Some geometric considerations related to transition densities of jump-type Markov processes.

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Some Geometric Considerations Related to Transition Densities of Jump-Type Markov Processes

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Abstract

Several non-trivial case studies of Levy and Levy-type generators (with variable coefficients) are investigated in order to shed some light on the behaviour of general Levy-type processes generated by pseudo-differential operators.

Chapter 1

Introduction

Feller semigroups and sub-Markovian semigroups are basic objects to construct stochastic processes. In case that the transition functions of these semigroups have a density with respect to Lebesgue measure, estimates for these transition functions lead immediately to estimates for probabilities. However, in general not much is known about off-diagonal estimates for these transition functions when dealing with Levy-type processes the generator of which are pseudo-differential operators with negative definite symbols. We refer to [17]-[19] and in particular to [12] and [13] where basic existence results and some properties were discussed. The situation for diagonal estimates is better . Since general results as presented in [34] and comparison results, see for example [33], could be used.

Very recently, in [20], a geometric interpretation for the transition function of a Levy process was suggested. The basic idea is to try to get bounds of Gaussian type but with different metrics. Metric measure spaces are since some time employed to study diffusions, see [10] or [32] and the references therein, more recently also jump processes where considered, see [2] and the reference there in. However in these cases the metric is given and the process already relates to that metric. A basic reference for the analysis on metric measure spaces in [11].

The idea in [20] is different. Using the fact that the square root of a (nice) negative definite function gives rise to a metric and that these are the only metrics which induce a metric space isometric to a Hilbert space, see [30], [31], or [5], in [20] it was suggested to express the diagonal estimate in terms of the metric induced by the symbol. Moreover, since Fourier transform of "Gaussians" should be "Gaussians" it was also suggested to express the off-diagonal term as a function decaying exponential with respect to (the square of) another metric. All this was discussed for Lévy processes, only vague indications were made for generators with variable coefficients.

The original aim of our investigations was to settle the case with variable coefficients along the lines as elliptic diffusions can be treated using the Riemannian metric associated with the principal symbol of the generator. This programme was too ambitious since almost all basic concepts break down when trying to transfer from the local to the non-local case.

However, we succeeded to make several case studies in non-trivial situations which we think shed some light on the problems we encounter in the general situations.

In Chapter 6 we discussed subordinate diffusions for diffusions allowing Aronson estimates. Here, given the Aronson estimates, the estimates are trivial, however combined with the geometric interpretation for diagonal terms in case of Levy processes, in Chapter 7 we derive a type of geometric interpretation of the diagonal term of the subordinate elliptic diffusions. This shows in particular that certain concepts of comparability shall carry over to the general situation. Briefly, as a by-product we also give a geometric interpretation of a passage-time result published in [25].

In Chapter 9 we construct a new class of examples of variable order symbols which leads to diagonal estimates with natural geometric interpretations.

In Chapter 10 we pick up a different problem: What can we do with these geometric interpretations. We take a symbol of a Levy process and discuss the probability $P(X_{t_1} \in C_1, X_{t_2} \in C_2)$ as an example. Assuming a representation of the transition function as in [20] we see rather concrete how probabilities are related to geometric notions such as balls in the metrics determining the transition function. In this part we also start to investigate the effect of non-isotropy which is the standard case whenever the process is not a subordinate Brownian motion.

While Chapter 10 handles Levy processes, the idea to handle processes generated by an operator with $x-$dependent symbol is to try some approximation procedure by freezing coefficients. Here non-isotropy becomes even more important. In Chapter 11 we made some graphical experiments. Starting with a symbol $q(x, y, \xi, \eta)$ we look at the corresponding metrics and transition functions for the Levy process associated with $q(x_0, y_0, \xi, \eta)$ and study the change of geometric features in dependence of x_0 and y_0. Of course this serves as a first hint what we have to expect when treating state space dependent metrics. In Chapter 12 we outline some ideas to proceed further in the study of processes with variable coefficient symbol.

As already stated, we could not yet establish a complete geometric theory to handle the general case, however we believe to have obtained some interesting non-trivial insights in what a general theory shoud cover.

Let us briefly describe the remaining chapter not get covered.

In the second chapter we collect several auxiliary results from analysis.

We introduce basic notions, for example function spaces and norms, discuss the Fourier transform and in particular we discuss the convolution theorem. We also discuss the Lax-Milgram theorem as a useful tool needed later on.

In Chapter 3 positive and negative definite functions in the sense of Bochner and Schoenberg are introduced, their properties, especially Peetre's inequality, are treated and relations to convolution semigroups of measures discussed. We also provide several examples of continuous negative definite functions and we discuss the Lévy-Khintchine formula. Subordination in the sense of Bochner is an important tool for us, so we introduce Bernstein functions and subordinate convolution semigroups. Eventually we discuss function spaces related to continuous negative definite functions and the Sobolev embedding theorem.

Next, in Chapter 4 we shift our focus to one parameter operator semigroups. The basic notions and properties are introduced. Furthermore, two very important classes of semigroups, Feller semigroup and sub-Markovian semigroup, are introduced and examples provided. We aslo discuss symmetric Feller semigroups and extensions of symmetric Feller semigroups. Then we handle the two important notions generator and resolvent which have strong relationships to each other. In the final part we discuss the famous Hille-Yosida theorem which builds the bridge between generators and semigroups of operators.

In Chapter 5 we introduce Dirichlet form which serves as a tool for estimating heat kernel in Chapter 6.

In Chapter 8 we discuss the pseudo-differential operators generating Markov processes under certain conditions.

Chapter 2

Essentials from Analysis

In this chapter, we recall some useful fact from analysis that we will need in later chapters. As general reference for these results we refer to these results from [17].

For a function $u : G \to \mathbb{C}$, $G \subset \mathbb{R}^n$ open, we say that u vanishes at infinity if for every $\epsilon > 0$ there exists a compact set $K \subset G$ such that $|u(x)| < \epsilon$, if $x \in K^c$. We define

$$C_\infty(G) := \{u \in C(G) \mid u \text{ vanishes at infinity}\} \tag{2.1}$$

where $C(G)$ means all continuous functions from G to $\mathbb{C}$. For any measure space $(\Omega, \mathcal{A}, \mu)$ the spaces $L^p(\Omega, \mu)$, $1 \leq p \leq \infty$, are the usual *Lebesgue spaces (of equivalence classes) of measurable functions*
$f : \Omega \to \mathbb{C}$ *with finite norm*

$$\|f\|_{L^p} := \left(\int_{\mathbb{R}^n} |f(x)|^p \mu(dx)\right)^{1/p} \quad, 1 \leq p < \infty.$$

The ***Schwartz space*** $\mathbf{S}(\mathbb{R}^n)$ *consists of all functions* $u \in C^\infty(\mathbb{R}^n)$ *such that for all* $m_1, m_2 \in \mathbb{N}_0$

$$p_{m_1,m_2}(u) := \sup_{x \in \mathbb{R}^n}((1 + |x|^2)^{m_1/2} \sum_{|\alpha| \leq m_2} |\partial^\alpha u(x)|) < \infty.$$

The family $(p_{m_1,m_2})_{m_1,m_2 \in \mathbb{N}_0}$ *forms a family of separating seminorms. The family* $(p_{\alpha,\beta})_{\alpha,\beta \in \mathbb{N}_0^n}$ *given by*

$$p_{\alpha,\beta}(u) := \sup_{x \in \mathbb{R}^n} |x^\beta \partial^\alpha u(x)|$$

is equivalent to the family $(p_{m_1,m_2})_{m_1,m_2 \in \mathbb{N}_0}$.

Definition 2.1. Let $G \subset \mathbb{R}^n$ be an open set, $C_0^\infty(G)$ be functions which are arbitrarily often differentiable and have compact support in G. The topological dual space $D'(G)$ of $C_0^\infty(G)$ is the space of **distributions** on G.

Definition 2.2. The topological dual space $S'(\mathbb{R}^n)$ of the Schwartz space $S(\mathbb{R}^n)$ is called the space of **tempered distributions**. It consists of all distributions $u \in D'(G)$ having a continuous extension to $S(\mathbb{R}^n)$, i.e. $S'(\mathbb{R}^n) \subset D'(\mathbb{R}^n)$.

Let $(\Omega, \mathcal{A})$ be an arbitrary measurable space. The total mass of a measure μ on $(\Omega, \mathcal{A})$ is denoted by $\|\mu\| := \mu(\Omega)$, and $\mathcal{M}_b^+(\Omega)$ is the set of all bounded measures on $(\Omega, \mathcal{A})$, i.e. $\mu \in \mathcal{M}_b^+(\Omega)$ implies $\|\mu\| < \infty$. By $\mathcal{M}_b^1(\Omega)$ we denote the set of all probability measures, i.e. measure μ with total mass $\|\mu\| = 1$.

Definition 2.3. Let $(A, D(A))$ be a linear operator from X to Y, both being topological vector spaces.
i). We call A continuous when $A : D(A) \to Y$ is continuous where $D(A)$ carries the topology induced by X.
ii). We call A a closed operator, if $\Gamma(A)$ is closed in $X \times Y$ where $\Gamma(A)$ is the graph of A, i.e.

$$\Gamma(A) := \{(x, y) \in X \times Y \mid y = Ax \quad \text{for some } x \in D(A)\}.$$

iii). The operator A is closable if it has a closed extention.

*The **graph norm** for a linear operator $(A, D(A))$ from a normed space $(X, \|.\|_X)$ into the normed space $(Y, \|.\|_Y)$ is denoted by*

$$\|x\|_{A;X,Y} := \|x\|_X + \|Ax\|_Y.$$

*The **strong operator topology** on $L(X, Y)$ where X and Y are normed spaces is the topology of pointwise convergence, i.e. this topology is defined by the seminorms indexed by $x \in X$:*

$$f \mapsto \|f(x)\|_Y.$$

Theorem 2.4. *(Lax-Milgram theorem) Let H be a Hilbert space, whose inner product and norm will be denoted by $(\cdot, \cdot)$ and $\|\cdot\|$, respectively. Assume that $\mathcal{E}(\cdot, \cdot)$ is a quardratic form defined on $H \times H$ and that there exist positive constants C and c such that*

$$|\mathcal{E}(u, v)| \leq C\|u\|\|v\| \tag{2.2}$$

and

$$|\mathcal{E}(u, u)| \geq c\|u\|^2 \tag{2.3}$$

holds for all $u \in H$. Under these conditions, if $F \in H^*$, i.e. if F is a continuous antilinear functional on H, there exists an element $u \in H$ such that $F(v) = \mathcal{E}(u, v)$ for all $v \in H$. Furthermore, u is uniquely determined by F.

Next, we collect some results from Fourier analysis without proofs but first we introduce the definition of Fourier transform on $S(\mathbb{R}^n)$.

Definition 2.5. Let $u \in S(\mathbb{R}^n)$. The Fourier transform of u is defined by

$$\hat{u}(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)dx.$$

Sometimes we will write $F_{x\mapsto\xi}(u)(\xi)$ or $F(u)(\xi)$ for $\hat{u}(\xi)$.

Definition 2.6. On $S(\mathbb{R}^n)$ we define the inverse Fourier transform by

$$F^{-1}u(\eta) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\eta\cdot y} u(y)dy. \tag{2.4}$$

We also will use $F^{-1}_{y\mapsto\eta}(u)(\eta)$ for denoting $(F^{-1}u)(\eta)$.

Theorem 2.7. *The Fourier transform F and its inverse F^{-1} are continuous linear mappings from $S(\mathbb{R}^n)$ into itself.*

Definition 2.8. For $u \in L^1(\mathbb{R}^n)$ the function $x \mapsto e^{-ix\cdot\xi}u(x)$ is an element of $L^1(\mathbb{R}^n)$, its integral is well defined and therefore we have Fourier transform on L^1, i.e.

$$\hat{u}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)dx. \tag{2.5}$$

Furthermore, we state Lemma of Riemann-Lebesgue

Theorem 2.9. *The Fourier transform is a continuous linear operator from $L^1(\mathbb{R}^n)$ into $C_\infty(\mathbb{R}^n)$ and*

$$\|\hat{u}\|_\infty \leq (2\pi)^{-n/2}\|u\|_{L^1}$$

holds for all $u \in L^1(\mathbb{R}^n)$.

The following result is called ***Theorem of Plancherel***

Theorem 2.10. *For all $u \in S(\mathbb{R}^n)$*

$$\|u\|_0 = \|\hat{u}\|_0$$

hold where $\|\cdot\|_0$ is L^2-norm.

Remark 2.11. The theorem implies that we can extend the Fourier transform from $S(\mathbb{R}^n)$ to a bijective isometry on $L^2(\mathbb{R}^n)$.

We now introduce the convolution of two functions $u, v \in \mathbf{S}(\mathbb{R}^n)$*, i.e. the function*

$$(u * v)(x) = \int_{\mathbb{R}^n} u(x-y)v(y)dy$$

which is again an element in $\mathbf{S}(\mathbb{R}^n)$*. The convolution theorem states:*

Theorem 2.12. *Let* $u, v \in \mathbf{S}(\mathbb{R}^n)$*. Then we have*

$$(u \cdot v)^\wedge(\xi) = (2\pi)^{-n/2}(\hat{u} * \hat{v})(\xi)$$

and

$$(u * v)^\wedge(\xi) = (2\pi)^{n/2}\hat{u}(\xi) \cdot \hat{v}(\xi).$$

Remark 2.13. Analogous results hold for F^{-1}.

Chapter 3

Convolution Semigroups and Negative Definite Functions

In this chapter we collect basic definitions and results about convolution semigroups of sub-probability measures on $\mathbb{R}^n$ and negative definite functions. If not otherwise stated, our basic reference in [17]. We start with:

Definition 3.1. A family $(\mu_t)_{t\geq 0}$ of bounded Borel measures on $\mathbb{R}^n$ is called **convolution semigroup** on $\mathbb{R}^n$ if the following conditions are fulfilled

(3.1) $\mu_t(\mathbb{R}^n) \leq 1$ for all $t \geq 0$

(3.2) $\mu_s * \mu_t = \mu_{t+s} \quad s,t \geq 0$ and $\mu_0 = \epsilon_0$;

(3.3) $\mu_t \to \epsilon_0$ vaguely as $t \to 0$;

where $(\mu_s * \mu_t)(B) = \int_{\mathbb{R}^n} \mu_s(B-y)\mu_t(dy)$ for all $B \in \mathcal{B}^{(n)}$.

Definition 3.2. A function $u : \mathbb{R}^n \to \mathbb{C}$ is called **positive definite** if for any choice of $k \in \mathbb{N}$ and vectors $\xi^1, \ldots, \xi^k \in \mathbb{R}^n$ the matrix $(u(\xi^j - \xi^l))_{j,l=1,\ldots,k}$ is positive Hermitian, i.e. for all $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ we have

$$\sum_{j,l=1}^{k} u(\xi^j - \xi^l)\lambda_j\overline{\lambda}_l \geq 0.$$

Definition 3.3. A function $\psi : \mathbb{R}^n \to \mathbb{C}$ is called **negative definite** if

(3.4) $\psi(0) \geq 0$

and

(3.5) $\xi \mapsto (2\pi)^{-n/2}e^{-t\psi(\xi)}$ is positive definite for $t \geq 0$.

Theorem 3.4. *Let $(\mu_t)_{t\geq 0}$ be a convolution semigroup on $\mathbb{R}^n$. Then there exists a negative definite function $\psi : \mathbb{R}^n \to \mathbb{C}$ such that*

$$\hat{\mu}_t(\xi) = (2\pi)^{-n/2} e^{-t\psi(\xi)} \tag{3.6}$$

holds for all $\xi \in \mathbb{R}^n$ and $t \geq 0$.

Example 3.5. a where $a \geq 0$, ξ^2, $|\xi|^\alpha$ where $\alpha \in (0,2)$ and $|\xi|$ are all continuous negative definite functions.

Lemma 3.6. For any locally bounded negative definite function ψ there exists a constant $c_\psi > 0$ such that for all $\xi \in \mathbb{R}^n$

$$|\psi(\xi)| \leq c_\psi(1 + |\xi|^2).$$

The next result is an inequality which in case of the function $\xi \mapsto |\xi|^2$ is often called ***Peetre's inequality****.*

Lemma 3.7. Let $\psi : \mathbb{R}^n \to \mathbb{C}$ be a negative definite function. Then we have

$$\frac{1 + |\psi(\xi)|}{1 + |\psi(\eta)|} \leq 2(1 + |\psi(\xi - \eta)|).$$

The famous Levy-Khnichin formula states:

Theorem 3.8. *Every continous negative definite function $\psi : \mathbb{R}^n \to \mathbb{C}$ has the representation*

$$\begin{aligned}\psi(\xi) &= c + i(d \cdot \xi) + q(\xi) \\ &\quad + \int_{\mathbb{R}^n \setminus \{0\}} \left(1 - e^{-ix\cdot\xi} - \frac{ix \cdot \xi}{1 + |x|^2}\right) \frac{1 + |x|^2}{|x|^2} \mu(dx)\end{aligned}$$

with a non-negative constant $c \geq 0$, a vector $d \in \mathbb{R}^n$, a symmetric positive semidefinite quadratic form q, and a finite Borel measure μ on $\mathbb{R}^n \setminus \{0\}$.

We want to introduce subordination of convolution semigroup. For this we need the definition of Bernstein functions. A comprehensive monograph about Bernstein function is [28]

Definition 3.9. A real-valued function $f \in C^\infty((0,\infty))$ is called a **Bernstein function** if

$$f \geq 0 \quad \text{and} \quad (-1)^k \frac{d^k f(x)}{dx^k} \leq 0 \tag{3.7}$$

holds for all $k \in \mathbb{N}$.

Example 3.10. The functions $f(x) = 1-e^{-xs}$ where $s \geq 0$, $f(x) = \log(1+x)$ and $f_\alpha(x) = x^\alpha$ where $\alpha \in [0,1]$ are Bernstein functions.

Theorem 3.11. *Let f be a Bernstein function. Then there exist constants $a, b \geq 0$ and a measure μ on $(0,\infty)$ verifying*

$$\int_{0+}^{\infty} \frac{s}{1+s}\mu(ds) < \infty \tag{3.8}$$

such that

$$f(x) = a + bx + \int_{0+}^{\infty}(1-e^{-xs})\mu(ds),\ x > 0. \tag{3.9}$$

The triple (a, b, μ) is uniquely determined by f. Conversely, given $a, b \geq 0$ and a measure μ on $(0,\infty)$ satisfying (3.8), then (3.9) defines a Bernstein function.

Next, we want to relate Bernstein functions to certain convolution semigroups of measures.

Definition 3.12. Let $(\eta_t)_{t\geq 0}$ be a convolution semigroup of measures on $\mathbb{R}$. It is said to be supported by $[0,\infty)$ if $\operatorname{supp} \eta_t \subset [0,\infty)$ for all $t \geq 0$.

Theorem 3.13. *Let $f : (0,\infty) \to \mathbb{R}$ be a Bernstein function. Then there exists a unique convolution semigroup $(\eta_t)_{t\geq 0}$ supported by $[0,\infty)$ such that*

$$\mathcal{L}(\eta_t)(x) = e^{-tf(x)}, \quad x > 0 \text{ and } t > 0, \tag{3.10}$$

holds where $\mathcal{L}$ stands for Laplace transform. Conversely, for any convolution semigroup $(\eta_t)_{t\geq 0}$ supported by $[0,\infty)$ there exists a unique Bernstein function f such that (3.10) holds.

Lemma 3.14. For any Bernstein function f and any continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{C}$, the function $f \circ \psi$ is also continuous and negative definite.

Since $f \circ \psi$ is a continuous negative definite function, there exists a convolution semigroup $(\mu_t^f)_{t\geq 0}$ associated with $f \circ \psi$.

Proposition 3.15. Let ψ be a continuous negative definite function with associated convolution semigroup $(\mu_t)_{t\geq 0}$ on $\mathbb{R}^n$. Further let f be a Bernstein function with associated semigroup $(\eta_t)_{t\geq 0}$ supported on $[0,\infty)$. The convolution semigroup $(\mu_t^f)_{t\geq 0}$ on $\mathbb{R}^n$ associated with the continuous negative definite function $f \circ \psi$ is given by

$$\int_{\mathbb{R}^n} \phi(x)\mu_t^f(dx) = \int_0^\infty \int_{\mathbb{R}^n} \phi(x)\mu_s(dx)\eta_t(ds), \quad \phi \in C_0(\mathbb{R}^n).$$

Definition 3.16. In the situation of Proposition 3.15 we call the convolution semigroup $(\mu_t^f)_{t\geq 0}$ the semigroup **subordinate** (in the sense of Bochner) to $(\mu_t)_{t\geq 0}$ with respect to $(\eta_t)_{t\geq 0}$.

In the following, we will discuss function spaces related to continuous negative definite functions.

Definition 3.17. The space $H^{\psi,s}(\mathbb{R}^n)$ consists of all tempered distributions $u \in S'(\mathbb{R}^n)$ such that

$$\|u\|_{\psi,s,2} = \|(1+|\psi(.)|)^{s/2}\hat{u}(.)\|_{L^2(\mathbb{R}^n)} < \infty.$$

Now, we will establish some relations to classical Sobolev spaces.

Definition 3.18. Let $s \in \mathbb{R}$. The classical Sobolev spaces $H^r(\mathbb{R}^n)$ are defined by

$$H^r(\mathbb{R}^n) := \{u \in L^2(\mathbb{R}^n) \mid F^{-1}((1+|.|^2)^{s/2}\hat{u}) \in L^2(\mathbb{R}^n)\}.$$

These spaces are Hilbert spaces with respect to the norm

$$\begin{aligned}\|u\|_{H^s} :&= \|F^{-1}((1+|.|^2)^{s/2}\hat{u})\|_{L^2} \\ &= \|((1+|.|^2)^{s/2}\hat{u})\|_{L^2}.\end{aligned}$$

The last equality is satisfied because of Plancherel's theorem.

The classical ***Sobolev embedding theorem*** *reads as follows*

Theorem 3.19. ***A.*** *Let $t > 0$. If $t > \frac{n}{2} + k$, $k \in \mathbb{N}_0$, then for all $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leq k$, we have*

$$\|D^\alpha u\|_\infty \leq C_{k,n,t}\|u\|_{H^t} \tag{3.11}$$

and $H^t(\mathbb{R}^n) \subset C_\infty^k(\mathbb{R}^n) := \{u \in C^k(\mathbb{R}^n) \mid D^\alpha u \in C_\infty(\mathbb{R}^n),\ |\alpha| \leq k\}$.
B. *If for some $r > 0$ it holds*

$$(1+|\xi|^2)^r \leq c_{r,\psi}(1+\psi(\xi)),\ c_{r,\psi} > 0, \tag{3.12}$$

then for $s > 0$ it follows that

$$\|u\|_{H^{rs}} \leq C\|u\|_{H^{\psi,s}} \tag{3.13}$$

and $H^{\psi,s}(\mathbb{R}^n) \subset H^{rs}(\mathbb{R}^n)$.
C. *if (3.12) holds and $rs > \frac{n}{2}$ then*

$$\|u\|_\infty \leq C_{r,s,n,\psi}\|u\|_{H^{\psi,s}} \tag{3.14}$$

and $H^{\psi,s}(\mathbb{R}^n) \subset C_\infty(\mathbb{R}^n)$.

Note that (3.12) is equivalent to the existence of $R \geq 0$ *and* $C_{r,R,\psi} > 0$ *such that*

$$|\xi|^{2r} \leq C_{r,R,\psi}\psi(\xi) \tag{3.15}$$

for all $|\xi| \geq R$.

Chapter 4

A Brief Introduction of One Parameter Semigroups

In this chapter we will give a short introduction to the theory of one parameter semigroups of operators which we will use in the rest of the thesis. We mainly take the material from [17] and [24]. First of all we introduce the notion of a (one parameter) semigroup of operators.

Definition 4.1. **A**. A one parameter family $(T_t)_{t\geq 0}$ of bounded linear operators $T_t : X \to X$ (where X is a real or complex Banach space with norm $\|.\|_X$) is called a (one parameter) **semigroup** of operators, if $T_0 = \mathrm{id}$ and $T_{s+t} = T_s \circ T_t$ holds for all $s, t \geq 0$. **B**. $(T_t)_{t\geq 0}$ is called **strongly continuous** if $\lim_{t\to 0} \|T_t u - u\|_X = 0$ for all $u \in X$. **C**. $(T_t)_{t\geq 0}$ is called a **contraction** semigroup if $\|T_t\| \leq 1$ for all $t \geq 0$, where $\|T_t\|$ denotes the operator norm $\|T_t\|_{X,X}$. **D**. Furthermore, if X is an ordered Banach space, then $(T_t)_{t\geq 0}$ is **positivity preserving** if $T_t u \geq 0$ for any $u \geq 0$.

Now we give two examples of semigroups constructed by convolution semigroups which are important in stochastics.

Example 4.2. Let $(\mu_t)_{t\geq 0}$ be a convolution semigroup on $\mathbb{R}^n$. On the Banach space $(C_\infty(\mathbb{R}^n), \|.\|_\infty)$ we define the operator

$$T_t u(x) := \int_{\mathbb{R}^n} u(x-y)\mu_t(dy).$$

We claim $(T_t)_{t\geq 0}$ is a strongly continous contraction semigroup. First, since

$u \in C_\infty(\mathbb{R}^n)$ is bounded we find

$$\begin{aligned}|T_t u(x)| &\le \left|\int_{\mathbb{R}^n} u(x-y)\mu_t(dy)\right| \\ &\le \int_{\mathbb{R}^n} |u(x-y)|\mu_t(dy) \\ &\le \|u\|_\infty \mu_t(\mathbb{R}^n) \le \|u\|_\infty.\end{aligned}$$

The last inequality holds because $\mu_t(\mathbb{R}^n) \le 1$. Therefore, $\|T_t u\|_\infty = \sup_{x\in\mathbb{R}^n} |T_t u(x)| \le \|u\|_\infty$ which implies that T_t, $t \ge 0$, is bounded and a contraction on $C_\infty(\mathbb{R}^n)$. From the definition of the convolution of measures, we have

$$\begin{aligned}(T_{t+s}u)^\wedge(\xi) &= (u * \mu_{t+s})^\wedge(\xi) = (2\pi)^{\frac{n}{2}}\hat{u}(\xi)\hat{\mu}_{t+s}(\xi) \\ &= (2\pi)^{\frac{n}{2}}\hat{u}(\xi)(2\pi)^{-\frac{n}{2}}e^{-(t+s)\psi(\xi)} \\ &= \hat{u}(\xi)e^{-(t+s)\psi(\xi)} \\ &= e^{-t\psi(\xi)}(T_s u)^\wedge(\xi) \\ &= (T_t \circ T_s u)^\wedge(\xi)\end{aligned}$$

which implies $T_{t+s}u = T_t \circ T_s u$. Since $\mu_0 = \varepsilon_0$, we have immediately $T_0 = \mathrm{id}$. Thus $(T_{t\ge 0})_{t\ge 0}$ has the semigroup property. Finally, we prove that $(T_t)_{t\ge 0}$ is strongly continuous for $t \to 0$. For this we use that any function in $C_\infty(\mathbb{R}^n)$ is uniformly continuous. Hence, for $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|u(x) - u(x-y)| < \varepsilon \quad \text{for} \quad |y| < \delta.$$

Moreover, the continuity of $(\mu_t)_{t\ge 0}$ in the Bernoulli topology implies that

$$\lim_{t\to 0} \mu_t(B_\delta(0)) = \varepsilon_0(B_\delta(0)) = 1,$$

i.e. $\mu_t(B_\delta^c(0)) < \varepsilon$ and $1 - \mu_t(\mathbb{R}^n) < \varepsilon$ for $0 < t \le t_0$. Now we find

$$\begin{aligned}|T_t u(x) - u(x)| &\le \left|\int_{\mathbb{R}^n} \{u(x-y) - u(x)\}\mu_t(dy)\right| + |u(x)|(1 - \mu_t(\mathbb{R}^n)) \\ &\le \int_{B_\delta(0)} |u(x-y) - u(x)|\mu_t(dy) \\ &\quad + \int_{B_\delta^c(0)} |u(x-y) - u(x)|\mu_t(dy) + \|u\|_\infty(1 - \mu_t(\mathbb{R}^n)) \\ &\le \varepsilon + 2\varepsilon\|u\|_\infty + \varepsilon\|u\|_\infty = \varepsilon(1 + 3\|u\|_\infty),\end{aligned}$$

which implies that $(T_t)_{t\ge 0}$ is strongly continuous as $t \to 0$. Furthermore, $(T_t)_{t\ge 0}$ is obviously positivity preserving.

This shows that $(T_t)_{t\ge 0}$ is a Feller semigroup in the sense of the following definition.

Definition 4.3. $(T_t)_{t\geq 0}$ is a **Feller semigroup** if $(T_t)_{t\geq 0}$ is a strongly continous contraction semigroup on $(C_\infty(\mathbb{R}^n), \|.\|_\infty)$ which is positivity preserving.

Example 4.4. Let $(\mu_t)_{t\geq 0}$ be a convolution semigroup on $\mathbb{R}^n$ as in the above example. For $u \in S(\mathbb{R}^n)$ we define as before

$$T_t u(x) = \int_{\mathbb{R}^n} u(x-y)\mu_t(dy)$$

and using Plancherel's theorem, we have

$$\begin{aligned}\|T_t u\|_0 = \|(T_t u)^\wedge\|_0 &= \|e^{-t\psi(\cdot)}\hat{u}(\cdot)\|_0 \\ &\leq \|e^{-t\psi(\cdot)}\|_0 \|\hat{u}\|_0 \\ &\leq \|\hat{u}\|_0 \\ &\leq \|u\|_0\end{aligned}$$

where $\|.\|_0$ denotes the norm in $L^2(\mathbb{R}^n)$. Since $S(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, we find that each of the operators T_t has an extension to $L^2(\mathbb{R}^n)$ and this extension is a contraction. We denote this extention once again by T_t, $t \geq 0$. Furthermore, we find

$$\begin{aligned}\|T_t u - u\|_0^2 &= \int_{\mathbb{R}^n} \left|e^{-t\psi(\xi)}\hat{u}(\xi) - \hat{u}(\xi)\right|^2 d\xi \\ &= \int_{\mathbb{R}^n} \left|e^{-t\psi(\xi)} - 1\right|^2 |\hat{u}(\xi)|^2 d\xi,\end{aligned}$$

implying the strong continuity of $(T_t)_{t\geq 0}$ as t tends to 0. As in the previous example using the convolution theorem we can prove the semigroup property on $S(\mathbb{R}^n)$. This extends to the L^2-extension. Moreover, we have $0 \leq T_t u \leq 1$ almost everywhere for $0 \leq u \leq 1$ almost everywhere.

