Grothendieck categories of enriched functors.

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Grothendieck Categories of Enriched Functors

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Abstract

It is shown that the category of enriched functors $[\mathcal{C}, \mathcal{V}]$ is Grothendieck whenever $\mathcal{V}$ is a closed symmetric monoidal Grothendieck category and $\mathcal{C}$ is a category enriched over $\mathcal{V}$. Localizations in $[\mathcal{C}, \mathcal{V}]$ associated to collections of objects of $\mathcal{C}$ are studied. Also, the category of chain complexes of generalized modules $\mathbf{Ch}(\mathcal{C}_R)$ is shown to be identified with the Grothendieck category of enriched functors $[\operatorname{mod} R, \mathbf{Ch}(\operatorname{Mod} R)]$ over a commutative ring R, where the category of finitely presented R-modules $\operatorname{mod} R$ is enriched over the closed symmetric monoidal Grothendieck category $\mathbf{Ch}(\operatorname{Mod} R)$ as complexes concentrated in zeroth degree. As an application, it is proved that $\mathbf{Ch}(\mathcal{C}_R)$ is a closed symmetric monoidal Grothendieck model category with explicit formulas for tensor product and internal Hom-objects. Furthermore, the class of unital algebraic almost stable homotopy categories generalizing unital algebraic stable homotopy categories of Hovey–Palmieri–Strickland [29] is introduced. It is shown that the derived category of generalized modules $\mathcal{D}(\mathcal{C}_R)$ over commutative rings is a unital algebraic almost stable homotopy category which is not an algebraic stable homotopy category.

Chapter 1

Introduction

In category theory, an enriched category generalizes the idea of a category by replacing Hom-sets with objects from a general monoidal category. It is motivated by the observation that, in many practical applications, the Hom-set often has additional structure that should be respected, e.g., that of being a vector space of morphisms, or a chain complex of morphisms.

Enriched categories have a multitude of uses and applications, that makes studying their general theory quite worthwhile. For example, Bondal–Kapranov [3] construct enrichments of some triangulated categories over chain complexes ("DG-categories") to study exceptional collections of coherent sheaves on projective varieties. Today DG-categories have become an important tool in many branches of algebraic geometry, non-commutative algebraic geometry, representation theory, and mathematical physics (see a survey by Keller [34]). Garkusha–Panin [19, 20, 21] enrich smooth algebraic varieties over symmetric spectra in order to develop the theory of "K-motives" and solve some problems for the motivic spectral sequence.

In the present project we study categories of enriched functors

$$[\mathcal{C}, \mathcal{V}],$$

where $\mathcal{V}$ is a closed symmetric monoidal Grothendieck category and $\mathcal{C}$ is a category enriched over $\mathcal{V}$ (i.e. a $\mathcal{V}$-category). The main result here states that the category $[\mathcal{C}, \mathcal{V}]$ is Grothendieck with an explicit collection of generators. Namely, the following theorem is true.

Theorem. *Let $\mathcal{V}$ be a closed symmetric monoidal Grothendieck category with a set of generators $\{g_i\}_I$. If $\mathcal{C}$ is a small $\mathcal{V}$-category, then the category of enriched functors $[\mathcal{C}, \mathcal{V}]$ is a Grothendieck $\mathcal{V}$-category with the set of generators $\{\mathcal{V}(c, -) \oslash g_i \mid c \in \operatorname{Ob} \mathcal{C}, i \in I\}$. Moreover, if $\mathcal{C}$ is a small symmetric monoidal $\mathcal{V}$-category, then $[\mathcal{C}, \mathcal{V}]$ is closed symmetric monoidal with explicit formulas for monoidal product and internal* Hom*-object.*

Taking into account this theorem, we refer to $[\mathcal{C}, \mathcal{V}]$ as a *Grothendieck category of enriched functors.* The usual Grothendieck category of additive functors

$$(\mathcal{B}, \mathsf{Ab})$$

from a pre-additive category $\mathcal{B}$ to abelian groups Ab is recovered from the preceding theorem in the case when $\mathcal{V} = \mathsf{Ab}$ ($\mathcal{B}$ is a $\mathcal{V}$-category). Further examples on how the category $[\mathcal{C}, \mathcal{V}]$ recovers some Grothendieck categories are given in Chapter 5.

Another virtue of this theorem is that $\mathcal{V}$ can have homological or homotopical information and this information is carried over enriched functors $[\mathcal{C}, \mathcal{V}]$. This will be used later when discussing model categories, but now let us discuss some localizations in Grothendieck category of enriched functors.

In [14, 17, 18] Garkusha and Generalov study localizations in Grothendieck categories with respect to projective objects. They apply the results to the study of absolutely pure rings, fp-flat and fp-injective modules (see [16, 17] for details). Some of these results have recently been used by Hovey–Lockridge–Puninski [28] for the Freyd Generating Hypothesis.

Garkusha–Generalov's results [14, 17, 18] for localizations can be generalized to enriched categories as follows.

Theorem. *Suppose $\mathcal{V}$ is a closed symmetric monoidal Grothendieck category. Let $\mathcal{C}$ be a $\mathcal{V}$-category and let $\mathcal{P}$ consist of a collection of objects of $\mathcal{C}$. Let $\mathcal{S}_{\mathcal{P}} = \{G \in [\mathcal{C}, \mathcal{V}] \mid G(p) = 0$ for all $p \in \mathcal{P}\}$. Then $\mathcal{S}_{\mathcal{P}}$ is a localizing subcategory of $[\mathcal{C}, \mathcal{V}]$ and $[\mathcal{P}, \mathcal{V}]$ is equivalent to the quotient category $[\mathcal{C}, \mathcal{V}]/\mathcal{S}_{\mathcal{P}}$.*

We apply Grothendieck categories of enriched functors to study homological algebra for generalized modules. The category of generalized modules

$$\mathcal{C}_R = (\operatorname{mod} R, \mathsf{Ab})$$

consists of the additive functors from the category of finitely presented R-modules, $\operatorname{mod} R$, to the category of abelian groups, Ab. Its morphisms are the natural transformations of functors. It is called the category of generalized R-modules for the reason that there is a fully faithful, right exact functor

$$M \mapsto - \otimes_R M$$

from the category of all R-modules to $\mathcal{C}_R$.

The category $\mathcal{C}_R$ has a number of remarkable properties which led to powerful applications in ring and module theory and representation theory (see, e.g., the books by Prest [45, 46]). After the work of Herzog [25] in the the early 90's, the category $\mathcal{C}_R$ provides a natural connecting langauge between algebra and model theory of modules. Also, in $\mathcal{C}_R$ one of fundamental model-theoretic concepts is realized: the Ziegler spectrum of a ring. This is a topological space which was first constructed by Ziegler [53] using model theory. Later Herzog [25] and Krause [35, 36] gave a purely algebraic approach for the Ziegler spectrum by using properties of $\mathcal{C}_R$. There are further applications in algebraic geometry (see Garkusha and Prest [15, 22]), where properties of $\mathcal{C}_R$, model theory of modules and the Ziegler spectrum are of great utility.

