x_{0,2}$ *and* $x_0 = x_{0,3}$ *are complex conjugate pairs for positions on the caustic where*

$$\lambda < -\frac{\varpi}{4}\cot\left(\frac{\varpi t}{2}\right)\cot\left(\frac{\Omega t}{2}\right).$$

Proof. Defined in Proposition (4.3.4), $\eta(\lambda) < 0$ if $\lambda < -\frac{\varpi}{4}\cot(\frac{\varpi t}{2})\cot(\frac{\Omega t}{2})$. Consequently, $x_{0,2} = \overline{x_{0,3}} \in \mathbb{C}$, where $\overline{x_{0,3}}$ is the complex conjugate of $x_{0,2}$. □

Corollary 4.4.4. *If* $\tan\left(\frac{t\varpi}{2}\right) \geq 0$ *then the caustic is cool at all parts of the caustic where*

$$\lambda < -\frac{\varpi}{4}\cot(\frac{\varpi t}{2})\cot(\frac{\Omega t}{2}).$$

Proof. For $\tan\left(\frac{t\varpi}{2}\right) \geq 0$, by Proposition (4.4.1), $f_{(x_t(\lambda),t)}(x_0^r) \leq f_{(x_t(\lambda),t)}(x_{0,1})$. Moreover, if

$$\lambda < -\frac{\varpi}{4}\cot(\frac{\varpi t}{2})\cot(\frac{\Omega t}{2}),$$

then by Lemma (4.4.3) there can exist only two real valued stationary points of $f_{(x_t(\lambda),t)}(x_0)$, namely, $x_0 = x_{0,1}$ and the repeated root $x_0 = x_0^r$. The result follows. □

Remark 4.4.4. At the point on the caustic where

$$\lambda = \tilde{\lambda} = -\frac{\varpi}{4}\cot(\frac{\varpi t}{2})\cot(\frac{\Omega t}{2})$$

the critical points $x_{0,2}$ and $x_{0,3}$ are real and equal (see Proposition (4.3.4)). In fact, $x_{0,2} = x_{0,3} = x_0^b$, where $x_0 = x_0^b$ is the value of the vertical asymptote of $f_{(x_t(\lambda),t)}(x_0)$. Then

$$\lim_{\lambda\to\tilde{\lambda}}\left(f_{(x_t(\lambda),t)}(x_{0,2}) - f_{(x_t(\lambda),t)}(x_0^r)\right) = +\infty.$$

By definition, this point on the caustic is designated as cool.

Using this method of analysis, without direct numeric computation, the nature of the caustic has been indeterminable for

$$\lambda > -\frac{\varpi}{4}\cot\left(\frac{\varpi t}{2}\right)\cot\left(\frac{\Omega t}{2}\right) \qquad \text{and} \qquad \tan\left(\frac{t\varpi}{2}\right) \geq 0.$$

Expressions such as

$$f_{(x_t(\lambda),t)}(x_{0,2}) - f_{(x_t(\lambda),t)}(x_0^r) \qquad \text{and} \qquad f_{(x_t(\lambda),t)}(x_{0,3}) - f_{(x_t(\lambda),t)}(x_0^r),$$

or even

$$f_{(x_t(\lambda),t)}(x_{0,3}) - f_{(x_t(\lambda),t)}(x_{0,2}),$$

turn out to be highly non-trivial. To make progress in classifying these remaining parts of the caustic as hot or cool we resort to a geometric argument.

4.5 Relative ordering of critical values

So far we have been unable to classify parts of the caustic where

$$\lambda > -\frac{\varpi}{4}\cot\left(\frac{\varpi t}{2}\right)\cot\left(\frac{\Omega t}{2}\right) \qquad \text{and} \qquad \tan\left(\frac{t\varpi}{2}\right) \geq 0.$$

Under the above conditions $f'_{(x_t(\lambda),t)}(x_0) = 0$ has precisely four roots (one of which is repeated) and a quadratic singularity. Thus, five critical points. Hence the task of trying to determining whether

$$f_{(x_t(\lambda),t)}(x_0^r) \leq \min_{i=1,2,3} f_{(x_t(\lambda),t)}(x_{0,i})$$

more difficult. Here we rely on geometrical arguments concerning the relative ordering of the five critical points. The break point

$$x_0 = x_0^b = \frac{\varpi}{2}\cot\left(\frac{\Omega t}{2}\right)\cot\left(\frac{\varpi t}{2}\right)$$

is extremely important in our study, allowing us the following lemmas:-

Remark 4.5.1. For the following analysis we assume $\tan\left(\frac{t\varpi}{2}\right) \neq 0$ and $\tan\left(\frac{t\Omega}{2}\right) \neq 0$. Comments shall be made about times t satisfying the contrary towards the end of the section.

Lemma 4.5.1.

$$\left.\begin{array}{l}\lim_{x_0 \nearrow x_0^b} f(x_0) \\ \lim_{x_0 \searrow x_0^b} f(x_0)\end{array}\right\} = +\infty \qquad \Leftrightarrow \tan\left(\frac{\varpi t}{2}\right) > 0. \tag{4.5.1}$$

Proof. As $x_0 \to x_0^b$ the numerator of $f_{(x_t(\lambda),t)}(x_0)$ tends to

$$\frac{1}{4}\varpi^2 \operatorname{cosec}\left(\frac{\varpi t}{2}\right)^3 \sec\left(\frac{\varpi t}{2}\right)\left(\varpi\cos\left(\frac{\varpi t}{2}\right)\cot\left(\frac{\Omega t}{2}\right) - 2\lambda\sin\left(\frac{\varpi t}{2}\right)\right)^4$$
$$\times\left(\varpi\cos\left(\frac{\varpi t}{2}\right)\cos\left(\frac{\Omega t}{2}\right) + 4\lambda\sin\left(\frac{\varpi t}{2}\right)\sin\left(\frac{\Omega t}{2}\right)\right)^2,$$

or equivalently,

$$\frac{\varpi^2\operatorname{cosec}\left(\frac{\varpi t}{2}\right)^2}{4\cos\left(\frac{\varpi t}{2}\right)\sin\left(\frac{\varpi t}{2}\right)}\left(\varpi\cos\left(\frac{\varpi t}{2}\right)\cot\left(\frac{\Omega t}{2}\right) - 2\lambda\sin\left(\frac{\varpi t}{2}\right)\right)^4$$
$$\times\left(\varpi\cos\left(\frac{\varpi t}{2}\right)\cos\left(\frac{\Omega t}{2}\right) + 4\lambda\sin\left(\frac{\varpi t}{2}\right)\sin\left(\frac{\Omega t}{2}\right)\right)^2.$$

Which is of course is positive as long as $\tan\left(\frac{\varpi t}{2}\right) > 0$. Result follows. ☐

Lemma 4.5.2. *Assuming* $\tan(\frac{\varpi t}{2}) > 0$ *and* $\tan(\frac{\Omega t}{2}) \neq 0$, *the tails of* $f_{(x_t(\lambda),t)}(x_0)$ *can be categorised as follows* *(i) For* $\tan(\frac{\Omega t}{2}) > 0$,

$$\lim_{x_0 \to \infty} f_{(x_t(\lambda),t)}(x_0) = +\infty \qquad \textit{and} \qquad \lim_{x_0 \to -\infty} f_{(x_t(\lambda),t)}(x_0) = -\infty.$$

(ii) For $\tan(\frac{\Omega t}{2}) < 0$,

$$\lim_{x_0 \to \infty} f_{(x_t(\lambda),t)}(x_0) = -\infty \qquad \textit{and} \qquad \lim_{x_0 \to -\infty} f_{(x_t(\lambda),t)}(x_0) = +\infty.$$

Proof. By computing the limit we discover

$$\lim_{x_0 \to \infty} f_{(x_t(\lambda),t)}(x_0) = \frac{1}{\operatorname{sign}\left\{\tan\left(\frac{t\Omega}{2}\right)\right\}} \times \infty$$

and

$$\lim_{x_0 \to -\infty} f_{(x_t(\lambda),t)}(x_0) = -\frac{1}{\operatorname{sign}\left\{\tan\left(\frac{t\Omega}{2}\right)\right\}} \times \infty.$$

Hence the result. □

Lemma 4.5.3. *If* $\tan(\frac{\varpi t}{2}) > 0$ *then*

(i) $x_0^b > 0$ *if* $\tan\left(\frac{\Omega t}{2}\right) > 0$,

(ii) $x_0^b < 0$ *if* $\tan\left(\frac{\Omega t}{2}\right) < 0$.

Proof. A simple consequence of its structure in Equation (4.3.3). □

Lemma 4.5.4. *Assume* $\lambda > -\frac{\varpi}{4}\cot\left(\frac{\varpi t}{2}\right)\cot\left(\frac{\Omega t}{2}\right)$ *so that* $x_{0,2}, x_{0,3} \in \mathbb{R}$. *Then, regardless of* $\operatorname{sgn}\left\{\tan\left(\frac{t\varpi}{2}\right)\right\}$ *and* $\operatorname{sgn}\left\{\tan\left(\frac{t\Omega}{2}\right)\right\}$, *the critical values* $x_0 = x_{0,2}$ *and* $x_0 = x_{0,3}$, *lie to the left and right of* $x_0 = x_0^b$ *respectively. That is to say* $x_{0,2} \leq x_0^b \leq x_{0,3}$.

Proof. Recall from Equation (4.3.3)

$$x_0^b = \frac{\varpi}{2}\cot\left(\frac{\Omega t}{2}\right)\cot\left(\frac{t\varpi}{2}\right) \tag{4.5.2}$$

and the critical values from Proposition (4.3.4)

$$x_{0,2} = \nu - \kappa\left|\xi(\lambda)\right|\sqrt{\eta(\lambda)},$$
$$x_{0,3} = \nu + \kappa\left|\xi(\lambda)\right|\sqrt{\eta(\lambda)}.$$

However, $\nu = x_0^b$ and so these two critical points can be expressed as

$$x_{0,2} = x_0^b - \kappa\left|\xi(\lambda)\right|\sqrt{\eta(\lambda)}$$
$$x_{0,3} = x_0^b + \kappa\left|\xi(\lambda)\right|\sqrt{\eta(\lambda)}.$$

The result follows. □

The following lemmas detail which, and more importantly how many, stationary points one can expect to obtain either side of the break point $x_0 = x_0^b$.

Lemma 4.5.5. *Let* $\tan(\frac{\varpi t}{2}) > 0$ *and* $\tan(\frac{\Omega t}{2}) > 0$. *Then*

1. *for* $\lambda > 0$ *the possible relative orderings are*

$$x_{0,i} < x_{0,j} < x_0^b < x_{0,3} < x_0^r,$$

$$x_{0,i} < x_{0,j} < x_0^b < x_0^r < x_{0,3},$$

$$x_{0,1} < x_0^r < x_{0,2} < x_0^b < x_{0,3},$$

$$x_{0,i} < x_{0,j} < x_0^r < x_0^b < x_{0,3},$$

   for $i, j = 1, 2,\ i \neq j$.

2. *for* $0 > \lambda > -\frac{1}{4}\cot(\frac{\varpi t}{2})\cot(\frac{\Omega t}{2})$ *the possibilities for relative ordering are*

$$x_0^r < x_{0,i} < x_{0,j} < x_0^b < x_{0,3}, \qquad \text{for} \quad i, j = 1, 2, \quad i \neq j.$$

Proof. Firstly, note that a point of inflexion such as $x_0 = x_0^r$ will not change the sign of the gradient and hence the vertical direction of the function. Therefore, if $\tan(\frac{t\varpi}{2}) > 0$ and $\tan(\frac{t\Omega}{2}) > 0$ then by Lemmas (4.5.1) and (4.5.2) geometry dictates that the number of *turning points* to the left and right of the vertical asymptote, $x_0 = x_0^b$, must be even and odd respectively. Moreover, observe by Lemma (4.5.4) $x_{0,2} < x_0^b < x_{0,3}$ and by Lemma (4.5.3), $x_0^b > 0$.

1. For $\lambda > 0$ we have $x_{0,1} < 0 < x_0^r$ and so the possible orderings are

$$x_{0,i} < x_{0,j} < x_0^b < x_{0,3} < x_0^r,$$

$$x_{0,i} < x_{0,j} < x_0^b < x_0^r < x_{0,3},$$

$$x_{0,1} < x_0^r < x_{0,2} < x_0^b < x_{0,3},$$

$$x_{0,i} < x_{0,j} < x_0^r < x_0^b < x_{0,3},$$

   for $i, j = 1, 2,\ i \neq j$.

2. For $0 > \lambda > -\frac{\varpi}{4}\cot(\frac{\varpi t}{2})\cot(\frac{\Omega t}{2})$, $x_0^r < 0 < x_{0,1}$ and so in line with the above results geometry dictates the only possible orderings to be

$$x_0^r < x_{0,i} < x_{0,j} < x_0^b < x_{0,3}, \qquad \text{for} \quad i, j = 1, 2, \quad i \neq j.$$

□

Lemma 4.5.6. *Let* $\tan(\frac{\varpi t}{2}) > 0$ *and* $\tan(\frac{\Omega t}{2}) < 0$. *Then for* $\lambda > -\frac{\varpi}{4}\cot(\frac{\varpi t}{2})\cot(\frac{\Omega t}{2}) > 0$ *the possibilities for relative ordering are*

$$x_{0,2} < x_0^b < x_{0,i} < x_{0,j} < x_0^r \qquad \text{or} \qquad x_{0,2} < x_0^b < x_{0,1} < x_0^r < x_{0,3},$$

for $i, j = 1$ *or* $3,\ i \neq j$.

Proof. Firstly, if $\tan(\frac{\varpi t}{2}) > 0$ and $\tan(\frac{\Omega t}{2}) < 0$ then by Lemmas (4.5.1) and (4.5.2) geometry dictates that the number of *turning points* to the left and right of the vertical asymptote, $x_0 = x_0^b$, must be odd and even respectively. By Lemma (4.5.4) $x_{0,2} < x_0^b < x_{0,3}$ and by Lemma (4.5.3) $x_0^b < 0$. If we insist $\lambda > -\frac{\varpi}{4}\cot(\frac{\varpi t}{2})\cot(\frac{\Omega t}{2}) > 0$ then $x_{0,1} < 0 < x_0^r$ and the only possible orderings are

$$x_{0,2} < x_0^b < x_{0,i} < x_{0,j} < x_0^r,$$

$$x_{0,2} < x_0^b < x_{0,1} < x_0^r < x_{0,3}.$$

for $i, j = 1$ or 3, $i \neq j$. □

Remark 4.5.2. For times $t = \frac{2k\pi}{\Omega}$, $k \in \mathbb{N}$, we can no longer use this method of analysis since $\tan(\frac{\Omega t}{2}) = 0$ and consequently, by Equation (4.3.3), $\lim_{t \to \frac{2k\pi}{\Omega}} x_0^b = +\infty$. Likewise, for times $t = \frac{2r\pi}{\varpi}$, $r \in \mathbb{N}$.

Lemma 4.5.7. *If* $\tan(\frac{\varpi t}{2}) > 0$ *the nature of the caustic indeterminable by relative ordering of critical values when*

1. $\tan\left(\frac{\Omega t}{2}\right) > 0$ *and the critical value* x_0^r *is left most*
2. $\tan\left(\frac{\Omega t}{2}\right) < 0$ *and the critical value* x_0^r *is right most.*

Proof. For $\tan\left(\frac{\Omega t}{2}\right) > 0$ lemma (4.5.2) states

$$\lim_{x_0 \to -\infty} f_{(x_t(\lambda),t)}(x_0) = -\infty.$$

When x_0^r is the left most critical point no minimum bound can be set on the critical value $f_{(x_t(\lambda),t)}(x_0^r)$ in order to make comparisons with other critical values which give rise to local minimums. A similar argument applies for $\tan\left(\frac{\Omega t}{2}\right) < 0$. □

Proposition 4.5.8. *For* $\tan\left(\frac{t\varpi}{2}\right) > 0$ *and* $\lambda > -\frac{1}{4}\cot(\frac{t\varpi}{2})\cot(\frac{t\Omega}{2})$ *all relative orderings of critical values stated in Lemmas* (4.5.5) *and* (4.5.6) *give rise to hot parts of the caustic except for the two cases described in Lemma* (4.5.7) *where the nature of the caustic is indeterminable.*

Proof. Referring to the following series of figures and as outlined by Lemmas (4.5.5) and (4.5.6) there exist 9 possible cases detailing different relative orderings . In this proof each ordering will be considered independently. Firstly, the reader is asked to consider the aforementioned Lemmas to familiarise themselves with geometrical properties of $f_{(x_t(\lambda),t)}(x_0)$.