We have now proved that $(T_t)_{t\geq 0}$ is an L^2-sub-Markovian semigroup in the sense of the following definition.

Definition 4.5. $(T_t)_{t\geq 0}$ is a **sub-Markovian semigroup** on L^p, $1 \leq p < \infty$, if $(T_t)_{t\geq 0}$ is a strongly continuous contraction semigroup on $L^p(\mathbb{R}^n)$ which satisfies $0 \leq T_t u \leq 1$ almost everywhere for $u \in L^p(\mathbb{R}^n)$ such that $0 \leq u \leq 1$.

Now we come to the definition of symmetric Feller semigroups which we will use in the Chapter 5

Definition 4.6. Let $(T_t)_{t\geq 0}$ be a Feller semigroup on $C_\infty(\mathbb{R})$. We call $(T_t)_{t\geq 0}$ a symmetric semigroup if $(T_t u, v)_0 = (u, T_t v)_0$ i.e.

$$\int_{\mathbb{R}^n} (T_t u)(x) v(x) dx = \int_{\mathbb{R}^n} u(x) T_t v(x) dx$$

holds for all $u, v \in C_\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.

The following theorem tells us about extensions of symmetric Feller semigroups

Theorem 4.7. *Let $(T_t)_{t\geq 0}$ be a symmetric Feller semigroup. Then for any $1 \leq p < \infty$ there is a strongly continuous contraction semigroup $(T^{(p)})_{t\geq 0}$ on $L^p(\mathbb{R}^n)$, such that $T^{(p)}$ coincides on $L^p(\mathbb{R}^n) \cap C_\infty(\mathbb{R}^n)$ with T_t, and for $p > 1$, $\frac{1}{p} + \frac{1}{p'} = 1$, this implies that*

$$(T_t^{(p)})^* = T_t^{(p')}.$$

Furthermore, each of the semigroups $(T_t^{(p)})_{t\geq 0}$ is sub-Markovian.

Next, we define the generator of a strongly continuous semigroup which plays an important part in the theory of semigroups of operators.

Definition 4.8. Let $(T_t)_{t\geq 0}$ be a strongly continuous semigroup of operators on a Banach space $(X, \|.\|_X)$. The **generator** A of $(T_t)_{t\geq 0}$ is given by

$$Au := \lim_{t\to 0^+} \frac{T_t u - u}{t} \quad \text{(strong limit)}$$

with domain

$$D(A) := \{u \in X : \lim_{t\to 0^+} \frac{T_t u - u}{t} \quad \text{exists in X}\}$$

Corollary 4.9. Let A be the generator of a strongly continuous semigroup $(T_t)_{t\geq 0}$ on the Banach space $(X, \|\cdot\|_X)$. Then $D(A) \subset X$ is a dense subspace and A is a closed operator. Moreover, $(T_t)_{t\geq 0}$ is a strongly continuous semigroup on $D(A)$ when $D(A)$ is equipped with the graph norm $\|u\|_{A,X} = \|Au\|_X + \|u\|_X$ is Banach space.

We will need

Lemma 4.10. For all $a \geq 0$ and $t \geq 0$ we have

$$\frac{at}{1+at} \leq 1 - e^{-at} \leq at$$

and

$$\left|\frac{e^{-at} - 1 + at}{t}\right| \leq \frac{1}{2}a^2 t$$

Example 4.11. Consider the Feller semigroup defined in Example 4.2 we have $\hat{\mu}_t(\xi) = (2\pi)^{-\frac{n}{2}} e^{-t\psi(\xi)}$ where ψ is a continous negative definitive function defined on $\mathbb{R}^n$. For any $u \in S(\mathbb{R}^n)$, we have

$$\frac{T_t u - u}{t} = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \frac{e^{-t\psi(\xi)} - 1}{t} \hat{u}(\xi) d\xi.$$

Since $\hat{u} \in S(\mathbb{R}^n)$ and $|\psi(\xi)| \leq C_\psi(1 + |\xi|^2)$ from Lemma 3.6, we can define the operator

$$\psi(D)u(x) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \psi(\xi) \hat{u}(\xi) d\xi.$$

We want to prove that $S(\mathbb{R}^n) \subset D(A^{(\infty)})$ and $A^{(\infty)}u = -\psi(D)u$ for $u \in S(\mathbb{R}^n)$, where $A^{(\infty)}$ is the generator of the (Feller) semigroup $(T_t)_{t\geq 0}$ from Lemma 4.10. We have

$$\left| \frac{e^{-t\psi(\xi)} - 1 + t\psi(\xi)}{t} \right| \leq t|\psi(\xi)|^2 \leq tc_\psi(1 + |\xi|^2)^2;$$

which implies

$$\left\| \frac{T_t u - u}{t} + \psi(D)u \right\|_\infty \leq tc_\psi \int_{\mathbb{R}^n} (1 + |\xi|^2)^2 |\hat{u}(\xi)| d\xi$$

where we use Theorem 2.9 in the above inequality which gives

$$\lim_{t\to 0^+} \frac{T_t u - u}{t} = -\psi(D)u.$$

Generally speaking we cannot characterise $D(A^\infty)$ using function spaces. But, this is sometimes possible for sub-Markovian semigroup on $L^2(\mathbb{R}^n)$ related to $(\mu_t)_{t\geq 0}$.

Example 4.12. We consider the operator semigroup $(T_t)_{t\geq 0}$ related to a convolution semigroup $(\mu_t)_{t\geq 0}$ with associated continuous negative definite function ψ. As in Example 4.11 proved this is a sub-Markovian semigroup on $L^2(\mathbb{R}^n)$. In order to find its generator we get first for $u \in S(\mathbb{R}^n)$

$$\begin{aligned} \left\| \frac{T_t u - u}{t} + \psi(D)u \right\|_0^2 &= (2\pi)^{-n} \int_{\mathbb{R}^n} \left| \frac{e^{-t\psi(\xi)} - 1}{t} + \psi(\xi) \right|^2 |\hat{u}(\xi)|^2 d\xi \\ &\leq t^2 c_\psi^2 \int_{\mathbb{R}^n} (1 + |\xi|^2)^4 |\hat{u}(\xi)|^2 d\xi \\ &= t^2 c_\psi^2 \|u\|_{H^4}^2. \end{aligned}$$

which implies that

$$\lim_{t\to 0} \left\| \frac{T_t u - u}{t} + \psi(D)u \right\|_0 = 0$$

for all $u \in S(\mathbb{R}^n)$. We denote the generator of the sub-Markovian semigroup in $L^2(\mathbb{R}^n)$ by $A^{(2)}$ and we have shown that on $S(\mathbb{R}^n)$ it holds $A^{(2)} = -\psi(D)u$. Next, we prove the operator $-\psi(D)$ defined on $S(\mathbb{R}^n)$ is closable in $L^2(\mathbb{R}^n)$. To this end, we consider a sequence $(u_\nu)_{\nu\in\mathbb{N}}$, $u_\nu \in S(\mathbb{R}^n)$, converging in $L^2(\mathbb{R}^n)$ to zero. Furthermore, assume that $(-\psi(D)u_\nu)_{\nu\in\mathbb{N}}$ converges in $L^2(\mathbb{R}^n)$ to some element $-g \in L^2(\mathbb{R}^n)$. Our aim is to show that $-g = 0$. For this take $\phi \in S(\mathbb{R}^n)$ and we find

$$\int_{\mathbb{R}^n} \psi(D)u_\nu\overline{\phi}dx = \int_{\mathbb{R}^n} u_\nu\overline{\overline{\psi}(D)\phi}dx,$$

which implies $(g,\phi)_0 = 0$ for all $\phi \in S(\mathbb{R}^n)$, therefore $-\psi(D)$ is closable. Moreover, we know that the norm $\|.\|_{\psi,2}$ on $S(\mathbb{R}^n)$ is equivalent to the graph norm $\|u\|_{\psi(D),L^2} = \|u\|_0 + \|\psi(D)u\|_0$. We know

$$\|u\|_{\psi,2}^2 = \int_{\mathbb{R}^n}(1+|\psi(\xi)|)^2|\hat{u}(\xi)|^2d\xi$$

Therefore we have

$$\|u\|_{\psi,2}^2 = \|\psi(D)u\|_0^2 + \|u\|_0^2,$$

which tells that $\|.\|_{\psi,2}$ is equivalent to $\|.\|_{\psi(D),L^2}$. Therefore this implies that the domain of the closure A of $-\psi(D)$ is given by

$$D(A) = \overline{S(\mathbb{R}^n)}^{\|.\|_{\psi(D),L^2}} = \overline{S(\mathbb{R}^n)}^{\|.\|_{\psi,2}} = H^{\psi,2}(\mathbb{R}^n).$$

The notion we are going to introduce next is that of the ***resolvent*** *which has strong connections to both the generator and the semigroup.*

Definition 4.13. Let $(A, D(A))$ be a linear operator on a complex Banach space $(X, \|.\|_X)$. The **resolvent set** $\rho(A)$ of A is defined to be the set of all $\lambda \in \mathbb{C}$ such that $(\lambda - A) : D(A) \to X$ is one-to-one mapping and its inverse $(\lambda - A)^{-1}$ satisfies the following two conditions:

i) $D((\lambda - A)^{-1}) = X$;

ii) $(\lambda - A)^{-1}$ is bounded on X.

The set $\sigma(A) := \mathbb{C} \setminus \rho(A)$ is called the **spectrum** of A and $\{(\lambda - A)^{-1} \mid \lambda \in \rho(A)\}$ the **resolvent** of A. We denote resolvent by $(R_\lambda)_{\lambda\in\rho(A)}$ and $R_\lambda := (\lambda - A)^{-1}$.

The following proposition shows connections between the resolvent and the semigroup.

Proposition 4.14. Let $(T_t)_{t\geq 0}$ be a strongly continuous contraction semigroup on the Banach space $(X, \|.\|_X)$ with generator $(A, D(A))$. Then $\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > 0\} \subset \rho(A)$ and

$$R_\lambda u = (\lambda - A)^{-1}u = \int_0^\infty e^{-\lambda t} T_t u dt, \quad u \in X, \operatorname{Re}\lambda > 0.$$

Lemma 4.15. (resolvent equation) Let A be a linear operator on X and $\lambda, \mu \in \rho(A)$. Then the resolvent equation

$$R_\lambda R_\mu = R_\mu R_\lambda = (\lambda - \mu)^{-1}(R_\mu - R_\lambda)$$

holds.

To have a concrete understanding of resolvent, we give a short example as follows

Example 4.16. Consider the sub-Markovian semigroup on $L^2(\mathbb{R}^n)$ constructed using a convolution semigroup $(\mu_t)_{t\geq 0}$ related to the continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{C}$. We have

$$T_t u(x) = \int_{\mathbb{R}^n} u(x-y)\mu_t(dy) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-t\psi(\xi)} \hat{u}(\xi) d\xi$$

for $u \in S(\mathbb{R}^n)$. For $\lambda > 0$ and $u \in S(\mathbb{R}^n)$ we have

$$\begin{aligned} R_\lambda u(x) &= \int_0^\infty e^{-\lambda t} T_t u(x) dt \\ &= (2\pi)^{-\frac{n}{2}} \int_0^\infty e^{-\lambda t} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-t\psi(\xi)} \hat{u}(\xi) d\xi dt. \end{aligned}$$

Due to the reason that $(t, \xi) \mapsto e^{-(\lambda+\psi(\xi))t}\hat{u}(\xi)e^{ix\cdot\xi}$ is $L^1((0,\infty) \times \mathbb{R}^n)$ we get from Fubini's theorem

$$\begin{aligned} R_\lambda u(x) &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \{\int_0^\infty e^{-(\lambda+\psi(\xi))t} dt\} e^{ix\xi} \hat{u}(\xi) d\xi \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \frac{1}{\lambda + \psi(\xi)} \hat{u}(\xi) d\xi, \end{aligned}$$

and this relation we can extend by continuously to $L^2(\mathbb{R}^n)$ since $\|R_\lambda u\|_{L^2} \leq \frac{1}{|\lambda|}\|u\|_{L^2}$ and $S(\mathbb{R}^n)$ is dense in L^2.

To formulate the Hille-Yosida theorem we introduce the notion of dissipativity.

Definition 4.17. A linear operator $A : D(A) \to X$, $D(A) \subset X$, is called $(X-)$**dissipative** if $\|\lambda u - Au\|_X \geq \lambda \|u\|_X$ holds for all $\lambda > 0$ and $u \in D(A)$.

Here is an example of a dissipative operator.

Example 4.18. Let $\psi : \mathbb{R}^n \to \mathbb{C}$ be a continuous negative definitve function. On $S(\mathbb{R}^n)$ which we know is a dense subspace of $L^2(\mathbb{R}^n)$ we introduce the following pseudo-differential operator

$$\psi(D)u(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\xi}\psi(\xi)\hat{u}(\xi)d\xi,$$

and we claim $-\psi(D)$ is $L^2(\mathbb{R}^n)-$dissipative. In fact, for $\lambda > 0$ and $u \in S(\mathbb{R}^n)$ we find

$$\begin{aligned}\|\lambda u + \psi(D)u\|_0^2 &= \lambda^2\|u\|_0^2 + 2\lambda \int_{\mathbb{R}^n} \operatorname{Re}\psi(\xi)|\hat{u}(\xi)|^2 d\xi + \|\psi(D)u\|_0^2 \\ &\geq \lambda^2\|u\|_0^2,\end{aligned}$$

which implies the L^2-dissipativity of $-\psi(D)$.

Now, we come to the famous Hille-Yosida theorem which gives the connection between generators and semigroups of operators.

Theorem 4.19. *A necessary and sufficient condition that a densely defined linear operator $(A, D(A))$ on a Banach space $(X, \|.\|_X)$ is the generator of a strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ is that*

a) *A is dissipative;*

b) $R(\lambda - A) = X$ *for some* $\lambda > 0$

Note that an operator satisfying the conditions of Theorem 4.19 is necessarily closed. In application, the operators related to concrete examples are often not closed. Therefore we have a more general version of Hille-Yosida theorem:

Theorem 4.20. *A necessary and sufficient condition that a densely defined linear operator $(A, D(A))$ is closable on a Bananch space $(X, \|.\|_X)$ and the closure $\bar{A}$ is the generator of a strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ is that*

a) *A is dissipative;*

b) $R(\lambda - A)$ *is dense in* X *for some* $\lambda > 0$

Now we come to a characterisation of the generator of Feller semigroups which we call positive maximum principle.

Definition 4.21. Let $A : D(A) \to B(\mathbb{R}^n, \mathbb{R})$ be a linear operator, $D(A) \subset B(\mathbb{R}^n, \mathbb{R})$ where $B(\mathbb{R}^n, \mathbb{R})$ is a bounded linear operator from $\mathbb{R}^n$ into $\mathbb{R}$. We say that $(A, D(A))$ satisfies the **positive maximum principle** if for any $u \in D(A)$ such that for some $x_0 \in \mathbb{R}^n$ the fact that $u(x_0) = \sup_{x\in\mathbb{R}^n} u(x) \geq 0$ implies that $Au(x_0) \leq 0$.

We claim that the generator $(A, D(A))$ of a Feller semigroup satisfies the positive maximum principle. To see this, we assume that $(T_t)_{t\geq 0}$ is a Feller semigroup on $C_\infty(\mathbb{R}^n)$ with generator $(A, D(A))$, $D(A) \subset C_\infty(\mathbb{R}^n)$. Suppose that $u \in D(A)$ and that for some $x_0 \in \mathbb{R}^n$ we have $u(x_0) = \sup_{x\in\mathbb{R}^n} u(x) \geq 0$. Since each of the operators $T_t, t \geq 0$, is positivity preserving we have

$$(T_t u)(x_0) = (T_t u^+)(x_0) - (T_t u^-)(x_0) \leq (T_t u^+)(x_0) \leq \|u^+\|_\infty = u(x_0)$$

which implies

$$Au(x_0) = \lim_{t\to 0} \frac{T_t u(x_0) - u(x_0)}{t} \leq 0.$$

It happens that the positive maximum principle is a characteristic property of generators of Feller semigroups. To this end, we need

Lemma 4.22. Suppose that a linear operator $(A, D(A))$, $D(A) \subset C_\infty(\mathbb{R}^n)$, on $C_\infty(\mathbb{R}^n)$ satisfies the positive maximum principle on $D(A)$. Then A is dissipative.

Theorem 4.23. *A linear operator $(A, D(A))$, $D(A) \subset C_\infty(\mathbb{R}^n)$, on $C_\infty(\mathbb{R}^n)$ is closable and its closure is the generator of a Feller semigroup if and only if the three following conditions hold:*

i) $D(A) \subset C_\infty(\mathbb{R}^n)$ *is dense;*

ii) $(A, D(A))$ *satisfies the positive maximum principle;*

iii) $R(\lambda - A)$ *is dense in* $C_\infty(\mathbb{R}^n)$ *for some* $\lambda > 0$.

Example 4.24. Let $\psi : \mathbb{R}^n \to \mathbb{C}$ be a continuous negative definite function. On $C_0^\infty(\mathbb{R}^n)$ we define the operator

$$-\psi(D)u(x) = -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\psi(\xi)\hat{u}(\xi)d\xi.$$

From Example 4.2 we know that $(-\psi(D), C_0^\infty(\mathbb{R}^n))$ has an extension generating a Feller semigroup, hence on $C_0^\infty(\mathbb{R}^n)$ the operator $-\psi(D)$ satisfies the positive maximum principle.

In order to introduce the notion of subordination for operator semigroup of operators we need to state the following theorem first

Theorem 4.25. *Let $(T_t)_{t\geq 0}$ be a strongly continous contraction semigroup on Banach space $(X, \|.\|)$ and let $(\eta_t)_{t\geq 0}$ be a convolution semigroup supported on $[0,\infty)$ and related to Bernstein function f. Define $T_t^f u$ for $u \in X$ by the Bochner integral*

$$T_t^f u = \int_0^\infty T_s u \, \eta_t(ds).$$

The integral is well-defined and $(T_t^f)_{t\geq 0}$ is a strongly continuous contraction semigroup on X.

Definition 4.26. The semigroup $(T_t^f)_{t\geq 0}$ defined in Theorem 4.25 is called **subordinate**(in the sense of Bochner) to $(T_t)_{t\geq 0}$ with respect to $(\eta_t)_{t\geq 0}$ or equivalently with respect to f.

As always, we illustrate this construction by our two examples.

Example 4.27. Let $(\mu_t)_{t\geq 0}$ and $(\eta_t)_{t\geq 0}$, supp $\eta_t \subset [0,\infty)$ be the two convolution semigroup related to a continuous negative definitive function $\psi : \mathbb{R}^n \to \mathbb{C}$ and a Bernstein function $f : (0,\infty) \to \mathbb{R}$ respectively. Consider the Feller semigroup $(T_t^{(\infty)})_{t\geq 0}$ on the Banach space $(C_\infty(\mathbb{R}^n; \mathbb{R}), \|.\|_\infty)$ or sub-Markovian semigroups $(T_t^{(2)})_{t\geq 0}$ on the spaces $L^2(\mathbb{R}^n)$, constructed by $(\mu_t)_{t\geq 0}$. On $S(\mathbb{R}^n)$ which is dense both in $C_\infty(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$ we have always

$$T_t u(x) = \int_{\mathbb{R}^n} u(x-y)\mu_t(dy) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\xi} e^{-t\psi(\xi)} \hat{u}(\xi) d\xi,$$

where T_t is the restriction of any of the operators $T_t^{(2)}$ or $T_t^{(\infty)}$ to $S(\mathbb{R}^n)$. From the definition of the subordinate semigroup $T_t^f u(x)$, $t \geq 0$, we have

$$\begin{aligned}
T_t^f u &= \int_0^\infty T_s u \, \eta_t(ds) \\
&= \int_0^\infty \int_{\mathbb{R}^n} u(x-y)\mu_s(dy)\eta_t(ds) \\
&= (2\pi)^{-\frac{n}{2}} \int_0^\infty \int_{\mathbb{R}^n} e^{ix\xi} e^{-s\psi(\xi)} \hat{u}(\xi) d\xi \eta_t(ds) \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\xi} \hat{u}(\xi) d\xi \int_0^\infty e^{-s\psi(\xi)} \eta_t(ds) \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\xi} \mathcal{L}(\eta_t)(\psi(\xi)) \hat{u}(\xi) d\xi \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\xi} e^{-tf(\psi(\xi))} \hat{u}(\xi) d\xi \qquad (4.1)
\end{aligned}$$

We use Fubini's theorem in the fourth equality above and we always denote $\int_0^\infty \mu_s(dy)\eta_t(ds)$ by $\mu_t^f(dy)$ where $(\mu_t^f)_{t\geq 0}$ is the convolution semigroup on $\mathbb{R}^n$ related to the continuous negative definite function $f \circ \psi$. Obviously, (4.1) is the subordinate semigroups to $(T_t^{(p)})_{t\geq 0}$, $p = 2, \infty$, related to f. Furthermore, we have $(T_t^{(\infty),f})_{t\geq 0}$ ($(T_t^{(2),f})_{t\geq 0}$) is also a Feller semigroup(sub-Markovian semigroup on $L^2(\mathbb{R};\mathbb{R})$)

We can now find exactly the generator of $(T_t^f)_{t\geq 0}$. We only illustrate the example of a subordination of a sub-Markovian semigroup associated with a convolution semigroup, subordination of the Feller-semigroup follows along the same line, however in this case we can not characterize the domain in terms of a function space.

Example 4.28. We contruct sub-Markovian semigroup $(T_t)_{t\geq 0}$ as usual by a convolution semigroup $(\mu_t)_{t\geq 0}$ on R^n associated to the continuous negative definitive function $\psi : \mathbb{R}^n \to \mathbb{C}$ through $\hat{\mu}(\xi) = (2\pi)^{-\frac{n}{2}} e^{-t\psi(\xi)}$ i.e. the sub-Markovian semigroup $(T_t)_{t\geq 0}$ on $L^2(\mathbb{R}^n;\mathbb{R})$ is given as follows

$$T_t u(x) = \int_{\mathbb{R}^n} u(x-y)\mu_t(dy) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\xi} e^{-t\psi(\xi)} \hat{u}(\xi) d\xi.$$

The corresponding generator of $(T_t)_{t\geq 0}$ is

$$Au(x) = -\psi(D)u(x) = -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\xi} \psi(\xi)\hat{u}(\xi) d\xi$$

which has the space $H^{\psi,2}(\mathbb{R}^n)$ as domain. We know from Example 4.27 that the subordinate semigroup $(T_t^f)_{t\geq 0}$ is

$$T_t^f u(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\xi} e^{-tf(\psi(\xi))} \hat{u}(\xi) d\xi$$

and from the definition of the generator, we have

$$\begin{aligned} A^f u(x) &= \lim_{t\to\infty} \frac{T_t^f u(x) - u(x)}{t} \\ &= \lim_{t\to\infty} (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \frac{e^{-tf(\psi(\xi))} - 1}{t} \hat{u}(\xi) d\xi \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \lim_{t\to\infty} \frac{e^{-tf(\psi(\xi))} - 1}{t} \hat{u}(\xi) d\xi \\ &= -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\psi(\xi)) \hat{u}(\xi) d\xi \end{aligned} \tag{4.2}$$

with $D(A^f) = H^{f\circ\psi,2}(\mathbb{R}^n)$. Moreover, we know the Bernstein function $f : (0,\infty) \to \mathbb{R}$ has the representation

$$f(x) = c_0 + c_1 x + \int_0^\infty (1 - e^{-xs})\mu(ds), \quad x > 0, \tag{4.3}$$

where $c_0, c_1 \geq 0$ and μ is a measure on $(0, \infty)$ such that $\int_0^\infty \frac{s}{1+s}\mu(ds) < \infty$. Therefore, substituting (4.3) into (4.2) we have for $u \in S(\mathbb{R})$

$$\begin{aligned} A^f u(x) &= -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\xi} \left(c_0 + c_1\psi(\xi) + \int_0^\infty (1 - e^{-\psi(\xi)s})\mu(ds)\right) \hat{u}(\xi)d\xi \\ &= -c_0 u(x) + c_1 Au(x) - (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \int_0^\infty e^{ix\cdot\xi}(1 - e^{-\psi(\xi)s})\hat{u}(\xi)\mu(ds)d\xi \\ &= -c_0 u(x) + c_1 Au(x) - \int_0^\infty (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}(1 - e^{-\psi(\xi)s})\hat{u}(\xi)d\xi\mu(ds) \\ &= -c_0 u(x) + c_1 Au(x) + \int_0^\infty (T_s u(x) - u(x))\mu(ds), \end{aligned}$$

where in the third equality above we can change the order of integration since for $u \in S(\mathbb{R}^n)$ we find

$$\left|e^{ix\cdot\xi}(1 - e^{\psi(\xi)s})\hat{u}(\xi)\right| \leq (s|\psi(\xi)| \wedge 2)|\hat{u}(\xi)| \leq (s \wedge 2)(1 + |\psi(\xi)|)|\hat{u}(\xi)|,$$

and the function $(s, \xi) \mapsto (s \wedge 2)(1 + |\psi(\xi)|)|\hat{u}(\xi)|$ is $\mu(ds) \otimes d\xi$ integrable. Thus the generator of $(T_t^f)_{t\geq 0}$ is

$$A^f u = -c_0 u + c_1 Au + \int_0^\infty (T_s u - u)\mu(ds).$$

Finally, we introduce analytic semigroups. In this section, X denotes a complex Banach space.

Definition 4.29. Let A be a closed operator densely defined in X. The operator A is said to be of type (ω, M) if there exist $0 < \omega < \pi$ and $M \geq 1$ such that $\rho(A) \supset \{\lambda\colon |\arg\lambda| > \omega\}$ and $\|\lambda(A - \lambda)^{-1}\| \leq M$ for $\lambda < 0$, and if there exists a number M_ϵ such that $\|\lambda(A - \lambda)^{-1}\| \leq M_\epsilon$ holds in $|\arg\lambda| > \omega + \epsilon$ for all $\epsilon > 0$.

Theorem 4.30. *Suppose there exist real numbers β, M and an angle $\omega \in [0, \frac{\pi}{2}]$ such that $-A + \beta$ is of the type (ω, M), then A is the generator of a semigroup $\{T(t)\}$. $T(t)$ can be continued holomorphically with respect to t into the sector $\{t : |\arg t| \leq \frac{\pi}{2} - \omega\}$, where $T(t + s) = T(t)T(s)$ holds. Let θ be an arbitrary angle satisfying $0 < \theta < \frac{\pi}{2} - \omega$, then $e^{-\beta t}T(t)$ is uniformly bounded in the closed subsector $\{t| \arg t| \leq \theta\}$ and, when t approaches 0 inside this subsector, $T(t)$ converges strongly to I, for any natural number n we have*

$$\limsup_{t\to+0} t^n\|(\frac{d}{dt})^n T(t)\| = \limsup_{t\to+0} t^n\|A^n T(t)\| < \infty \tag{4.4}$$

Definition 4.31. The semigroup described in Theorem 4.30 is called an **analytic semigroup** or a parabolic semigroup.

Note that analytic semigroups have the important property that

$$T_t u \in \bigcap_{k \geq 1} D(A^k) \tag{4.5}$$

for all $u \in X$.

Chapter 5

Dirichlet Forms

In this section we want to introduce Dirichlet forms.

Definition 5.1. Let H be a real Hilbert space with inner product $(\ ,\ )$. $\mathcal{E}$ is called a **symmetric form** on H if the following conditions are satisfied:

i) $\mathcal{E}$ is defined on $D(\mathcal{E}) \times D(\mathcal{E})$ with values in $\mathbb{R}^1$, $D(\mathcal{E})$ being a dense linear subspace of H;

ii) $\mathcal{E}(u,v) = \mathcal{E}(v,u), \quad \mathcal{E}(u+v,w) = \mathcal{E}(u,w) + \mathcal{E}(v,w),$
$a\mathcal{E}(u,v) = \mathcal{E}(au,v), \quad \mathcal{E}(u,u) \geq 0,\ u,v,w \in D(\mathcal{E}),\ a \in \mathbb{R}^1.$

We call $D(\mathcal{E})$ the domain of $\mathcal{E}$.

Given a symmetric form $\mathcal{E}$ on H

$$\mathcal{E}_\alpha(u,v) = \mathcal{E}(u,v) + \alpha(u,v), \quad u,v \in D(\mathcal{E}) \qquad (5.1)$$
$$D(\mathcal{E}_\alpha) = D(\mathcal{E})$$

defines a new symmetric form on H for each $\alpha > 0$. We know the space $D(\mathcal{E})$ is a pre-Hilbert space with inner product $\mathcal{E}_\alpha$. Furthermore $\mathcal{E}_\alpha$ and $\mathcal{E}_\beta$ determine equivalent metrics on $D(\mathcal{E})$ for different $\alpha, \beta > 0$.

