Stability analysis of stochastic differential delay equations with jumps.

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Stability Analysis of Stochastic Differential Delay Equations with Jumps

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Ph.D.

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2011

DECLARATION

This work has not previously been accepted in substance for any degree and is not being concurrently submitted in candidature for any degree.

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Contents

Abstract

Nowadays, stochastic differential delay equations with jump processes are playing an important role in science and industry, particularly, in economics, finance and engineering. This thesis focuses on studying the strong convergence and almost sure stability of many kinds of stochastic differential delay equations with jumps and its approximations. First of all, we introduce the uniqueness and existence of the global solution of neutral stochastic differential delay equations with jumps under the local Lipschitz condition by using the Lyapunov function and semi-martingale convergence theorem. Then the convergence of Euler-Maruyama method for neutral stochastic differential delay equations with jumps has been proved and the rate has been estimated both under global Lipschitz condition and local one, for this purpose we also define the continuous approximate solution from the discrete approximation. Meanwhile, we analyze the almost sure exponential stability of Euler-Maruyama method for neutral stochastic differential delay equations with jumps, which is derived from the moment stability. Furthermore, we study the strong convergence and almost sure stability of theta Euler-Maruyama method for neutral stochastic differential delay equations with jumps under the local Lipschitz condition and the monotone conditions using the discrete semi-martingale convergence theorem. Then we introduce the Skorokhod problem, and estimate the uniqueness and existence of the solution of reflected stochastic differential delay equations with jumps, especially, we separate the jumps into the large and small jumps and study the stability in distribution under the local Lipschitz condition. At last we estimate the strong convergence of implicit balanced methods for neutral stochastic differential delay equations with jumps.

Acknowledgements

In following I honestly appreciate those people who guide and help the completion of the dissertation. There are too many of them to thank one by one.

First and foremost I extend my sincere thanks to my supervisors Doctor Chenggui Yuan and Professor Feng-yu Wang, I have learned a lot of knowledge from their supervision. They are not only good teachers but also helpful friends that I shall never forget their patience to me. Next I want to thank my friend Jianhai Bao, who always enlightens my thinking and provides grateful help for my dissertation. I also want to thank all of staffs who are working in the mathematics department of Swansea University. Without the knowledge taught by them, I could not have done what I am able to do now.

My family has also given me many helps. My parents support me a lot by allowing me to study and live abroad. They always encourage me and give me confidence when I have trouble in study or living.

Finally, I would like to thank everyone else who gave me help during my dissertation time as well.

Chapter 1

Introduction

1.1 Background

Stochastic differential equations (SDEs) are increasingly playing a significant role in many branches of science and industry. Such models have been used with great success in a variety of areas, including biology, epidemiology, mechanics, economics and finance. The models have been well developed in extensive literature, for example, [5, 15, 17, 20, 31, 34, 38]. The numerical methods on SDEs have been further established and they can be referred to, for instance, [8, 9, 11, 26, 29, 37, 39, 44, 56, 40, 55]. The importance of stochastic differential delay equations (SDDEs) derives from the fact that many of the phenomena around us do not have an immediate effect at the time when they occur. A patient, for example, may show symptoms of illness days (or even weeks) after he or she was infected. Generally speaking, in almost any area of science (medicine, physics, ecology, biology, economics, etc.) we can end many systems for which the principle of causality, i.e., the future state of a system is independent of the past states and is determined solely by the present, does not apply. Many dynamical systems do not only depend on present and past states but also involve derivatives with delays. In order to incorporate this time lag (between the moment an action takes place and the moment its effect is observed) into our models, it is necessary to include an extra term which is called time delay.

Furthermore, the neutral stochastic functional differential equations are important for their applications to chemical engineering systems and aeroelasticity [31, 32]. In [40], neutral stochastic differential delay equations (NSDDEs) depending on past and present values but that involves derivatives with delay as well as the function itself, such equations are difficult to motivate but often arise in the study of two or more simple oscillatory systems with some interconnections between them. Mao investigated existence and uniqueness, moment and path-wise estimates, exponential stability of neutral stochastic functional differential equations. In the past few decades, the theory of NSDDEs has also received a great deal of attention.

Moreover, stochastic differential delay equations with jumps (SDDEwJs) have been widely used in many areas of science and industry, especially, in economics, finance and engineering, for example [13, 36, 52, 49]. Since most SDDEwJs cannot be solved explicitly, numerical methods have become essential. There is extensive literature on the numerical simulation for stochastic differential equations with jumps (SDEwJs) [23, 24, 12] and for SDDEwJs [35, 53, 27].

On the other hand, most of the existing results of the solutions for SDEs are proved under the global Lipschitz condition. However, there are many SDEs that only satisfy the local Lipschitz condition. It is very useful to establish solutions for them.

1.2 Overview of Study

In this paper we focus on the analysis of neutral stochastic differential delay equations with jumps (NSDDEwJs). Chapter 2 shows some notions for instance Brownian motion, jump process and some useful inequalities. For SDEs, there are two very natural concepts, namely mean-square stability and asymptotic stability. Asymptotic stability is more amenable to analyze, and hence this property dominates in the literature, especially [43] works in the almost surely asymptotic stability of NSDDEs with Markovian switching, so in chapter 3 we investigate the almost sure asymptotic stability of NSDDEwJs under the local Lipschitz conditions by using Lyapunov function and semi-martingale convergence theorem. By the elicitation of [27] and [54] which study the strong convergence of Euler-Maruyama method for stochastic differential delay equations with jumps (EMSDDEwJs) under the local Lipschitz conditions and calculate the convergence rate, we change the model to the NSDDEwJs in chapter 4 which analyze the strong convergence of EMNSDDEwJs, under the global Lipschitz conditions as well as the local one, and obtain the rate of convergence as well. Under the local Lipschitz conditions, [54] studies the almost sure exponential stability of Euler-Maruyama method for neutral stochastic differential delay equations with jumps (EMNSDDEwJs), which is derived from the moment stability; and [51] uses the monotone conditions to solve the strong convergence and almost sure stability of several EM type methods for SDEs. In chapter 5 we continue to work with the almost sure exponential stability of EMNSDDEwJs and strong convergence and almost sure stability of theta-Euler-Maruyama method for neutral stochastic differential delay equations with jumps (TEMNSDDEwJs) respectively under the local Lipschitz conditions based on the continuous and discrete semi-martingale convergence theorems.

Chapter 6 focuses on the stability in distribution of reflected stochastic differential delay equations with jumps (RSDDEwJs) under local Lipschitz conditions by [10] which using the notion of Skorokhod problem. The last chapter proves the order of strong convergence of implicit balanced method for neutral stochastic differential delay equations with jumps (BNSDDEwJs) which can be referred in [46] by using an important theorem in [45].

Chapter 2

Notation

2.1 Basic Probability Theory

The outcome of studying mathematical of trials depend on chance by using probability theory. All the possible outcome which we call the elementary events are grouped together to form a set Ω with an typical element, $\omega \in \Omega$. Usually we only group the observable or interesting events together as a family $\mathcal{F}$ of subsets of Ω because which are not include every subset of Ω. For the purpose of probability theory, $\mathcal{F}$ should have the following properties:

- $\emptyset \in \mathcal{F}$ denotes the empty set;
- $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$, where $A^c = \Omega - A$ is the complement of A in Ω;
- $\{A_i\}_{i\geq 1} \subset \mathcal{F} \Rightarrow \cup_{i=1}^{\infty} A_i \in \mathcal{F}$.

A family $\mathcal{F}$ with these properties is known as a σ-algebra. The pair $(\Omega, \mathcal{F})$ is called a measurable space, and the elements of $\mathcal{F}$ is called $\mathcal{F}$-measurable sets.

A real-valued function $X : \Omega \to \mathbb{R}$ is said to be $\mathcal{F}$-measurable if

$$\{\omega : X(\omega) \le a\} \in \mathcal{F} \quad \text{for all } a \in \mathbb{R}.$$

The function X is also called a real-valued random variable. An $\mathbb{R}^n$-valued function $X(\omega) = (X_1(\omega), \ldots, X_n(\omega))$ is said to be $\mathcal{F}$-measurable if all the elements X_i are $\mathcal{F}$-measurable. Similarly, a $n \times m$-matrix-valued function $X(\omega) = (X_{ij}(\omega))_{n\times m}$ is said to be $\mathcal{F}$-measurable if all the elements X_{ij} are $\mathcal{F}$-measurable. The indicator function I_A of a set $A \subset \Omega$ is defined by

$$I_A(\omega) = \begin{cases} 1 & \text{for} \quad \omega \in A, \\ 0 & \text{for} \quad \omega \notin A. \end{cases}$$

The indicator function I_A is $\mathcal{F}$-measurable if and only if A is an $\mathcal{F}$-measurable set, i.e. $A \in \mathcal{F}$.

A probability measure $\mathbb{P}$ on a measurable space$(\Omega, \mathcal{F})$ is a function $\mathbb{P}$: $\mathcal{F} \to [0, 1]$ such that

- $\mathbb{P}(\Omega) = 1$;
- for any disjoint sequence $\{A_i\}_{i\ge 1} \subset \mathcal{F}$,

$$\mathbb{P}(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a probability space. It is called a complete probability space if we set

$$\bar{\mathcal{F}} = \{A \subset \Omega : \exists B, C \in \mathcal{F} \quad \text{such that} \quad B \subset A \subset C, \mathbb{P}(B) = \mathbb{P}(C)\}.$$

Then $\bar{\mathcal{F}}$ is a σ-algebra and is called the completion of $\mathcal{F}$.

In the sequence of this section, we let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. If X is a real-valued random variable and is integrable with respect to the probability measure $\mathbb{P}$, then the number

$$\mathbb{E}X = \int_{\Omega} X(\omega) d\mathbb{P}(\omega)$$

is called the expectation of X with respect to $\mathbb{P}$. For $p \in (0, \infty)$, let $L^p = L^p(\Omega; \mathbb{R}^n)$ be the family of $\mathbb{R}^n$-valued random variables with $\mathbb{E}|X|^p < \infty$. In L^1, we have $\mathbb{E}X \leq \mathbb{E}|X|$.

Moreover, the following two inequalities are very useful:

Hölder's inequality

$$|\mathbb{E}(X^T Y)| \leq (\mathbb{E}|X|^p)^{\frac{1}{p}} (\mathbb{E}|Y|^q)^{\frac{1}{q}} \tag{2.1}$$

if $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $X \in L^p$, $Y \in L^q$;

Chebyschev's inequality

$$\mathbb{P}\{\omega : |X(\omega)| \geq c\} \leq c^{-p}\mathbb{E}|X|^p \tag{2.2}$$

if $c, p > 0$, $X \in L^p$.

Let X and X_k, $k \geq 1$, be $\mathbb{R}^n$-valued random variables. The following four convergence concepts are important:

- If there exists a $\mathbb{P}$-null set $\Omega_0 \in \mathcal{F}$ such that for every ω not in Ω_0, the sequence $\{X_k(\omega)\}$ converges to $X(\omega)$ in the usual sense in $\mathbb{R}^n$, then $\{X_k\}$ is said to converges to X almost surely or with probability 1, and we write $\lim_{k\to\infty} X_k = X$ a.s.

- If for every $\varepsilon > 0$, $\mathbb{P}\{\omega : |X_k(\omega) - X(\omega)| > \varepsilon\} \to 0$ as $k \to \infty$, then $\{X_k\}$ is said to converge to X stochastically or in probability.

- If X_k, $X \in L^p$ and $\mathbb{E}|X_k - X|^p \to 0$, then $\{X_k\}$ is said to converge to X in L^p.

- If for every real-valued continuous bounded function g defined on $\mathbb{R}^n$, $\lim_{k\to\infty} \mathbb{E}g(X_k) = \mathbb{E}g(X)$, then $\{X_k\}$ is said to converge to X in distribution.

Both the convergence in L^p and the a.s. convergence implies the convergence in probability which leads the convergence in distribution. There are some important integration convergence theorems and lemmas in the follows (see [55, P8 Theorem 1.1, 1.2] and [1, P7 Corollary 1.1.2] respectively) that

Theorem 2.1 ***(Monotonic convergence theorem)*** *If $\{X_k\}$ is an increasing sequence of non-negative random variables, then*

$$\lim_{k\to\infty} \mathbb{E}X_k = \mathbb{E}\Big(\lim_{k\to\infty} X_k\Big).$$

From this we easily deduce

Lemma 2.1 ***(Fatou lemma)*** *If $\{X_k\}$ is a sequence of non-negative random variables, if there exists a integrable random variable Y such that $X_k \le Y$ for all k, then*

$$\mathbb{E}\Big(\limsup_{k\to\infty} X_k\Big) \ge \limsup_{k\to\infty} \mathbb{E}X_k.$$

Theorem 2.2 ***(Dominated convergence theorem)*** *Let $p \ge 1$, $\{X_k\} \subset L^p(\Omega;\mathbb{R}^n)$ and $Y \in L^p(\Omega;\mathbb{R})$. Assume that $|X_k| \le Y$ a.s. and $\{X_k\}$ converges to X in probability. Then $X \in L^p(\Omega;\mathbb{R}^n)$, $\{X_k\}$ converges to $X \in L^p$, and*

$$\lim_{k\to\infty} \mathbb{E}X_k = \mathbb{E}X.$$

Now let $\{A_k\}$ be a sequence of sets in $\mathcal{F}$. The set of all those points which belong to all A_k is called the inferior limit of A_k, and is denoted by $\liminf_{k\to\infty} A_k$. Clearly,

$$\liminf_{k\to\infty} A_k = \cup_{i=1}^{\infty} \cap_{k=i}^{\infty} A_k.$$

The set of all those points which belong to infinitely many A_k is called the superior limit of A_k and is denoted by $\limsup_{k\to\infty} A_k$. It is easy to see

$$\limsup_{k\to\infty} A_k = \cap_{i=1}^{\infty} \cup_{k=i}^{\infty} A_k.$$

Moreover,

$$\liminf_{k\to\infty} A_k \subset \limsup_{k\to\infty} A_k.$$

With regard to their probabilities, we have the following well-known lemma [55, P10 Lemma 1.2].

Lemma 2.2 ***(Borel-Cantelli lemma)***

- *If $\{A_k\} \subset \mathcal{F}$ and $\sum_{k=1}^{\infty} \mathbb{P}(A_k) < \infty$, then*

$$\mathbb{P}(\limsup_{k\to\infty} A_k) = 0.$$

That is, there exists a set $\Omega_1 \in \mathcal{F}$ with $\mathbb{P}(\Omega_1) = 1$ and an integer-valued random variable k_1 such that for every $\omega \in \Omega_1$ we have ω is not in A_k whenever $k \geq k_1(\omega)$.

- *If the sequence $\{A_k\} \subset \mathcal{F}$ is independent and $\sum_{k=1}^{\infty} \mathbb{P}(A_k) = \infty$, then*

$$\mathbb{P}(\limsup_{k\to\infty} A_k) = 1.$$

That is, there exists a set $\Omega_2 \in \mathcal{F}$ with $\mathbb{P}(\Omega_2) = 1$ such that for every $\omega \in \Omega_2$, there exists a sub-sequence $\{A_k\}$ such that the ω belongs to every $\{A_{k_i}\}$.

Let $A, B \in \mathcal{F}$ with $\mathbb{P}(B) > 0$. The conditional probability of A under condition B is

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

We also need the more general concept of condition expectation. Let $X \in L^1(\Omega; \mathbb{R})$. Let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra of $\mathcal{F}$ so $(\Omega, \mathcal{G})$ is a measurable space. In general, X is not $\mathcal{G}$-measurable. We now seek an integrable $\mathcal{G}$-measurable random variable Y such that it has the same values as X on the average in the following sense

$$\mathbb{E}(I_G Y) = \mathbb{E}(I_G X) \quad \text{i.e.} \quad \int_G Y(\omega)\mathbb{P}(\omega) = \int_G X(\omega)\mathbb{P}(\omega) \quad \forall G \in \mathcal{G}.$$

$$\mathbb{E}(I_G Y) = \mathbb{E}(I_G X) \quad \text{i.e.} \quad \int_G Y(\omega)\mathbb{P}(\omega) = \int_G X(\omega)\mathbb{P}(\omega) \quad \forall G \in \mathcal{G}.$$

Any such random variable Y is called the conditional expectation of X given $\mathcal{G}$ and it is written

$$Y = \mathbb{E}(X|\mathcal{G}).$$

2.2 Stochastic Processes

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A filtration is a family $\{\mathcal{F}_t\}_{t\geq 0}$ of increasing sub-σ-algebra of $\mathcal{F}$. (i.e. $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ for all $0 \leq s \leq t \leq \infty$). The filtration is said to be right continuous if $\mathcal{F}_t = \cap_{s>t}\mathcal{F}_s$ for all $t \geq 0$. If the probability space is complete, we say the filtration satisfies the usual conditions if it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets.

From now on, unless otherwise specified, we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ which satisfies the usual conditions and any stochastic processes will be defined in this space.

A stochastic process is a family of $\mathbb{R}^n$-random variables $\{X_t\}_{t\geq 0}$, indexed by a real parameter t and defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The parameter set t usually represents time and so its parameter set I is usually the half line $\mathbb{R}_+ = [0, \infty)$, but it may also be an interval $[t_1, t_2]$ or $\{0, 1, 2, \ldots\}$. It is worth noting that for each fixed $t \in I$ we have a random variable

$$\Omega \ni \omega \rightarrow X_t(\omega) \in \mathbb{R}^n.$$

Meanwhile, for each fixed $\omega \in \Omega$ we have a function

$$I \ni t \rightarrow X_t(\omega) \in \mathbb{R}^n,$$

which is called a sample path of the process. Sometimes we write $X(t, \omega)$ instead of $X_t(\omega)$, and the stochastic process may be regarded as a function of the two variables t and ω from $I \times \Omega$ to $\mathbb{R}^n$. A stochastic process $\{X_t\}_{t\geq 0}$ may sometimes be written as $\{X_t\}$, X_t or $X(t)$.

An $\mathbb{R}^n$-valued stochastic process $\{X_t\}_{t\geq 0}$ is said to be continuous (respectively right continuous, left continuous) if for almost all $\omega \in \Omega$ the function $X_t(\omega)$ is continuous (respectively right continuous, left continuous) on $t \geq 0$. It is said to be integrable if X_t is an integrable random variable for every $t \geq 0$. It is said to be $\{\mathcal{F}_t\}$-adapted (or more simply, adapted) if, for every t, X_t is $\mathcal{F}_t$ measurable.

An $\{\mathcal{F}_t\}$-stopping time is a random variable $\tau : \Omega \to [0, \infty]$ (it may take the value ∞) for which $\{\omega : \tau(\omega) \leq t\} \in \mathcal{F}_t$ for any $t \geq 0$.

An important type of random process which we must introduce is the martingale process. A martingale with respect to $\{\mathcal{F}_t\}$ (or more simply, martingale) is an $\mathbb{R}^n$-valued $\{\mathcal{F}_t\}$-adapted integrable process $\{M_t\}_{t\geq 0}$ satisfying

$$\mathbb{E}(M_t|\mathcal{F}_s) = M_s \quad \text{a.s. for all} \quad 0 \leq s < t < \infty.$$

The process is said to be a super-martingale (respectively sub-martingale) if equality is replaced by $\leq$ (respectively $\geq$). A right continuous adapted process $M = \{M_t\}_{t\geq 0}$ is called a local martingale if there exists a nondecreasing sequence $\{\tau_k\}_{k\geq 1}$ of stopping times with $\tau_k \uparrow \infty$ a.s. such that every $\{M_{t\wedge\tau_k} - M_0\}_{t\geq 0}$ is a martingale. While every martingale is a local martingale by [55, P15 Theorem 1.5], the opposite is not true.

We can describe a stochastic process $X = \{X_t\}_{t\geq 0}$ as being square-integrable if $\mathbb{E}|X_t|^2 < \infty$ for every $t \geq 0$. Let $M = \{M_t\}_{t\geq 0}$ be a real-valued square-integrable continuous martingale, there exists a unique continuous integrable adapted increasing process denoted by $\{\langle M, M\rangle_t\}$, and called the quadratic variation of M, such that $\{M_t^2 - \langle M, M\rangle_t\}$ is a continuous martingale vanishing at $t = 0$. If $N = \{N_t\}_{t\geq 0}$ is another real-valued square-integrable continuous martingale, we define $\langle M, N\rangle_t = \frac{1}{2}\big(\langle M+N, M+N\rangle_t - \langle M, M\rangle_t - \langle N, N\rangle_t\big)$, and call $\{\langle M, N\rangle_t\}$ the joint quadratic variation of M

and N. It is useful to know that $\{\langle M,N\rangle_t\}$ is the unique continuous integrable adapted process of finite variation such that $\{M_tN_t - \langle M,N\rangle_t\}$ is a continuous martingale vanishing at $t=0$. For two real-valued continuous local martingales $M=\{M_t\}_{t\geq 0}$ and $N=\{N_t\}_{t\geq 0}$, their joint quadratic variation $\{\langle M,N\rangle_t\}$ is the unique continuous adapted process of finite vanishing at $t=0$.

For $\mathbb{R}^n$-valued martingale we have the following well-known Doob's martingale inequalities [55, P18 Theorem 1.11].

Theorem 2.3 ***(Doob's martingale inequalities)*** *Let $\{M_t\}_{t\geq 0}$ be an $\mathbb{R}^n$-valued martingale. Let $[a,b]$ be a bounded interval in $\mathbb{R}_+$,*

- *If $p\geq 1$, $c>0$ and $M_t\in L^p(\Omega;\mathbb{R}^n)$, then*

$$\mathbb{P}\left\{\omega : \sup_{a\leq t\leq b}|M_t(\omega)|\geq c\right\}\leq \frac{\mathbb{E}|M_b|^p}{c^p}. \tag{2.3}$$

- *If $p>1$, and $M_t\in L^p(\Omega;\mathbb{R}^n)$, then*

$$\mathbb{E}\left(\sup_{a\leq t\leq b}|M_t|^p\right)\leq\left(\frac{p}{p-1}\right)^p\mathbb{E}|M_b|^p. \tag{2.4}$$

2.3 Brownian Motion

Brownian motion is the name given to the irregular movement of pollen, suspended in the water. It was first observed by the Scottish botanist Robert Brown in 1828. The range of applications of Brownian motion covers areas such as physics, biology, economics and many more. The first quantitative work on Borwnian motion is due to Bachelier (1990), but Einstein (1905) derived the transition density for it. A rigorous treatment of Brownian motion began with Wiener (1923), who gave a mathematical representation.

Definition 2.1 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$. A 1-dimensional Brownian motion is a real-valued continuous $\{\mathcal{F}_t\}$-adapted process $\{B(t)\}_{t\geq 0}$ with the following properties:*

- *$B(0) = 0$ a.s.;*
- *the increment $B(t) - B(s)$ follows a normal distribution with mean zero and variance $(t - s)$ for $0 \leq s < t \leq \infty$;*
- *for $0 \leq s < t \leq \infty$, the increment $B(t) - B(s)$ is independent of $\mathcal{F}_s$.*

Definition 2.2 *A d-dimensional stochastic process $\{B(t) = (B^1(t), \ldots, B^d(t))\}$ is called a d-dimensional Brownian motion if every $B^i(t)$, with $1 \leq i \leq d$, is a 1-dimensional Brownian motion and $\{B^1(t)\}, \ldots, \{B^d(t)\}$ are independent.*

The increment $B(t) - B(s)$ is normally distributed with mean zero and covariance matrix $(t - s)I_d$, where I_d is the $d \times d$ identity matrix.

2.4 Stochastic Integrals

In this section we shall define the stochastic integral

$$\int_0^t f(t)dB(t)$$

with respect to a d-dimensional Bro$\int_0^t f(t)dB(t)$ $\{B(t)\}$ for a class of $m \times d$-matrix-valued stochastic process $\{f(t)\}$. Since $\{B(t)\}$ has unbounded variations, for almost all $\omega \in \Omega$, the Brownian sample path $B(\omega)$ is nowhere differentiable, and so the integral cannot be defined in the usual sense. However, K. Itô succeeded to give a definition of the stochastic integral using the stochastic nature of Brownian motion.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Let $B = \{B(t)\}_{t\geq 0}$ be a 1-dimensional Brownian motion. Let $L^2(\Omega, \mathbb{R})$ denote the space of all $\mathcal{F}_t$-adapted stochastic processes $f_t = f_t(\omega)$ such that

$$\mathbb{E}|f_t|^2 < \infty$$

The space of all $\mathbb{R}$-valued, $\mathcal{F}_t$-adapted stochastic processes $f_t = f_t(\omega)$ such that

$$\|f\|_{a,b}^2 = \mathbb{E}\int_a^b |f(t)|^2 dt < \infty$$

is denoted by $\mathcal{M}^2([a,b];\mathbb{R})$.

A real-valued stochastic process $f = \{f(t)\}_{t\geq 0}$ is called a step process if there exist a partition $a = t_0 < t_1 < \cdots < t_n = b$ of $[a,b]$, and bounded random variables ξ_i, $0 \leq i \leq n-1$ such that ξ_i is a $\mathcal{F}_{t_i}$-measurable and

$$f(t) = \xi_0 I_{[t_0,t_1]}(t) + \sum_{i=1}^{n-1} \xi_i I_{(t_i,t_{i+1}]}(t).$$

For this step process f, the stochastic integral of f with respect to $B(t)$ is defined as a random variable

$$\int_a^b f(t)dB(t) = \sum_{i=0}^{n-1} \xi_i (B_{t_i+1} - B_{t_i}).$$

Definition 2.3 *Let $f \in \mathcal{M}^2([a,b];\mathbb{R})$. The Itô integral of f with respect to $B(t)$ is defined by*

$$\int_a^b f(t)dB(t) = \lim_{n\to\infty} \int_a^b g_n(t)dB(t) \quad in \quad L^2(\Omega,\mathbb{R}),$$

where $\{g_n\}$ is a sequence of step process such that

$$\lim_{n\to\infty} \mathbb{E}\int_a^b |f(t) - g_n(t)|^2 dt = 0.$$

Definition 2.4 *The total variation of a real-valued stochastic process f, defined on an interval $[a,b] \subset \mathbb{R}$ is the quantity,*

$$V_a^b(f) = \sup_{p \in \mathcal{P}} \sum_{i=0}^{n_p - 1} |f(X_{i+1}) - f(X_i)|.$$

where the $\mathcal{P} = \{p = \{X_0, \dots, X_{n_p}\} | p$ is a partition of $[a,b]\}$. A real-valued f defined on $[a,b]$, if its total variation is finite.

Definition 2.5 *Let $X = \{X_t\}_{t \geq 0}$, $Y = \{Y_t\}_{t \geq 0}$ be two real-valued stochastic processes, the quadratic variation of X is*

$$\langle X, X \rangle_t = \lim_{\|p\| \to 0} \sum_{k=1}^{n} (X_{t_k} - X_{t_{k-1}})^2,$$

where p ranges over partitions of the interval of $[0,t]$, the $\|p\|$ is the longest of these subintervals, that is $\max\{|X_j - X_{j-1}| : j = 1, \dots, n\}$;

$$\langle X, Y \rangle_t = \lim_{\|p\| \to 0} \sum_{k=1}^{n} (X_{t_k} - X_{t_{k-1}})(Y_{t_k} - Y_{t_{k-1}}).$$

We shall now extend the Itô stochastic integral to the multi-dimensional case. Let $\{B(t) = (B^1(t), \dots, B^d(t))^T\}_{t \geq 0}$ be a d-dimensional Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ adapted to the filtration $\mathcal{F}_t$. Let $\mathcal{M}^2([a,b]; \mathbb{R}^{n \times d})$ denote the family of all $m \times d$-matrix-valued measurable $\mathcal{F}_t$-adapted processes $f = \{(f_{ij}(t))_{m \times d}\}_{0 \leq t \leq T}$ such that

$$\mathbb{E} \int_a^b |f(s)|^2 ds < \infty.$$

Here, and throughout this thesis, $|A|$ will denote the trace norm for matrix A, i.e. $|A| = \sqrt{\text{trace}(A^T A)}$.

Definition 2.6 *Let $f \in \mathcal{M}^2([0,T]; \mathbb{R}^{n \times d})$, define Itô integral*

$$\int_0^t f(s) dB(s) = \int_0^t \begin{pmatrix} f_{11}(s) & \dots & f_{1d}(s) \\ \vdots & \dots & \vdots \\ f_{m1}(s) & \dots & f_{md}(s) \end{pmatrix} \begin{pmatrix} dB^1(s) \\ \vdots \\ dB^d(s) \end{pmatrix}$$

to be the n-column-vector-valued process whose i-th component is the following sum of 1*-dimensional Itô integrals*

$$\sum_{j=1}^{d} \int_0^t f_{ij}(s)dB^j(s).$$

The Itô integral has a number of nice properties. For $f, g \in \mathcal{M}^2([a,b];\mathbb{R}^{n\times d})$ and α, β are two real numbers, we note the following

- $\int_a^b f(t)dB(t)$ is $\mathcal{F}_b$-measurable;
- $\mathbb{E}\int_a^b f(t)dB(t) = 0$;
- $\mathbb{E}|\int_a^b f(t)dB(t)|^2 = \mathbb{E}\int_a^b |f(t)|^2dt$;
- $\int_a^b[\alpha f(t) + \beta g(t)]dB(t) = \alpha\int_a^b f(t)dB(t) + \beta\int_a^b g(t)dB(t)$.

Let $\mathcal{L}^2(\mathbb{R}_+;\mathbb{R}^{n\times d})$ denote the family of all $m \times d$-matrix-valued measurable $\{\mathcal{F}_t\}$-adapted processes $f = \{f(t)\}_{t\geq 0}$ such that

$$\int_0^T |f(t)|^2dt < \infty \quad \text{a.s. for every} \quad T > 0.$$

The following theorem is known as the Burkholder-Davis-Gundy inequality [55, P70 Theorem 2.13].

Theorem 2.4 ***(Burkholder-Davis-Gundy inequality)*** *Let $g \in \mathcal{M}^2(\mathbb{R}_+;\mathbb{R}^{n\times d})$. Define, for $t \geq 0$,*

$$X(t) = \int_0^t g(s)dB(s) \quad \text{and} \quad A(t) = \int_0^t |g(s)|^2ds.$$

Then for every $p > 0$, there exist universal positive constants c_p, C_p depending only on p, such that

$$c_p\mathbb{E}|A(t)|^{\frac{p}{2}} \leq \mathbb{E}\left(\sup_{0\leq s\leq t}|X(s)|^p\right) \leq C_p\mathbb{E}|A(t)|^{\frac{p}{2}} \tag{2.5}$$

for all $t \geq 0$. In particular, one may take

$$\begin{aligned}
c_p &= (p/2)^p, & C_p &= (32/p)^{p/2} & &\text{if} \quad 0 < p < 2; \\
c_p &= 1, & C_p &= 4 & &\text{if} \quad p = 2; \\
c_p &= (2p)^{-p/2}, & C_p &= (p^{p+1}/2(p-1)^{p-1})^{p/2} & &\text{if} \quad p > 2.
\end{aligned}$$

Especially, for $t \geq 0$,

$$\mathbb{E}\left(\sup_{0\leq s\leq t}|X(s)|\right) \leq 3\mathbb{E}(|A(t)|^{\frac{1}{2}}). \tag{2.6}$$

Theorem 2.5 ***(Gronwall's inequality)*** *[55, P54 Theorem 2.2] Let $T > 0$ and $c > 0$. Let $u(\cdot)$ be a Borel measurable bounded nonnegative function on $[0, T]$, and let $v(\cdot)$ be a nonnegative integrable function on $[0, T]$. If*

$$u(t) \leq c + \int_0^t v(s)u(s)ds \quad \text{for all} \quad 0 \leq t \leq T,$$

then

$$u(t) \leq c \exp\left\{\int_0^t v(s)ds\right\} \quad \text{for all} \quad 0 \leq t \leq T. \tag{2.7}$$

(Discrete Gronwall's inequality) *[55, P56 Theorem 2.5] Let M be a positive integer. Let u_k and v_k be non-negative numbers for $k = 0, 1, \ldots, M$. If*

$$u_k \leq u_0 + \sum_{j=0}^{k-1} v_j u_j, \quad \text{for all} \quad k = 1, \ldots, M,$$

then

$$u_k \leq u_0 \exp\left\{\sum_{j=0}^{k-1} v_j\right\}, \quad \text{for all} \quad k = 1, \ldots, M. \tag{2.8}$$

2.5 Poisson Stochastic Integrals

Let $X = \{X_t\}_{t\geq 0}$ be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. We say that it has independent increments if for each

$n \in N$ and each $0 \leq t_1 \leq t_2 \leq \ldots \leq t_{n+1} < \infty$ the random variables $(X(t_{j+1}) - X(t_j), 1 \leq j \leq n)$ are independent and that it has stationary increments if each $X(t_{j+1} - X(t_j))$ is equal in distribution to $X(t_{j+1} - t_j) - X(0)$. Then we say that X is a Lévy process if

- $X(0) = 0$ a.s.;
- X has independent and stationary increments;
- X is stochastically continuous, i.e. for all $a \geq 0$ and for all $s \geq 0$

$$\lim_{t \to s} \mathbb{P}(|X(t) - X(s)| > a) = 0.$$

Here we need to mention a notion, which would be useful in the following definitions that let S be a subsets of $\mathbb{R}^d$. We equip S with the relative topology induced from $\mathbb{R}^d$, so that $U \subseteq S$ is open in S if $U \cap S$ is open in $\mathbb{R}^d$. Let $\mathcal{B}(S)$ denote the smallest σ-algebra of subsets of S that contains every open set in S. We call $\mathcal{B}(S)$ the Borel σ-algebra of S. Elements of $\mathcal{B}(S)$ are called Borel sets and any measure on $(s, \mathcal{B}(S))$ is called a Borel measure.