Assume $\tan(\frac{t\varpi}{2}) > 0$ and $\tan(\frac{t\Omega}{2}) > 0$ so that the possible orderings are

Case 1: If $x_{0,i} < x_{0,j} < x_0^b < x_0^r < x_{0,3}$ then necessarily $f(x_{0,3}) < f(x_0^r)$.

Case 2: If $x_{0,i} < x_{0,j} < x_0^b < x_{0,3} < x_0^r$ then necessarily $f(x_{0,3}) < f(x_0^r)$.

Case 3: If $x_{0,1} < x_0^r < x_{0,2} < x_0^b < x_{0,3}$ then necessarily $f(x_{0,2}) < f(x_0^r)$.

Case 4: If $x_{0,i} < x_{0,j} < x_0^r < x_0^b < x_{0,3}$ then necessarily $f(x_{0,j}) < f(x_0^r)$.

Case 5: If $x_0^r < x_{0,1} < x_{0,2} < x_0^b < x_{0,3}$ no comparisons with $f(x_0^r)$ can be made.

Assume $\tan(\frac{t\varpi}{2}) > 0$ and $\tan(\frac{t\Omega}{2}) < 0$ so that the possible orderings are

Case 6: If $x_{0,2} < x_0^b < x_{0,i} < x_{0,j} < x_0^r$ no comparisons with $f(x_0^r)$ can be made.

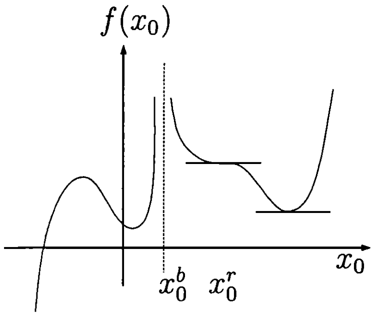
Case 7: If $x_{0,2} < x_0^b < x_{0,1} < x_0^r < x_{0,3}$ then necessarily $f(x_{0,1}) < f(x_0^r)$.

By definition, for those orderings where $f(x_0^r)$ is *not* the minimum over all extremum values the caustic is defined as hot. □

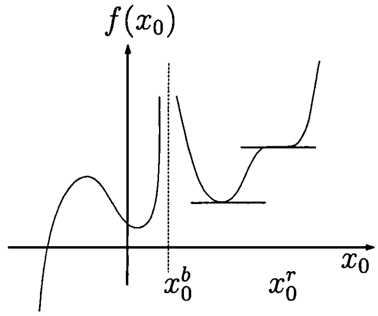
As yet we have neglected to mention the consequences of the sequence of times $\{t_k\}_{k\in\mathbb{N}}$ such that $\tan\left(\frac{\varpi t}{2}\right) = 0$ or $\tan\left(\frac{\Omega t}{2}\right) = 0$. In fact, for $t \longrightarrow \frac{2k\pi}{\varpi}$ we have by virtue of Equation (4.1.17)

$$\lim_{t\longrightarrow\frac{2k\pi}{\varpi}} \mathcal{A}(x_0, y_0, x, y, t) = \pm\infty.$$

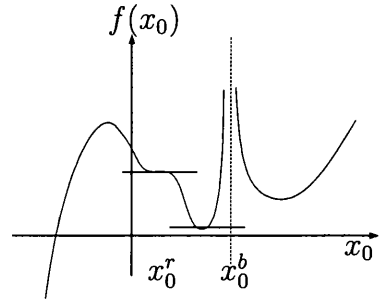
Computing the limit as $t \longrightarrow \frac{2k\pi}{\Omega}$, so that the second condition is satisfied, reveals the existence of only three roots of $f'_{(x_t(\lambda),t)}(x_0)$. Those being $x_0 = x_{0,1}$ and the repeated root $x_0 = x_0^r$. Consequently, for these times Proposition (4.4.1) can be used to trivially determine the nature of the caustic.



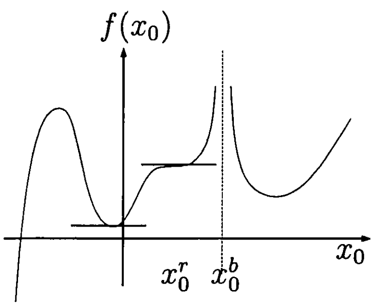
Case 1: Hot.



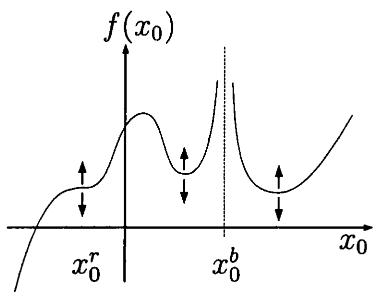
Case 2: Hot.



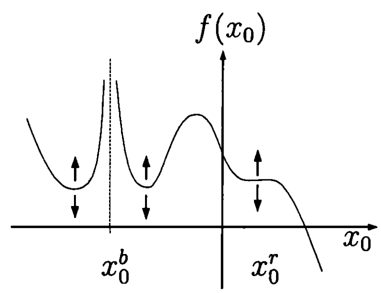
Case 3: Hot.



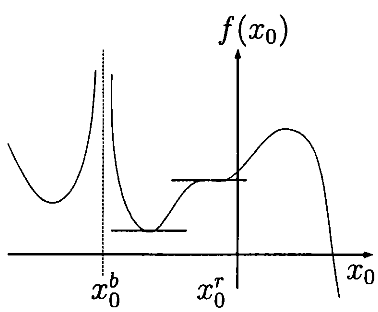
Case 4: Hot.



Case 5: Indeterminable.



Case 6: Indeterminable.



Case 7: Hot.

Summary

To classify the caustics for a Burgers fluid with vorticity and harmonic oscillator potential we have relied on two very different methods. The critical values of the reduced action function $f_{(x_t(\lambda),t)}(x_0)$ have been very important in determining the hot and cool parts of the caustic.

It turned out that $f_{(x_t(\lambda),t)}(x_0)$ had only two critical values $x_0 = x_0^r$ and $x_0 = x_{0,1}$ for those parts of the parameterised caustic where $\lambda < -\frac{\varpi}{4}\cot\left(\frac{\Omega t}{2}\right)\cot\left(\frac{\varpi t}{2}\right) = \tilde{\lambda}$. Using algebraic techniques it was discovered

$$f_{(x_t(\lambda),t)}(x_0^r) < f_{(x_t(\lambda),t)}(x_{0,1}) \qquad \Leftrightarrow \qquad \tan\left(\frac{\varpi t}{2}\right) > 0.$$

Consequently,

1. If $\tan\left(\frac{\varpi t}{2}\right) < 0$ the whole of the caustic is hot.
2. If $\tan\left(\frac{\varpi t}{2}\right) > 0$ then parts of the parameterised caustic such that $\lambda < \tilde{\lambda}$ are cool.

For $\tan\left(\frac{\varpi t}{2}\right) > 0$ and $\lambda > \tilde{\lambda}$ the reduced action function $f_{(x_t(\lambda),t)}(x_0)$ had five critical points x_0^r, $x_{0,1}$, $x_{0,2}$, $x_{0,3}$ and x_0^b. To make progress in this situation we resorted to a geometrical argument concerning the *relative ordering* of these critical values. Under the aforementioned conditions it was discovered that, for non-trivial times, all parts of the caustic are hot. However, in a handful of circumstances the geometrical argument broke down and failed to provide a conclusive result. Thus, in the following situations we have been unable to classify the caustic for all generality

(a) $\tan\left(\frac{\varpi t}{2}\right) > 0$, $\tan\left(\frac{\Omega t}{2}\right) > 0$ and $\lambda > -\frac{1}{4}\cot\left(\frac{\varpi t}{2}\right)\cot\left(\frac{\Omega t}{2}\right)$ where $x_0^r < x_{0,i}$ for $i = 1, 2, 3$.

(b) $\tan\left(\frac{\varpi t}{2}\right) > 0$, $\tan\left(\frac{\Omega t}{2}\right) < 0$ and $\lambda > -\frac{1}{4}\cot\left(\frac{\varpi t}{2}\right)\cot\left(\frac{t\Omega}{2}\right)$ where $x_{0,j} < x_0^r$ for $j = 1, 2, 3$.

If either of these two situations occur direct numerical computation is needed to determine whether the specified part of the caustic is hot or cool.

Chapter 5

A Stochastic Burgers Fluid in a Rotating Bucket under a Harmonic Oscillator potential

Summary

In this chapter we continue our theme by considering an example of the stochastic inviscid Burgers equation with vorticity. To achieve a stochastic mechanical system a noisy external force $f(t)$ has been introduced to the classical set-up. Here we calculate the action of the stochastic classical mechanical path, illustrate geometrical relationships and discuss possible intermittence of stochastic turbulence.

5.1 The set-up

Formally write

$$H(q,p,t) = 2^{-1}(p - a(q))^2 + V(q) + \varepsilon q.f(t)\dot{W}_t, \tag{5.1.1}$$

where $\dot{W}_t$ is white noise. Defining $V(q) := \frac{1}{2}q\omega^2 q^T$, where ω^2 is a real symmetric positive definite 3×3 matrix with

$$(\omega)_{ij} = \begin{cases} \omega & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

and the vector potential $a(q) := -\frac{1}{2}q \wedge \Omega$ mean that Hamilton's equations applied to equation (5.1.1) yield

$$d\dot{q} = -\Omega \wedge dq - \omega^2 q ds + \varepsilon f(t) dW_t. \tag{5.1.2}$$

Here $q = (x,y,z)$, $f(t) = (f_x(t), f_y(t), f_z(t))$ and $\Omega = (0,0,\Omega)$, ω are constants. The above equation is the stochastic analogue of Equation (4.1.5) from the preceding chapter and corresponds to the Burgers equation for $v(x,t) = v$,

$$dv + (v.\nabla)vdt + v \wedge \text{curl } vdt = -\nabla V(x)dt - \varepsilon \nabla k(x,t)dW_t$$

where $k(x,t) \equiv x.f(t)$.

We proceed by following the deterministic example of the previous chapter by setting $\varsigma^\varepsilon(s) = x(s) + iy(s)$ and setting $f_\varsigma(t) = f_x(t) + if_y(t)$. Equation (5.1.2) reduces to

$$\left(\ddot{\varsigma}^\varepsilon + i\Omega\dot{\varsigma}^\varepsilon + \omega^2\varsigma^\varepsilon\right) ds = \varepsilon f_\varsigma(t) dW_t. \tag{5.1.3}$$

Now consider the identity

$$g(D)e^{\alpha x} f(x) = e^{\alpha x} g(D+\alpha) f(x)$$

where D is the derivative operator. By setting $\alpha = \frac{i\Omega}{2}$ this identity yields

$$\begin{aligned}\varepsilon D^2 e^{\frac{i\Omega t}{2}} \varsigma^\varepsilon(t) &= \varepsilon e^{\frac{i\Omega t}{2}} \left(D + \frac{i\Omega}{2}\right)^2 \varsigma^\varepsilon(t), \\ &= \varepsilon e^{\frac{i\Omega t}{2}} \left(D^2 + i\Omega D - \frac{\Omega^2}{4}\right) \varsigma^\varepsilon(t).\end{aligned}$$

Equation (5.1.3) now becomes

$$\left(D^2 + \frac{\Omega^2}{4}\right) e^{\frac{i\Omega t}{2}} \varsigma^\varepsilon(t) ds + \omega^2 e^{\frac{i\Omega t}{2}} \varsigma^\varepsilon(t) ds = \varepsilon e^{\frac{i\Omega t}{2}} f_\varsigma(t) dW_t$$

$$\left[D^2 + \left(\frac{\Omega^2}{4} + \omega^2\right)\right] e^{\frac{i\Omega t}{2}} \varsigma^\varepsilon(t) ds = \varepsilon e^{\frac{i\Omega t}{2}} f_\varsigma(t) dW_t. \tag{5.1.4}$$

The general solution of this non-homogeneous ODE is given by

$$g(t) = g_h(t) + g_p(t)$$

where $g_h(t)$ is the general solution of the homogeneous (deterministic) problem studied in Chapter 4 and $g_p(t)$ is the particular solution of the non-homogeneous differential equation (5.1.4). Using the Green's function, we find

$$g_p(t) = e^{\frac{i\Omega t}{2}} \varsigma_p^\varepsilon(t) = \varepsilon \int_0^t e^{\frac{i\Omega(u)}{2}} \frac{\sin\left[(u-t)\sqrt{\frac{\Omega^2}{4} + \omega^2}\right]}{\sqrt{\frac{\Omega^2}{4} + \omega^2}} f_\varsigma(u) dW_u.$$

or

$$\varsigma_p^\varepsilon(t) = \varepsilon \int_0^t e^{\frac{i\Omega(u-t)}{2}} \frac{2\sin\left[\frac{(u-t)}{2}\varpi\right]}{\varpi} f_\varsigma(u) dW_u,$$

on setting $\varpi = \sqrt{4\omega^2 + \Omega}$. If $\varsigma(t)$ is our solution to the deterministic problem, it follows that

$$\begin{aligned}\varsigma^\varepsilon(t) &= \varsigma(t) + \varsigma_p(t) \\ &= e^{-\frac{i\Omega}{2}t}\left[(x_0 + iy_0)\cos\left(\frac{t\varpi}{2}\right) + \frac{(2x_0y_0 + ix_0^2)}{\varpi}\sin\left(\frac{t\varpi}{2}\right)\right] \\ &+ \frac{2\varepsilon}{\varpi}\int_0^t e^{\frac{i\Omega(u-t)}{2}} \sin\left[\frac{(u-t)}{2}\varpi\right] f_\varsigma(u) dW_u.\end{aligned} \tag{5.1.5}$$

Remark 5.1.1. Observe the consequence of adding time dependent, but space independent, noise to our dynamical system means the deterministic solution $\varsigma(t)$ has been perturbed by an amount equal to that of the integral term above.

Remark 5.1.2. By a similar argument motion along the z-axis is described by

$$Z(t) = z_0 \cos(\omega t) + \varepsilon \int_0^t \frac{\sin\left[\omega(u-t)\right]}{\omega} f_z(u) dW_u.$$

Once again, it is the integral term that signifies the displacement in this stochastic case.

5.2 The stochastic mechanical path X_s^ε

Following the deterministic example, we examine the fluid flow in the more interesting first two dimensions. For this we require the stochastic mechanical path $X^\varepsilon(s) = X_s^\varepsilon = (X_s^\varepsilon, Y_s^\varepsilon)$ given by

$$\begin{aligned} X_s^\varepsilon &= (\text{Re } \varsigma_s^\varepsilon, \text{Im } \varsigma_s^\varepsilon), \\ &= (\text{Re } (\varsigma(s) + \varsigma_p(s)), \text{Im } (\varsigma(s) + \varsigma_p(s))) \end{aligned}$$

where *Re* and *Im* have been used to denote the real and imaginary parts respectively.

Lemma 5.2.1. *The stochastic caustic is displaced bodily by an amount*

$$Re\ \varsigma_p(t) = \frac{2\varepsilon}{\varpi} Re \int_0^t e^{\frac{i\Omega(u-t)}{2}} \sin\left[\frac{(u-t)}{2}\varpi\right] f_\varsigma(u) dW_u \tag{5.2.1}$$

along the x-axis and

$$Im\ \varsigma_p(t) = \frac{2\varepsilon}{\varpi} Im \int_0^t e^{\frac{i\Omega(u-t)}{2}} \sin\left[\frac{(u-t)}{2}\varpi\right] f_\varsigma(u) dW_u \tag{5.2.2}$$

along the y-axis.