The following theorem states that the category $\mathbf{Ch}(\mathcal{C}_R)$ of chain complexes of $\mathcal{C}_R$ over a commutative ring can be regarded as a Grothendieck category of enriched functors.

Theorem. *Suppose R is a commutative ring. Then the category of chain complexes of generalized R-modules $\mathbf{Ch}(\mathcal{C}_R)$ can naturally be identified with the Grothendieck category of enriched functors* $[\operatorname{mod} R, \mathbf{Ch}(\operatorname{Mod} R)]$*, where the category of finitely presented modules* $\operatorname{mod} R$ *is naturally enriched over* $\mathbf{Ch}(\operatorname{Mod} R)$ *as complexes concentrated in zeroth degree.*

A Grothendieck category of enriched functors $[\mathcal{C}, \mathcal{V}]$ can also contain a homotopy information whenever $\mathcal{V}$ is a reasonable model category in the sense of Quillen [47]. As an application of the preceding theorem, we show the following

Theorem. *Let R be a commutative ring, then $\mathbf{Ch}(\mathcal{C}_R)$ is a left and right proper closed symmetric monoidal $\mathcal{V}$-model category, where $\mathcal{V} = \mathbf{Ch}(\operatorname{Mod} R)$. The tensor* *product of two complexes* $F_\bullet, G_\bullet \in \mathbf{Ch}(\mathcal{C}_R)$ *is given by*

$$F_\bullet \odot G_\bullet = \int^{(M,N)\in \operatorname{mod} R \otimes \operatorname{mod} R} F_\bullet(M) \otimes_R G_\bullet(N) \otimes_R \operatorname{Hom}_R(M \otimes_R N, -).$$

Here $\operatorname{Hom}_R(M \otimes_R N, -)$ *is regarded as a complex concentrated in zeroth degree. The internal* Hom-*object is defined as*

$$\underline{\operatorname{Hom}}(F_\bullet, G_\bullet)(M) = \int_{N \in \operatorname{mod} R} \underline{\operatorname{Hom}}_{\mathbf{Ch}(\operatorname{Mod} R)}(F_\bullet(N), G_\bullet(M \otimes_R N)).$$

There are various ways to construct closed symmetric monoidal structures on the derived category of a reasonable closed symmetric abelian category (see [26, 52] for some examples). For the case of the derived category $\mathcal{D}(\mathcal{C}_R)$ of generalized modules with R a commutative ring we apply the preceding theorem as well as some facts for compactly generated triangulated categories to establish the following

Theorem. *Let* R *be a commutative ring. Then the derived category* $\mathcal{D}(\mathcal{C}_R)$ *of the Grothendieck category* $\mathcal{C}_R$ *is a compactly generated triangulated closed symmetric monoidal category, where the above formulas yield the derived tensor product* $F_\bullet \odot^L G_\bullet$ *and derived internal* Hom-*object* $R\underline{\operatorname{Hom}}(F_\bullet, G_\bullet)$. *The compact objects of* $\mathcal{D}(\mathcal{C}_R)$ *are the complexes isomorphic to bounded complexes of coherent functors in* $\operatorname{coh}\mathcal{C}_R$.

In the classical stable homotopy theory (see, for example, Hovey–Palmieri–Strickland [29]) the category of compact objects of a stable homotopy category possesses a duality, which in some cases is also known as the Spanier–Whitehead Duality. In order to find a duality on the category of compact objects $\mathcal{D}(\mathcal{C}_R)^c$ of $\mathcal{D}(\mathcal{C}_R)$, we use the Auslander–Gruson–Jensen Duality [1, 23, 25] for coherent objects $\operatorname{coh}\mathcal{C}_R$. In the model theory of modules, this duality corresponds to elementary duality, introduced by Prest [45, Chapter 8] and developed by Herzog in [24], for positive-primitive formulas. We show that the Auslander–Gruson–Jensen Duality makes sense for compact objects of $\mathcal{D}(\mathcal{C}_R)$. More precisely, the following result is true.

Theorem (Auslander–Gruson–Jensen Duality for compact objects). *Let* $\mathcal{D}(\mathcal{C}_R)^c$ *be the full triangulated subcategory of* $\mathcal{D}(\mathcal{C}_R)$ *of compact objects. Then there is a* *duality*

$$D : (\mathcal{D}(\mathcal{C}_R)^c)^{\mathrm{op}} \to \mathcal{D}(\mathcal{C}_R)^c$$

that takes a compact object $C_\bullet$ to

$$DC_\bullet := R\underline{\mathrm{Hom}}(C_\bullet, - \otimes_R R).$$

Basing on the above results for $\mathcal{D}(\mathcal{C}_R)$, we introduce the class of unital algebraic almost stable homotopy categories. These essentially the same with unital algebraic stable homotopy categories in the sense of Hovey–Palmieri–Strickland [29] except that the compact objects do not have to be strongly dualizable, but must have a duality. We finish the project by proving the following

Theorem. *Let R be a commutative ring. Then $\mathcal{D}(\mathcal{C}_R)$ is a unital algebraic almost stable homotopy category, which is not an algebraic stable homotopy category in the sense of Hovey–Palmieri–Strickland.*

This thesis is organized as follows. It consists of seven chapters. In Chapters 2, 3 and 4 we collect all the information which is necessary for the results of the project. More precisely, Chapter 2 is devoted to basic facts and constructions from category theory. In Chapter 3 we collect necessary facts from enriched category theory. In Chapter 4, we recall necessary facts from model categories used in the project. In Chapter 5 we prove our main results for Grothendieck categories of enriched functors. In Chapter 6 we apply results of Chapter 5 to construct a closed symmetric monoidal model category structure on the category of chain complexes of generalized modules over commutative rings. Other applications are given in Chapter 7, in which we introduce unital algebraic almost stable homotopy categories. The main result of this chapter states that the derived category of generalized modules is such a category.

Chapter 2

Preliminaries

In this chapter we collect basic facts from the theory of categories. We mostly follow the books of Mac Lane [39], Neeman [43], Popescu [44], Stenstroem [50] and Weibel [52].

2.1 Category Theory

Definition 2.1.1. A *category* $\mathcal{C}$ consists of the following data:

- a class $\mathrm{Ob}(\mathcal{C})$ of objects of $\mathcal{C}$;
- for any objects C, C', a set $\mathrm{Hom}_{\mathcal{C}}(C, C')$, whose elements are called morphisms from C to C';
- for any objects C, C', C'', a composition

$$\mathrm{Hom}_{\mathcal{C}}(C', C'') \times \mathrm{Hom}_{\mathcal{C}}(C, C') \to \mathrm{Hom}_{\mathcal{C}}(C, C'').$$

Before stating the axioms for categories we introduce a useful notation. To indicate that $\alpha \in \mathrm{Hom}_{\mathcal{C}}(C, C')$ we write $\alpha : C \to C'$. The composition of $\alpha : C \to C'$ and $\beta : C' \to C''$ is denoted $\beta\alpha$. The axioms for categories can now be given:

- if $\alpha : C \to C', \beta : C' \to C''$ and $\gamma : C'' \to C'''$ are morphisms, then $\gamma(\beta\alpha) = (\gamma\beta)\alpha$;

- ⋄ for each object C there exists $1_C \in \mathrm{Hom}_{\mathcal{C}}(C, C)$ such that $1_C\alpha = \alpha$ and $\beta 1_C = \beta$ for all $\alpha : C' \to C$ and $\beta : C \to C'$.