Let $0 \leq t < \infty$ and $A \in \mathcal{B}(\mathbb{R}^d - \{0\})$, define

$$N(t, A) := \sharp\{0 \leq s \leq t; \Delta X(s) \in A\} = \sum_{0 \leq s \leq t} \chi_A(\Delta X(s)).$$

Note that for each $\omega \in \Omega$, $t \geq 0$, the set function $A \to N(t, A)(\omega)$ is a counting measure on $\mathcal{B}(\mathbb{R}^d - \{0\})$ and hence

$$\mathbb{E}(N(t, A)) = \int N(t, A)(\omega) d\mathbb{P}(\omega)$$

is a Borel measure on $\mathcal{B}(\mathbb{R}^d - \{0\})$. We write $\nu(\cdot) = \mathbb{E}(N(1, \cdot))$ and call it the intensity measure associated with X.

Let ν be a Borel measure defined on $\mathbb{R}^d - \{0\}$, we say that it is a Lévy measure if

$$\int_{\mathbb{R}^d - \{0\}} (|x|^2 \wedge 1) \nu(dx) < \infty.$$

Let N be a Poisson random measure on $\mathbb{R}^+ \times (\mathbb{R}^n - \{0\})$ with intensity measure ν. We assume that ν is a Lévy measure. Let f be a Borel measurable function from $\mathbb{R}^n \to \mathbb{R}^n$ and let $A \in \mathcal{B}(\mathbb{R}^n - \{0\})$; then for each $t > 0$, $\omega \in \Omega$, we may define the poisson integral of f as a random finite sum by

$$\int_A f(x)N(t, dx)(\omega) = \sum_{x \in A} f(x)N(t, \{x\})(\omega).$$

Note that each $\int_A f(x)N(t, dx)$ is $\mathbb{R}^n$-valued random variable and gives rise to a càdlàg (right-continuous) stochastic process as we vary t. Since $N(t, \{x\}) \neq 0 \leftrightarrow \Delta X(u) = x$ for at least one $0 \leq u \leq t$, we have [1, (2.5) pp91]

$$\int_A f(x)N(t, dx) = \sum_{0 \leq u \leq t} f(\Delta X(u))\chi_A(\Delta X(u)).$$

Now taking $\tilde{N}$ be the compensated Poisson processes, then in [1, P207]

$$\int_0^T \int_A f(x)\tilde{N}(dt, dx) = \int_0^T \int_A f(x)N(dt, dx) - \int_0^T \int_A f(x)\nu(dx)dt.$$

2.6 Stochastic Stability Theory

A.M. Lyapunov introduced the concept of stability of a dynamic system in 1892. Generally speaking, a system is stable if it is insensitive to small changes in the initial state or the parameters of the system. Lyapunov developed a method for determining stability without solving the equation, but as the theory of SDEs developed it became clear that a similar method of stochastic concept was necessary. In this section we shall introduce various type of stability for the n-dimensional SDEs

$$dX(t) = f(X(t), t)dt + g(X(t), t)dB(t) \tag{2.9}$$

on $t \geq t_0$. As a standing hypothesis we assume that both f and g are sufficiently smooth so that (2.9) has a unique solution. Moreover, assume that

$f(0,t) = 0$ and $g(0,t) = 0$ so (2.9) admits the trivial solution $X(t;t_0,0) \equiv 0$. Let us now present the different kinds of stochastic stability.

Definition 2.7 ***(Almost sure asymptotic stability)***

The trivial solution of equation (2.9) *is said to be almost sure asymptotically stable if*

$$\mathbb{P}(\lim_{t\to\infty} |X(t;t_0,X_0)| = 0) = 1 \quad a.s.$$

for all $X_0 \in \mathbb{R}^n$.

Definition 2.8 ***(Almost sure exponential stability)***

The trivial solution of equation (2.9) *is said to be almost sure exponentially stable if*

$$\limsup_{t\to\infty} \frac{1}{t}\log|X(t;t_0,X_0)| < 0 \quad a.s.$$

for all $X_0 \in \mathbb{R}^n$.

Definition 2.9 ***(Moment exponential stability)***

The trivial solution of equation (2.9) *is said to be p-th moment exponentially stable if there is a pair of positive constants* C_1, C_2 *such that*

$$\mathbb{E}|X(t;t_0,X_0)|^p \leq C_1|X_0|^p e^{-C_2(t-t_0)}$$

on $t \geq t_0$*, for all* $X_0 \in \mathbb{R}^n$*. It is usually said to be exponentially stable in mean square when* $p = 2$.

Definition 2.10 ***(Asymptotic stability in distribution)***

The process $X(t)$ *is said to be asymptotically stable in distribution if there exists a probability measure* $\pi(\cdot)$ *on* $\mathbb{R}^n$ *such that the transition probability* $p(t,\xi,d\zeta)$ *of* $X(t)$ *converges weakly to* $\pi(d\zeta)$ *as* $t \to \infty$ *for every* $\xi \in \mathbb{R}^n$.

Chapter 3

Almost Sure Asymptotic Stability of Neutral Stochastic Differential Delay Equations with Jumps

3.1 Introduction

Many dynamical systems not only depend on present and past states but also involve derivatives with delays. Hale and Lune [18] have studied deterministic NSDDEs and their stability. Taking the environmental disturbances into account, Kolmanovskii and Nosov [30] and Mao [40] discussed the NSDDEs. Kolmanovskii and Nosov [30] not only establish the theory of existence and uniqueness of the solution but also investigate the stability and asymptotic stability of the equations, while Mao [40] studied the exponential stability of the equations. [18] studied deterministic NSDDEs and their stability. For NSDDEs, [43] studied the almost sure asymptotic stability of the equations.

In this chapter we will mainly discuss the almost sure asymptotic stability of NSDDEwJs.

3.2 Non-Linear Neutral Stochastic Differential Delay Equations with Jumps

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$, which is right-continuous having left-limit and satisfies that each $\{\mathcal{F}_t\}_{t\geq 0}$ contains all P-null set in $\mathcal{F}$.

Let $\{B(t) := \left(B^1(t), B^2(t), \dots, B^m(t)\right)^T,\ t \in [0,T]\}$ be a m-dimensional $\mathcal{F}_t$-adapted standard Brownian motion independent of $\mathcal{F}_0$.

Let $N(dt, dz)$ be a d-dimensional Poisson processes and denote the compensated Poisson processes by

$$\begin{aligned}\tilde{N}(dt, dz) &= \left(\tilde{N}_1(dt, dz_1), \dots, \tilde{N}_d(dt, dz_d)\right)^T \\ &= \left(N_1(dt, dz_1) - \nu_1(dz_1)dt, \dots, N_d(dt, dz_d) - \nu_d(dz_d)dt\right)^T,\end{aligned}$$

where $\{N_j, j = 1, 2, \dots, d\}$ are independent d-dimensional Poisson random measures with intensity measure $\{\nu_j, j = 1, 2, \dots, d\}$. We assume that $B(t)$ and $N(dt, dz)$ are independent. For more details regarding possion stochastic integral, see e.g. [1, P207].

Let $|\cdot|$ be the Euclidean norm as well as the matrix trace norm. Let $\tau > 0$ and denote $D([-\tau, 0]; \mathbb{R}^n)$ be the family of all right-continuous and left-limit $\mathbb{R}^n$-valued functions ϕ from $[-\tau, 0]$ to $\mathbb{R}^n$ with the norm $\|\phi\| = \sup_{-\tau\leq t\leq 0} |\phi(t)|$, and $D^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ be the family of all $\mathcal{F}_0$-measurable bounded $D([-\tau, 0]; \mathbb{R}^n)$-valued random variables $\xi = \{\xi(t), t \in [-\tau, 0]\}$.

Consider the nonlinear n-dimensional NSDDEwJs

$$d[X(t) - G(X(t-\tau))] = f(X(t), X(t-\tau))dt + g(X(t), X(t-\tau))dB(t) + \int_{\mathbb{R}^d} h(X(t^-), X((t-\tau)^-), z)\tilde{N}(dt, dz), \tag{3.1}$$

with the given initial segment

$$\xi = \{\xi(t), t \in [-\tau, 0]\} \in D^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n), \tag{3.2}$$

for any $t \geq 0$, $\tau > 0$ and $\nu = (\nu_1, \ldots, \nu_d)^T$ are bounded Lévy measures, i.e., $\nu(\mathbb{R}^d) < \infty$ and $\nu(A) = \nu(-A)$ for all Borel sets $A \in \mathbb{R}^d$, where $X(t^-) = \lim_{s\uparrow t} X(s)$, and $G : \mathbb{R}^n \to \mathbb{R}^n$, $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ as well as $h : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^{n\times d}$. We denote that each column $h^{(k)}$ of the $n \times d$ matrix $h = [h_{ij}]$ depends on z only through the k-th coordinate z_k, i.e., $h^{(k)}(x, y, z) = h^{(k)}(x, y, z_k)$, $z = (z_1, \ldots, z_d) \in \mathbb{R}^d$. Furthermore, all values $\xi(t)$ of the initial segment are assumed to be $\mathcal{F}_0$-measurable for $t \in [-\tau, 0]$.

Using the notation above, we can rewrite the components $X_i(t)$, $i = 1, \ldots, n$, in (3.1) that is

$$d[X_i(t) - G_i(X(t-\tau))] = f_i(X(t), X(t-\tau))dt + \sum_{j=1}^{m} g_{ij}(X(t), X(t-\tau))dB_j(t) + \sum_{j=1}^{d} \int_{\mathbb{R}} h_{ij}(X(t^-), X((t-\tau)^-), z_j)\tilde{N}_j(dt, dz_j) \tag{3.3}$$

Assumption 3.1 ***(Local Lipschitz condition)*** *For each integer $R \geq 1$ there exists a positive constant K_R, such that*

$$|f(x_1, y_1) - f(x_2, y_2)|^2 + |g(x_1, y_1) - g(x_2, y_2)|^2 + \sum_{k=1}^{d} \int_{\mathbb{R}} |h^{(k)}(x_1, y_1, z_k) - h^{(k)}(x_2, y_2, z_k)|^2 \nu_k(dz_k) \leq K_R(|x_1 - x_2|^2 + |y_1 - y_2|^2), \tag{3.4}$$

for $x_1, x_2, y_1, y_2 \in \mathbb{R}^n$ *and* $t \in [0, T]$*, where* $|x_1| \vee |y_1| \vee |x_2| \vee |y_2| \leq R$*. We also assume that there is a constant* $\tilde{K} \in (0, 1)$ *such that*

$$|G(y_1) - G(y_2)| \leq \tilde{K}|y_1 - y_2|, \quad \textit{for all} \quad y_1, y_2 \in \mathbb{R}^n. \tag{3.5}$$

Let $C^2(\mathbb{R}^n; \mathbb{R}_+)$ be the family of all nonnegative functions $V(x)$ on $\mathbb{R}^n$ that are continuously twice differentiable in x. If $V \in C^2(\mathbb{R}^n; \mathbb{R}_+)$, define an operator LV from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$ by

$$\begin{aligned} LV(x, y) = {} & V_x(x - G(y))f(x, y) + \frac{1}{2}\text{trace}[g^T(x, y)V_{xx}(x - G(y))g(x, y)] \\ & + \sum_{k=1}^{d} \int_{\mathbb{R}} [V\big((x - G(y)) + h^{(k)}(x, y, z_k)\big) - V(x - G(y)) \\ & - V_x(x - G(y))h^{(k)}(x, y, z_k)]\nu_k(dz_k), \end{aligned} \tag{3.6}$$

where

$$V_x(x) = \left(\frac{\partial V(x)}{\partial x_1}, \ldots, \frac{\partial V(x)}{\partial x_n}\right), \quad V_{xx}(x) = \left(\frac{\partial^2 V(x)}{\partial x_i \partial x_j}\right)_{n \times n},$$

and we will use $X(t)$ instead of $X(t^-)$ sometimes in the following because this will not effect on the Lebesgue integrals involved. Then the Itô formula gives that if $V \in C^2(\mathbb{R}^n; \mathbb{R}_+)$, then for any $t \geq 0$

$$\begin{aligned} & V\big(X(t) - G(X(t - \tau))\big) - V\big(X(0) - G(\xi(-\tau))\big) \\ = & \int_0^t LV(X(s), X(s - \tau))ds \\ & + \int_0^t V_x\big(X(s) - G(X(s - \tau))\big)g(X(s), X(s - \tau))dB(s) \\ & + \sum_{k=1}^{d} \int_0^t \int_{\mathbb{R}} [V\big((X(s^-) - G(X((s - \tau)^-))) + h^{(k)}(X(s^-), X(s - \tau)^-), z_k)\big) \\ & - V\big(X(s^-) - G(X((s - \tau)^-))\big)]\tilde{N}_k(ds, dz_k). \end{aligned} \tag{3.7}$$

Lemma 3.1 *(**Semi-martingale convergence theorem**) Let $A_1(t)$ and $A_2(t)$ be two adapted increasing processes on $t \geq 0$ with $A_1(0) = A_2(0) = 0$ a.s. Let $M(t)$ be a real-valued local martingale with $M(0) = 0$ a.s. Let ζ be a nonnegative $\mathcal{F}_0$-measurable random variable such that $\mathbb{E}\zeta < \infty$. Define*

$$\tilde{X}(t) = \zeta + A_1(t) - A_2(t) + M(t) \quad \textit{for any} \quad t \geq 0.$$

If $\tilde{X}(t)$ is nonnegative, then

$$\left\{\lim_{t\to\infty} A_1(t) < \infty\right\} \subset \left\{\lim_{t\to\infty} \tilde{X}(t) < \infty\right\} \cap \left\{\lim_{t\to\infty} A_2(t) < \infty\right\} \quad \textit{a.s.},$$

where $C \subset D$ a.s. means $\mathbb{P}(C \cap D^c) = 0$. In particular, if $\lim_{t\to\infty} A_1(t) < \infty$ a.s. then, with probability 1,

$$\lim_{t\to\infty} \tilde{X}(t) < \infty, \quad \lim_{t\to\infty} A_2(t) < \infty$$

and

$$-\infty < \lim_{t\to\infty} M(t) < \infty.$$

That is, all of the three processes $\tilde{X}(t), A_2(t)$ and $M(t)$ converge to finite random variables.

3.3 Almost Sure Asymptotic Stability for Neutral Stochastic Differential Delay Equations with Jumps

Let $D(\mathbb{R}^n; \mathbb{R}_+)$ be the family of all right-continuous and left-limit nonnegative functions defined on $\mathbb{R}^n$. If K is a subset of $\mathbb{R}^n$, denote by $d(x, K)$ the Haussdorff semi-distance between $x \in \mathbb{R}^n$ and the set K, namely $d(x, K) =$

$\inf_{y\in K}|x-y|$, If W is a real-valued function defined on $\mathbb{R}^n$, then its kernel is denoted by $Ker(W)$, namely $Ker(W)=\{x\in\mathbb{R}^n : W(x)=0\}$.
For the purposes of stability, we shall assume that for all $t\in\mathbb{R}_+$,

$$G(0)\equiv 0,\quad f(0,0)\equiv 0,\quad g(0,0)\equiv 0,\quad h(0,0,z)\equiv 0, \tag{3.8}$$

which admits the trivial solution $\xi\equiv 0$ for (3.1).

Theorem 3.1 *Let the assumption 3.1 holds. Assume that there are three functions $V\in C^2(\mathbb{R}^n;\mathbb{R}_+)$, $U\in C(\mathbb{R}^n;\mathbb{R}_+)$ and $W\in C(\mathbb{R}^n;\mathbb{R}_+)$ such that*

$$LV(x,y)\le -\lambda_1 U(x)+\lambda_2 U(y)-W(x-G(y)), \tag{3.9}$$

for $(x,y)\in\mathbb{R}^n\times\mathbb{R}^n$ with $\lambda_1>\lambda_2>0$, and

$$\lim_{|x|\to\infty} V(x)=\infty. \tag{3.10}$$

Then for any initial data $\xi=\{\xi(t),t\in[-\tau,0]\}\in D^b_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n)$, the (3.1) *has a unique global solution which is denoted by $X(t;\xi)$. Moreover, the solution obeys that*

$$\limsup_{t\to\infty} V\big(X(t;\xi)-G(X(t-\tau;\xi))\big)<\infty \quad a.s. \tag{3.11}$$

and $Ker(W)\neq\emptyset$ and

$$\lim_{t\to\infty} d\big(X(t;\xi)-G(X(t-\tau;\xi)),Ker(W)\big)=0 \quad a.s. \tag{3.12}$$

In particular, if W moreover has the property that

$$W(x)=0 \quad \text{if and only if} \quad x=0, \tag{3.13}$$

then the solution further obeys that

$$\lim_{t\to\infty} X(t;\xi)=0 \quad a.s. \tag{3.14}$$

Before we prove the theorem 3.1, we now first show the uniqueness and existence of the global solution to (3.1) as follows.

Theorem 3.2 *Let the assumption 3.1 holds. Assume that function* $V \in C^2(\mathbb{R}^n; \mathbb{R}_+)$*, there exists a positive constant* C*, which may different line by line, such that*

$$LV(x,y) \leq -\lambda_1 U(x) + \lambda_2 U(y), \tag{3.15}$$

for $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$ *with* $\lambda_1 > \lambda_2 > 0$*, and*

$$\lim_{|x| \to \infty} V(x) = \infty. \tag{3.16}$$

Then for any initial data $\xi = \{\xi(t), t \in [-\tau, 0]\} \in D^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$*, the* (3.1) *exists a unique global solution which is denoted by* $X(t)$ *on* $t \geq -\tau$*.*

Proof. Let $\bar{\xi}$ be the bound for ξ. For each integer $R \geq \bar{\xi}$, define

$$f^{(R)}(x,y) = f\left(\frac{|x| \wedge R}{|x|}x, \frac{|y| \wedge R}{|y|}y\right),$$

where we set $(|x| \wedge R/|x|)x = 0$ when $x = 0$. Define $g^{(R)}(x,y)$ and $h^{(R)}(x,y,z)$ similarly. Consider the NSDDEwJs

$$\begin{aligned} d[X_R(t) - G(X_R(t-\tau))] &= f^{(R)}(X_R(t), X_R(t-\tau))dt \\ &+ g^{(R)}(X_R(t), X_R(t-\tau))dB(t) + \int_{\mathbb{R}^d} h^{(R)}(X_R(t^-), X_R((t-\tau)^-), z)\tilde{N}(dt, dz), \end{aligned} \tag{3.17}$$

on $t \geq 0$ with initial ξ. By the assumption 3.1, we observe that $f^{(R)}$, $g^{(R)}$ and $h^{(R)}$ satisfy the global Lipschitz condition and the linear growth condition. By the known uniqueness and existence theorem [1, Theorem 6.2.3 P304], there exists a unique global solution $X_R(t)$ on $t \in [0, \tau]$ to the equation

$$X_R(t) = \xi(0) + G(\xi(t-\tau)) - G(\xi(-\tau)) + \int_0^t f^{(R)}(X_R(s), \xi(s-\tau))ds$$
$$+ \int_0^t g^{(R)}(X_R(s), \xi(s-\tau))dB(s)$$
$$+ \int_0^t \int_{\mathbb{R}^d} h^{(R)}(X_R(s^-), \xi((s-\tau)^-), z)\tilde{N}(ds, dz). \tag{3.18}$$

Once we obtain the unique solution on $[0, \tau]$ we can regard them as the initial data and consider (3.17) on $t \in [\tau, 2\tau]$. In this case, (3.17) can be written as

$$X_R(t) = \xi(\tau) + G(\xi(t-\tau)) - G(\xi(0)) + \int_\tau^t f^{(R)}(X_R(s), \xi(s-\tau))ds$$
$$+ \int_\tau^t g^{(R)}(X_R(s), \xi(s-\tau))dB(s)$$
$$+ \int_\tau^t \int_{\mathbb{R}^d} h^{(R)}(X_R(s^-), \xi((s-\tau)^-), z)\tilde{N}(ds, dz).$$

Again, (3.17) has a unique solution $X_R(t)$ on $[\tau, 2\tau]$. Repeating this procedure on intervals $[2\tau, 3\tau]$, $[3\tau, 4\tau]$ and so on we obtain the unique solution $X_R(t)$ to (3.17) on $t \geq -\tau$. Let us now define a stopping time

$$\sigma_R = \inf\{t \geq 0 : |X_R(t)| > R\}.$$

Clearly, $|X_R(s)| \vee |X_R(s-\tau)| < R$ for $0 \leq s \leq \sigma_R$. Therefore

$$f^{(R)}(X_R(s), X_R(s-\tau)) = f^{(R+1)}(X_R(s), X_R(s-\tau)),$$
$$g^{(R)}(X_R(s), X_R(s-\tau)) = g^{(R+1)}(X_R(s), X_R(s-\tau)),$$
$$h^{(R)}(X_R(s^-), X_R(s-\tau)^-), z) = h^{(R+1)}(X_R(s^-), X_R(s-\tau)^-), z),$$

on $\tau \leq s \leq \sigma_R$. These implies

$$X_R(t \wedge \sigma_R) = \xi(0) + G(X_R(t \wedge \sigma_R - \tau)) - G(\xi(-\tau))$$
$$+ \int_0^{t\wedge\sigma_R} f^{(R+1)}(X_R(s), X_R(s-\tau))ds + \int_0^{t\wedge\sigma_R} g^{(R+1)}(X_R(s), X_R(s-\tau))dB(s)$$
$$+ \int_0^{t\wedge\sigma_R} \int_{\mathbb{R}^d} h^{(R+1)}(X_R(s^-), X_R((s-\tau)^-), z)\tilde{N}(ds, dz).$$

So we have

$$X_R(t) = X_{R+1}(t) \quad \text{if} \quad 0 \le t \le \sigma_R.$$

This implies that σ_R is increasing in R. Let $\sigma = \lim_{R\to\infty} \sigma_R$. The property above also enables us to define $X(t)$ for $t \in [-\tau, \sigma)$ as follows

$$X(t) = X_R(t) \quad \text{if} \quad -\tau \le t \le \sigma_R.$$

It is clear that $X(t)$ is a unique solution to (3.1) for $t \in [-\tau, \sigma)$. To complete the proof of global solution, we need to show that $\mathbb{P}\{\sigma = \infty\} = 1$. By the generalized Itô formula (3.7), we have that for any $t > 0$,

$$\begin{aligned} &\mathbb{E}V[X_R(t \wedge \sigma_R) - G(X_R(t \wedge \sigma_R - \tau))] \\ &= \mathbb{E}V[\xi(0) - G(\xi(-\tau))] + \mathbb{E}\int_0^{t\wedge\sigma_R} L^{(R)}V(X_R(s), X_R(s-\tau))ds, \end{aligned} \tag{3.19}$$

where the operator $L^{(R)}V$ is defined similarly as LV which replaced f, g and h by $f^{(R)}$, $g^{(R)}$ and $h^{(R)}$, respectively. Due to the definitions of $f^{(R)}$, $g^{(R)}$ and $h^{(R)}$, we hence observe that

$$L^{(R)}V(X_R(s), X_R(s-\tau)) = LV(X_R(s), X_R(s-\tau)) \quad \text{if } 0 \le s \le t \wedge \sigma_R.$$

Using (3.15), we can then derive easily from (3.19) that

$$\begin{aligned} &\mathbb{E}\big(V[X_R(t \wedge \sigma_R) - G(X_R(t \wedge \sigma_R - \tau))]I_{\{\sigma_R \le t\}}\big) \\ &\le \mathbb{E}V[X_R(t \wedge \sigma_R) - G(X_R(t \wedge \sigma_R - \tau))] \\ &\le \mathbb{E}V[\xi(0) - G(\xi(-\tau))] + \lambda_2 \mathbb{E}\int_{-\tau}^0 U(\xi(s))ds \\ &\quad - (\lambda_1 - \lambda_2)\int_0^{t\wedge\sigma_R} \mathbb{E}U(X_R(s))ds \\ &\le \psi := \mathbb{E}V[\xi(0) - G(\xi(-\tau))] + \lambda_2 \mathbb{E}\int_{-\tau}^0 U(\xi(s))ds. \end{aligned} \tag{3.20}$$

On the other hand, for any $\omega \in \{\sigma_R \le t\}$, we have $|X(\sigma_R)| > R$ and $X(\sigma_R - \tau) < R$ so

$$\begin{aligned} &|X_R(t \wedge \sigma_R) - G(X_R(t \wedge \sigma_R - \tau))| \\ &\ge |X_R(t \wedge \sigma_R)| - |G(X_R(t \wedge \sigma_R - \tau))| \\ &\ge (1 - \tilde{K})R. \end{aligned}$$

It then follows from (3.20) and (3.16) that

$$\mathbb{P}\{\sigma_R \le t\} \le \frac{\psi}{V((1-\tilde{K})R)}.$$

Letting $R \to \infty$, we obtain that $\mathbb{P}\{\sigma \le t\} = 0$. Since t is arbitrary, we must have

$$\mathbb{P}\{\sigma = \infty\} = 1$$

as desired.

Now let us divide the proof of assertions in theorem 3.1 for five steps.

Step 1. Let us first show assertion (3.11). For fixing any initial data ξ and writing $X(t;\xi) = X(t)$ for simplicity, by the generalized Itô formula (3.7) and condition (3.9) we have

$$\begin{aligned} &V\big(X(t) - G(X(t-\tau))\big) \le V\big(\xi(0) - G(\xi(-\tau))\big) + M(t) \\ &+ \int_0^t \big[- \lambda_1 U(X(s)) + \lambda_2 U(X(s-\tau)) - W\big(X(s) - G(X(s-\tau))\big)\big] ds \\ &\le V\big(\xi(0) - G(\xi(-\tau))\big) + \lambda_2 \int_{-\tau}^0 U(\xi(s)) ds - \int_0^t W\big(X(s) - G(X(s-\tau))\big) ds + M(t), \end{aligned} \tag{3.21}$$

where

$$\begin{aligned} M(t) = &\int_0^t V_x\big(X(s) - G(X(s-\tau))\big) g(X(s), X(s-\tau)) dB(s) \\ &+ \sum_{k=1}^d \int_0^t \int_{\mathbb{R}} \big[V\big((X(s^-) - G(X((s-\tau)^-))) + h^{(k)}(X(s^-), X(s-\tau)^-), z_k)\big) \\ &- V\big(X(s^-) - G(X((s-\tau)^-))\big)\big] \tilde{N}_k(ds, dz_k), \end{aligned}$$

which is a local martingale with $M(0) = 0$ a.s. Applying lemma 3.1 we immediately obtain that

$$\limsup_{t \to \infty} V\big(X(t) - G(X(t-\tau))\big) < \infty \quad \text{a.s.}$$

which is the required assertion (3.11). It then follows easily that

$$\sup_{0 \le t < \infty} V\big(X(t) - G(X(t-\tau))\big) < \infty \quad \text{a.s.}$$

This, together with (3.10), yields

$$\sup_{0 \le t < \infty} |X(t) - G(X(t-\tau))| < \infty. \tag{3.22}$$

But for any $T > 0$, by assumption 3.1, we have if $0 \le t \le T$

$$\begin{aligned} |X(t)| &\le |G(X(t-\tau))| + |X(t) - G(X(t-\tau))| \\ &\le \tilde{K}|X(t-\tau)| + |X(t) - G(X(t-\tau))|. \end{aligned}$$

This implies

$$\begin{aligned} \sup_{0 \le t \le T} |X(t)| &\le \tilde{K} \sup_{0 \le t \le T} |X(t-\tau)| + \sup_{0 \le t \le T} |X(t) - G(X(t-\tau))| \\ &\le \tilde{K}\bar{\xi} + \tilde{K} \sup_{0 \le t \le T} |X(t)| + \sup_{0 \le t \le T} |X(t) - G(X(t-\tau))|, \end{aligned}$$

where $\bar{\xi}$ is the bound for the initial data ξ. Hence

$$\sup_{0 \le t \le T} |X(t)| \le \frac{1}{1 - \tilde{K}} \left(\tilde{K}\bar{\xi} + \sup_{0 \le t \le T} |X(t) - G(X(t-\tau))| \right).$$

Letting $T \to \infty$ and using (3.22) we obtain that

$$\sup_{0 \le t < \infty} |X(t)| < \infty \quad \text{a.s.} \tag{3.23}$$

Step 2. Taking the expectations on both sides of (3.21) and letting $t \to \infty$, we obtain that

$$\mathbb{E} \int_0^\infty W(X(s) - G(X(s-\tau))ds < \infty. \tag{3.24}$$

This of course implies

$$\int_0^\infty W(X(s) - G(X(s-\tau))ds < \infty \quad \text{a.s.} \tag{3.25}$$

Set $y(t) = X(t) - G(X(t-\tau))$ for $t \geq 0$. It is straightforward to see from (3.25) that

$$\liminf_{t\to\infty} W(y(t)) = 0 \quad \text{a.s.} \tag{3.26}$$

We now claim that

$$\lim_{t\to\infty} W(y(t)) = 0 \quad \text{a.s.} \tag{3.27}$$

If this is false, then

$$\mathbb{P}\bigg\{ \limsup_{t\to\infty} W(y(t)) > 0 \bigg\} > 0.$$

Hence there is a number $\varepsilon > 0$ such that

$$\mathbb{P}(\Omega_1) \geq 3\varepsilon, \tag{3.28}$$

where

$$\Omega_1 = \bigg\{ \limsup_{t\to\infty} W(y(t)) > 2\varepsilon \bigg\}.$$

Recalling (3.23) as well as the boundedness of the initial data ξ, we can find a positive number R, which depends on ε, sufficiently large for

$$\mathbb{P}(\Omega_2) \geq 1 - \varepsilon, \tag{3.29}$$

where

$$\Omega_2 = \bigg\{ \sup_{-\tau\leq t<\infty} |y(t)| < R \bigg\}.$$

It is easy to see from (3.28) and (3.29) that

$$\mathbb{P}(\Omega_1 \cap \Omega_2) \geq 2\varepsilon. \tag{3.30}$$

We now define a sequence of stopping times,

$$\begin{aligned}
\tau_R &= \inf\{t \geq 0 : |y(t)| \geq R\}, \\
\sigma_1 &= \inf\{t \geq 0 : W(y(t)) \geq 2\varepsilon\}, \\
\sigma_{2l} &= \inf\{t \geq \sigma_{2l-1} : W(y(t)) \leq \varepsilon\}, \quad l = 1, 2, \ldots, \\
\sigma_{2l+1} &= \inf\{t \geq \sigma_{2l} : W(y(t)) \geq 2\varepsilon\}, \quad l = 1, 2, \ldots,
\end{aligned}$$

where throughout this chapter we set $\inf \emptyset = \infty$. From (3.26) and the definitions of Ω_1 and Ω_2 we observe that if $\omega \in \Omega_1 \cap \Omega_2$, then

$$\tau_R = \infty \quad \text{and} \quad \sigma_l < \infty, \quad \forall l \geq 1. \tag{3.31}$$

Let I_A denote the indicator function of set A. Noting the fact that $\sigma_{2l} < \infty$ whenever $\sigma_{2l-1} < \infty$, we derive from (3.24) that

$$\begin{aligned}
\infty > &\mathbb{E} \int_0^\infty W(y(t))dt \\
\geq &\sum_{l=1}^{\infty} \mathbb{E}\left[I_{\{\sigma_{2l-1}<\infty, \sigma_{2l}<\infty, \tau_R=\infty\}} \int_{\sigma_{2l-1}}^{\sigma_{2l}} W(y(t))dt \right] \\
\geq &\varepsilon \sum_{l=1}^{\infty} \mathbb{E}[I_{\{\sigma_{2l-1}<\infty, \tau_R=\infty\}}(\sigma_{2l} - \sigma_{2l-1})].
\end{aligned} \tag{3.32}$$

On the other hand, by assumption 3.1, there exists a constant $L_R > 0$ such that

$$|f(x,y)|^2 \vee |g(x,y)|^2 \vee \sum_{k=1}^{d} \int_{\mathbb{R}} |h^{(k)}(x,y,z_k)|^2 \nu_k(dz_k) \leq L_R \tag{3.33}$$

whenever $|x| \vee |y| \leq R$. By the Hölder inequality (2.1), the Doob martingale inequality (2.4) and (3.33), we compute that, for any $T > 0$ and $l = 1, 2, \ldots$,

$$\begin{aligned}
&\mathbb{E}\Big[I_{\{\tau_R \wedge \sigma_{2l-1} < \infty\}} \sup_{0 \leq t \leq T} |y(\tau_R \wedge (\sigma_{2l-1} + t)) - y(\tau_R \wedge \sigma_{2l-1})|^2\Big] \\
&\leq 3\mathbb{E}\Big[I_{\{\tau_R \wedge \sigma_{2l-1} < \infty\}} \sup_{0 \leq t \leq T} \Big|\int_{\tau_R \wedge \sigma_{2l-1}}^{\tau_R \wedge (\sigma_{2l-1}+t)} f(X(s), X(s-\tau))ds\Big|^2\Big] \\
&\quad + 3\mathbb{E}\Big[I_{\{\tau_R \wedge \sigma_{2l-1} < \infty\}} \sup_{0 \leq t \leq T} \Big|\int_{\tau_R \wedge \sigma_{2l-1}}^{\tau_R \wedge (\sigma_{2l-1}+t)} g(X(s), X(s-\tau))dB(s)\Big|^2\Big] \\
&\quad + 3\mathbb{E}\Big[I_{\{\tau_R \wedge \sigma_{2l-1} < \infty\}} \sup_{0 \leq t \leq T} \Big|\int_{\tau_R \wedge \sigma_{2l-1}}^{\tau_R \wedge (\sigma_{2l-1}+t)} \int_{\mathbb{R}^d} h(X(s^-), X((s-\tau)^-), z)\tilde{N}(ds, dz)\Big|^2\Big] \\
&\leq 3T\mathbb{E}\Big[I_{\{\tau_R \wedge \sigma_{2l-1} < \infty\}} \int_{\tau_R \wedge \sigma_{2l-1}}^{\tau_R \wedge (\sigma_{2l-1}+T)} |f(X(s), X(s-\tau))|^2 ds\Big] \\
&\quad + 12\mathbb{E}\Big[I_{\{\tau_R \wedge \sigma_{2l-1} < \infty\}} \int_{\tau_R \wedge \sigma_{2l-1}}^{\tau_R \wedge (\sigma_{2l-1}+T)} |g(X(s), X(s-\tau))|^2 ds\Big] \\
&\quad + C_T \sum_{k=1}^{d} \mathbb{E}\Big[I_{\{\tau_R \wedge \sigma_{2l-1} < \infty\}} \int_{\tau_R \wedge \sigma_{2l-1}}^{\tau_R \wedge (\sigma_{2l-1}+T)} \int_{\mathbb{R}} |h^{(k)}(X(s), X(s-\tau), z_k)|^2 \nu_k(dz_k)ds\Big] \\
&\leq C_T L_R,
\end{aligned} \tag{3.34}$$

where we use $X(t)$ instead of $X(t^-)$ because this will not efect on the Lebesgue integrals involved, and C_T is a positive constant which may different line by line. Since W is continuous in $\mathbb{R}^n$, it must be uniformly continuous in the closed ball $\bar{S}_R = \{x \in \mathbb{R}^n : |x| \leq R\}$. We can therefore choose $\delta = \delta(\varepsilon) > 0$ so small such that

$$|W(x) - W(y)| < \frac{\varepsilon}{2} \quad \text{whenever} \quad x, y \in \bar{S}_R, \quad |x - y| < \delta. \tag{3.35}$$

We furthermore choose $T = T(\varepsilon, \delta, R) > 0$ sufficiently small for $\frac{C_T L_R}{\delta^2} < \varepsilon$. It then follows from (3.34) that

$$\mathbb{P}\Big(\{\tau_R \wedge \sigma_{2l-1} < \infty\} \cap \Big\{\sup_{0 \leq t \leq T} |y(\tau_R \wedge (\sigma_{2l-1} + t)) - y(\tau_R \wedge \sigma_{2l-1})| \geq \delta\Big\}\Big) < \varepsilon.$$

Noting that

$$\{\tau_R = \infty, \sigma_{2l-1} < \infty\} = \{\tau_R \wedge \sigma_{2l-1} < \infty, \tau_R = \infty\} \subset \{\tau_R \wedge \sigma_{2l-1} < \infty\},$$

we hence have

$$\mathbb{P}\bigg(\{\tau_R = \infty, \sigma_{2l-1} < \infty\} \cap \bigg\{\sup_{0\le t\le T} |y(\sigma_{2l-1}+t) - y(\sigma_{2l-1})| \ge \delta\bigg\}\bigg) < \varepsilon.$$

By (3.30) and (3.31), we further compute

$$\begin{aligned}
&\mathbb{P}\bigg(\{\tau_R = \infty, \sigma_{2l-1} < \infty\} \cap \bigg\{\sup_{0\le t\le T} |y(\sigma_{2l-1}+t) - y(\sigma_{2l-1})| < \delta\bigg\}\bigg) \\
&= \mathbb{P}(\{\tau_R = \infty, \sigma_{2l-1} < \infty\}) \\
&\quad - \mathbb{P}\bigg(\{\tau_R = \infty, \sigma_{2l-1} < \infty\} \cap \bigg\{\sup_{0\le t\le T} |y(\sigma_{2l-1}+t) - y(\sigma_{2l-1})| \ge \delta\bigg\}\bigg) \\
&> \varepsilon.
\end{aligned}$$

By (3.35) we hence obtain that

$$\mathbb{P}\bigg(\{\tau_R = \infty, \sigma_{2l-1} < \infty\} \cap \bigg\{\sup_{0\le t\le T} |W(y(\sigma_{2l-1}+t)) - W(y(\sigma_{2l-1}))| < \varepsilon\bigg\}\bigg) > \varepsilon. \tag{3.36}$$

Set

$$\bar{\Omega}_l = \bigg\{\sup_{0\le t\le T} |W(y(\sigma_{2l-1}+t)) - W(y(\sigma_{2l-1}))| < \varepsilon\bigg\}.$$

Noting that

$$\sigma_{2l}(\omega) - \sigma_{2l-1}(\omega) \ge T \quad \text{if } \omega \in \{\tau_R = \infty, \sigma_{2l-1} < \infty\} \cap \bar{\Omega}_l,$$

we derive from (3.32) and (3.36) that

$$\begin{aligned}\infty &> \varepsilon \sum_{l=1}^{\infty} \mathbb{E}[I_{\{\tau_R=\infty, \sigma_{2l-1}<\infty\}}(\sigma_{2l}-\sigma_{2l-1})] \\ &\geq \varepsilon \sum_{l=1}^{\infty} \mathbb{E}[I_{\{\tau_R=\infty, \sigma_{2l-1}<\infty\}\cap\bar{\Omega}_l}(\sigma_{2l}-\sigma_{2l-1})] \\ &\geq T\varepsilon \sum_{l=1}^{\infty} \mathbb{P}(\{\tau_R=\infty, \sigma_{2l-1}<\infty\}\cap\bar{\Omega}_l) \\ &\geq T\varepsilon \sum_{l=1}^{\infty} \varepsilon = \infty,\end{aligned}$$

which is a contradiction. So (3.27) must holds.