Proof. Using the above notation the deterministic classical mechanical path is defined by

$$X_s = (\text{Re } \varsigma(s), \text{Im } \varsigma(s)),$$

where as the stochastic classical mechanical path is

$$X_s^\varepsilon = (\text{Re}\,(\varsigma(s) + \varsigma_p(s)), \text{Im}\,(\varsigma(s) + \varsigma_p(s))).$$

The result follows. □

If we are later to calculate the stochastic action we must first find an explicit representation for the stochastic mechanical path $X_s^\varepsilon = (X_s^\varepsilon, Y_s^\varepsilon)$. Given that $f_\varsigma(u) = f_x(u) + i f_y(u)$ it follows from Equations (5.2.2) and (5.2.1) that

$$\begin{aligned} X_s^\varepsilon &= \text{Re}\,(\varsigma^\varepsilon(s)) \\ &= \frac{x_0}{\varpi} \sin(\frac{s\varpi}{2}) \left(2y_0 \cos(\frac{s\Omega}{2}) + x_0 \sin(\frac{s\Omega}{2})\right) + \cos(\frac{s\varpi}{2}) \left(x_0 \cos(\frac{s\Omega}{2}) + y_0 \sin(\frac{s\Omega}{2})\right) \\ &+ \frac{2\varepsilon}{\varpi} \int_0^s \sin\left(\frac{(u-s)\,\varpi}{2}\right) \left(\cos\left(\frac{(u-s)\,\Omega}{2}\right) f_x(u) - \sin\left(\frac{(u-s)\,\Omega}{2}\right) f_y(u)\right) dW_u \end{aligned}$$

and

$$\begin{aligned} Y_s^\varepsilon &= \mathrm{Im}\,(\varsigma^\varepsilon(s)) \\ &= \cos(\frac{s\varpi}{2})\left(y_0\cos(\frac{s\Omega}{2}) - x_0\sin(\frac{s\Omega}{2})\right) + \frac{x_0}{\varpi}\sin(\frac{s\varpi}{2})\left(x_0\cos(\frac{s\Omega}{2}) - 2y_0\sin(\frac{s\Omega}{2})\right) \\ &+ \frac{2\varepsilon}{\varpi}\int_0^s \sin\left(\frac{(u-s)\,\varpi}{2}\right)\left(\cos\left(\frac{(u-s)\,\Omega}{2}\right)f_y(u) + \sin\left(\frac{(u-s)\,\Omega}{2}\right)f_x(u)\right)dW_u. \end{aligned}$$

Corollary 5.2.2. *The stochastic cusped caustic for a Burgers fluid with vorticity and a harmonic oscillator potential is given by the parametric equations*

$$\begin{aligned} x(x_0,t) = &\frac{4}{\varpi^2}\cos(\frac{t\Omega}{2})\sin(\frac{t\varpi}{2})\tan(\frac{t\varpi}{2})x_0^3 + \frac{3}{\varpi}\sin(\frac{t\varpi}{2})\sin(\frac{t\Omega}{2})x_0^2 \\ &- \frac{\varpi}{2}\left(\cos(\frac{t\varpi}{2})\cot(\frac{t\varpi}{2})\sin(\frac{t\Omega}{2})\right) + \varepsilon\mathcal{W}_x(t) \end{aligned}$$

and

$$\begin{aligned} y(x_0,t) = &-\frac{4}{\varpi^2}\sin(\frac{t\varpi}{2})\sin(\frac{t\Omega}{2})\tan(\frac{t\varpi}{2})x_0^3 + \frac{3}{\varpi}\cos(\frac{t\Omega}{2})\sin(\frac{t\varpi}{2})x_0^2 \\ &- \frac{\varpi}{2}\left(\cos(\frac{t\varpi}{2})\cos(\frac{t\Omega}{2})\cot(\frac{t\varpi}{2})\right) + \varepsilon\mathcal{W}_y(t), \end{aligned}$$

where $\mathcal{W}_x(t)$ and $\mathcal{W}_y(t)$ are the stochastic displacements along the x and y axes respectively given by

$$\mathcal{W}_x(t) = \frac{2}{\varpi}\int_0^t \sin\left(\frac{(u-t)\,\varpi}{2}\right)\left(\cos\left(\frac{(u-t)\,\Omega}{2}\right)f_x(u) - \sin\left(\frac{(u-t)\,\Omega}{2}\right)f_y(u)\right)dW_u \tag{5.2.3}$$

and

$$\mathcal{W}_y(t) = \frac{2}{\varpi}\int_0^t \sin\left(\frac{(u-t)\,\varpi}{2}\right)\left(\cos\left(\frac{(u-t)\,\Omega}{2}\right)f_y(u) + \sin\left(\frac{(u-t)\,\Omega}{2}\right)f_x(u)\right)dW_u. \tag{5.2.4}$$

Proof. Follows immediately from Lemma (5.2.1) and the representation for the deterministic caustic. □

Theorem 5.2.3 (McKean). *Consider a stochastic integral $I(t) = \int_0^t f(s)dW_s$ where $f(s)$ is a non-anticipating, possibly random function. Assume that, with probability one, $\int_0^t f^2(s)ds < \infty$, then*

$$I(t) = \widetilde{W}\left(\int_0^t f^2(s)ds\right)$$

where $\widetilde{W}$ is a new Brownian motion.

Proof. See McKean in [22]. □

Proposition 5.2.4. *$\mathcal{W}_x(t)$ and $\mathcal{W}_y(t)$ can be written as a sum of non-independent correlated Brownian motions. That is*

$$\mathcal{W}_x(t) = \frac{2}{\varpi}\Sigma_i f_i(t)\widetilde{W}_i\left(\int_0^t g_i^2(u)du\right) \qquad i = 1, 2, \ldots, 8.$$

and

$$\mathcal{W}_y(t) = \frac{2}{\varpi}\Sigma_i h_i(t)\widetilde{W}_i\left(\int_0^t j_i^2(u)du\right) \qquad i = 1, 2, \ldots, 8.$$

where $f_i(t)$, $h_i(t)$, $g_i(t)$ and $j_i(t)$ are continuous non-anticipating functions and $\widetilde{W}_i$ is a particular Brownian motion.

Proof. Firstly, observe

$$\begin{aligned}
\frac{\varpi}{2}\mathcal{W}_x(t) = &-\sin(\frac{\varpi t}{2})\cos(\frac{\Omega t}{2})\int_0^t \cos(\frac{u\varpi}{2})\cos(\frac{u\Omega}{2})f_x(u)dW_u \\
&+\cos(\frac{\varpi t}{2})\cos(\frac{\Omega t}{2})\int_0^t \sin(\frac{u\varpi}{2})\cos(\frac{u\Omega}{2})f_x(u)dW_u \\
&-\sin(\frac{\varpi t}{2})\sin(\frac{\Omega t}{2})\int_0^t \cos(\frac{u\varpi}{2})\cos(\frac{u\Omega}{2})f_y(u)dW_u \\
&+\cos(\frac{\varpi t}{2})\sin(\frac{\Omega t}{2})\int_0^t \sin(\frac{u\varpi}{2})\cos(\frac{u\Omega}{2})f_y(u)dW_u \\
&+\sin(\frac{\varpi t}{2})\cos(\frac{\Omega t}{2})\int_0^t \cos(\frac{u\varpi}{2})\sin(\frac{u\Omega}{2})f_y(u)dW_u \\
&-\cos(\frac{\varpi t}{2})\cos(\frac{\Omega t}{2})\int_0^t \sin(\frac{u\varpi}{2})\sin(\frac{u\Omega}{2})f_y(u)dW_u \\
&-\sin(\frac{\varpi t}{2})\sin(\frac{\Omega t}{2})\int_0^t \cos(\frac{u\varpi}{2})\sin(\frac{u\Omega}{2})f_x(u)dW_u \\
&+\cos(\frac{\varpi t}{2})\sin(\frac{\Omega t}{2})\int_0^t \sin(\frac{u\varpi}{2})\sin(\frac{u\Omega}{2})f_x(u)dW_u
\end{aligned}$$

and

$$\begin{aligned}\frac{\varpi}{2}\mathcal{W}_y(t) = &-\sin(\frac{\varpi t}{2})\cos(\frac{\Omega t}{2})\int_0^t \cos(\frac{u\varpi}{2})\cos(\frac{u\Omega}{2})f_y(u)dW_u \\ &+\cos(\frac{\varpi t}{2})\cos(\frac{\Omega t}{2})\int_0^t \sin(\frac{u\varpi}{2})\cos(\frac{u\Omega}{2})f_y(u)dW_u \\ &+\sin(\frac{\varpi t}{2})\sin(\frac{\Omega t}{2})\int_0^t \cos(\frac{u\varpi}{2})\cos(\frac{u\Omega}{2})f_x(u)dW_u \\ &-\cos(\frac{\varpi t}{2})\sin(\frac{\Omega t}{2})\int_0^t \sin(\frac{u\varpi}{2})\cos(\frac{u\Omega}{2})f_x(u)dW_u \\ &-\sin(\frac{\varpi t}{2})\cos(\frac{\Omega t}{2})\int_0^t \cos(\frac{u\varpi}{2})\sin(\frac{u\Omega}{2})f_x(u)dW_u \\ &+\cos(\frac{\varpi t}{2})\cos(\frac{\Omega t}{2})\int_0^t \sin(\frac{u\varpi}{2})\sin(\frac{u\Omega}{2})f_x(u)dW_u \\ &-\sin(\frac{\varpi t}{2})\sin(\frac{\Omega t}{2})\int_0^t \cos(\frac{u\varpi}{2})\sin(\frac{u\Omega}{2})f_y(u)dW_u \\ &+\cos(\frac{\varpi t}{2})\sin(\frac{\Omega t}{2})\int_0^t \sin(\frac{u\varpi}{2})\sin(\frac{u\Omega}{2})f_y(u)dW_u.\end{aligned}$$

Using McKean's result, it follows that $\mathcal{W}_x(t)$ and $\mathcal{W}_y(t)$ can be expressed as a sum of particular, non-independent and correlated Brownian motions, i.e.

$$\begin{aligned}\mathcal{W}_x(t) &= \frac{2}{\varpi}\Sigma_i f_i(t)\int_0^t g_i(u)dW_u, \\ &= \frac{2}{\varpi}\Sigma_i f_i(t)\widetilde{W}_i\left(\int_0^t g_i^2(u)du\right) \qquad i = 1, 2, ..., 8.\end{aligned}$$

and

$$\begin{aligned}\mathcal{W}_y(t) &= \frac{2}{\varpi}\Sigma_i h_i(t)\int_0^t j_i(u)dW_u, \\ &= \frac{2}{\varpi}\Sigma_i h_i(t)\widetilde{W}_i\left(\int_0^t j_i^2(u)du\right) \qquad i = 1, 2, ..., 8.\end{aligned}$$

□

Remark 5.2.1. If the first and second components of the noisy force $f(t)$ where equal, i.e $f_x(t) = f_y(t)$ then necessarily $g_i(t) = j_i(t)$, for $i = 1, 2, ...8$.

5.3 The stochastic action

We wish to calculate the action for the stochastic mechanical path X_s^ϵ starting at a point x_0 and reaching x in a time t. To realise this path we require X_s^ϵ in terms of some initial momentum $p(x_0, x, t) = (p_1(0), p_2(0))$. This is calculated as follows. Resorting to canonical coordinates $q = (q_1, q_2)$ and $p = (p_1, p_2)$ the stochastic Hamiltonian $H(q,p,t)$ is defined by

$$H(q,p,t) = 2^{-1}(p - a(q))^2 + V(q) + \varepsilon q.f(t)\dot{W}_t$$

where, as before, $f(t)$ is independent of q. Write

$$\varsigma^\varepsilon(t) = q_1 + iq_2$$

to represent our equation of motion (5.1.5), then

$$\dot{\varsigma}^\varepsilon(t) = \dot{q}_1 + i\dot{q}_2.$$

From Hamilton's equations $\dot{q} = p - a(q)$. This would mean

$$\begin{aligned}\dot{\varsigma}^\varepsilon(0) &= \dot{q}_1(0) + i\dot{q}_2(0)\\ &= \big(p_1(0) - a_1(q(0))\big) + i\big(p_2(0) - a_2(q(0))\big),\end{aligned}$$

where $a = (a_1, a_2)$ and $p(x_0, x, t) = (p_1(0), p_2(0))$ is the initial momentum vector. Using the above initial condition when solving Equation (5.1.3) yields $\varsigma^\varepsilon(x_0, p_0(x_0, x, t), t)$ which allows us to find $(p_1(0), p_2(0))$ explicitly

$$p_1(0) = -\frac{\varpi}{2}\operatorname{cosec}(\frac{\varpi t}{2})\left(x_0\cos(\frac{\varpi t}{2}) + \cos(\frac{\Omega t}{2})\,(\varepsilon\mathcal{W}_x(t) - x) + \sin(\frac{\Omega t}{2})\,(y - \varepsilon\mathcal{W}_y(t))\right) \tag{5.3.1}$$

and

$$p_2(0) = -\frac{\varpi}{2}\operatorname{cosec}(\frac{\varpi t}{2})\left(y_0\cos(\frac{\varpi t}{2}) + \sin(\frac{\Omega t}{2})\,(\varepsilon\mathcal{W}_x(t) - x) + \cos(\frac{\Omega t}{2})\,(\varepsilon\mathcal{W}_y(t) - y)\right). \tag{5.3.2}$$

Here $\mathcal{W}_x(t)$ and $\mathcal{W}_y(t)$ are the noise terms detailed by Equations (5.2.3) and (5.2.4). Following the procedure outline in the deterministic example we find the stochastic mechanical path $X_s^\varepsilon = (X_s^\varepsilon, Y_s^\varepsilon)$ in terms of the initial momentum as

$$\begin{aligned}X^\varepsilon(s, x_0, p_0(x_0, x, t)) &= x_0\cos(\frac{s\varpi}{2})\cos(\frac{s\Omega}{2}) + \frac{2p_1(0)\cos(\frac{s\Omega}{2})\sin(\frac{s\varpi}{2})}{\varpi} + y_0\cos(\frac{s\varpi}{2})\sin(\frac{s\Omega}{2})\\ &+ \frac{2p_2(0)\sin(\frac{s\varpi}{2})\sin(\frac{s\Omega}{2})}{\varpi} + \varepsilon\mathcal{W}_x(s)\end{aligned} \tag{5.3.3}$$

and

$$\begin{aligned}Y^\varepsilon(s, x_0, p_0(x_0, x, t)) &= y_0\cos(\frac{s\varpi}{2})\cos(\frac{s\Omega}{2}) + \frac{2p_2(0)\cos(\frac{s\Omega}{2})\sin(\frac{s\varpi}{2})}{\varpi} - x_0\cos(\frac{s\varpi}{2})\sin(\frac{s\Omega}{2})\\ &- \frac{2p_1(0)\sin(\frac{s\varpi}{2})\sin(\frac{s\Omega}{2})}{\varpi} + \varepsilon\mathcal{W}_y(s).\end{aligned} \tag{5.3.4}$$

It is indeed this path which will be used to calculate the action for a stochastic mechanical path starting at a point x_0 and finishing at a point x in a time t. Recall from Equation

(3.1.2) of Chapter 3 that the stochastic action with initial condition $S_0(X_0^\varepsilon)$ was defined as

$$\mathcal{A}(X_0^\varepsilon, X_t^\varepsilon, t) := A[X_0^\varepsilon, X_t^\varepsilon, t] + S_0(X_0^\varepsilon)$$

where

$$A[X_0^\varepsilon, X_t^\varepsilon, t] = \int_0^t \left[\frac{1}{2}\dot{X}_s^{\varepsilon\,2} - V(X_s^\varepsilon)\right] ds + \int_0^t a(X_s^\varepsilon, s).dX_s^\varepsilon - \varepsilon \int_0^t k(X_s^\varepsilon, s) dW_s. \quad (5.3.5)$$

As before, we take $S_0(X_0^\varepsilon)$ to be the generic cusp initial condition $S_0(X_0^\varepsilon) = \frac{x_0^2 y_0}{2}$. Before we reveal the expression of $\mathcal{A}[X_0^\varepsilon, X_t^\varepsilon, t]$ consider the following proposition

Proposition 5.3.1. *The stochastic integral terms $\mathcal{W}_x(t)$ and $\mathcal{W}_y(t)$, defined by Equations* (5.2.3) *and* (5.2.4), *are differentiable functions with derivatives*

$$\frac{d\mathcal{W}_x(t)}{dt} = \frac{2}{\varpi}\int_0^t \frac{d}{dt}\sin\left(\frac{(u-t)\varpi}{2}\right)\left[\cos\left(\frac{(u-t)\Omega}{2}\right)f_x(u) - \sin\left(\frac{(u-t)\Omega}{2}\right)f_y(u)\right] dW_u$$

and

$$\frac{d\mathcal{W}_y(t)}{dt} = \frac{2}{\varpi}\int_0^t \frac{d}{dt}\sin\left(\frac{(u-t)\varpi}{2}\right)\left[\cos\left(\frac{(u-t)\Omega}{2}\right)f_y(u) + \sin\left(\frac{(u-t)\Omega}{2}\right)f_x(u)\right] dW_u.$$