Definition 5.2. A symmetric form $\mathcal{E}$ is said to be **closed** if

$$u_n \in D(\mathcal{E}),\ \mathcal{E}_1(u_n - u_m, u_n - u_m) \to 0, n, m \to \infty$$
$$\Rightarrow \quad \exists u \in D(\mathcal{E}), \mathcal{E}_1(u_n - u, u_n - u) \to 0, n \to \infty.$$

Definition 5.3. We say that a symmetric form $\mathcal{E}$ is **closable** if the following condition is fulfilled:

$$u_n \in D(\mathcal{E}),\ \mathcal{E}(u_n - u_m, u_n - u_m) \to 0, n, m \to \infty, \qquad (5.2)$$
$$(u_n, u_n) \to 0, n \to \infty \Rightarrow \mathcal{E}(u_n, u_n) \to 0, n \to \infty.$$

*A **core** of a symmetric form $\mathcal{E}$ defined on some $L^2(X,\mu)$ space with X being a locally compact Hausdorff space is by definition a subset $\mathcal{C}$ of $D(\mathcal{E}) \cap C_0(X)$ such that $\mathcal{C}$ is dense in $D(\mathcal{E})$ with $\mathcal{E}_1-$norm and dense in $C_0(X)$ with uniform norm.*

Lemma 5.4. A symmetric form $(\mathcal{E}, D(\mathcal{E}))$ is closed if and only if $D(\mathcal{E})$ equipped with the scalar product $\mathcal{E}_\alpha$, $\alpha > 0$, is a Hilbert space, i.e. complete.

Definition 5.5. A symmetric form $\mathcal{E}$ is called **regular** if $\mathcal{E}$ possesses a core

It's clear that $\mathcal{E}$ is regular if and only if the space $D(\mathcal{E}) \cap C_0(X)$ is a core of $\mathcal{E}$.

Definition 5.6. A. Let $(\mathcal{E}, D(\mathcal{E}))$ be a closed bilinear form on $L^2(\mathbb{R}^n)$. If it is symmetric and

$$\mathcal{E}(u^+ \wedge 1, u^+ \wedge 1) \leq \mathcal{E}(u,u) \quad \text{for all } u \in D(\mathcal{E})),$$

then $(\mathcal{E}, D(\mathcal{E}))$ is said to be a **(symmetric)Dirichlet form**.
B. A Dirichlet form $(\mathcal{E}, D(\mathcal{E})$ having the property that for two functions $u, v \in C_0^\infty(\mathbb{R}^n)$ with disjoint support it yield $\mathcal{E}(u,v) = 0$, is called a **local Dirichlet form**.

In comparison to the Feller semigroups, we cannot make use of similar pointwise statements such as the positive maximum principle in an L^p setting. It turns out that the following concept is essential to obtain a characterisation of generators of sub-Markovian semigroups, see e.g. Chapter 4.6 in [17]

Definition 5.7. A closed, densely defined linear operator $A : D(A) \to L^p(\mathbb{R}^n)$, $1 < p < \infty$, with domain $D(A) \subset L^p(\mathbb{R}^n)$, is called a Dirichlet operator if

$$\int_{\mathbb{R}^n} (Au)((u-1)^+)^{p-1} dx \leq 0$$

for all $u \in D(A)$.

Lemma 5.8. A Dirichlet operator A on $L^p(\mathbb{R}^n)$ is dissipative, i.e.
$\|(\lambda - A)u\|_{L^p} \geq \lambda \|u\|_{L^p}$.

For the rest of discussion related to sub-Markovian case we only consider $p = 2$ and work with symmetric sub-Markovian semigroups $(T_t^{(2)})_{t\geq 0}$ on $L^2(\mathbb{R}^n)$. We know the generator $(A, D(A))$ of a symmetric L^2-sub-Markovian semigroup is a self-adjoint Dirichlet operator. The following theorem gives us also the converse.

Theorem 5.9. *A self-adjoint operator $A : D(A) \to L^2(\mathbb{R}^n)$, $D(A) \subset L^2(\mathbb{R}^n)$ dense, be a self-adjoint linear operator. Then it is a Dirichlet operator if and only if it generates a symmetric sub-Markovian semigroup on $L^2(\mathbb{R}^n)$.*

The fact that A is a Dirichlet operator implies

$$\int_{\mathbb{R}^n}(-Au)u dx = (-Au, u)_{L^2} \geq 0. \tag{5.3}$$

Hence $-A$ is a non-negative and self-adjoint operator. The following theorem ensures the existence of $(-A)^{\frac{1}{2}}$ and of a positive semidefinite bilinear form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(\mathbb{R}^n)$ defined by

$$\mathcal{E}(u,v) := (-Au, v)_{L^2} = ((-A)^{\frac{1}{2}}u, (-A)^{\frac{1}{2}}v)_{L^2} \tag{5.4}$$

for $v \in D(\mathcal{E}) := D((-A)^{\frac{1}{2}})$ and $u \in D(A) \cap D(\mathcal{E})$. We quote the result from [17], Theorem 4.7.5.

Theorem 5.10. *Let $(A, D(A))$ be a closed and densely defined self-adjoint operator on $L^2(\mathbb{R}^n)$, which satisfies (5.3). Then there exists a closed, symmetric and positive semidefinite bilinear form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(\mathbb{R}^n)$ defined by (5.4), which satisfies the Cauchy-Schwarz inequality*

$$|(-Au, v)_{L^2}| \leq \|(-A)^{\frac{1}{2}}u\|_{L^2}\|(-A)^{\frac{1}{2}}v\|_{L^2}, \quad u \in D(A), v \in D(A^{\frac{1}{2}}).$$

Further, we have $D(A) \hookrightarrow D(\mathcal{E}) \hookrightarrow L^2(\mathbb{R}^n)$, where these continuous embeddings are dense. In particular, $(D(A), \|\cdot\|_A)$ and $(D(\mathcal{E}), \|\cdot\|_{\mathcal{E}})$ are Hilbert spaces equipped with the graph norms $\|u\|_A = \|u\|_{L^2} + \|(-A)u\|_{L^2}$ and $\|u\|_{\mathcal{E}} = \|u\|_{L^2} + \sqrt{\mathcal{E}(u,u)}$, respectively. Moreover, $D(\mathcal{E}) = D((-A)^{1/2})$

In the following we will give an example of a symmetric translation invariant Dirichlet form involving a real-valued continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{R}$.

Example 5.11. Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a continuous negative definite function. Then the pseudo-differential operator $-\psi(D)$ defined on $C_0^\infty(\mathbb{R}^n)$ by

$$-\psi(D)u(x) = -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\psi(\xi)\hat{u}(\xi)d\xi \tag{5.5}$$

extends to a self-adjoint Dirichlet operator $(A, H^{\psi,2}(\mathbb{R}^n))$, see Exampl 4.12. Further, for $u \in H^{\psi,2}(\mathbb{R}^n)$ we find

$$\begin{aligned}\mathcal{E}(u,u) &:= \int_{\mathbb{R}^n}(-Au)(x)u(x)dx \\ &= \int_{\mathbb{R}^n}\psi(\xi)\hat{u}(\xi)\overline{\hat{u}(\xi)}d\xi = \int_{\mathbb{R}^n}\psi(\xi)|\hat{u}(\xi)|^2 d\xi,\end{aligned}$$

which implies that the symmetric Dirichlet form related to $(A, H^{\psi,2}(\mathbb{R}^n))$ has the domain $D(\mathcal{E}) = H^{\psi,1}(\mathbb{R}^n)$ and it is given by

$$\mathcal{E}(u,v) = \int_{\mathbb{R}^n} \psi(\xi)\hat{u}(\xi)\overline{\hat{v}(\xi)}d\xi = \int_{\mathbb{R}^n} [\psi(D)]^{\frac{1}{2}}u \cdot [\psi(D)]^{\frac{1}{2}}v dx, \tag{5.6}$$

where $-[\psi(D)]^{\frac{1}{2}}u$ is given on $C_0^\infty(\mathbb{R}^n)$ by (5.5) with $\psi(\xi)^{\frac{1}{2}}$ instead of $\psi(\xi)$. In particular, since $(1+\psi(\cdot))^{\frac{1}{2}}$ is also a continuous negative definitive function with values only in $\mathbb{R}$, we see that for any continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{R}$ the space $(H^{\psi,1}(\mathbb{R}^n), (\cdot,\cdot)_{\psi,1})$ is a symmetric Dirichlet space and therefore $u \in H^{\psi,1}(\mathbb{R}^n)$ implies always that $u^+ \wedge \lambda$ and $u \wedge \lambda$, $\lambda \geq 0$, belongs to $H^{\psi,1}(\mathbb{R}^n)$ too.

The operator $-\psi(D)$ is invariant under translation, i.e. for the operator $\tau_y : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto x - y$, we have

$$\begin{aligned}\tau_y(-\psi(D)u)(x) &= -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\psi(\xi)\hat{u}(\xi)d\xi \\ &= -(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\psi(\xi)(\tau_y u)^\wedge(\xi)d\xi = -\psi(D)(\tau_y u)(x).\end{aligned}$$

Clearly this calculation applies also to $-[\psi(D)]^{1/2}$ which yields

$$\begin{aligned}\mathcal{E}(\tau_y(u), \tau_y(v)) &= \left([\psi(D)]^{1/2}(\tau_y u), [\psi(D)]^{1/2}(\tau_y v)\right)_0 \\ &= \left(\tau_y\left([\psi(D)]^{1/2}u\right), \tau_y\left([\psi(D)]^{1/2}v\right)\right)_0 \\ &= ([\psi(D)]^{1/2}u, [\psi(D)]^{1/2}v)_0 = \mathcal{E}(u,v),\end{aligned}$$

thus $\mathcal{E}$ is **translation invariant** in the sense that

$$\mathcal{E}(\tau_y(u), \tau_y(v)) = \mathcal{E}(u,v),$$

for all $u, v \in D(\mathcal{E})$ and $y \in \mathbb{R}^n$.

We are looking for another representation of $\mathcal{E}(u,v)$ for smooth functions. We know the continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{R}$ has a Levy-Khinchin representation

$$\psi(\xi) = c + q(\xi) + \int_{\mathbb{R}\setminus\{0\}} (1 - \cos(x\cdot\xi))\nu(dx) \tag{5.7}$$

Now, substituting (5.7) into (5.6) we can get

$$\begin{aligned}\mathcal{E}(u,v) = c\int_{\mathbb{R}^n} u(x)v(x)dx + \int_{\mathbb{R}^n} \sum_{k,l=1}^n q_{kl}\frac{\partial u(x)}{\partial x_k}\frac{\partial v(x)}{\partial x_l}dx \\ + \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (u(x+y) - u(x))v(x+y) - v(x))\nu(dy)dx.\end{aligned} \tag{5.8}$$

Let us consider (5.8) for $u, v \in C_0^\infty(\mathbb{R}^n)$ (or $H^{\psi,1}(\mathbb{R}^n)$ with $\operatorname{supp} u \cap \operatorname{supp} v = \emptyset$. This implies $\operatorname{supp} \frac{\partial u}{\partial x_k} \cap \operatorname{supp} \frac{\partial v}{\partial x_l} = \emptyset$ for $1 \leq k, l \leq n$, by the locality of the differential operator $\frac{\partial}{\partial x_k}$, and therefore we find

$$\mathcal{E}(u,v) = \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (u(x+y) - u(x))(v(x+y) - v(x))\nu(dy)dx.$$

Example 5.12. Let $\psi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be a continuous negative definite function with Levy-Khinchine representation

$$\psi(\xi,\eta) = \iint_{(\mathbb{R}^n \times \mathbb{R}^m)\setminus\{0\}} (1 - \cos(\xi \cdot x + \eta \cdot y))\nu(dx,dy) \tag{5.9}$$

where ν is the corresponding Levy measure on $(\mathbb{R}^n \times \mathbb{R}^m) \setminus \{0\}$. Associated with ψ we introduce the scale of spaces $H^{\psi,s}(\mathbb{R}^n \times \mathbb{R}^m)$, $s \geq 0$, by

$$H^{\psi,s}(\mathbb{R}^n \times \mathbb{R}^m) = \{u \in L^2(\mathbb{R}^n \times \mathbb{R}^m) \mid \|u\|_{\psi,s} < \infty\} \tag{5.10}$$

where

$$\|u\|_{\psi,s}^2 = \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} (1 + \psi(\xi,\eta))^s |\hat{u}(\xi,\eta)|^2 d\eta d\xi. \tag{5.11}$$

Here $\hat{u}$ denotes the Fourier transform of u, i.e.

$$\hat{u}(\xi,\eta) = (2\pi)^{-\frac{(n+m)}{2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} e^{-ix\xi - iy\eta} u(x,y) dy dx. \tag{5.12}$$

The Dirichlet form associated with ψ has domain $H^{\psi,1}(\mathbb{R}^n \times \mathbb{R}^m)$ and is given by

$$\begin{aligned}\mathcal{E}^\psi(u,v) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} \psi(\xi,\eta)\hat{u}(\xi,\eta)\overline{\hat{v}(\xi,\eta)} d\eta d\xi \\ &= \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} \int_{\mathbb{R}^n \times \mathbb{R}^m \setminus \{0\}} (u(x_1 - x_2, y_1 - y_2) - u(x_1,y_1))(v(x_1 - x_2, y_1 - y_2) \\ &\quad - v(x_1,y_1))\nu(dx_2,dy_2)dy_1dx_1.\end{aligned}$$

Chapter 6

Heat kernels and some of their estimates

In this chapter, we will discuss estimates for the densities of transition functions, i.e. the Markov kernels representing the semigroups under consideration. We discuss mainly symmetric Feller semigroups since we can use Hilbert space methods to obtain regularity results.

First of all, let $(T_t)_{t\geq 0}$ *be a symmetric Feller semigroup which we consider on* $C_\infty(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$. *From the definition of Feller semigroup, for* $x\in\mathbb{R}^n$ *and* $t\geq 0$ *fixed, the mapping* $u\mapsto T_tu(x)$ *is a linear continuous and positive functional on* $C_\infty(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$. *From Riesz's representation theorem, it follows that for* x *fixed there exists a Borel measure* $p_t(x,dy)$ *on* $\mathcal{B}^{(n)}$ *which is uniquely determined and*

$$T_tu(x)=\int_{\mathbb{R}^n}u(y)p_t(x,dy) \tag{6.1}$$

holds for $u\in C_\infty(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$. *We can extend* $(T_t)_{t\geq 0}$ *to all constants using the lemma below.*

Lemma 6.1. Let $(T_t)_{t\geq 0}$ be a Feller semigroup. Then we may extend T_t, $t>0$, to all constant functions $x\mapsto a\in\mathbb{R}$, and for $a\geq 0$ we have

$$T_ta\leq a. \tag{6.2}$$

By monotone convergence we find using (6.2)

$$1\geq (T_t1)(x)=\lim_{\nu\to\infty}\int_{\mathbb{R}^n}u_\nu(y)p_t(x,dy)=\int_{\mathbb{R}^n}1p_t(x,dy).$$

Therefore each of the measures $p_t(x,dy)$, $x\in\mathbb{R}^n$, $t\geq 0$, *is a sub-probability measure. Hence we may extend* T_t *to* $B_b(\mathbb{R}^n)$ *just by defining the operator* $\tilde{T}_t$

by

$$\tilde{T}_t u(x) := \int_{\mathbb{R}^n} u(y) p_t(x, dy) \tag{6.3}$$

for all $u \in B_b(\mathbb{R}^n)$. Moreover, we find that

$$\left|\tilde{T}_t u(x)\right| \leq \int_{\mathbb{R}^n} |u(y)| p_t(x, dy) \leq \|u\|_\infty,$$

or

$$\|\tilde{T}_t u\|_\infty \leq \|u\|_\infty,$$

i.e. $\tilde{T}_t : B_b(\mathbb{R}^n) \to \{u : \mathbb{R}^n \to \mathbb{R} \mid \|u\|_\infty < \infty\}$ is a positive preserving contraction. So we claim

Theorem 6.2. *Let $(T_t)_{t\geq 0}$ be a symmetric Feller semigroup and define $p_t(x, dy)$ by (6.1). Then $p_t(x, dy)$ is a sub-Markovian kernel for every $t \geq 0$.*

Now we are in the position to prove also the semigroup property of the extended family $(\tilde{T}_t)_{t\geq 0}$ of the Feller semigroup $(T_t)_{t\geq 0}$. For $u \in C_\infty(\mathbb{R}^n)$ and $t, s \geq 0$ we can use Fubini's theorem

$$\begin{aligned}
\int_{\mathbb{R}^n} p_{t+s}(x, dz) u(z) &= T_{t+s} u(x) \\
&= T_t(T_s u)(x) = \int_{\mathbb{R}^n} p_t(x, dy) \int_{\mathbb{R}^n} p_s(y, dz) u(z) \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} p_t(x, dy) p_s(y, dz) u(z)
\end{aligned}$$

and the uniqueness part of Riesz's representation theorem gives

$$p_{t+s}(x, A) = \int_A p_t(x, dy) p_s(y, A) \tag{6.4}$$

for all $t, s \geq 0$, $x \in \mathbb{R}^n$ and $A \in \mathcal{B}^{(n)}$. The equations (6.4) are called the ***Chapman-Kolmogorov equations****, and it follows that*

$$\tilde{T}_{t+s} = \tilde{T}_t \circ \tilde{T}_s = \tilde{T}_s \circ \tilde{T}_t. \tag{6.5}$$

In addition, since $T_0 = id$, we find that

$$u(x) = T_0 u(x) = \int_{\mathbb{R}^n} u(y) p_0(x, dy), \tag{6.6}$$

implying that

$$p_0(x, dy) = \varepsilon_x(dy) \tag{6.7}$$

for all $x \in \mathbb{R}^n$, i.e. $\tilde{T}_0 = id$. Therefore we have proved

Theorem 6.3. *Let $(T_t)_{t\geq 0}$ be a Feller semigroup. Then there exists a family $(p_t(\cdot,\cdot))_{t\geq 0}$ of sub-Markovian kernels on $\mathbb{R}^n \times \mathcal{B}^{(n)}$ satisfying the Chapman-Kolmogorov equations (6.4).*

For L^2-sub-Markovian semigroups the situation is more complicated since $x \mapsto T_t^{(2)}u(x)$ in this case is only almost everywhere defined. In the situations we are interested in however, we can assume the existence of a sub-Markovian kernels with similar properties as discussed for Feller semigroups. For details we refer to Chapter 3 *in [18] and Chapter 6 in [19]*

Definition 6.4. An L^2-sub-Markovian semigroup $(T_t)_{t\geq 0}$ is **conservative** if $T_t 1 = 1$ a.e. for all $t > 1$.

Let $(T_t)_{t\geq 0}$ be a symmetric Feller semigroup or an L^2-sub-Markovian semigroup with representing kernels $p_t(x, dy)$, i.e.

$$T_t u(x) = \int u(y) p_t(x, dy).$$

We want to find condtions for $p_t(x, dy)$ having a density with respect to Lebesgue measure, i.e.

$$p_t(x, dy) = p_t(x, y)\lambda^{(n)}(dy),$$

and then we long for estimates for the density $p_t(x, y)$. The following result, due to Dunford and Pettis, is a tool for getting the existence of densities:

Theorem 6.5. *Let $K_{op} : L^p(G) \to L^\infty(G)$, $1 \leq p < \infty$, be a bounded linear operator. Then there exists a kernel function $K : G \times G \to \mathbb{C}$, $K \in L^\infty(G) \otimes L^{p'}(G)$, $\frac{1}{p} + \frac{1}{p'} = 1$, such that*

$$K_{op}u(x) = \int_G K(x, y) u(y) dy. \tag{6.8}$$

Conversely, every operator defined by (6.8) with a kernel function $K : G \times G \to \mathbb{C}$, $K \in L^\infty(G) \otimes L^{p'}(G)$, $\frac{1}{p} + \frac{1}{p'} = 1$, defines a bounded linear operator from $L^p(G)$ into $L^\infty(G)$. Furthermore, the operator norm of K_{op} is given by

$$\|K_{op}\| = \operatorname{ess\,sup}_{x\in G} \|K(x,\cdot)\|_{L^{p'}}.$$

In case that the semigroup under consideration is also a conservative symmetric L^2-Markovian semigroup the next result taken from N. Varopoulos[34] gives estimates for the operator norms of T_t, and these estimates are diagonal estimates for the corresponding densities.

Theorem 6.6. *Let $(T_t)_{t\geq 0}$ be a symmetric conservative sub-Markovian semigroup on $L^2(\mathbb{R}^n)$ with related regular Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$. Moreover, let $p > 2$ and $N := \frac{2p}{p-2} > 2$. The following estimates are equivalent*

$$\|u\|_{L^p}^2 \leq c\mathcal{E}(u,u) \quad \text{for all } u \in D(\mathcal{E}); \tag{6.9}$$

$$\|u\|_{L^2}^{2+4/N} \leq c\mathcal{E}(u,u)\|u\|_{L^1}^{4/N} \quad \text{for all } u \in D(\mathcal{E}) \cap L^1(\mathbb{R}^n) \tag{6.10}$$

$$\|T_t\|_{L^1-L^\infty} \leq c' t^{-N/2} \quad \text{for all } t > 0, \tag{6.11}$$

where $\|B\|_{X-Y}$ denotes the operator norm of $B : X \to Y$.

Note that (6.9) is a Sobolev-type inequality and (6.10) is a Nash-type inequality.

Note that the symmetry of $p_t(x,y)$ implies by the Chapman-Kolmogorov equations and the Cauchy-Schwarz inequality

$$p_t(x,y) \leq p_t^{\frac{1}{2}}(x,x) p_t^{\frac{1}{2}}(y,y),$$

which gives

$$\begin{aligned} \operatorname{ess\,sup}_{x,y\in\mathbb{R}^n} p_t(x,y) &\leq (\operatorname{ess\,sup}_{x\in\mathbb{R}^n} p_t^{\frac{1}{2}}(x,x))(\operatorname{ess\,sup}_{y\in\mathbb{R}^n} p_t^{\frac{1}{2}}(y,y)) \\ &= (\operatorname{ess\,sup}_{x\in\mathbb{R}^n} p_t(x,x)) \leq (\operatorname{ess\,sup}_{x,y\in\mathbb{R}^n} p_t(x,y)) \end{aligned}$$

i.e.

$$(\operatorname{ess\,sup}_{x,y\in\mathbb{R}^n} p_t(x,y)) = (\operatorname{ess\,sup}_{x\in\mathbb{R}^n} p_t(x,x)). \tag{6.12}$$

Therefore (6.11) is a diagonal estimate for $p_t(x,y)$, i.e. an estimate for $p_t(x,x)$ which controls also $p_t(x,y)$.

In case we have instead of (6.9) a Gårding-type inequality

$$\|u\|_{L^p}^2 \leq c_2 \mathcal{E}_\lambda(u,u) = c_2(\mathcal{E}(u,u) + \lambda(u,u)_0), \quad \lambda > 0,$$

we obtain instead of (6.11)

$$\|T_t\|_{L^1-L^\infty} \leq c_3(\lambda) \frac{e^{\lambda t}}{t^{N/2}}.$$

and

$$\operatorname{ess\,sup}_{x,y\in\mathbb{R}^n} p_t(x,y) \leq c_4(\lambda) \frac{e^{\lambda t}}{t^{N/2}}.$$

The following result which is due to R.Schilling and J.Wang[29] relates Theorem 6.6 to subordiante semigroups.

Theorem 6.7. *Let $(T_t)_{t\geq 0}$ be a strongly continuous contraction semigroup of symmetric opertators on $L^2(X,m)$ and assume that for each $t \geq 0$, $T_t|_{L^2(X,m)\cap L^1(X,m)}$ has an extension which is a contraction on $L^1(X,m)$, i.e. $\|T_t u\|_1 \leq \|u\|_1$ for all $u \in L^1(X,m) \cap L^2(X,m)$. Suppose that the generator $(A, D(A))$ satisfies the following Nash-type inequality:*

$$\|u\|_2^2 B(\|u\|_2^2) \leq \langle Au, u\rangle, \quad u \in D(A),\ \|u\|_1 = 1, \tag{6.13}$$

where $B : (0,\infty) \to (0,\infty)$ is an increasing function. Then, for any Bernstein function f, the generator $f(A)$ of the subordinate semigroup satisfies

$$\frac{\|u\|_2^2}{2} f\left(B\left(\frac{\|u\|_2^2}{2}\right)\right) \leq \langle f(A)u, u\rangle, \quad u \in D(f(A)),\ \|u\|_1 = 1. \tag{6.14}$$

The following example gives a diagonal estimate of transition function in the context of the Cauchy semigroup. The aim is to have diagonal estimates of transition function of subordinate Cauchy semigroups in the end.

Example 6.8. Let $(X_t)_{t\geq 0}$ be the two-dimensional Cauchy process whose symbol is $\psi(\xi,\eta) = \sqrt{\xi^2+\eta^2}$ and the transition function is

$$p_t^{\psi}(x,y) = \frac{1}{2\pi}\frac{t}{((x^2+y^2)+t^2)^{3/2}}. \tag{6.15}$$

Therefore we have $\|T_t\|_{L^1-L^\infty} = p_t(0) = \frac{1}{2\pi}\frac{t}{((x^2+y^2)+t^2)^{3/2}}\Big|_{x=0,y=0} = \frac{1}{2\pi}t^{-2}$. Using Theorem 6.6, we know from (6.15) that $N = 4$ in the inequality (6.11) which implies

$$\|u\|_{L^2}^3 \leq c\mathcal{E}(u,u), \quad u \in D(\mathcal{E}) \cap L^1(\mathbb{R}^2),\ \|u\|_{L^1} = 1. \tag{6.16}$$

In order to apply Theorem 6.7, we rewrite (6.16) as

$$\|u\|_{L^2}^2 B(\|u\|_{L^2}^2) \leq \langle Au, u\rangle \quad \text{where } B(x) = \frac{1}{c}x^{1/2},$$

therefore we can deduce from Theorem 6.7 that if we let $f(x) = x^\alpha$, $(0 < \alpha < 1)$, we have

$$\|u\|_{L^2}^2 f\left(B\left(\frac{\|u\|_2^2}{2}\right)\right) \leq \langle A^f u, u\rangle, \quad u \in D(A^f),\ \|u\|_{L^1} = 1$$

$$\|u\|_{L^2}^2 f\left(\frac{1}{c}\left(\frac{\|u\|_2^2}{2}\right)^{\frac{1}{2}}\right) \leq \langle A^f u, u\rangle$$

$$\|u\|_{L^2}^2 (1/c^\alpha)\left(\frac{\|u\|_2^2}{2}\right)^{\frac{\alpha}{2}} \leq \langle A^f u, u\rangle$$

$$\|u\|_{L^2}^{2+\alpha} \leq c'\langle A^f u, u\rangle. \tag{6.17}$$

Using Theorem 6.6 this implies

$$\|T_t^f\|_{L^1 \to L^\infty} \le c'' t^{-\frac{N'}{2}}, \quad t > 0,\ N' = 4/\alpha$$

which is equivalent to $p_t^f(0) \le c'' t^{-\frac{N'}{2}}$.

We want to prepare some considerations for following sections. Let

$$L(x, D) = \sum_{k,l=1}^{n} \frac{\partial}{\partial x_k}(a_{kl}(x)\frac{\partial}{\partial x_l}) \tag{6.18}$$

be a uniformly elliptic differential operators with coefficients $a_{kl} = a_{lk} \in C_b^3(\mathbb{R}^n)$. *Thus with* $0 < \lambda_0$ *we have*

$$\lambda_0 |\xi|^2 \le \sum_{k,l=1}^{n} a_{kl}(x)\xi_k \xi_l \le \lambda_0^{-1}|\xi|^2. \tag{6.19}$$

It is well known that in this case there exists a ***fundamental solution*** $\Gamma : (0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *to the operator* $\frac{\partial}{\partial t} - L(x, D)$ *such that* $\Gamma(t, x, y) > 0$, $x \mapsto \Gamma(t, x, y)$ *is a* C^2-*function,* $t \mapsto \Gamma(t, x, y)$ *is a* C^1-*function,* $\Gamma(t, x, y) = \Gamma(t, y, x)$ *and*

$$\frac{\partial \Gamma(t, x, y)}{\partial t} - L(x, D_x)\Gamma(t, x, y) = 0 \tag{6.20}$$

and

$$\lim_{t \to 0} \int_{\mathbb{R}^n} \Gamma(t, x, y) f(y) dy = f(x) \tag{6.21}$$

for $f \in C_b(\mathbb{R}^n)$. *Moreover it holds*

$$T_t f(x) = \int_{\mathbb{R}^n} \Gamma(t, x, y) f(y) dy, \quad t > 0, \tag{6.22}$$

where $(T_t)_{t \ge 0}$ *is the Feller (or* L^p-*sub-Markovian) semigroup generated by an extension of* $L(x, D)$. *(Compare Theorem 2.9 in [18] or A.Friedman[8], or S.Ito[14].)*

Thus $\Gamma(t, x, y)$ *is a heat kernel and we long to estimate it. Under our assumptions two-sided estimates were obtained by D.Aronson[1]:*

Theorem 6.9. *For* Γ *as above it holds*

$$\kappa_1 \Gamma^{\gamma_1}(t, x, y) \le \Gamma(t, x, y) \le \kappa_2 \Gamma^{\gamma_2}(t, x, y) \tag{6.23}$$

where

$$\Gamma^\gamma(t, x, y) = \frac{1}{(4\pi t\gamma)^{n/2}} e^{-|x-y|/4\gamma t} \tag{6.24}$$

is the heat kernel associated with $\gamma\Delta_{(n)}$, $\Delta_{(n)}$ *being the Laplace operator in* $\mathbb{R}^n$.

Now let f be a Bernstein function associated with the convolution semigroup $(\eta_t)_{t\geq 0}$, $\operatorname{supp}\eta_t([0,\infty\})$. We consider the subordinate semigroup

$$T_t^f u(x) = \int_0^\infty \int_{\mathbb{R}^n} \Gamma(s,x,y)u(y)dy\eta_t(ds). \tag{6.25}$$

With

$$\Gamma^f(t,x,y) = \int_0^\infty \Gamma(s,x,y)\eta_t(ds) \tag{6.26}$$

we find

$$T_t^f u(x) = \int_{\mathbb{R}^n} \Gamma^f(t,x,y)u(y)dy. \tag{6.27}$$

Now (6.23) implies with

$$\Gamma^{\gamma,f}(t,x,y) = \int_0^\infty \Gamma^\gamma(s,x,y)\eta_t(ds) \tag{6.28}$$

the estimates

$$\kappa_1 \Gamma^{\gamma_1,f}(t,x,y) \leq \Gamma^f(t,x,y) \leq \kappa_2 \Gamma^{\gamma_2,f}(t,x,y). \tag{6.29}$$

We will try to interpret (6.29) in geometric terms later on, see Chapter 7.