Step 3. Let us now show that $Ker(W) \neq \emptyset$. From (3.27) and (3.22) we see that there is an $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that

$$\lim_{t\to\infty} W(y(t,\omega)) = 0 \quad \text{and} \quad \sup_{0\leq t<\infty} |y(t,\omega)| < \infty \text{ for all } \omega \in \Omega_0. \tag{3.37}$$

Choose any $\omega \in \Omega_0$. Then $\{y(t,\omega)\}_{t\geq 0}$ is bounded in $\mathbb{R}^n$ so there must be an increasing sequence $\{t_l\}_{l\geq 1}$ such that $t_l \to \infty$ and $\{y(t_l,\omega)\}_{l\geq 1}$ converges to some $\bar{y} \in \mathbb{R}^n$. Thus $W(\bar{y}) = \lim_{l\to\infty} W(y(t_l,\omega)) = 0$, which implies that $\bar{y} \in Ker(W)$ whence $Ker(W) \neq \emptyset$.

Step 4. We can now show assertion (3.12). It is clearly sufficient if we could show that

$$\lim_{t\to\infty} d(y(t,\omega), Ker(W)) = 0 \quad \text{for all} \quad \omega \in \Omega_0. \tag{3.38}$$

If this is false, then there are some $\bar{\omega} \in \Omega_0$ such that

$$\limsup_{t\to\infty} d(y(t,\bar{\omega}), Ker(W)) > 0.$$

Hence there is a subsequence $\{y(t_l,\bar{\omega})\}_{l\geq 0}$ of $\{y(t,\bar{\omega})\}_{t\geq 0}$ such that

$$\limsup_{l\to\infty} d(y(t_l,\bar{\omega}), Ker(W)) > \bar{\varepsilon}$$

for some $\bar{\varepsilon} > 0$. Since $\{y(t_l, \bar{\omega})\}_{l\geq 0}$ is bounded, we can find its subsequence $\{y(\bar{t}_l, \bar{\omega})\}_{l\geq 0}$ which converges to some $\hat{y} \in \mathbb{R}^n$. Clearly, $\hat{y}$ is not in $Ker(W)$ so $W(\hat{y}) > 0$. But, by (3.37), $W(\hat{y}) = \lim_{l\to\infty} W(y(\bar{t}_l, \bar{\omega})) = 0$, It is a contradiction. Hence (3.38) must holds.

Step 5. Finally, let us show assertion (3.14) under the additional condition (3.13). Clearly, (3.13) implies that $Ker(W) = \{0\}$. It then follows from (3.12) that

$$\lim_{t\to\infty} [X(t) - G(X(t-\tau))] = \lim_{t\to\infty} y(t) = 0 \quad \text{a.s.}$$

But, by (3.5),

$$\begin{aligned} |X(t)| &\leq |G(X(t-\tau))| + |X(t) - G(X(t-\tau))| \\ &\leq \tilde{K}|X(t-\tau)| + |X(t) - G(X(t-\tau))|. \end{aligned}$$

Letting $t \to \infty$ we obtain that

$$\limsup_{t\to\infty} |X(t)| \leq \tilde{K} \limsup_{t\to\infty} |X(t)| \quad \text{a.s.}$$

This, together with (3.23), yields

$$\limsup_{t\to\infty} |X(t)| = 0 \quad \text{a.s.}$$

which is the required assertion (3.14). The proof is therefore complete.

3.4 Example

Let $B(t)$ be a 1-dimensional standard Brownian motion, $N(t, z)$ be a 1-dimensional Poisson process and denote the compensated Poisson process by

$$\tilde{N}(dt, dz) = (N(dt, dz) - \nu(dz)dt).$$

We assume that $B(t)$ and $N(dt, dz)$ are independent. Consider a scalar non-linear NSDDEwJs of the form

$$d[X(t) - G(X(t-\tau))] = f(X(t), X(t-\tau))dt + g(X(t), X(t-\tau))dB(t) + \int_{\mathbb{R}} h(X(t^-), X((t-\tau)^-), z)\tilde{N}(dt, dz).$$

Suppose that

$$G(y) = -0.1y, \quad f(x) = -x^3 - 2x, \quad g(y) = y^2 \sin t$$
$$h(x, y, z) = A(z)h'(x, y), \quad h'(x, y) = 0.1(x^2 + y^2),$$
$$\int_{\mathbb{R}} A^2(z)\nu(dz) = 1.$$

Define $V(x) = x^2$, therefore the operator

$$LV : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

takes the form

$$LV(x, y) \leq -1.83x^4 - 2.3x^2 + 1.07y^4 + 0.53y^2 - 0.1(x + 0.1y)^2,$$

By defining $U(x) = x^4 + 1.2x^2$, $W(x) = 0.1x^2$, we hence have

$$LV(x, y) \leq -1.83U(x) + 1.07U(y) - W(x - G(y))$$

which follows the theorem 3.1.

Chapter 4

Convergence of Euler-Maryuama Method for Neutral Stochastic Differential Delay Equations with Jumps

4.1 Introduction

Most of NSDDEwJs do not have explicit solutions and hence require numerical solutions. However, there are seldom explicit formula for solutions to NSDDEwJs, and several numerical schemes have been developed to produce approximate solutions. Mao and Sabanis in [42] have proved that the numerical solution of the Euler scheme converges to the true solution in the sense of strong convergence for SDDEs under a local Lipschitz condition and a linear growth condition, in [24], Higham and Kloeden have investigated the strong convergence of numerical solutions for SDEwJs. Moreover, in [27], Jacob, Wang and Yuan estimated the rate convergence of numerical solutions of SDDEwJs. The main aim of this chapter is to investigate that the Euler-Maruyama numerical solutions will converge to the true solutions of NSDDEwJs under the local Lipschitz condition, and we do not only show the convergence of the Euler scheme, but also reveal the rate of the convergence. It is the first time that the rate of convergence has been obtained under local Lipschitz condition for NSDDEs to the best of our knowledge.

4.2 Euler-Maryuama Method for Neutral Stochastic Differential Delay Equations with Jumps

Let the time step-size $\Delta \in (0,1)$, and $\Delta = \frac{T}{N} = \frac{\tau}{m}$ for the sufficiently large integer N, we also let $t_n = n\Delta$ for $n = 0, 1, \ldots, N$. Compute the discrete approximations $Y_n \approx X(t_n)$ by setting $Y_0 = X(t_0)$ and performing

$$\begin{aligned} Y_{n+1} - G(Y_{n+1-m}) = Y_n - G(Y_{n-m}) + f(Y_n, Y_{n-m})\Delta + g(Y_n, Y_{n-m})\Delta B_n \\ + \int_{\mathbb{R}^d} h(Y_n, Y_{n-m}, z)\Delta\tilde{N}_n(dz) \end{aligned} \tag{4.1}$$

where $\Delta B_n = B(t_{n+1}) - B(t_n)$ and $\Delta\tilde{N}_n(dz) = \tilde{N}(t_{n+1}, dz) - \tilde{N}(t_n, dz)$.

Let us introduce the following notations $\bar{Y}(t) = Y_n$, $\bar{Y}(t-\tau) = Y_{n-m}$, for $t \in [t_n, t_{n+1})$, with the initial value $\bar{Y}(0) = \xi(0)$. The continuous EM approximate solution $Y(t)$ is to be interpreted as the stochastic integral

$$\begin{aligned} Y(t) = \xi(0) - G(\xi(-\tau)) + G(\bar{Y}(t-\tau)) + \int_0^t f(\bar{Y}(s), \bar{Y}(s-\tau))ds \\ + \int_0^t g(\bar{Y}(s), \bar{Y}(s-\tau))dB(s) + \int_0^t \int_{\mathbb{R}^d} h(\bar{Y}(s^-), \bar{Y}((s-\tau)^-), z)\tilde{N}(ds, dz), \end{aligned} \tag{4.2}$$

for $t \in [0, T]$ with initial data $\xi = \{\xi(t), t \in [-\tau, 0]\}$. Therefore we have for any $t \geq 0$,

$$
\begin{aligned}
Y(t) = \xi(0) - G(\xi(-\tau)) + G(\bar{Y}(t-\tau)) + \int_0^{n\Delta} f(\bar{Y}(s), \bar{Y}(s-\tau))ds \\
+ \int_0^{n\Delta} g(\bar{Y}(s), \bar{Y}(s-\tau))dB(s) + \int_0^{n\Delta} \int_{\mathbb{R}^d} h(\bar{Y}(s^-), \bar{Y}((s-\tau)^-), z)\tilde{N}(ds, dz) \\
+ \int_{n\Delta}^t f(\bar{Y}(s), \bar{Y}(s-\tau))ds + \int_{n\Delta}^t g(\bar{Y}(s), \bar{Y}(s-\tau))dB(s) \\
+ \int_{n\Delta}^t \int_{\mathbb{R}^d} h(\bar{Y}(s^-), \bar{Y}((s-\tau)^-), z)\tilde{N}(ds, dz).
\end{aligned}
\tag{4.3}
$$

Clearly, $Y(t_n) = X(t_n) = Y_n = \bar{Y}(t_n)$, that is, $Y(t)$ and $\bar{Y}(t)$ coincide with the discrete approximate solution at the grid-points.

Assume that f, g, h satisfy the linear growth condition that

$$
|f(x,y)|^2 + |g(x,y)|^2 + \sum_{k=1}^{d} \int_{\mathbb{R}} |h^{(k)}(x,y,z_k)|^2 \nu_k(dz_k) \leq K(1 + |x|^2 + |y|^2),
\tag{4.4}
$$

and

$$
\sum_{k=1}^{d} \int_{\mathbb{R}} |h^{(k)}(x,y,z_k)|^p \nu_k(dz_k) \leq K(1 + |x|^p + |y|^p),
\tag{4.5}
$$

for $x, y \in \mathbb{R}^n$, $p > 2$.

Lemma 4.1 *Under the linear growth condition* (4.4) *for any* $p \leq 2$ *and* (4.5) *for any* $p > 2$, *there exists a positive constant* H_p *which is independent of* Δ *such that*

$$
\mathbb{E}\Big(\sup_{0 \leq t \leq T} |X(t)|^p\Big) \vee \mathbb{E}\Big(\sup_{0 \leq t \leq T} |Y(t)|^p\Big) \leq H_p.
\tag{4.6}
$$

Proof. Consider the proof of the first part is more simple which can be contained by the second one, hence we here only prove the complicated one as follow.

By using (3.5) and the inequality $|a+b|^p \le (1+\alpha^{\frac{1}{p-1}})^{p-1}(|a|^p + \frac{1}{\alpha}|b|^p)$ we have, let $\alpha = (\frac{\tilde{K}}{1-\tilde{K}})^{p-1}$,

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\le s\le t}|Y(s)|^p\right)\\
&\le \left(1+\alpha^{\frac{1}{p-1}}\right)^{p-1}\left(\mathbb{E}\sup_{0\le s\le t}|Y(s)-G(\bar{Y}(s-\tau))|^p + \frac{1}{\alpha}\mathbb{E}\sup_{0\le s\le t}|G(\bar{Y}(s-\tau))|^p\right)\\
&\le \left(\frac{1}{1-\tilde{K}}\right)^{p-1}\mathbb{E}\sup_{0\le s\le t}|Y(s)-G(\bar{Y}(s-\tau))|^p + \tilde{K}\mathbb{E}\sup_{0\le s\le t}|\bar{Y}(s-\tau)|^p\\
&\le \left(\frac{1}{1-\tilde{K}}\right)^{p-1}\mathbb{E}\sup_{0\le s\le t}|Y(s)-G(\bar{Y}(s-\tau))|^p + \tilde{K}\mathbb{E}\sup_{-\tau\le s\le 0}|\bar{Y}(s)|^p + \tilde{K}\mathbb{E}\sup_{0\le s\le t}|\bar{Y}(s)|^p\\
&\le \left(\frac{1}{1-\tilde{K}}\right)^{p}\mathbb{E}\sup_{0\le s\le t}|Y(s)-G(\bar{Y}(s-\tau))|^p + \frac{\tilde{K}}{1-\tilde{K}}\mathbb{E}\sup_{-\tau\le s\le 0}|\xi(s)|^p.
\end{aligned}$$

Using the form of EM approximation (4.2) and Hölder's inequality (2.1), we compute straightforward

$$\begin{aligned}
\mathbb{E}\left(\sup_{0\le s\le t}|Y(s)|^p\right) &\le \frac{5^{p-1}}{(1-\tilde{K})^p}\mathbb{E}|\xi(0)|^p + \frac{5^{p-1}}{(1-\tilde{K})^p}\mathbb{E}|G(\xi(-\tau))|^p\\
&+ \frac{\tilde{K}}{1-\tilde{K}}\mathbb{E}\sup_{-\tau\le s\le 0}|\xi(s)|^p + \frac{5^{p-1}}{(1-\tilde{K})^p}\mathbb{E}\left(\sup_{0\le s\le t}\left|\int_0^s f(\bar{Y}(u),\bar{Y}(u-\tau))du\right|^p\right)\\
&+ \frac{5^{p-1}}{(1-\tilde{K})^p}\mathbb{E}\left(\sup_{0\le s\le t}\left|\int_0^s g(\bar{Y}(u),\bar{Y}(u-\tau))dB(u)\right|^p\right)\\
&+ \frac{5^{p-1}}{(1-\tilde{K})^p}\mathbb{E}\left(\sup_{0\le s\le t}\left|\int_0^s\int_{\mathbb{R}^d} h(\bar{Y}(u^-),\bar{Y}((u-\tau)^-),z)\tilde{N}(du,dz)\right|^p\right)\\
&\le \frac{5^{p-1}}{(1-\tilde{K})^p}\mathbb{E}|\xi(0)|^p + \frac{5^{p-1}\tilde{K}^p}{(1-\tilde{K})^p}\mathbb{E}|\xi(-\tau)|^p + \frac{\tilde{K}}{1-\tilde{K}}\mathbb{E}\sup_{-\tau\le s\le 0}|\xi(s)|^p\\
&+ \frac{(5T)^{p-1}}{(1-\tilde{K})^p}\int_0^s \mathbb{E}|f(\bar{Y}(u),\bar{Y}(u-\tau))|^p du\\
&+ \frac{5^{p-1}}{(1-\tilde{K})^p}\mathbb{E}\left(\sup_{0\le s\le t}\left|\int_0^s g(\bar{Y}(u),\bar{Y}(u-\tau))dB(u)\right|^p\right)\\
&+ \frac{5^{p-1}}{(1-\tilde{K})^p}\mathbb{E}\left(\sup_{0\le s\le t}\left|\int_0^s\int_{\mathbb{R}^d} h(\bar{Y}(u^-),\bar{Y}((u-\tau)^-),z)\tilde{N}(du,dz)\right|^p\right).
\end{aligned}$$

By using Burkholder-Davis-Gundy's inequality (2.5) we have

$$\begin{aligned}
&\mathbb{E}\bigg(\sup_{0\le s\le t}\bigg|\int_0^s g(\bar{Y}(u),\bar{Y}(u-\tau))dB(u)\bigg|^p\bigg)\\
&\le C_p\mathbb{E}\bigg(\int_0^s |g(\bar{Y}(u),\bar{Y}(u-\tau))|^2du\bigg)^{\frac{p}{2}}\\
&\le C_pT^{\frac{p}{2}-1}\mathbb{E}\int_0^s |g(\bar{Y}(u),\bar{Y}(u-\tau))|^pdu,
\end{aligned}$$

where C_p is a positive constant, which depends only on p. We also have

$$\begin{aligned}
&\mathbb{E}\bigg(\sup_{0\le s\le t}\bigg|\int_0^s\int_{\mathbb{R}^d} h(\bar{Y}(u^-),\bar{Y}((u-\tau)^-),z)\tilde{N}(du,dz)\bigg|^p\bigg)\\
&\le C_p\mathbb{E}\int_0^s\bigg(\sum_{k=1}^d\int_{\mathbb{R}}|h^{(k)}(\bar{Y}(u),\bar{Y}(u-\tau),z_k)|^2\nu_k(dz_k)\bigg)^{\frac{p}{2}}du\\
&\quad + C_{p,t}\sum_{k=1}^d\mathbb{E}\int_0^s\int_{\mathbb{R}}|h^{(k)}(\bar{Y}(u),\bar{Y}(u-\tau),z_k)|^p\nu_k(dz_k)du,
\end{aligned}$$

where $C_{p,t}$ is also a positive constant, which depends only on p and t, which follows from [3, Theorem 2.4] (see also [28]). Then we have by using (4.4) and (4.5)

$$\begin{aligned}
&\mathbb{E}\bigg(\sup_{0\le s\le t}|Y(s)|^p\bigg)\\
&\le \frac{5^{p-1}}{(1-\tilde{K})^p}\mathbb{E}|\xi(0)|^p + \bigg(\frac{\tilde{K}^p5^{p-1}}{(1-\tilde{K})^p}+\frac{\tilde{K}}{1-\tilde{K}}\bigg)\mathbb{E}\sup_{-\tau\le s\le 0}|\xi(s)|^p\\
&\quad + \bigg(\frac{K^p(5T)^{p-1}}{(1-\tilde{K})^p}+\frac{K^p5^{p-1}C_pT^{\frac{p}{2}-1}}{(1-\tilde{K})^p}+\frac{C_{p,t}K^p5^{p-1}}{(1-\tilde{K})^p}\bigg)\\
&\quad\times\bigg(\int_0^s\mathbb{E}|\bar{Y}(u)|^p+\mathbb{E}|\bar{Y}(u-\tau)|^pdu\bigg)\\
&\le \frac{5^{p-1}}{(1-\tilde{K})^p}\mathbb{E}|\xi(0)|^p + \bigg(\frac{\tilde{K}^p5^{p-1}}{(1-\tilde{K})^p}+\frac{\tilde{K}}{1-\tilde{K}}\bigg)\mathbb{E}\sup_{-\tau\le s\le 0}|\xi(s)|^p\\
&\quad + \bigg(\frac{K^p(5T)^{p-1}}{(1-\tilde{K})^p}+\frac{K^p5^{p-1}C_pT^{\frac{p}{2}-1}}{(1-\tilde{K})^p}+\frac{C_{p,t}K^p5^{p-1}}{(1-\tilde{K})^p}\bigg)\\
&\quad\times\bigg(T\mathbb{E}\sup_{-\tau\le s\le 0}|\xi(s)|^p+2\int_0^s\mathbb{E}\sup_{0\le r\le u}|\bar{Y}(r)|^pdu\bigg)\\
&\le C_1+C_2\int_0^s\mathbb{E}\sup_{0\le r\le u}|Y(r)|^pdu,
\end{aligned}$$

where C_1 and C_2 are two different positive constants depend on p. Then the Gronwall inequality (2.7) shows that

$$\mathbb{E}\left(\sup_{0\le s\le t}|Y(s)|^p\right)\le C_1e^{C_2T}:=H_p,$$

which complete the proof.

4.3 Convergence of Euler-Maryuama Method for Neutral Stochastic Differential Delay Equations with Jumps

Assumption 4.1 ***(Global Lipschitz condition)*** *For a positive constant K, we have*

$$|f(x_1,y_1)-f(x_2,y_2)|^2+|g(x_1,y_1)-g(x_2,y_2)|^2$$
$$+\sum_{k=1}^{d}\int_{\mathbb{R}}|h^{(k)}(x_1,y_1,z_k)-h^{(k)}(x_2,y_2,z_k)|^2\nu_k(dz_k)\le K^2(|x_2-x_1|^2+|y_2-y_1|^2),$$

for $x_1,x_2,y_1,y_2\in\mathbb{R}^n$ and $t\in[0,T]$.

Moreover there is a positive constant $\hat{K}$ such that the initial data ξ obeys

$$\mathbb{E}|\xi(v)-\xi(u)|\le\hat{K}|v-u|, \tag{4.7}$$

where $-\tau\le v\le u\le 0$.

Theorem 4.1 *Under the the assumption 4.1, we have*

$$\lim_{\Delta\to 0}\mathbb{E}\left(\sup_{0\le t\le T}|X(t)-Y(t)|^2\right)=0 \quad \text{for all} \quad t\in[0,T]. \tag{4.8}$$

Proof.

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{0\le s\le t}|X(s)-Y(s)|^2\right)\\
&\le \frac{1}{1-\tilde{K}}\mathbb{E}\sup_{0\le s\le t}|X(s)-G(X(s-\tau))-Y(s)+G(\bar{Y}(s-\tau))|^2\\
&\quad+\frac{1}{\tilde{K}}\mathbb{E}\sup_{0\le s\le t}|G(X(s-\tau))-G(\bar{Y}(s-\tau))|^2\\
&\le \frac{1}{1-\tilde{K}}\mathbb{E}\sup_{0\le s\le t}|X(s)-G(X(s-\tau))-Y(s)+G(\bar{Y}(s-\tau))|^2\\
&\quad+\tilde{K}\mathbb{E}\sup_{0\le s\le t}|X(s-\tau)-\bar{Y}(s-\tau)|^2\\
&\le \frac{1}{1-\tilde{K}}\mathbb{E}\sup_{0\le s\le t}|X(s)-G(X(s-\tau))-Y(s)+G(\bar{Y}(s-\tau))|^2\\
&\quad+\sqrt{\tilde{K}}\mathbb{E}\sup_{0\le s\le t}|X(s-\tau)-Y(s-\tau)|^2+\frac{\tilde{K}}{1-\sqrt{\tilde{K}}}\mathbb{E}\sup_{0\le s\le t}|Y(s-\tau)-\bar{Y}(s-\tau)|^2\\
&\le \frac{1}{(1-\tilde{K})(1-\sqrt{\tilde{K}})}\mathbb{E}\sup_{0\le s\le t}|X(s)-G(X(s-\tau))-Y(s)+G(\bar{Y}(s-\tau))|^2\\
&\quad+\frac{\tilde{K}}{(1-\sqrt{\tilde{K}})^2}\mathbb{E}\sup_{0\le s\le t}|Y(s-\tau)-\bar{Y}(s-\tau)|^2+\frac{\sqrt{\tilde{K}}}{1-\sqrt{\tilde{K}}}\mathbb{E}\sup_{-\tau\le s\le 0}|X(s)-Y(s)|^2\\
&\le \frac{\tilde{K}}{(1-\sqrt{\tilde{K}})^2}\mathbb{E}\sup_{0\le s\le t}|Y(s-\tau)-\bar{Y}(s-\tau)|^2\\
&\quad+\frac{3}{(1-\tilde{K})(1-\sqrt{\tilde{K}})}\mathbb{E}\sup_{0\le s\le t}\left|\int_0^s f(X(u),X(u-\tau))-f(\bar{Y}(u),\bar{Y}(u-\tau))du\right|^2\\
&\quad+\frac{3}{(1-\tilde{K})(1-\sqrt{\tilde{K}})}\mathbb{E}\sup_{0\le s\le t}\left|\int_0^s g(X(u),X(u-\tau))-g(\bar{Y}(u),\bar{Y}(u-\tau))dB(u)\right|^2\\
&\quad+\frac{3}{(1-\tilde{K})(1-\sqrt{\tilde{K}})}\mathbb{E}\sup_{0\le s\le t}\left|\int_0^s\int_{\mathbb{R}^d} h(X(u^-),X((u-\tau)^-),z)\right.\\
&\quad\left.-h(\bar{Y}(u^-),\bar{Y}((u-\tau)^-),z)\tilde{N}(du,dz)\right|^2.
\end{aligned}
$$

By Hölder's inequality (2.1), Doob's martingale inequality (2.4) and the assumption 4.1, we estimate the following three terms that

$$\mathbb{E}\sup_{0\le s\le t}\left|\int_0^s f(X(u),X(u-\tau))-f(\bar{Y}(u),\bar{Y}(u-\tau))du\right|^2$$
$$\le TK^2\int_0^t \mathbb{E}|X(u)-\bar{Y}(u)|^2+\mathbb{E}|X(u-\tau)-\bar{Y}(u-\tau)|^2du,$$
$$\mathbb{E}\sup_{0\le s\le t}\left|\int_0^s g(X(u),X(u-\tau))-g(\bar{Y}(u),\bar{Y}(u-\tau))dB(u)\right|^2$$
$$\le 4K^2\int_0^t \mathbb{E}|(X(u)-\bar{Y}(u)|^2+\mathbb{E}|X(u-\tau)-\bar{Y}(u-\tau)|^2du$$
$$\mathbb{E}\sup_{0\le s\le t}\left|\int_0^s\int_{\mathbb{R}^d} h(X(u^-),X((u-\tau)^-),z)-h(\bar{Y}(u^-),\bar{Y}((u-\tau)^-),z)\tilde{N}(du,dz)\right|^2$$
$$\le 4K^2\int_0^t \mathbb{E}|X(u)-\bar{Y}(u)|^2+\mathbb{E}|X(u-\tau)-\bar{Y}(u-\tau)|^2du.$$

By combination we have

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0\le s\le t}|X(s)-Y(s)|^2\Big)\\
&\le \frac{\tilde{K}}{(1-\sqrt{\tilde{K}})^2}\mathbb{E}\sup_{0\le s\le t}|Y(t-\tau)-\bar{Y}(t-\tau)|^2\\
&\quad+\frac{3K^2(T+8)}{(1-\tilde{K})(1-\sqrt{\tilde{K}})}\int_0^t \mathbb{E}|X(u)-\bar{Y}(u)|^2+\mathbb{E}|X(u-\tau)-\bar{Y}(u-\tau)|^2du\\
&\le \frac{\tilde{K}}{(1-\sqrt{\tilde{K}})^2}\mathbb{E}\sup_{0\le s\le t}|Y(t-\tau)-\bar{Y}(t-\tau)|^2\\
&\quad+\frac{6K^2(T+8)}{(1-\tilde{K})(1-\sqrt{\tilde{K}})}\int_0^t \mathbb{E}|X(u)-Y(u)|^2+\mathbb{E}|X(u-\tau)-Y(u-\tau)|^2du\\
&\quad+\frac{6K^2(T+8)}{(1-\tilde{K})(1-\sqrt{\tilde{K}})}\int_0^t \mathbb{E}|Y(u)-\bar{Y}(u)|^2+\mathbb{E}|Y(u-\tau)-\bar{Y}(u-\tau)|^2du.
\end{aligned}$$

For any $t\in[0,T]$ choose n such that $t\in[n\Delta,(n+1)\Delta)$. Then

$$\begin{aligned}
Y(t)-\bar{Y}(t)=Y(t)-Y(n\Delta)&=\int_{n\Delta}^t f(\bar{Y}(s),\bar{Y}(s-\tau))ds\\
&+\int_{n\Delta}^t g(\bar{Y}(s),\bar{Y}(s-\tau))dB(s)+\int_{n\Delta}^t\int_{\mathbb{R}^d} h(\bar{Y}(s^-),\bar{Y}((s-\tau)^-),z)\tilde{N}(ds,dz),
\end{aligned}$$

by the linear growth condition (4.4) and (4.7)

$$\begin{aligned}\mathbb{E}|Y(t)-\bar{Y}(t)|^2 &\le \mathbb{E}\sup_{0\le s\le t}|Y(s)-\bar{Y}(s)|^2\\ &\le (3\Delta+24)K^2\int_{n\Delta}^{s}(\mathbb{E}|\bar{Y}(u)|^2+\mathbb{E}|\bar{Y}(u-\tau)|^2)du \qquad (4.9)\\ &\le K'\Delta,\end{aligned}$$

as well as

$$\begin{aligned}\mathbb{E}|Y(t-\tau)-\bar{Y}(t-\tau)|^2 &\le \mathbb{E}\sup_{-\tau\le s\le 0}|Y(s)-\bar{Y}(s)|^2+\mathbb{E}\sup_{0<s\le t}|Y(s)-\bar{Y}(s)|^2\\ &\le \hat{K}^2|t-\tau-(n-m)\Delta|^2+\mathbb{E}\sup_{0<s\le t}|Y(s)-\bar{Y}(s)|^2\\ &\le K''\Delta+\mathbb{E}\sup_{0<s\le t}|Y(s)-\bar{Y}(s)|^2\\ &\le K''\Delta\end{aligned} \qquad (4.10)$$

holds for all $t\in[0,T]$ as required, where K' and K'' are two different positive constants.

Then we can come back to the proof of theorem 4.1 that

$$\begin{aligned}&\mathbb{E}\left(\sup_{0\le s\le t}|X(s)-Y(s)|^2\right)\\ &\le \frac{\tilde{K}}{(1-\sqrt{\tilde{K}})^2}K''\Delta+\frac{6K^2(T+8)T(K'\Delta+K''\Delta)}{(1-\tilde{K})(1-\sqrt{\tilde{K}})}\\ &\quad+\frac{6K^2(T+8)}{(1-\tilde{K})(1-\sqrt{\tilde{K}})}\int_0^s \mathbb{E}|X(u)-Y(u)|^2+\mathbb{E}|X(u-\tau)-Y(u-\tau)|^2du\\ &\le \frac{\tilde{K}}{(1-\sqrt{\tilde{K}})^2}K''\Delta+\frac{6K^2(T+8)T(K'\Delta+K''\Delta)}{(1-\tilde{K})(1-\sqrt{\tilde{K}})}\\ &\quad+\frac{12K^2(T+8)}{(1-\tilde{K})(1-\sqrt{\tilde{K}})}\int_0^s \mathbb{E}\sup_{0\le r\le u}|X(r)-Y(r)|^2du,\end{aligned}$$

that is by using Gronwall's inequality (2.7)

$$\mathbb{E}\left(\sup_{0\le t\le T}|X(t)-Y(t)|^2\right)\le C_3e^{C_4}\le C_5\Delta, \qquad (4.11)$$

where $C_3 = \frac{\tilde{K}}{(1-\sqrt{\tilde{K}})^2} K''\Delta + \frac{6K^2(T+8)T(K'\Delta + K''\Delta)}{(1-\tilde{K})(1-\sqrt{\tilde{K}})}$ and $C_4 = \frac{12K^2(T+8)}{(1-\tilde{K})(1-\sqrt{\tilde{K}})}$.