The identity morphism 1_C is uniquely determined by $\mathcal{C}$, for if also $1'_C$ satisfies the last axiom, then $1_C = 1_C \cdot 1'_C = 1'_C$.

A *subcategory* $\mathcal{D}$ of $\mathcal{C}$, also written as $\mathcal{D} \subset \mathcal{C}$, can be defined simply as a category in itself inheriting its structure from $\mathcal{C}$. So objects and morphisms of $\mathcal{D}$ are objects and morphisms in $\mathcal{C}$ and the identities and compositions of morphisms stay the same. Formally, $\mathcal{D}$ is given by a subcollection $\mathrm{Ob}\,\mathcal{D}$ of objects. A subcollection $\mathrm{Hom}\,\mathcal{D}$ of arrows is such that for all $D \in \mathcal{D}$ the identity morphism 1_D is in $\mathrm{Hom}\,\mathcal{D}$. We also require for all morphisms in $\mathcal{D}$ that both the source and target are in $\mathrm{Ob}\,\mathcal{D}$ and for any pair of morphisms in the subcategory, the composite morphism is in the subcategory, too.

Here are some useful categories:

- ⋄ The category **Set**: the objects are the sets, and the morphisms are the set maps;

- ⋄ The category Ab: the objects are the abelian groups, and the morphisms are the group homomorphisms;

- ⋄ The category of right R-modules $\mathrm{Mod}\,R$ for an arbitrary ring R: the objects are the right R-modules, and the morphisms are the module homomorphisms. This category will play a key role throughout the thesis. We can similarly write $R\,\mathrm{Mod}$ for the category of left R-modules;

- ⋄ The category $\mathrm{mod}\,R$: the objects are finitely presented right R-modules, and the morphisms are R-module homomorphisms.

The category $\mathrm{Mod}\,R$ has subcategories of finitely generated, finitely presented, and coherent R-modules respectively.

For each category $\mathcal{C}$ there is a *dual category* $\mathcal{C}^{\mathrm{op}}$, whose objects are those of $\mathcal{C}$ but with

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(C, C') = \mathrm{Hom}_{\mathcal{C}}(C', C)$$

and $\alpha * \beta = \beta \circ \alpha$, where $*$ denotes composition in $\mathcal{C}^{\mathrm{op}}$ and $\circ$ denotes composition in $\mathcal{C}$. Every definition or theorem in $\mathcal{C}$ has a dual definition or theorem in $\mathcal{C}^{\mathrm{op}}$.

2.1.1 Functors

Definition 2.1.2. A *functor* $T : \mathcal{B} \to \mathcal{C}$ between categories $\mathcal{B}$ and $\mathcal{C}$ is defined as follows. T assigns to each object B in $\mathcal{B}$ an object $T(B)$ in $\mathcal{C}$, and assigns to each morphism $\alpha : B \to B'$ in $\mathcal{B}$ a morphism $T(\alpha) : T(B) \to T(B')$ in $\mathcal{C}$ in such a way that:

- ◇ $T(\beta\alpha) = T(\beta)T(\alpha)$ for any morphisms $\alpha : B \to B', \beta : B' \to B''$ in $\mathcal{B}$;
- ◇ $T(1_B) = 1_{T(B)}$.

A functor $T : \mathcal{B} \to \mathcal{C}$ thus defines a map

$$\mathrm{Hom}_{\mathcal{B}}(B, B') \to \mathrm{Hom}_{\mathcal{C}}(T(B), T(B'))$$

for each pair B, B' of objects in $\mathcal{B}$. T is said to be *faithful* if these are injective maps and it is *full* if they are surjective.

Definition 2.1.3. A *natural transformation* $\mu : S \to T$ between two functors $S, T : \mathcal{B} \to \mathcal{C}$ is defined by associating to each object B in $\mathcal{B}$ a morphism $\mu_B : S(B) \to T(B)$ in $\mathcal{C}$ so that for every morphism $\alpha : B \to B'$ in $\mathcal{B}$ one gets a commutative diagram

$$\begin{array}{ccc} S(B) & \xrightarrow{\mu_B} & T(B) \\ {\scriptstyle S(\alpha)}\downarrow & & \downarrow{\scriptstyle T(\alpha)} \\ S(B') & \xrightarrow[\mu_{B'}]{} & T(B') \end{array}$$

μ is an *equivalence of functors* if each μ_B is an isomorphism in $\mathcal{C}$.

Definition 2.1.4. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and consider two functors $S : \mathcal{C} \to \mathcal{D}$ and $T : \mathcal{D} \to \mathcal{C}$. We say that T is a *right adjoint* of S (and symmetrically S is a *left adjoint* of T) if there is an equivalence

$$\eta : \mathrm{Hom}_{\mathcal{C}}(-, T(-)) \to \mathrm{Hom}_{\mathcal{D}}(S(-), -)$$

of functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathbf{Set}$, i.e. for each pair of objects $C \in \mathcal{C}$, $D \in \mathcal{D}$ there is an isomorphism

$$\eta_{C,D} : \mathrm{Hom}_{\mathcal{C}}(C, T(D)) \to \mathrm{Hom}_{\mathcal{D}}(S(C), D)$$

which is natural in $\mathcal{C}$ and $\mathcal{D}$. A right (left) adjoint is uniquely determined up to a natural equivalence of functors.

Example 2.1.5. Over a commutative ring R, in the category of R-modules, we have an adjunction pair

$$(\mathrm{Hom}_R(M,-), - \otimes_R M),$$

that is $\mathrm{Hom}_R(M,-)$ is right adjoint of $-\otimes_R M$ and $-\otimes_R M$ is left adjoint of $\mathrm{Hom}_R(M,-)$.

2.1.2 Limits and Colimits

Let $\mathcal{C}$ be a category and let I be a small category. We propose to define the notion of "limit" of a functor

$$F : I \to \mathcal{C}.$$

If X is an object of $\mathcal{C}$ and if we are given a morphism $\alpha_i : X \to F(i)$ for each $i \in \mathrm{Ob}(I)$, then the family (α_i) is called *compatible* if for every $\lambda : i \to j$ in I one has $\alpha_j = F(\lambda)\alpha_i$.

A *limit* or *projective limit* of the functor $F : I \to \mathcal{C}$ is an object $\varprojlim F$ in $\mathcal{C}$ together with a compatible family of morphisms $\pi_i : \varprojlim F \to F(i)$ such that for each other compatible family $\xi_i : X \to F(i)$ there exists a unique $\xi : X \to \varprojlim F(i)$ with $\pi_i \xi = \xi_i$.

The limit of F solves a universal problem and is therefore unique up to isomorphism. The category $\mathcal{C}$ is called *complete* if the limit exists for every functor $F : I \to \mathcal{C}$ when I is small.

Colimits are defined in a dual fashion. If we take I as a directed set, the colimit of a direct system $F : I \to \mathcal{C}$ is called a *direct limit*, while the limit of an inverse system $F : I^{\mathrm{op}} \to \mathcal{C}$ is an *inverse limit*.