Proof. Recall

$$\frac{\varpi}{2}\mathcal{W}_x(t) = \int_0^t \sin\left(\frac{(u-t)\varpi}{2}\right)\left[\cos\left(\frac{(u-t)\Omega}{2}\right)f_x(u) - \sin\left(\frac{(u-t)\Omega}{2}\right)f_y(u)\right] dW_u$$

and

$$\frac{\varpi}{2}\mathcal{W}_y(t) = \int_0^t \sin\left(\frac{(u-t)\varpi}{2}\right)\left[\cos\left(\frac{(u-t)\Omega}{2}\right)f_y(u) + \sin\left(\frac{(u-t)\Omega}{2}\right)f_x(u)\right] dW_u.$$

Then $d_t\mathcal{W}_x(t)$ and $d_t\mathcal{W}_y(t)$ may be expressed as

$$\begin{aligned}\frac{\varpi}{2}d_t\mathcal{W}_x(t) &= \underbrace{\sin\left(\frac{(t-t)\varpi}{2}\right)}_{=0}\left[\cos\left(\frac{(t-t)\Omega}{2}\right)f_x(t) - \sin\left(\frac{(t-t)\Omega}{2}\right)f_y(t)\right] dW_t \\ &\quad + \int_0^t \frac{d}{dt}\sin\left(\frac{(u-t)\varpi}{2}\right)\left[\cos\left(\frac{(u-t)\Omega}{2}\right)f_x(u) - \sin\left(\frac{(u-t)\Omega}{2}\right)f_y(u)\right] dW_u dt\end{aligned}$$

and

$$\begin{aligned}\frac{\varpi}{2}d_t\mathcal{W}_y(t) &= \underbrace{\sin\left(\frac{(t-t)\varpi}{2}\right)}_{=0}\left[\cos\left(\frac{(t-t)\Omega}{2}\right)f_y(t) + \sin\left(\frac{(t-t)\Omega}{2}\right)f_x(t)\right] dW_t \\ &\quad + \int_0^t \frac{d}{dt}\sin\left(\frac{(u-t)\varpi}{2}\right)\left[\cos\left(\frac{(u-t)\Omega}{2}\right)f_y(u) + \sin\left(\frac{(u-t)\Omega}{2}\right)f_x(u)\right] dW_u dt.\end{aligned}$$

Result follows. □

It turns out from Equations (5.3.3) and (5.3.4) that $\mathcal{A}[X_0^\varepsilon, X_t^\varepsilon, t]$ may be expressed as

$$\begin{aligned}
&\mathcal{A}[X_0^\varepsilon, X_t^\varepsilon, t]\\
&= \frac{1}{2}\Big(p_1(0)x_0 + p_2(0)y_0\Big)\Big(\cos(\varpi t) - 1\Big)\\
&+ \frac{\sin(t\varpi)}{2\varpi}\Big(4\left(p_1(0)^2 + p_2(0)^2\right) - \left({x_0}^2 + {y_0}^2\right)\varpi^2\Big)\\
&+ \frac{\varepsilon}{4}\int_0^t \Big\{\varpi \sin(\frac{s\varpi}{2})\Big[\sin(\frac{s\Omega}{2})\Big(\left(x_0\Omega - 2p_2(0)\right)\mathcal{W}_x(s) + \left(2p_1(0) + y_0\Omega\right)\mathcal{W}_y(s)\Big)\\
&+ 2\sin(\frac{s\Omega}{2})\left(x_0\mathcal{W}_y'(s) - y_0\mathcal{W}_x'(s)\right) - 2\cos(\frac{s\Omega}{2})(x_0\mathcal{W}_x'(s) + y_0\mathcal{W}_y'(s))\\
&\cos(\frac{s\Omega}{2})\Big(\left(2p_1(0) + y_0\Omega\right)\mathcal{W}_x(s) + 2p_2(0)\mathcal{W}_y(s) - x_0\Omega\mathcal{W}_y(s)\Big)\Big]\\
&- \cos(\frac{s\varpi}{2})\Big[\sin(\frac{s\Omega}{2})\Big(\left(y_0\varpi^2 + 2p_1(0)\Omega\right)\mathcal{W}_x(s) + \left(2p_2(0)\Omega - x_0\varpi^2\right)\mathcal{W}_y(s)\Big)\\
&+ 4\sin(\frac{s\Omega}{2})\left(p_1(0)\mathcal{W}_y'(s) - p_2(0)\mathcal{W}_x'(s)\right) - 4\cos(\frac{s\Omega}{2})\left(p_1(0)\mathcal{W}_x'(s) + p_2(0)\mathcal{W}_y'(s)\right)\\
&+ \cos(\frac{s\Omega}{2})\Big(\left(x_0\varpi^2 - 2p_2(0)\Omega\right)\mathcal{W}_x(s) + \left(y_0\varpi^2 + 2p_1(0)\Omega\right)\mathcal{W}_y(s)\Big)\Big]\Big\}\, ds\\
&+ \frac{\varepsilon^2}{2}\int_0^t \Big(\mathcal{W}_y'(s)^2 + \mathcal{W}_x'(s)^2 - \omega^2\left(\mathcal{W}_x(s)^2 + \mathcal{W}_y(s)^2\right)\\
&- \Omega\mathcal{W}_y(s)\mathcal{W}_x'(s) + 2\Omega\mathcal{W}_x(s)\mathcal{W}_y'(s)\Big)ds - \varepsilon\int_0^t k(X_s^\varepsilon, s)dW_s.
\end{aligned} \tag{5.3.6}$$

where $\mathcal{W}_x'(s) = \frac{d\mathcal{W}_x(s)}{ds}$ and $\mathcal{W}_y'(s) = \frac{d\mathcal{W}_y(s)}{ds}$ are well-defined time derivatives of a stochastic integral.

Remark 5.3.1. Inherent in the initial momentum vector $(p_1(0), p_2(0))$ are terms of order ε.

In a later section we consider the stochastic set-up where the noise is one dimensional and parallel to the x coordinate axis, i.e setting $f(t) = (f_x(t), 0, 0)$ so that $k(X_s^\varepsilon, s) = X_s^\varepsilon(x_0, p_0(x_0, x, t), x, t)f_x(s)$. In this instance, the final term of the stochastic action equation (5.3.6) would be

$$\begin{aligned}
&\int_0^t k(X_s^\varepsilon, s)dW_s\\
&= \int_0^t X_s^\varepsilon f_x(s)dW_s\\
&= \int_0^t \Big(x_0\cos\left(\frac{s\Omega}{2}\right)\cos\left(\frac{s\varpi}{2}\right) + y_0\sin\left(\frac{s\Omega}{2}\right)\cos\left(\frac{s\varpi}{2}\right)\\
&+ \frac{2p_1(0)}{\varpi}\cos\left(\frac{s\Omega}{2}\right)\sin\left(\frac{s\varpi}{2}\right) + \frac{2p_2(0)}{\varpi}\sin\left(\frac{s\Omega}{2}\right)\sin\left(\frac{s\varpi}{2}\right) + \varepsilon\mathcal{W}_x(s)\Big)f_x(s)dW_s
\end{aligned} \tag{5.3.7}$$

where $p(x_0, x, t) = (p_1(0), p_2(0))$ is the initial momentum vector outlined in Equations (5.3.1) and (5.3.2) and $\mathcal{W}_x(s)$ is defined by (5.2.3).

Remark 5.3.2. Once the initial momentum term is substituted into the stochastic action, Equation (5.3.6), the complexity of the expression increases dramatically.

5.4 Simulating noise terms in the stochastic action

For the calculation of random quantities the computer software package *Mathematica* was used. The foundation of the study involves simulating a random walk to approximate Brownian motion. The basis of this work began with the following module devised by Reynolds in [25].

```
SeedRandom[n]

Brownian1[T_, a_, Optional[s_, PlotStyle -> RGBColor[0, 0, 0]]] :=

Module[{k, S, t, X, n, d},

S[0] = 0; t[0] = 0; d = Sqrt[a]; n = Floor[T/a];

Do[k = Random[Integer]; If[k == 1, X = d, X = -d];

S[i] = S[i - 1] + X; t[i] = t[i - 1] + a, {i, n}];

W1[r] := S[Floor[r/a]];

W2[r] := Sum[S[i - 1](t[i] - t[i - 1]), {i, Floor[r/a]}];

W3[r] := Sum[S[i - 1]^2(t[i] - t[i - 1]), {i, Floor[r/a]}];

ListPlot[Table[{t[i], S[i]}, {i, 0, n}], PlotJoined -> True, u]]
```

In the above module the user selects the time (`T`) that the process will run for and the time step size (a) to be used. This produces a sample path and calculates the following quantities

$$\begin{aligned} \texttt{W1[t]} &\approx W(t)\ , \\ \texttt{W2[t]} &\approx \int_0^t W(u)\mathrm{d}u\ , \\ \texttt{W3[t]} &\approx \int_0^t W(u)^2\mathrm{d}u\ . \end{aligned}$$

Finally the command `ListPlot` produces a picture of the sample path.

The command `SeedRandom[n]` has been added to the module to ensure one obtains the same sequence of pseudorandom numbers on different occasions (the value n is to be chosen by the user). For a specific time, it is necessary to compute the stochastic action numerically before prelevel and level surface equations may be calculated. Therefore, in order to build a portfolio of meaningful pictures of the evolution of the stochastic mechanical system it is imperative that the **same** Brownian sample paths are constructed. The `seedRandom[n]` command takes care of this as long as the path is constructed for the same finite time and mesh size.

Step 1

The first step is to decide in what manner to add noise to our underlying classical mechanical system. For our illustration we choose to add the noise parallel to the x axis. Consequently, we set $k(X_s^\epsilon, s) = X_s^\epsilon$ which implies $f_y(s) = 0$ and $f_x(s) = 1$.

Step 2

Under the assumptions of the previous step we define, for $u < s$,

$$\begin{aligned}\frac{\varpi}{2}\mathcal{W}_x(s) :=& -\sin(\frac{\varpi s}{2})\cos(\frac{\Omega s}{2})\int_0^s \cos(\frac{u\varpi}{2})\cos(\frac{u\Omega}{2})f_x(u)dW_u \\ &+\cos(\frac{\varpi s}{2})\cos(\frac{\Omega s}{2})\int_0^s \sin(\frac{u\varpi}{2})\cos(\frac{u\Omega}{2})f_x(u)dW_u \\ &-\sin(\frac{\varpi s}{2})\sin(\frac{\Omega s}{2})\int_0^s \cos(\frac{u\varpi}{2})\sin(\frac{u\Omega}{2})f_x(u)dW_u \\ &+\cos(\frac{\varpi s}{2})\sin(\frac{\Omega s}{2})\int_0^s \sin(\frac{u\varpi}{2})\sin(\frac{u\Omega}{2})f_x(u)dW_u\end{aligned}$$

and

$$\begin{aligned}\frac{\varpi}{2}\mathcal{W}_y(s) :=& +\sin(\frac{\varpi s}{2})\sin(\frac{\Omega s}{2})\int_0^s \cos(\frac{u\varpi}{2})\cos(\frac{u\Omega}{2})f_x(u)dW_u \\ &-\cos(\frac{\varpi s}{2})\sin(\frac{\Omega s}{2})\int_0^s \sin(\frac{u\varpi}{2})\cos(\frac{u\Omega}{2})f_x(u)dW_u \\ &-\sin(\frac{\varpi s}{2})\cos(\frac{\Omega s}{2})\int_0^s \cos(\frac{u\varpi}{2})\sin(\frac{u\Omega}{2})f_x(u)dW_u \\ &+\cos(\frac{\varpi s}{2})\cos(\frac{\Omega s}{2})\int_0^s \sin(\frac{u\varpi}{2})\sin(\frac{u\Omega}{2})f_x(u)dW_u.\end{aligned}$$

Both these terms occur frequently in the expression for the stochastic action. Moreover, they occur within the integrand of time integrals. If we consider individual terms of $\mathcal{W}_x(s)$ and $\mathcal{W}_y(s)$ the question arises of how one can approximate integrals of the following structure?

$$\int_0^t \sin(\frac{\varpi s}{2})\cos(\frac{\Omega s}{2})\left(\int_0^s \cos(\frac{u\varpi}{2})\cos(\frac{u\Omega}{2})dW_u\right) ds. \tag{5.4.1}$$

Firstly, observe that all such integrands in $\mathcal{W}_x(s)$ and $\mathcal{W}_y(s)$ are non-anticipating. Let $0 = s_0 < s_1 < \ldots < s_n = s$ be a partition of s, then for any non-anticipating function $f(u)$

$$\int_0^s f(u)dW_u \approx \sum_{i=1}^{n} f(s_{i-1})\left(W(s_i) - W(s_{i-1})\right).$$

Hence, if we let $f(s) := \cos(\frac{s\varpi}{2})\cos(\frac{s\Omega}{2})$, the code we would add to the above module to approximate

$$\begin{aligned}\int_0^s \cos(\frac{u\varpi}{2})\cos(\frac{u\Omega}{2})dW_u &= \int_0^s f(u)dW_u \\ &\approx I[s]\end{aligned}$$

would be

```
I[r]:= Sum[f[t[i - 1]]( W1[t[i]] - W1[t[i - 1]] ), {i,
Floor[r/a]}];
```

Essentially, `I[s]` would then be a sum of functions dependent on s. Consequently, for any specified value s, `I[s]` is a real number. In light of this, Equation (5.4.1) can now be approximated using numerical integration techniques. The generic command `NIntegrate` will do this. However, there are more sophisticated techniques available to reduce the associated errors. Taking the naive option, the code to approximate Equation (5.4.1) is

```
NIntegrate[ g[s]I[s], {s, 0, t} ]
```

Where $g(s) := \sin(\frac{\varpi s}{2})\cos(\frac{\Omega s}{2})$ and t is the time for which the stochastic action is to be calculated.

Step 3

Following the method outlined in Step 2 one may calculate virtually all noise terms encountered in the stochastic action. However, there exists a double stochastic integral arising from

$$\begin{aligned}\int_0^t k(X_s^\varepsilon, s)dW_s &:= \int_0^t X_s^\varepsilon f_x(s)dW_s \\ &= \int_0^t X_s^\varepsilon dW_s.\end{aligned}$$

The integrand of this expression may be separated into deterministic and random functions. Write these as $h(s)$ and $\mathcal{W}_x(s)$ respectively, i.e.,

$$\int_0^t k(X_s^\varepsilon, s)dW_s = \int_0^t \left(h(s) + \mathcal{W}_x(s)\right)dW_s.$$

The integral $\int_0^t h(s)dW_s$ can be handled following the procedure in step 2, but the series of double stochastic integrals inherent in $\int_0^t \mathcal{W}_x(s)dW_s$ requires a slightly different approach.

Observing the expression for $\mathcal{W}_x(s)$, consider a typical double stochastic integral from this series, say,

$$\int_0^t \sin(\frac{\varpi s}{2}) \cos(\frac{\Omega s}{2}) \left(\int_0^s \cos(\frac{u\varpi}{2}) \cos(\frac{u\Omega}{2}) dW_u \right) dW_s$$

for $u < s$. From the work in step 2 we can immediately say

$$\int_0^t \sin(\frac{\varpi s}{2}) \cos(\frac{\Omega s}{2}) \left(\int_0^s \cos(\frac{u\varpi}{2}) \cos(\frac{u\Omega}{2}) dW_u \right) dW_s$$
$$\approx \int_0^t \sin(\frac{\varpi s}{2}) \cos(\frac{\Omega s}{2}) I[s] dW_s,$$

but $\sin(\frac{\varpi s}{2}) \cos(\frac{\Omega s}{2}) I[s]$ is itself a non-anticipating function. Consequently, once this fact is observed, the method in part 2 is applicable.

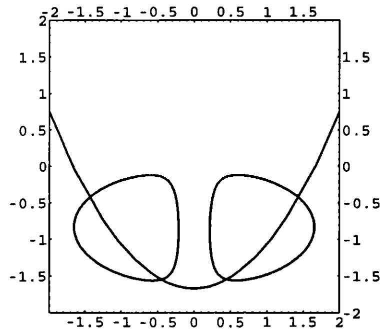
Step 4

We have succeeded in finding a method to simulate all noise terms of any order for any specified time t. Direct numerical computation of the stochastic action is needed for each of these times t before we are able to use methods outlined in previous chapters to produce equations, and hence, pictures. The following catalogue illustrates how the prelevel and level surfaces deform in the presence of noise.