Remark 4.1 *Since proving as several cases in [27, Lemma 2.4] is not necessary for our fixed time delay models, we choose more easily way as the proof of* (4.9) *and* (4.10) *to deal with* $\mathbb{E}|Y(t) - \bar{Y}(t)|^2$ *and* $\mathbb{E}|Y(t-\tau) - \bar{Y}(t-\tau)|^2$.

Theorem 4.2 *Let the assumption 3.1 and* (4.5) *hold, the EM approximate solution converges to the exact solution of the NSDDEwJs* (3.1)*, in the sense that*

$$\lim_{\Delta\to 0} \mathbb{E}\left(\sup_{0\le t\le T} |X(t) - Y(t)|^2 \right) = 0.$$

Proof. Define two stopping times

$$\sigma_R = \inf\{t \ge 0 : |X(t)| > R\},$$
$$\delta_R = \inf\{t \ge 0 : |Y(t)| > R\},$$

and write $\rho_R = \sigma_R \wedge \delta_R$. Recall the Young inequality for $a^{-1} + b^{-1} = 1$, and all $\alpha, \beta, \gamma > 0$

$$\alpha\beta \le \frac{\gamma}{a}\alpha^a + \frac{1}{b\gamma^{\frac{b}{a}}}\beta^b.$$

Thus, for any $\gamma > 0$,

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\le t\le T} |X(t) - Y(t)|^2 \right) \\
&= \mathbb{E}\left(\sup_{0\le t\le T} |X(t) - Y(t)|^2 \mathbf{1}_{\{\sigma_R > T, \delta_R > T\}} \right) + \mathbb{E}\left(\sup_{0\le t\le T} |X(t) - Y(t)|^2 \mathbf{1}_{\{\sigma_R \le T \text{ or } \delta_R \le T\}} \right) \\
&\le \mathbb{E}\left(\sup_{0\le t\le T} |X(t\wedge\rho_R) - Y(t\wedge\rho_R)|^2 \mathbf{1}_{\{\rho_R > T\}} \right) + \frac{2\gamma}{p}\mathbb{E}\left(\sup_{0\le t\le T} |X(t) - Y(t)|^p \right) \\
&\quad + \frac{1 - \frac{2}{p}}{\gamma^{\frac{2}{p-2}}}\mathbb{P}(\sigma_R \le T \quad \text{or} \quad \delta_R \le T),
\end{aligned}$$

then we deduce that

$$\mathbb{P}(\sigma_R \le T) = \mathbb{E}(\mathbf{1}_{\{\sigma_R \le T\}}) \le \mathbb{E}\left(\mathbf{1}_{\{\sigma_R \le T\}} \frac{|X(\sigma_R)|^p}{R^p} \right) \le \frac{1}{R^p}\mathbb{E}\left(\sup_{0\le t\le T} |X(t)|^p \right) \le \frac{H_p}{R^p},$$

and we have

$$\mathbb{P}(\sigma_R \le T \quad \text{or} \quad \delta_R \le T) \le \frac{H_p}{R^p},$$

These bounds give

$$\mathbb{E}\left(\sup_{0\le t\le T} |X(t) - Y(t)|^2\right) \le \mathbb{E}\left(\sup_{0\le t\le T} |X(t\wedge\rho_R) - Y(t\wedge\rho_R)|^2\right) + \frac{2^{(p+1)}\gamma H_p}{p} + \frac{2(p-2)H_p}{p\gamma^{\frac{2}{p-2}}R^p}.$$

Then given any $\epsilon > 0$, we can choose γ such that $\frac{2^{(p+1)}\gamma H_p}{p} < \frac{\epsilon}{3}$, and then we choose R such that $\frac{2(p-2)H_p}{p\gamma^{\frac{2}{p-2}}R^p} < \frac{\epsilon}{3}$. Note for any sufficiently small Δ it follows that $C_5\Delta < \frac{\epsilon}{3}$,combining the bounds above we finally get

$$\mathbb{E}\left(\sup_{0\le t\le T} |X(t) - Y(t)|^2\right) < \epsilon,$$

which complete the proof.

4.4 Convergence Rate of Euler-Maryuama Method for Neutral Stochastic Differential Delay Equations with Jumps

Now let us consider the rate of convergence.

Assumption 4.2 *(Logarithm growth condition) For each $R \in N$ which is in the definition of stopping time, and K_R in the assumption 3.1, there exist positive constants ϱ and α such that*

$$\alpha K_R^2 \le \varrho \log R, \tag{4.12}$$

where $\alpha = \frac{24(T+8)}{(1-\check{K})(1-\sqrt{\check{K}})}$.

Theorem 4.3 *If the assumption 3.1 and 4.2 hold, then the order of convergence of the EM approximation is* $1/2$.

Proof. For each $R \geq 1$, define the function

$$f^{(R)}(x,y) = f\left(\frac{|x| \wedge R}{|x|}x, \frac{|y| \wedge R}{|y|}y\right),$$

where $(|x| \wedge R/|x|)x = 0$ when $x = 0$ and $g^{(R)}(x,y)$ and $h^{(R)}(x,y,z)$ are similar. Let $Y_R(t)$ be the EM approximation to the following NSDDEwJs

$$d[X_R(t) - G(X_R(t-\tau))] = f^{(R)}(X_R(t), X_R(t-\tau))dt + g^{(R)}(X_R(t), X_R(t-\tau))dB(t) + \int_{\mathbb{R}^d} h^{(R)}(X_R(t^-), X_R((t-\tau)^-), z)\tilde{N}(dt, dz)$$

with $Y_R(0) = X(0)$ and

$$Y_R(t) - G(\bar{Y}_R(t-\tau)) = \bar{Y}_R(0) - G(\bar{Y}_R(-\tau)) + \int_0^t f^{(R)}(\bar{Y}_R(s), \bar{Y}_R(s-\tau))ds + \int_0^t g^{(R)}(\bar{Y}_R(s), \bar{Y}_R(s-\tau))dB(s) + \int_0^t \int_{\mathbb{R}^d} h^{(R)}(\bar{Y}_R(s^-), \bar{Y}_R((s-\tau)^-), z)\tilde{N}(ds, dz).$$

By (4.11) we have

$$\begin{aligned}
\mathbb{E} \sup_{0 \leq t \leq T} |X_R(t) - Y_R(t)|^2 &\leq [\beta + T(K' + K'')\frac{1}{4}\alpha K_R^2]\Delta e^{\frac{1}{2}\alpha K_R^2} \\
&\leq [\beta \frac{1}{4}\alpha K_R^2 + T(K' + K'')\frac{1}{4}\alpha K_R^2]\Delta e^{\frac{1}{2}\alpha K_R^2} \\
&\leq [\beta + T(K' + K'')]e^{\alpha K_R^2}\Delta,
\end{aligned}$$

where $\beta = \frac{\tilde{K}K''}{(1-\sqrt{\tilde{K}})^2}$ and $\alpha = \frac{24(T+8)}{(1-\tilde{K})(1-\sqrt{\tilde{K}})}$ Let

$$\hat{X}(t) = \sup_{0 \leq t \leq T} |X(t)|, \quad \hat{Y}(t) = \sup_{0 \leq t \leq T} |Y(t)|.$$

Define the stopping time

$$\rho_R = T \wedge \inf\{t \in [0,T] : |X_R(t)| \vee |Y_R(t)| > R\}.$$

Clearly, $|X_R(t)| \vee |X_R(t-\tau)| \le R$ for $0 \le t \le \rho_R$, hence

$$f^{(R)}(X_R(t), X_R(t-\tau)) = f^{(R+1)}(X_R(t), X_R(t-\tau)),$$
$$g^{(R)}(X_R(t), X_R(t-\tau)) = g^{(R+1)}(X_R(t), X_R(t-\tau)),$$
$$h^{(R)}(X_R(s^-), X_R((s-\tau)^-), z) = h^{(R+1)}(X_R(s^-), X_R((s-\tau)^-), z)$$

on $-\tau \le t \le \rho_R$. Therefore,

$$X_R(t) = X_{R+1}(t) \quad \text{and} \quad Y_R(t) = Y_{R+1}(t) \quad \text{if} \quad 0 \le t < \rho_R.$$

This implies that ρ_R is increasing in R, and let $\rho = \lim_{R\to\infty} \rho_R$, the property above also enables us to define $X(t)$ for $t \in [-\tau, \rho)$ as follows

$$X(t) = X_R(t) \quad \text{if} \quad -\tau \le t < \rho_R.$$

It is clear that $X(t)$ is the unique solution to equation (3.1) for $t \in [-\tau, \rho)$. On the other hand, for $t \in [0, T]$, we compute

$$\begin{aligned}
&\mathbb{E} \sup_{0\le s\le t\wedge\rho_R} |X(s)| = \mathbb{E}|\xi(0)| - \mathbb{E}|\xi(-\tau)| \\
&\quad + \tilde{K}\mathbb{E} \sup_{0\le s\le t\wedge\rho_R} |X(s-\tau)| + \mathbb{E} \sup_{0\le s\le t\wedge\rho_R} \int_0^s |f(X(u), X(u-\tau))|du \\
&\le \frac{\mathbb{E}|\xi(0)| + (2KT - (1-\tilde{K}))\mathbb{E}|\xi(-\tau)|}{1-\tilde{K}} + \frac{2K}{1-\tilde{K}}\mathbb{E} \sup_{0\le s\le t\wedge\rho_R} \int_0^s |X(u)|du,
\end{aligned}$$

and by the Gronwall inequality (2.7), we get

$$\mathbb{E}|X(t\wedge\rho_R)| \le \frac{\mathbb{E}|\xi(0)| + (2KT - (1-\tilde{K}))\mathbb{E}|\xi(-\tau)|}{1-\tilde{K}} e^{\frac{2K}{1-\tilde{K}}}.$$

Noting that $|X(\rho_R)| > R$, and therefore we derive

$$R\mathbb{P}(\rho_R < T) \le \mathbb{E}|X(t\wedge\rho_R)I_{\{\rho_R<T\}}| \le \frac{\mathbb{E}|\xi(0)| + (2KT - (1-\tilde{K}))\mathbb{E}|\xi(-\tau)|}{1-\tilde{K}} e^{\frac{2K}{1-\tilde{K}}}$$

that is

$$\mathbb{P}(\rho_R < T) \le \frac{\mathbb{E}|\xi(0)| + (2KT - (1-\tilde{K}))\mathbb{E}|\xi(-\tau)|}{(1-\tilde{K})R} e^{\frac{2K}{1-\tilde{K}}}.$$

Letting $R \to \infty$, we obtain $\mathbb{P}(\rho_R < T) = 0$, this implies $\lim_{R\to\infty} \rho_R = T$ a.s. We compute, for $t \in [0, T)$

$$\begin{aligned}|X(t) - Y(t)| &= \sum_{R=1}^{\infty} |X(t) - Y(t)| \mathbf{1}_{\{R-1 \le \hat{X}(T) \vee \hat{Y}(T) \le R\}} \\ &= \sum_{R=1}^{\infty} |X_R(t) - Y_R(t)| \mathbf{1}_{\{R-1 \le \hat{X}(T) \vee \hat{Y}(T) \le R\}}.\end{aligned}$$

Therefore

$$\mathbb{E} \sup_{0 \le t \le T} |X(t) - Y(t)| \le \sum_{R=1}^{\infty} (\mathbb{E} \sup_{0 \le t \le T} |X_R(t) - Y_R(t)|^2)^{\frac{1}{2}} (\mathbb{E}\mathbf{1}_{\{R-1 \le \hat{X}(T) \vee \hat{Y}(T) \le R\}})^{\frac{1}{2}}$$

$$= \sum_{R=1}^{\infty} (\mathbb{E} \sup_{0 \le t \le T} |X_R(t) - Y_R(t)|^2)^{\frac{1}{2}} \sqrt{\mathbb{P}(R - 1 \le \hat{X}(T) \vee \hat{Y}(T) \le R)}$$

and from the logarithm growth condition (4.12), it follows that

$$\mathbb{E} \sup_{0 \le t \le T} |X_R(t) - Y_R(t)|^2 \le [\beta + T(K' + K'')] R^{\varrho} \Delta.$$

On the other hand, if $q \ge 2$ then by Chebyschev's inquality (2.2)

$$\begin{aligned}&\mathbb{P}(R - 1 \le |\hat{X}(T) \vee \hat{Y}(T)| \le R) \le \mathbb{P}(R - 1 \le |\hat{X}(T) \vee \hat{Y}(T)|) \\ &\le \frac{\mathbb{E}|\hat{X}(T)|^q + \mathbb{E}|\hat{Y}(T)|^q}{(R-1)^q} \le 2 \frac{H_p}{(R-1)^q},\end{aligned}$$

and therefore,

$$\begin{aligned}\mathbb{E} \sup_{0 \le t \le T} |X(t) - Y(t)| &\le \sum_{R=1}^{\infty} \sqrt{[\beta + T(K' + K'')] R^{\varrho}} \frac{\sqrt{2H_p}}{\sqrt{(R-1)^q}} \sqrt{\Delta} \\ &\le \sum_{R=1}^{\infty} \sqrt{[\beta + T(K' + K'')] R^{\varrho}} \frac{\sqrt{2^{q+1} H_p}}{\sqrt{R^q}} \sqrt{\Delta}.\end{aligned}$$

Let q be sufficiently large such that $q > \varrho$, we see the right hand side is convergent, whence we get the rate of convergence is $\frac{1}{2}$, which complete the proof.

Chapter 5

Almost Sure Stability of Numerical Methods for Neutral Stochastic Differential Delay Equations with Jumps

5.1 Introduction

The stability theory of numerical solutions is one of the central problems in numerical analysis. Stability analysis of numerical methods for NSDDEs has recently received a great deal of attention. The stability concepts of numerical schemes for NSDDEs are included due to the stochastic nature, for example, moment stability (M-stability) and almost sure stability or trajectory stability (T-stability). Regarding the almost sure stability of numerical methods for SDEs, it was shown, by the Chebyshev inequality and the Borel-Cantelli lemma, that the moment exponential stability implies almost sure exponential stability under certain conditions. Using the technique based on the continuous semi-martingale convergence theorem in [54], the stability of SDEs has been examined. Note that there are similar expressions for the continuous and discrete semi-martingale convergence theorems. To our knowledge, there is no similar result using martingale techniques for numerical solutions of nonlinear NSDDEwJs. In the last section we shall use the martingale techniques to investigate whether numerical methods can reproduce the almost sure exponential ability of the exact solutions to nonlinear NSDDEwJs.

Recently, most of the existing convergence theory for numerical methods requires the global Lipschitz condition. However, it was observed that the classical one, which guarantees the strong uniform convergence of the Euler-Maruyama method to the true solution, can be significantly relaxed. They proved that under the local Lipschitz condition the uniform bounded-ness of moments of the true solution and its approximations are sufficient for strong convergence. They immediately raises the question that which type of conditions can guarantee such a uniform bounded-ness of moments. It is well known that the classical linear growth condition is sufficient to bound the moments for both SDEs and EM methods, and it is also known that in the case of a continuous solution, the first useful step to relax the linear growth conditions is to apply the Lyapunov function technique. All these above lead us to the monotone condition, and the introduction of the monotone condition will be given in section 2. To our best knowledge, there is no result about numerical approximation of NSDDEwJs under the monotone condition. It allows us to develop bounds for polynomial coefficients. In this chapter, we will investigate the almost sure stability of theta method of NSDDEwJs under monotone condition, and give a particularly introduction of the theta Euler-Maruyama method in the following section.

5.2 Almost Sure Exponential Stability for Euler-Maryuama Method for Neutral Stochastic Differential Delay Equations with Jumps

Theorem 5.1 *Let the assumption 3.1 holds. Assume that there are five nonnegative constants H_1-H_5 such that*

$$\begin{aligned} 2\langle x - G(y), f(x,0)\rangle &\le -H_1|x|^2 + H_2|y|^2, \\ |f(x,y) - f(x,0)| &\le H_3|y| \\ |g(x,y)|^2 + \sum_{k=1}^{d} \int_{\mathbb{R}} |h^{(k)}(x,y,z_k)|^2 \nu_k(dz_k) &\le H_4|x|^2 + H_5|y|^2 \end{aligned} \tag{5.1}$$

for all $x, y \in \mathbb{R}$. If

$$H_1 > H_2 + 2H_3 + H_4 + H_5 \tag{5.2}$$

then for any given initial data $\xi \in D^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$, there exists a unique global solution to (3.1) *and this solution denoted by $X(t;\xi)$, has property that*

$$\limsup_{t\to\infty} \frac{1}{t} \log(|X(t;\xi)|) \le -\frac{\bar{\alpha}}{2} \quad a.s. \tag{5.3}$$

where $\bar{\alpha} > 0$ is the unique positive root of

$$H_1 - H_3 - H_4 - \bar{\alpha} = (H_2 + H_3 + H_5)e^{\bar{\alpha}\tau} \tag{5.4}$$

Proof: Let $V(x) = |x|^2$. Using the assumption (5.1), we compute

$$
\begin{aligned}
LV(x,y) &= 2\langle x - G(y), f(x,y)\rangle + |g(x,y)|^2 + \sum_{k=1}^{d} \int_{\mathbb{R}} |h^{(k)}(x,y,z_k)|^2 \nu_k(dz_k) \\
&\le 2\langle x - G(y), f(x,0)\rangle + 2|x||f(x,y) - f(x,0)| \\
&\quad + |g(x,y)|^2 + \sum_{k=1}^{d} \int_{\mathbb{R}} |h^{(k)}(x,y,z_k)|^2 \nu_k(dz_k) \\
&\le -H_1|x|^2 + H_2|y|^2 + 2H_3|x||y| + H_4|x|^2 + H_5|y|^2 \\
&\le -H_1|x|^2 + H_2|y|^2 + H_3(|x|^2 + |y|^2) + H_4|x|^2 + H_5|y|^2 \\
&= -(H_1 - H_3 - H_4)|x|^2 + (H_2 + H_3 + H_5)|y|^2.
\end{aligned}
$$

Now the conclusion follows from [41, Corollary 3.2], which complete the proof.

Let $\Delta B_n = B(t_{n+1}) - B(t_n)$ and $\Delta \tilde{N}_n(dz) = \tilde{N}(t_{n+1}, dz) - \tilde{N}(t_n, dz)$ with $\Delta = \frac{T}{N} = \frac{\tau}{m}$ for sufficiently large integer N, where $n = 1, 2, \ldots, N$, and $Y_n \approx X(t_n)$ by setting $Y_0 = X(0)$ that

$$
\begin{aligned}
Y_{n+1} - G(Y_{n+1-m}) = Y_n - G(Y_{n-m}) + f(Y_n, Y_{n-m})\Delta + g(Y_n, Y_{n-m})\Delta B_n \\
+ \int_{\mathbb{R}^d} h(Y_n, Y_{n-m}, z)\Delta \tilde{N}_n(dz).
\end{aligned}
\tag{5.5}
$$

Theorem 5.2 *Let the assumptions 3.1 and* (5.1) *hold. Assume that f satisfies the linear growth condition, namely, there exists a constant $K > 0$ such that*

$$
|f(x,y)|^2 \le K(|x|^2 + |y|^2). \tag{5.6}
$$

Let $\bar{\alpha}$ be the positive number which is defined by (5.4) *and $\varepsilon \in (0, \frac{\bar{\alpha}}{2})$ be arbitrary. Then there exists a $\Delta^* > 0$ such that if $\Delta < \Delta^*$, then for any given finite-valued $\mathcal{F}_0$-measurable random variables $\xi(n\Delta)$, $n = -m, -m+1 \ldots, 0$, the EM approximate solution of* (5.5) *obeys*

$$
\limsup_{n\to\infty} \frac{1}{n\Delta} \log(|Y_n|) \le -\frac{\bar{\alpha}}{2} + \varepsilon \quad a.s. \tag{5.7}
$$

Lemma 5.1 ***(Discrete semi-martingale convergence theorem)*** *Let $\{A_i\}$, $\{U_i\}$ be two sequences of nonnegative random variables such that both A_i and U_i are $\mathcal{F}_{i-1}$-measurable for $i = 1, 2, \ldots$, and $A_0 = U_0 = 0$ a.s. Let M_i be a real-value local martingale with $M_0 = 0$ a.s. Let ζ be a nonnegative $\mathcal{F}_0$-measurable random variable. Assume that $\{X_i\}$ is a nonnegative semi-martingale with the Doob-Mayer decomposition*

$$X_i = \zeta + A_i - U_i + M_i.$$

If $\lim_{i\to\infty} A_i < \infty$ a.s. then, for $\omega \in \Omega$,

$$\lim_{i\to\infty} X_i < \infty, \quad \lim_{i\to\infty} U_i < \infty,$$

for all of the three processes X_i, U_i and M_i converge to finite random variables.

Proof of theorem: Noting that

$$\begin{aligned}
&|Y_{n+1} - G(Y_{n+1-m})|^2 \\
&= |Y_n - G(Y_{n-m})|^2 + 2\langle Y_n - G(Y_{n-m}), f(Y_n, Y_{n-m})\Delta\rangle + |f(Y_n, Y_{n-m})\Delta|^2 \\
&\quad + |g(Y_n, Y_{n-m})|^2\Delta + \sum_{k=1}^{d} \int_{\mathbb{R}} |h^{(k)}(Y_n, Y_{n-m}, z_k)|^2 \Delta\nu_k(dz_k) + \Delta M_{n+1},
\end{aligned}$$

where

$$\begin{aligned}
\Delta M_{n+1} := {}& |g(Y_n, Y_{n-m})|^2(|\Delta B|_n^2 - \Delta) + 2\langle Y_n - G(Y_{n-m}) \\
& + f(Y_n, Y_{n-m})\Delta, g(Y_n, Y_{n-m})\rangle\Delta B_n \\
& + 2\Big\langle g(Y_n, Y_{n-m})\Delta B_n, \int_{\mathbb{R}^d} h(Y_n, Y_{n-m}, z)\Delta\tilde{N}_n(dz)\Big\rangle \\
& + 2\Big\langle Y_n - G(Y_{n-m}) + f(Y_n, Y_{n-m})\Delta, \int_{\mathbb{R}^d} h(Y_n, Y_{n-m}, z)\Delta\tilde{N}_n(dz)\Big\rangle \\
& + \left(\left|\int_{\mathbb{R}^d} h(Y_n, Y_{n-m}, z)\Delta\tilde{N}_n(dz)\right|^2 - \sum_{k=1}^{d}\int_{\mathbb{R}} |h^{(k)}(Y_n, Y_{n-m}, z_k)|^2\Delta\nu_k(dz_k)\right).
\end{aligned}$$

Then we have

$$
\begin{aligned}
&|Y_{n+1} - G(Y_{n+1-m})|^2 \\
&= |Y_n - G(Y_{n-m})|^2 + 2\langle Y_n - G(Y_{n-m}), f(Y_n, Y_{n-m})\Delta\rangle + |f(Y_n, Y_{n-m})|^2\Delta^2 \\
&\quad + |g(Y_n, Y_{n-m})|^2\Delta + \sum_{k=1}^{d}\int_{\mathbb{R}} |h^{(k)}(Y_n, Y_{n-m}, z_k)|^2\Delta\nu_k(dz_k) + \Delta M_{n+1} \\
&\le |Y_n - G(Y_{n-m})|^2 + 2\langle Y_n - G(Y_{n-m}), f(Y_n, 0)\rangle\Delta \\
&\quad + 2\langle Y_n - G(Y_{n-m}), f(Y_n, Y_{n-m}) - f(Y_n, 0)\rangle\Delta \\
&\quad + |f(Y_n, Y_{n-m})|^2\Delta^2 + |g(Y_n, Y_{n-m})|^2\Delta + \sum_{k=1}^{d}\int_{\mathbb{R}} |h^{(k)}(Y_n, Y_{n-m}, z_k)|^2\Delta\nu_k(dz_k) + \Delta M_{n+1} \\
&\le |Y_n - G(Y_{n-m})|^2 - H_1|Y_n|^2\Delta + H_2|Y_{n-m}|^2\Delta \\
&\quad + H_3|Y_n|^2\Delta + H_3|Y_{n-m}|^2\Delta + K(|Y_n|^2 + |Y_{n-m}|^2)\Delta^2 \\
&\quad + H_4|Y_n|^2\Delta + H_5|Y_{n-m}|^2\Delta + \Delta M_{n+1}
\end{aligned}
$$

and for a positive constant $C > 1$, we have

$$
\begin{aligned}
&C^{(n+1)\Delta}|Y_{n+1} - G(Y_{n+1-m})|^2 - C^{n\Delta}|Y_n - G(Y_{n-m})|^2 \\
&= C^{(n+1)\Delta}(|Y_{n+1} - G(Y_{n+1-m})|^2 - |Y_n - G(Y_{n-m})|^2) \\
&\quad + (C^{(n+1)\Delta} - C^{n\Delta})|Y_n - G(Y_{n-m})|^2 \\
&\le C^{(n+1)\Delta}[K\Delta^2 - H_1\Delta + H_3\Delta + H_4\Delta]|Y_n|^2 \\
&\quad + C^{(n+1)\Delta}[H_2\Delta + H_3\Delta + H_5\Delta + K\Delta^2]|Y_{n-m}|^2 \\
&\quad + C^{(n+1)\Delta}\Delta M_{n+1} + (C^{(n+1)\Delta} - C^{n\Delta})|Y_n - G(Y_{n-m})|^2,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&C^{n\Delta}|Y_n - G(Y_{n-m})|^2 \le |Y_0 - G(\xi(-\tau))|^2 + \sum_{i=0}^{n-1} C^{(i+1)\Delta}\Delta M_{i+1} \\
&\quad + \left[\frac{1}{1-\tilde{K}}(1 - C^{-\Delta}) - H_1\Delta + H_3\Delta + H_4\Delta + K\Delta^2\right]\sum_{i=0}^{n-1} C^{(i+1)\Delta}|Y_i|^2 \\
&\quad + [\tilde{K}(1 - C^{-\Delta}) + H_2\Delta + H_3\Delta + H_5\Delta + K\Delta^2]\sum_{i=0}^{n-1} C^{(i+1)\Delta}|Y_{i-m}|^2.
\end{aligned}
$$

Since

$$\sum_{i=0}^{n-1} C^{(i+1)\Delta}|Y_{i-m}|^2 = \sum_{i=-m}^{-1} C^{(i+m+1)\Delta}|Y_i|^2 + \sum_{i=0}^{n-1} C^{(i+m+1)\Delta}|Y_i|^2 - \sum_{i=n-m}^{n-1} C^{(i+m+1)\Delta}|Y_i|^2,$$

we have

$$\begin{aligned}&[\tilde{K}(1-C^{-\Delta}) + H_2\Delta + H_3\Delta + H_5\Delta + K\Delta^2] \sum_{i=n-m}^{n-1} C^{(i+m+1)\Delta}|Y_i|^2 \\ &+ C^{n\Delta}|Y_n - G(Y_{n-m})|^2 \le O_n,\end{aligned} \tag{5.8}$$

where

$$\begin{aligned}O_n : = &|Y_0 - G(\xi(-\tau))|^2 + \Big[(\tilde{K}(1-C^{-\Delta}) + H_2\Delta + H_3\Delta + H_5\Delta + 2K\Delta^2)C^{m\Delta} \\ &+ \Big(- H_1\Delta + 2H_3\Delta + H_4\Delta + K\Delta^2 + \frac{1}{1-\tilde{K}}(1-C^{-\Delta})\Big)\Big] \sum_{i=0}^{n-1} C^{(i+1)\Delta}|Y_i|^2 \\ &+ (\tilde{K}(1-C^{-\Delta}) + H_2\Delta + H_3\Delta + H_5\Delta + K\Delta^2) \sum_{i=-m}^{-1} C^{(i+m+1)\Delta}|Y_i|^2 \\ &+ \sum_{i=0}^{n-1} C^{(i+1)\Delta}\Delta M_{i+1}.\end{aligned}$$

Now we need to show $\bar{M}(N) := \sum_{i=0}^{N-1} C^{(i+1)\Delta}\Delta M_{i+1}$ is a local martingale, which is equivalent to prove

$$\mathbb{E}(\bar{M}(N)|\mathcal{F}_{t_{N-1}}) = \bar{M}(N-1). \tag{5.9}$$

Indeed,

$$\begin{aligned}\mathbb{E}(\bar{M}(N)|\mathcal{F}_{t_{N-1}}) &= \bar{M}(N-1) + \mathbb{E}(\Delta M_N|\mathcal{F}_{t_{N-1}}) \\ &= \bar{M}(N-1),\end{aligned}$$

thanks to the fact that

$$\begin{aligned}&\mathbb{E}(|g(Y_{N-1}, Y_{N-m-1})|^2(|\Delta B_{N-1}|^2 - \Delta)|\mathcal{F}_{t_{N-1}}) \\ &= |g(Y_{N-1}, Y_{N-m-1})|^2\mathbb{E}((|\Delta B_{N-1}|^2 - \Delta)|\mathcal{F}_{t_{N-1}}) = 0,\end{aligned}$$

since $|g(Y_{N-1},Y_{N-m-1})|^2$ is $\mathcal{F}_{t_{N-1}}$-measurable, and $|\Delta B_{N-1}|^2-\Delta$ is independent of $\mathcal{F}_{t_{N-1}}$, we can obtain the following results similarly

$$\mathbb{E}(\langle F(Y_{N-1}),g(Y_{N-1},Y_{N-m-1})\rangle\Delta B_{N-1}|\mathcal{F}_{t_{N-1}})=0,$$

$$\mathbb{E}(\langle f(Y_{N-1},Y_{N-m-1})\Delta,g(Y_{N-1},Y_{N-m-1})\rangle\Delta B_{N-1}|\mathcal{F}_{t_{N-1}})=0,$$

and we can also obtain

$$\mathbb{E}\bigg(\int_{\mathbb{R}^d}\langle F(Y_{N-1}),h(Y_{N-1},Y_{N-m-1},z)\rangle\Delta\tilde{N}_{N-1}(dz)\bigg|\mathcal{F}_{t_{N-1}}\bigg)=0,$$

$$\mathbb{E}\bigg(\int_{\mathbb{R}^d}\langle f(Y_{N-1},Y_{N-m-1})\Delta,h(Y_{N-1},Y_{N-m-1},z)\rangle\Delta\tilde{N}_{N-1}(dz)\bigg|\mathcal{F}_{t_{N-1}}\bigg)=0,$$

where $\Delta\tilde{N}_{N-1}$ is independent of $\mathcal{F}_{t_{N-1}}$ as well, and

$$\begin{aligned}\mathbb{E}\bigg(\bigg|\int_{\mathbb{R}^d}&h(Y_{N-1},Y_{N-m-1},z)\Delta\tilde{N}_{N-1}(dz)\bigg|^2\\&-\sum_{k=1}^d\int_{\mathbb{R}}|h^{(k)}(Y_{N-1},Y_{N-m-1},z_k)|^2\Delta\nu_k(dz_k)\bigg|\mathcal{F}_{t_{N-1}}\bigg)=0.\end{aligned}$$

Then we prove the following form for the reason of $\Delta\tilde{N}_{N-1}$ independent of ΔB_{N-1} that

$$\mathbb{E}\bigg(\int_{\mathbb{R}^d}\langle g(Y_{N-1},Y_{N-m-1}),h(Y_{N-1},Y_{N-m-1},z)\rangle\Delta\tilde{N}_{N-1}(dz)\Delta B_{N-1}\bigg|\mathcal{F}_{t_{N-1}}\bigg)=0.$$

Since we have

$$\begin{aligned}&\mathbb{E}(\Delta\tilde{N}_{N-1}\Delta B_{N-1}|\mathcal{F}_{t_{N-1}})\\&=\mathbb{E}((\tilde{N}_N-\tilde{N}_{N-1})(B_N-B_{N-1})|\mathcal{F}_{N-1})\\&=\mathbb{E}(\tilde{N}_NB_N|\mathcal{F}_{N-1})+\tilde{N}_{N-1}B_{N-1}\\&\quad-\mathbb{E}(\tilde{N}_NB_{N-1}|\mathcal{F}_{N-1})-\mathbb{E}(\tilde{N}_{N-1}B_N|\mathcal{F}_{N-1})\\&=\mathbb{E}(\tilde{N}_NB_N|\mathcal{F}_{N-1})-\tilde{N}_{N-1}B_{N-1},\end{aligned}$$

where $\mathbb{E}(\tilde{N}_N|\mathcal{F}_{N-1}) = \tilde{N}_{N-1}$ and $\mathbb{E}(B_N|\mathcal{F}_{N-1}) = B_{N-1}$ by using the martingale property. Now we only need to check

$$\mathbb{E}(\tilde{N}_N B_N|\mathcal{F}_{N-1}) = \tilde{N}_{N-1}B_{N-1}. \tag{5.10}$$

Noting that, for two martingale processes $\tilde{N}(t)$ and $B(t)$, $\tilde{N}(t)B(t)-[\tilde{N}, B](t)$ is a martingale, where $[\tilde{N}, B]$ is the corresponding quadratic covariance process (see [55, P15]). Hence, if we can show $[\tilde{N}, B](t) = 0$, then (5.10) follows immediately. Moreover, for two semi-martingales X and Y

$$[X, Y](t) = \sum_{s\leq t} \Delta X(s)\Delta Y(s),$$

where $\Delta X(s) = X(s) - X(s^-)$, provided that one of the processes $\tilde{N}(t)$ or $B(t)$ is of finite variation [55, P16] and [1, P233]. Compute

$$\begin{aligned}
[\tilde{N}, \tilde{N}](t) &= [N(t) - \lambda t, N(t) - \lambda t] \\
&= [N, N](t) - 2\lambda[N(t), t] + \lambda^2[t, t] \\
&= \sum_{s\leq t}(\Delta N(s))^2 - 2\lambda\sum_{s\leq t}(\Delta N(s)\Delta s) + \lambda^2\sum_{s\leq t}(\Delta s)^2 \\
&= \sum_{s\leq t}(\Delta N(s))^2 - \max(\Delta t)2\lambda\sum_{s\leq t}\Delta N(s) + \max(\Delta t)\lambda^2\sum_{s\leq t}(\Delta s) \\
&= \sum_{s\leq t}(\Delta N(s)) = N(t),
\end{aligned}$$

which means $\tilde{N}(t)$ is the finite variation, since $\Delta \to 0$ and $\Delta N(t) = 0$ or 1. Then $[\tilde{N}, B](t) = 0$. Consequently, $B_N\tilde{N}_N$ is a martingale, and

$$\mathbb{E}(\tilde{N}_N B_N|\mathcal{F}_{N-1}) = \tilde{N}_{N-1}B_{N-1}$$

as required. So $\sum_{i=0}^{n-1} C^{(i+1)\Delta}\Delta M_{i+1}$ is a local martingale. Let us now introduce the function

$$\begin{aligned}
h(C) :&= (\tilde{K}(1 - C^{-\Delta}) + H_2\Delta + H_3\Delta + H_5\Delta + K\Delta^2)C^{(m+1)\Delta} \\
&+ \Big(- H_1\Delta + H_3\Delta + H_4\Delta + K\Delta^2 + \frac{1}{1-\tilde{K}}\Big)C^{\Delta} - \frac{1}{1-\tilde{K}}.
\end{aligned} \tag{5.11}$$

Choose $\Delta_1^* > 0$, such that for any $\Delta < \Delta_1^*$, $\frac{1}{1-\tilde{K}} - H_1\Delta + H_3\Delta + H_4\Delta + K\Delta^2 > 0$. We therefore have $h'(C) > 0$ for any $C \geq 1$. Clearly,

$$h(1) = 2K\Delta^2 + (-H_1 + H_2 + 2H_3 + H_4 + H_5)\Delta.$$

Hence, there exist two solutions that Δ_1 and Δ_2, and then for any $\Delta < \Delta_2^* = (H_1 - H_2 - 2H_3 - H_4 - H_5)/2K$, $h(1) < 0$, where $\Delta_1 < \Delta_2^* < \Delta_2$, which implies that for any $\Delta < \Delta_1^* \wedge \Delta_2^*$, there exists a unique $C_\Delta^* > 1$ such that $h(C_\Delta^*) = 0$. Choosing $C = C_\Delta^*$, we therefore have

$$\begin{aligned} O_n = & |Y_0 - G(\xi(-\tau))|^2 + (\tilde{K}(1 - C^{-\Delta}) + H_2\Delta + H_3\Delta \\ & + H_5\Delta + K\Delta^2) \sum_{i=-m}^{-1} C^{(i+m+1)\Delta}|Y_i|^2 + \sum_{i=0}^{n-1} C^{(i+1)\Delta}\Delta M_{i+1}. \end{aligned}$$

Noting that the initial sequence $Y_i < \infty$ for all $i = -m, \ldots, 0$, by the lemma 5.1, for $C = C_\Delta^*$, $\lim_{n\to\infty} |Y_n - G(Y_{n-m})| < \infty$ a.s. By (5.8), therefore we have

$$\begin{aligned} & \limsup_{n\to\infty} C_\Delta^{*n\Delta}|Y_n - G(Y_{n-m})|^2 \\ & \leq \limsup_{n\to\infty} \Big(C_\Delta^{*n\Delta}|Y_n - G(Y_{n-m})|^2 + (\tilde{K}(1 - C^{-\Delta}) + H_2\Delta \\ & \quad + H_3\Delta + H_5\Delta + 2K\Delta^2) \sum_{i=n-m}^{n-1} C^{(i+m+1)\Delta}|Y_i|^2 \Big) \\ & \leq \lim_{n\to\infty} O_n < \infty \quad \text{a.s.} \end{aligned} \tag{5.12}$$

Noting that $m\Delta = \tau$, by (5.11),

$$\begin{aligned} & \left(\frac{\tilde{K}(1 - C^{*-\Delta})}{\Delta} + H_2 + H_3 + 2H_5 + K\Delta \right) C_\Delta^{*\tau} \\ & + \frac{1 - C^{*-\Delta}}{(1 - \tilde{K})\Delta} - H_1 + H_3 + 2H_4 + K\Delta = 0. \end{aligned} \tag{5.13}$$

Choose the constant μ such that $C = e^\mu$ and hence $1 - C^{-\Delta} = 1 - e^{-\mu\Delta}$.