Finally, we want to state a comparison theorem on Dirichlet forms which we take from the paper [33]. For $i=1,2$, let $(\mathcal{E}^{(i)}, D(\mathcal{E}^{(i)}))$ be a symmetric Dirichlet form on L^2 and $\{T_t^{(i)} : t>0\}$ the symmetric strongly continuous Markovian semigroup on L^2 associated with $\mathcal{E}^{(i)}$.

Theorem 6.10. *Assume that $D(\mathcal{E}^{(1)}) \supset D(\mathcal{E}^{(2)})$ and that there is a positive number C such that*

$$\mathcal{E}^{(1)}(u,u) \leq C\mathcal{E}^{(2)}(u,u), \quad u \in D(\mathcal{E}^{(2)}), \tag{6.30}$$

and moreover suppose

$$\|T_t^{(1)}\|_{1\to\infty} \leq g(t), \quad t>0, \tag{6.31}$$

where g is a right continuous nonincreasing function and satisfies the following condition: $H(\xi) \equiv \int_\xi^\infty \{G(\eta)/g(G(\eta))\}d\eta < +\infty$, $\xi>0$, G being the left continuous inverse function of g. If $T_t^{(2)}1 = 1$ $m-a.e.$, $t>0$, then it holds that

$$\|T_t^{(2)}\|_{1\to\infty} \leq 2h(t/2C), \quad t>0, \tag{6.32}$$

where h is the inverse function to H.

In particular, if tg(t) is continuous and nondecreasing, then

$$\|T_t^{(2)}\|_{1\to\infty} \le 2g(t/2C), \quad t > 0. \tag{6.33}$$

Therefore, if $t\|T_t^{(1)}\|_{1\to\infty}$ *is continuous and nondecreasing in t, then*

$$\|T_t^{(2)}\|_{1\to\infty} \le 2\|T_{t/2C}^{(1)}\|_{1\to\infty}, \quad t > 0.$$

We immediately obtain the following

Corollary 6.11. Let $D(\mathcal{E}^{(1)}) = D(\mathcal{E}^{(2)})$ and assume that

$$C_1\mathcal{E}^{(2)}(u,u) \le \mathcal{E}^{(1)}(u,u) \le C_2\mathcal{E}^{(2)}(u,u), \quad u \in \mathcal{E}^{(1)},$$

for some $C_i > 0$, $i = 1, 2$. If $T_t^{(i)}1 = 1$ $m-$a.e. , $t > 0$, and $t\|T_t^{(i)}\|_{1\to\infty}$ is continuous and nondecreasing in t for each i, then

$$(1/2)\|T_{2C_2t}^{(2)}\|_{1\to\infty} \le \|T_t^{(1)}\|_{1\to\infty} \le 2\|T_{(C_1/2)t}^{(2)}\|_{1\to\infty}, \quad t > 0 \tag{6.34}$$

Since we know $\|T_t\|_{1\to\infty} = p_t(0)$ *in case of a Levy process, we can write (6.34) as*

$$\frac{1}{2}p_{2C_2t}^{(2)}(0) \le p_t^{(1)}(0) \le 2p_{(C_1/2)t}^{(2)}(0) \tag{6.35}$$

in the translation invariant case or

$$\frac{1}{2}p_{2C_2t}^{(2)}(0) \le p_t^{(1)}(x,x) \le 2p_{(C_1/2)t}^{(2)}(0) \tag{6.36}$$

in the general case.

Remark 6.12. $p_t^{(2)}$ is still translation invariant in the general case.

Chapter 7

A Geometric Interpretation of the Transition Density of a Levy process

In this chapter, we mainly summarize some results from the paper [20] which will be used in the next chapters. From now on, $(X_t^\psi)_{t\geq 0}$ *is a symmetric Levy process, i.e.* ψ *is real-valued, in addition, in Levy-Khintchine formula, we let* $c = d_j = q_{kl} = 0$ *which means*

$$\psi(\xi) = \int_{\mathbb{R}^n\setminus\{0\}} (1 - \cos y\xi)\nu(dy), \quad \nu \text{ Levy measure}$$

and $\psi(\xi) = 0$ *if and only if* $\xi = 0$*. Then*

$$d_\psi(\xi, \eta) := \psi^{1/2}(\xi - \eta)$$

is a metric on $\mathbb{R}^n$.

First of all, we introduce the notion of a metric measure space.

Definition 7.1. A **metric measure space** is a triple (X, d, μ) where (X, d) is a metric space and μ is a measure on the Borel sets of the space X.

We are mainly interested in metric measure spaces whose metric is induced by a negative definite function. In particular, every locally bounded, non-periodic negative definite function with $\psi(0) = 0$ *induces a metric on* $\mathbb{R}^n$ *by*

$$d_\psi : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty), \quad d_\psi(\xi, \eta) := \sqrt{\psi(\xi - \eta)} \tag{7.1}$$

The metric d_ψ *is invariant under translations, i.e.*

$$d_\psi(\xi + \zeta, \eta + \zeta) = d_\psi(\xi, \eta).$$

To proceed further, consider $d_\psi(\xi,\eta) = \psi^{1/2}(\xi-\eta)$ *and*

$$\begin{aligned} B^{d_\psi}(\xi,r) :&= \{\eta \in \mathbb{R}^n \mid d_\psi(\xi,\eta) < r\} \\ &= \{\eta \in \mathbb{R}^n \mid \psi(\xi-\eta) < r^2\} \end{aligned}$$

Notice that $B^{d_\psi}(\xi,r) = \xi + B^{d_\psi}(0,r)$. *We set*

$$\begin{aligned} m(r) &:= \inf\{|\eta| \mid \psi^{1/2}(\eta) = r\}, \\ M(r) &:= \sup\{|\eta| \mid \psi^{1/2}(\eta) = r\}. \end{aligned}$$

When $B(x,\rho)$ *denotes the Euclidean ball with center* x *and radius* ρ,

$$B(\xi, m(r)) \subset B^{d_\psi}(\xi,r) \subset B(\xi, M(r)).$$

Lemma 7.2. The metric d_ψ generates the Euclidean topology if and only if $\lim_{|\xi|\to\infty}\psi(\xi) > 0$.

Definition 7.3. Let $\psi : \mathbb{R}^n \to \mathbb{R}$ non-periodic, continuous negative definite function such that d_ψ is a metric which generates the Euclidean topology. Then we call ψ to be **metric generating** on $\mathbb{R}^n$ and this class is denoted by $\mathcal{MCN}(\mathbb{R}^n)$, i.e. $\psi \in \mathcal{MCN}(\mathbb{R}^n)$.

In the following we assume always $\psi \in \mathcal{MCN}(\mathbb{R}^n)$.

We want to study the metric measure space $(\mathbb{R}^n, d_\psi, \lambda^{(n)})$. *For this we need to introduce the notion of volume doubling which plays a central role in the analysis on metric measure spaces.*

Definition 7.4. Let (X,d,μ) be a metric measure space. We say that (X,d,μ) or μ has the **volume doubling property** if there exists a constant c_2 such that

$$\mu(B^d(x,2r)) \le c_2\mu(B^d(x,r)) \tag{7.2}$$

holds for all metric balls $B^d(x,r) = \{y \in X : d(y,x) < r\} \subset X$. If (7.2) holds only for all balls with radii $r < \rho$ for some fixed $\rho > 0$, we say that (X,d,μ) (or μ) is **locally** volume doubling. If volume doubling holds, then for $R \ge 1$,

$$\mu(B^d(x,R)) \le c_2^{\log_2 R}\mu(B^d(x,1)) = R^{\log_2 c_2}\mu(B^d(x,1)),$$

i.e. $R \mapsto \mu(B^d(x,R))$ has at most power growth.

An important class of metric measure space are so called ***homogeneous spaces*** *(in the sense of Coifman and Weiss): (X,d,μ) is a homogeneous space if there exists $N \geq 1$ such that for all $x \in X$ and all radii $r > 0$ the ball $B^d(x,r)$ contains at most N points $x_1, \cdots, x_N$ such that $d(x_j,x_k) > r/2$ for $j \neq k$.*

Lemma 7.5. If $(\mathbb{R}^n, d_\psi, \lambda^{(n)})$ has the volume doubling property, where $\psi \in \mathcal{MCR}(\mathbb{R}^n)$, then it is a homogeneous space.

The aim is to understand the transition density $p_t(x)$ of a given Levy process in geometric terms using the fact that $\psi^{1/2}$ is a metric. We assume in the following always that $(X_t^\psi)_{t\geq 0}$ is a symmetric Levy process, that its transition function has a density p_t given by

$$p_t(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-ix\xi} e^{-t\psi(\xi)} d\xi, \tag{7.3}$$

and that $d_\psi(\xi,\eta) = \psi^{1/2}(\xi - \eta)$ is a metric on $\mathbb{R}^n$ which has with respect to the Lebesgue measure $\lambda^{(n)}$ the volume doubling property.

The first observation is that

$$p_t(0) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-t\psi(\xi)} d\xi = (2\pi)^{-n} \int_0^\infty \lambda^{(n)}(B^{d_\psi}(0, \sqrt{\frac{r}{t}}) e^{-r} dr \tag{7.4}$$

which yields under the assumption of volume doubling:

Theorem 7.6. *([20] or [22]) If the metric measure space $(\mathbb{R}^n, d_\psi, \lambda^{(n)})$ has the volume doubling property then we have with constants $0 < \gamma_1 \leq \gamma_2$*

$$\gamma_1 \lambda^{(n)}(B^{d_\psi}(0, \frac{1}{\sqrt{t}})) \leq p_t(0) \leq \gamma_2 \lambda^{(n)}(B^{d_\psi}(0, \frac{1}{\sqrt{t}})). \tag{7.5}$$

Thus we can interpret the diagonal term in the setting of the metric measure space $(\mathbb{R}^n, d_\psi, \lambda^{(n)})$.

Following further [20] we note that Theorem 7.6 also extend to $f \circ \psi$ where f is a Bernstein function with corresponding convolution semigroup $(\eta_t)_{t\geq 0}$, i.e. we have

$$\begin{aligned} \tilde{\gamma}_1 \int_0^\infty \lambda^{(n)}(B^{d_\psi}(0, \frac{1}{\sqrt{s}}))\eta_t(ds) &\leq \lambda^{(n)}(B^{d_{f\circ\psi}}(0, \frac{1}{\sqrt{t}})) \\ &\leq \tilde{\gamma}_2 \int_0^\infty \lambda^{(n)}(B^{d_\psi}(0, \frac{1}{\sqrt{s}}))\eta_t(ds) \end{aligned} \tag{7.6}$$

provided the metric $d_{f\circ\psi}$ has again the volumn doubling property.

We can apply these results immediately to the Aronson estimates (6.17) and (6.23). First we observe that (6.17) implies

$$\kappa_1 \Gamma^{\lambda_1}(t,x,x) \leq \Gamma(t,x,x) \leq \kappa_2 \Gamma^{\lambda_2}(t,x,x) \tag{7.7}$$

but

$$\Gamma^{\gamma}(t,x,x) = p_t^{\gamma}(0) = (4\pi\gamma t)^{-n/2} \tag{7.8}$$

where $p_t^{\gamma}(x)$ is the transition density corresponding to $\gamma\Delta_{(n)}$.

Thus we get

$$\kappa_1 p_t^{\lambda_1}(0) \leq \Gamma(t,x,x) \leq \kappa_2 p_t^{\gamma_2}(0). \tag{7.9}$$

Note that $x \mapsto \Gamma(t,x,x)$ is in general non-constant, i.e. it is $x-$dependent, but the lower and upper bounds are independent of x and are understood in the metric measure spaces $(\mathbb{R}^n, d_{f(\gamma_1|\cdot|^2)}, \lambda^{(n)})$ and $(\mathbb{R}^n, d_{f(\gamma_2|\cdot|^2)}, \lambda^{(n)})$. Note that by Lemma 3.9.34.A in [17] the two metrics $d_{f(\gamma_1|\cdot|^2)}$ and $d_{f(\gamma_2|\cdot|^2)}$ are equivalent. In case that f is a Bernstein function such that $f(|\cdot|^2)$ induces a metric on $\mathbb{R}^n$ having the volume doubling property with respect to Lebesgue measure, using (6.23), we first arrive at

$$\kappa_1 \Gamma^{\gamma_1,f}(t,x,x) \leq \Gamma^f(t,x,x) \leq \kappa_2 \Gamma^{\gamma_2,f}(t,x,x) \tag{7.10}$$

and since

$$\Gamma^{\gamma,f}(t,x,x) = \int_0^\infty \Gamma^{\gamma}(s,x,x)\eta_t(ds) \tag{7.11}$$

it follows further

$$\kappa_1 \int_0^\infty p_s^{\gamma_1}(0)\eta_t(ds) \leq \Gamma^f(t,x,x) \leq \kappa_2 \int_0^\infty p_s^{\gamma_2}(0)\eta_t(ds). \tag{7.12}$$

Now we can use (7.5) and (7.6) to deduce

$$\kappa_1^* \lambda^{(n)}(B^{f(\gamma_1|\cdot|^2)}(0, \frac{1}{\sqrt{t}})) \leq \Gamma^f(t,x,x) \leq \kappa_2^* \lambda^{(n)}(B^{f(\gamma_2|\cdot|^2)}(0, \frac{1}{\sqrt{t}})). \tag{7.13}$$

Again we conclude that $\Gamma^f(t,x,x)$ is controlled in terms of two metric measure spaces carrying equivalent metrics and these controls are $x-$independent.

So far we dealt only with $p_t(0) = p_t^{\psi}(0)$. As discussed in [20] there are good reasons to conjecture that in many cases there exists a second metric, in general depending on t, such that

$$p_t(x-y) = p_t(0)e^{-\delta_{t,\psi}^2(x,y)}. \tag{7.14}$$

Clearly the metric must be translation invariant too.

Here are some examples:

In case of the Brownian motion we have

$$\delta_{t,|\cdot|^2}(x,y) = \frac{1}{\sqrt{2t}}|x-y|, \tag{7.15}$$

for the Cauchy process we have in one dimension

$$\delta_{t,|\cdot|}(x,y) = \sqrt{\ln[\frac{|x-y|^2+t^2}{t^2}]}, \tag{7.16}$$

for the symmetric one-dimensional Meixner process we have

$$\delta_{t,M}(x,y) = -\ln\left|\frac{\Gamma(\frac{t+i(x-y)}{2})}{\Gamma(\frac{t}{2})}\right|^2 = \sum_{j=1}^{\infty}\ln(1+\frac{|x-y|^2}{(t+2j)^2}). \tag{7.17}$$

Moreover, Theorem 7.1 in [20] gives classes of examples where (7.14) holds for subordinate Brownian motion in dimension $n = 1, 2, 3$.

Again we can apply this to the Aronson estimates, assuming that $f \circ |\cdot|^2$ *leads to a representation (7.14). In this case we can apply (6.23). For this we note first that now*

$$\begin{aligned}\Gamma^{\gamma,f}(t,x,y) &= \int_0^\infty \Gamma^\gamma(s,x,y)\eta_t(ds)\\ &= \int_0^\infty p_s(0)e^{-\frac{|x-y|^2}{4\gamma s}}\eta_t(ds)\\ &= p_t^{f(\gamma|\cdot|^2)}(0)e^{-\delta^2_{t,f(\gamma|\cdot|)}(x,y)},\end{aligned}$$

which yields

(7.18)

$$\kappa_1 p_t^{f(\gamma_1|\cdot|^2)}(0)e^{-\delta^2_{t,f(\gamma_1|\cdot|^2)}(x,y)} \leq \Gamma^f(t,x,y) \leq \kappa_2 p_t^{f(\gamma_2|\cdot|^2)}(0)e^{-\delta^2_{t,f(\gamma_2|\cdot|^2)}(x,y)},$$

and hence we have in this case a rather uniform geometric control on $\Gamma^f(t,x,y)$.

Eventually we want to give a probabilistic application of these results. For this let $(X_t^\psi)_{t\geq 0}$ *be a symmetric Levy process such that the metric measure space* $(\mathbb{R}^n, d_\psi, \lambda^{(n)})$ *has the doubling property. For* $R > 0$ *and* $x \in \mathbb{R}^n$ *we introduce the first passage time of* $B_R(x) = \{y \in \mathbb{R}^n \mid |x-y| \leq R\}$ *as*

$$\sigma_R := \inf\{t \geq 0 \mid |X_t^\psi - x| > R\}. \tag{7.19}$$

Following R.Schilling[25] the estimates

$$\frac{c_n}{\sup_{|\xi|\leq 1}\psi(\frac{\xi}{R})} \leq E^x(\sigma_R) \leq \frac{c_n}{\sup_{|\xi|\leq 1}\psi(\frac{\pi}{8}\frac{\xi}{R})} \tag{7.20}$$

hold. However, within the metric measure space $(\mathbb{R}^n, d_\psi, \lambda^{(n)})$ *these estimates become*

$$\frac{c_n}{\sup_{|\xi|\leq 1} d_\psi^2(0,\frac{\xi}{R})} \leq E^x(\sigma_x) \leq \frac{c_n}{\sup_{|\xi|\leq 1} d_\psi^2(0,\frac{\pi}{8}\frac{\xi}{R}}, \tag{7.21}$$

i.e. they have a geometric meaning.

Note that (7.21) is an estimate for an expectation, hence the metric should be that we encounter in the Fourier-transformed space: $E^x(\sigma_R)$ *is the Fourier transform of the distribution of* σ_R *at* $0 \in \mathbb{R}^n$.

Chapter 8

On Pseudo-Differential Operators Generating Markov Processes

Already in the previous two sections we discussed non-translation invariant generators of Feller and L^2−sub-Markovian semigroups, namely the generators of the subordinate semigroups $(T_t^f)_{t\geq 0}$ where f is a Bernstein function and $(T_t)_{t\geq 0}$ is generated by (an extension of) $L(x,D)$ as given by (6.18). In the section we want to summarize some basic results on pseudo-differential operators generating Markov processes. Our main reference is W.Hoh[13] and [17]-[19].

Let $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ be a locally bounded function such that for any $x \in \mathbb{R}^n$ the function $q(x,\cdot) : \mathbb{R}^n \to \mathbb{C}$ is continuous and negative definite. On $C_0^\infty(\mathbb{R}^n)$ or $\mathcal{S}(\mathbb{R}^n)$ the operator

$$q(x,D)u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^N} e^{ix\xi} q(x,\xi)\hat{u}(\xi)d\xi \tag{8.1}$$

is defined and is called a ***pseudo-differential operator*** *with* ***negative definite symbol*** *$q(x,\xi)$.*

As a first result we state

Theorem 8.1. *Let $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ be a locally bounded function such that for any $x \in \mathbb{R}^n$ the function $q(x,\cdot) : \mathbb{R}^n \to \mathbb{C}$ is continuous and negative definite. Define on $C_0^\infty(\mathbb{R}^n)$ the operator*

$$-q(x,D)u(x) := -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} q(x,\xi)\hat{u}(\xi)d\xi. \tag{8.2}$$

Then the operator $(-q(x,D), C_0^\infty(\mathbb{R}^n))$ satisfies the positive maximum principle.

Proof. First note that by Lemma 3.6 we have

$$|q(x,\xi)| \le \tilde{c}(x)(1+|\xi|^2)$$

for all $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$, which implies that the operator $q(x,D)$ is well defined on $C_0^\infty(\mathbb{R}^n)$. Now let $u \in C_0^\infty(\mathbb{R}^n)$ and $x_0 \in \mathbb{R}^n$ such that $u(x_0) = \sup_{x\in\mathbb{R}^n} u(x) \ge 0$. We have to prove that $-[q(x,D)u](x_0) \le 0$ holds. Consider the function $\psi_{x_0} : \mathbb{R}^n \to \mathbb{C}$, $\psi_{x_0}(\xi) = q(x_0,\xi)$. By our assumptions $\xi \to \psi_{x_0}(\xi)$ is a continuous negative definite function and the operator $-\psi_{x_0}(\xi)$ is a continuous negative definite function and the operator $-\psi_{x_0}(D)$ defined on $C_0^\infty(\mathbb{R}^n)$ by

$$\begin{aligned}-\psi_{x_0}(D)u(x) &= -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\psi_{x_0}(\xi)\hat{u}(\xi)d\xi \\ &= -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}q(x_0,\xi)\hat{u}(\xi)d\xi\end{aligned}$$

satisfies the positive maximum principle, thus we have

$$-\psi_{x_0}(D)u(x_0) \le 0$$

for $u(x_0) = \sup_{x\in\mathbb{R}^n} u(x) \ge 0$. But for any $u \in C_0^\infty(\mathbb{R}^n)$ we have

$$\begin{aligned}-[q(x,D)u](x_0) &= -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix_0\cdot\xi}q(x_0,\xi)\hat{u}(\xi)d\xi \\ &= -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix_0\cdot\xi}\psi_{x_0}(\xi)\hat{u}(\xi)d\xi \\ &= -\psi_{x_0}(D)u(x_0)\end{aligned}$$

which implies the theorem. □

Having the Hille-Yosida-Ray theorem in mind or the results about symmetric Dirichlet forms we may ask when is is possible to extend $-q(x,D)$ as in Theorem 8.1 to a generator of a Feller semigroup or L^2-sub-Markovian semigroup.

In the affirmative case we may ask whether the transition functions of these semigroups have densities and we may try to estimate these densities and to give these estimates a geometric interpretation.

For constructing semigroups with a pseudo-differential operator as generator we introduce two approaches. The first uses Hilbert space methods and a pertubation argument and is taken from [15], see also [17].

Note that we just recall the results in a straightforward way, neither do we go into detailed proofs nor do we optimize conditions on the symbol class.

Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a continuous negative definite function and assume

$$\psi(\xi) \geq c|\xi|^r \tag{8.3}$$

for all $|\xi| \geq 1$ with some $c > 0$ and $r > 0$. We consider now a symbol

$$q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \tag{8.4}$$

with decompostion

$$q(x,\xi) = q_1(\xi) + q_2(x,\xi) \tag{8.5}$$

where q_1 and $q_2(x,\cdot)$ are continuous negative definite functions.
In addition we assume for some constants $0 < c_0 < c_1$

(A.1) $$c_0(1+\psi(\xi)) \leq q_1(\xi) \leq c_1(1+\psi(\xi))$$

for all $\xi \in \mathbb{R}^n$, $|\xi| \geq 1$.
We define the constant γ_M as

$$\gamma_M := \left(8C_M(2(1+c_\psi))^{1/2} \int_{\mathbb{R}} (1+|\xi|^2)^{-M+1} d\xi\right)^{-1} \tag{8.6}$$

where c_ψ is such that

$$1+\psi(\xi) \leq (1+c_\psi)(1+|\xi|^2) \tag{8.7}$$

holds for all $\xi \in \mathbb{R}^n$, and C_M is a constant such that

$$(1+|\eta|^2)^{M/2} \leq C_M \sum_{|\beta|\leq M} |\eta^\beta| \quad ,\eta \in \mathbb{N}_0^n. \tag{8.8}$$

We need the following assumptions
(A.2.M) *The symbol $q_2(x,\xi)$ is M-times continuously differentiable with respect to x and for $\beta \in \mathbb{N}_0^n$, $|\beta| \leq M$, there are functions $\Phi_\beta \in L^1(\mathbb{R}^n)$ such that for $M > n+1$*

$$|\partial_x^\beta q_2(x,\xi)| \leq \Phi_\beta(x)(1+\psi(\xi)) \tag{8.9}$$

(A.3.M) *With the constant $\gamma_M > 0$ as in (8.6) we have*

$$\sum_{|\beta|\leq M} \|\Phi_\beta\|_{L^1} \leq \gamma_M \cdot c_0, \tag{8.10}$$

where c_0 is taken from (A.1)

Theorem 8.2. *Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a continuous negative definite function satisfying (8.3) for some $r > 0$. Assume that (A.1), (A.2.M) and (A.3.M) holds with an integer $M > (\frac{n}{r} \vee 1) + n$. Then $(-q(x, D), C_0^\infty(\mathbb{R}^n))$ is closable in $C_\infty(\mathbb{R}^n)$ and the closure is the generator of a Feller semigroup.*

Remark 8.3. This theorem is due to Jacob [15], we followed in our presentation Hoh [13]. The interpretation of the result is that certain perturbation $-q_2(x, D)$ of $-q_1(D)$ are admissible in the sense that $-q(x, D) = -q_1(D) - q_2(x, D)$ is also a generator of a Feller semigroup.

Let $q(x, D)$ be as in Theorem 8.2 and denote by B_λ the corresponding bilinear form

$$B_\lambda(u, v) = (-q(x, D)u, v)_0 + \lambda(u, v)_0, \quad \lambda \geq 0. \tag{8.11}$$

Originally defined on $C_0^\infty(\mathbb{R}^n)$ one can prove, see [12], [13], that B_λ has a continuous extension to $H^{\psi,1}(\mathbb{R}^n)$ and that $-q(x, D)$ extends to a Dirichlet operator with domain $H^{\psi,2}(\mathbb{R}^n)$.

The following resulte is taken from [13], Corollary 8.4, however, it is essentially contained in [12]. We simply write B for B_0:

Theorem 8.4. *Under the assumption of Theorem 8.2, and in case $q(x, D)$ is a symmetric operator on $L^2(\mathbb{R}^n)$, then $(B, H^{\psi,1}(\mathbb{R}^n))$ is a regular symmetric Dirichlet form.*

The results of Theorem 8.2 and 8.4 depend on certain estimates which we collect in the following. Details are given in [12] and [13]. Under the assumptions of Theorem 8.2 we have

$$\|q(x, D)u\|_{\psi,s} \leq c\|u\|_{\psi,s+2} \tag{8.12}$$

for $s \in \mathbb{R}$, $|s - 1| + 1 + n < M$, and $u \in H^{\psi,s+2}(\mathbb{R}^n)$;

$$|B_\lambda(u, v)| \leq c\|u\|_{\psi,1}\|v\|_{\psi,1} \tag{8.13}$$

for $M > n + 1$ and all $u, v \in H^{\psi,1}(\mathbb{R}^n)$.

Moreover, using (A.3.M) it holds for all $\lambda \geq \lambda_0$ for some $\lambda_0 \in \mathbb{R}$

$$B_\lambda(u, u) \geq \frac{c_0}{2}\|u\|_{\psi,1}^2 \quad , u \in H^{\psi,1}(\mathbb{R}^n). \tag{8.14}$$

Finally for $s \geq 0$, $M > |s - 1| + 1 + n$ and $u \in H^{\psi,s+2}(\mathbb{R}^n)$ we have

$$\|u\|_{\psi,s+2} \leq c(\|q(x, D)u\|_{\psi,s} + \|u\|_0). \tag{8.15}$$

The second method of constructing Feller semigroups starting with a pseudo-differential operator with a negative definite symbol is using a symbolic calculus introduced by W.Hoh[12], see also W.Hoh[13].

Definition 8.5. **A.** A C^∞ function $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ is said to be a symbol in the class $S_\rho^{m,\langle\psi\rangle}$ if for all $\alpha, \beta \in \mathbb{N}_0^n$ there are constants $c_{\alpha,\beta} \geq 0$ such that

$$(8.16) \qquad |\partial_\xi^\alpha \partial_x^\beta q(x,\xi)| \leq c_{\alpha,\beta}(1+\psi(\xi))^{\frac{m-\rho(|\alpha|)}{2}}$$

holds for all $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$ with $m \in \mathbb{R}$ called the order of q. Here the function ρ is defined as $\rho : \mathbb{N}_0 \to \mathbb{N}_0$, $\rho(k) = k \wedge 2$.
B. If instead of (8.16) we have

$$(8.17) \qquad |\partial_\xi^\alpha \partial_x^\beta q(x,\xi)| \leq c_{\alpha,\beta}(1+\psi(\xi))^{m/2}$$

we called q a symbol in the class $S_0^{m,\langle\psi\rangle}$.

The following result is due to W.Hoh, see also [18].

Theorem 8.6. *Assume that (8.3) holds. Let $q \in S_\rho^{2,\langle\psi\rangle}$ and assume that $\xi \mapsto q(x,\xi)$ is for every $x \in \mathbb{R}^n$ continuous and negative definite. If*

$$(8.18) \qquad q(x,\xi) \geq \delta(1+\psi(\xi))$$

for some $\delta > 0$ and all $|\xi| \geq R \geq 0$, then $-q(x,D)$ defined on $C_0^\infty(\mathbb{R}^n)$ is closable in $C_\infty(\mathbb{R}^n)$ and its closure generates a Feller semigroup.

Note that (8.18) implicitely requires $q(x,\xi)$ to be real-valued.

Theorem 8.7. *If q is as in Theorem 8.6 and $q(x,D)$ is a symmetric operator on $L^2(\mathbb{R}^n)$, then $(B, H^{\psi,1}(\mathbb{R}^n))$ is a regular symmetric Dirichlet form.*

Using Hoh's calculus, in his thesis [4] , see also [5] , B. Böttcher succeeded to construct a fundamental solution to the problem

$$\frac{\partial u}{\partial t} + q(x,D)u = f$$
$$\lim_{t\downarrow 0} u(\cdot,t) = u_0 \in L^2(\mathbb{R}^n).$$

His results allow to consider the expression

$$(8.19) \qquad (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\xi} e^{-q(x,\xi)} \hat{u}(\xi) d\xi$$

as approximation (for $t > 0$ small) of $T_t u$, $(T_t)_{t\geq 0}$ being generated by $-q(x,D)$, compare [5], p.1241.

We will return to this result later on.