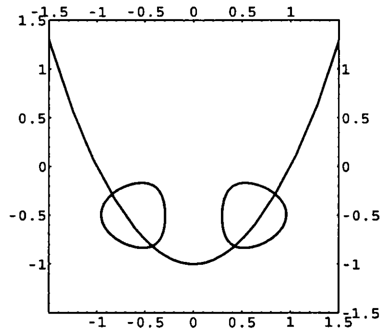
Remark 5.4.1. Little thought has been given to computing time in the construction of this method to approximate the necessary random quantities. Undoubtedly, there exist approaches and techniques to give better approximations. The obvious being a smaller time-step in the random walk and a more advanced integration technique. In a further publication these issues could be addressed.

Remark 5.4.2. Perhaps one way of obtaining a better approximation would be to use the Brownian Bridge technique with the random walk.

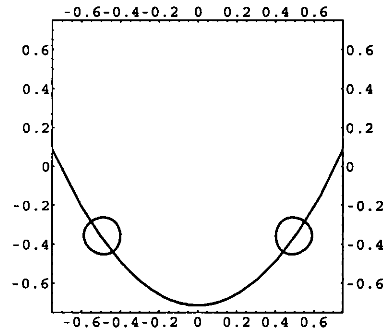
Figure 5.1: Pre curves for $c = -0.01$.



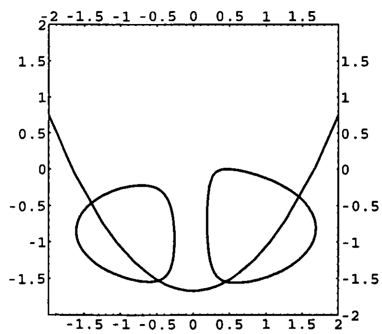
$\varepsilon = 0$ and $t = 0.6$.



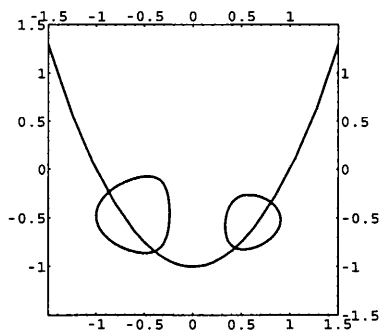
$\varepsilon = 0$ and $t = 1$.



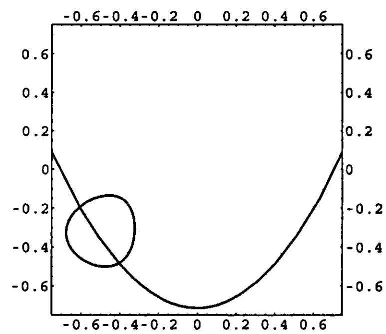
$\varepsilon = 0$ and $t = 1.4$.



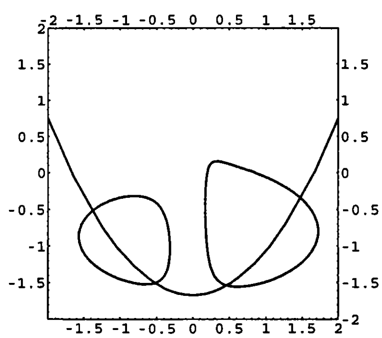
$\varepsilon = 0.05$ and $t = 0.6$.



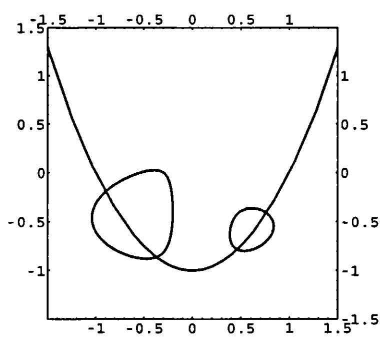
$\varepsilon = 0.05$ and $t = 1$.



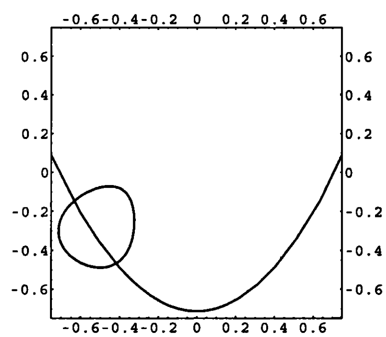
$\varepsilon = 0.05$ and $t = 1.4$.



$\varepsilon = 0.1$ and $t = 0.6$.

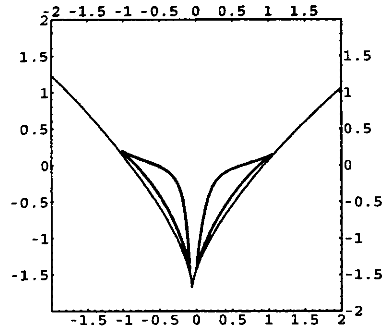


$\varepsilon = 0.1$ and $t = 1$.

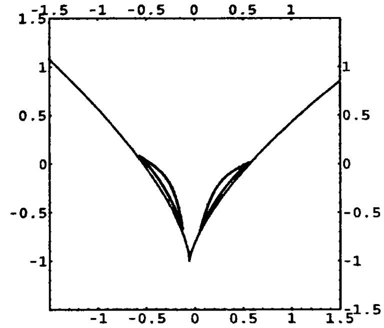


$\varepsilon = 0.1$ and $t = 1.4$.

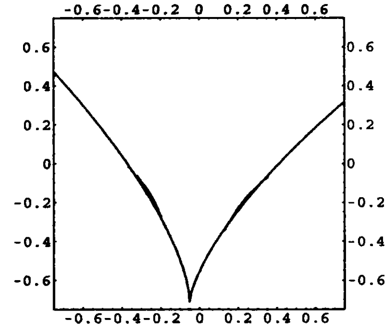
Figure 5.2: Image curves for $c = -0.01$.



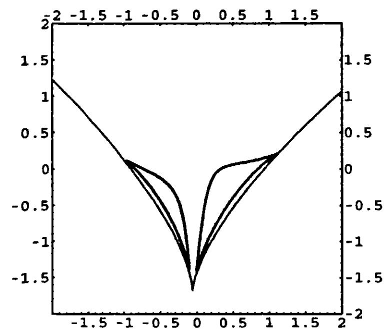
$\varepsilon = 0$ and $t = 0.6$.



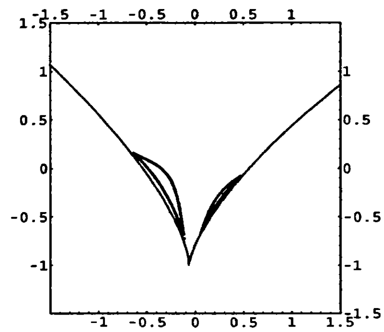
$\varepsilon = 0$ and $t = 1$.



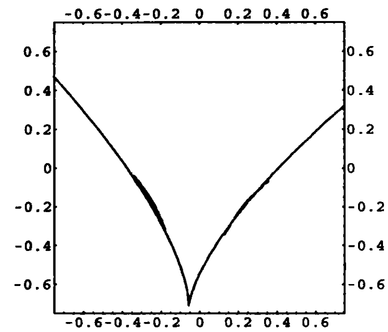
$\varepsilon = 0$ and $t = 1.4$.



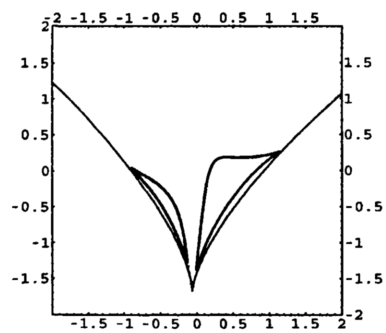
$\varepsilon = 0.05$ and $t = 0.6$.



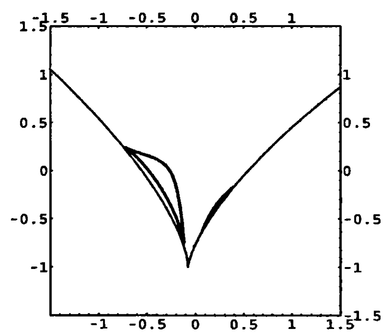
$\varepsilon = 0.05$ and $t = 1$.



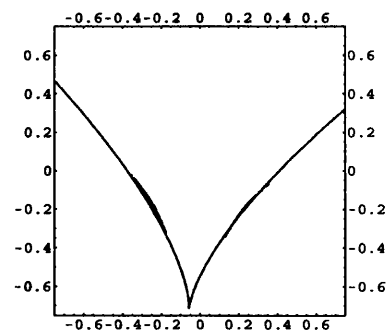
$\varepsilon = 0.05$ and $t = 1.4$.



$\varepsilon = 0.1$ and $t = 0.6$.

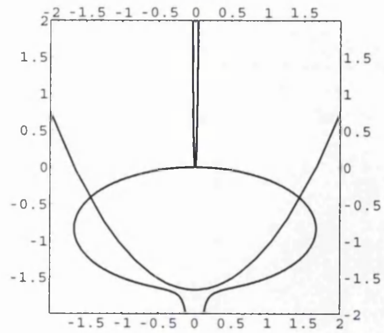


$\varepsilon = 0.1$ and $t = 1$.

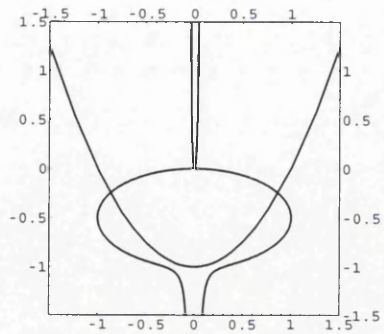


$\varepsilon = 0.1$ and $t = 1.4$.

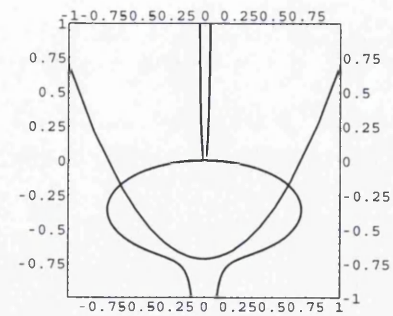
Figure 5.3: Pre curves for $c = 0$.



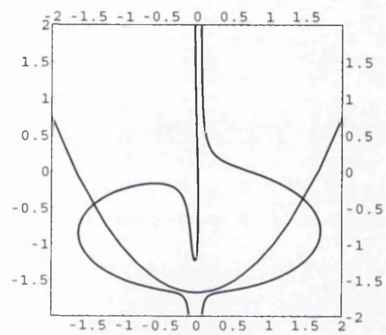
$\varepsilon = 0$ and $t = 0.6$.



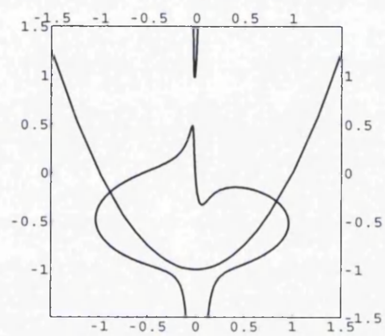
$\varepsilon = 0$ and $t = 1$.



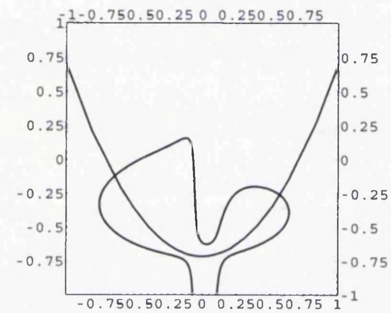
$\varepsilon = 0$ and $t = 1.4$.



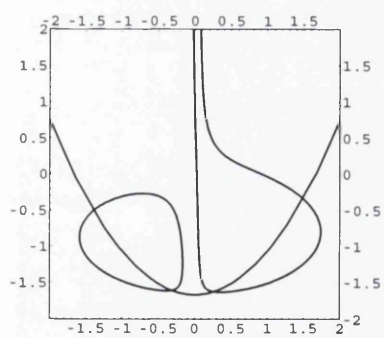
$\varepsilon = 0.05$ and $t = 0.6$.



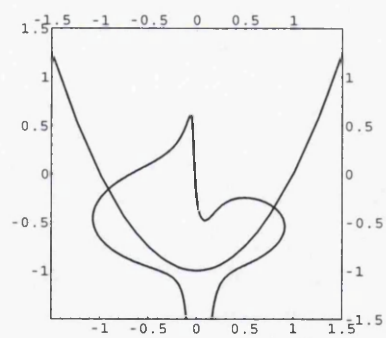
$\varepsilon = 0.05$ and $t = 1$.



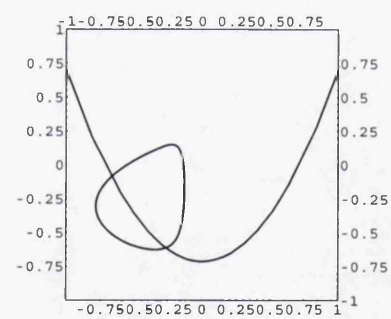
$\varepsilon = 0.05$ and $t = 1.4$.



$\varepsilon = 0.1$ and $t = 0.6$.

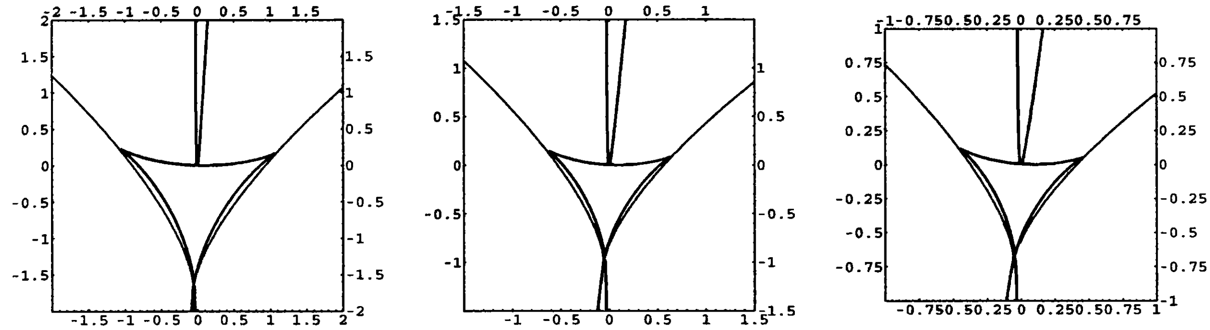


$\varepsilon = 0.1$ and $t = 1$.

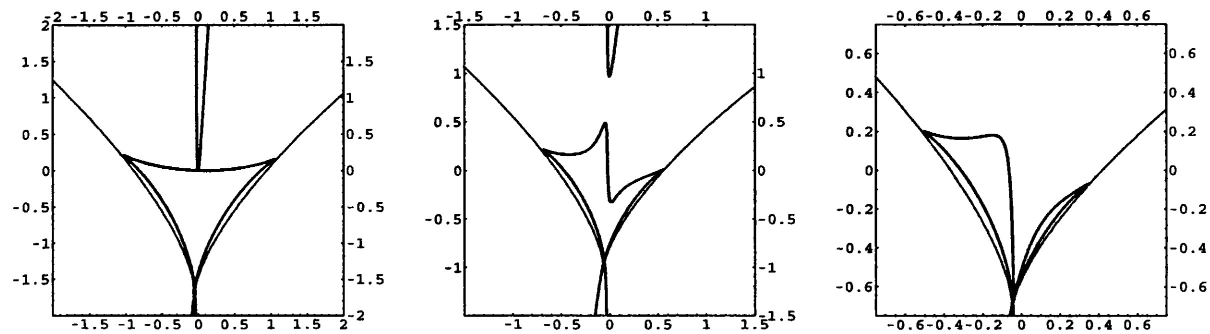


$\varepsilon = 0.1$ and $t = 1.4$.

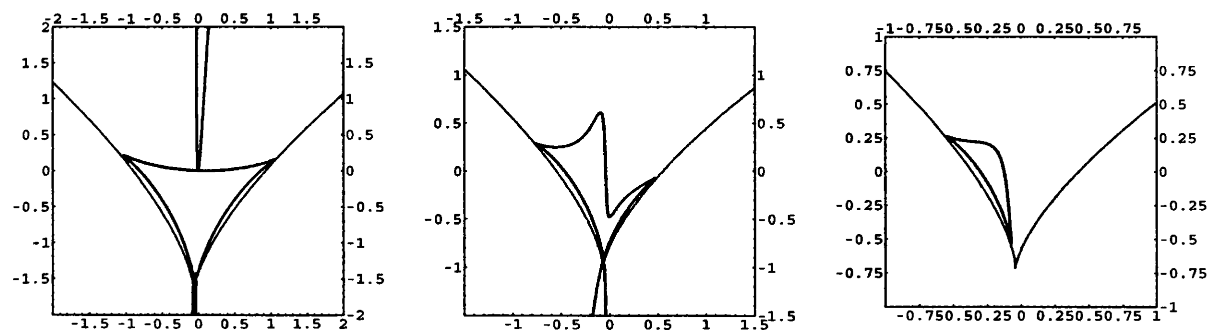
Figure 5.4: Image curves for $c = 0$.