Define

$$\bar{h}_\Delta(\mu) = \left(\frac{\tilde{K}(1-e^{-\mu\Delta})}{\Delta} + H_2 + H_3 + H_5 + K\Delta\right)e^{\mu\tau} + \frac{1-e^{-\mu\Delta}}{(1-\tilde{K})\Delta} - H_1 + H_3 + H_4 + K\Delta.$$

Letting $\mu^*_\Delta = \log C^*_\Delta$, by (5.11), for any $\Delta < \Delta^*_1 \wedge \Delta^*_2$, we have

$$\bar{h}_\Delta(\mu^*_\Delta) = 0. \tag{5.14}$$

Noting that $\lim_{\Delta\to 0}(1-e^{-\tau\Delta})/\Delta = \mu$, we have

$$\lim_{\Delta\to 0} \bar{h}_\Delta(\mu) = (\tilde{K}\mu + H_2 + H_3 + H_5)e^{\mu\tau} + \frac{\mu}{1-\tilde{K}} - H_1 + H_3 + H_4. \tag{5.15}$$

By definition of $\bar{\alpha}$,(5.13) and (5.15) we have

$$\lim_{\Delta\to 0} \mu^*_\Delta = \bar{\alpha},$$

which implies that for any positive $\varepsilon \in (0, \frac{\bar{\alpha}}{2})$, there exists a $\Delta^*_3 > 0$, such that for any $\Delta < \Delta^*_3$, we have

$$\mu^*_\Delta > \bar{\alpha} - 2\varepsilon.$$

Note that (5.13), together with the definition of μ^*_Δ shows that

$$\limsup_{n\to\infty} e^{\mu^*_\Delta n\Delta}|Y_n - G(Y_{n-m})|^2 < \infty.$$

Because we have

$$|Y_n|^2 \le \tilde{K}|Y_{n-m}|^2 + \frac{1}{1-\tilde{K}}|Y_n - G(Y_{n-m})|^2,$$

so,

$$\begin{aligned}
&\limsup_{n\to\infty} e^{\mu^*_\Delta n\Delta}|Y_n|^2 \\
&\le \tilde{K}\limsup_{n\to\infty} e^{\mu^*_\Delta n\Delta}|Y_{n-m}|^2 + \frac{1}{1-\tilde{K}}\limsup_{n\to\infty} e^{\mu^*_\Delta n\Delta}|Y_n - G(Y_{n-m})|^2 \\
&\le \frac{1}{(1-\tilde{K})^2}\limsup_{n\to\infty} e^{\mu^*_\Delta n\Delta}|Y_n - G(Y_{n-m})|^2 \\
&< \infty.
\end{aligned}$$

We therefore obtain that for any $\Delta < \Delta_1^* \wedge \Delta_2^* \wedge \Delta_3^*$,

$$\limsup_{n\to\infty} \log|Y_n| < -\frac{\bar{\alpha}}{2} + \varepsilon \quad \text{a.s.}$$

5.3 Theta-Euler-Maryuama Method for Neutral Stochastic Differential Delay Equations with Jumps

In this section we will define the TEMNSDDEwJs (3.1). Let $\Delta B_n = B(t_{n+1}) - B(t_n)$ and $\Delta\tilde{N}_n(dz) = \tilde{N}(t_{n+1}, dz) - \tilde{N}(t_n, dz)$ with $\Delta = \frac{T}{N} = \frac{\tau}{m}$ for sufficiently large integer N, where $n = 1, 2, \dots, N$, and $Y_n \approx X(t_n)$ by setting $Y_0 = X(0)$ that

$$\begin{aligned}
Y_{n+1} - G(Y_{n+1-m}) =& Y_n - G(Y_{n-m}) + \theta f(Y_{n+1}, Y_{n+1-m})\Delta \\
&+ (1-\theta) f(Y_n, Y_{n-m})\Delta + g(Y_n, Y_{n-m})\Delta B_n \qquad (5.16)\\
&+ \int_{\mathbb{R}^d} h(Y_n, Y_{n-m}, z)\Delta\tilde{N}_n(dz)
\end{aligned}$$

where $\theta \in (0,1)$.

Now, we need to ensure this scheme is well defined. For this purpose, we impose the following one-sided Lipschitz condition on f in x: there exists a constant K such that for any $x_1, x_2, y \in \mathbb{R}^n$ and $t \geq 0$,

$$\langle x_1 - x_2, f(x_1, y) - f(x_2, y) \rangle \leq K|x_1 - x_2|^2. \tag{5.17}$$

Under this condition, let $\Delta < K^{-1}$ and for $n \in [-m, 0]$ we can rewrite (5.16) as

$$\begin{aligned} Y_{n+1} - G(\xi(n+1-m)) = \xi(n) - G(\xi(n-m)) + \theta f(Y_{n+1}, \xi(n+1-m))\Delta \\ + (1-\theta)f(\xi(n), \xi(n-m))\Delta + g(\xi(n), \xi(n-m))\Delta B_n \\ + \int_{\mathbb{R}^d} h(\xi(n), \xi(n-m), z)\Delta \tilde{N}_n(dz). \end{aligned}$$

Then let $Y_{n+1} = c$, and

$$\begin{aligned} &G(\xi(n+1-m)) - G(\xi(n-m)) + \xi(n) \\ &+ (1-\theta)f(\xi(n), \xi(n-m))\Delta + g(\xi(n), \xi(n-m))\Delta B_n \\ &+ \int_{\mathbb{R}^d} h(\xi(n), \xi(n-m), z)\Delta \tilde{N}_n(dz) = d. \end{aligned}$$

Define $Z : \mathbb{R}^n \to \mathbb{R}^n$ by

$$Z(c) := c - d - \theta f(c, \xi(n+1-m))\Delta$$

In the follows we shall use the uniform monotonicity theorem [50, Theorem C.2 in Appendix C, P656] to show that the equation $Z(c) = 0$ has a unique solution.

Theorem 5.3 ***(Uniform monotonicity theorem)*** *Suppose that $G : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and for a positive constant C that*

$$\langle G(v) - G(w), v - w \rangle \geq C\|v - w\|^2 \quad \forall\, v, w \in \mathbb{R}^n.$$

Then there exists a unique $u \in \mathbb{R}^n$ such that $G(u) = 0$ and, furthermore,

$$\|u - v\| \leq \frac{1}{C}\|G(v)\| \quad \forall\, v \in \mathbb{R}^n.$$

Then by one-sided Lipschitz condition (5.17) we have

$$\begin{aligned}
&\langle Z(c) - Z(\tilde{c}), c - \tilde{c}\rangle \\
&= \langle (c-d) - \theta f(c, \xi(n+1-m))\Delta - (\tilde{c} - d) + \theta f(\tilde{c}, \xi(n+1-m))\Delta, c - \tilde{c}\rangle \\
&= \langle c - \tilde{c}, c - \tilde{c}\rangle - \langle \theta f(c, \xi(n+1-m)) - \theta f(\tilde{c}, \xi(n+1-m)), c - \tilde{c}\rangle\Delta \\
&\geq (1 - K\theta\Delta)|c - \tilde{c}|^2,
\end{aligned}$$

for $n \in [-m, 0]$. Then repeating this procedure we can obtain the same result for $n \in [0, m]$ and so on. Then combine all integrals from $[-m, 0]$ to $[(N-1)\Delta, N\Delta]$ we have the following result that for $n \in [-m, N\Delta]$ the existence of a unique solution $c \in \mathbb{R}^n$ such that $Z(c) = 0$ follows from the uniform monotonicity theorem. Hence there exists a unique solution to (5.16).

(**Monotone condition**) Let the assumption 3.1 holds, for all $x, y \in \mathbb{R}^n$ and $t \in [0, T]$, there exist positive constants α and β such that

$$\begin{aligned}
&\langle x - G(y), f(x,y)\rangle + \frac{1}{2}|g(x,y)|^2 + \frac{1}{2}\sum_{k=1}^{d}\int_{\mathbb{R}} |h^{(k)}(x,y,z_k)|^2 \nu_k(dz_k) \\
&\quad + \frac{1-2\theta}{2}|f(x,y)|^2\Delta \leq \alpha + \beta(|x|^2 + |y|^2), \quad \forall \Delta \in \left(0, \max\left\{\frac{1}{K\theta}, \frac{(1-\tilde{K})^2}{4\beta\theta}\right\}\right]
\end{aligned} \tag{5.18}$$

for all $x, y \in \mathbb{R}^n$. Clearly, for $\theta \geq \frac{1}{2}$ the above condition does not need to add $\frac{1-2\theta}{2}|f(x,y)|^2\Delta$.

Theorem 5.4 *Let the assumptions 3.1 and* (5.18) *hold. There exists a unique, global solution $X(t)$ to* (3.1). *Moreover, the solution has the properties that for any $T > 0$,*

$$\mathbb{E}|X(T)|^2 < C,$$

where C is a positive constant.

Proof: Applying the Itô formula to the function $V(x) = |x|^2$, we have

$$LV(x,y) \leq 2\alpha + 2\beta(|x|^2 + |y|^2).$$

Define a stopping time

$$\sigma_R = \inf\{t \geq 0 : |X(t)| > R\},$$

therefore

$$\begin{aligned}\mathbb{E}|X(t\wedge\sigma_R) - G(X(t\wedge\sigma_R - \tau))|^2 &\leq |X_0 - G(X(-\tau))|^2 + 2T\alpha \\ &+ 2\beta\int_0^{t\wedge\sigma_R}\mathbb{E}|X(s)|^2ds + 2\beta\int_0^{t\wedge\sigma_R}\mathbb{E}|X(s-\tau)|^2ds,\end{aligned}$$

where

$$\begin{aligned}2\beta\int_0^{t\wedge\sigma_R}\mathbb{E}|X(s-\tau)|^2ds &= 2\beta\int_{-\tau}^{t\wedge\sigma_R-\tau}\mathbb{E}|X(s)|^2ds \\ &\leq 2\beta\int_{-\tau}^{0}\mathbb{E}|\xi(s)|^2ds + 2\beta\int_0^{t\wedge\sigma_R}\mathbb{E}|X(s)|^2ds.\end{aligned}$$

Therefore

$$\begin{aligned}\mathbb{E}|X(t\wedge\sigma_R) - G(X(t\wedge\sigma_R - \tau))|^2 &\leq |X_0 - G(X(-\tau))|^2 + 2T\alpha \\ &+ 2\tau\beta\sup_{-\tau\leq s\leq 0}\mathbb{E}|\xi(s)|^2 + 4\beta\int_0^t\mathbb{E}|X(s\wedge\sigma_R)|^2ds.\end{aligned}$$

By using $(a+b)^2 \leq \frac{a^2}{\tilde{K}} + \frac{b^2}{1-\tilde{K}}$ with $0 < \tilde{K} < 1$ and (3.5), we have

$$\begin{aligned}|X(t\wedge\sigma_R)|^2 &= |X(t\wedge\sigma_R) - G(X(t\wedge\sigma_R-\tau)) + G(X(t\wedge\sigma_R-\tau))|^2 \\ &\leq \tilde{K}|X(t\wedge\sigma_R-\tau)|^2 + \frac{1}{1-\tilde{K}}|X(t\wedge\sigma_R) - G(X(t\wedge\sigma_R-\tau))|^2.\end{aligned}$$

Taking the expectation in both sides, we have

$$\begin{aligned}
&\sup_{0\le s\le t}\mathbb{E}|X(s\wedge\sigma_R)|^2\\
&\le \tilde{K}\sup_{0\le s\le t}\mathbb{E}|X(s\wedge\sigma_R-\tau)|^2+\frac{1}{1-\tilde{K}}\sup_{0\le s\le t}\mathbb{E}|X(s\wedge\sigma_R)-G(X(s\wedge\sigma_R-\tau))|^2\\
&\le \tilde{K}\sup_{-\tau\le s\le 0}\mathbb{E}|\xi(s)|^2+\tilde{K}\sup_{0\le s\le t}\mathbb{E}|X(s\wedge\sigma_R)|^2\\
&\quad+\frac{1}{1-\tilde{K}}\sup_{0\le s\le t}\mathbb{E}|X(s\wedge\sigma_R)-G(X(s\wedge\sigma_R-\tau))|^2\\
&\le \frac{1}{(1-\tilde{K})^2}|X_0-G(\xi(-\tau))|^2+\frac{2T\alpha}{(1-\tilde{K})^2}\\
&\quad+\frac{(1-\tilde{K})\tilde{K}+2\tau\beta}{(1-\tilde{K})^2}\sup_{-\tau\le s\le 0}\mathbb{E}|\xi(s)|^2+\frac{4\beta}{(1-\tilde{K})^2}\int_0^t\sup_{0\le u\le s}\mathbb{E}|X(u\wedge\sigma_R)|^2du.
\end{aligned}$$

Then by using Gronwall's inequality (2.7) we obtain

$$\begin{aligned}
&\sup_{0\le t\le T}\mathbb{E}|X(t\wedge\sigma_R)|^2\\
&\le C:=\left(\frac{2\mathbb{E}|X_0|^2+2T\alpha}{(1-\tilde{K})^2}+\frac{(1-\tilde{K})\tilde{K}+2\tilde{K}^2+2\tau\beta}{(1-\tilde{K})^2}\sup_{-\tau\le s\le 0}\mathbb{E}|\xi(s)|^2\right)\\
&\quad\exp\left\{\frac{4T\beta}{(1-\tilde{K})^2}\right\}.
\end{aligned}$$

Hence for $\sigma_R\le t$ we have

$$\mathbb{P}(\sigma_R\le T)R^2=\mathbb{E}(\mathbf{1}_{\{\sigma_R\le t\}})R^2\le\mathbb{E}(|X(t\wedge\sigma_R)|^2\mathbf{1}_{\{\sigma_R\le t\}})\le C.$$

It implies that $\mathbb{P}(\sigma_R\le T)\longrightarrow 0$ as $R\to\infty$, that is $\sigma_R\to\infty$ as $R\to\infty$. Next, let $R\to\infty$ and applying Fatou's lemma 2.1, we obtain $\mathbb{E}|X(T)|^2\le C$, which completes the proof. From now on we always assume that $\Delta\in\left(0,\max\left\{\frac{1}{K\theta},\frac{(1-\tilde{K})^2}{4\beta\theta}\right\}\right]$.

(Polynomial growth condition) For three positive constants C, p and q, we have

$$|f(x,y)|\vee|g(x,y)|\vee\sum_{k=1}^{d}\int_{\mathbb{R}}|h^{(k)}(x,y,z_k)|\nu_k(dz_k)\le C(1+|x|^p+|y|^q),\quad\forall x,y\in\mathbb{R}^n.\tag{5.19}$$

Theorem 5.5 *Let the assumption 3.1,* (5.18) *and* (5.19) *hold, and let* $\Delta^* \in \left(0, \max\left\{\frac{1}{K\theta}, \frac{(1-\tilde{K})^2}{4\beta\theta}\right\}\right]$ *be sufficiently small such that whenever* $\Delta \le \Delta^*$ *we have*

$$\frac{[e^{\Delta}-1]}{\Delta} \le 2 \quad and \quad e^{\Delta} \le 2.$$

Then, for$\Delta \le \Delta^*$ *and* $T > 0$ *there exists a constant* $C > 0$*, such that* $\mathbb{E}|Y_n|^2 \le C$*.*

Proof: Let $\sigma_R = \inf\{n : |Y_n| > R\}$ be a stopping time with respect to $\{\mathcal{F}_{t_n}\}_{n\ge 0}$. We then define a function

$$F(x) = x - G(y) - \theta f(x,y)\Delta,$$

such that we represent (5.16) as

$$\begin{aligned} F(Y_{n+1}) =& F(Y_n) + f(Y_n, Y_{n-m})\Delta + g(Y_n, Y_{n-m})\Delta B_n \\ &+ \int_{\mathbb{R}^d} h(Y_n, Y_{n-m}, z)\Delta\tilde{N}_n(dz). \end{aligned}$$

Now we have the following equation

$$\begin{aligned} |F(Y_{n+1})|^2 =& |F(Y_n)|^2 + |f(Y_n, Y_{n-m})|^2\Delta^2 + |g(Y_n, Y_{n-m})|^2\Delta + \Delta M_{n+1} \\ &+ 2\langle F(Y_n), f(Y_n, Y_{n-m})\rangle\Delta + \sum_{k=1}^{d}\int_{\mathbb{R}} |h^{(k)}(Y_n, Y_{n-m}, z_k)|^2\Delta\nu_k(dz_k) \\ =& |F(Y_n)|^2 + \Delta M_{n+1} + \Big(2\langle Y_n - G(Y_n), f(Y_n, Y_{n-m})\rangle + |g(Y_n, Y_{n-m})|^2 \\ &+ \sum_{k=1}^{d}\int_{\mathbb{R}} |h^{(k)}(Y_n, Y_{n-m}, z_k)|^2\nu_k(dz_k) + (1-2\theta)|f(Y_n, Y_{n-m})|^2\Delta\Big)\Delta, \end{aligned}$$

where

$$\Delta M_{n+1} = |g(Y_n, Y_{n-m})|^2(\Delta B_n^2 - \Delta) + 2\langle F(Y_n), g(Y_n, Y_{n-m})\rangle \Delta B_n$$
$$+ 2\langle f(Y_n, Y_{n-m})\Delta, g(Y_n, Y_{n-m})\rangle \Delta B_n$$
$$+ 2\left\langle g(Y_n, Y_{n-m})\Delta B_n, \int_{\mathbb{R}^d} h(Y_n, Y_{n-m}, z)\Delta \tilde{N}_n(dz)\right\rangle$$
$$+ 2\left\langle F(Y_n), \int_{\mathbb{R}^d} h(Y_n, Y_{n-m}, z)\Delta \tilde{N}_n(dz)\right\rangle$$
$$+ 2\left\langle f(Y_n, Y_{n-m})\Delta, \int_{\mathbb{R}^d} h(Y_n, Y_{n-m}, z)\Delta \tilde{N}_n(dz)\right\rangle$$
$$+ \left(\left|\int_{\mathbb{R}^d} h(Y_n, Y_{n-m}, z)\Delta \tilde{N}_n(dz)\right|^2 - \sum_{k=1}^{d}\int_{\mathbb{R}} |h^{(k)}(Y_n, Y_{n-m}, z_k)|^2 \Delta\nu_k(dz_k)\right).$$

It is possible to rewrite as

$$|F(Y_{n+1})|^2 = |F(Y_n)|^2 + A(Y_n)\Delta + \Delta M_{n+1},$$

where

$$A(Y_n) = 2\langle Y_n - G(Y_n), f(Y_n, Y_{n-m})\rangle + |g(Y_n, Y_{n-m})|^2$$
$$+ \sum_{k=1}^{d}\int_{\mathbb{R}} |h^{(k)}(Y_n, Y_{n-m}, z_k)|^2 \nu_k(dz_k) + (1 - 2\theta)|f(Y_n, Y_{n-m})|^2\Delta.$$

Therefore, let N be any nonnegative integer such that $N\Delta \le T$. Summing up the inequalities above from $n = 0$ to $N \wedge \sigma_R$, we get

$$|F(Y_{N\wedge\sigma_R+1})|^2 = |F(Y_0)|^2 + A(Y_0)\Delta + \Delta M_0 + \sum_{n=1}^{N\wedge\sigma_R} A(Y_n)\Delta + \sum_{n=0}^{N\wedge\sigma_R} \Delta M_{n+1},$$

then we have

$$
\begin{aligned}
|Y_{N\wedge\sigma_R+1}|^2 \le \frac{1}{1-\tilde{K}}\Bigg(&|F(Y_0)|^2 + A(Y_0)\Delta + \Delta M_0 + \sum_{n=1}^{N\wedge\sigma_R} A(Y_n)\Delta + \sum_{n=0}^{N\wedge\sigma_R} \Delta M_{n+1} \\
&+ \sum_{n=1}^{(N\wedge\sigma_R)+1} 2\theta\langle Y_n - G(Y_{n-m}), f(Y_n, Y_{n-m})\rangle\Delta\Bigg) + \sum_{n=1}^{(N\wedge\sigma_R)+1} \frac{1}{\tilde{K}}|G(Y_{n-m})|^2 \\
\le \frac{1}{1-\tilde{K}}\Bigg(&|F(Y_0)|^2 + A(Y_0)\Delta + \Delta M_0 + \sum_{n=1}^{N} A(Y_n)\mathbf{1}_{[0,\sigma_R]}(n)\Delta \\
&+ \sum_{n=0}^{N} \Delta M_{n+1}\mathbf{1}_{[0,\sigma_R]}(n) + \sum_{n=1}^{N} 2\theta\langle Y_n - G(Y_{n-m}), f(Y_n, Y_{n-m})\rangle\mathbf{1}_{[0,\sigma_R]}(n)\Delta \\
&+ 2\theta\langle Y_{N\wedge\sigma_R+1} - G(Y_{N\wedge\sigma_R+1-m}), f(Y_{N\wedge\sigma_R+1}, Y_{N\wedge\sigma_R+1-m})\rangle\Delta\Bigg) \\
&+ \sum_{n=1}^{N} \frac{1}{\tilde{K}}|G(Y_{n-m})|^2\mathbf{1}_{[0,\sigma_R]}(n) + \frac{1}{\tilde{K}}|G(Y_{N\wedge\sigma_R+1-m})|^2.
\end{aligned}
$$

Applying the assumptions 3.1, (5.18) and (5.9), we then take the expectation on both sides of the inequality above and get

$$
\begin{aligned}
\mathbb{E}|Y_{N\wedge\sigma_R+1}|^2 \le \frac{1}{1-\tilde{K}}\mathbb{E}\Bigg(&|F(Y_0)|^2 + A(Y_0)\Delta + \mathbb{E}\sum_{n=1}^{N}\left(4\alpha + 4\beta(|Y_n|^2 + |Y_{n-m}|^2)\right)\mathbf{1}_{[0,\sigma_R]}(n)\Delta \\
&+ 2\theta\langle Y_{N\wedge\sigma_R+1} - G(Y_{N\wedge\sigma_R+1-m}), f(Y_{N\wedge\sigma_R+1}, Y_{N\wedge\sigma_R+1-m})\rangle\Delta\Bigg) \\
&+ \mathbb{E}(\tilde{K}|Y_{n-m}|^2\mathbf{1}_{[0,\sigma_R]}(n) + \tilde{K}|Y_{N\wedge\sigma_R+1-m}|^2) \\
\le \frac{1}{1-\tilde{K}}&\mathbb{E}(|F(Y_0)|^2 + A(Y_0)\Delta + 8N\alpha) + \frac{4\beta\Delta}{1-\tilde{K}}\sum_{n=1}^{N}\mathbb{E}(|Y_n|^2\mathbf{1}_{[0,\sigma_R]}(n)) \\
&+ \tilde{K}\sum_{n=1}^{N}\mathbb{E}(|Y_{n-m}|^2\mathbf{1}_{[0,\sigma_R]}(n)) + \frac{4\beta\Delta}{1-\tilde{K}}\sum_{n=1}^{N}\mathbb{E}(|Y_{n-m}|^2\mathbf{1}_{[0,\sigma_R]}(n)) \\
&+ \frac{2\beta\Delta}{1-\tilde{K}}\left(\mathbb{E}|Y_{(N\wedge\sigma_R)+1}|^2 + \mathbb{E}|Y_{(N\wedge\sigma_R)+1-m}|^2\right) + \tilde{K}\mathbb{E}|Y_{N\wedge\sigma_R+1-m}|^2.
\end{aligned}
$$

Then

$$\mathbb{E}\sup_{0\le n\le N\wedge\sigma_R+1}|Y_n|^2$$
$$\le \frac{C}{(1-\tilde{K})^2-4\beta\theta\Delta}+\frac{4\beta\Delta}{(1-\tilde{K})^2-4\beta\theta\Delta}\sup_{0\le n\le N\wedge\sigma_R}\sum_{n=1}^{N}\mathbb{E}[|Y_n|^2\mathbf{1}_{[0,\sigma_R]}(n)]$$
$$+\frac{\tilde{K}(1-\tilde{K})+4\beta\Delta}{(1-\tilde{K})^2-4\beta\theta\Delta}\sup_{0\le n\le N\wedge\sigma_R}\sum_{n=1}^{N}\mathbb{E}[|Y_{n-m}|^2\mathbf{1}_{[0,\sigma_R]}(n)],$$

where C is a positive constant may different line by line. Now by discrete Gronwall's inequality lemma 5.1,

$$\mathbb{E}\sup_{0\le n\le N\wedge\sigma_R+1}[|Y_n|^2\mathbf{1}_{[0,\sigma_R]}(n)]\le\frac{C}{(1-\tilde{K})^2-4\beta\theta\Delta}\exp\left\{\frac{8\beta\Delta+\tilde{K}(1-\tilde{K})}{(1-\tilde{K})^2-4\beta\theta\Delta}\right\},$$

where we use the fact that $N\Delta\le T$. Thus, letting $R\to\infty$ and applying Fatou's lemma 2.1, we have $\mathbb{E}|Y_{N+1}|^2\le C$, which complete the proof.

5.4 Almost Sure Stability for Theta-Euler-Maryuama Method for Neutral Stochastic Differential Delay Equations with Jumps

Theorem 5.6 *Let the assumptions 3.1and* (5.18) *hold. Assume that there exist two functions $\omega\in C(\mathbb{R}^n;\mathbb{R}_+)$ and $W\in C(\mathbb{R}^n;\mathbb{R}_+)$ such that*

$$\langle(x-G(y)),f(x,y)\rangle+\frac{1}{2}|g(x,y)|^2+\frac{1-2\theta}{2}|f(x,y)|^2\Delta$$
$$+\frac{1}{2}\sum_{k=1}^{d}\int_{\mathbb{R}}|h^{(k)}(Y_n,Y_{n-m},z_k)|^2\nu_k(dz_k)\le-\lambda_1\omega(x)+\lambda_2\omega(y)-W(x-G(y)),\tag{5.20}$$

for all $x,y\in\mathbb{R}^n$ and $\Delta\in\left(0,\max\left\{\frac{1}{K\theta},\frac{(1-\tilde{K})}{4\beta\theta}\right\}\right]$ with $\lambda_1>\lambda_2>0$. Then the solution of the theta EM method (5.16) *obeys*

$$\limsup_{n\to\infty}|Y_n|^2<\infty,\quad a.s.,$$

$$\lim_{n\to\infty} \omega(Y_n) = 0, \quad a.s.$$

and

$$\sum_{n=0}^{\infty} \mathbb{E}(\omega(Y_n))\Delta < \infty.$$

If additionally $\omega \in \mathcal{K}$

$$\lim_{n\to\infty} Y_n = 0, \quad a.s.$$

where $\mathcal{K}$ *denotes the class of continuous, non-decreasing functions* $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ *with* $\mu(0) = 0$.