As outlined in [16] or [20], see also R.Schilling[26], given a "nice" Feller process, we can calculate the symbol of its generator as

$$-q(x,\xi) = \lim_{t\to 0} \frac{E^x(e^{i(X_t-x)\cdot\xi} - 1)}{t}. \tag{8.20}$$

This formula enables us to identify many subordinate processes as being generated by a pseudo-differential operator. This applies for example to the processes associated with $(T_t^f)_{t\geq 0}$, T_t^f *given as in (6.19).*

Chapter 9

An Example for Diagonal Estimates in Case of a Simple Generator with Variable Coefficients

A major aim of our thesis, as indicated in the introduction, is to explore in geometric terms how the non-isotropy with respect to the co-variables of a negative definite symbol is reflected in the behavior of the transition density of the corresponding process. Moreover, we want to include the case of non-translation invariant generators and use the idea of freezing coefficients to get some insights. In this section we want to provide a class of toy-examples.

Let $\psi_1 : \mathbb{R}^n \to \mathbb{R}$ *and* $\psi_2 : \mathbb{R}^m \to \mathbb{R}$ *be two continuous negative definite functions with Levy-Khinchine representations*

$$\psi_1(\xi) = \int_{\mathbb{R}^n \setminus \{0\}} (1 - \cos(x\xi))\nu_1(dx) \tag{9.1}$$

and

$$\psi_2(\eta) = \int_{\mathbb{R}^m \setminus \{0\}} (1 - \cos(y\eta))\nu_2(dy), \tag{9.2}$$

respectively. Let $a_1 : \mathbb{R}^n \to \mathbb{R}$ *and* $a_2 : \mathbb{R}^m \to \mathbb{R}$ *be two continuous functions such that* $0 \le a_1(x) \le \|a_1\|_{L^\infty} < \infty$ *and* $0 \le a_2(y) \le \|a_2\|_{L^\infty} < \infty$*. Then on* $\mathbb{R}^n \times \mathbb{R}^m$ *a negative definite symbol is given by*

$$q(x, y, \xi, \eta) = \psi(\xi, \eta) + a_2(y)\psi_1(\xi) + a_1(x)\psi_2(\eta), \tag{9.3}$$

where $\psi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ *is a further continuous negative definite function with corresponding Levy measure* ν*.*

We assume with a constant κ_1 that

$$1 + \psi_1(\xi) + \psi_2(\eta) \leq \kappa_1(1 + \psi(\xi, \eta))^\alpha$$

for some $\alpha \in (0, 1]$.

Let $q(x, y, D_x, D_y)$ denote the pseudo-differential operator

$$q(x, y, D_x, D_y)u(x, y) = (2\pi)^{-\frac{n+m}{2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} e^{ix\xi} e^{iy\eta} q(x, y, \xi, \eta)\hat{u}(\xi, \eta) d\eta d\xi.$$

Proposition 9.1. The operator $q(x, y, D_x, D_y)$ maps $H^{\psi,2}(\mathbb{R}^n \times \mathbb{R}^m)$ continuously into $L^2(\mathbb{R}^n \times \mathbb{R}^m)$.

Proof. We need to show

$$\|q(x, y, D_x, D_y)u\|_{L^2} \leq C\|u\|_{\psi,2}.$$

First we note

$$\begin{aligned}\|q(x, y, &D_x, D_y)u\|_{L^2} \\ &\leq \|\psi(D_x, D_y)u\|_{L^2} + \|a_2(\cdot)\psi_1(D_x)u\|_{L^2} + \|a_1(\cdot)\psi_2(D_y)u\|_{L^2}.\end{aligned}$$

Now, by the definition of $\psi(D_x, D_y)$ it follows from Plancherel's theorem that

$$\|\psi(D_x, D_y)u\|_{L^2} = \|F(\psi(D_x, D_y)u)\|_{L^2}$$

and since $\psi(D_x, D_y)u = F^{-1}(\psi(\cdot,\cdot)\hat{u}(\cdot,\cdot))$ we find

$$\begin{aligned}\|\psi(D_x, D_y)u\|_{L^2} &= \|F(F^{-1}(\psi(\cdot,\cdot)\hat{u}(\cdot,\cdot)))\|_{L^2} \\ &= \|\psi(\cdot,\cdot)\hat{u}(\cdot,\cdot)\|_{L^2} = \Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^m} \psi^2(\xi,\eta)|\hat{u}(\xi,\eta)|^2 d\eta d\xi\Big)^{\frac{1}{2}} \\ &\leq \Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^m} (1 + \psi(\xi,\eta))^2|\hat{u}(\xi,\eta)|^2 d\eta d\xi\Big)^{\frac{1}{2}} \\ &= \|u\|_{\psi,2},\end{aligned}$$

Next we find

$$\|a_2(\cdot)\psi_1(D_x)u\|_{L^2} \leq \|a_2\|_{L^\infty}\|\psi_1(D_x)u\|_{L^2}. \tag{9.4}$$

Note that

$$\psi_1(D_x)u(x, y) = F^{-1}(\psi_1(\cdot)\hat{u}(\cdot,\cdot)) \tag{9.5}$$

and again, by Planchrel's theorem we get

$$\begin{aligned}\|\psi_1(D_x)u\|_{L^2} &= \|\psi_1(\cdot)\hat{u}(\cdot,\cdot)\|_{L^2} \\ &\le \kappa_1\|(1+\psi)^\alpha\hat{u}(\cdot,\cdot)\|_{L^2} \le \kappa_1\|(1+\psi)\hat{u}(\cdot,\cdot)\|_{L^2} \\ &= \kappa_1\|u\|_{\psi,2},\end{aligned}$$

implying

$$\|a_2(\cdot)\psi_1(D_x)u\|_{L^2} \le \kappa_1\|a_2\|_\infty\|u\|_{\psi,2}. \tag{9.6}$$

Analogously we find

$$\|a_1(\cdot)\psi_2(D_y)u\|_{L^2} \le \kappa_1\|a_1\|_\infty\|u\|_{\psi,2} \tag{9.7}$$

proving the proposition. □

We define on $S(\mathbb{R}^n \times \mathbb{R}^m)$ the quadratic form

$$\mathcal{E}(u,v) = (q(x,y,D_x,D_y)u,v)_{L^2} \tag{9.8}$$

Proposition 9.2. The quadratic form $\mathcal{E}$ has a continuous extension to $H^{\psi,1}(\mathbb{R}^n \times \mathbb{R}^m)$, i.e. for all $u, v \in H^{\psi,1}(\mathbb{R}^n \times \mathbb{R}^m)$ it holds

$$|\mathcal{E}(u,v)| \le \kappa_2\|u\|_{\psi,1}\|v\|_{\psi,1}. \tag{9.9}$$

Proof. Since

(9.10)
$$\begin{aligned}\mathcal{E}(u,v) &= (q(x,y,D_x,D_y)u,v)_{L^2} \\ &= (\psi(D_x,D_y)u,v)_{L^2} + (a_2(\cdot)\psi_1(D_x)u,v)_{L^2} + (a_1(\cdot)\psi_2(D_y)u,v)_{L^2}\end{aligned}$$

We can treat each term separately. First we note

$$\begin{aligned}|(\psi(D_x,D_y)u,v)_{L^2}| &= |(F^{-1}(\psi(\cdot,\cdot)\hat{u}(\cdot,\cdot)),v(\cdot,\cdot))_{L^2}| \\ &= |(\psi(\cdot,\cdot)\hat{u}(\cdot,\cdot),\hat{v}(\cdot,\cdot))_{L^2}| \\ &= |(\psi^{\frac{1}{2}}(\cdot,\cdot)\hat{u}(\cdot,\cdot),\psi^{\frac{1}{2}}(\cdot,\cdot)\hat{v}(\cdot,\cdot))_{L^2}| \\ &\le \|\psi^{\frac{1}{2}}\hat{u}\|_{L^2}\|\psi^{\frac{1}{2}}\hat{v}\|_{L^2} \\ &\le \|u\|_{\psi,1}\|v\|_{\psi,1}. \end{aligned} \tag{9.11}$$

The second equality derives from Plancherel's theorem.

Next we consider $(a_2(\cdot)\psi_1(D_x)u, v)_{L^2}$:

$$
\begin{aligned}
&(a_2(\cdot)\psi_2(D_x)u(\cdot,\cdot), v(\cdot,\cdot))_{L^2} \\
&= \int_{\mathbb{R}^m} a_2(y)(\int_{\mathbb{R}^n} \psi_1(D_x)u(x,y)v(x,y)dx)dy \\
&= \int_{\mathbb{R}^m} a_2(y)\left(\int_{\mathbb{R}^n} \psi_1(\xi)(F_{x\mapsto\xi}u)(\xi,y)\overline{(F_{x\mapsto\xi}v)(\xi,y)}d\xi\right)dy \\
&= \int_{\mathbb{R}^m} a_2(y)\left(\int_{\mathbb{R}^n} \psi_1^{\frac{1}{2}}(\xi)(F_{x\mapsto\xi}u)(\xi,y)\overline{\psi_1^{\frac{1}{2}}(\xi)(F_{x\mapsto\xi}v)(\xi,y)}d\xi\right)dy \\
&= \int_{\mathbb{R}^m} a_2(y)\left(\int_{\mathbb{R}^n} \psi_1^{\frac{1}{2}}(D_x)u(x,y)\psi_1^{\frac{1}{2}}(D_x)v(x,y)dx\right)dy
\end{aligned}
$$

which implies

$$
\begin{aligned}
|(a_2(\cdot)\psi_1(D_x)u, v)_{L^2}| &\leq \|a_2\|_{L^\infty} \int_{\mathbb{R}^m}\int_{\mathbb{R}^n} |\psi_1^{\frac{1}{2}}(D_x)u(x,y)||\psi_1^{\frac{1}{2}}(D_x)v(x,y)|dxdy \\
&\leq \|a_2\|_{L^\infty}\|\psi_1^{\frac{1}{2}}(D_x)u\|_{L^2}\|\psi_1^{\frac{1}{2}}(D_x)v\|_{L^2},
\end{aligned}
$$

Since

$$
\begin{aligned}
\|\psi_1^{\frac{1}{2}}(D_x)u\|_{L^2}^2 &= \int_{\mathbb{R}^m}\int_{\mathbb{R}^n} \psi_1(\xi)|\hat{u}(\xi,\eta)|^2 d\xi d\eta \\
&\leq \kappa_1 \int_{\mathbb{R}^m}\int_{\mathbb{R}^n} (1+\psi(\xi,\eta))^\alpha |\hat{u}(\xi,\eta)|^2 d\xi d\eta \\
&\leq \kappa_1 \|u\|_{\psi,1}^2,
\end{aligned}
\tag{9.12}
$$

we arrive at

$$
|(a_2(\cdot)\psi_1(D_x)u, v)_{L^2}| \leq \kappa_1 \|a_2\|_{L^\infty}\|u\|_{\psi,1}\|v\|_{\psi,1}
\tag{9.13}
$$

Analogously we get

$$
|(a_1(\cdot)\psi_2(D_y)u, v)_{L^2}| \leq \kappa_1 \|a_1\|_{L^\infty}\|u\|_{\psi,1}\|v\|_{\psi,1}.
\tag{9.14}
$$

Combining (9.11), (9.13) and (9.14) gives (9.9) with

$$
\kappa_2 \geq 1 + \kappa_1(\|a_1\|_\infty + \|a_2\|_\infty).
\tag{9.15}
$$

□

Theorem 9.3. *The quadratic form $(\mathcal{E}, H^{\psi,1}(\mathbb{R}^n \times \mathbb{R}^m))$ is a closed and symmetric form in particular $\mathcal{E}(u,u) \geq 0$, and sub-Markovian. Hence $(\mathcal{E}, H^{\psi,1}(\mathbb{R}^n \times \mathbb{R}^m))$ is a symmetric Dirichlet form*

Proof. It remains to prove that $\mathcal{E}(u,u) \geq 0$ and is sub-Markovian. We claim that each of the three terms in the decomposition (9.10) is non-negative. For each term we use the Levy-Khinchin representation of ψ, ψ_1 and ψ_2 respectively to find

(9.16)

$$(\psi(D_x, D_y)u, v)_{L^2} = \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^m}\int_{\mathbb{R}^n\times\mathbb{R}^m\setminus\{0\}}(u(x_1-x_2,y_1-y_2)-u(x_1,y_1)) \times (v(x_1-x_2,y_1-y_2)-v(x_1,y_1))\nu(dx_2,dy_2)dy_1\,dx_1$$

and

$$(a_2(\cdot)\psi_1(D_x)u,v)_{L^2} = \frac{1}{2}\int_{\mathbb{R}^m} a_2(y)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n\setminus\{0\}}(u(x_1-x_2,y)-u(x_1)) \times (v(x_1-x_2,y)-v(x_1))\nu_1(dx_2)dx_1dy_1 \tag{9.17}$$

as well as

(9.18)

$$(a_1(\cdot)\psi_2(D_y)u,v)_{L^2} = \frac{1}{2}\int_{\mathbb{R}^n} a_1(x)\int_{\mathbb{R}^m}\int_{\mathbb{R}^n\setminus\{0\}}(u(x,y_1-y_2)-u(x,y_1)) \times (v(x,y_1-y_2)-v(x,y_1))\nu_2(dy_2)dy_1dx$$

Since $a_1(x) \geq 0$ and $a_2(y) \geq 0$ it follows that $\mathcal{E}(u,u) \geq 0$. Furthermore, from the book [17] or [9] we also know $\mathcal{E}$ is sub-Markovian. □

Thus we can associate with $(\mathcal{E}, H^{\psi,1}(\mathbb{R}^n\times\mathbb{R}^m))$ or with $-q(x,y,D_x,D_y)$ a symmetric sub-Markovian semigroup. Clearly, a further symmetric Dirichlet form is $(\mathcal{E}^\psi, H^{\psi,1}(\mathbb{R}^n\times\mathbb{R}^m))$ the generator of which is $-\psi(D_x,D_y)$ and

$$\mathcal{E}^\psi(u,u) \leq c\mathcal{E}(u,u). \tag{9.19}$$

Thus adding the assumptions $\psi(0,0) = 0$, and the conservativeness of the semigroup $(T_t)_{t\geq 0}$ generated by $-q(x,y,D_x,D_y)$, we obtain by Theorem 6.6 diagonal estimates for $(T_t)_{t\geq 0}$ in terms of geometric conditions related to $(T_t^\psi)_{t\geq 0}$.

To provide an example we need the following result taken from Page 132 in [18]

Theorem 9.4. *Let $(T_t)_{t\geq 0}$ be a Feller semigroup with generator $(A, D(A))$ such that $C_c^\infty(\mathbb{R}^n) \subset D(A)$ and $A|C_c^\infty(\mathbb{R}^n) = -q(x,D)$ with symbol $q(x,\xi)$ in the similar form as (9.3) satisfying $\sup_{x\in\mathbb{R}^n}|q(x,\xi)| \leq c(1+|\xi|^2)$. The semigroup $\{T_t\}_{t\geq 0}$ is conservative, if $q(x,0) \equiv 0$.*

Conversely, if $\{T_t\}_{t\geq 0}$ is conservative and $q(\cdot,0)$ continuous, then $q(x,0) \equiv 0$.

Example 9.5. In the following example we do not use the result of Theorem 8.2 and 8.4, but we rely on their proofs. The structure of the coefficients imply that all commutators of importance are zero, hence no regularities of the coefficients is needed to get estimates. Take $\psi(\xi,\eta) = (|\xi|^2 + |\eta|^2)^{1/2}$, $\psi_1(\xi) = |\xi|^{1/2}$, $\psi_2(\eta) = |\eta|^{1/2}$, first of all, we need to check that $(T_t)_{t\geq 0}$ is a Feller semigroup using Theorem 8.6. We let $\psi(\xi,\eta) = (|\xi|^2 + |\eta|^2)^{\frac{1}{2}}$ in Theorem 8.6. Since $(|\xi|^2 + |\eta|^2)^{\frac{1}{2}}$ is equivalent to $(1 + |\xi|^2 + |\eta|^2)^{\frac{1}{2}} - 1$, we have $q(x,y,\xi,\eta) = (1+|\xi|^2+|\eta|^2)^{\frac{1}{2}} - 1 + a_2(y)|\xi|^{\frac{1}{2}} + a_1(x)|\eta|^{\frac{1}{2}}$. Moreover, we can choose $|\xi|$ and $|\eta|$ large enough so that we have $a_2(y)|\xi|^{\frac{1}{2}} + a_1(x)|\eta|^{\frac{1}{2}} - 1 \geq 0$. Furthermore, $q(x,y,\xi,\eta) \geq (1 + (|\xi|^2 + |\eta|^2)^{\frac{1}{2}})$ satisfies condition (8.18). Therefore, $-q(x,y,D_\xi,D_\eta)$ is a generator of a Feller semigroup. Secondly, we can use Theorem 9.4 to verify that the Feller semigroup associated with our generator $-q(x,y,D_\xi,D_\eta)$ is conservative. To this end, we need to verify that $\sup_{x,y\in\mathbb{R}} |q(x,y,\xi,\eta)| \leq c(1 + |\xi|^2 + |\eta|^2)$ which is obvious since $0 \leq a_1(x) \leq \|a_1\|_{L^\infty} < \infty$ and $0 \leq a_2(y) \leq \|a_2\|_{L^\infty} < \infty$. Then we can use Theorem 6.10 to get an estimate of our symbol we stated beforehand. In Theorem 6.10 we take $\mathcal{E}^{(1)}$ as the Dirichlet form associated with $\psi(\xi,\eta)$ and $\mathcal{E}^{(2)}$ as the Dirichlet form associated with $q(x,y,\xi,\eta)$. From Example 6.8 we can get that $\|T_t\|_{1\to\infty} = \frac{1}{\pi}t^{-2}$ which means we can take $g(t) = \frac{1}{\pi}t^{-2}$ and $H(\xi) = \frac{1}{2}\xi^{-1/2}$ and $h(x) = \frac{1}{4}x^{-1/2}$. Therefore, $\|T_t^{(2)}\| \leq 2C^2t^{-2}$. Then we can use Theorem 6.6 and Theorem 6.7 to get the estimate of subordination of our symbol which is the same as Example 6.8 except the coefficients.

Unfortunately, so far this approach does not provide examples for off-diagonal estimates.

Chapter 10

An Application of the Geometric Interpretation of the Transition Function

With this chapter we start the second part of our thesis where we want to show how a geometric interpretation of transition functions might become useful in probability theory. Our contributions are more case studies than complete theories, but we believe that they still give some new insights.

In this chapter we study first transition functions of certain Levy processes with state space $\mathbb{R}^n$ *and then we move to a more concrete (class of) example(s) in order to get a better understanding of the non-isotropic behaviour of certain (classes of) Levy process(es).*

Let $(X_t^{\psi})_{t\geq 0}$ *be a symmetric Levy process with state space* $\mathbb{R}^n$ *and transition function* p_t *which we assume to exist as element in* $C_\infty(\mathbb{R}^n)$ *given by*

$$p_t(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\xi} e^{-t\psi(\xi)} d\xi. \tag{10.1}$$

Furthermore, we assume that p_t *has the form*

$$p_t(x-y) = p_t(0) e^{-\delta^2_{\psi,t}(x,y)} \tag{10.2}$$

with $p_t(0)$ *as in (7.4), i.e.*

$$p_t(0) = (2\pi)^{-n} \int_0^\infty \lambda^{(n)}(B^{d_\psi}(0, \sqrt{\frac{r}{t}})) e^{-r} dr, \tag{10.3}$$

where $d_\psi(\xi,\eta) = \psi^{1/2}(\xi-\eta)$ *is assumed to be a metric, in fact we assume that the metric measure space* $(\mathbb{R}^n, d_\psi, \lambda^{(n)})$ *has the volume doubling property, and hence we have, compare Chapter 6, that*

$$p_t(0) \asymp \lambda^{(n)}(B^{d_\psi}(0, \frac{1}{\sqrt{t}})) \quad t>0. \tag{10.4}$$

Moreover we assume that $\delta_{\psi,t}(\cdot,\cdot)$ is a metric on $\mathbb{R}^n$.

In order to get some idea how to use the geometric interpretation in the following we pick up one simple problem, namely to estimate some probability.

Let the process start at $z \in \mathbb{R}^n$ and try to find, i.e. estimate, the probability

$$P^z(X_{t_1} \in C_1, X_{t_2} \in C_2) \tag{10.5}$$

for Borel sets C_1, $C_2 \subset \mathbb{R}^n$ and $t_1 \leq t_2$. For reasons which become clear in Chapter 12 we prefer here to have an arbitrary starting point $z \in \mathbb{R}^n$, not just $O \in \mathbb{R}^n$. Using the very definition and properties of (Levy) processes we find, see R.Schilling [27],

$$P^z(X_{t_1} \in C_1, X_{t_2} \in C_2) = T_{t_1}(\chi_{C_1} T_{t_2-t_1}\chi_{C_2})(z) \tag{10.6}$$

or

$$\begin{aligned} &P^z(X_{t_1} \in C_1, X_{t_2} \in C_2) \\ &= \int_{\mathbb{R}^n} p_{t_1}(z-x)\chi_{C_1}(x)\left(\int_{\mathbb{R}^n} p_{t_2-t_1}(x-y)\chi_{C_2}(y)dy\right)dx. \end{aligned} \tag{10.7}$$

Now we use the representation (10.2) for p_t to find

$$\begin{aligned} &P^z(X_{t_1} \in C_1, X_{t_2} \in C_2) \\ &= \int_{\mathbb{R}^n} \chi_{C_1}(x)p_{t_1}(0)e^{-\delta^2_{\psi,t_1}(z,x)}\left(\int_{\mathbb{R}^n} \chi_{C_2}(y)p_{t_2-t_1}(0)e^{-\delta^2_{\psi,t_2-t_1}(x,y)}dy\right)dx \\ &= p_{t_1}(0)p_{t_2-t_1}(0)\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} \chi_{C_1}(x)\chi_{C_2}(y)e^{-\delta^2_{\psi,t_1}(z,x)}e^{-\delta^2_{\psi,t_2-t_1}(x,y)}dy\right)dx \\ &= p_{t_1}(0)p_{t_2-t_1}(0)\int_{C_1}\int_{C_2} e^{-\delta^2_{\psi,t_1}(z,x)}e^{-\delta^2_{\psi,t_2-t_1}(x,y)}dydx. \end{aligned} \tag{10.8}$$

For bounded Borel sets C_1 and C_2 the formula (10.8) allows us to obtain further estimates. For this observe that

$$\begin{aligned} &\int_{C_1}\int_{C_2} e^{-\delta^2_{\psi,t_1}(z,x)}e^{-\delta^2_{\psi,t_2-t_1}(x,y)}dydx \\ &\geq e^{-\sup_{x\in C_1,\,y\in C_2}(\delta^2_{\psi,t_1}(z,x)+\delta^2_{\psi,t_2-t_1}(x,y))}\lambda^{(n)}(C_1)\lambda^{(n)}(C_2) \end{aligned}$$

With (10.4) we obtain (with constants depending only on ψ and the volume growth function)

$$\begin{aligned} &P^z(X_{t_1} \in C_1, X_{t_2} \in C_2) \\ &\geq c_0\lambda^{(n)}(B^{d_\psi}(0,\frac{1}{\sqrt{t_1}}))\lambda^{(n)}(B^{d_\psi}(0,\frac{1}{\sqrt{t_2-t_1}}))\cdot \\ &e^{-\sup_{x\in C_1,\,y\in C_2}(\delta^2_{\psi,t_1}(z,x)+\delta^2_{\psi,t_2-t_1}(x,y))}\lambda^{(n)}(C_1)\lambda^{(n)}(C_2). \end{aligned}$$

Furthermore we have

$$P^z(X_{t_1} \in C_1, X_{t_1} \in C_2) \leq c_1 \lambda^{(n)}(B^{d_\psi}(0, \frac{1}{\sqrt{t_1}}))\lambda^{(n)}(B^{d_\psi}(0, \frac{1}{\sqrt{t_2 - t_1}}))\cdot$$
$$e^{-\inf_{x \in C_1, y \in C_2}(\delta^2_{\psi,t_2}(z,x) + \delta^2_{\psi,t_2-t_1}(x,y))}\lambda^{(n)}(C_1)\lambda^{(n)}(C_2).$$

Of course we now can specific the sets C_1 and C_2. In particular when prescribing both sets with the metric d_ψ or with the metrices δ_{ψ,t_2} and δ_{ψ,t_2-t_1}, respectively, we obtain purely geometric bounds.

As C_1, C_2 are considered as sets in the state space it would be natural to characterise both sets with the metric $\delta_{\psi,\cdot}$. For example we may take

$$C_1 = B^{\delta_{\psi,t_1}}(x_0, r_1), \tag{10.9}$$

$$C_2 = B^{\delta_{\psi,t_2-t_1}}(y_0, r_2), \tag{10.10}$$

and we may assume $C_1 \cap C_2 = \emptyset$, $z \notin C_1 \cup C_2$.

We want to use these considerations to get some more ideas on what happens in non-isotropic situations. For this we split $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and consider the symbol

$$\begin{aligned} \psi : & \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R} \\ & \psi(\xi, \eta) = \psi_1(\xi) + \psi_2(\eta) \end{aligned} \tag{10.11}$$

Now, to ψ_j corresponds a Levy process $(X_t^{\psi_j})_{t \geq 0}$ with state space $\mathbb{R}^{n_j}$, $j = 1, 2$, and for the corresponding transition functions we assume representations analogous to (10.2), i.e.,

$$P_t^{(j)}(x_j - y_j) = P_t^{(j)}(0) e^{-\delta^2_{\psi_j,t}(x_j, y_j)}. \tag{10.12}$$

For this case it follows now with $C_1, C_2 \subset \mathbb{R}^{n_1+n_2}$

$$\begin{aligned} P^z(X_{t_1} \in C_1, X_{t_2} \in C_2) &= T_{t_1}(\chi_{C_1} T_{t_2-t_1} \chi_{C_2})(z) \\ &= \int_{\mathbb{R}^n} p_{t_1}(z,x) \chi_{C_1}(x) \left(\int_{\mathbb{R}^n} p_{t_2-t_1}(x,y) \chi_{C_2}(y) dy \right) dx. \end{aligned} \tag{10.13}$$

Using our previous estimates we obtain

(10.14)

$$P^z(X_{t_1} \in C_1, X_{t_2} \in C_2) \geq$$
$$p_{t_1}(0,0)p_{t_2-t_1}(0,0) \int_{\mathbb{R}^{n_1+n_2}} \chi_{C_1}(x_1,x_2) e^{-\delta_{t_1}^{(1)2}(z_1,x_1)-\delta_{t_1}^{(2)2}(z_2,x_2)}$$

$$\left(\int_{\mathbb{R}^{n_1+n_2}} e^{-\sup_{x\in C_1,\, y\in C_2}(\delta_{t_2-t_1}^{(1)2}(x_1,y_1)+\delta_{t_2-t_1}^{(2)2}(x_2,y_2))} \chi_{C_2}(y_1,y_2)dy_1dy_2 \right) dx_1dx_2$$
$$\geq p_{t_1}(0,0)p_{t_2-t_1}(0,0) e^{-\sup_{x\in C_1,\, y\in C_2}(\delta_{t_2-t_1}^{(1)2}(x_1,y_1)+\delta_{t_2-t_1}^{(2)2}(x_2,y_2))} \lambda^n(C_2)$$

$$\int_{\mathbb{R}^{n_1+n_2}} \chi_{C_1}(x_1,x_2) e^{-\delta_{t_1}^{(1)2}(z_1,x_1)-\delta_{t_1}^{(2)2}(z_2,x_2)} dx_1dx_2$$
$$\geq p_{t_1}(0,0)p_{t_2-t_1}(0,0) e^{-\sup_{x\in C_1,\, y\in C_2}(\delta_{t_2-t_1}^{(1)2}(x_1,y_1)+\delta_{t_2-t_1}^{(2)2}(x_2,y_2))}$$

$$e^{-\sup_{x\in C_1}(\delta_{t_2-t_1}^{(1)2}(z_1,x_1)+\delta_{t_2-t_1}^{(2)2}(z_2,x_2))} \lambda^n(C_1)\lambda^n(C_2),$$

and

(10.15)

$$P^z(X_{t_1} \in C_1, X_{t_2} \in C_2) \leq$$
$$p_{t_1}(0,0)p_{t_2-t_1}(0,0) \int_{\mathbb{R}^{n_1+n_2}} \chi_{C_1}(x_1,x_2) e^{-\delta_{t_1}^{(1)2}(z_1,x_1)-\delta_{t_1}^{(2)2}(z_2,x_2)}$$

$$\left(\int_{\mathbb{R}^{n_1+n_2}} e^{-\inf_{x\in C_1,\, y\in C_2}(\delta_{t_2-t_1}^{(1)2}(x_1,y_1)+\delta_{t_2-t_1}^{(2)2}(x_2,y_2))} \chi_{C_2}(y_1,y_2)dy_1dy_2 \right) dx_1dx_2$$
$$\leq p_{t_1}(0,0)p_{t_2-t_1}(0,0) e^{-\inf_{x\in C_1,\, y\in C_2}(\delta_{t_2-t_1}^{(1)2}(x_1,y_1)+\delta_{t_2-t_1}^{(2)2}(x_2,y_2))} \lambda^n(C_2)$$

$$\int_{\mathbb{R}^{n_1+n_2}} \chi_{C_1}(x_1,x_2) e^{-\delta_{t_1}^{(1)2}(z_1,x_1)-\delta_{t_1}^{(2)2}(z_2,x_2)} dx_1dx_2$$
$$\leq p_{t_1}(0,0)p_{t_2-t_1}(0,0) e^{-\inf_{x\in C_1,\, y\in C_2}(\delta_{t_2-t_1}^{(1)2}(x_1,y_1)+\delta_{t_2-t_1}^{(2)2}(x_2,y_2))}$$

$$e^{-\inf_{x\in C_1}(\delta_{t_1}^{(1)2}(z_1,x_1)+\delta_{t_1}^{(2)2}(z_2,x_2))} \lambda^n(C_1)\lambda^n(C_2)$$

We now want to specialize further. We consider the case where $C_j = C_{j,1} \times C_{j,2}$, $j = 1, 2$, and $C_{j,i} \subset \mathbb{R}^{n_i}$

$$
\begin{aligned}
&P^z(X_{t_1} \in C_1, X_{t_2} \in C_2) = \\
&p_{t_1}(0,0)p_{t_2-t_1}(0,0) \int_{\mathbb{R}^{n_1+n_2}} \chi_{C_1}(x_1,x_2) e^{-\delta_{t_1}^{(1)2}(z_1,x_1) - \delta_{t_1}^{(2)2}(z_2,x_2)} \\
&\left(\int_{\mathbb{R}^{n_1+n_2}} e^{-\delta_{t_2-t_1}^{(1)2}(x_1,y_1)} e^{-\delta_{t_2-t_1}^{(2)2}(x_2,y_2)} \chi_{C_2}(y_1,y_2) dy_1 dy_2 \right) dx_1 dx_2 \\
&= p_{t_1}(0,0)p_{t_2-t_1}(0,0) \int_{\mathbb{R}^{n_1+n_2}} \chi_{C_{11}}(x_1)\chi_{C_{12}}(x_2) e^{-\delta_{t_1}^{(1)2}(z_1,x_1) - \delta_{t_1}^{(2)2}(z_2,x_2)} \\
&\left(\int_{\mathbb{R}^{n_1+n_2}} e^{-\delta_{t_2-t_1}^{(1)2}(x_1,y_1)} e^{-\delta_{t_2-t_1}^{(2)2}(x_2,y_2)} \chi_{C_{21}}(y_1)\chi_{C_{22}}(y_2) dy_1 dy_2 \right) dx_1 dx_2 \\
&= p_{t_1}(0,0)p_{t_2-t_1}(0,0) \int_{\mathbb{R}^{n_1}} \chi_{C_{11}}(x_1) e^{-\delta_{t_1}^{(1)2}(z_1,x_1)} \int_{\mathbb{R}^{n_2}} \chi_{C_{12}}(x_2) e^{-\delta_{t_1}^{(2)2}(z_2,x_2)} \\
&\left(\int_{\mathbb{R}^{n_1}} e^{-\delta_{t_2-t_1}^{(1)2}(x_1,y_1)} \chi_{C_{21}}(y_1) \int_{\mathbb{R}^{n_2}} e^{-\delta_{t_2-t_1}^{(2)2}(x_2,y_2)} \chi_{C_{22}}(y_2) dy_1 dy_2 \right) dx_1 dx_2 \\
&= P^{z_1}(X_{t_1}^{\psi_1} \in C_{11}, X_{t_2}^{\psi_1} \in C_{12}) P^{z_2}(X_{t_1}^{\psi_2} \in C_{21}, X_{t_2}^{\psi_2} \in C_{22}).
\end{aligned}
$$

Now for $P^{z_1}(X_{t_1}^{\psi_1} \in C_{11}, X_{t_2}^{\psi_1} \in C_{12})$ and $P^{z_2}(X_{t_1}^{\psi_2} \in C_{21}, X_{t_2}^{\psi_2} \in C_{22})$ we can apply the previous results.