$\varepsilon = 0$ and $t = 0.6$. $\varepsilon = 0$ and $t = 1$. $\varepsilon = 0$ and $t = 1.4$.

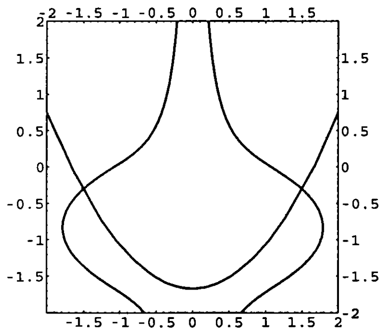


$\varepsilon = 0.05$ and $t = 0.6$. $\varepsilon = 0.05$ and $t = 1$. $\varepsilon = 0.05$ and $t = 1.4$.

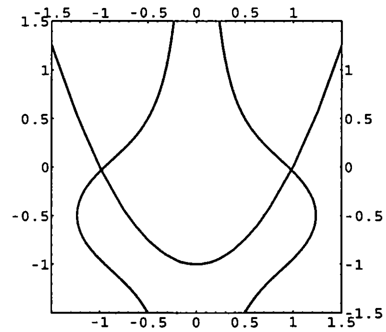


$\varepsilon = 0.1$ and $t = 0.6$. $\varepsilon = 0.1$ and $t = 1$. $\varepsilon = 0.1$ and $t = 1.4$.

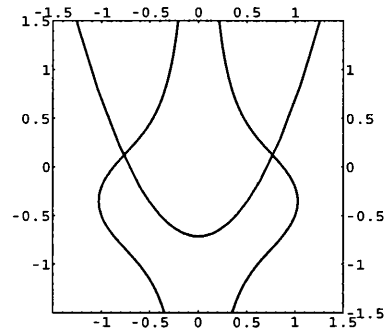
Figure 5.5: Pre curves for $c = 0.1$.



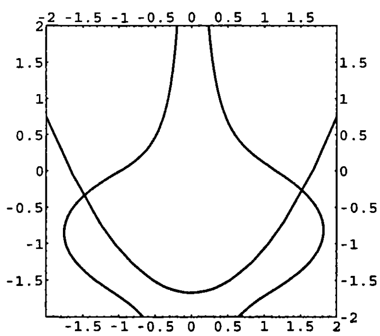
$\varepsilon = 0$ and $t = 0.6$.



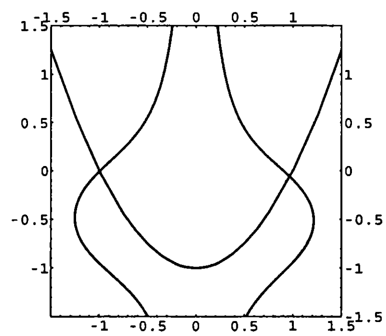
$\varepsilon = 0$ and $t = 1$.



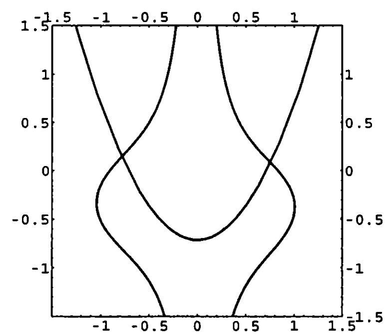
$\varepsilon = 0$ and $t = 1.4$.



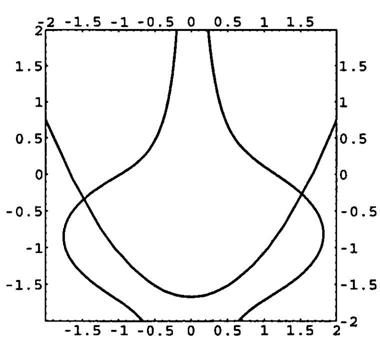
$\varepsilon = 0.05$ and $t = 0.6$.



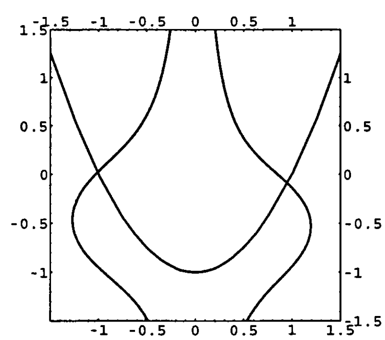
$\varepsilon = 0.05$ and $t = 1$.



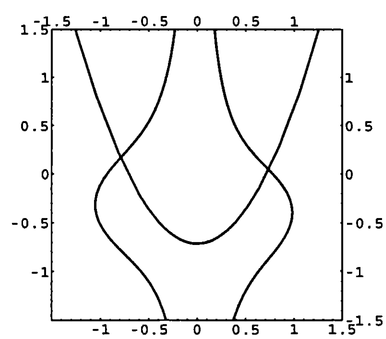
$\varepsilon = 0.05$ and $t = 1.4$.



$\varepsilon = 0.1$ and $t = 0.6$.

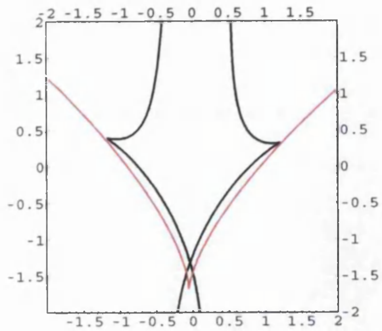


$\varepsilon = 0.1$ and $t = 1$.

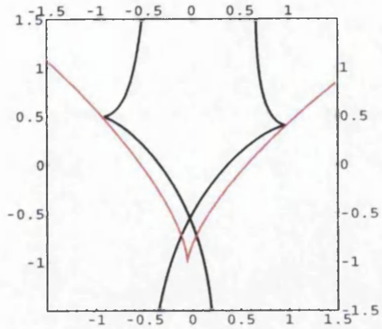


$\varepsilon = 0.1$ and $t = 1.4$.

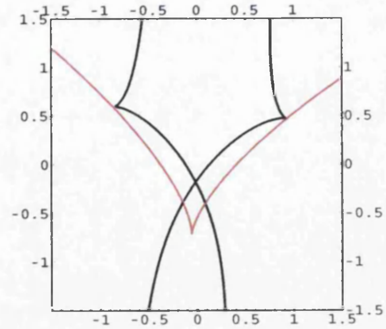
Figure 5.6: Image curves for $c = 0.1$



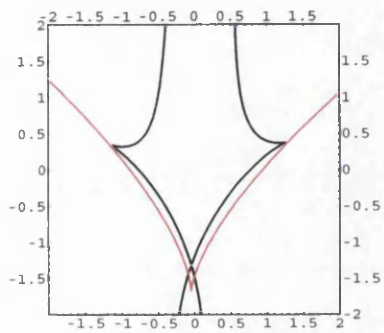
$\varepsilon = 0$ and $t = 0.6$.



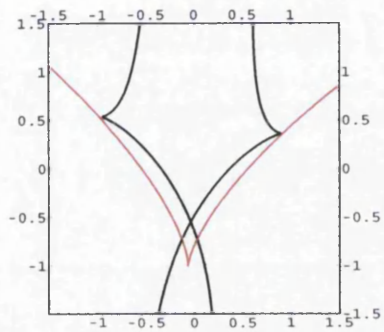
$\varepsilon = 0$ and $t = 1$.



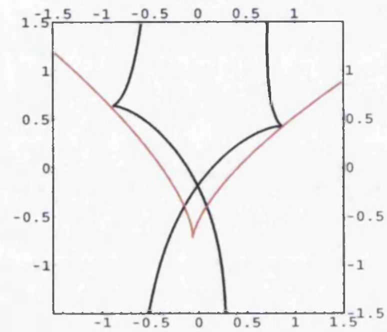
$\varepsilon = 0$ and $t = 1.4$.



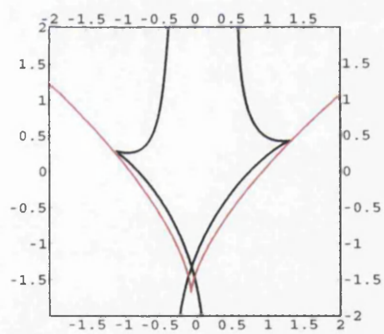
$\varepsilon = 0.05$ and $t = 0.6$.



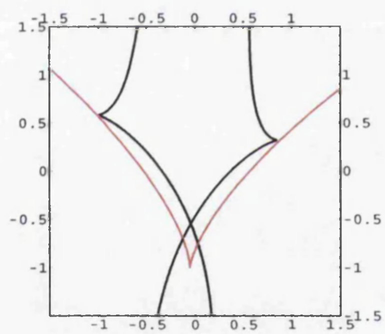
$\varepsilon = 0.05$ and $t = 1$.



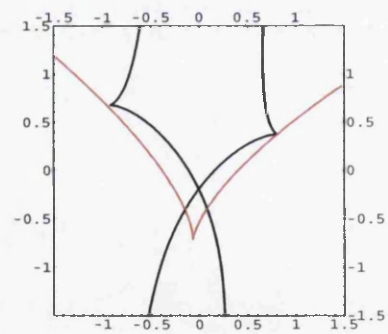
$\varepsilon = 0.05$ and $t = 1.4$.



$\varepsilon = 0.1$ and $t = 0.6$.



$\varepsilon = 0.1$ and $t = 1$.



$\varepsilon = 0.1$ and $t = 1.4$.

5.5 Intermittence of stochastic turbulence

Following work by RTW in [26], we saw in Section (3.6) that ideas pertaining to the reduced stochastic action functional $f_{(x_t(\lambda),t)}$ can be extended to obtain the turbulent times of the mechanical system. Recall that the notion of *turbulent times* was defined to be the times for which the precaustic and prelevel surface touch. The turbulent times will be the zeros of a stochastic process which we name zeta. One can determine whether the turbulence is intermittent by analysing the recurrent nature of the zeta process.

In this section we illustrate the idea of intermittence of stochastic turbulence by using a harmonic oscillator potential and adding the noise potential term parallel with the x-axis, i.e. setting $f(s) = (f_x(s), f_y(s)) = (f_x(s), 0)$ so that $k(x,s) = xf_x(s)$. Let $\lambda \longmapsto x_t(\lambda)$ be a parameterisation of the caustic where λ is also the first coordinate of a point on the precaustic. For $x = (x^1, x^2, ...)$ and $x_0 = (x_0^1, x_0^2, ...)$, a point $(\lambda, x_0^2(\lambda))$ maps under the stochastic mechanical flow map Φ_t to $x_t(\lambda)$, i.e, $x_t(\lambda) = \Phi_t(\lambda, x_0^2(\lambda))$. Hence,

$$f_{(x_t(\lambda),t)}(x_0^1) = f_{\Phi_t(\lambda,x_0^2(\lambda))}(x_0^1).$$

Therefore, when $x_0^1 = \lambda$ we must have

$$f'_{(x_t(\lambda),t)}(x_0^1) = 0 \quad \text{and} \quad f''_{(x_t(\lambda),t)}(x_0^1) = 0.$$

Thus, $x_0^1 = \lambda$ is a repeated root of $f'_{(x_t(\lambda),t)}(x_0^1) = 0$.

Proposition 5.5.1. *Let $k(X_s^\varepsilon, s) = xf_x(s)$ and $V = \frac{1}{2}(x,y)\omega^2(x,y)^T$, where ω^2 is a real symmetric positive definite 2×2 matrix with*

$$(\omega)_{ij} = \begin{cases} \omega & \textit{if } i = j, \\ 0 & \textit{otherwise.} \end{cases}$$

and $X^{\varepsilon}_s = x$. *Consider* $S_0(x_0, y_0) = \frac{x_0^2 y_0}{2}$. *Then the zeros* $t(\tilde{\omega})$ *of the stochastic process*

$$\begin{aligned}
&\zeta(t)\\
&= -c - \frac{\varpi^3}{16}\cos(\frac{t\varpi}{2})^2 \cot(\frac{t\varpi}{2})\\
&+ \frac{\varepsilon\varpi}{2}\cos(\frac{t\varpi}{2})\left(\sin(\frac{t\Omega}{2})\mathcal{W}_x(t) + \cos(\frac{t\Omega}{2})\mathcal{W}_y(t)\right)\\
&- \frac{\varepsilon\varpi}{2}\cot(\frac{t\varpi}{2})\int_0^t \sin(\frac{s\Omega}{2})\cos(\frac{s\varpi}{2}) f_x(s)dW_s\\
&+ \frac{\varepsilon^2}{4\varpi}\Bigg[-2\varpi\Omega\int_0^t \mathcal{W}_y(s)\mathcal{W}_x'(s)ds + 2\varpi\Omega\int_0^t \mathcal{W}_x(s)\mathcal{W}_y'(s)ds + 2\varpi\int_0^t \mathcal{W}_x'(s)^2 ds\\
&+ 2\varpi\int_0^t \mathcal{W}_y'(s)^2 ds + \frac{\varpi}{2}\left(-\varpi^2 + \Omega^2\right)\int_0^t \mathcal{W}_x(s)^2 ds + \int_0^t \mathcal{W}_y(s)^2 ds\Bigg]\\
&+ \frac{\varepsilon^2 \operatorname{cosec}(\frac{t\varpi}{2})}{4\varpi}\Bigg[4\cos(\frac{t\varpi}{2})\left(\mathcal{W}_x(t)^2 + \mathcal{W}_y(t)^2\right)\\
&- 4\cos(\frac{t\varpi}{2})\left(\varpi^2\cos(\frac{t\varpi}{2})\mathcal{W}_x(t)^2 + \varpi^2\cos(\frac{t\varpi}{2})\mathcal{W}_y(t)^2\right)\\
&+ (-2+\varpi)\varpi^2\left(-1+\varpi^2+3\Omega^2\right)\cos(\frac{t\Omega}{2})\mathcal{W}_x(t)\int_0^t \cos(\frac{s\Omega}{2})\sin(\frac{s\varpi}{2})\mathcal{W}_x(s)ds\\
&- (-2+\varpi)\varpi^2\left(-1+\varpi^2+3\Omega^2\right)\sin(\frac{t\Omega}{2})\mathcal{W}_y(t)\int_0^t \cos(\frac{s\Omega}{2})\sin(\frac{s\varpi}{2})\mathcal{W}_x(s)ds\Bigg]\\
&+ \varepsilon^2 \operatorname{cosec}(\frac{t\varpi}{2})\Bigg[\cos(\frac{t\Omega}{2})\mathcal{W}_x(t)\int_0^t \cos(\frac{s\Omega}{2})\sin(\frac{s\varpi}{2}) f_x(s)dW_s\\
&+ \frac{2}{\varpi}\cos(\frac{t\Omega}{2})\mathcal{W}_x(t)\int_0^t \cos(\frac{s\Omega}{2})\sin(\frac{s\varpi}{2}) f_x(s)dW_s - \mathcal{W}_x(t)\int_0^t \sin(\frac{s\varpi}{2})\sin^2(\frac{s\Omega}{2}) f_x(s)dW_s\\
&+ \frac{2}{\varpi}\mathcal{W}_x(t)\int_0^t \sin(\frac{s\varpi}{2})\sin^2(\frac{s\Omega}{2}) f_x(s)dW_s + \cos(\frac{t\Omega}{2})\mathcal{W}_y(t)\int_0^t \sin(\frac{s\varpi}{2})\sin(\frac{s\Omega}{2}) f_x(s)dW_s\\
&- \frac{2}{\varpi}\cos(\frac{t\Omega}{2})\mathcal{W}_y(t)\int_0^t \sin(\frac{s\varpi}{2})\sin(\frac{s\Omega}{2}) f_x(s)dW_s\\
&- \sin(\frac{t\Omega}{2})\mathcal{W}_y(t)\int_0^t \cos(\frac{s\Omega}{2})\sin(\frac{s\varpi}{2}) f_x(s)dW_s\\
&+ \frac{2}{\varpi}\sin(\frac{t\Omega}{2})\mathcal{W}_y(t)\int_0^t \cos(\frac{s\Omega}{2})\sin(\frac{s\varpi}{2}) f_x(s)dW_s\Bigg]
\end{aligned}$$

will be turbulent times. Moreover, the turbulence occurs at $\Phi_t\left(0, -\frac{\varpi}{2}\cot(\frac{\varpi t}{2})\right)$.