The category $\mathcal{C}$ is called *bicomplete* if both limit and colimit exist for every functor $F : I \to \mathcal{C}$ whenever I is small.

Example 2.1.6. The simplest examples of limits and colimits are those of *pushouts* and *pullbacks*.

A *pushout* is a colimit (if it exists) of a diagram of the form

$$A \leftarrow C \to B.$$

Similarly, a *pullback* is a limit (if it exists) of a diagram of the form

$$A \to C \leftarrow B.$$

Example 2.1.7 ([50] IV. 3). Further important examples of limits and colimits are those of *products* and *coproducts*.

Definition 2.1.8. A *product* of a family $(C_i)_{i\in I}$ of objects of $\mathcal{C}$ is an object C together with morphisms $\pi_i : C \to C_i, i \in I$, such that for each object X and morphisms $\eta_i : X \to C_i, i \in I$, there is a unique morphism $\eta : X \to C$ with $\pi_i\eta = \eta_i$ for all $i \in I$.

Here we are facing a universal problem and so the product is unique up to isomorphism, and we denote it by $\prod_I C_i$. The canonical morphisms $\pi_i : \prod_I C_i \to C_i$ are called *projections*. This definition gives rise to a canonical isomorphism

$$\operatorname{Hom}(X, \prod_I C_i) \cong \prod_I \operatorname{Hom}(X, C_i),$$

where the second product is taken in the category of sets.

We can dually define the *coproduct*, and it is denoted by $\coprod_I C_i$. It yields a canonical isomorphism

$$\operatorname{Hom}(\coprod_I C_i, X) \cong \prod_I \operatorname{Hom}(C_i, X).$$

2.2 Abelian Categories

This section contains basic facts from the theory of abelian categories. We shall mostly follow [44] and [50].

Definition 2.2.1. By a *preadditive* category we mean a category $\mathcal{C}$ together with an abelian group structure on each set $\operatorname{Hom}_{\mathcal{C}}(A, B)$ of morphisms, in such a way that the composition mappings

$$\alpha_{ABC} : \operatorname{Hom}_{\mathcal{C}}(A, B) \times \operatorname{Hom}_{\mathcal{C}}(B, C) \to \operatorname{Hom}_{\mathcal{C}}(A, C), \ (f, g) \to g \circ f$$

are group homomorphisms in each variable. We shall write the group structure additively. Clearly, the category of abelian groups or, more generally, any category of modules over a ring is preadditive. As a consequence, every full subcategory of a category of modules is preadditive as well.

Example 2.2.2. $\operatorname{Mod} R$ is a preadditive category. An abelian group structure on each Hom-set $\operatorname{Hom}_R(M, N)$, where M and N are two right R-modules, is defined as follows. Given two homomorphisms $f, g \in \operatorname{Hom}_R(M, N)$,

$$(f+g)(m) := f(m) + g(m)$$

for any element $m \in M$.

Remark 2.2.3. If $\mathcal{C}$ is preadditive, then we shall also write $\bigoplus_I C_i$ to denote the coproduct, if it exists, and call it the *direct sum* of the family $(C_i)_{i \in I}$.

Definition 2.2.4. If $\mathcal{C}$ is a preadditive category with zero object 0, then the *cokernel* of an arrow $f : A \to B$, denoted by $\operatorname{Coker} f$, is a pushout (if it exists) of the diagram

$$0 \leftarrow A \xrightarrow{f} B.$$

The canonical morphism $B \to \operatorname{Coker} f$ will also be denoted by $\operatorname{coker} f$.

Similarly, the *kernel* of an arrow $A \to B$, denoted by $\operatorname{Ker} f$, is the pullback (if it exists) of the diagram

$$A \xrightarrow{f} B \leftarrow 0.$$

The canonical morphism $\operatorname{Ker} f \to A$ will also be denoted by $\ker f$.

By definition, both kernels and cokernels satisfy natural universal properties, and therefore are unique up to canonical isomorphism.

Let $\mathcal{C}$ be a preadditive category with the property that every morphism has a kernel and a cokernel. For a morphism $\alpha : B \to C$ there is a canonical factorization as indicated by the commutative diagram

$$\begin{array}{ccccccc} \operatorname{Ker} \alpha & \longrightarrow & B & \xrightarrow{\quad \alpha \quad} & C & \longrightarrow & \operatorname{Coker} \alpha \\ & & {\scriptstyle \lambda}\downarrow & & \uparrow{\scriptstyle \eta} & & \\ & & \operatorname{Coker}(\ker \alpha) & \xrightarrow[\overline{\alpha}]{} & \operatorname{Ker}(\operatorname{coker} \alpha) & & \end{array}$$

where $\overline{\alpha}$ is obtained as follows. $\operatorname{coker} \alpha \cdot \alpha = 0$ implies $\alpha = \eta\beta$ for some $\beta : B \to \operatorname{Ker}(\operatorname{coker} \alpha)$. Then $\eta\beta \cdot \ker \alpha = \alpha \cdot \ker \alpha = 0$, which implies $\beta \cdot \ker \alpha = 0$ since η is a monomorphism, and hence β factors as $\beta = \overline{\alpha}\lambda$.

Consider Ab as an example. Here $\operatorname{Coker}(\ker \alpha) \cong B / \operatorname{Ker} \alpha$ and $\operatorname{Ker}(\operatorname{coker} \alpha) \cong \operatorname{Im} \alpha$, and $\overline{\alpha}$ is an isomorphism. This allows us to make the following definition:

Definition 2.2.5. A category $\mathcal{C}$ is *abelian* if

- $\diamond$ $\mathcal{C}$ is preadditive.
- $\diamond$ Every finite family of objects has a product (and a coproduct).
- $\diamond$ Every morphism has a kernel and a cokernel.
- $\diamond$ Either $\overline{\alpha} : \mathrm{Coker}(\ker \alpha) \to \mathrm{Ker}(\mathrm{coker}\, \alpha)$ is an isomorphism for every morphism α,
- $\diamond$ or every morphism α has a factorization $\alpha = \mu\beta$, where β is a cokernel and μ is a kernel.

For every morphism α of an abelian category define the *image* of α as $\mathrm{Im}\, \alpha := \mathrm{Ker}(\mathrm{coker}\, \alpha)$.

Consider two preadditive categories $\mathcal{C}$ and $\mathcal{D}$. A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be *additive* if

$$F(f + g) = F(f) + F(g)$$

for any morphisms $f, g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and any objects $X, Y \in \mathcal{C}$.

Definition 2.2.6. Let $\mathcal{C}$ be an abelian category. An object C of $\mathcal{C}$ is a *generator* for $\mathcal{C}$ if $\mathrm{Hom}(C, -)$ is faithful, and is a *cogenerator* if $\mathrm{Hom}(-, C)$ is faithful.

Example 2.2.7. A module M is a generator for $\mathrm{Mod}\, R$ if and only if R is a direct summand of some direct sum of copies of M (see [50, Proposition IV.6.2] for details).