Remark 10.1. The process $(X^{\psi}_{t\geq 0})$ associated with the characteristic exponent, i.e. symbol, (10.11) splits into the processes $(X_t^{\psi_1})_{t\geq 0}$ and $(X_t^{\psi_2})_{t\geq 0}$, i.e.

$$(X_t^{\psi})_t = ((X_t^{\psi_1}, X_t^{\psi_2}))_{t\geq 0}$$

In case we consider product sets $C_1 \times C_2 \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ the process $(X_t^{\psi})_{t\geq 0}$ is decoupled, we need only consider the components to obtain results for $(X_t^{\psi})_{t\geq 0}$. This is of course the content of the last calculation. From the geometric point of view matters are easy too, we are dealing with products only. However, in case that $C \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ is not of product structure, we start to observe a type of coupling of $(X_t^{\psi_1})_{t\geq 0}$ and $(X_t^{\psi_2})_{t\geq 0}$ of both processes. In particular, due to the non-isotropy we discover that while the products of balls in $(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \delta_t^{\psi_1} \otimes \delta_t^{\psi_2})$ and the balls in $(\mathbb{R}^n, \delta_t^{\psi})$ give rise to the same topological space, their geometric impact (for example on estimates) is quite different.

If $C_j \neq C_{j1} \times C_{j2}$, $j = 1, 2$, in general, it's complicated to estimate if we take arbitrary C_j, but if we take $C_1 = B_{r_1}^{\delta_{t_1}}(0)$, $C_2 = B_{r_2}^{\delta_{t_1-t_2}}(0)$, where $\delta_{t_1}^2 = \delta_{t_1}^{(1)2} + \delta_{t_1}^{(2)2}$ and $\delta_{t_1-t_2}^2 = \delta_{t_1-t_2}^{(1)2} + \delta_{t_1-t_2}^{(2)2}$, therefore, we have

$$
\begin{aligned}
P^z(X_{t_1} \in C_1, X_{t_2} \in C_2) &= \int_{\mathbb{R}^n} p_{t_1}^{\psi}(z,x)\chi_{C_1}(x) \int_{\mathbb{R}^n} p_{t_2-t_1}^{\psi} \chi_{C_2}(y) dy dx \\
&= p_{t_1}(0)p_{t_2-t_1}(0) \int_{\mathbb{R}^n} e^{-\delta_{t_1}^2(z,x)} \chi_{C_1}(x) \int_{\mathbb{R}^n} e^{-\delta_{t_2-t_1}^2(x,y)} \chi_{C_2}(x) dy dx
\end{aligned}
$$

So we have upper bound and lower bound of $P^z(X_{t_1} \in C_1, X_{t_2} \in C_2)$,

$$(10.16)\quad p_{t_1}(0)p_{t_2-t_1}(0)\lambda^n(B^{\delta_{t_1}}_{r_1}(0))\lambda^n(B^{\delta_{t_2-t_1}}_{r_2}(0)) \le P^z(X_{t_1} \in C_1, X_{t_2} \in C_2) \le p_{t_1}(0)p_{t_2-t_1}(0)e^{-r_1^2}\lambda^n(B^{\delta_{t_1}}_{r_1}(0))e^{-r_2^2}\lambda^n(B^{\delta_{t_2-t_1}}_{r_2}(0))$$

Example 10.2. If we take $C_1 = B^{\psi_1}_1(0) \times B^{\psi_2}_1(0)$, $C_2 = B^{\psi_1}_1(a_1) \times B^{\psi_2}_1(a_2)$, where $\psi_1(\xi) = |\xi|^2$ and $\psi_2(\eta) = |\eta|$, therefore, from (10.14) and (10.15), we have

$$P^z(X_{t_1} \in C_1, X_{t_2} \in C_2) \ge$$
$$p_{t_1}(0,0)p_{t_2-t_1}(0,0)e^{-\sup_{x\in C_1, y\in C_2}(\delta^{(1)2}_{t_2-t_1}(x_1,y_1)+\delta^{(2)2}_{t_2-t_1}(x_2,y_2))}\lambda^2(C_2)$$
$$e^{-\sup_{x\in C_1}(\delta^{(1)2}_{t_2-t_1}(z_1,x_1)+\delta^{(2)2}_{t_2-t_1}(z_2,x_2))}\lambda^2(C_1)$$
$$\ge \frac{1}{2\sqrt{\pi t_1}} \times \frac{1}{\pi} \times \frac{1}{t_1} \times \frac{1}{2\sqrt{\pi(t_2-t_1)}} \times \frac{1}{\pi} \times \frac{1}{t_2-t_1}$$
$$\times\, e^{-\left(\frac{2}{4(t_2-t_1)}-2\ln\left((t_2-t_1)^2\right)+\ln\left((t_2-t_1)^2+\max\{|a_1-1|,|1+a_1|\}^2\right)+\ln\left((t_2-t_1)^2+\max\{|a_2-1|,|a_2+1|\}^2\right)\right)}$$
$$\times\, 4 \times e^{-\left(\frac{z_1^2+z_2^2}{4(t_2-t_1)}+2\ln\left(\frac{1+(t_2-t_1)^2}{(t_2-t_1)^2}\right)\right)} \times 4$$

and

$$P^z(X_{t_1} \in C_1, X_{t_2} \in C_2) \le$$
$$p_{t_1}(0,0)p_{t_2-t_1}(0,0)e^{-\inf_{x\in C_1, y\in C_2}(\delta^{(1)2}_{t_2-t_1}(x_1,y_1)+\delta^{(2)2}_{t_2-t_1}(x_2,y_2))}\lambda^2(C_2)$$
$$e^{-\inf_{x\in C_1}(\delta^{(1)2}_{t_2-t_1}(z_1,x_1)+\delta^{(2)2}_{t_2-t_1}(z_2,x_2))}\lambda^2(C_1)$$
$$\le \frac{1}{2\sqrt{\pi t_1}} \times \frac{1}{\pi} \times \frac{1}{t_1} \times \frac{1}{2\sqrt{\pi(t_2-t_1)}} \times \frac{1}{\pi} \times \frac{1}{t_2-t_1}$$
$$\times\, e^{-\left(\frac{2}{4(t_2-t_1)}-2\ln\left((t_2-t_1)^2\right)+\ln\left((t_2-t_1)^2+\max\{\mathrm{sgn}((a_1-1)(1+a_1)),0\}\min\{|a_1-1|,|1+a_1|\}^2\right)\right)}$$
$$\times\, e^{-\ln\left((t_2-t_1)^2+\max\{\mathrm{sgn}((a_2-1)(a_2+1)),0\}\min\{|a_2-1|,|a_2+1|\}^2\right)}$$
$$\times\, 4 \times e^{-\left(\frac{z_1^2+z_2^2}{4(t_2-t_1)}+2\ln\left(\frac{1+(t_2-t_1)^2}{(t_2-t_1)^2}\right)\right)} \times 4$$

Example 10.3. If we take $C_1 = B^{\psi}_1(0,0)$, $C_2 = B^{\psi}_1(a_1,a_2)$, where $\psi(\xi,\eta) = |\xi|^2 + |\eta|$, therefore, from (10.14) and (10.15), we have

$$P^z(X_{t_1} \in C_1, X_{t_2} \in C_2) \geq$$

$$p_{t_1}(0,0)p_{t_2-t_1}(0,0)e^{-\sup_{x\in C_1, y\in C_2}(\delta^{(1)2}_{t_2-t_1}(x_1,y_1)+\delta^{(2)2}_{t_2-t_1}(x_2,y_2))}\lambda^2(C_2)$$

$$e^{-\sup_{x\in C_1}(\delta^{(1)2}_{t_2-t_1}(z_1,x_1)+\delta^{(2)2}_{t_2-t_1}(z_2,x_2))}\lambda^2(C_1)$$

$$\geq \frac{1}{2\sqrt{\pi t_1}} \times \frac{1}{\pi} \times \frac{1}{t_1} \times \frac{1}{2\sqrt{\pi(t_2-t_1)}} \times \frac{1}{\pi} \times \frac{1}{t_2-t_1}$$

$$\times e^{-\left(\frac{2}{4(t_2-t_1)}-2\ln\left((t_2-t_1)^2\right)+\ln\left((t_2-t_1)^2+\max\{|a_1-1|,|1+a_1|\}^2\right)+\ln\left((t_2-t_1)^2+\max\{|a_2-1|,|a_2+1|\}^2\right)\right)}$$

$$\times \frac{8\sqrt{2}}{3} \times e^{-\left(\frac{z_1^2+z_2^2}{4(t_2-t_1)}+2\ln\left(\frac{1+(t_2-t_1)^2}{(t_2-t_1)^2}\right)\right)} \times \frac{8\sqrt{2}}{3}$$

and

$$P^z(X_{t_1} \in C_1, X_{t_2} \in C_2) \leq$$

$$p_{t_1}(0,0)p_{t_2-t_1}(0,0)e^{-\inf_{x\in C_1, y\in C_2}(\delta^{(1)2}_{t_2-t_1}(x_1,y_1)+\delta^{(2)2}_{t_2-t_1}(x_2,y_2))}\lambda^2(C_2)$$

$$e^{-\inf_{x\in C_1}(\delta^{(1)2}_{t_2-t_1}(z_1,x_1)+\delta^{(2)2}_{t_2-t_1}(z_2,x_2))}\lambda^2(C_1)$$

$$\leq \frac{1}{2\sqrt{\pi t_1}} \times \frac{1}{\pi} \times \frac{1}{t_1} \times \frac{1}{2\sqrt{\pi(t_2-t_1)}} \times \frac{1}{\pi} \times \frac{1}{t_2-t_1}$$

$$\times e^{-\left(\frac{2}{4(t_2-t_1)}-2\ln\left((t_2-t_1)^2\right)+\ln\left((t_2-t_1)^2+\max\{\operatorname{sgn}((a_1-1)(1+a_1)),0\}\min\{|a_1-1|,|1+a_1|\}^2\right)\right)}$$

$$\times e^{-\ln\left((t_2-t_1)^2+\max\{\operatorname{sgn}((a_2-1)(a_2+1)),0\}\min\{|a_2-1|,|a_2+1|\}^2\right)}$$

$$\times \frac{8\sqrt{2}}{3} \times e^{-\left(\frac{z_1^2+z_2^2}{4(t_2-t_1)}+2\ln\left(\frac{1+(t_2-t_1)^2}{(t_2-t_1)^2}\right)\right)} \times \frac{8\sqrt{2}}{3}$$

Example 10.4. If we take $C_1 = B_1^{\delta_{t_1}}(0)$, $C_2 = B_1^{\delta_{t_1-t_2}}(0)$, where δ is related to $\psi(\xi,\eta) = |\xi|^2 + |\eta|$, we calculate δ as follows.

$$\begin{aligned} p_t(x_1,x_2) &= (2\pi)^{-(n_1+n_2))}\int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} e^{i(x_1,x_2)\cdot(\xi,\eta)}d\xi d\eta \\ &= (2\pi)^{-(n_1+n_2)}\int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} e^{-i(x_1\xi+x_2\eta)}e^{-t(|\xi|^2+|\eta|)}d\xi d\eta \\ &= (2\pi)^{-n_1}\int_{\mathbb{R}^{n_1}} e^{-ix_1\xi}e^{-t|\xi|^2}d\xi \cdot (2\pi)^{-n_2}\int_{\mathbb{R}^{n_2}} e^{-ix_2\eta}e^{-t|\eta|}d\eta \\ &= p_t(x_1)p_t(x_2) \end{aligned}$$

Since we know the δ values of Gaussian and Cauchy process from the paper [20], therefore we can deduce that

$$\delta(x_1,x_2) = \sqrt{\frac{x_1^2}{4t} - \ln(t^2) + \ln(t^2+x_2^2)},$$

$from (10.16), we have P^z(X_{t_1} \in C_1, X_{t_2} \in C_2) \geq$

$$\frac{1}{2\sqrt{\pi t_1}} \times \frac{1}{\pi} \times \frac{1}{t_1} \times \frac{1}{2\sqrt{\pi(t_2 - t_1)}} \times \frac{1}{\pi} \times \frac{1}{t_2 - t_1} \lambda^2(B_1^{\delta_{t_1}}(0)) \lambda^2(B_1^{\delta_{t_2 - t_1}}(0))$$

and

$P^z(X_{t_1} \in C_1, X_{t_2} \in C_2) \leq$

$$\frac{1}{2\sqrt{\pi t_1}} \times \frac{1}{\pi} \times \frac{1}{t_1} \times \frac{1}{2\sqrt{\pi(t_2 - t_1)}} \times \frac{1}{\pi} \times \frac{1}{t_2 - t_1} e^{-1} \lambda^2(B_1^{\delta_{t_1}}(0)) e^{-1} \lambda^2(B_1^{\delta_{t_2 - t_1}}(0))$$

Since the volume of the metric balls are hard to calculate even using Mathematica, we can not calculate the exact estimates of the above.

Chapter 11

Some Graphical Experiments

We have seen that in many cases of Levy processes there is a natural geometric interpretation of the transition functions involving two $t-$dependent families of metrics. These metrics are in general non-isotropic.

In case we are dealing with Markov processes generated by non-translation invariant operators, i.e. by pseudo-differential operators having an $x-$dependent symbol, in general we do have neither explicit formula for the transition function nor so far a geometric interpretations of these transition functions. However, using some symbolic calculus or other approximation procedures, compare [29] or [6], it is possible for small time and locally with respect to some point x_0 in the state space to compare such transition function with those of the corresponding Levy process obtained when freezing the coefficients of the generator, i.e. the $x-$dependence, at x_0. Thus one might have the idea to approximate a probability such as

$$P^{x_0}(X_{t_1} \in G_1, X_{t_2} \in G_2)$$

by probabilities calculated with the help of transition functions of certain Levy processes. Thus it is a natural question to study, i.e. to compare the transition functions (and the underlying metrics) of Levy processes obtained by freezing the coefficients of a given generator at different point, i.e. to compare transition functions

$$P_t^{x_j}(x-y) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i(x-y)\xi} e^{-tq(x^j,\xi)} d\xi$$

for a sequence $(x_j)_{j\in\mathbb{N}}$ in state space.

We know that in general we have to handle non-isotropicc operators (metrics, transition functions) and hence it is important to have some knowledge how the non-isotropy develops when x_j runs through a set of points.

In order to have concrete functions and an affordable amount of calculations we restrict the case study to the symbol

$$q(x, y, \xi, \eta) = a(y)|\xi|^2 + b(x)|\eta|$$

where $x, y \in \mathbb{R}$ as are $\xi, \eta \in \mathbb{R}$. Freezing the coefficients leads to consider the characteristic functions

$$a(y_0)|\xi|^2 + b(x_0)|\eta|$$

and the corresponding Levy processes. In this case we find

$$p_t^{(x_0,y_0)}(x_1 - x_2, y_1 - y_2) = p_t^{(x_0,y_0)}(0,0)e^{-\delta^2_{t,(x_0,y_0)}((x_1,y_1),(x_2,y_2))}$$

where

$$p_t^{(x_0,y_0)}(0,0) = \frac{1}{2\pi^{3/2}t^{3/2}\sqrt{a(y_0)}\sqrt{b(x_0)}}$$

with corresponding metric $d_{t,\psi}^{(x_0,y_0)}((\xi_1, \eta_1), (\xi_2, \eta_2)) := \sqrt{t\psi(\xi - \eta)}$ and

$$\delta^2_{t,(x_0,y_0)}((x_1, y_1), (x_2, y_2)) = \frac{(x_2 - x_1)^2}{4ta(y_0)} - \ln(t^2 b(x_0)) + \ln(t^2 b(x_0)^2 + (y_2 - y_1)^2)$$

The following plots show the balls with respect to the metric $d_{t,\psi}^{(x_j,y_j)}$, the balls with respect to the metric $\delta_{t,\psi}^{(x_j,y_j)}$ and the transition function $p_t^{(x_j,y_j)}$ for different values of t and x_0, y_0.

In order to have a good understanding of the Levy-type transition function $p(x_1, x_2) = p(0,0)e^{-\delta_t^2(x_1,x_2)}$, we need to understand the metric ball

$$B^{\psi}_{\sqrt{t}}(0) = \{(\xi_1, \xi_2) \in \mathbb{R}^2 \mid a(y_0)|\xi_1|^2 + b(x_0)|\xi_2| \le t\}$$

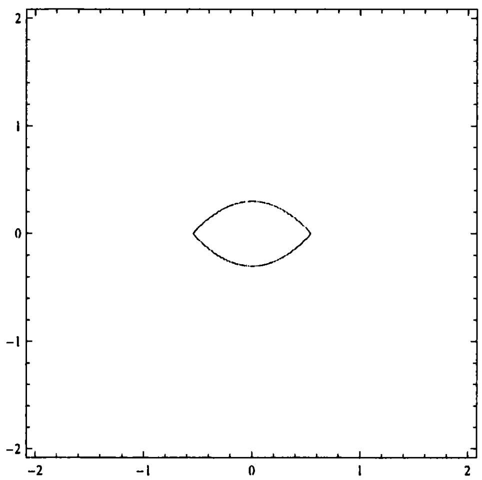
and the metric ball

$$B_{\delta_t(x_0,x_0)}(0, 1) = \{(\eta_1, \eta_2) \in \mathbb{R}^2 \mid \frac{\eta_1^2}{4ta(y_0)} - \ln(t^2 b(x_0)) + \ln(t^2 b(x_0)^2 + \eta_2^2) \le 1\}.$$

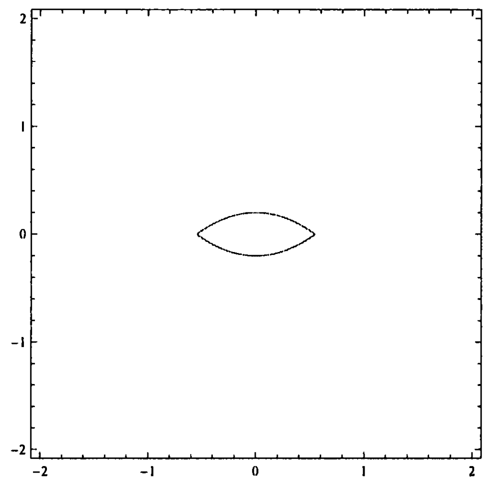
The first set of plots shows $B^{\psi}_{\sqrt{t}}(0)$ for $t = 0.3$, $a = 1$ and b running through a set of values for b, $b = 1, 1.5, 2, 2.5, 3, 3.5$. The second set of plots shows $B^{\psi}_{\sqrt{t}}(0)$ for $t = 0.3$, $b = 1$ and a set of values for a, $a = 1, 1.5, 2, 2.5, 3, 3.5$.

Remark 11.1. In the following graphics we replace $a(y_0)$ and $b(x_0)$ by a and b since we want to see the effects of isotropy and simplify the notation.

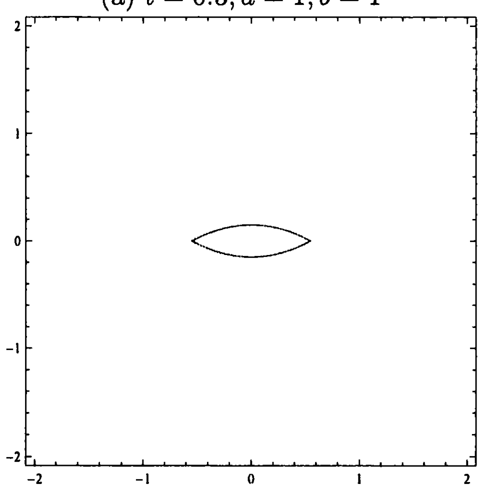
First Set of Plots for $B_t^{\psi}(0)$



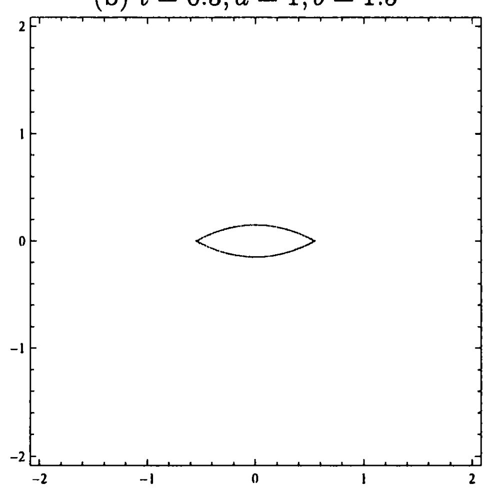
(a) $t = 0.3, a = 1, b = 1$



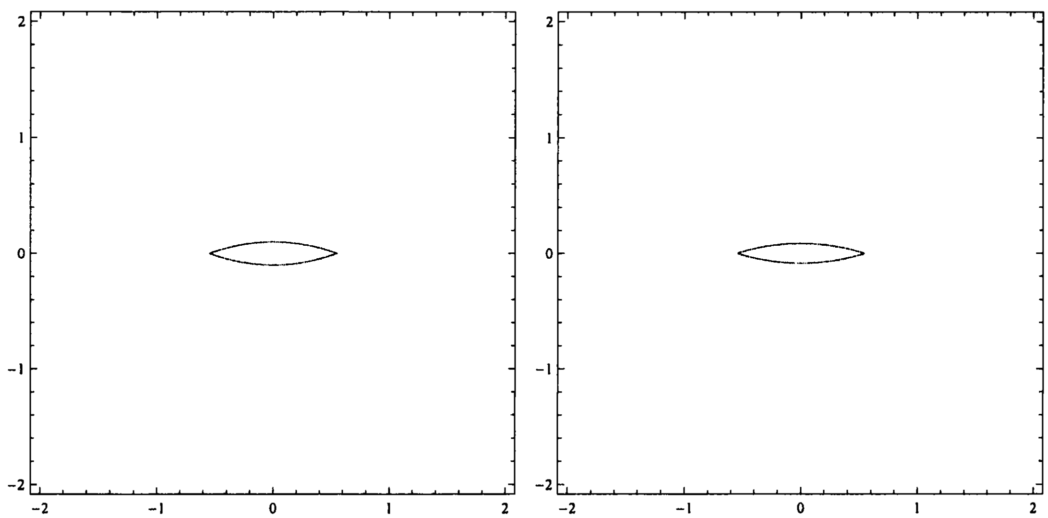
(b) $t = 0.3, a = 1, b = 1.5$



(c) $t = 0.3, a = 1, b = 2$



(d) $t = 0.3, a = 1, b = 2.5$

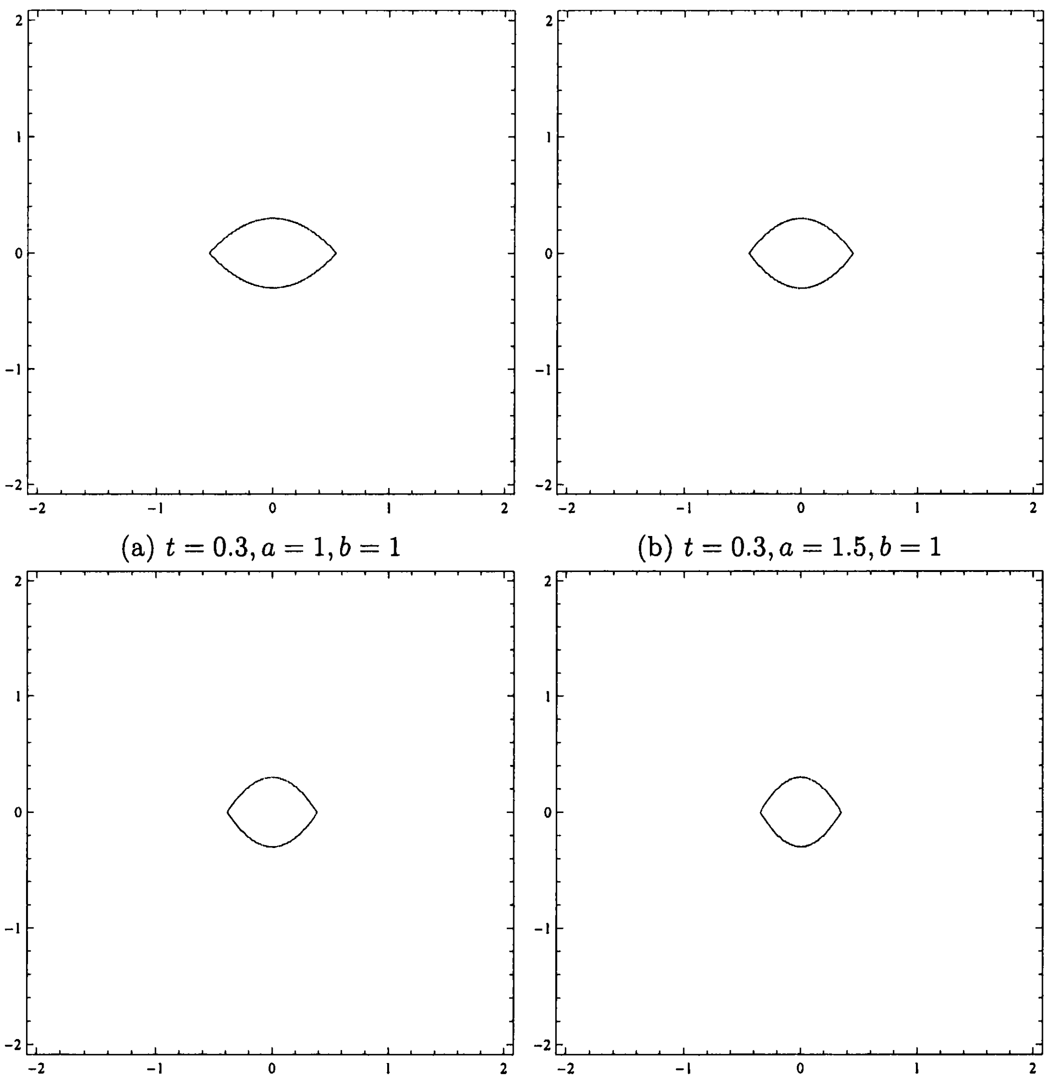


(e) $t = 0.3, a = 1, b = 3$

(f) $t = 0.3, a = 1, b = 3.5$

Figure 11.1: $t = 0.3$, $a = 1$, b takes the values from 1 to 3.5 in the interval of 0.5

Second Set of Plots for $B_t^{\psi}(0)$

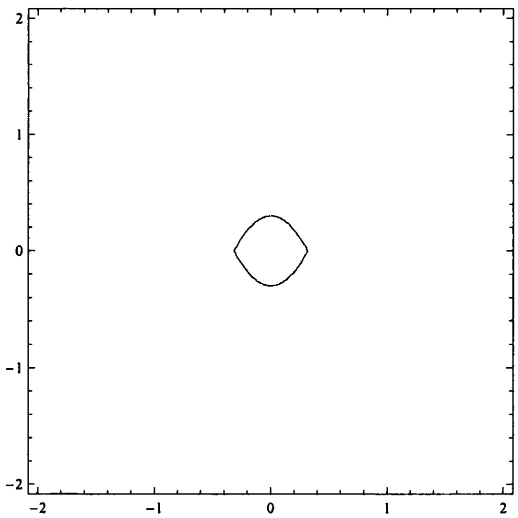


(a) $t = 0.3, a = 1, b = 1$

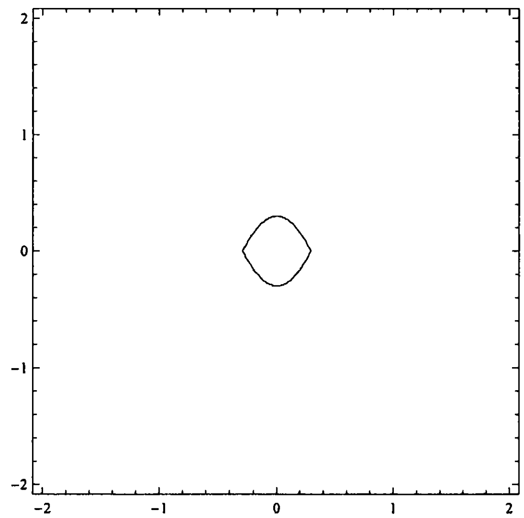
(b) $t = 0.3, a = 1.5, b = 1$

(c) $t = 0.3, a = 2, b = 1$

(d) $t = 0.3, a = 2.5, b = 1$



(e) $t = 0.3, a = 1, b = 3$



(f) $t = 0.3, a = 1, b = 3.5$

Figure 11.2: $t = 0.3$, $b = 1$, a takes the values from 1 to 3.5 in the interval of 0.5

The second set of plots for $B_t^{\psi}(0)$ shows the more interesting phenomenon, namely a more visible change of the non-isotropic character of the ball $B_t^{\psi}(0)$. Note that we can consider each plot as belonging to one fixed Levy process, but we can also consider the series as change of local approximation for the transition function associated with a generator $-q(x, y, D_x, D_y)$ with symbol $q(x, y, \xi, \eta) = a(y)|\xi|^2 + tb(x)|\eta|$ where a and b are nice functions taking the appropriate values for $x, y \in \{1, 1.5, 2, 2.5, 3, 3.5\}$.