Proof: Let $F(x, y) = x - G(y) - \theta f(x, y)\Delta$ and define $\sigma_R = \inf\{n : |Y_n| > R\}$ is a stopping time with respect to $\{\mathcal{F}_{t_n}\}_{n\geq 0}$. We have the following inequality

$$\begin{aligned}
|F(Y_{n+1})|^2 &= |F(Y_n)|^2 + |f(Y_n, Y_{n-m})|^2\Delta^2 + |g(Y_n, Y_{n-m})|^2\Delta \\
&\quad + 2\langle F(Y_n), f(Y_n, Y_{n-m})\rangle\Delta + \sum_{k=1}^{d}\int_{\mathbb{R}} |h^{(k)}(Y_n, Y_{n-m}, z_k)|^2\Delta\nu_k(dz_k) + \Delta M_{n+1} \\
&= |F(Y_n)|^2 - A(Y_n)\Delta + \Delta M_{n+1},
\end{aligned} \tag{5.21}$$

where

$$\begin{aligned}
A(Y_n) := -\Big(&2\langle Y_n - G(Y_n), f(Y_n, Y_{n-m})\rangle + |g(Y_n, Y_{n-m})|^2 \\
&+ \sum_{k=1}^{d}\int_{\mathbb{R}} |h^{(k)}(Y_n, Y_{n-m}, z_k)|^2\nu_k(dz_k) + (1 - 2\theta)|f(Y_n, Y_{n-m})|^2\Delta\Big),
\end{aligned}$$

and

$$\begin{aligned}\Delta M_{n+1} := &|g(Y_n, Y_{n-m})|^2(\Delta B_n^2 - \Delta) + 2\langle F(Y_n), g(Y_n, Y_{n-m})\rangle \Delta B_n \\ &+ 2\langle f(Y_n, Y_{n-m})\Delta, g(Y_n, Y_{n-m})\rangle \Delta B_n \\ &+ 2\Big\langle g(Y_n, Y_{n-m})\Delta B_n, \int_{\mathbb{R}^d} h(Y_n, Y_{n-m}, z)\Delta \tilde{N}_n(dz)\Big\rangle \\ &+ 2\Big\langle F(Y_n), \int_{\mathbb{R}^d} h(Y_n, Y_{n-m}, z)\Delta \tilde{N}_n(dz)\Big\rangle \\ &+ 2\Big\langle f(Y_n, Y_{n-m})\Delta, \int_{\mathbb{R}^d} h(Y_n, Y_{n-m}, z)\Delta \tilde{N}_n(dz)\Big\rangle \\ &+ \bigg(\Big|\int_{\mathbb{R}^d} h(Y_n, Y_{n-m}, z)\Delta \tilde{N}_n(dz)\Big|^2 - \sum_{k=1}^{d}\int_{\mathbb{R}} |h^{(k)}(Y_n, Y_{n-m}, z_k)|^2 \Delta\nu_k(dz_k)\bigg).\end{aligned}$$

For any integer $N \geq 1$, by (5.21) it is easy to see that

$$|F(Y_{t_{N+1}})|^2 = |F(Y_0)|^2 - \sum_{n=0}^{N} A(Y_n)\Delta + \sum_{n=0}^{N} \Delta M_{n+1}. \tag{5.22}$$

It suffices to verify that

$$\sum_{n=0}^{N} A(Y_n) \geq 0 \text{ and } \bar{M}(N) := \sum_{n=0}^{N} \Delta M_{n+1} \text{ is a local martingale.}$$

In what follows we shall show both of them one by one. Note by (5.20) and (3.2) that

$$\begin{aligned}\sum_{n=0}^{N} A(Y_n) &\geq \lambda_1 \sum_{n=0}^{N} \omega(Y_n) - \lambda_2 \sum_{n=0}^{N} \omega(Y_{n-m}) \\ &\geq \lambda_1 \sum_{n=0}^{N} \omega(Y_n) - \lambda_2 \sum_{n=0}^{m} \omega(Y_{n-m}) - \lambda_2 \sum_{n=m+1}^{N} \omega(Y_{n-m}) \\ &\geq \lambda_1 \sum_{n=0}^{N} \omega(Y_n) - \lambda_2 \sum_{n=0}^{m} \omega(Y_{n-m}) - \lambda_2 \sum_{n=0}^{N} \omega(Y_n) \\ &\geq (\lambda_1 - \lambda_2) \sum_{n=0}^{N} \omega(Y_n) - \lambda_2 \sum_{n=0}^{m} \omega(Y_{n-m}) \\ &\geq (\lambda_1 - \lambda_2) \sum_{n=0}^{N} \omega(Y_n) - \lambda_2 \sup_{-\tau \leq s \leq 0} \omega(\xi(s))\tau.\end{aligned}$$

Due to $\lambda_1 > \lambda_2$, we have $(\lambda_1 - \lambda_2)\sum_{n=0}^{N}\omega(Y_n) \geq 0$, and by (5.9) $\bar{M}(N)$ is a local martingale. Now the (5.22) can be rewritten as

$$|F(Y_{t_{N+1}})|^2 \leq |F(Y_0)|^2 + \lambda_2 \sup_{-\tau\leq s\leq 0} \omega(\xi(s))\tau \\ - (\lambda_1 - \lambda_2)\sum_{n=0}^{N}\omega(Y_n)\Delta + \sum_{n=0}^{N}\Delta M_{n+1}. \tag{5.23}$$

Then we are in a position to apply the lemma 5.1 to obtain

$$\lim_{n\to\infty} |F(Y_n)|^2 < \infty.$$

Observing the fundamental inequality [7] that for any $a, b \in \mathbb{R}$ and $\alpha > 0$,

$$2ab \leq \alpha a^2 + \frac{b^2}{\alpha},$$

and let $\alpha = \tilde{K}$ and by the Assumption (5.18), it is easy to show that

$$\begin{aligned}|Y_n - G(Y_{n-m})|^2 &= |Y_n|^2 - 2\langle Y_n, G(Y_{n-m})\rangle + |G(Y_{n-m})|^2 \\ &\geq |Y_n|^2 - \tilde{K}|Y_n|^2 - \frac{1}{\tilde{K}}|G(Y_{n-m})|^2 + |G(Y_{n-m})|^2 \\ &\geq (1-\tilde{K})|Y_n|^2 - \tilde{K}(1-\tilde{K})|Y_{n-m}|^2,\end{aligned}$$

and

$$\begin{aligned}|F(Y_n)|^2 &= |Y_n - G(Y_{n-m})|^2 - 2\theta\langle Y_n - G(Y_{n-m}), f(Y_n, Y_{n-m})\rangle\Delta \\ &\quad + \theta^2|f(Y_n, Y_{n-m})|^2\Delta^2 \\ &\geq |Y_n - G(Y_{n-m})|^2 - 2\beta\theta\Delta(|Y_n|^2 + |Y_{n-m}|^2) \\ &\quad + \theta^2|f(Y_n, Y_{n-m})|^2\Delta^2 - 2\alpha\theta\Delta \\ &\geq (1-\tilde{K}-2\beta\theta\Delta)|Y_n|^2 - (\tilde{K}(1-\tilde{K}) + 2\beta\theta\Delta)|Y_{n-m}|^2 - 2\alpha\theta\Delta,\end{aligned}$$

taking $\limsup_{n\to\infty}$ on both sides that

$$\begin{aligned}
&\limsup_{n\to\infty}[|F(Y_n)|^2] \\
&\geq \limsup_{n\to\infty}[(1-\tilde{K}-2\beta\theta\Delta)|Y_n|^2] \\
&\quad + \limsup_{n\to\infty}[-(\tilde{K}(1-\tilde{K})+2\beta\theta\Delta)|Y_{n-m}|^2] + \limsup_{n\to\infty}[-2\alpha\theta\Delta] \\
&\geq (1-\tilde{K}-2\beta\theta\Delta)\limsup_{n\to\infty}[|Y_n|^2] \\
&\quad - (\tilde{K}(1-\tilde{K})+2\beta\theta\Delta)\liminf_{n\to\infty}[|Y_{n-m}|^2] - \liminf_{n\to\infty}[2\alpha\theta\Delta] \\
&\geq (1-\tilde{K}-2\beta\theta\Delta)\limsup_{n\to\infty}[|Y_n|^2] \\
&\quad - (\tilde{K}(1-\tilde{K})+2\beta\theta\Delta)\limsup_{n\to\infty}[|Y_{n-m}|^2] - \liminf_{n\to\infty}[2\alpha\theta\Delta] \\
&\geq ((1-\tilde{K})^2-4\beta\theta\Delta)\limsup_{n\to\infty}[|Y_n|^2] - \liminf_{n\to\infty}[2\alpha\theta\Delta].
\end{aligned}$$

Choosing Δ sufficiently small which further yields that

$$\limsup_{n\to\infty}|Y_n|^2 < \infty, \quad \text{a.s.}$$

By lemma 5.1,

$$\sum_{n=0}^{\infty}\omega(Y_n)\Delta < \infty, \quad \text{a.s..}$$

which implies

$$\lim_{n\to\infty}\omega(Y_n) = 0, \quad \text{a.s.}$$

If additionally $\omega \in \mathcal{K}$

$$\lim_{n\to\infty} Y_n = 0, \quad \text{a.s.}$$

as required. Summing up the both sides of (5.21) we have

$$\sum_{n=0}^{\infty}\omega(Y_n)\Delta \leq |F(Y_0)|^2 + \lambda_2 \sup_{-\tau\leq s\leq 0}\omega(\xi(s))\tau + \sum_{n=0}^{N}\Delta M_n.$$

Then taking expectation on both sides, the proof completed.

Chapter 6

Stability in Distribution for Reflected Stochastic Differential Delay Equations with Jumps

6.1 Introduction

Since the importance of delay equations derived from the fact that many phenomena witnessed around us do not have an immediate effect at the moment of their occurrence, delay dynamical systems are used in a lot of models of science and engineering. Moreover, in applications of some quantities of interest needed to be positive, Kinnally and Williams [33], Bo and Yuan [10] present the reflection which is positively constraint. Furthermore, [30] has not only established the existence-and-uniqueness theory, but also investigated moment asymptotic bounded-ness and moment exponential stability of the equations. Most of the existing papers are concerned with the stability of RSDDEs in respect of sample paths or moments. However, in many practical systems, such stability is sometimes too strong and in this case it is useful to know whether or not the probability distribution of the solution will converge weakly to some distributions (but not necessarily to zero). In this chapter, we will couple the Bownian motion, small jump and large jump terms (referred to [4]) separately to the system and study the stability in distribution of RSDDEwJs by using the Skorokhod problem which will be introduced in section 2.

6.2 Reflected Stochastic Differential Delay Equations with Jumps

Consider the nonlinear n-dimensional RSDDEwJs

$$\begin{aligned} dX(t) = f(X(t), X(t-\tau))dt + g(X(t), X(t-\tau))dB(t) \\ + \int_{|z|<r} h(X(t^-), X((t-\tau)^-), z)\tilde{N}(dt, dz) \\ + \int_{|z|\geq r} h'(X(t^-), X((t-\tau)^-), z)N(dt, dz) + dL(t), \end{aligned} \tag{6.1}$$

for any $t \geq 0$. Here the given initial segment

$$\xi = \{\xi(t), t \in [-\tau, 0]\} \in D^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n), \tag{6.2}$$

where $X(t^-) = \lim_{s\uparrow t} X(s)$, $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ as well as $h, h' : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^{n\times d}$ the positive number r plays the role here of separating small jumps (which are compensated) from large jumps (which are not) [4]. We denote that each column $h^{(k)}$ of the $n \times d$ matrix $h = [h_{ij}]$ depends on z only through the k-th coordinate z_k, i.e., $h^{(k)}(x, y, z) = h^{(k)}(x, y, z_k)$, $z = (z_1, \ldots, z_d) \in \mathbb{R}^d$. Furthermore, all values $\xi(t)$ of the initial segment are assumed to be $\mathcal{F}_0$-measurable for $t \in [-\tau, 0]$.

The $\mathbb{R}^n_+$-valued processes $L(t) = (L_1(t), L_2(t), \ldots, L_n(t))^T$ is called the regulator of the solution processes $X = (X(t); t \geq -\tau)$. Specially, the i-th component of $L(t)$ can only increase when the i-th component of $X(t)$ reaches the point zero. Hence, the action of $L(t)$ is termed reflection at the boundary of the orthant $\mathbb{R}^n_+$. Further, the regulator $L(t)$ satisfy the following properties: [19] The i-th component of $L(t)$ is continuous, nondecreasing $\mathcal{F}_t$-measurable and $L_i(0) = 0, i = 1, 2, \ldots, n$, and for all $t > 0$, it holds that P-a.s.

$$\int_0^t X(s)^T dL(s) = 0.$$

Then we will refer to [33, Definition 2.1.1] for the definition of solution to equation (6.1). For the existence and uniqueness of the solution we shall impose a hypothesis:

Assumption 6.1 *Assume that f, g, h, h' satisfies the local Lipschitz condition and the linear growth condition, i.e. for each integer $R \geq 1$ there exist a positive constant K_R and another positive constant K, such that*

$$\begin{aligned}
&|f(x_1, y_1) - f(x_2, y_2)|^2 + |g(x_1, y_1) - g(x_2, y_2)|^2 \\
&\quad + \sum_{k=1}^d \int_{|z_k|<r} |h^{(k)}(x_1, y_1, z_k) - h^{(k)}(x_2, y_2, z_k)|^2 \nu_k(dz_k) \\
&\quad + \sum_{k=1}^d \int_{|z_k|\geq r} |h'^{(k)}(x_1, y_1, z_k) - h'^{(k)}(x_2, y_2, z_k)|^2 \nu_k(dz_k) \\
&\leq K_R(|x_2 - x_1|^2 + |y_2 - y_1|^2), \\
&|f(x, y)|^2 + |g(x, y)|^2 + \sum_{k=1}^d \int_{|z_k|<r} |h^{(k)}(x, y, z_k)|^2 \nu_k(dz_k) \\
&\quad + \sum_{k=1}^d \int_{|z_k|\geq r} |h'^{(k)}(x, y, z_k)|^2 \nu_k(dz_k) \leq K(|x|^2 + |y|^2),
\end{aligned} \tag{6.3}$$

for $x_1, x_2, y_1, y_2 \in \mathbb{R}^n$ and $t \in [0, T]$, with $|x_1| \vee |y_1| \vee |x_2| \vee |y_2| \leq R$.

Let $C^2(\mathbb{R}^n, \mathbb{R}_+)$ denote the family of all nonnegative functions $V(x)$ on $\mathbb{R}^n$ that are continuously twice differentiable in x. If $V \in C^2(\mathbb{R}^n, \mathbb{R}_+)$, define an operator LV from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$ by

$$\begin{aligned}LV(x,y) &= V_x(x)f(x,y) + \frac{1}{2}\text{trace}[g^T(x,y)V_{xx}(x)g(x,y)]\\ &\quad+\sum_{k=1}^{d}\int_{|z_k|<r}[V(x+h^{(k)}(x,y,z_k)) - V(x) - V_x(x)h^{(k)}(x,y,z_k)]\nu_k(dz_k)\\ &\quad+\sum_{k=1}^{d}\int_{|z_k|\geq r}[V(x+h'^{(k)}(x,y,z_k)) - V(x)]\nu_k(dz),\end{aligned}$$

where

$$V_x(x) = \left(\frac{\partial V(x)}{\partial x_1}, \ldots, \frac{\partial V(x)}{\partial x_n}\right), \quad V_{xx}(x) = \left(\frac{\partial^2 V(x)}{\partial x_i \partial x_j}\right)_{n\times n}.$$

Then the Itô formula gives that if $V \in C^2(\mathbb{R}^n, \mathbb{R}_+)$, then for any $t \geq 0$

$$\begin{aligned}&V(X(t)) - V(X(0))\\ &= \int_0^t LV(X(s), X(s-\tau))ds + \int_0^t V_x(X(s))g(X(s), X(s-\tau))dB(s)\\ &\quad+\sum_{k=1}^{d}\int_0^t\int_{|z_k|<r}[V(X(s^-) + h^{(k)}(X(s^-), X((s-\tau)^-), z_k)) - V(X(s^-))]\tilde{N}_k(ds, dz_k)\\ &\quad+\sum_{k=1}^{d}\int_0^t\int_{|z_k|\geq r}[V(X(s^-) + h'^{(k)}(X(s^-), X((s-\tau)^-), z_k)) - V(X(s^-))]\tilde{N}_k(ds, dz_k)\\ &\quad+\int_0^t V_x(X(s))dL(s).\end{aligned}$$

Then let us come to discuss the existence and uniqueness of (6.1).

Theorem 6.1 *Let $\xi = (\xi(t); t \in [-\tau, 0])$ be an element of $D^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n_+)$. Assume the assumption 6.1 holds. Then there exists a unique solution $X(t)$ to equation* (6.1) *with initial data $X_0 = \xi$ and $X_0(t) = \xi(0)$.*

Proof At first we shall give some definitions about the Skorokhod problem and its solutions from [48] and [14]. Let $\mathcal{B}$ be a closed set in $\mathbb{R}^n$, and for each

$x \in \partial\mathcal{B}$, let $d(x)$ denote a set of unit vectors, which is called directions of reflection. For $\psi \in D([0,T];\mathbb{R}^n)$, let $|\psi|(T)$ denote the total variation of ψ on $[0,T]$. For a set $\mathcal{B} \subset \mathbb{R}^n$, and $\alpha \in \mathbb{R}$, we define $\alpha\mathcal{B} = \{\alpha x : x \in \mathcal{B}\}$.

Definition 6.1 *(Skorokhod problem) Let $\varphi \in D([0,T];\mathbb{R}^n)$ with $\varphi(0) \in \mathcal{B}$ be given. Then (ϕ, φ, ψ) solves the Skorokhod problem if*

- $\phi = \varphi + \psi$, $\phi(0) = \varphi(0)$;
- $\phi(t) \in \mathcal{B}$ *for* $t \in [0,T]$;
- $|\psi|(T) < \infty$;
- $|\psi|(t) = \int_{(0,t]} I_{\{\phi(s)\in\partial\mathcal{B}\}} d|\psi|(s)$;
- *There exists measurable* $\gamma : [0,T] \to \mathbb{R}^n$ *such that* $\gamma(s) \in d(\phi(s))$ *($d|\psi|$ a.s.) and* $\psi(t) = \int_{(0,t]} \gamma(s) d|\psi|(s)$.

Hence ϕ never leaves $\mathcal{B}$, and ψ changes only when $\phi \in \partial\mathcal{B}$, in which case the change points in one of the directions $d(\phi)$. The ψ term is related to the local time spent on $\partial\mathcal{B}$.

By [33, Appendix A], given an adapted stochastic process $\{\zeta(t), t \geq 0\}$ taking values in $\mathbb{R}^n_+$, all defined on some filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, we define

$$\begin{aligned} S(\zeta)(t) = \zeta(0) &+ \int_0^t f(\zeta(s), \zeta(s-\tau))ds + \int_0^t g(\zeta(s), \zeta(s-\tau))dB(s) \\ &+ \int_0^t \int_{|z|<r} h(\zeta(s^-), \zeta((s-\tau)^-), z)\tilde{N}(ds, dz) \\ &+ \int_0^t \int_{|z|\geq r} h'(\zeta(s^-), \zeta((s-\tau)^-), z)N(ds, dz), \end{aligned} \tag{6.4}$$

For a solution X of the (6.1),

$$X(t) = S(X)(t) + L(t)$$

for $t \in [0, \tau]$, where the regulator term L has the following explicit formula in terms of $S(X)$: for each $i = 1, \ldots, n$,

$$L^i(t) = \max_{s \in [0,t]} ((S(X))^i(s))^-, \quad t \geq 0.$$

For any adapted process $\zeta = (\zeta(t); t \geq 0)$ taking values in $\mathbb{R}^n_+$. Using the form of solution to Skorokhod problem, there exist Lipschitz functions ϕ and ψ with $K_\phi > 0$ and $K_\psi > 0$ respectively, such that

$$X(t) = \phi(S(X))(t), \quad \text{and } L(t) = \psi(S(X))(t), \quad \text{on} \quad t \in [0, T]. \tag{6.5}$$

Because of the uniqueness of solution to the Skorokhod problem; L is a function of X that $L^i(t) = \max_{0 \leq s \leq t}(X^i(s))^-$, $i = 1, \ldots, n$. Then as a consequence of [33, Proposition A.0.1(i)], we have the following lemma.

Lemma 6.1 *For any* $0 \leq a < b < \infty$

$$\begin{aligned}
&\max_{i=1}^{n} \sup_{s,t \in [a,b]} |X^i(t) - X^i(s)| \\
&\leq K_\phi \max_{i=1}^{n} \sup_{s,t \in [a,b]} |(S(X))^i(t) - (S(X))^i(s)|.
\end{aligned}$$

Now let $\bar{\xi}$ be the bound for ξ. For each integer $R \geq \bar{\xi}$, define

$$f^{(R)}(x, y) = f\left(\frac{|x| \wedge R}{|x|} x, \frac{|y| \wedge R}{|y|} y\right),$$

where we set $(|x| \wedge R/|x|)x = 0$ when $x = 0$. Define $g^{(R)}(x, y)$, $h^{(R)}(x, y, z)$ and $h'^{(R)}(x, y, z)$ similarly. Consider (6.4)

$$\begin{aligned}
S(X_R)(t) &= X_R(0) + \int_0^t f^{(R)}(X_R(s), X_R(s - \tau))ds \\
&+ \int_0^t g^{(R)}(X_R(s), X_R(s - \tau))dB(s) \\
&+ \int_0^t \int_{|z|<r} h^{(R)}(X_R(s^-), X_R((s - \tau)^-), z)\tilde{N}(ds, dz) \\
&+ \int_0^t \int_{|z| \geq r} h'^{(R)}(X_R(s^-), X_R((s - \tau)^-), z)N(ds, dz),
\end{aligned} \tag{6.6}$$

on $t \geq 0$ with initial ξ. By assumption 6.1, we observe that $f^{(R)}$, $g^{(R)}$, $h^{(R)}$ and $h'^{(R)}$ satisfy the global Lipschitz condition and the linear growth condition.

By [2, Theorem 6.2.9 P374] (also [47, Proposition 3.1]), the uniqueness and existence on $t \in [0, \tau]$ of (6.6) can be proved. Once we obtain the unique solution on $[0, \tau]$ we can regard them as the initial data and consider (6.6) on $[\tau, 2\tau]$. Repeating this procedure on intervals $[2\tau, 3\tau]$, $[3\tau, 4\tau]$ and so on, we obtain the unique solution $S(X_R)(t)$ to (6.6) on $t \geq -\tau$. Then it follows from the proof of (3.17), which is showing that $\mathbb{P}\{\sigma = \infty\} = 1$, by defining a stopping time and using the Lipschitz functions ϕ, ψ and (6.5), we obtain that (6.1) exists a unique, $\{\mathcal{F}_t\}_{t\geq 0}$-measurable, right continuous and left limited global solution $X(t)$.

6.3 Stability in Distribution for Reflected Stochastic Differential Delay Equations with Jumps

The main purpose of this section is to discuss the stability in distribution of the solution (6.1). We now first give the definition of stability in distribution. For the segment processes $X_t = (X(t+s); -\tau \leq s \leq 0)$. Let $X^{\xi}(t)$ denote the solution of (6.1) with initial data $X_0 = \xi \in D^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n_+)$ and $X^{\xi}_t = (X^{\xi}(t+s); -\tau \leq s \leq 0)$ for $t \geq 0$. Let $p(t, \xi, d\zeta)$ denote the transition probability [40, P84] of the process $X(t)$. Denote by $\mathbb{P}(t, \xi, \Gamma)$ the probability of event $\{X(t) \in \Gamma\}$, i.e. with $\Gamma \in \mathcal{B}(\mathbb{R}^n_+)$, which denotes the Borel σ-algebra of $\mathbb{R}^n_+$, $\mathbb{P}(t, \xi, \Gamma) = \int_\Gamma p(t, \xi, d\zeta)$.

Definition 6.2 *The processes X_t is said to be stable in distribution if there exists a probability measure $\pi(\cdot)$ on $D([-\tau, 0]; \mathbb{R}^n_+)$ such that the transition* *probability* $p(t,\xi,d\zeta)$ *converges weakly to* $\pi(d\zeta)$ *as* $t \to \infty$ *for every* $\xi \in D([-\tau,0];\mathbb{R}^n_+)$. *In this case,* (6.1) *is said to be stable in distribution.*

Theorem 6.2 *Suppose that the segment* X_t *satisfies the following properties:*

(P1) For all $\xi \in D([-\tau,0];\mathbb{R}^n_+)$,

$$\sup_{0\le t<\infty} \mathbb{E}\|X_t^\xi\|^2 < \infty. \tag{6.7}$$

(P2) For any compact subset M *of* $D([-\tau,0];\mathbb{R}^n_+)$,

$$\lim_{t\to\infty} \mathbb{E}\|X_t^\xi - X_t^\eta\|^2 = 0 \tag{6.8}$$

uniformly in $\xi,\eta \in M$. *Then* $X(t)$ *is stable in distribution.*

To prove this theorem we need to introduce more notations. Let $\mathbb{P}(\mathbb{R}^n_+)$ denote all probability measures on $\mathbb{R}^n$. For $\mathbb{P}_1, \mathbb{P}_2 \in \mathbb{P}(\mathbb{R}^n)$ define metric $d_{\mathbb{L}}$ as follows:

$$d_{\mathbb{L}}(\mathbb{P}_1,\mathbb{P}_2) = \sup_{f\in\mathbb{L}} \left| \int_{\mathbb{R}^n} f(\xi)\mathbb{P}_1(d\xi) - \int_{\mathbb{R}^n} f(\xi)\mathbb{P}_2(d\xi) \right|$$

and

$$\mathbb{L} = \{f : D([-\tau,0];\mathbb{R}^n) \to \mathbb{R} : |f(\xi)-f(\eta)| \le \|\xi-\eta\| \text{ and } |f(\cdot)| \le 1\}.$$

Let us now present three lemmas.

Lemma 6.2 *Under the assumption 6.1, for every* $p > 0$ *and any compact subset* M *of* $\mathbb{R}^n$,

$$\sup_{\xi\in M} \mathbb{E}\left(\sup_{0\le s\le t} |X^\xi(s)|^p \right) < \infty \quad \forall t \ge 0.$$

Lemma 6.3 *Let the assumption 6.1 holds and* (6.1) *have property P2. Then, for any compact subset* M *of* $\mathbb{R}^n$,

$$\lim_{t\to\infty} d_{\mathbb{L}}(p(t,\xi,\cdot), p(t,\eta,\cdot)) = 0$$

uniformly in $\xi,\eta \in M$.

Lemma 6.4 *Let the assumption 6.1 holds. If* (6.1) *has properties P1 and P2, then for any* $x \in \mathbb{R}^n$, $\{p(t,\xi,\cdot) : t \geq 0\}$ *is Cauchy sequence in the space* $\mathbb{P}(\mathbb{R}^n)$ *with metric* $d_{\mathbb{L}}$.

The proof of lemma 6.3, 6.4 and theorem 6.2 are similar with [19, Lemma 3.3, Lemma 3.4,Theorem 3.1] without Markovian switching.

Then let us derive results on the stability in distribution for the processes $X(t)$, i.e. we shall establish some sufficient criteria for (P1) and (P2) of Theorem 6.2. We first proof (P1).

Proposition 6.1 *Let the assumption 6.1 holds and* c_1 *be a positive number and* $\lambda_1 > \lambda_2 \geq 0$. *Assume that there exist functions* $V(x) \in C^2(\mathbb{R}^n_+;\mathbb{R}_+)$ *and* $\omega_1(x) \in C(\mathbb{R}^n_+;\mathbb{R}_+)$ *such that*

$$c_1|x|^2 \leq V(x) \leq \omega_1(x), \tag{6.9}$$

for all $x \in \mathbb{R}^n_+$ *and*

$$\frac{\partial V(x_l^0)}{\partial x_l} \leq 0, \quad l = 1, 2, \dots, n, \tag{6.10}$$

where

$$x_l^0 := (x_1, \dots, x_{l-1}, 0, x_{l+1}, \dots, x_n)$$

with $x_i \geq 0$ $(i \neq l)$, *and*

$$LV(x,y) \leq -\lambda_1\omega_1(x) + \lambda_2\omega_1(y), \tag{6.11}$$

for all $(x,y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+$, *then*

$$\sup_{0\leq t<\infty} \mathbb{E}\|X_t^\xi\|^2 < \infty, \quad \forall \xi \in D^b_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n_+). \tag{6.12}$$

Proof. For convenience, we write $X^{\xi}(t) = X(t)$ for any initial data $X(0) = \xi$. Define a stopping time by

$$\sigma_R = \inf(t \geq 0; |X(t)| > R),$$

where we set $\inf \emptyset = \infty$ as usual. By the generalized Itô formula, we have that for any $t > 0$,

$$\begin{aligned}
&\mathbb{E}V(X(t \wedge \sigma_R)) - \mathbb{E}V(X(0)) \\
&= \mathbb{E}\int_0^{t\wedge\sigma_R} LV(X(s), X(s-\tau))ds + \mathbb{E}\int_0^{t\wedge\sigma_R} V_x(X_l^0(s))dL(s) \\
&\quad + \mathbb{E}\bigg(\int_0^{t\wedge\sigma_R} V_x(X(s))g(X(s), X(s-\tau))dB(s) \\
&\quad + \sum_{k=1}^{d}\int_0^{t\wedge\sigma_R}\int_{|z_k|<r} [V(X(s^-) + h^{(k)}(X(s^-), X((s-\tau)^-), z_k)) \\
&\quad - V(X(s^-))]\tilde{N}_k(ds, dz_k) \\
&\quad + \sum_{k=1}^{d}\int_0^{t\wedge\sigma_R}\int_{|z_k|\geq r} [V(X(s^-) + h'^{(k)}(X(s^-), X((s-\tau)^-), z_k)) \\
&\quad - V(X(s^-))]N_k(ds, dz_k)\bigg).
\end{aligned} \tag{6.13}$$

By using (6.10) and the fact of $dL(t) \geq 0$ for all $t \geq 0$, P-a.s.

$$\mathbb{E}\int_0^{t\wedge\sigma_R} V_x(X_l^0(s))dL(s) \leq 0.$$

Furthermore, by (6.11), we then derive from (6.13) that

$$\mathbb{E}V(X(t\wedge\sigma_R)) \leq \mathbb{E}V(\xi(0)) + \lambda_2\int_{-\tau}^{0} \omega_1(\xi(s))ds. \tag{6.14}$$

Letting $R \to \infty$ we have

$$\mathbb{E}V(X(t)) \leq \mathbb{E}V(\xi(0)) + \lambda_2\int_{-\tau}^{0} \omega_1(\xi(s))ds. \tag{6.15}$$

This implies

$$\sup_{0\le t<\infty} \mathbb{E}V(X(t)) < \infty. \tag{6.16}$$

Hence by the condition (6.9),

$$\sup_{0\le t<\infty} \mathbb{E}|X(t)|^2 \le \frac{1}{c_1}\left(\mathbb{E}V(\xi(0)) + \lambda_2 \int_{-\tau}^{0} \omega_1(\xi(s))ds\right) < \infty.$$

In what follows, we shall estimate the segment processes X_t. Let $t > \tau$ and $\theta \in [0,\tau]$. From (6.1) and the Itô formula, it follows that

$$\begin{aligned}
&|X(t-\theta)|^2 - |X(t-\tau)|^2 \\
&= \int_{t-\tau}^{t-\theta} |g(X(s),X(s-\tau))|^2 ds + 2\int_{t-\tau}^{t-\theta} X(s)^T f(X(s),X(s-\tau))ds \\
&\quad + 2\int_{t-\tau}^{t-\theta} X(s)^T g(X(s),X(s-\tau))dB(s) + 2\int_{t-\tau}^{t-\theta} X(s)^T dL(s) \\
&\quad + 2\sum_{k=1}^{d}\int_{t-\tau}^{t-\theta}\int_{|z_k|<r} |h^{(k)}(X(s),X(s-\tau),z_k)|^2 \nu_k(dz_k)ds \\
&\quad + \sum_{k=1}^{d}\int_{t-\tau}^{t-\theta}\int_{|z_k|<r} [|h^{(k)}(X(s^-),X((s-\tau)^-),z_k)|^2 \\
&\qquad + 2X(s^-)^T h^{(k)}(X(s^-),X((s-\tau)^-),z_k)]\tilde{N}_k(ds,dz_k) \\
&\quad + \sum_{k=1}^{d}\int_{t-\tau}^{t-\theta}\int_{|z_k|\ge r} [|h'^{(k)}(X(s^-),X((s-\tau)^-),z_k)|^2 \\
&\qquad + 2X(s^-)^T h'^{(k)}(X(s^-),X((s-\tau)^-),z_k)]\tilde{N}_k(ds,dz_k) \\
&\quad + \sum_{k=1}^{d}\int_{t-\tau}^{t-\theta}\int_{|z_k|\ge r} |h'^{(k)}(X(s),X(s-\tau),z_k)|^2 \nu_k(dz_k)ds \\
&\quad + \sum_{k=1}^{d}\int_{t-\tau}^{t-\theta}\int_{|z_k|\ge r} [|X(s)+h'^{(k)}(X(s),X(s-\tau),z_k)|^2 - |X(s)|^2]\nu_k(dz_k)ds.
\end{aligned} \tag{6.17}$$

By the property of regulator $L(t)$, we have P-a.s.

$$\int_{t-\tau}^{t-\theta} X(s)^T dL(s) = \int_{0}^{t-\theta} X(s)^T dL(s) - \int_{0}^{t-\tau} X(s)^T dL(s) = 0.$$

Then we estimate the terms as follow

$$
\begin{aligned}
&\mathbb{E}\sup_{0\le\theta\le\tau}\Bigg|\sum_{k=1}^{d}\int_{t-\tau}^{t-\theta}\int_{|z_k|<r}|h^{(k)}(X(s),X(s-\tau),z_k)|^2\nu_k(dz_k)ds\\
&\quad+\sum_{k=1}^{d}\int_{t-\tau}^{t-\theta}\int_{|z_k|<r}[|h^{(k)}(X(s^-),X((s-\tau)^-),z_k)|^2\\
&\qquad+2X(s^-)^Th^{(k)}(X(s^-),X((s-\tau)^-),z_k)]\tilde{N}_k(ds,dz_k)\Bigg|\\
&=\mathbb{E}\sup_{0\le\theta\le\tau}\Bigg|\sum_{k=1}^{d}\int_{t-\tau}^{t-\theta}\int_{|z_k|<r}|h^{(k)}(X(s^-),X((s-\tau)^-),z_k)|^2N_k(ds,dz_k)\\
&\quad+\sum_{k=1}^{d}\int_{t-\tau}^{t-\theta}\int_{|z_k|<r}2X(s^-)^Th^{(k)}(X(s^-),X((s-\tau)^-),z_k)\tilde{N}_k(ds,dz_k)\Bigg|,
\end{aligned}
\tag{6.18}
$$

where

$$
\begin{aligned}
&\mathbb{E}\sup_{0\le\theta\le\tau}\Bigg|\sum_{k=1}^{d}\int_{t-\tau}^{t-\theta}\int_{|z_k|<r}|h^{(k)}(X(s^-),X((s-\tau)^-),z_k)|^2N_k(ds,dz_k)\Bigg|\\
&=\mathbb{E}\sum_{k=1}^{d}\int_{t-\tau}^{t}\int_{|z_k|<r}|h^{(k)}(X(s^-),X((s-\tau)^-),z_k)|^2N_k(ds,dz_k)\\
&=\mathbb{E}\sum_{k=1}^{d}\int_{t-\tau}^{t}\int_{|z_k|<r}|h^{(k)}(X(s^-),X((s-\tau)^-),z_k)|^2\tilde{N}_k(ds,dz_k) \qquad (6.19)\\
&\quad+\mathbb{E}\sum_{k=1}^{d}\int_{t-\tau}^{t}\int_{|z_k|<r}|h^{(k)}(X(s^-),X((s-\tau)^-),z_k)|^2\nu_k(dz_k)ds\\
&=\sum_{k=1}^{d}\int_{t-\tau}^{t}\int_{|z_k|<r}\mathbb{E}|h^{(k)}(X(s^-),X((s-\tau)^-),z_k)|^2\nu_k(dz_k)ds,
\end{aligned}
$$

so does the large jumps term. In the meanwhile,

$$
\begin{aligned}
&\mathbb{E}\sup_{0\le\theta\le\tau}\Bigg|\sum_{k=1}^{d}\int_{t-\tau}^{t-\theta}\int_{|z_k|\ge r}[|X(s)+h'^{(k)}(X(s),X(s-\tau),z_k)|^2-|X(s)|^2]\nu_k(dz_k)ds\Bigg|\\
&\le 2\mathbb{E}\sum_{k=1}^{d}\int_{t-\tau}^{t}\int_{|z_k|\ge r}|h'^{(k)}(X(s),X(s-\tau),z_k)|^2\nu_k(dz_k)ds\\
&\quad+\mathbb{E}\sum_{k=1}^{d}\int_{t-\tau}^{t}\int_{|z_k|\ge r}|X(s)|^2\nu_k(dz_k)ds
\end{aligned}
$$

which implies that

$$\sum_{k=1}^{d}\int_{t-\tau}^{t}\int_{|z_k|\geq r}|X(s)|^2\nu_k(dz_k)ds = \int_{t-\tau}^{t}|X(s)|^2\sum_{k=1}^{d}\int_{|z_k|\geq r}\nu_k(dz_k)ds$$
$$= \int_{t-\tau}^{t}|X(s)|^2\sum_{k=1}^{d}\nu_k(|z_k|\geq r)ds = \sum_{k=1}^{d}\nu_k(|z_k|\geq r)\int_{t-\tau}^{t}|X(s)|^2ds,$$

where $\sum_{k=1}^{d}\nu_k(|z_k|\geq r) < \infty$. By the Burkholder-Davis-Gundy inequality (2.6) and $2ab \leq \frac{1}{\alpha}a^2 + \alpha b^2$ for $\alpha > 0$,

$$\mathbb{E}\sup_{0\leq\theta\leq\tau}\left|\int_{t-\tau}^{t-\theta}X(s)^Tg(X(s),X(s-\tau))dB(s)\right|$$
$$\leq 3\mathbb{E}\bigg(\sup_{0\leq\theta\leq\tau}|X(t-\theta)|^2\int_{t-\tau}^{t}|g(X(s),X(s-\tau))|^2ds\bigg)^{\frac{1}{2}}$$
$$\leq \frac{1}{12}\mathbb{E}\sup_{0\leq\theta\leq\tau}|X(t-\theta)|^2 + 27\mathbb{E}\int_{t-\tau}^{t}|g(X(s),X(s-\tau))|^2ds,$$

as well as

$$\mathbb{E}\sup_{0\leq\theta\leq\tau}\left|\sum_{k=1}^{d}\int_{t-\tau}^{t-\theta}\int_{|z_k|<r}X(s^-)^Th^{(k)}(X(s^-),X((s-\tau)^-),z_k)\tilde{N}_k(ds,dz_k)\right|,$$
$$\leq \frac{1}{12}\mathbb{E}\sup_{0\leq\theta\leq\tau}|X(t-\theta)|^2$$
$$+ 27\mathbb{E}\sum_{k=1}^{d}\int_{t-\tau}^{t}\int_{|z_k|<r}|h^{(k)}(X(s^-),X((s-\tau)^-),z_k)|^2\nu_k(dz_k)ds$$

and the large jumps term. Then we have

$$
\begin{aligned}
&\mathbb{E}\sup_{0\le\theta\le\tau}|X(t-\theta)|^2 \\
&\le 2\mathbb{E}|X(t-\tau)|^2 + \left(2+\sum_{k=1}^{d}\nu_k(|z_k|\ge r)\right)\int_{t-\tau}^{t}\mathbb{E}|X(s)|^2ds \\
&\quad + 2\int_{t-\tau}^{t}\mathbb{E}|f(X(s),X(s-\tau))|^2ds + 110\int_{t-\tau}^{t}\mathbb{E}|g(X(s),X(s-\tau))|^2ds \\
&\quad + C\int_{t-\tau}^{t}\mathbb{E}\sum_{k=1}^{d}\int_{|z_k|<r}|h^{(k)}(X(s),X(s-\tau),z_k)|^2\nu_k(dz_k)ds \\
&\quad + C'\int_{t-\tau}^{t}\mathbb{E}\sum_{k=1}^{d}\int_{|z_k|\ge r}|h'^{(k)}(X(s),X(s-\tau),z_k)|^2\nu(dz_k)ds.
\end{aligned}
\tag{6.20}
$$

Therefore by linear growth condition (6.3) we have

$$
\begin{aligned}
&\mathbb{E}\sup_{0\le\theta\le\tau}|X(t-\theta)|^2 \\
&\le 2\mathbb{E}|X(t-\tau)|^2 + C\int_{t-\tau}^{t}\mathbb{E}|X(s)|^2ds + C'\int_{t-\tau}^{t}\mathbb{E}(|X(s)|^2+|X(s-\tau)|^2)ds \\
&\le C + C'\mathbb{E}\sup_{0\le t<\infty}|X(t)|^2,
\end{aligned}
$$

where $C, C' > 0$ are constants which may different line by line. Using (6.16) we obtain (6.12), which complete the proof.