Of particular interest are the functor categories $(\mathcal{B}, \mathsf{Ab})$, where $\mathcal{B}$ is a preadditive category. By definition, its objects are the covariant additive functors from $\mathcal{B}$ to Ab. The morphisms are the natural transformations of functors. A typical example of an object in $(\mathcal{B}, \mathsf{Ab})$ is the *representable functor*

$$h^B := \mathrm{Hom}_{\mathcal{B}}(B, -)$$

associated with an object $B \in \mathcal{B}$.

The following statement is also known as the Yoneda Lemma.

Proposition 2.2.8 (Yoneda Lemma). *Let $\mathcal{B}$ be a small preadditive category. For every object B of $\mathcal{B}$ and every additive functor $T : \mathcal{B} \to \mathsf{Ab}$ there is an isomorphism*

$$\mathrm{Hom}_{(\mathcal{B},\mathsf{Ab})}(h^B, T) \cong T(B),$$

which is natural in T and B.

We also collect some useful facts from Bucur–Deleanu [6].

Definition 2.2.9. A sequence of morphisms of an abelian category $\mathcal{C}$

$$A \xrightarrow{u} B \xrightarrow{v} C$$

is said to be *exact*, if $\mathrm{Im}\, u = \mathrm{Ker}\, v$. An arbitrary sequence of consecutive morphisms is said to be exact, if the subsequence formed by any couple of consecutive morphisms is an exact sequence.

The following propositions are straightforward.

Proposition 2.2.10. *The necessary and sufficient condition that the sequence*

$$0 \to A \xrightarrow{u} B \xrightarrow{v} C$$

be exact is that the sequence of abelian groups and homomorphisms of abelian groups

$$0 \to \mathrm{Hom}_{\mathcal{C}}(X, A) \to \mathrm{Hom}_{\mathcal{C}}(X, B) \to \mathrm{Hom}_{\mathcal{C}}(X, C)$$

be exact for any object X of $\mathcal{C}$.

Proposition 2.2.11. *The necessary and sufficient condition that the sequence*

$$C \xrightarrow{u} B \xrightarrow{v} A \to 0$$

be exact is that the sequence of abelian groups and homomorphisms of abelian groups

$$0 \to \mathrm{Hom}_{\mathcal{C}}(A, X) \to \mathrm{Hom}_{\mathcal{C}}(B, X) \to \mathrm{Hom}_{\mathcal{C}}(C, X)$$

be exact for any object X of $\mathcal{C}$.

Proposition 2.2.12. *In order the sequence*

$$0 \to A' \xrightarrow{u} A \xrightarrow{v} A'' \to 0$$

be exact it is necessary and sufficient that (A', u) *be a kernel of* v *and* (A'', v) *be a cokernel of* u.

Definition 2.2.13. If $\mathcal{C}$ and $\mathcal{D}$ are abelian categories and if $F : \mathcal{C} \to \mathcal{D}$ is an additive covariant functor, we say that F is *left exact* if for any exact sequence

$$0 \to A' \xrightarrow{\alpha'} A \xrightarrow{\alpha''} A'' \to 0$$

in the category $\mathcal{C}$, the sequence

$$0 \to F(A') \xrightarrow{F(\alpha')} F(A) \xrightarrow{F(\alpha'')} F(A'')$$

is exact in the category $\mathcal{D}$. If for any exact sequence

$$0 \to A' \xrightarrow{\alpha'} A \xrightarrow{\alpha''} A'' \to 0$$

the sequence,

$$0 \to F(A') \xrightarrow{F(\alpha')} F(A) \xrightarrow{F(\alpha'')} F(A'') \to 0$$

is exact, then we say that F is *exact.*

We can extrapolate from this, by means of dualization, the notion of a left exact (or right exact) contravariant functor and that of an exact contravariant functor.

Proposition 2.2.14. *The necessary and sufficient condition that the functor* $F : \mathcal{C} \to \mathcal{D}$ *be exact is that it transforms any exact sequence in the category* $\mathcal{C}$ *into an exact sequence in the category* $\mathcal{D}$.

The proof of this statement can be found in [6, Proposition 5.16].

Proposition 2.2.15. *If* $\mathcal{C}$ *and* $\mathcal{D}$ *are abelian categories and if* $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ *are covariant functors such that* G *is an adjoint of the functor* F, *then* G *is left exact and* F *is right exact.*

Given a commutative ring R, recall that $\mathrm{Hom}_R(M, -)$ is adjoint to $- \otimes_R M$. It follows from the preceding proposition that $\mathrm{Hom}_R(M, -) : \mathrm{Mod}\, R \to \mathrm{Mod}\, R$ is left exact and $- \otimes_R M : \mathrm{Mod}\, R \to \mathrm{Mod}\, R$ is right exact.

2.3 Grothendieck Categories

We mostly follow Garkusha [14] and Herzog [25] to collect some basic facts on *Grothendieck categories.* Recall that an abelian category is *cocomplete* or an *Ab3-category* if it has arbitrary direct sums. The cocomplete abelian category $\mathcal{C}$ is said to be an *Ab5-category* if for any directed family $\{A_i\}_{i\in I}$ of subobjects of A and for any subobject B of A, the relation

$$(\sum_{i\in I} A_i) \cap B = \sum_{i\in I}(A_i \cap B)$$

holds.

The condition $Ab3$ is equivalent to the existance of arbitrary direct limits. Also $Ab5$ is equivalent to the fact that there exist inductive limits and the inductive limits over directed families of indices are exact, i.e. if I is a directed set and

$$0 \longrightarrow A_i \longrightarrow B_i \longrightarrow C_i \longrightarrow 0$$

is an exact sequence for any $i \in I$, then

$$0 \longrightarrow \varinjlim A_i \longrightarrow \varinjlim B_i \longrightarrow \varinjlim C_i \longrightarrow 0$$

is an exact sequence.

Let $\mathcal{C}$ be a category and $\mathcal{U} = \{U_i\}_{i\in I}$ a family of objects of $\mathcal{C}$. The family $\mathcal{U}$ is said to be a *family of generators* of the category $\mathcal{C}$ if for any object A of $\mathcal{C}$ and any subobject B of A distinct from A there exists an index $i \in I$ and a morphism $u : U_i \to A$ which cannot be factorized through the canonical injection $i : B \to A$ of B into A. An object U of $\mathcal{C}$ is said to be a *generator* of the category $\mathcal{C}$ provided that the family $\{U\}$ is a family of generators of the category $\mathcal{C}$.

Let $\mathcal{C}$ be a cocomplete abelian category; then $\mathcal{U} = \{U_i\}_{i\in I}$ is a family of generators for $\mathcal{C}$ if and only if the object $\bigoplus_{i\in I} U_i$ is a generator of $\mathcal{C}$ [6]. According to [6, Prop. 5.35] the cocomplete abelian category $\mathcal{C}$ which possesses a family of generators $\mathcal{U}$ is locally small and it can be proved that any object of $\mathcal{C}$ is isomorphic to a quotient of an object $\bigoplus_{j\in J} U_j$, where J is some set of indices, $U_j \in \mathcal{U}$ for any $j \in J$.

An abelian category which satisfies the condition $Ab5$ and which possesses a family of generators is called a *Grothendieck category.*

Example 2.3.1. Given any ring R (associative with identity), the category $\operatorname{Mod} R$ of R-modules is a Grothendieck category, where R is a generator.