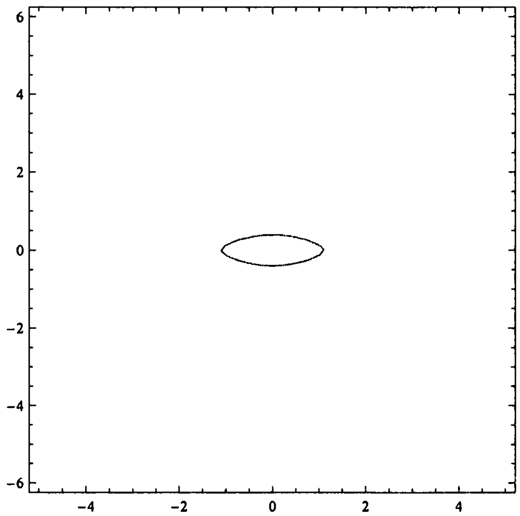
Now we turn to the ball $B_{\delta_t(x_0,y_0)}(0,0)$ where

$$\delta_t((x_0, y_0), (0,0)) = \sqrt{\frac{x_1^2}{4ta(y_0)} - \ln(t^2 b(x_0)^2) + \ln(t^2 b(x_0)^2 + x_2^2)}. \tag{11.1}$$

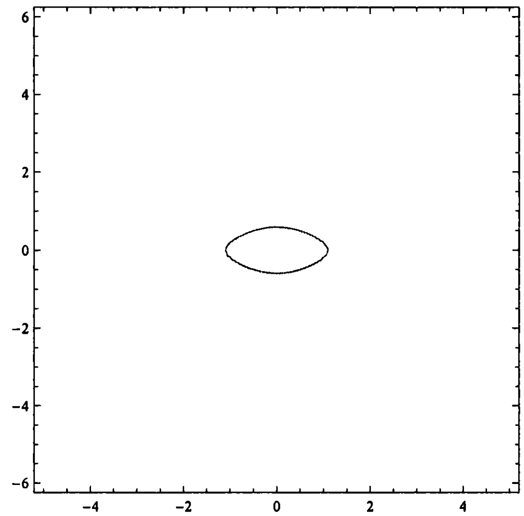
The first set of plots refer to $t = 0.3, a = 1$ and $b \in \{1, 1.5, 2, 2.5, 3, 3.5\}$, i.e. this corresponds to the first set of values consider for $B_t^{\psi}(0)$. For the second set of plots in this case we have chosen $t = 0.5$, $a = 1$ and $b \in \{1, \dots, 3.5\}$; a third set is considered with $t = 0.75$, $a = 1$ and $b \in \{1, \dots, 3.5\}$. Especially the second and the third series of plots show very clearly how the non-isotropy of the balls change. Next we consider the some three values of t, i.e. $t = 0.3, 0.5$ and 0.75, we fix $b = 1$ and let run now a through the set $\{1, \dots, 3.5\}$.

Having in mind for example the estimates in the last chapter, we get a better understanding by these plots to which extent the non-isotropy geometry will affect the probabilities.

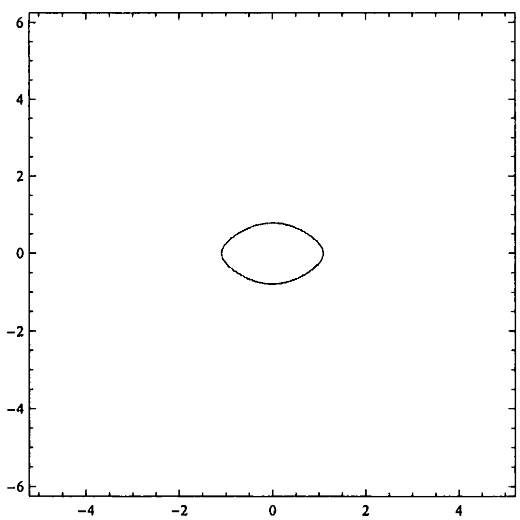
First Set of Plots for B_{δ_t}



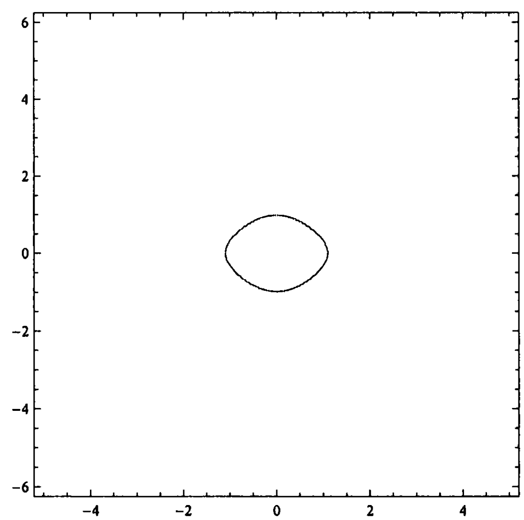
(a) $t = 0.3, a = 1, b = 1$



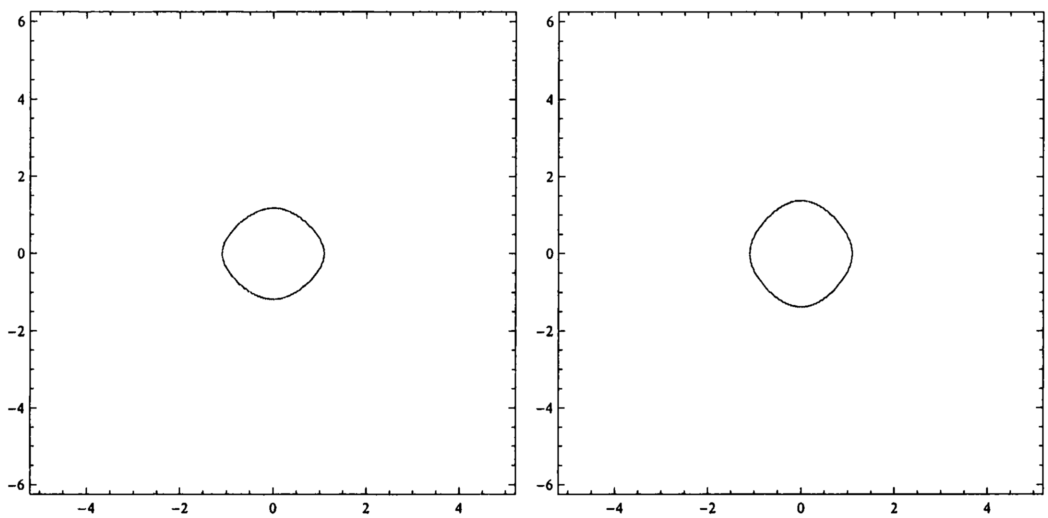
(b) $t = 0.3, a = 1, b = 1.5$



(c) $t = 0.3, a = 1, b = 2$



(d) $t = 0.3, a = 1, b = 2.5$

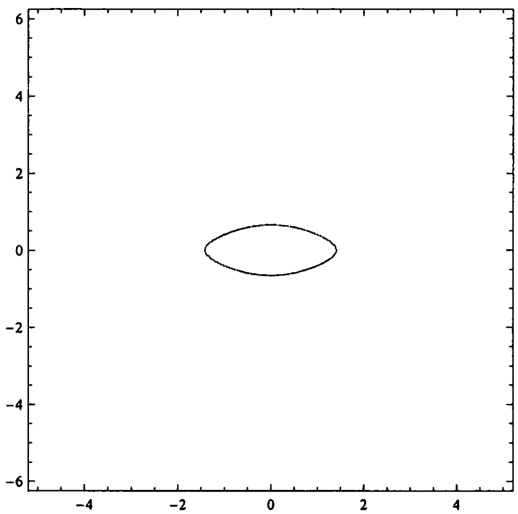


(e) $t = 0.3, a = 1, b = 3$

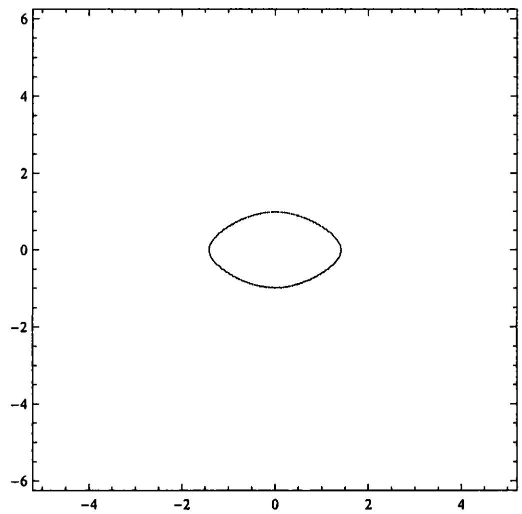
(f) $t = 0.3, a = 1, b = 3.5$

Figure 11.3: $t = 0.3$, $a = 1$, b changes from 1 to 3.5 in the interval of 0.5

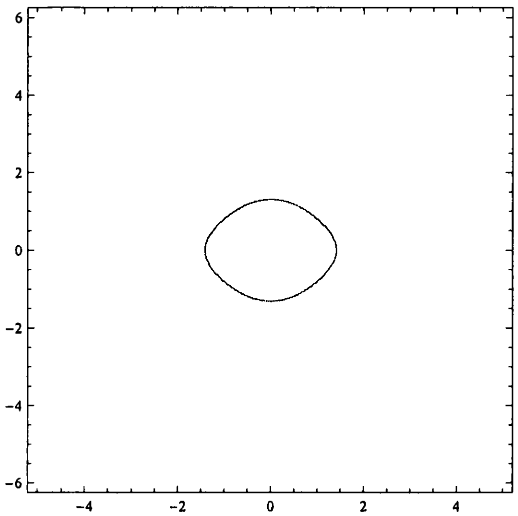
Second Set of Plots for B_{δ_t}



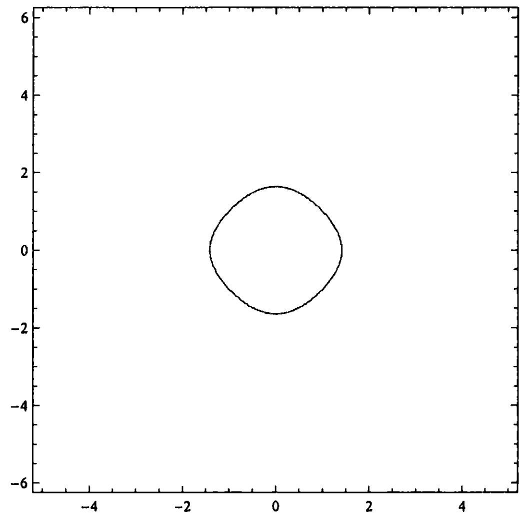
(a) $t = 0.5, a = 1, b = 1$



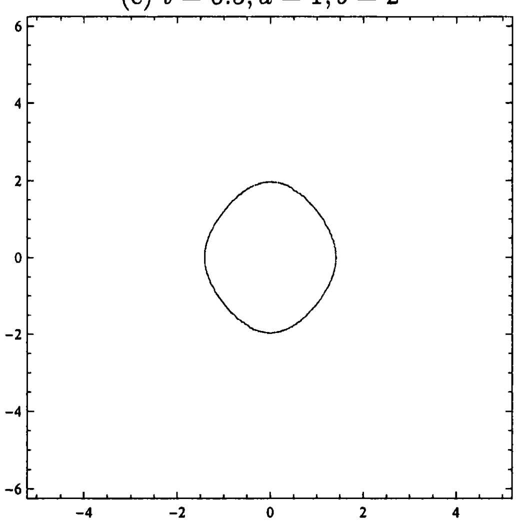
(b) $t = 0.5, a = 1, b = 1.5$



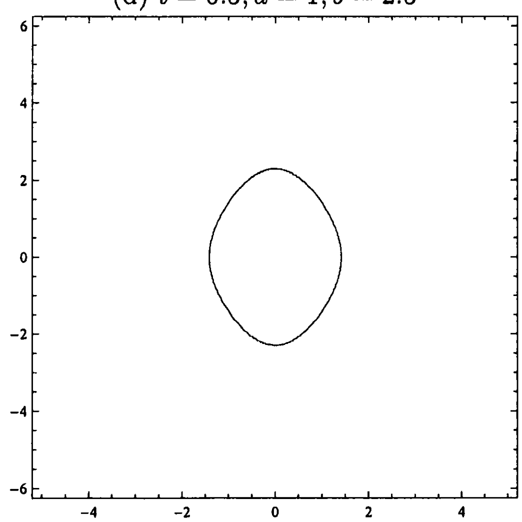
(c) $t = 0.5, a = 1, b = 2$



(d) $t = 0.5, a = 1, b = 2.5$



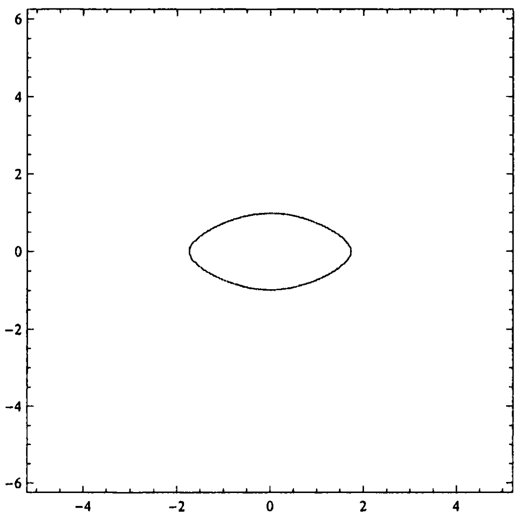
(e) $t = 0.5, a = 1, b = 3$



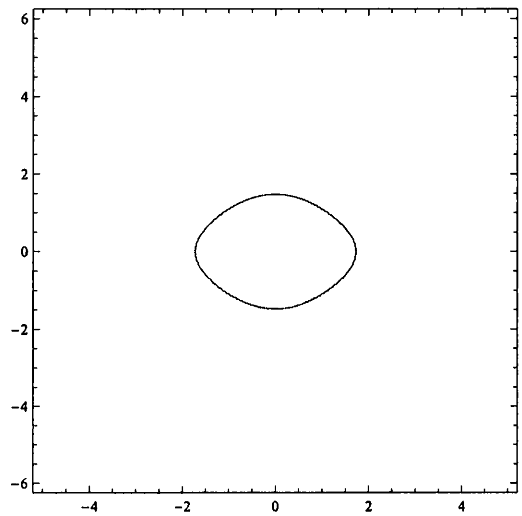
(f) $t = 0.5, a = 1, b = 3.5$

Figure 11.4: $t = 0.5$, $a = 1$, b changes from 1 to 3.5 in the interval of 0.5

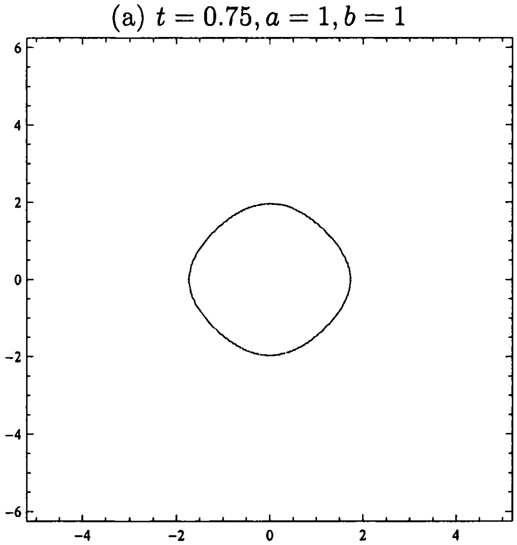
Third Set of Plots for B_{δ_t}



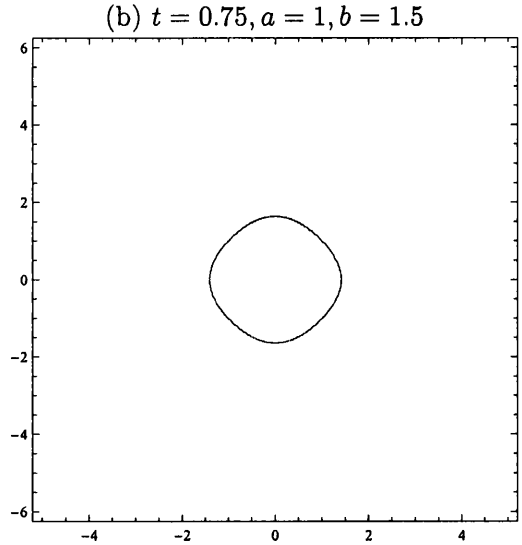
(a) $t = 0.75, a = 1, b = 1$



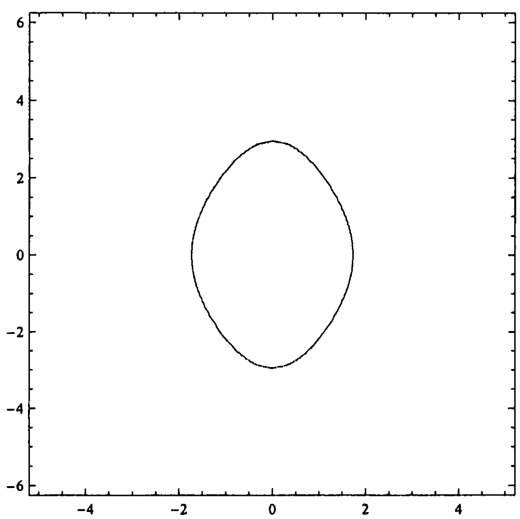
(b) $t = 0.75, a = 1, b = 1.5$



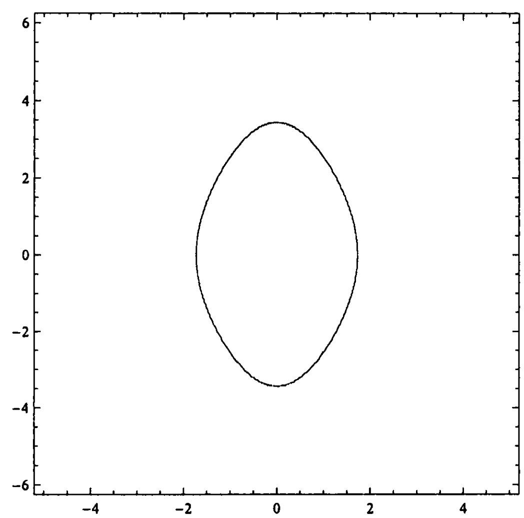
(c) $t = 0.75, a = 1, b = 2$



(d) $t = 0.75, a = 1, b = 2.5$



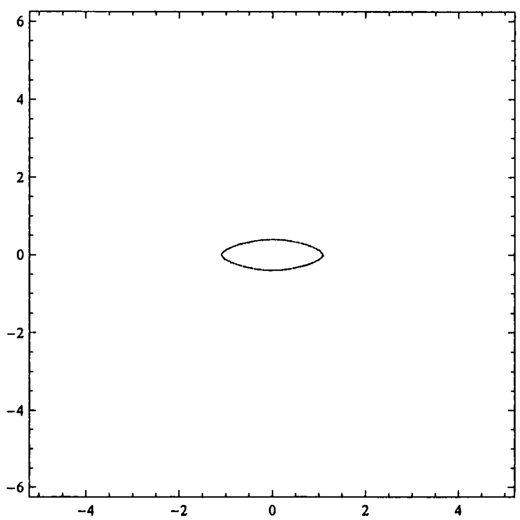
(e) $t = 0.75, a = 1, b = 3$



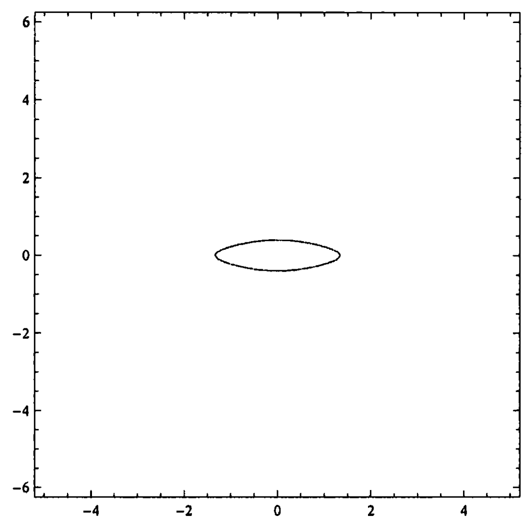
(f) $t = 0.75, a = 1, b = 3.5$

Figure 11.5: $t = 0.75$, $a = 1$, b changes from 1 to 3.5 in the interval of 0.5

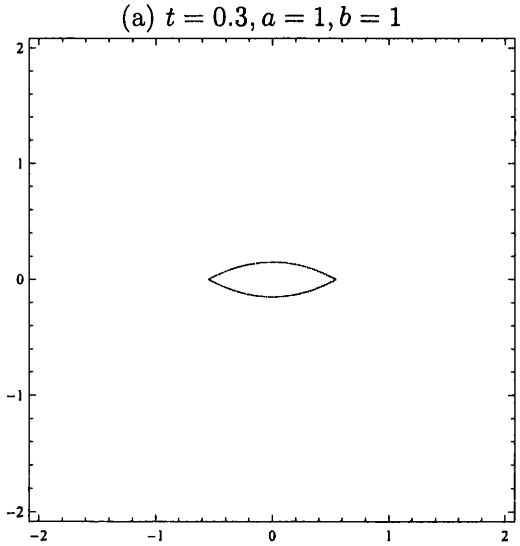
Fourth Set of Plots for B_{δ_t}



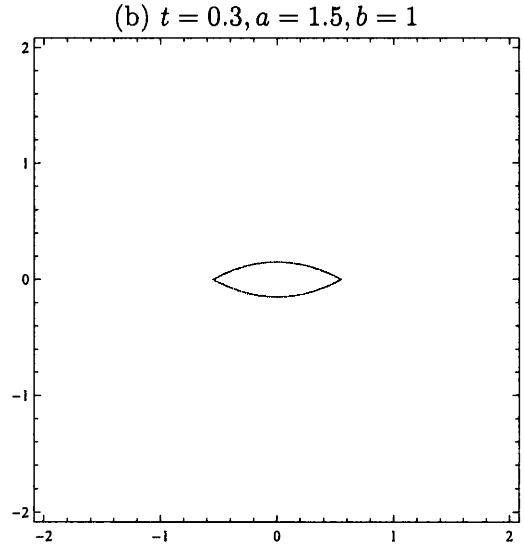
(a) $t = 0.3, a = 1, b = 1$



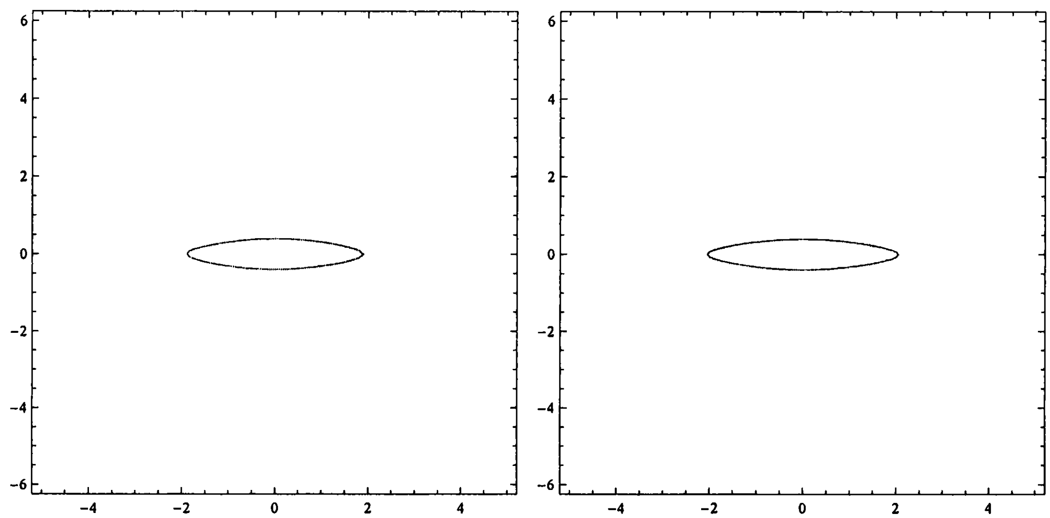
(b) $t = 0.3, a = 1.5, b = 1$



(c) $t = 0.3, a = 2, b = 1$



(d) $t = 0.3, a = 2.5, b = 1$

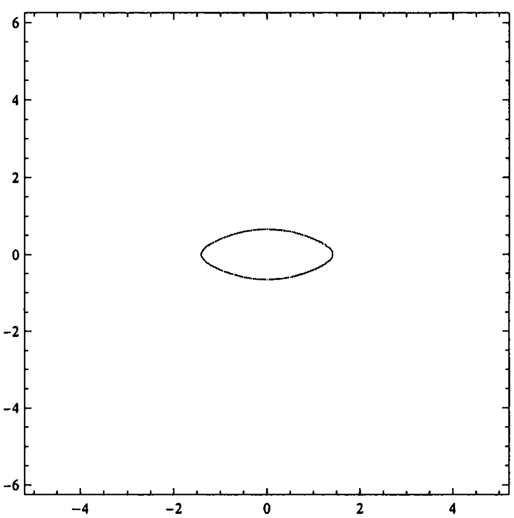


(e) $t = 0.3, a = 3, b = 1$

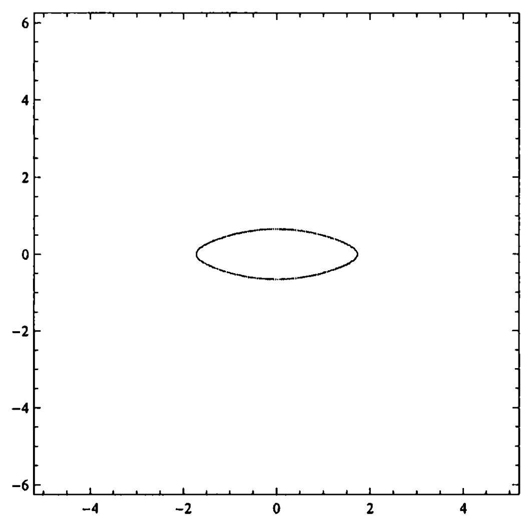
(f) $t = 0.3, a = 3.5, b = 1$

Figure 11.6: $t = 0.3$, $b = 1$, a varies from 1 to 3.5 in the interval of 0.5

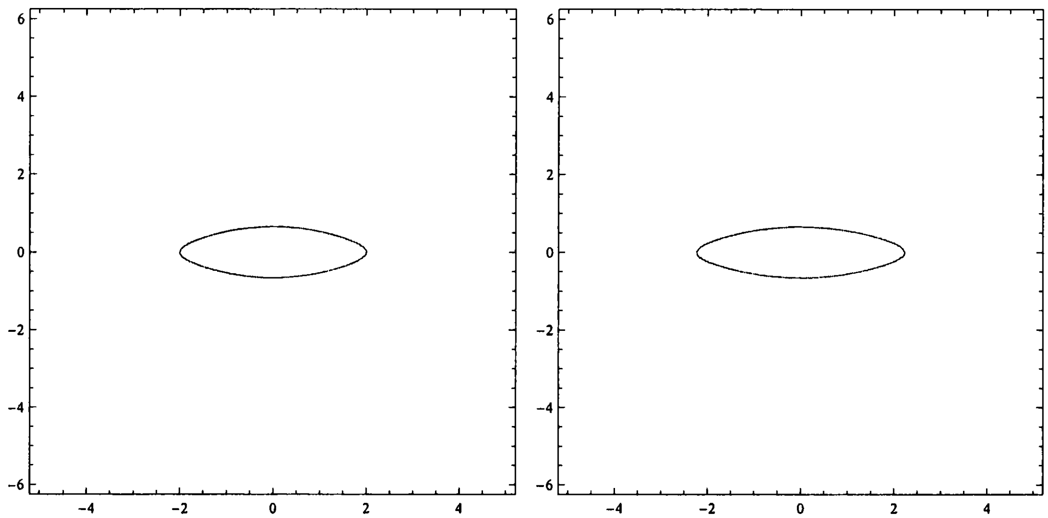
Fifth Set of Plots for B_{δ_t}



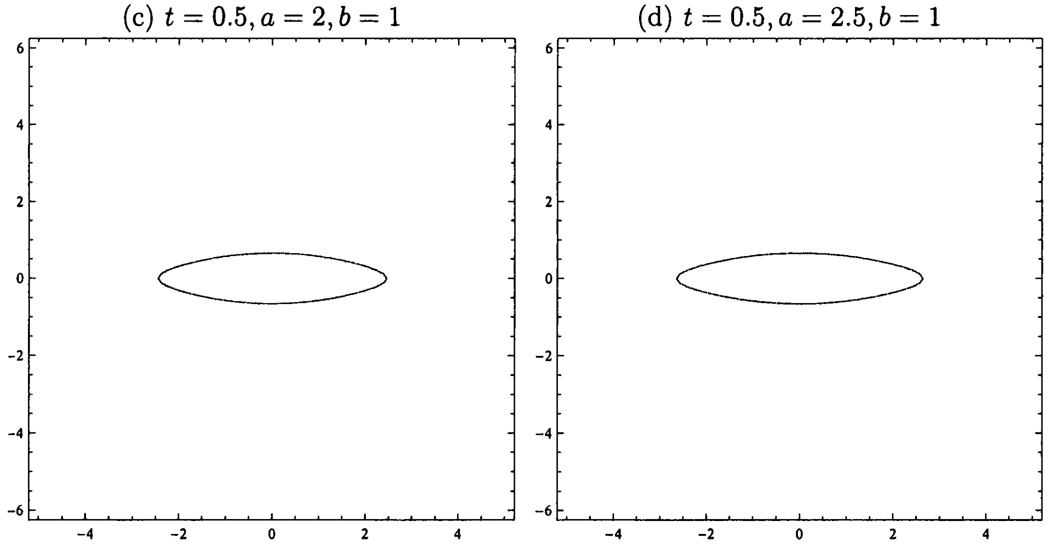
(a) $t = 0.5, a = 1, b = 1$



(b) $t = 0.5, a = 1.5, b = 1$



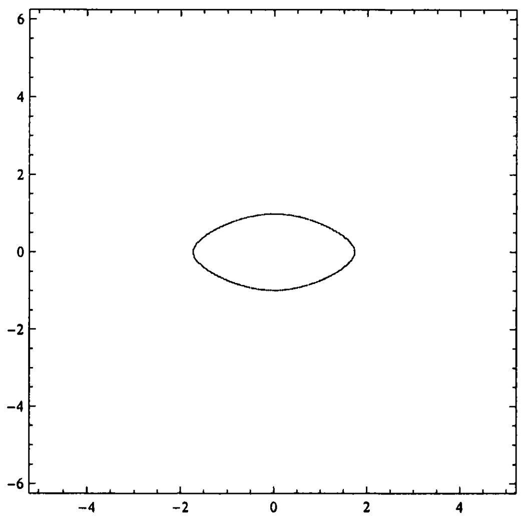
(c) $t = 0.5, a = 2, b = 1$ (d) $t = 0.5, a = 2.5, b = 1$