Remark 6.1 *Using the definition of compensate Poisson processes twice to prove the terms of* (6.18) *and* (6.19) *and the fact of Poisson measure is a counting measure to take off the superior are more convenient than the proof of [47, Proposition 3.1] and [6, Proof of (4.7)].*

To solve property (P2), we need to consider the difference between two solutions of (6.1) starting from different initial values, namely

$$
\begin{aligned}
&X^{\xi}(t) - X^{\eta}(t) - (\xi(0) - \eta(0)) \\
&= \int_0^t [f(X^{\xi}(s), X^{\xi}(s-\tau)) - f(X^{\eta}(s), X^{\eta}(s-\tau))]ds \\
&\quad + \int_0^t [g(X^{\xi}(s), X^{\xi}(s-\tau)) - g(X^{\eta}(s), X^{\eta}(s-\tau))]dB(s) \\
&\quad + \int_0^t \int_{|z|<r} [h(X^{\xi}(s^-), X^{\xi}((s-\tau)^-), z) - h(X^{\eta}(s^-), X^{\eta}((s-\tau)^-), z)]\tilde{N}(ds, dz) \\
&\quad + \int_0^t \int_{|z|\geq r} [h'(X^{\xi}(s^-), X^{\xi}((s-\tau)^-), z) - h'(X^{\eta}(s^-), X^{\eta}((s-\tau)^-), z)]N(ds, dz) \\
&\quad + \int_0^t d(L^{\xi} - L^{\eta})(s),
\end{aligned} \tag{6.21}
$$

where the initial datum $\xi, \eta \in D([-\tau, 0]; \mathbb{R}^n_+)$. In addition, the pair of processes $(X^{\xi}|_{\mathbb{R}^n_+} L^{\xi})$ and $(X^{\eta}|_{\mathbb{R}^n_+} L^{\eta})$ solve the Skorokhod problem $X(t) = S(X)(t) + L(t)$ with different initial datum, respectively. For given function $U \in C^2(\mathbb{R}^n; \mathbb{R}_+)$, we define an operator $L'U : \mathbb{R}^{n\times 4} \to \mathbb{R}$ associated with (6.21) by

$$
\begin{aligned}
L'U(x_1, y_1, x_2, y_2) &= U_x(x_1 - x_2)(f(x_1, y_1) - f(x_2, y_2)) \\
&\quad + \frac{1}{2}\text{trace}[(g(x_1, y_1) - g(x_2, y_2))^T U_{xx}(x_1 - x_2)(g(x_1, y_1) - g(x_2, y_2))] \\
&\quad + \sum_{k=1}^{d} \int_{|z_k|<r} \big[U\big(x_1 - x_2 + (h^{(k)}(x_1, y_2, z_k) - h^{(k)}(x_2 - y_2, z_k))\big) \\
&\qquad - U(x_1 - x_2) - U_x(x_1 - x_2)(h^{(k)}(x_1, y_2, z_k) - h^{(k)}(x_2 - y_2, z_k))\big]\nu_k(dz_k) \\
&\quad + \sum_{k=1}^{d} \int_{|z_k|\geq r} \big[U\big(x_1 - x_2 + (h'^{(k)}(x_1, y_2, z_k) - h'^{(k)}(x_2 - y_2, z_k))\big) - U(x_1 - x_2)\big]\nu_k(dz_k).
\end{aligned}
$$

For the future use, we shall impose anther hypothesis:

Assumption 6.2 *There is a $\bar{K} > 0$ which may different of the previous one such that*

$$\langle x_1 - x_2, f(x_1, y_1) - f(x_2, y_2)\rangle + |g(x_1, y_1) - g(x_2, y_2)|^2$$
$$+ \sum_{k=1}^{d} \int_{|z_k|<r} |h^{(k)}(x_1, y_1, z_k) - h_{(k)}(x_2, y_2, z_k)|^2 \nu_k(dz_k)$$
$$+ \sum_{k=1}^{d} \int_{|z|\geq r} |h'^{(k)}(x_1, y_1, z_k) - h_{(k)}(x_2, y_2, z_k)|^2 \nu_k(dz_k)$$
$$\leq \bar{K}(|x_2 - x_1|^2 + |y_2 - y_1|^2)$$

for $x_1, x_2, y_1, y_2 \in \mathbb{R}^n_+$.

Proposition 6.2 *Let the condition of proposition 6.1 and the assumption 6.2 hold. Assume that there exist positive numbers c_2 and $\lambda_3 > \lambda_4 \geq 0$ and $U(x) \in C^2(\mathbb{R}^n; \mathbb{R}_+)$, $\omega_2(x) \in C(\mathbb{R}^n; \mathbb{R})$ such that*

$$U(0, 0) = 0.$$

$$c_2|x| \leq \omega_2(x) \wedge U(x), \tag{6.22}$$

and

$$\frac{\partial U}{\partial x_l}(x_l^-) \leq 0, \quad \frac{\partial U}{\partial x_l}(x_l^+) \geq 0, \quad l = 1, 2, \ldots, n, \tag{6.23}$$

for

$$x_l^{\mp} := (x_1, \ldots, x_{l-1}, \mp x_l, x_{l+1}, \ldots, x_n)$$

with all $x_i \in \mathbb{R}(i \neq l)$ and $x_l \in \mathbb{R}_+$, and

$$L'U(x_1, x_2, y_1, y_2) \leq -\lambda_3\omega_2(x_1 - x_2) + \lambda_4\omega_2(y_1 - y_2), \tag{6.24}$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{R}^n_+$, then

$$\lim_{t\to\infty} \|X_t^{\xi} - X_t^{\eta}\|^2 = 0 \quad \textit{uniformly in} \quad \xi, \eta \in M, \tag{6.25}$$

for any compact subset M of $D([-\tau, 0]; \mathbb{R}^n_+)$.

Proof. First we prove

$$\lim_{t\to\infty} |X^\xi(t) - X^\eta(t)|^2 = 0 \quad \text{uniformly in} \quad \xi, \eta \in M. \tag{6.26}$$

Let R be positive number and define a stopping time by

$$\varrho_R = \inf\{t > 0; |X^\xi(t) - X^\eta(t)| > R\}.$$

Setting $T_R = \varrho_R \wedge t$ and applying the Itô formula to (6.21) we conclude that

$$\begin{aligned} \mathbb{E}U[X^\xi(T_R)) - X^\eta(T_R)] \le \mathbb{E}U(\xi(0) - \eta(0)) &+ \lambda_4 \tau \int_{-\tau}^0 \omega_2(\xi(s) - \eta(s))ds \\ &- (\lambda_3 - \lambda_4)\mathbb{E}\int_0^{T_R} |X^\xi(s) - X^\eta(s)|^2 ds \\ &+ \mathbb{E}\int_0^{T_R} U_x(X^\xi(s) - X^\eta(s))d(L^\xi - L^\eta)(s). \end{aligned} \tag{6.27}$$

We conclude that by employing the condition (6.23) and property $\int_0^t X(s)^T dL(s) = 0$, P-a.s.

$$\begin{aligned} &\int_0^{T_R} U_x(X^\xi - X^\eta)(s) d(L^\xi - L^\eta)(s) \\ &= \int_0^{T_R} U_x(X^\xi - X^\eta)(s) d(L^\xi)(s) - \int_0^{T_R} U_x(X^\xi - X^\eta)(s) d(L^\eta)(s) \\ &= \int_0^{T_R} U_x\big((X_1^\xi - X_1^\eta)(s), \dots, (X_{l-1}^\xi - X_{l-1}^\eta)(s), -X_l^\eta(s), \\ &\quad (X_{l+1}^\xi - X_{l+1}^\eta)(s), \dots, (X_n^\xi - X_n^\eta)(s)\big) d(L^\xi)(s) \\ &\quad - \int_0^{T_R} U_x\big((X_1^\xi - X_1^\eta)(s), \dots, (X_{l-1}^\xi - X_{l-1}^\eta)(s), X_l^\xi(s), \\ &\quad (X_{l+1}^\xi - X_{l+1}^\eta)(s), \dots, (X_n^\xi - X_n^\eta)(s)\big) d(L^\eta)(s) \le 0, \end{aligned}$$

since $dL^\xi(t) \ge 0$ and $dL^\eta(t) \ge 0$. This implies

$$\begin{aligned} &\int_0^\infty \mathbb{E}|X^\xi(s) - X^\eta(s)|^2 ds \\ &\le \frac{1}{\lambda_3 - \lambda_4} U(\xi(0) - \eta(0)) + \frac{\lambda_4}{\lambda_3 - \lambda_4} \tau \int_{-\tau}^0 \omega_2(\xi(s) - \eta(s))ds < \infty. \end{aligned} \tag{6.28}$$

Then we claim

$$\lim_{t\to\infty} \mathbb{E}|X^{\xi}(t) - X^{\eta}(t)|^2 = 0. \tag{6.29}$$

If (6.29) false, then there exists a constant $\Lambda > 0$ such that

$$\limsup_{t\to\infty} \mathbb{E}|X^{\xi}(t) - X^{\eta}(t)|^2 = \Lambda.$$

So there is a positive number $\epsilon < \frac{\Lambda}{11}$ and a sequence $(t_n, n = 1, 2, \dots,)$, $t_n \to \infty$ as $n \to \infty$ such that

$$\mathbb{E}|X^{\xi}(t_n) - X^{\eta}(t_n)|^2 \geq \Lambda - \epsilon.$$

Let $t > t_n$ and $|t - t_n| < 1$. Then by (6.21)

$$\begin{aligned}
&X^{\xi}(t) - X^{\eta}(t) \\
&= X^{\xi}(t_n) - X^{\eta}(t_n) + \int_{t_n}^{t} d(L^{\xi} - L^{\eta})(s) \\
&\quad + \int_{t_n}^{t} [f(X^{\xi}(s), X^{\xi}(s-\tau)) - f(X^{\eta}(s), X^{\eta}(s-\tau))]ds \\
&\quad + \int_{t_n}^{t} [g(X^{\xi}(s), X^{\xi}(s-\tau)) - g(X^{\eta}(s), X^{\eta}(s-\tau))]dB(s) \\
&\quad + \int_{t_n}^{t}\int_{|z|<r} [h(X^{\xi}(s^-), X^{\xi}((s-\tau)^-), z) \\
&\quad - h(X^{\eta}(s^-), X^{\eta}((s-\tau)^-), z)]\tilde{N}(ds, dz) \\
&\quad + \int_{t_n}^{t}\int_{|z|\geq r} [h'(X^{\xi}(s^-), X^{\xi}((s-\tau)^-), z) \\
&\quad - h'(X^{\eta}(s^-), X^{\eta}((s-\tau)^-), z)]N(ds, dz).
\end{aligned}$$

As a consequence

$$
\begin{aligned}
&|X^{\xi}(t) - X^{\eta}(t)|^2 \\
&\geq \frac{1}{5}|X^{\xi}(t_n) - X^{\eta}(t_n)|^2 - \bigg| \int_{t_n}^{t} d(L^{\xi} - L^{\eta})(s) \\
&\quad + \int_{t_n}^{t} [f(X^{\xi}(s), X^{\xi}(s-\tau)) - f(X^{\eta}(s), X^{\eta}(s-\tau))]ds \bigg|^2 \\
&\quad - \bigg| \int_{t_n}^{t} [g(X^{\xi}(s), X^{\xi}(s-\tau)) - g(X^{\eta}(s), X^{\eta}(s-\tau))]dB(s) \bigg|^2 \\
&\quad - \bigg| \int_{t_n}^{t} \int_{|z|<r} [h(X^{\xi}(s^-), X^{\xi}((s-\tau)^-), z) - h(X^{\eta}(s^-), X^{\eta}((s-\tau)^-), z)]\tilde{N}(ds, dz) \bigg|^2 \\
&\quad - \bigg| \int_{t_n}^{t} \int_{|z|\geq r} [h'(X^{\xi}(s^-), X^{\xi}((s-\tau)^-), z) - h'(X^{\eta}(s^-), X^{\eta}((s-\tau)^-))]N(ds, dz) \bigg|^2.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\mathbb{E}|X^{\xi}(t) - X^{\eta}(t)|^2 \\
&\geq \frac{1}{5}\mathbb{E}|X^{\xi}(t_n) - X^{\eta}(t_n)|^2 - 2\mathbb{E}\bigg| \int_{t_n}^{t} d(L^{\xi} - L^{\eta})(s) \bigg|^2 \\
&\quad - 2\mathbb{E}\bigg| \int_{t_n}^{t} [f(X^{\xi}(s), X^{\xi}(s-\tau)) - f(X^{\eta}(s), X^{\eta}(s-\tau))]ds \bigg|^2 \\
&\quad - \mathbb{E}\bigg| \int_{t_n}^{t} [g(X^{\xi}(s), X^{\xi}(s-\tau)) - g(X^{\eta}(s), X^{\eta}(s-\tau))]dB(s) \bigg|^2 \\
&\quad - \mathbb{E}\bigg| \int_{t_n}^{t} \int_{|z|<r} [h(X^{\xi}(s^-), X^{\xi}((s-\tau)^-), z) - h(X^{\eta}(s^-), X^{\eta}((s-\tau)^-), z)]\tilde{N}(ds, dz) \bigg|^2 \\
&\quad - \mathbb{E}\bigg| \int_{t_n}^{t} \int_{|z|\geq r} [h'(X^{\xi}(s^-), X^{\xi}((s-\tau)^-), z) - h'(X^{\eta}(s^-), X^{\eta}((s-\tau)^-), z)]N(ds, dz) \bigg|^2.
\end{aligned}
\tag{6.30}
$$

Note that

$$
\begin{aligned}
&2\mathbb{E}\bigg| \int_{t_n}^{t} d(L^{\xi} - L^{\eta})(s) \bigg|^2 \\
=&2\mathbb{E}|(L^{\xi}(t) - L^{\xi}(t_n)) - (L^{\eta}(t) - L^{\eta}(t_n))|^2 \\
\leq&4\mathbb{E}|(L^{\xi}(t) - L^{\xi}(t_n))|^2 + 4\mathbb{E}|(L^{\eta}(t) - L^{\eta}(t_n))|^2.
\end{aligned}
\tag{6.31}
$$

From (6.1), for any $t > t_n$, we have

$$L(t) - L(t_n) = X(t) - X(t_n) - \int_{t_n}^{t} f(X(s), X(s-\tau))ds$$
$$- \int_{t_n}^{t} g(X(s), X(s-\tau))dB(s) - \int_{t_n}^{t}\int_{|z|<r} h(X(s^-), X((s-\tau)^-), z)\tilde{N}(ds, dz)$$
$$- \int_{t_n}^{t}\int_{|z|\geq r} h'(X(s^-), X((s-\tau)^-), z)N(ds, dz).$$

Hence

$$\mathbb{E}|L(t) - L(t_n)|^2 = 5\mathbb{E}|X(t) - X(t_n)|^2 + 5\mathbb{E}\left|\int_{t_n}^{t} f(X(s), X(s-\tau))ds\right|^2$$
$$+ 5\mathbb{E}\left|\int_{t_n}^{t} g(X(s), X(s-\tau))dB(s)\right|^2 + 5\mathbb{E}\left|\int_{t_n}^{t}\int_{|z|<r} h(X(s^-), X((s-\tau)^-), z)\tilde{N}(ds, dz)\right|^2$$
$$+ 5\mathbb{E}\left|\int_{t_n}^{t}\int_{|z|\geq r} h'(X(s^-), X((s-\tau)^-), z)N(ds, dz)\right|^2.$$

Substituting this into (6.31) yields

$$2\mathbb{E}\left|\int_{t_n}^{t} d(L^{\xi} - L^{\eta})(s)\right|^2$$
$$\leq 20\mathbb{E}|X^{\xi}(t) - X^{\xi}(t_n)|^2 + 20\mathbb{E}|X^{\eta}(t) - X^{\eta}(t_n)|^2$$
$$+ 20\mathbb{E}\left|\int_{t_n}^{t} f(X^{\xi}(s), X^{\xi}(s-\tau))ds\right|^2 + 20\mathbb{E}\left|\int_{t_n}^{t} f(X^{\eta}(s), X^{\eta}(s-\tau))ds\right|^2$$
$$+ 20\mathbb{E}\left|\int_{t_n}^{t} g(X^{\xi}(s), X^{\xi}(s-\tau))dB(s)\right|^2 + 20\mathbb{E}\left|\int_{t_n}^{t} g(X^{\eta}(s), X^{\eta}(s-\tau))dB(s)\right|^2$$
$$+ 20\mathbb{E}\left|\int_{t_n}^{t}\int_{|z|<r} h(X^{\xi}(s^-), X^{\xi}((s-\tau)^-), z)\tilde{N}(ds, dz)\right|^2$$
$$+ 20\mathbb{E}\left|\int_{t_n}^{t}\int_{|z|<r} h(X^{\eta}(s^-), X^{\eta}((s-\tau)^-), z)\tilde{N}(ds, dz)\right|^2$$
$$+ 20\mathbb{E}\left|\int_{t_n}^{t}\int_{|z|\geq r} h'(X^{\xi}(s^-), X^{\xi}((s-\tau)^-)), z)N(ds, dz)\right|^2$$
$$+ 20\mathbb{E}\left|\int_{t_n}^{t}\int_{|z|\geq r} h'(X^{\eta}(s^-), X^{\eta}((s-\tau)^-), z)N(ds, dz)\right|^2. \tag{6.32}$$

In what follows it is sufficient to estimate the following two terms $\mathbb{E}|X^\xi(t) - X^\xi(t_n)|^2$ and $\mathbb{E}|X^\eta(t) - X^\eta(t_n)|^2$. Next applying the Itô formula

$$\begin{aligned}
&|X^\xi(t) - X^\xi(t_n)|^2 \\
&= 2\int_{t_n}^t [X^\xi(s) - X^\xi(t_n)]^T f(X^\xi(s), X^\xi(s-\tau))ds \\
&\quad + 2\int_{t_n}^t [X^\xi(s) - X^\xi(t_n)]^T g(X^\xi(s), X^\xi(s-\tau))dB(s) \\
&\quad + \sum_{k=1}^d \int_{t_n}^t \int_{|z_k|<r} [|X^\xi(s^-) - X^\xi(t_n^-) + h^{(k)}(X^\xi(s^-), X^\xi((s-\tau)^-), z_k)|^2 \\
&\quad - |X^\xi(s^-) - X^\xi(t_n^-)|^2]\tilde{N}_k(ds, dz_k) + \int_{t_n}^t |g(X^\xi(s), X^\xi(s-\tau))|^2 ds \\
&\quad + \sum_{k=1}^d \int_{t_n}^t \int_{|z_k|\geq r} [|X^\xi(s^-) - X^\xi(t_n^-) + h'^{(k)}(X^\xi(s^-), X^\xi((s-\tau)^-), z_k)|^2 \\
&\quad - |X^\xi(s^-) - X^\xi(t_n^-)|^2]\tilde{N}_k(ds, dz_k) + 2\int_{t_n}^t [X^\xi(s) - X^\xi(t_n)]^T dL(s) \\
&\quad + \sum_{k=1}^d \int_{t_n}^t \int_{|z_k|<r} |h^{(k)}(X^\xi(s), X^\xi(s-\tau), z_k)|^2 \nu_k(dz_k)ds \\
&\quad + \sum_{k=1}^d \int_{t_n}^t \int_{|z_k|\geq r} [|X^\xi(s) - X^\xi(t_n) + h'^{(k)}(X^\xi(s), X^\xi(s-\tau), z_k)|^2 \\
&\quad - |X^\xi(s) - X^\xi(t_n)|^2]\nu_k(dz_k)ds.
\end{aligned} \tag{6.33}$$

Observe that

$$-2\int_{t_n}^t \langle X^\xi(t_n), dL(s)\rangle \leq 0$$

since from the definition and the increasing property of the regulators we have

$$2\int_{t_n}^t \langle X^\xi(s), dL(s)\rangle = 2\int_0^t \langle X^\xi(s), dL(s)\rangle - 2\int_0^{t_n} \langle X^\xi(s), dL(s)\rangle = 0.$$

Thus, taking expectations on both sides of (6.33) and estimate it using the method, which is same as the proof from (6.17) to (6.20), we obtain

$$
\begin{aligned}
&\mathbb{E}\sup_{0\le s\le t}|X^{\xi}(s)-X^{\xi}(t_n)|^2\\
&\le\left(2+\sum_{k=1}^{d}\nu_k(|z_k|\ge r)\right)\int_{t_n}^{t}\mathbb{E}|X^{\xi}(s)-X^{\xi}(t_n)|^2ds\\
&\quad+2\int_{t_n}^{t}\mathbb{E}|f(X^{\xi}(s),X^{\xi}(s-\tau))|^2ds+110\int_{t_n}^{t}\mathbb{E}|g(X^{\xi}(s),X^{\xi}(s-\tau))|^2ds\\
&\quad+C\int_{t_n}^{t}\mathbb{E}\sum_{k=1}^{d}\int_{|z_k|<r}|h^{(k)}(X^{\xi}(s),X^{\xi}(s-\tau),z_k)|^2\nu_k(dz_k)ds\\
&\quad+C'\int_{t_n}^{t}\mathbb{E}\sum_{k=1}^{d}\int_{|z_k|\ge r}|h'^{(k)}(X^{\xi}(s),X^{\xi}(s-\tau),z_k)|^2\nu(dz_k)ds.
\end{aligned}
$$

Consequently, thanks to the Hölder inequality (2.1) and Doob's martingale inequality (2.4), (6.32) gives

$$
\begin{aligned}
&2\mathbb{E}\left|\int_{t_n}^{t}d(L^{\xi}-L^{\eta})(s)\right|^2\\
&\le 20\left(2+\sum_{k=1}^{d}\nu_k(|z_k|\ge r)\right)\int_{t_n}^{t}\mathbb{E}|X^{\xi}(s)-X^{\xi}(t_n)|^2+|X^{\eta}(s)-X^{\eta}(t_n)|^2ds\\
&\quad+40(t-t_n)\int_{t_n}^{t}\mathbb{E}|f(X^{\xi}(s),X^{\xi}(s-\tau))|^2+|f(X^{\eta}(s),X^{\eta}(s-\tau))|^2ds\\
&\quad+80\times 110\int_{t_n}^{t}\mathbb{E}|g(X^{\xi}(s),X^{\xi}(s-\tau))|^2+|g(X^{\eta}(s),X^{\eta}(s-\tau))|^2ds\\
&\quad+C\int_{t_n}^{t}\mathbb{E}\sum_{k=1}^{d}\int_{|z_k|<r}|h^{(k)}(X^{\xi}(s),X^{\xi}(s-\tau),z_k)|^2\\
&\qquad+|h^{(k)}(X^{\eta}(s),X^{\eta}(s-\tau),z_k)|^2\nu_k(dz_k)ds\\
&\quad+C'\int_{t_n}^{t}\mathbb{E}\sum_{k=1}^{d}\int_{|z_k|\ge r}|h'^{(k)}(X^{\xi}(s),X^{\xi}(s-\tau),z_k)|^2\\
&\qquad+|h'^{(k)}(X^{\eta}(s),X^{\eta}(s-\tau),z_k)|^2\nu_k(dz_k)ds.
\end{aligned}
$$

Putting this into (6.30), and using the linear growth condition to f, g, h, h', we have

$$\mathbb{E}|X^\xi(t) - X^\eta(t)|^2 \geq \frac{1}{5}\mathbb{E}|X^\xi(t_n) - X^\eta(t_n)|^2$$
$$- C_K\mathbb{E}\int_{t_n}^{t}(|(X^\xi(s)|^2 + |X^\xi(s-\tau)|^2 + |X^\eta(s)|^2 + |X^\eta(s-\tau)|^2)ds, \tag{6.34}$$

where C_K is a positive constant which may different line by line. From proposition 6.1 it follows that there exists $0 \leq \delta \leq 1$ such that

$$C_K\mathbb{E}\int_{t_n}^{t}(|(X^\xi(s)|^2 + |X^\xi(s-\tau)|^2 + |X^\eta(s)|^2 + |X^\eta(s-\tau)|^2)ds \leq \epsilon,$$

whenever $|t - t_n| < \delta$. This, together with (6.34), yields

$$\mathbb{E}|X^\xi(t) - X^\eta(t)|^2 \geq \frac{\Lambda}{11}, \tag{6.35}$$

whenever $|t - t_n| < \delta$. It follows that

$$\int_0^\infty \mathbb{E}|X^\xi(t) - X^\eta(t)|^2 dt = \infty. \tag{6.36}$$

Since this contradicts with (6.28), we must have (6.29). We finally can follow the desired (6.26) by a similar argument of [58]. Next let $t \geq \tau$ and $\theta \in [0, \tau]$, by Itô's formula and (6.21) we obtain

$$
\begin{aligned}
&|X^{\xi}(t-\theta)-X^{\eta}(t-\theta)|^2\\
&=|X^{\xi}(t-\tau)-X^{\eta}(t-\tau)|^2+2\int_{t-\tau}^{t-\theta}(X^{\xi}(s)-X^{\eta}(s))^T d(L^{\xi}-L^{\eta})(s)\\
&\quad+\int_{t-\tau}^{t-\theta}|g(X^{\xi}(s),X^{\xi}(s-\tau))-g(X^{\eta}(s),X^{\eta}(s-\tau))|^2 ds\\
&\quad+\sum_{k=1}^{d}\int_{t-\tau}^{t-\theta}\int_{|z_k|<r}|h^{(k)}(X^{\xi}(s),X^{\xi}(s-\tau),z_k)-h^{(k)}(X^{\eta}(s),X^{\eta}(s-\tau),z_k)|^2\nu_k(dz_k)ds\\
&\quad+2\int_{t-\tau}^{t-\theta}(X^{\xi}(s)-X^{\eta}(s))^T[f(X^{\xi}(s),X^{\xi}(s-\tau))-f(X^{\eta}(s),X^{\eta}(s-\tau))]ds\\
&\quad+2\int_{t-\tau}^{t-\theta}(X^{\xi}(s)-X^{\eta}(s))^T[g(X^{\xi}(s),X^{\xi}(s-\tau))-g(X^{\eta}(s),X^{\eta}(s-\tau))]dB(s)\\
&\quad+\sum_{k=1}^{d}\int_{t-\tau}^{t-\theta}\int_{|z_k|<r}[|h^{(k)}(X^{\xi}(s^-),X^{\xi}((s-\tau)^-),z_k)-h^{(k)}(X^{\eta}(s^-),X^{\eta}((s-\tau)^-),z_k)|^2\\
&\quad+2(X^{\xi}(s^-)-X^{\eta}(s^-))^T\big(h^{(k)}(X^{\xi}(s^-),X^{\xi}((s-\tau)^-),z_k)\\
&\qquad-h^{(k)}(X^{\eta}(s^-),X^{\eta}((s-\tau)^-),z_k)\big)]\tilde{N}_k(ds,dz_k)\\
&\quad+\sum_{k=1}^{d}\int_{t-\tau}^{t-\theta}\int_{|z_k|\geq r}[|h'^{(k)}(X^{\xi}(s^-),X^{\xi}((s-\tau)^-),z_k)-h'^{(k)}(X^{\eta}(s^-),X^{\eta}((s-\tau)^-),z_k)|^2\\
&\quad+2(X^{\xi}(s^-)-X^{\eta}(s^-))^T\big(h'^{(k)}(X^{\xi}(s^-),X^{\xi}((s-\tau)^-),z_k)\\
&\qquad-h'^{(k)}(X^{\eta}(s^-),X^{\eta}((s-\tau)^-),z_k)\big)]\tilde{N}_k(ds,dz_k)\\
&\quad+\sum_{k=1}^{d}\int_{t-\tau}^{t-\theta}\int_{|z_k|\geq r}[|h'^{(k)}(X^{\xi}(s),X^{\xi}(s-\tau),z_k)-h'^{(k)}(X^{\eta}(s),X^{\eta}(s-\tau),z_k)|^2\\
&\quad+2(X^{\xi}(s)-X^{\eta}(s))^T\big(h'^{(k)}(X^{\xi}(s),X^{\xi}(s-\tau),z_k)\\
&\qquad-h'^{(k)}(X^{\eta}(s),X^{\eta}(s-\tau),z_k)\big)]\nu_k(dz_k)ds
\end{aligned}
$$

.

$$(6.37)$$

Compare with the calculation from (6.17) to (6.20), and the assumption 6.2 we have

$$\begin{aligned}
&\mathbb{E}\sup_{0\le\theta\le T}|X^{\xi}(t-\theta)-X^{\eta}(t-\theta)|^2\\
&\le 5\mathbb{E}|X^{\xi}(t-\tau)-X^{\eta}(t-\tau)|^2+5\mathbb{E}\int_{t-\tau}^{t-\theta}(X^{\xi}(s)-X^{\eta}(s))^T d(L^{\xi}-L^{\eta})(s)\\
&\quad+C\bar{K}\mathbb{E}\int_{t-\tau}^{t}(|X^{\xi}(s)-X^{\eta}(s)|^2+|X^{\xi}(s-\tau))-X^{\eta}(s-\tau)|^2)ds\\
&\le 5\mathbb{E}|X^{\xi}(t-\tau)-X^{\eta}(t-\tau)|^2\\
&\quad+C\bar{K}\mathbb{E}\int_{t-\tau}^{t}(|X^{\xi}(s)-X^{\eta}(s)|^2+|X^{\xi}(s-\tau))-X^{\eta}(s-\tau)|^2)ds,
\end{aligned}\tag{6.38}$$

because we can use a fact therein, P-a.s.

$$\begin{aligned}
&\int_0^t (X^{\xi}(s)-X^{\eta}(s))^T d(L^{\xi}-L^{\eta})(s)\\
&=\int_0^t X^{\xi}(s)^T d(L^{\xi}-L^{\eta})(s)-\int_0^t X^{\eta}(s)^T d(L^{\xi}-L^{\eta})(s)\\
&=-\int_0^t X^{\xi}(s)^T dL^{\eta}(s)-\int_0^t X^{\eta}(s)^T dL^{\xi}(s)\le 0,\quad \forall t>0,
\end{aligned}$$

since $dL^{\eta}(t)\ge 0$, $dL^{\xi}(t)\ge 0$, $X_l^{\eta}(t)\ge 0$ and $X_l^{\xi}(t)\ge 0$ for all $l=1,2,\dots,n$. The required assertion (6.25) finally follows from (6.26) and (6.38). The proof is hence complete.