We describe the subcategories consisting of finitely generated, finitely presented and coherent objects respectively. These categories are ordered by inclusion as follows:

$$\mathcal{C} \supseteq \operatorname{fg}\mathcal{C} \supseteq \operatorname{fp}\mathcal{C} \supseteq \operatorname{coh}\mathcal{C}.$$

Recall an object $A \in \mathcal{C}$ is *finitely generated* if whenever there are subobjects $A_i \subseteq A$ for $i \in I$ satisfying $A = \sum_{i\in I} A_i$, then there is a finite subset $J \subset I$ such that $A = \sum_{i\in J} A_i$. The category of finitely generated subobjects of $\mathcal{C}$ is denoted by $\operatorname{fg}\mathcal{C}$. The category is *locally finitely generated* provided that every object $X \in \mathcal{C}$ is a directed sum $X = \sum_{i\in I} X_i$ of finitely generated subobjects X_i, or equivalently, $\mathcal{C}$ possesses a family of finitely generated generators.

Theorem 2.3.2. *[50, V.3.2] An object $C \in \mathcal{C}$ is finitely generated if and only if the canonical homomorphism*

$$\Phi : \operatorname{colim} \operatorname{Hom}_{\mathcal{C}}(C, D_i) \to \operatorname{Hom}_{\mathcal{C}}(C, \sum D_i)$$

is an isomorphism for every object $D \in \mathcal{C}$ and every directed family $\{D_i\}_I$ of subobjects of D.

A finitely generated object $B \in \mathcal{C}$ is *finitely presented* provided that every epimorphism $\eta : A \to B$ with A finitely generated has a finitely generated kernel $\operatorname{Ker}\eta$. The subcategory of finitely presented objects of $\mathcal{C}$ is denoted by $\operatorname{fp}\mathcal{C}$. The corresponding categories of finitely presented left and right R-modules over the ring R are denoted by $R\operatorname{mod} = \operatorname{fp}(R\operatorname{Mod})$ and $\operatorname{mod} R = \operatorname{fp}(\operatorname{Mod} R)$, respectively. Note that the subcategory $\operatorname{fp}\mathcal{C}$ of $\mathcal{C}$ is closed under extensions. Moreover, if

$$0 \to A \to B \to C \to 0$$

is a short exact sequence in $\mathcal{C}$ with B finitely presented, then C is finitely presented if and only if A is finitely generated.

Definition 2.3.3. An object P of an abelian category $\mathcal{A}$ is said to be *projective* if the functor

$$\operatorname{Hom}_{\mathcal{A}}(P, -) : \mathcal{A} \to \mathsf{Ab}$$

is exact.

The most obvious example of a finitely presented object of $\mathcal{C}$ is a finitely generated projective object P. We say that $\mathcal{C}$ has *enough finitely generated projectives* provided that every finitely presented object $A \in \mathcal{C}$ admits an epimorphism $\eta : P \to A$ with P a finitely generated projective object. If $\mathcal{C}$ has enough finitely generated projectives, then by the previous remarks, every finitely presented object $B \in \mathcal{C}$ is isomorphic to the cokernel of a morphism between finitely generated projective objects. This is expressed by an exact sequence

$$P_1 \longrightarrow P_0 \longrightarrow B \longrightarrow 0$$

called a *projective presentation* of B.

Example 2.3.4. The category $\operatorname{Mod} R$ of right R-modules has enough finitely generated projectives.

Example 2.3.5. By [25], another example of a category having enough finitely generated projective objects is the category of additive functors $(\mathcal{B}, \mathsf{Ab})$ from a small preadditive category $\mathcal{B}$ to the category of abelian groups Ab. This category is a Grothendieck category, in which limits and colimits of functors are defined objectwise. A family of projective generators for $(\mathcal{B}, \mathsf{Ab})$ is given by the collection of representable functors $\{h^B\}_{B \in \operatorname{Ob} \mathcal{B}}$. In what follows we shall also write $(B, -)$ to denote the representable functor h^B, $B \in \operatorname{Ob} \mathcal{B}$.

In this category every finitely generated projective object is a coproduct factor of a finite coproduct of representable objects $\bigoplus_{i=1}^{n} (B_i, -)$. In addition, if $\mathcal{B}$ is an additive category, that is $\mathcal{B}$ is preadditive, has finite products/coproducts and idempotents split in $\mathcal{B}$, then every finitely generated projective object in $(\mathcal{B}, \mathsf{Ab})$ is representable by [25, Proposition 2.1].

The category $\mathcal{C}$ is *locally finitely presented* provided that every object $B \in \mathcal{C}$ is a direct limit $B = \varinjlim B_i$ of finitely presented objects B_i, or equivalently, $\mathcal{C}$ possesses a family of finitely presented generators. As an example (see [37]), any locally finitely generated Grothendieck category having enough finitely presented projectives $\{P_i\}_{i \in I}$ is locally finitely presented. In this case, $\{P_i\}_{i \in I}$ are generators for $\mathcal{C}$. For instance, the set of representable functors $\{(B, -)\}_{B \in \mathcal{B}}$ of the functor category $(\mathcal{B}, \mathsf{Ab})$ with $\mathcal{B}$ as a small preadditive category form a family of finitely generated projective generators for $(\mathcal{B}, \mathsf{Ab})$. Therefore $(\mathcal{B}, \mathsf{Ab})$ is a locally finitely presented Grothendieck category (see [25, Proposition 1.3]).

Definition 2.3.6. A finitely presented object C of a locally finitely presented Grothendieck categoy $\mathcal{C}$ is *coherent* if every finitely generated subobject B of C is finitely presented. Equivalently, every epimorphism $h : C \to A$ with A finitely presented has a finitely presented kernel. Evidently, a finitely generated subobject of coherent object is also coherent. The subcategory of coherent objects of $\mathcal{C}$ is denoted by $\operatorname{coh}\mathcal{C}$.

Definition 2.3.7. By [35], a Grothendieck category $\mathcal{C}$ is said to be *locally coherent Grothendieck* provided that $\mathcal{C}$ has a generating set of finitely presented objects and the full subcategory $\operatorname{fp}\mathcal{C}$ of finitely presented objects in $\mathcal{C}$ is abelian.

The main Grothendieck category we work with is the category of generalized R-modules $\mathcal{C}_R$, which we define below. It is a locally coherent Grothendieck category with enough finitely generated projectives.

Example 2.3.8. $\operatorname{Mod} R$ is locally coherent if and only if R is right coherent.

Recall from [50] that a ring R is *right coherent* if it satisfies any of the below equivalent conditions:

- ◇ Every direct product of flat left-R modules is flat.
- ◇ R^I is a flat left R-module for every set I.
- ◇ Every finitely presented right R-module is coherent.
- ◇ R is coherent as a right R-module.