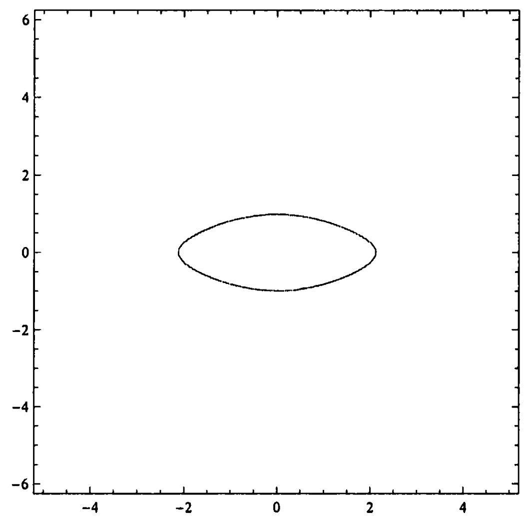
(e) $t = 0.5, a = 3, b = 1$ (f) $t = 0.5, a = 3.5, b = 1$

Figure 11.7: $t = 0.5$, $b = 1$, a varies from 1 to 3.5 in the interval of 0.5

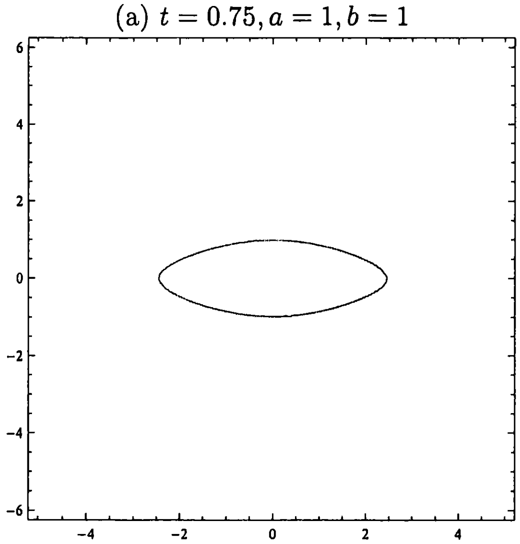
Sixth Set of Plots for B_{δ_t}



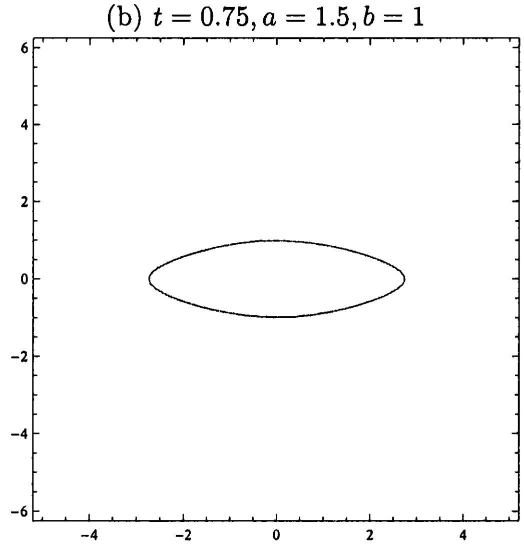
(a) $t = 0.75, a = 1, b = 1$



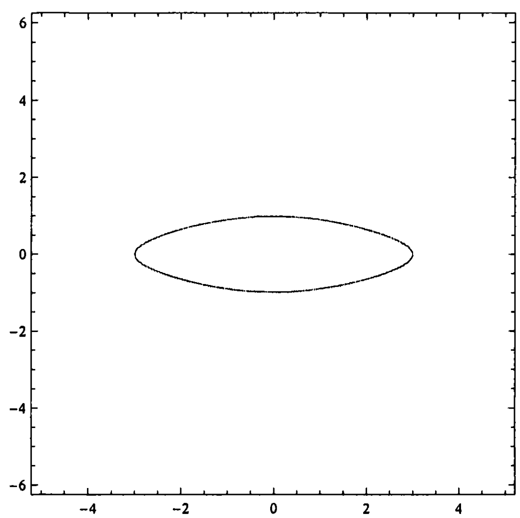
(b) $t = 0.75, a = 1.5, b = 1$



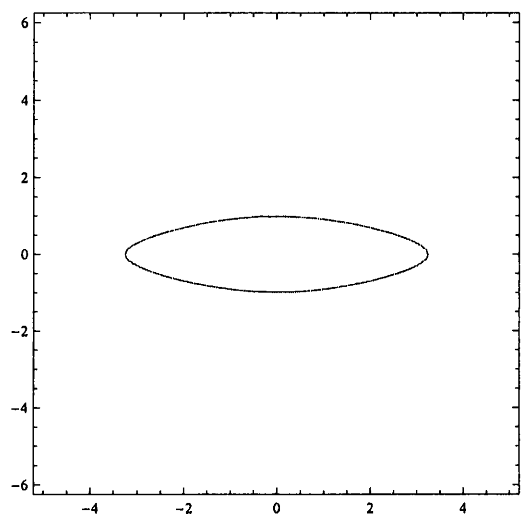
(c) $t = 0.75, a = 2, b = 1$



(d) $t = 0.75, a = 2.5, b = 1$



(e) $t = 0.75, a = 3, b = 1$



(f) $t = 0.75, a = 3.5, b = 1$

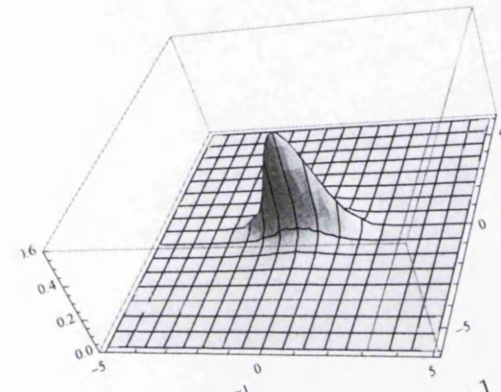
Figure 11.8: $t = 0.75$, $b = 1$, a varies from 1 to 3.5 in the interval of 0.5

We come to the graphic study of $p(x_1, x_2)$. We know

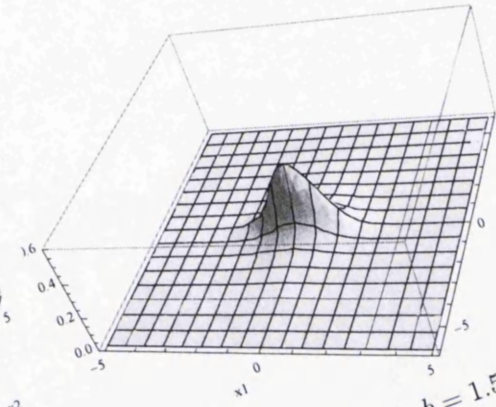
$$p_t(x_1, x_2) = \frac{1}{2\pi t b\sqrt{\pi a t}} \exp\left(-\frac{x_1^2}{4ta} + \ln(t^2b^2) - \ln(t^2b^2 + x_2^2)\right)$$

The following plots corresponding in the first set to $t = 0.3$, $a = 1$ and $b \in \{1, \ldots, 3.5\}$, while in the second set to $t = 0.3, b = 1$ and $a \in \{1, \ldots, 3.5\}$. It is the first series which shows best character the non-isotropy may develop.

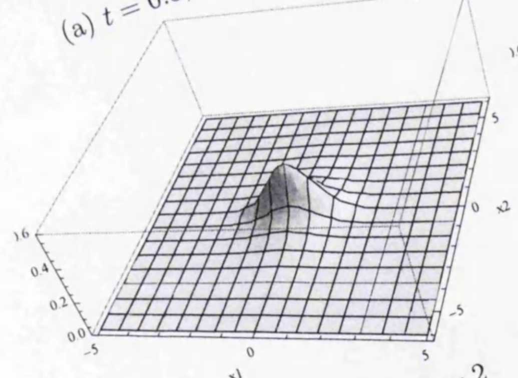
First Set of Plots for $p_t(x_1, x_2)$



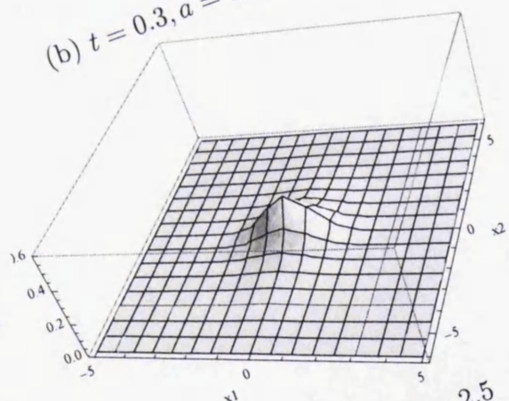
(a) $t = 0.3, a = 1, b = 1$



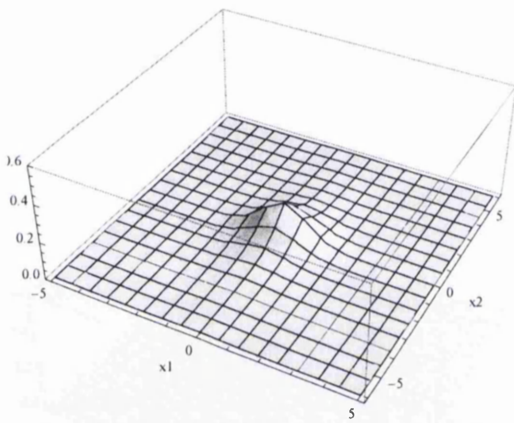
(b) $t = 0.3, a = 1, b = 1.5$



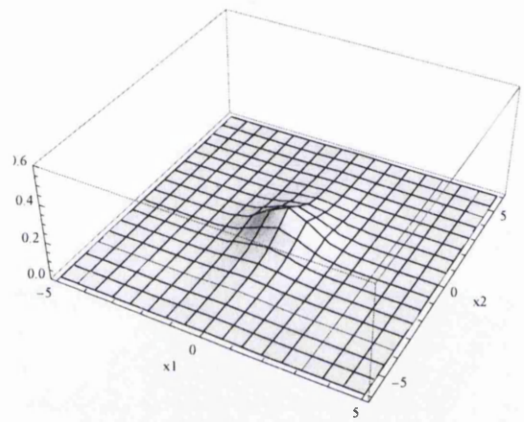
(c) $t = 0.3, a = 1, b = 2$



(d) $t = 0.3, a = 1, b = 2.5$



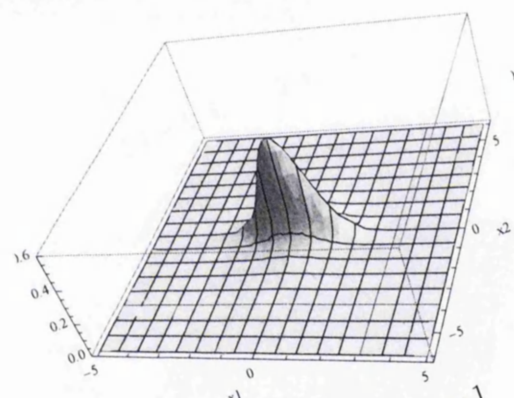
(e) $t = 0.3, a = 1, b = 3$



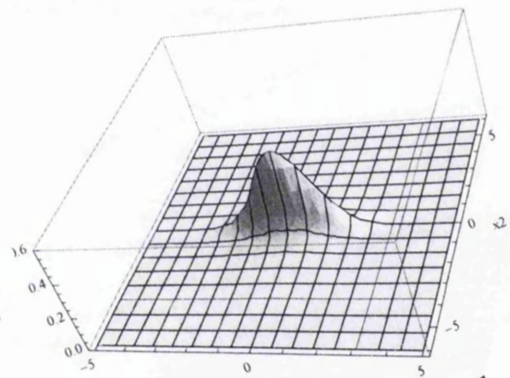
(f) $t = 0.3, a = 1, b = 3.5$

Figure 11.9: $t = 0.3$, $a = 1$, b changes from 1 to 3.5 in the interval of 0.5

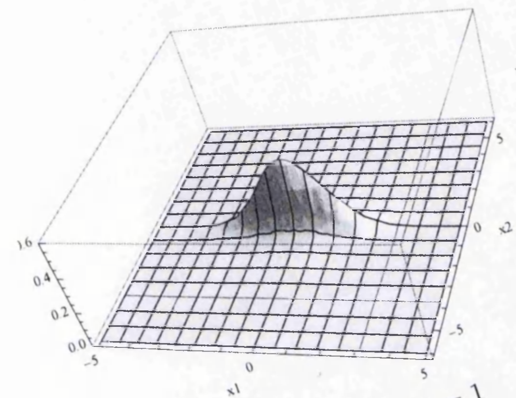
Second Set of Plots for $p_t(x_1, x_2)$



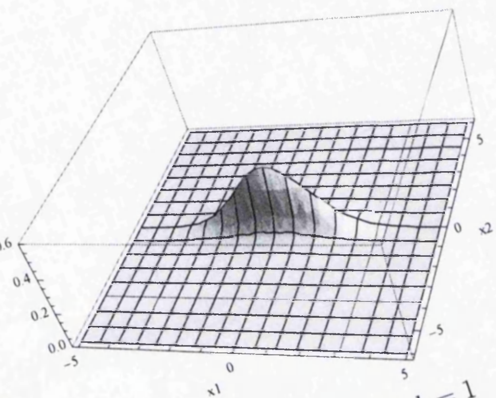
(a) $t = 0.3, a = 1, b = 1$



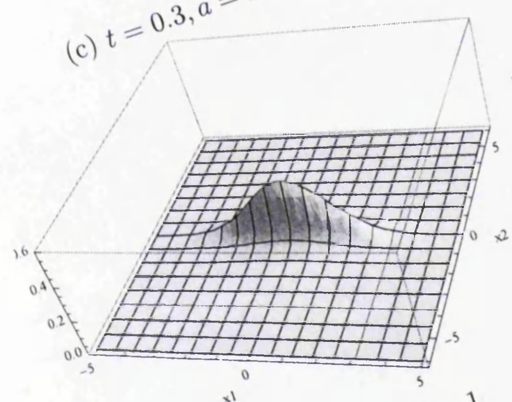
(b) $t = 0.3, a = 1.5, b = 1$



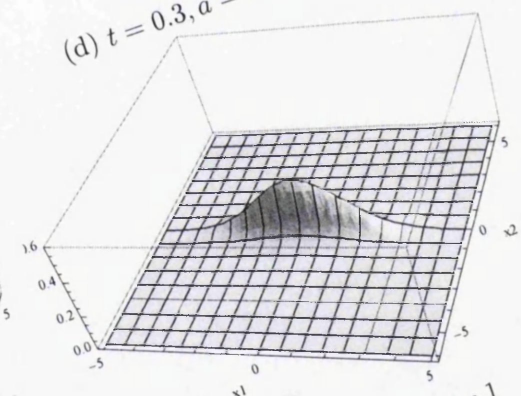
(c) $t = 0.3, a = 2, b = 1$



(d) $t = 0.3, a = 2.5, b = 1$



(e) $t = 0.3, a = 3, b = 1$



(f) $t = 0.3, a = 3.5, b = 1$

Figure 11.10: $t = 0.3$, $b = 1$, a changes from 1 to 3.5 in the interval of 0.5

Chapter 12

Towards a Study of Transition Functions for Levy-type Processes

Before we start to address how we may study and interpret the transition function of a stochastic process generated by a pseudo-differential operator with variable coefficients, i.e. an operator $-q(x, D)$ with a symbol $q(x, \xi)$ which is with respect to ξ a continuous negative definite function and not constant with respect to x, we need to collect more information on the translation invariant case. In this case $q(x, \xi) = \psi(\xi)$ is just a continuous negative definite function. As such ψ need not be smooth, take as example the function $\psi(\xi, \eta) = |\xi|^\alpha + |\eta|^\beta$, $0 < \alpha < \beta \leq 2$, $\xi \in \mathbb{R}^n$, $\eta \in \mathbb{R}^m$. Further ψ need not be convex and in addition, in general we can not decompose in a series of homogeneous terms, and even in case where this is possible, the highest order form does in general not provide us with a reasonable principal symbol. Just have a look at $\psi(\xi, \eta) = |\xi|^\alpha + |\eta|^\beta$, $0 < \alpha < \beta \leq 2$, $\xi \in \mathbb{R}^n$, $\eta \leq \mathbb{R}^m$. The leading homogeneous term is $|\eta|^\beta$, but it does not give any control on the $\xi-$dependence. This example highlights also a further problem, namely the fact that in general a continuous negative definite function is non-isotropic.

Thus the idea to transfer "standard" techniques used to handle elliptic operators, i.e. operators with a well-defined homogeneous, isotropic principle symbol which is eventually smooth (everywhere but at the origin) and which controls the full symbol, hence the operator, this idea does not look promising, not even in case of translation invariant operators, i.e. generators of Levy processes. In fact already the definition of the spaces $H^{\psi,s}(\mathbb{R}^n)$ takes this into account, we use the full symbol, not a part of the symbol. Note that we may decompose ψ according to $\psi = \psi_1 + \psi_2$ and assume that $\lim_{|\xi|\to\infty} \frac{\psi_2(\xi)}{\psi_1(\xi)} = 0$.

In this case $H^{\psi_1,s}(\mathbb{R}^n)$ and the two norm $\|\cdot\|_{\psi_1,s}$ and $\|\cdot\|_{\psi,s}$ are equivalent, however this decomposition is in general not unique.

Further, when looking at symbols depending on x, i.e. $q(x,\xi)$, for different $x_1, x_2 \in \mathbb{R}^n$ the symbols $q(x_1,\xi)$ and $q(x_2,\xi)$ with "frozen" coefficients may have different order which need not be varied in discrete steps, consider the symbol $q(x,\xi) = |\xi|^{\alpha(x)}$, $0 < \alpha(x) \leq 2$ for all $x \in \mathbb{R}^n$. Thus, even in case we can develop a "good" theory for the operator $-\psi(D)$, and the associated semigroup and stochastic process, we can not expect to transfer it to operators $-q(x,D)$ and their associated semigroups and processes since the properties of $-q(x,D)$ and $-q(x_2,D)$ may change and become quite unrelated. Subordination of variable order, i.e. symbols of type of $f(x,\psi(\xi))$ or even $f(x,q(x,\xi))$ give examples, compare [6] or [7] . Here for every x the function $r \mapsto f(x,r)$ is supposed to be a Bernstein function.

Despite all these obstacles, there are some thoughts we may follow up and for this we first want to sketch some basic ideas, some of which we indicated already in Chapter 7. Let $q(x,\xi)$ be a negative definite symbol, i.e. for every $x \in \mathbb{R}^n$ fixed $\xi \mapsto q(x,\xi)$ is a continuous negative definite function. Suppose $-q(x,D)$ is (or extends to) a generator of a Markov or Feller semigroup $(T_t)_{t\geq 0}$ and assume that T_t has a density, i.e.

Denote the corresponding process by $((\tilde{X}_t)_{t\geq 0}, P^z)_{z\in\mathbb{R}^n}$. Thus

$$P^z(\tilde{X}_t \in C) = T_t\chi_C(z) = \tag{12.1}$$
$$= \int_{\mathbb{R}^n} \chi_C(y) p_t(z,y)\,dy = \int_C p_t(z,y)\,dy.$$
$$= \int_{\mathbb{R}^n} \chi_C(y) p_t(z,y)\,dy = \int_C p_t(z,y)\,dy.$$

Next we freeze the coefficient of $q(x,D)$ at $x \in \mathbb{R}^n$, hence we obtain a continuous negative definite function $\psi_x(\xi) = q(x,\xi)$. Denote by $((Y_t^x)_{t\geq 0}, \Pi_x^z)_{z\in\mathbb{R}^n}$ the corresponding Levy process i.e.

$$E^z(e^{iY_t^x\cdot\xi}) = (2\pi)^{-n/2} \int e^{iw\cdot\xi}\Pi^z_{x,Y_t^x}(dw) \tag{12.2}$$
$$= (2\pi)^{-n/2} \int e^{i(w-z)\cdot\xi}\mu_t(dw)$$

with

$$\hat{\mu}_t(\xi) = e^{-tq(x,\xi)}. \tag{12.3}$$

Here Π^z_{x,Y^x_t} denotes the distribution of Y^x_t under Π^z_x. Thus, from (12.3) we deduce for the density p^x_t of $(S^x_t)_{t\geq 0}$, where

$$S^x_t u(z) = \int_{\mathbb{R}^n} u(z-w)\mu_t(dw) = \int_{\mathbb{R}^n} u(w)p^x_t(z-w)dw, \tag{12.4}$$

that

$$p^x_t(y) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{iy\xi}e^{-tq(x,\xi)}d\xi. \tag{12.5}$$

We may agree that the process $((\tilde{X}_t)_{t\geq 0}, P^z)_{z\in\mathbb{R}^n}$ and the processes $((Y^x_t)_{t>0}, \Pi^z_x)_{z\in\mathbb{R}^n}$ are all defined on the Skorohod space $\mathcal{D}$ and that the random variables are just the projections, which allows us to write X_t for $\tilde{X}_t$ as well as for Y^x_t, $t > 0$. Thus we only need to deal with the probability measures P^z and Π^z_x defined on $\mathcal{D}$.

Following B.Böttcher, compare [4] and [5], see also Chapter 7, we know that under suitable conditions on the symbol $q(x,\xi)$ we have for the symbol $\sigma(T_t)(x,\xi)$ of the semigroup generated by $-q(x,D)$ the asymptotic formula

$$\sigma(T_t)(x,\xi) = e^{-tq(x,\xi)} + r_0(t,0;x,\xi) \tag{12.6}$$

with $r_0(t,0;x,\xi)\to 0$ as $t\to 0$, weakly in some locally convex topology induced on an appropriate symbol class.

Thus we may try to get an approximation for $p_t(z,y)$ by $p^x_t(z-y)$. We may even think P^z, $z\in\mathbb{R}^n$ fixed, "embedded" into a field of probability measures $(\Pi^z_x)_{x\in\mathbb{R}^n}$. While Böttcher's result is precise it does not allow direct calculations of approximate probabilities so far. Nonetheless the idea is striking to find for example

$$P^z(X_{t_1}\in C_1, X_{t_2}\in C_2) \tag{12.7}$$

by using the probabilities $(\Pi^z_x)_{x\in\mathbb{R}^n}$ and we want now explore on an heuristical level how this may work out.

The starting point is of course

$$\begin{aligned} P^z(X_{t_1}\in C_1, X_{t_2}\in C_2) &= T_{t_1}X_{C_1}(T_{t_2-t_1}\chi_{C_2})(z) \\ &= \int_{\mathbb{R}^n} p_{t_1}(z,y)\chi_{C_1}(y)\int_{\mathbb{R}^n} p_{t_2-t_1}(y,w)\chi_{C_2}(w)dwdy \\ &= \int_{C_1} p_{t_1}(z,y)\int_{C_2} p_{t_2-t_1}(y,w)dwdy. \end{aligned} \tag{12.8}$$

Clearly, we have to assume t_1 and t_2-t_1 to be small, and Böttcher's result also suggest that $|x-y|$ and $|y-w|$ are small. However, y varies in C_1 and w in C_2.

Suppose we can get a good approximation (given some error bound ϵ) where $|z-y|$ and $|y-w|$ are small. Thus assume the existence of $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$ such that

(12.9)

$$\max\left(\sup\{|a-y| \mid y \in C_1\}, |a-z|, \sup\{|b-y| \mid y \in C_1\}, \sup\{|b-w| \mid w \in C_2\}\right) < \delta$$

implies we get a good error bound, then we may replace

$$\int_{C_1} p_{t_1}(z,y) \int_{C_2} p_{t_2-t_1}(y,w)\, dw dy \tag{12.10}$$

by

$$\int_{C_1} p^a_{t_1}(z-y) \int_{C_2} p^b_{t_2-t_1}(y-w)\, dw dy. \tag{12.11}$$

Before we discuss further the problems of getting estimates, let us discuss (12.11) in light of our non-isotropy problems and the geometric interpretation of the transition function of a Levy process.

Suppose that the transition density of a Levy process is given by

$$p_t(x-y) = p_t(0) e^{-\delta_t^2(x,y)} \tag{12.12}$$

with a metric δ_t on $\mathbb{R}^n$. Due to the translation invariance, in the following it is sufficient to restrict ourselves to the case $x \in \mathbb{R}^n$ and $y = 0$. Since

$$\int_{\mathbb{R}^n} e^{-\delta_t^2(x,0)} dx = \frac{1}{p_t(0)} \tag{12.13}$$

we know that given $\epsilon > 0$ there exists $R = R(\epsilon)$ such that

$$\int_{B_R^c(0)} e^{-\delta_t^2(x,0)} dx < \epsilon \tag{12.14}$$

However our studies in Section 10 suggest that not the Euclidean balls $B_R^c(0)$ captures the shape of $p_t(\cdot)$ best.

In case we know that

$$\lim_{|x|\to\infty} \delta_t^2(x,0) = \infty, \tag{12.15}$$

then for every $\epsilon > 0$ we can find some $r = r(\epsilon)$ such that

$$\int_{(B_r^{\delta_t}(0))^c} e^{-\delta_t^2(x,0)} dx < \epsilon. \tag{12.16}$$

In this case we find

$$P^0(X_t \in C) - \int_{(B_r^{\delta_t}(0))\cap C} p_t(0)e^{-\delta_t^2(x,0)}dx < \epsilon, \tag{12.17}$$

and it is the knowledge of the metric geometry, i.e. the shape of the ball $B_r^{\delta_t}(0)$*, which allows to get an approximation of* $P^0(X_t \in C)$*. Clearly, a more detailed knowledge of* C *will lead to an improved estimate.*

Now we return to (12.11) and under the assumptions made above we note that given $\epsilon > 0$ *we can find* r_1 *and* r_2 *such that*

$$p_{t_1}(0)p_{t_2-t_1}(0)\int_{C_1\cap B_{r_1}^{\delta_{a,t_1}}(a)} e^{-\delta_{a,t_1}^2(y,0)}dy \int_{C_2\cap B_{r_2}^{\delta_{b,t_2-t_1}}(b)} e^{-\delta_{b,t_2-t_1}^2(y,0)}dy \tag{12.18}$$

is an approximation of (12.11). Note that we have to deal with two different metrices reflecting the fact that at different points in general we have to expect a different (metric) geometry governing the process.

So far the consideration in the section form a programme for future research. We pointed out in a heuristical manner what we expect to hold in lack of any knowledge of off-diagonal estimates for transition functions of processes generated by (non-isotropic) pseudo-differential operators $-q(x,D)$*. The emphasis is on the geometric structure of the transition function.*

Of course, the example used in Section 10 *may serve once again for calculation.*

A more challenging problem is to find first evidence in form of estimates which turn our ideas into theorems. While for the considerations leading to (12.18) estimates are attainable in such concrete case, it seems that the current state of the art does not allow to derive estimates for

$$\left|\int_{C_1} p_{t_1}(z,y)\int_{C_2} p_{t_2-t_1}(y,w)dwdy - \int_{C_1} p_{t_1}^a(z-y)\int_{C_2} p_{t_2-t_1}^b(y-w)dwdy\right|.$$

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