Proof. Let $f_{(x_t(\lambda),t)}(x_0)$ be the one dimension reduced stochastic action functional for the stochastic Burgers fluid with vorticity where $x_0 = x_0^1$. Here $x_t(\lambda)$ indicates the intrinsic paramterisation of the caustic. The turbulent times will be given by $t > 0$ satisfying $f_{(x_t(\lambda),t)}(x_0) = c$ and $\frac{\partial}{\partial x_0} f_{(x_t(\lambda),t)}(x_0) = 0$. By the above tautology $x_0 = \lambda$ is the repeated root of $\frac{\partial}{\partial x_0} f_{(x_t(\lambda),t)}(x_0)$. From Section (3.6), we know that one type of turbulence occurs at the cusps on the caustic. From Section (4.2) it is precisely $\lambda = 0$ that gives rise to cusps on the caustics. Therefore, by evaluating this function at the cusps on the caustic and their inverse images a stochastic process can be obtained whose zeros give the times at which the cusped turbulence occurs. Thus, following Section (4.2) the stochastic Zeta Process $\zeta_c(t)$ is found by computing

$$\zeta_c(t) := f_{(x_t(0),t)}(0,t) - c.$$

Moreover, observe from the precaustic equation

$$y_0(\lambda) = \frac{-\varpi^2 \cot(\frac{\varpi t}{2}) + 4\lambda^2 \tan(\frac{\varpi t}{2})}{2\varpi}.$$

If t represents the turbulent times then the turbulence must occur at $\Phi_t\left(0, -\frac{\varpi}{2}\cot(\frac{\varpi t}{2})\right)$. □

In the following example we show the stochastic turbulence to be intermittent for a certain class of functions $f_x(s)$ by proving the $\zeta(t)$ stochastic process is recurrent to zero.

Example 5.5.1. For $|f_x^2(s)| < \frac{1}{1+s^2}$ there exists an increasing sequence $\{t_i\}$ with $t_i \nearrow \infty$ such that $\zeta(t_i) = 0$ almost surely.

Solution 5.5.1. Observe that the stochastic process $\zeta(t)$ may be rewritten as

$$\begin{aligned}\zeta(t) = &- c - \frac{\varpi^3}{16}\cos\left(\frac{t\varpi}{2}\right)^3 \operatorname{cosec}\left(\frac{t\varpi}{2}\right) + \varepsilon\left(R(t) + \operatorname{cosec}(\frac{t\varpi}{2})Q(t)\right) \\ &+ \varepsilon^2\left(\operatorname{cosec}\left(\frac{t\varpi}{2}\right)U(t) + Z(t)\right)\end{aligned}$$

where $\varepsilon > 0$ denotes the strength of the noise and

$$R(t) := \frac{\varpi}{2}\cos\left(\frac{t\varpi}{2}\right)\left(\sin\left(\frac{t\Omega}{2}\right)\mathcal{W}_x(t) + \cos\left(\frac{t\Omega}{2}\right)\mathcal{W}_y(t)\right),$$

$$Q(t) := -\frac{\varpi}{2}\cos\left(\frac{t\varpi}{2}\right)\int_0^t \sin\left(\frac{s\Omega}{2}\right)\cos\left(\frac{s\varpi}{2}\right) f_x(s) dW_s,$$

$$
\begin{aligned}
U(t) :=& \frac{1}{4\varpi}\Bigg[4\cos(\frac{t\varpi}{2})\Big(\mathcal{W}_x(t)^2+\mathcal{W}_y(t)^2\Big)\\
&-4\cos(\frac{t\varpi}{2})\Big(\varpi^2\cos(\frac{t\varpi}{2})\mathcal{W}_x(t)^2+\varpi^2\cos(\frac{t\varpi}{2})\mathcal{W}_y(t)^2\Big)\\
&+(-2+\varpi)\,\varpi^2\left(-1+\varpi^2+3\Omega^2\right)\cos(\frac{t\Omega}{2})\mathcal{W}_x(t)\int_0^t\cos(\frac{s\Omega}{2})\sin(\frac{s\varpi}{2})\mathcal{W}_x(s)ds\\
&-(-2+\varpi)\,\varpi^2\left(-1+\varpi^2+3\Omega^2\right)\sin(\frac{t\Omega}{2})\mathcal{W}_y(t)\int_0^t\cos(\frac{s\Omega}{2})\sin(\frac{s\varpi}{2})\mathcal{W}_x(s)ds\Bigg]\\
&+\cos(\frac{t\Omega}{2})\mathcal{W}_x(t)\int_0^t\cos(\frac{s\Omega}{2})\sin(\frac{s\varpi}{2})f_x(s)dW_s\\
&+\frac{2}{\varpi}\cos(\frac{t\Omega}{2})\mathcal{W}_x(t)\int_0^t\cos(\frac{s\Omega}{2})\sin(\frac{s\varpi}{2})f_x(s)dW_s\\
&-\mathcal{W}_x(t)\int_0^t\sin(\frac{s\varpi}{2})\sin^2(\frac{s\Omega}{2})f_x(s)dW_s\\
&+\frac{2}{\varpi}\mathcal{W}_x(t)\int_0^t\sin(\frac{s\varpi}{2})\sin^2(\frac{s\Omega}{2})f_x(s)dW_s\\
&+\cos(\frac{t\Omega}{2})\mathcal{W}_y(t)\int_0^t\sin(\frac{s\varpi}{2})\sin(\frac{s\Omega}{2})f_x(s)dW_s\\
&-\frac{2}{\varpi}\cos(\frac{t\Omega}{2})\mathcal{W}_y(t)\int_0^t\sin(\frac{s\varpi}{2})\sin(\frac{s\Omega}{2})f_x(s)dW_s\\
&-\sin(\frac{t\Omega}{2})\mathcal{W}_y(t)\int_0^t\cos(\frac{s\Omega}{2})\sin(\frac{s\varpi}{2})f_x(s)dW_s\\
&+\frac{2}{\varpi}\sin(\frac{t\Omega}{2})\mathcal{W}_y(t)\int_0^t\cos(\frac{s\Omega}{2})f_x(s)\sin(\frac{s\varpi}{2})dW_s
\end{aligned}
$$

and

$$
\begin{aligned}
Z(t) :=& \frac{1}{4\varpi}\Bigg[-2\varpi\Omega\int_0^t\mathcal{W}_y(s)\mathcal{W}_x'(s)ds+2\varpi\Omega\int_0^t\mathcal{W}_x(s)\mathcal{W}_y'(s)ds+2\varpi\int_0^t\mathcal{W}_x'(s)^2ds\\
&+2\varpi\int_0^t\mathcal{W}_y'(s)^2ds+\frac{\varpi}{2}\left(-\varpi^2+\Omega^2\right)\int_0^t\mathcal{W}_x(s)^2ds+\int_0^t\mathcal{W}_y(s)^2ds\Bigg]
\end{aligned}
$$

are stochastic processes well defined for all t and $\mathcal{W}_x(t)$ and $\mathcal{W}_y(t)$ are a sum of stochastic integrals given by

$$\begin{aligned}
\frac{\varpi}{2}\mathcal{W}_x(t) := &- \sin(\frac{\varpi t}{2})\cos(\frac{\Omega t}{2})\int_0^t \cos(\frac{u\varpi}{2})\cos(\frac{u\Omega}{2})f_x(u)dW_u \\
&+ \cos(\frac{\varpi t}{2})\cos(\frac{\Omega t}{2})\int_0^t \sin(\frac{u\varpi}{2})\cos(\frac{u\Omega}{2})f_x(u)dW_u \\
&- \sin(\frac{\varpi t}{2})\sin(\frac{\Omega t}{2})\int_0^t \cos(\frac{u\varpi}{2})\sin(\frac{u\Omega}{2})f_x(u)dW_u \\
&+ \cos(\frac{\varpi t}{2})\sin(\frac{\Omega t}{2})\int_0^t \sin(\frac{u\varpi}{2})\sin(\frac{u\Omega}{2})f_x(u)dW_u
\end{aligned}$$

and

$$\begin{aligned}
\frac{\varpi}{2}\mathcal{W}_y(t) := &+ \sin(\frac{\varpi t}{2})\sin(\frac{\Omega t}{2})\int_0^t \cos(\frac{u\varpi}{2})\cos(\frac{u\Omega}{2})f_x(u)dW_u \\
&- \cos(\frac{\varpi t}{2})\sin(\frac{\Omega t}{2})\int_0^t \sin(\frac{u\varpi}{2})\cos(\frac{u\Omega}{2})f_x(u)dW_u \\
&- \sin(\frac{\varpi t}{2})\cos(\frac{\Omega t}{2})\int_0^t \cos(\frac{u\varpi}{2})\sin(\frac{u\Omega}{2})f_x(u)dW_u \\
&+ \cos(\frac{\varpi t}{2})\cos(\frac{\Omega t}{2})\int_0^t \sin(\frac{u\varpi}{2})\sin(\frac{u\Omega}{2})f_x(u)dW_u
\end{aligned}$$

with well defined time derivatives, as shown by Proposition (5.3.1). Here $f_x(u)$ denotes the deterministic function dependent on time and $\varpi(=\sqrt{4\omega^2+\Omega^2})$ and Ω are positive non-zero constants.

We remark that if $|f_x^2(s)| \leq \frac{1}{1+s^2}$ and $g(s)$ is continuous function such that $|g(s)| \leq 1$, then, for large time, stochastic integrals of the form

$$\begin{aligned}
\int_0^t g(s)f_x(s)dW_s &= \widetilde{W}\left(\int_0^t g^2(s)f_x^2(s)ds\right) \\
&\leq \sup\left|\widetilde{W}(v)\right|_{v\in\int_0^t g^2(s)f_x^2(s)ds} \\
&< \infty.
\end{aligned}$$

Such integrals are bounded by a constant to allow us to apply the following argument: Consider the increasing sequence of times $\{t_n\} = \frac{2\pi n}{\varpi}$ for which $\sin(\frac{\varpi t_n}{2}) = 0$. Since the $R(t)$ and $Z(t)$ are well defined stochastic processes that can be bounded, the dominant terms of $\zeta(t)$ for this sequence of times is

$$\begin{aligned}
\zeta(t_n) &= \left(-\frac{\varpi^3}{16}\cos\left(\frac{t_n\varpi}{2}\right)^3 + \varepsilon Q(t_n) + \varepsilon^2 U(t_n)\right)\operatorname{cosec}\left(\frac{\varpi t_n}{2}\right) \\
&= \left(\frac{\varpi^3}{16}(-1)^{3n+1} + \varepsilon Q(t_n) + \varepsilon^2 U(t_n)\right)\operatorname{cosec}\left(\frac{\varpi t_n}{2}\right)
\end{aligned}$$

Clearly, for $t = \{t_n\}_{n\in\mathbb{N}}$ and n sufficiently large, it would be absurd to suggest that the sign of $\varepsilon Q(t_n) + \varepsilon^2 U(t_n)$ remains unchanged. Thus, for an increasing subsequence $\{t_{n_k}\}$

we have $\zeta(t)$ continuous on $(t_{n_k}, t_{n_{k+1}})$ and that $\lim_{t \to t_{n_k}} \zeta(t)$ must successively switch between plus and minus infinity. So by continuity and the intermediate value theorem there will exist an increasing sequence $\{t_j\}$ with $t_j \nearrow \infty$ at which $\zeta(t_j) = 0$ almost surely.

Remark 5.5.1. Since the $Q(t)$ and $U(t)$ processes consist entirely of non-anticipating integrands it is possible to use Mckean's ideas to rewrite these processes as a series of time changed Brownian motions. The argument then becomes more intuitive.

Chapter 6

Remarks on Vorticity and its Influence on Incompressibility of Fluid Flow

Summary

To conclude the thesis, let us take a brief and very speculative excursion into conditions imposed by trying to make the Burgers fluid incompressible in the presence of vorticity. Compressibility is an important concept in modelling fluid behaviour. It is well known that the Burgers Equation is a simplified version of the Navier-Stokes equation. Burgers equation offers many advantages over the Navier-Stokes equation, particularly in aspects of turbulence. However, the price paid for this simplification is that no concept of incompressibility can usually be incorporated into the Burgers model because in effect $\phi = \mathrm{Det}(\nabla \Phi_t(x_0))$ is the fluid density. In this chapter we present a heuristic discussion on the effects of vorticity on the "incompressibility" of the Burgers fluid. This is inevitably a very preliminary and tentative study requiring further attention.

6.1 Introduction

Consider a fixed closed surface ∂S within a fluid moving at a velocity v. Assume this surface to have an outward facing normal vector n. Certain portions of the surface ∂S will have fluid entering whilst others have fluid exiting. Within this description the volume of fluid leaving via a surface element $vd\tau$ per unit time will be $v \cdot n d\tau$. This is essentially the volume flux of the fluid. Therefore, the volume rate at which fluid is leaving ∂S can be expressed as

$$\int\int_{\partial S} v \cdot n d\tau. = \int\int\int_S \nabla \cdot v dV = 0.$$

If we insist the fluid is incompressible then the volume flux rate must be identically zero. Since we cannot have fluid entering ∂S without the exact amount leaving. Consequently, for incompressible fluids we must demand that

$$\nabla \cdot v = 0.$$

6.2 Adding a noise potential term

In the endeavour to make our existing Burgers fluid incompressible consider the following argument. Recall Hamilton's characteristic function

$$S(t,x) = \inf_{x_0}\left[S_0(x_0) + \int_0^t \mathcal{L}(q(s),\dot{q}(s),s)ds\right]$$

where $q(s)$ is the classical mechanical path starting at x_0 and finishing at x in time t, that is to say $q(0) = x_0$ and $q(t) = x$. In other words

$$S(t,x) = \inf_{x_0}\left[S_0(x_0) + A(x_0,x,t)\right],$$

where $A(x_0,x,t)$ is the classical action of the classical mechanical path $q(s)$. Let us now consider the above Hamilton characteristic function when a noise potential term $\varepsilon k_t(x)dW_t$ has been added to the scalar potential $V(x)dt$. Here W_t is a one dimensional Wiener process on the probability space $(\Omega,\mathcal{F},P)$. Keeping the vector potential $a_t(x)$ fixed and setting $S(t,x) = S_t(x)$ we find to first order in ε

$$S_t(x) = S_t^0(x) - \varepsilon\int_0^t k(s,\Phi_s\Phi_t^{-1}x)dW_s. \tag{6.2.1}$$

Here Φ_s denotes the classical mechanical flow map in the absence of noise. $S_t^0(x) = S^0(t,x)$ is the above Hamilton principle function satisfying the deterministic Hamilton-Jacobi equation

$$\frac{\partial S_t^0(x)}{\partial t} + \frac{1}{2}\left|\nabla S_t^0(x) - a_t^0(x)\right|^2 + V_t(x) = 0.$$

$a_t^0(x)$ can be thought of as the vorticity potential for $k(x,t) = 0$. By taking the gradient of equation (6.2.1) we obtain

$$\nabla_x S_t(x) = \nabla_x S_t^0(x) - \varepsilon\int_0^t \nabla_x k(\Phi_s\Phi_t^{-1}x,s)dW_s. \tag{6.2.2}$$

Using Taylor's theorem on Equation (6.2.1), to first order in ε, we have

$$dS_t(x) = \left(\dot{S}_t^0(x) - \varepsilon\int_0^t \frac{\partial}{\partial t}k(\Phi_s\Phi_t^{-1}x,s)dW_s\right)dt - \varepsilon k_t(x)dW_t. \tag{6.2.3}$$

However, $S_t(x)$ has to satisfy the stochastic Hamilton-Jacobi equation

$$dS_t(x) + \frac{1}{2}\left|\nabla S_t(x) - a_t(x)\right|^2 dt + V(x)dt + \varepsilon k_t(x)dW_t = 0. \tag{6.2.4}$$

Hence, substituting expressions (6.2.2) and (6.2.3) into the above yields, for fixed $a_t(x)$,

$$\begin{aligned}&\left(\dot{S}_t^0(x) - \varepsilon\int_0^t \frac{\partial}{\partial t}k(\Phi_s\Phi_t^{-1}x,s)dW_s\right)dt - \varepsilon k(x,t)dW_t\\ &+ 2^{-1}\left|\nabla S_t^0(x) - \varepsilon\int_0^t \nabla k(\Phi_s\Phi_t^{-1}x,s)dW_s - a_t(x)\right|^2 dt + V(x)dt + \varepsilon k(x,t)dW_t = 0.\end{aligned}$$

We conclude to first order in ε and fixed $a_t(x)$

$$(\nabla S_t^0(x) - a_t(x)). \int_0^t \nabla_x k(\Phi_s \Phi_t^{-1} x, s) dW_s + \int_0^t \frac{\partial}{\partial t} k(\Phi_s \Phi_t^{-1} x, s) dW_s = 0. \tag{6.2.5}$$

We label this equation as the *rate of work equation*, which we shall discuss in more depth later. Recall that in our set up the velocity field associated with the deterministic Burgers equation was defined to be $v_t^0(x) = \nabla S_t^0(x) - a_t^0(x)$. In the presence of noise the corresponding change would be

$$\begin{aligned} v_t(x) &= v_t^0(x) - \varepsilon \int_0^t \nabla_x k(\Phi_s \Phi_t^{-1} x, s) dW_s \\ &= \nabla_x S_t^0(x) - a_t^0(x) - \varepsilon \int_0^t \nabla_x k(\Phi_s \Phi_t^{-1} x, s) dW_s. \end{aligned} \tag{6.2.6}$$

Taking the divergence of (6.2.6) and working in the gauge of $\nabla . a_t(x) = 0$ leads us to

$$\text{div } v_t(x) = \triangle_x S_t^0(x) - \varepsilon \int_0^t \triangle_x k(\Phi_s \Phi_t^{-1} x, s) dW_s. \tag{6.2.7}$$

If we try to demand the fluid is incompressible then

$$\text{div } v_t(x) = 0.$$

If this condition is satisfied equation (6.2.7) would imply that

$$\triangle_x S_t^0(x) = 0 \qquad \text{and} \qquad \triangle_x k_t(x) = 0, \tag{6.2.8}$$

since $\triangle S_t^0(x)$ is a deterministic quantity.