Theorem 2.3.9 ([25, 48])**.** *The following conditions on a locally finitely presented Grothendieck category $\mathcal{C}$ are equivalent:*

- ◇ *$\mathcal{C}$ is locally coherent;*
- ◇ $\operatorname{fp}\mathcal{C} = \operatorname{coh}\mathcal{C}$*;*
- ◇ $\operatorname{fp}\mathcal{C}$ *is an abelian category.*

Proof. We refer to [25, Theorem 1.6]. □

Proposition 2.3.10 ([1, 50]). *Let $\mathcal{B}$ be a small additive category, that is, $\mathcal{B}$ is preadditive, has finite products/coproducts and idempotents split in $\mathcal{B}$. Then every finitely generated projective object in $(\mathcal{B}, \mathsf{Ab})$ is representable. If $\mathcal{B}$ has cokernels, then $(\mathcal{B}, \mathsf{Ab})$ is locally coherent and* $\operatorname{coh}(\mathcal{B}, \mathsf{Ab})$ *has projective global dimension at most 2.*

Under the assumption of the preceding proposition this leads us to the fact that every finitely presented object $B \in (\mathcal{B}, \mathsf{Ab})$ is coherent, that is, B is finitely presented and every finitely presented subobject of B is also finitely presented.

2.4 The category of generalized modules $\mathcal{C}_R$

Following Herzog [25], we define the category $\mathcal{C}_R$ as

$$\mathcal{C}_R := (\operatorname{mod} R, \mathsf{Ab}),$$

whose objects are the additive functors $F : \operatorname{mod} R \to \mathsf{Ab}$ from the category of right finitely presented R-modules $\operatorname{mod} R$ to the category of abelian groups Ab. Its morphisms are the natural transformations of functors. Similarly, the category ${}_R\mathcal{C}$ consists of the additive functors from the category of left finitely presented R-modules to Ab. Since the category $\operatorname{mod} R$ has cokernels, it follows from Proposition 2.3.10 that $\mathcal{C}_R$ is a locally coherent Grothendieck category. Moreover, the category of coherent objects $\operatorname{coh} \mathcal{C}_R$ has projective global dimension at most two.

The latter fact means that every coherent object $C \in \operatorname{coh} \mathcal{C}_R$ has a resolution by representable functors

$$0 \to (M, -) \to (N, -) \to (L, -) \to C \to 0,$$

where M, N, L are finitely presented right R-modules.

The category $\mathcal{C}_R$ is also called the *category of generalized modules.* for the reason that there is a fully faithful, right exact functor

$$M \mapsto - \otimes_R M$$

from the category of all R-modules to $\mathcal{C}_R$. The category $\mathcal{C}_R$ has a number of remarkable properties which led to powerful applications in ring and module theory and representation theory (see [14, 24, 25, 35, 36, 45, 46, 53]).

2.5 Triangulated Categories

Triangulated categories are a convenient tool to describe the type of structure inherent in the derived category of an abelian category. We mostly follow the book of Neeman [43], Keller [33] and Weibel [52] for the following material.

Definition 2.5.1. Let $\mathcal{C}$ be an additive category and $\Sigma : \mathcal{C} \to \mathcal{C}$ be an additive endofunctor of $\mathcal{C}$. Asssume thoughout that the endofunctor Σ is invertible. A *candidate triangle* in $\mathcal{C}$ (with respect to Σ) is a diagram of the form:

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

such that the composition $v \circ u, w \circ v$ and $\Sigma u \circ w$ are the zero morphisms.

A morphism of candidate triangles is a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow \Sigma f \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

where each row is a candidate triangle.

Definition 2.5.2. A pre-triangulated category $\mathcal{T}$ is an additive category, together an additive automorphism Σ, and a class of candidate triangles (with respect to Σ) called *distinguished* triangles. The following conditions must hold:

TR0: Any candidate triangle which is isomorphic to a distinguished triangle is a distinguished trinagle. The candidate triangle

$$X \xrightarrow{1} X \longrightarrow 0 \longrightarrow \Sigma X$$

is distinguished.

TR1: For any morphism $f : X \to Y$ in $\mathcal{T}$ there exists a distinguished triangle of the form

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow \Sigma X$$

TR2: Consider the two candidate triangles

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

and

$$Y \xrightarrow{-v} Z \xrightarrow{-w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y$$

if one is distinguished triangle, then so is the other.

TR3: For any commutative diagram of the form

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & & & \downarrow{\scriptstyle \Sigma f} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

where the rows are distinguished triangles, there is a morphim $h : Z \to Z'$, not necessarily unique, which makes the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \Sigma f} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

commutative.

Definition 2.5.3. Let $\mathcal{T}$ be a pre-triangulated category. Suppose that we are given a morphism of candidate triangles

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \Sigma f} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

There is a way to form a new candidate triangle out of this data. It is the diagram

$$Y \oplus X' \xrightarrow{\begin{pmatrix} -v & 0 \\ g & u' \end{pmatrix}} Z \oplus Y' \xrightarrow{\begin{pmatrix} -w & 0 \\ h & v' \end{pmatrix}} \Sigma X \oplus Z' \xrightarrow{\begin{pmatrix} -\Sigma u & 0 \\ \Sigma f & w' \end{pmatrix}} \Sigma Y \oplus \Sigma X'.$$

This new candidate triangle is called the *mapping cone* on a map of candidate triangles.

Definition 2.5.4. Two maps of candidate triangles

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \Sigma f} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

and

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle \Sigma f'} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

are called *homotopic* if they differ by homotopy; that is, if there exist Θ, Φ and Ψ below

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ & \swarrow{\scriptstyle \Theta} & & \swarrow{\scriptstyle \Phi} & & \swarrow{\scriptstyle \Psi} & \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

with

$$f - f' = \Theta u + \Sigma^{-1}(w'\Psi), \quad g - g' = \Phi v + u'\Theta, \quad h - h' = \Psi w + v'\Phi.$$

Definition 2.5.5. Let $\mathcal{T}$ be a pre-triangulated category. Then $\mathcal{T}$ is *triangulated* if it satisfies the further hypothesis

TR4': Given any diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & & & \downarrow{\scriptstyle \Sigma f} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

where the rows are triangles, there is, by [TR3], a way to choose an $h : Z \to Z'$ to make the diagram commutative. This h may be chosen so that the mapping cone

$$Y \oplus X' \xrightarrow{\begin{pmatrix} -v & 0 \\ g & u' \end{pmatrix}} Z \oplus Y' \xrightarrow{\begin{pmatrix} -w & 0 \\ h & v' \end{pmatrix}} \Sigma X \oplus Z' \xrightarrow{\begin{pmatrix} -\Sigma u & 0 \\ \Sigma f & w' \end{pmatrix}} \Sigma Y \oplus \Sigma X'$$

is a triangle.

Definition 2.5.6. Let $\mathcal{T}$ be a triangulated category in which all set indexed direct sums exist. An object A of $\mathcal{T}$ is called *compact* if the canonical map

$$\bigoplus_{i\in I}\mathrm{Hom}(A,E_i)\to\mathrm{Hom}(A,\bigoplus_{i\in I}E_i)$$

is an isomorphism for any set of objects E_i in $\mathcal{T}$ and $i\in I$. A triangulated category $\mathcal{T}$ is *compactly generated* if there is a set S of compact objects with the following property, if for every object $E\in\mathcal{T}$ we have

$$\mathrm{Hom}(A,E)=0 \text{ for all } A\in S \text{ implies } E=0.$$

2.6 Derived categories

Definition 2.6.1. Let $\mathcal{A}$ be an additive category. A graded $\mathcal{A}$-object is a family $X=(X_n;n\in\mathbb{Z})$ of objects of $\mathcal{A}$. the object X_n is called the *homogeneous component of degree* n of X.