6.4 Examples

In this section we first show an Corollary that

Corollary 6.1 *Let Assumption 6.1 and 6.2 hold, and let c be a positive number, $\lambda_5>\lambda_6\ge 0$, $\lambda_7>\lambda_8\ge 0$. Assume that there exist functions $V(x)\in C^2(\mathbb{R}_+)$ satisfying $\frac{\partial V(x_l^0)}{\partial x_l}\le 0$, $U(x)\in C^2(\mathbb{R})$ obeying $\frac{\partial U}{\partial x_l}(x_l^-)\le 0$,* *and* $\frac{\partial U}{\partial x_l}(x_l^+) \geq 0$, *and* $\omega(x) \in C^2(\mathbb{R})$ *such that*

$$\begin{aligned} c|x|^2 &\leq V(x) \leq \omega(x), && \forall x \in \mathbb{R}_+, \\ c|x|^2 &\leq U(x), && \forall x \in \mathbb{R}, \\ LV(x,y) &\leq -\lambda_5\omega(x) + \lambda_6\omega(y), && \forall x, y \in \mathbb{R}_+, \\ LU(x_1 - y_1, x_2 - y_2) &\leq -\lambda_7\omega(x_1 - y_1) + \lambda_8\omega(x_2 - y_2), && \forall x_1, y_1, x_2, y_2 \in \mathbb{R}_+. \end{aligned}$$

Then

$$\sup_{\xi\in M}\left(\sup_{0\leq t<\infty}\mathbb{E}\|X^\xi(t)\|^2\right) < \infty,$$

and

$$\lim_{t\to\infty}\mathbb{E}|X_t^\xi - X_t^\eta|^2 = 0,$$

uniformly in $\xi, \eta \in M$, *where* M *is a compact subset of* $D([-\tau, 0]; \mathbb{R}_+)$.

Now we turn to give an simple example. Let $B(t)$ be a 1-dimensional standard Brownian motion, $N(t,z)$ be a 1-dimensional Poisson process and denote the compensated Poisson process by

$$\tilde{N}(dt, dz) = (N(dt, dz) - \nu(dz)dt).$$

We assume that $B(t)$ and $N(dt, dz)$ are independent. Consider a 1-dimensional linear RSDDEwJs of the form

$$\begin{aligned} dX(t) &= A(X(t) + X(t-1))dt + CX(t-1)dB(t) + \int_{|z|<r} DX(t^-)\tilde{N}(dt, dz) \\ &\quad + \int_{|z|\geq r} D'X(t^-)N(dt, dz) + dL(t), \\ X_0 &= \xi \in D^b_{\mathcal{F}_0}([-1, 0]; \mathbb{R}_+), \end{aligned}$$

on $t \geq 0$, where the $\mathbb{R}_+$-valued reflected term $L(t)$ satisfies the properties that $t \to L(t)$ is a nondecreasing and continuous random process and it satisfies $\int_0^\infty X(s)^T dL(s) = 0$. Define $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ and $g : \mathbb{R}_+ \to \mathbb{R}$ as well as $h, h' : \mathbb{R}_+ \to \mathbb{R}$ by $f(x,y) = A(x+y)$, $g(y) = Cy$ and $h(x) = Dx$, $h'(x) = D'x$.

Assume that there is a positive number β such that

$$3A\beta + D^2 \int_{|z|<r} \nu(dz) + D' \int_{|z|\geq r} \nu(dz) < 0,$$
$$A\beta + C^2\beta > 0.$$

Next we examine the stability in distribution. We first construct the functions $V : \mathbb{R}_+ \to \mathbb{R}_+$ and $U : \mathbb{R} \to \mathbb{R}_+$ by $V(x) = U(x) = \beta|x|^2$. It is easy to check that the operator LV from $\mathbb{R}_+ \to \mathbb{R}_+$ to $\mathbb{R}$ has the form

$$\begin{aligned} LV(x,y) &= 2\beta x A(x+y) + \beta C^2|y|^2 + \int_{|z|<r} D^2|x|^2\nu(dz) + \int_{|z|\geq r} 2(D'^2+1)|x|^2\nu(dz) \\ &\leq [3A\beta + D^2 \int_{|z|<r} \nu(dz) + 2(D'^2+1)\int_{|z|\geq r} \nu(dz)]|x|^2 + [A\beta + C^2\beta]|y|^2, \end{aligned}$$

similarly for $x_1, x_2, y_1, y_2 \in \mathbb{R}_+$,

$$\begin{aligned} &LU(x_1,x_2,y_1,y_2) \\ &= 2\beta(x_1-x_2)A(x_1-x_2+(y_1-y_2)) + \beta C^2|y_1-y_2|^2 + \int_{|z|<r} D^2|x_1-x_2|^2\nu(dz) \\ &\quad + \int_{|z|\geq r} 2(D'^2+1)|x_1-x_2|^2\nu(dz) \\ &\leq [3A\beta + D^2\int_{|z|<r}\nu(dz) + 2(D'^2+1)\int_{|z|\geq r}\nu(dz)]|x_1-x_2|^2 + [A\beta + C^2\beta]|y_1-y_2|^2. \end{aligned}$$

Suppose $\int_{\mathbb{Z}} \nu(dz) < \infty$, then the jump term satisfies the Assumption 6.1.

On the other hand, since $V_x(x) = 2\beta x$, for all the vectors having the form $x_l^0 := (x_1, \ldots, x_{l-1}, 0, x_{l+1}, \ldots, x_n)$ with $x_i \in \mathbb{R}_+$ $(i \neq l)$ and $l = 1, 2, \ldots, n$, it is easy to get $\frac{\partial V(x_l^0)}{\partial x_l} = 2\beta \times 0 = 0$, similarly, $x_l^{\mp} := (x_1, \ldots, x_{l_1}, \mp x_l, x_{l+1}, \ldots, x_n)$ with all $x_i \in \mathbb{R}$ $(i \neq l)$ and $x_l \in \mathbb{R}_+$, we have for $l = 1, 2, \ldots, n$, $\frac{\partial U}{\partial x_l}(x_l^-) = -2\beta x_l \leq 0$, and $\frac{\partial U}{\partial x_l}(x_l^+) = 2\beta x_l \geq 0$, because of $\beta > 0$. By Corollary we can conclude that the solution process X_t is stable in distribution.

Next we shall give another example considering an 1-dimensional non-linear RSDDEwJs,

$$\begin{aligned} dX(t) &= [-2X(t) - DX(t-1)]dt + g(X(t), X(t-1))dB(t) \\ &\quad + \int_{|z|<r} h(X(t), X(t-1), z)\tilde{N}(dt, dz) \\ &\quad + \int_{|z|\geq r} h'(X(t), X(t-1), z)N(dt, dz) + dL(t), \\ X_0 &= \xi \in D^b_{\mathcal{F}_0}([-1, 0]; \mathbb{R}_+). \end{aligned}$$

We now shall give some conditions to support the corollary above. Suppose that

$$\begin{aligned} xD(y) &\leq \frac{1}{16}|x|^2 + \frac{1}{16}|y|^2, \\ \int_{|z|<r} |h(x, y, z)|^2 \nu(dz) &\leq \frac{1}{16}|x|^2 + \frac{1}{16}|y|^2 \\ |g(x, y)|^2 &\leq \frac{1}{8}|x|^2 + \frac{1}{8}|y|^2 \\ \int_{|z|\geq r} |h'(x, y, z)|^2 + 2xh'(x, y, z)\nu(dz) &\leq \frac{1}{16}|x|^2 + \frac{1}{16}|y|^2, \end{aligned}$$

and

$$\begin{aligned} (x_1 - x_2)(D(y_1) - D(y_2)) &\leq \frac{1}{16}|x_1 - x_2|^2 + \frac{1}{16}|y_1 - y_2|^2, \\ \int_{|z|<r} |h(x_1, y_1, z) - h(x_2, y_2, z)|^2 \nu(dz) &\leq \frac{1}{16}|x_1 - x_2|^2 + \frac{1}{16}|y_1 - y_2|^2 \\ |g(x_1, y_1) - g(x_2, y_2)|^2 &\leq \frac{1}{8}|x_1 - x_2|^2 + \frac{1}{8}|y_1 - y_2|^2 \\ \int_{|z|\geq r} |h'(x_1, y_1, z) - h'(x_2, y_2, z)|^2 &+ 2(x_1 - x_2)(h'(x_1, y_1, z) \\ -h'(x_2, y_2, z))\nu(dz) &\leq \frac{1}{16}|x_1 - x_2|^2 + \frac{1}{16}|y_1 - y_2|^2. \end{aligned}$$

Let the functions $V : \mathbb{R}_+ \to \mathbb{R}_+$, $U : \mathbb{R} \to \mathbb{R}_+$, and let $V(x) = U(x) = |x|^2$, we have

$$LV(x, y) \leq -\frac{5}{8}|x|^2 + \frac{3}{8}|y|^2,$$

and

$$LU(x_1, x_2, y_1, y_2) \leq -\frac{5}{8}|x_1 - x_2|^2 + \frac{3}{8}|y_1 - y_2|^2$$

Then the conditions of the corollary hold.

Chapter 7

Convergence of Balance Method for Neutral Stochastic Differential Delay Equations with Jumps

7.1 Introduction

During the last few years several authors have proposed implicit numerical methods for stochastic differential equations in respect of strong and weak convergence criteria. The balanced method can be interpreted as a family of specific methods providing a kind of balance within approximating stochastic terms in the numerical scheme [46]. One can hope that by an appropriate choice of the parameters involved in these schemes one is able to find an acceptable combination suitable for the integration of a given stiff stochastic differential equation. Numerical experiments show a better behavior of the balanced method in comparison with the explicit Euler method. For example, the balanced method is having a larger range of suitable step sizes where they work without any numerical instability in contrast to the explicit Euler method. In this chapter we will investigate the convergence of balanced method for NSDDEwJs.

7.2 Balance Method for Neutral Stochastic Differential Delay Equations with Jumps

In this chapter we shall let the coefficients of (3.1) satisfies the global Lipschitz condition and linear growth condition.

Assumption 7.1 *(**Global Lipschitz condition**) There exist a positive constant K, such that*

$$|f(x_1,y_1)-f(x_2,y_2)|^2+|g(x_1,y_1)-g(x_2,y_2)|^2 \\ +\sum_{k=1}^{d}\int_{\mathbb{R}}|h^{(k)}(x_1,y_1,z_k)-h^{(k)}(x_2,y_2,z_k)|^2\nu_k(dz_k)\le K(|x_2-x_1|^2+|y_2-y_1|^2), \tag{7.1}$$

*for $x_1,x_2,y_1,y_2\in\mathbb{R}^n$ and $t\in[0,T]$. Assume moreover that f, g, h satisfy the **linear growth condition** that*

$$|f(x,y)|^2+|g(x,y)|^2+\sum_{k=1}^{d}\int_{\mathbb{R}}|h^{(k)}(x,y,z_k)|^2\nu_k(dz_k)\le K_1(1+|x|^2+|y|^2),$$

for $x,y\in\mathbb{R}^n$, $p\ge 1$. We also assume that there is a constant $\tilde{K}\in(0,1)$ such that

$$|G(y_1)-G(y_2)|\le\tilde{K}|y_1-y_2|,\quad \textit{for all}\quad y_1,y_2\in\mathbb{R}^n. \tag{7.2}$$

Now let us introduce the family of balanced methods. A balanced method applied to (3.1) can be written as the general form that

$$
\begin{aligned}
Y_{n+1} - G(Y_{n+1-m}) &= Y_n - G(Y_{n-m}) + f(Y_n, Y_{n-m})\Delta + g(Y_n, Y_{n-m})\Delta B_n \\
&+ \int_{\mathbb{R}^d} h(Y_n, Y_{n-m}, z)\Delta \tilde{N}_n(dz) + \big(c_0(Y_n, Y_{n-m})\Delta \\
&+ \sum_{i=1}^{n} c_i(Y_n, Y_{n-m})|\Delta B_n|\big)[Y_n - G(Y_{n-m}) - (Y_{n+1} - G(Y_{n+1-m}))],
\end{aligned}
\tag{7.3}
$$

where $\Delta B_n = B(t_{n+1}) - B(t_n)$, $\Delta \tilde{N}_n(dz) = \tilde{N}(t_{n+1}, dz) - \tilde{N}(t_n, dz)$ and $c_i := c_0, c_1, \ldots, c_n$ represent $n \times n$-matrix-valued functions. We assume that for any sequence of real numbers (α_i) with $\alpha_0 \in [0, \bar{\alpha}]$, $\alpha_1 \geq 0, \ldots, \alpha_n \geq 0$, where $\bar{\alpha} \geq \Delta$ for all step sizes Δ considered and $x, y \in \mathbb{R}^n$, the matrix

$$
M(x, y) = I + \alpha_0 c_0(x, y) + \sum_{i=1}^{n} \alpha_i c_i(x, y) \tag{7.4}
$$

has an inverse and satisfies the condition $|(M(x, y))^{-1}| \leq K < \infty$. Here I is the unit matrix. Obviously (7.4) can be easily fulfilled in keeping $c_0, c_1, \ldots, c_n$ all positive definite. Thus, under these conditions one obtains directly the one-step increment $Y_{n+1} - G(Y_{n+1-m}) - (Y_n - G(Y_{n-m}))$ of the balanced method via the solution of a system of linear algebraic equations. Furthermore, we suppose that the components of the matrices $c_0, c_1, \ldots, c_n$ are uniformly bounded. That is to say, there exists a positive constant B, such that

$$
|c_i| \leq B, \tag{7.5}
$$

where $i = 0, 1, \ldots, n$. Then we illustrate the following lemma before show the main result.

Lemma 7.1 *Let the assumption 7.1 holds, there is a positive constant H such that*

$$\mathbb{E}\sup_{0\le s\le t}|X_t^y|^2 \le H(1+|y|^2), \tag{7.6}$$

where X_t^y denote the value of a solution of (3.1) at time t which starts from $y\in\mathbb{R}^n$.

7.3 Convergence of Balance Method for Neutral Stochastic Differential Delay Equations with Jumps

Theorem 7.1 *Under the assumption 7.1 and (7.5), the balanced method (7.3) converges with strong order $\gamma = 0.5$, that is for all $k = 0,1,\ldots,n$, where $n = 0,1,\ldots,N$ and step size $\Delta = T/N = \tau/m$,*

$$(\mathbb{E}|X_{t_n} - Y_n|^2)^{\frac{1}{2}} \le C(1+|X_0|^2)^{\frac{1}{2}}\Delta^{\frac{1}{2}}, \tag{7.7}$$

where C does not depend on Δ.

To prove the theorem 7.1 we recall the following theorem concerning the order of strong convergence (see [44], [45]).

Theorem 7.2 *Assume for a one-step discrete time approximation Y that the local mean error and mean-square error for all $N = 1,2,\ldots,$ and $n = 0,1,\ldots,N$ satisfy the estimates*

$$|\mathbb{E}(X_{t_{n+1}}^{Y_n} - Y_{n+1})| \le C\Big(1+\mathbb{E}\sup_{k\in[-m,n]}|Y_k|^2\Big)^{\frac{1}{2}}\Delta^{p_1} \tag{7.8}$$

and

$$(\mathbb{E}|X_{t_{n+1}}^{Y_n} - Y_{n+1}|^2)^{\frac{1}{2}} \le C\Big(1+\mathbb{E}|Y_n|^2\Big)^{\frac{1}{2}}\Delta^{p_2} \tag{7.9}$$

with $p_2 \geq \frac{1}{2}$ and $p_1 \geq p_2 + \frac{1}{2}$, where the positive constant C may different line by line. Then

$$(\mathbb{E}|X_{t_n}^{X_0} - Y_n|^2)^{\frac{1}{2}} \leq C(1 + |X_0|^2)^{\frac{1}{2}} \Delta^{p_2 - \frac{1}{2}} \tag{7.10}$$

holds for each $k = 0, 1, \ldots, n$.

Proof of Theorem 7.1. At first, we show that the estimate (7.8) holds for the balanced method (7.3) with $p_1 = \frac{3}{2}$. For this purpose, the local Euler approximation step

$$Y_{n+1}^E = Y_n - G(Y_{n-m}) + G(Y_{n+1-m}) + f(Y_n, Y_{n-m})\Delta + g(Y_n, Y_{n-m})\Delta B_n + \int_{\mathbb{R}^d} h(Y_n, Y_{n-m}, z)\Delta \tilde{N}_n(dz), \tag{7.11}$$

which can be deduced for $n = 0, 1, \ldots, N-1$ that

$$\begin{aligned} H_1 :&= |\mathbb{E}(X_{t_{n+1}}^{Y_n} - G(X_{t_{n+1-m}}^{Y_n}) - (Y_{n+1} - G(Y_{n+1-m})))| \\ &\leq |\mathbb{E}(X_{t_{n+1}}^{Y_n} - G(X_{t_{n+1-m}}^{Y_n}) - (Y_{n+1}^E - G(Y_{n+1-m})))| \\ &\quad + |\mathbb{E}(Y_{n+1}^E - G(Y_{n+1-m}) - (Y_{n+1} - G(Y_{n+1-m})))|, \end{aligned}$$

by Hölder's inequality (2.1) and global Lipschitz condition we have

$$\begin{aligned} H_2 :&= |\mathbb{E}(X_{t_{n+1}}^{Y_n} - G(X_{t_{n+1-m}}^{Y_n}) - (Y_{n+1}^E - G(Y_{n+1-m})))| \\ &= 3\left|\mathbb{E}\int_{t_n}^{t_{n+1}} [f(X_{t_n}^{Y_n}, X_{t_{n-m}}^{Y_n}) - f(Y_n, Y_{n-m})]du\right| \\ &\leq 3\Delta^{\frac{1}{2}} \int_{t_n}^{t_{n+1}} (\mathbb{E}|f(X_{t_n}^{Y_n}, X_{t_{n-m}}^{Y_n}) - f(Y_n, Y_{n-m})|^2)^{\frac{1}{2}} du \\ &\leq 3K\Delta(\mathbb{E}(|X_{t_k}^{Y_k} - Y_k|^2 + |X_{t_{k-m}}^{Y_k} - Y_{k-m}|^2))^{\frac{1}{2}} \\ &\leq 3K\Delta\left(2\mathbb{E}\sup_{k\in[-m,n]} |X_{t_k}^{Y_k} - Y_k|^2\right)^{\frac{1}{2}} \\ &\leq C\Delta^2(1 + \mathbb{E}|Y_n|^2)^{\frac{1}{2}}, \end{aligned}$$

then by using $\mathbb{E}|\Delta B_n| \le \sqrt{m\Delta}$, and (7.5) we also have

$$\begin{aligned}
H_3 := &\Big|\mathbb{E}\big(Y^E_{n+1} - G(Y_{n+1-m}) - (Y_{n+1} - G(Y_{n+1-m}))\big)\Big| \\
= &\Big|\mathbb{E}\big((I + c_0(Y_n, Y_{n-m})\Delta + c(Y_n, Y_{n-m})|\Delta B_n|)^{-1} \\
&\times \Big(c_0(Y_n, Y_{n-m})\Delta + \sum_{i=1}^{n} c_i(Y_n, Y_{n-m})|\Delta B_n|\Big) f(Y_n, Y_{n-m})\Delta\Big)\Big| \\
\le &\, C\Delta^{\frac{3}{2}}\Big(1 + \mathbb{E}|Y_n|^2\Big)^{\frac{1}{2}},
\end{aligned}$$

and we obtain

$$H_3 \le C\Delta^{\frac{3}{2}}\Big(1 + \mathbb{E}|Y_n|^2|\Big)^{\frac{1}{2}},$$

and

$$H_2 \le C\Delta^{\frac{3}{2}}\Big(1 + \mathbb{E}|Y_n|^2|\Big)^{\frac{1}{2}}.$$

Since we have $H_1 = H_2 + H_3$, we deduce that

$$\begin{aligned}
\sup_{k\in[0,n]} |\mathbb{E}(X^{Y_k}_{t_{k+1}} - Y_{k+1})| = &\sup_{k\in[0,n]} |\mathbb{E}\big(X^{Y_k}_{t_{k+1}} - G(X^{Y_k}_{t_{k+1-m}}) - (Y_{k+1} - G(Y_{k+1-m}))\big) \\
&+ \mathbb{E}\big(G(X^{Y_k}_{t_{k+1-m}}) - G(Y_{k+1-m})\big)| \\
\le &\sup_{k\in[0,n]} \big(H_1 + |\mathbb{E}(G(X^{Y_k}_{t_{k+1-m}}) - G(Y_{k+1-m}))|\big) \\
\le &\sup_{k\in[0,n]} \big(H_1 + \tilde{K}|\mathbb{E}(X^{Y_k}_{t_{k+1-m}} - Y_{k+1-m})|\big) \\
\le &\sup_{k\in[-m,n]} \big(H_1 + \tilde{K}|\mathbb{E}(X^{Y_k}_{t_{k+1}} - Y_{k+1})|\big) \\
\le &\sup_{k\in[-m,n]} H_1 + \sup_{k\in[0,n]} \tilde{K}|\mathbb{E}(X^{Y_k}_{t_{k+1}} - Y_{k+1})| \\
&+ \sup_{k\in[-m,0]} \tilde{K}|\mathbb{E}(X^{Y_k}_{t_{k+1}} - Y_{k+1})| \\
\le &\sup_{k\in[-m,n]} \frac{1}{1-\tilde{K}} H_1 = C\Delta^{\frac{3}{2}}\Big(1 + \mathbb{E}\sup_{k\in[-m,n]} |Y_n|^2|\Big)^{\frac{1}{2}}.
\end{aligned}$$

Thus the assumption (7.8) with $p_1 = 1.5$ in theorem 7.2 is satisfied for the balanced method. Similarly, we check assumption (7.9) for the local mean-square error of the balanced method (7.3) and obtain by standard arguments

$$\begin{aligned} H_4 :&= \left(\mathbb{E}|X^{Y_n}_{t_{n+1}} - G(X^{Y_n}_{t_{n+1-m}}) - (Y_{n+1} - G(Y_{n+1-m}))|^2\right)^{\frac{1}{2}} \\ &\leq \left(\mathbb{E}|X^{Y_n}_{t_{n+1}} - G(X^{Y_n}_{t_{n+1-m}}) - (Y^E_{n+1} - G(Y_{n+1-m}))|^2\right)^{\frac{1}{2}} \\ &\quad + \left(\mathbb{E}|Y^E_{n+1} - G(Y_{n+1-m}) - (Y_{n+1} - G(Y_{n+1-m}))|^2\right)^{\frac{1}{2}}, \end{aligned}$$

by using Hölder's inequality (2.1), Doob's martingale inequality (2.4) and assumption 7.1 we have

$$\begin{aligned} H_5 :&= \left(\mathbb{E}|X^{Y_n}_{t_{n+1}} - G(X^{Y_n}_{t_{n+1-m}}) - (Y^E_{n+1} - G(Y_{n+1-m}))|^2\right)^{\frac{1}{2}} \\ &\leq \Bigg(\mathbb{E}\Bigg[\left|\int_{t_n}^{t_{n+1}} [f(X^{Y_n}_{t_n}, X^{Y_n}_{t_{n-m}}) - f(Y_n, Y_{n-m})]du\right|^2 \\ &\quad + \left|\int_{t_n}^{t_{n+1}} [g(X^{Y_n}_{t_n}, X^{Y_n}_{t_{n-m}}) - g(Y_n, Y_{n-m})]dB(u)\right|^2 \\ &\quad + \left|\int_{t_n}^{t_{n+1}} \int_{\mathbb{R}^d} [h(X^{Y_n}_{t_n}, X^{Y_n}_{t_{n-m}}, z) - h(Y_n, Y_{n-m}, z)]\tilde{N}(du, dz)\right|^2\Bigg]\Bigg)^{\frac{1}{2}} \\ &\leq C\Delta(1 + \mathbb{E}|Y_n|^2)^{\frac{1}{2}}, \end{aligned}$$

then by using (7.5) we have

$$\begin{aligned} H_6 :&= \left(\mathbb{E}|Y^E_{n+1} - G(Y_{n+1-m}) - (Y_{n+1} - G(Y_{n+1-m}))|^2\right)^{\frac{1}{2}} \\ &= \Bigg(\mathbb{E}\Bigg|(I + c_0(Y_n, Y_{n-m})\Delta + c(Y_n, Y_{n-m})|\Delta B_n|)^{-1} \\ &\qquad \times (c_0(Y_n, Y_{n-m})\Delta + c(Y_n, Y_{n-m})|\Delta B_n|)\Big(f(Y_n, Y_{n-m})\Delta \\ &\quad + g(Y_n, Y_{n-m})\Delta B_n + \int_{\mathbb{R}^d} h(Y_n, Y_{n-m}, z)\Delta\tilde{N}_n(dz)\Big)\Bigg|^2\Bigg)^{\frac{1}{2}} \\ &\leq C\Delta^{\frac{3}{2}}(1 + \mathbb{E}|Y_n|^2)^{\frac{1}{2}} + 2C\Delta(1 + \mathbb{E}|Y_n|^2)^{\frac{1}{2}} \\ &\leq C\Delta(1 + \mathbb{E}|Y_n|^2)^{\frac{1}{2}}, \end{aligned}$$

we also have since $H_4 = H_5 + H_6$,

$$\left(\mathbb{E}\sup_{k\in[0,n]}|X^{Y_k}_{t_{k+1}} - Y_{k+1}|^2\right)^{\frac{1}{2}}$$
$$\leq \left(\mathbb{E}\sup_{k\in[0,n]}|X^{Y_k}_{t_{k+1}} - G(X^{Y_k}_{t_{k+1-m}}) - (Y_{k+1} - G(Y_{k+1-m}))|^2\right)^{\frac{1}{2}}$$
$$+ \left(\mathbb{E}\sup_{k\in[0,n]}|G(X^{Y_k}_{t_{k+1-m}}) - G(Y_{k+1-m})|^2\right)^{\frac{1}{2}}$$
$$\leq H_4 + \left(\tilde{K}^2\mathbb{E}\sup_{k\in[0,n]}|X^{Y_k}_{t_{k+1-m}} - Y_{k+1-m}|^2\right)^{\frac{1}{2}}$$
$$\leq H_4 + \tilde{K}\left(\mathbb{E}\sup_{k\in[0,n]}|X^{Y_k}_{t_{k+1}} - Y_{k+1})|^2\right)^{\frac{1}{2}}$$
$$\leq \frac{1}{1-\tilde{K}}H_4.$$

Thus we can choose in theorem 7.2 the exponent $p_2 = 1.0$ together with $p_1 = 1.5$ and apply it to finally prove the strong order $\gamma = 0.5$ of the balanced method as follows with a useful lemma (see [45, pp-739-740]).

Lemma 7.2 *The following representation holds,*

$$[X^\xi(t+\Delta) - G(X^\xi(t+\Delta-\tau))] - [X^\eta(t-\Delta) - G(X^\eta(t-\Delta-\tau))] = \xi - \eta - Z, \tag{7.12}$$

where

$$\mathbb{E}|X^\xi(t+\Delta) - X^\eta(t+\Delta)|^2 \leq \|\xi-\eta\|^2(1 - C\Delta), \tag{7.13}$$

and

$$\mathbb{E}|Z|^2 \leq C\|\xi-\eta\|^2\Delta, \tag{7.14}$$

with two different initial data ξ and η, for a positive constant C may different line by line.

Proof of lemma 7.2.

$$
\begin{aligned}
&\mathbb{E}\sup_{t\le s\le t+\Delta}|X^{\xi}(s)-X^{\eta}(s)|^2\\
=&\mathbb{E}\sup_{t\le s\le t+\Delta}|[X^{\xi}(s)-G(X^{\xi}(s-\tau))]-[X^{\eta}(s)-G(X^{\eta}(s-\tau))]\\
&+[G(X^{\xi}(s-\tau))-G(X^{\eta}(s-\tau))]|^2\\
\le&\frac{1}{1-\tilde K}\mathbb{E}\sup_{t\le s\le t+\Delta}|[X^{\xi}(s)-G(X^{\xi}(s-\tau))]-[X^{\eta}(s)-G(X^{\eta}(s-\tau))]|^2\\
&+\frac{1}{\tilde K}\mathbb{E}\sup_{t\le s\le t+\Delta}|G(X^{\xi}(s-\tau))-G(X^{\eta}(s-\tau))|^2\\
\le&\frac{1}{(1-\tilde K)^2\tilde K}\mathbb{E}|\xi(0)-\eta(0)|^2+\frac{1}{1-\tilde K}\mathbb{E}|\xi(-\tau)-\eta(-\tau)|^2+\frac{\mathbb{E}\sup_{t\le s\le t+\Delta}|Z|^2}{(1-\tilde K)^2}\\
&+\tilde K\mathbb{E}\sup_{t\le s\le t+\Delta}|X^{\xi}(s-\tau)-X^{\eta}(s-\tau)|^2\\
\le&\frac{2}{(1-\tilde K)^2\tilde K}\mathbb{E}\|\xi-\eta\|^2+\tilde K\mathbb{E}\sup_{t-\tau\le s\le t+\Delta-\tau}|X^{\xi}(s)-X^{\eta}(s)|^2\\
&+\frac{2(\Delta+C)}{(1-\tilde K)^2}\int_t^{t+\Delta}\mathbb{E}\sup_{t-\tau\le s\le t+\Delta}|X^{\xi}(s)-X^{\eta}(s)|^2ds,
\end{aligned}
$$

where for two positive constant C and C'

$$
\begin{aligned}
&\mathbb{E}\sup_{t-\tau\le s\le t+\Delta}|X^{\xi}(s)-X^{\eta}(s)|^2\\
&\le\|\xi-\eta\|^2+\mathbb{E}\sup_{t\le s\le t+\Delta}|X^{\xi}(s)-X^{\eta}(s)|^2\\
&\le C\|\xi-\eta\|^2+C'\int_t^{t+\Delta}\mathbb{E}\sup_{t-\tau\le s\le t+\Delta}|X^{\xi}(s)-X^{\eta}(s)|^2ds,
\end{aligned}
$$

by Gronwall's inequality (2.7) we derive that

$$
\mathbb{E}\sup_{t-\tau\le s\le t+\Delta}|X^{\xi}(s)-X^{\eta}(s)|^2\le C\|\xi-\eta\|^2e^{C'}.
$$

So we have

$$
\mathbb{E}\sup_{t\le s\le t+\Delta}|Z|^2\le C\int_t^{t+\Delta}\mathbb{E}\sup_{t-\tau\le s\le t+\Delta}|X^{\xi}(s)-X^{\eta}(s)|^2ds\le C\|\xi-\eta\|^2\Delta.
$$

Then

$$
\mathbb{E}|X^{\xi}(t+\Delta)-X^{\eta}(t+\Delta)|^2\le C\|\xi-\eta\|^2+C'\|\xi-\eta\|^2\Delta\le\|\xi-\eta\|^2(1+C\Delta),
$$

which is the proof complete. Now we focus to proof the theorem 7.2, we have

$$X^{X_0}(t_{n+1}) - Y^{X_0}(t_{n+1}) = X^{X_{t_n}}(t_{n+1}) - Y^{Y_n}(t_{n+1})$$
$$= [X^{X_{t_n}}(t_{n+1}) - X^{Y_n}(t_{n+1})] - [X^{Y_n}(t_{n+1}) - Y^{Y_n}(t_{n+1})],$$

then

$$\mathbb{E}|X^{X_0}(t_{n+1}) - Y^{X_0}(t_{n+1})|^2$$
$$\leq 2\mathbb{E}|X^{X_{t_n}}(t_{n+1}) - X^{Y_n}(t_{n+1})|^2 + 2\mathbb{E}|X^{Y_n}(t_{n+1}) - Y^{Y_n}(t_{n+1})|^2,$$

where by lemma (7.2), (7.9) and lemma 7.1,

$$\mathbb{E}|X^{X_{t_n}}(t_{n+1}) - X^{Y_n}(t_{n+1})|^2 \leq \mathbb{E}|X_{t_n} - Y_n|^2(1 + C\Delta),$$
$$\mathbb{E}|X^{Y_n}(t_{n+1}) - Y^{Y_n}(t_{n+1})|^2 \leq C(1 + \mathbb{E}|Y_n|^2)\Delta^{2p_2}$$
$$\leq C(1 + \mathbb{E}|X_0|^2)\Delta^{2p_2}.$$

Then we introduce the notation $\varepsilon_n^2 = \mathbb{E}|X_{t_n} - Y_n|^2$, noting that the condition $p_1 \leq p_2 + \frac{1}{2}$, we have

$$\varepsilon_{n+1}^2 \leq \varepsilon_n^2(1 + C\Delta) + C'(1 + \mathbb{E}|X_0|^2)\Delta^{2p_2}.$$

By iteration for $n = 0, 1, \ldots$ we obtain that

$$\varepsilon_n^2 \leq C' \sum_{i=0}^{n-1}(1 + C\Delta)^i(1 + \mathbb{E}|X_0|^2)\Delta^{2p_2} \leq C(1 + \mathbb{E}|X_0|^2)\Delta^{2(p_2 - \frac{1}{2})}$$

as was claimed on theorem 7.2. Then theorem 7.1 follows.

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