Lemma 6.2.1. *The rate of work equation is trivially satisfied.*

Proof. In what follows revert to an earlier notation once used where

$$X(s, x_0, p_0) = \Phi_s x_0 \tag{6.2.9}$$

is the stochastic mechanical path with initial momentum $p_0(x_0) = \nabla S_0(x_0) - a(x_0)$. Here x_0 satisfies

$$x = X(t, x_0, p_0(x_0)) \quad \text{and} \quad x_0 = x_0(x, t) = \Phi_t^{-1} x.$$

Differentiating equation (6.2.9) with respect to time t we obtain

$$\left.\frac{\partial X^i}{\partial s}\right|_{s=t} + \frac{\partial X^i}{\partial x_0^j} \frac{dx_0^j}{dt} = 0.$$

But

$$\left.\frac{\partial X^i}{\partial s}\right|_{s=t} = \nabla_i S(x,t) - a_i(x,t),$$

and so

$$\frac{dx_0^j}{dt}(x,t) = -\frac{\partial x_0^j}{\partial X^i} \left(\nabla_i S - a_i\right)(x,t).$$

Defining the connected derivative $\frac{D}{Dt}$ as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \left(\nabla S_t^0 - a_t\right).\nabla \tag{6.2.10}$$

this is nothing but

$$\frac{D}{Dt} x_0^j(x,t) = 0.$$

For a function f of $x_0(x,t)$ it follows

$$\frac{D}{Dt} f(x_0^j(x,t)) = \frac{\partial f}{\partial x_0^i}\frac{\partial x_0^i}{\partial t} + (\nabla_j S - a_j)\frac{\partial f}{\partial x_0^i}\frac{\partial x_0^i}{\partial x^j}.$$

In other words

$$\frac{D}{Dt} f(x_0(x,t)) = \nabla_{x_0} f(x_0).\frac{D}{Dt} x_0.$$

This implies $\frac{D}{Dt} f(x_0(x,t)) = 0$ also. Consequently, the rate of work equation must be trivially satisfied. □

6.3 The implications of div $v_t(x) = 0$, i.e. $\triangle S_t^0(x) = 0$ and $\triangle k_t(x) = 0$

We know that $S_t^0(x)$ satisfies

$$\frac{\partial S_t^0(x)}{\partial t} + \frac{1}{2}\left|\nabla S_t^0(x) - a_t^0(x)\right|^2 + V(x) = 0.$$

Taking the Laplacian of the above equation yields

$$2^{-1}\triangle\left|\nabla S_t^0(x) - a_t^0(x)\right|^2 + \triangle V(x) = 0,$$

and so

$$2^{-1}\left|\nabla S_t^0(x) - a_t^0(x)\right|^2 = -\frac{1}{4\pi}\int_{\mathbb{R}} \frac{\triangle V(x')}{|x-x'|} d^3x'$$

for $x' \in \mathbb{R}^d$ in neighbourhood of x. This result is meaningful only for potentials whereupon $\triangle V_t(x) \leq 0$. Geometrically, at time t, $a_t^0(x)$ lies in a sphere of centre $\nabla S_t^0(x)$ and radius $\mathcal{R}_t(x)$ where

$$\mathcal{R}_t(x) = -\frac{1}{4\pi}\int_{\mathbb{R}} \frac{\triangle_{x'} V(x')}{|x-x'|} d^3x'.$$

Remark 6.3.1. This argument can be repeated with the stochastic Hamilton-Jacobi equation so that

$$\begin{aligned} 2^{-1}\left|\nabla S_t^0(x) - a_t^0(x)\right|^2 &= -\frac{1}{4\pi}\int_{\mathbb{R}} \frac{\triangle V(x')}{|x-x'|} d^3x \\ &= 2^{-1}\left|\nabla S_t(x) - a_t(x)\right|^2 \end{aligned} \tag{6.3.1}$$

Remark 6.3.2. At first glance there appears to be a high degree of non-uniqueness since $(\nabla S_t(x) - a_t(x))$ can be rotated in a random way (possibly to be consistent with div $a_t(x) = 0$).

6.4 The vorticity potential for an incompressible Burgers fluid

If we make the fluid incompressible what consequences are there for the so-called stochastic Burgers equation which reads

$$dv_t(x) + (v_t(x).\nabla)\, dt + (v_t(x) \wedge \text{curl} v_t(x))\, dt = -\nabla V(x) dt - \varepsilon \nabla_x k_t(x) dW_t - da_t(x)$$

where $a_t(x)$ has to satisfy the gauge condition div $a_t(x)$ and conditions (6.2.8) and (6.3.1)? Consider to first order in noise

$$\nabla_x S_t(x) - a_t(x) = \nabla_x S_t^0(x) - a_t^0(x) - \varepsilon \int_0^t \nabla_x k(s, \Phi_s \Phi_t^{-1} x) dW_s - \varepsilon a_t^1(x)$$

where it is assumed $a_t(x) = a_t^0(x) + \varepsilon a_t^1(x) + O(\varepsilon^2)$. Evidently, we need

$$\left(\int_0^t \nabla_x k(s, \Phi_s \Phi_t^{-1} x) dW_s - a_t^1(x) \right) \left(\nabla_x S_t^0(x) - a_t^0(x) \right) = 0 \qquad (6.4.1)$$

and div $(a_t^1(x)) = 0$. To make progress consider the deterministic Hamilton Jacobi equation where

$$\dot{S}_t^0(x) + \frac{1}{2} \left| v_t^0(x) \right|^2 + V(x) = 0,$$

where $v_t^0(x) = \nabla S_t^0(x) - a_t^0(x)$. Differentiating with respect to time t gives

$$\ddot{S}_t^0(x) + \frac{1}{2} \frac{\partial}{\partial t} \left| v_t^0(x) \right|^2 = 0$$

taking the Laplacian of this yields

$$\frac{1}{2} \Delta_x \frac{\partial}{\partial t} \left| v_t^0(x) \right|^2 = 0$$

which implies $|v_t^0(x)|^2$ is constant in time. This allows us to write

$$\begin{aligned} Z_t &= \int_0^t \nabla_x k(s, \Phi_s \Phi_t^{-1} x) dW_s - a_t^1(x) \\ &= \alpha_t(x) v_t^0(x) + \beta_t(x)\, \dot{v}_t^0(x) + \gamma_t(x) \left(v_t^0(x) \wedge \dot{v}_t^0(x) \right). \end{aligned}$$

Here we have written the vector Z_t in its most general form in terms of the three orthogonal components $v_t^0(x)$, $\dot{v}_t^0(x)$ and ($v_t^0(x) \wedge \dot{v}_t^0(x)$). From equation (6.4.1) we now see $Z_t.v_t^0 = 0$. This implies $Z_t = 0$ and consequently must imply that $\alpha_t(x) \equiv 0$. Taking the divergence of Z_t we have

$$\text{div } \left[\beta_t(x) \dot{v}_t^0(x) + \gamma_t(x) \left(v_t^0(x) \wedge \dot{v}_t^0(x) \right) \right] = 0.$$

Using the following vector identity for a vector A and scalar function ϕ

$$\text{div}(A\phi) = \phi \text{div } A + \text{grad } \phi . A$$

implies that

$$\nabla\beta_t(x).\dot{v}_t^0(x) + \nabla\gamma_t(x).\left(v_t^0(x)\wedge\dot{v}_t^0(x)\right) + \gamma_t(x)\text{div }\left((v_t^0(x)\wedge\dot{v}_t^0(x)\right) = 0.$$

In two dimension fluid flow we conclude that if $\gamma_t(x)$ is dependent only on the coordinates in the plane of motion the last two terms in the above equation have to be zero. Therefore, $\nabla\beta_t(x).\dot{v}_t^0(x) = 0$ which implies $\beta_t(x) \parallel v_t^0(x)$. A solution of this last requirement is provided by the fact that $\beta_t(x)$ can be expressed as $\beta_t(x) = f_t(|v(x)|)$ for almost any
provided by the fact that $\beta_t(x)$ can be expressed as $\beta_t(x) = f_t(|v(x)|)$ for almost any
function f_t. This is proved in the following lemma. Assuming the result to be true a general solution of Z_t is therefore

$$Z_t = f\left(t, |v_t^0(x)|\right)\dot{v}_t^0(x) + \gamma_t(x)\left(v_t^0(x)\wedge\dot{v}_t^0(x)\right).$$

Remark 6.4.1. It seems that $\beta_t(x)$ and $\gamma_t(x)$ can possibly be any arbitrary $\mathcal{F}_t$-measurable process.

Remark 6.4.2. For div $(v_t^0(x)) = 0$ for all time it is necessary that $\triangle V \leq 0$. We can argue this as follows. The deterministic Burgers equation reads

$$\frac{\partial}{\partial t}v_t^0(x) + (v_t^0.\nabla)v_t^0(x) - \text{curl } v_t^0\wedge v_t^0 = -\nabla V(x) - \frac{\partial}{\partial t}a_t^0(x).$$

This is nothing but

$$\frac{\partial}{\partial t}v_t^0(x) + \nabla\left(\frac{(v_t^0(x))^2}{2}\right) = -\nabla V(x) - \frac{\partial}{\partial t}a_t^0(x),$$

therefore

$$\triangle\left(\frac{(v_t^0(x))^2}{2}\right) + \triangle V(x) = 0$$

as before. Also note that by taking the curl, because forces are gradients,

$$\frac{\partial}{\partial t}\text{curl } v_t^0(x) = -\frac{\partial}{\partial t}\text{curl } a_t^0(x)$$

as is obvious anyway.

Lemma 6.4.1. *For two dimensional fluid flow for a fluid velocity whose speed remains constant in time if* $\nabla_x\beta_t(x) \parallel v_t(x)$ *then* $\beta_t(x) = f_t(|v(x)|)$ *where* f_t *is almost any arbitrary function.*

Sketch proof. Here we resort to a geometrical argument. Consider $v_t(x)$ as a velocity field constant in time. Interpret this as the collection of points x in space which have an identical velocity at some fixed time t . Connecting these points gives a level surface in 2 dimensions.

Let the velocity field $v(x)$ be given by

$$v(x) = (X(x), Y(x)),$$

for $x \in \mathbb{R}^2$. Then for fixed time

$$|v(\boldsymbol{x})| = \sqrt{X(\ x)^2 + Y(x)^2} = c$$
$$\Rightarrow X(x)^2 + Y(x)^2 = c^2 \tag{6.4.2}$$

where c is a constant. We must prove the normal to this curve always parallel to the velocity field $v_t(x)$. We know that $(X(x)+\delta X(x), Y(x)+\delta Y(x))$ is a point on the curve. Consequently, subtracting

$$(X(x) + \delta X(x))^2 + (Y(x) + \delta Y(x))^2 = c^2$$

from (6.4.2) yields

$$X(x)\delta X(x) + Y(x)\delta Y(x) = 0, \tag{6.4.3}$$

equivalently

$$(X(x), Y(x)).(\delta X(x), \delta Y(x)) = 0,$$

which implies

$$(X(x), Y(x)) \perp (\delta X(x), \delta Y(x)).$$

This proves our assertion in polar coordinates, but what about a more general coordinate system? Clearly, for $x = (x, y)$, $\delta X(x)$ and $\delta Y(x)$ can be expressed as

$$\delta X = \frac{\partial X}{\partial x}\delta x + \frac{\partial X}{\partial y}\delta y \quad \text{and} \quad \delta Y = \frac{\partial Y}{\partial x}\delta x + \frac{\partial Y}{\partial y}\delta y.$$

Substitution into Equation (6.4.3) yields

$$\left(X\frac{\partial X}{\partial x}\delta x + X\frac{\partial X}{\partial y}\delta y\right) + \left(Y\frac{\partial Y}{\partial x}\delta x + Y\frac{\partial Y}{\partial y}\delta y\right) = 0$$
$$\delta x\left(X\frac{\partial X}{\partial x} + Y\frac{\partial Y}{\partial x}\right) + \delta y\left(X\frac{\partial X}{\partial y} + Y\frac{\partial Y}{\partial y}\right) = 0.$$

It follows immediately that

$$(\delta x, \delta y).\left(X\frac{\partial X}{\partial x} + Y\frac{\partial Y}{\partial x}, X\frac{\partial X}{\partial y} + Y\frac{\partial Y}{\partial y}\right) = 0,$$

and so

$$(\delta x, \delta y) \perp \left(X\frac{\partial X}{\partial x} + Y\frac{\partial Y}{\partial x}, X\frac{\partial X}{\partial y} + Y\frac{\partial Y}{\partial y}\right).$$

The coordinate on the right hand side must therefore be on the normal to the curve $X(x)^2 + Y(x)^2 = c^2$. We must now determine if this is indeed parallel to the velocity field $v_t(x)$. From our assertion

$$\beta_t(x) = f_t\left(|v_t(x)|\right)$$

for almost any function f_t. So

$$\nabla_x\beta_t(x) = \frac{\partial}{\partial x}f\left(\sqrt{X^2 + Y^2}\right)$$

for fixed time t. Setting $R^2 = X^2 + Y^2$ gives

$$\nabla_x \beta_t(x) = f'(R) \left(\frac{\partial R}{\partial x}, \frac{\partial R}{\partial y} \right).$$

Differentiating the equation for R^2 with respect to x and y yields

$$\frac{\partial R}{\partial x} = \frac{X}{R}\frac{\partial X}{\partial x} + \frac{Y}{R}\frac{\partial Y}{\partial x}$$

and

$$\frac{\partial R}{\partial y} = \frac{X}{R}\frac{\partial X}{\partial y} + \frac{Y}{R}\frac{\partial Y}{\partial y}.$$

Hence

$$\begin{aligned} \nabla_x \beta_t(x) &= f'(R) \left(\frac{X}{R}\frac{\partial X}{\partial x} + \frac{Y}{R}\frac{\partial Y}{\partial x}, \frac{X}{R}\frac{\partial X}{\partial y} + \frac{Y}{R}\frac{\partial Y}{\partial y} \right) \\ &= \frac{1}{R} f'(R) \left(X\frac{\partial X}{\partial x} + Y\frac{\partial Y}{\partial x}, X\frac{\partial X}{\partial y} + Y\frac{\partial Y}{\partial y} \right). \end{aligned}$$

However, the right hand side is a constant multiple of the normal to the curve $X(x)^2 + Y(x)^2 = c^2$ and so must be parallel to $v_t(x) = (X(x), Y(x))$. □

This discussion has only scratched the surface of an area that is clearly important in developing a model to study physically meaningful fluids.

List of Figures

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