Let X and Y be two graded $\mathcal{A}$-objects and $n\in\mathbb{Z}$. We denote by $\mathrm{Hom}_p(X,Y)$ the set of all *graded morphisms of degree* p.

Definition 2.6.2. A *chain complex* of $\mathcal{A}$-objects is a pair (X,d_X) consisting of a graded $\mathcal{A}$-object X and a graded morphism $d_X\in\mathrm{Hom}_1(X,X)$ such that $d_X\circ d_X=0$. The morphism d_X is called the *differential* of the complex. We can view the complex as a diagram

$$\cdots\longrightarrow X_{n+1}\xrightarrow{d_{n+1}}X_n\xrightarrow{d_n}X_{n-1}\xrightarrow{d_{n-1}}\cdots.$$

If (X,d_X) and (Y,d_Y) are two complexes of $\mathcal{A}$-objects, a *morphism of complexes* $f:(X,d_X)\to(Y,d_Y)$ is a graded morphism $f\in\mathrm{Hom}_0(X,Y)$ such that

$$f\circ d_X=d_Y\circ f,$$

i.e., the diagram

$$\begin{array}{ccccccccc}\cdots\longrightarrow & X_{n+1} & \xrightarrow{d_{n+1}} & X_n & \xrightarrow{d_n} & X_{n-1} & \xrightarrow{d_{n-1}} & \cdots.\\ & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\ \cdots\longrightarrow & Y_{n+1} & \xrightarrow{d_{n+1}} & Y_n & \xrightarrow{d_n} & Y_{n-1} & \xrightarrow{d_{n-1}} & \cdots\end{array}$$

commutes.

The *category of complexes of $\mathcal{A}$-objects* is the category $\mathbf{Ch}(\mathcal{A})$ with complexes of $\mathcal{A}$-objects as objects and morphisms of complexes as morphisms.

Definition 2.6.3. Let $f : X \to Y$ be a morphism in $\mathbf{Ch}(\mathcal{A})$. Then f is homotopic to zero if there exist $h \in \mathrm{Hom}_1(X, Y)$ such that

$$f = d_Y \circ h + h \circ d_X.$$

Let $\mathrm{Ht}(X, Y)$ be the set of all morphisms in $\mathrm{Hom}_{\mathbf{Ch}(\mathcal{A})}(X, Y)$ which are homotopic to zero. We say that the morphisms $f : X \to Y$ and $g : X \to Y$ are *homotopic* if $f - g \in \mathrm{Ht}(X, Y)$.

Lemma 2.6.4. *The subset* $\mathrm{Ht}(X, Y)$ *is a subgroup of* $\mathrm{Hom}_{\mathbf{Ch}(\mathcal{A})}(X, Y)$.

Definition 2.6.5. Let $\mathcal{A}$ be an abelian category, and consider the category $\mathbf{Ch}(\mathcal{A})$ of chain complexes in $\mathcal{A}$. The quotient category $\mathbf{K}(\mathcal{A})$ of $\mathbf{Ch}(\mathcal{A})$ is defined as follows. The objects of $\mathbf{K}(\mathcal{A})$ are chain complexes (objects of $\mathbf{Ch}(\mathcal{A})$) and the morphisms of $\mathbf{K}(\mathcal{A})$ are the chain homotopy equivalence classes of maps in $\mathbf{Ch}(\mathcal{A})$.

Definition 2.6.6. Let $\mathcal{A}$ be an abelian category. For $n \in \mathbb{Z}$ and any complex X in $\mathbf{Ch}(\mathcal{A})$ we define the nth homology of X as follows

$$\mathbf{H}_n(X) = \mathrm{Ker}\, d_n / \mathrm{Im}\, d_{n+1}.$$

If $f : X \to Y$ is a morphism of complexes in $\mathbf{Ch}(\mathcal{A})$, f induces a morphism $\mathbf{H}_n(f) : \mathbf{H}_n(X) \to \mathbf{H}_n(Y)$. Therefore, $\mathbf{H}_n$ is a functor from the category $\mathbf{Ch}(\mathcal{A})$ into the category $\mathcal{A}$.

Definition 2.6.7. A morphism $f : X \to Y$ in $\mathbf{Ch}(\mathcal{A})$ is a *quasi-isomorphism* if $\mathbf{H}_n(f) : \mathbf{H}_n(X) \to \mathbf{H}_n(Y)$ are isomorphisms for all $n \in \mathbb{Z}$.

Definition 2.6.8. (Triangles in $\mathbf{K}(\mathcal{A})$) Let $u : A \to B$ be a morphism in $\mathbf{Ch}(\mathcal{A})$. Recall that the mapping cone of u fits into an exact sequence

$$0 \longrightarrow B \xrightarrow{v} \mathrm{cone}(u) \xrightarrow{\delta} \Sigma A \longrightarrow 0$$

in $\mathbf{Ch}(\mathcal{A})$ (see [52, 1.52]). The degree n part of cone(u) is $A^{n+1} \oplus B^n$ and A^{n+1} is the degree n part of ΣA. The *strict triangle* on u is the triple (u, v, δ) of maps in $\mathbf{K}(\mathcal{A})$; this data is usually written in the form

$$\begin{array}{ccc} & \mathrm{cone}(u) & \\ {}^{\delta}\swarrow & & \nwarrow^{v} \\ A & \xrightarrow{\quad u \quad} & B. \end{array}$$

Now consider three fixed chain complexes A, B and C. Suppose we are given three maps $u : A \to B, v : B \to C$, and $w : C \to \Sigma A$ in $\mathbf{K}(\mathcal{A})$. We say that (u, v, w) is an *exact triangle* (A, B, C) if it is "isomorphic" to a strict triangle (u', v', δ) on $u' : A' \to B'$ in the sense that there is a diagram of chain complexes,

$$\begin{array}{ccccccc} A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & \Sigma A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \Sigma f} \\ A' & \xrightarrow{u'} & B' & \xrightarrow{v'} & \mathrm{cone}(u') & \xrightarrow{\delta} & \Sigma A' \end{array},$$

commuting in $\mathbf{K}(\mathcal{A})$ (i.e., commuting in $\mathbf{Ch}(\mathcal{A})$ up to chain homotopy equivalences) and such that maps f, g, h are isomorphisms in $\mathbf{K}(\mathcal{A})$ (i.e., chain homotopy equivalences). If we replace u, v and w by chain homotopy equivalent maps, we get the same diagram in $\mathbf{K}(\mathcal{A})$. This allows us to think of (u, v, w) as a triangle in the category $\mathbf{K}(\mathcal{A})$. A triangle is usually written as follows:

$$\begin{array}{ccc} & C & \\ {}^{w}\swarrow & & \nwarrow^{v} \\ A & \xrightarrow{\quad u \quad} & B. \end{array}$$

Proposition 2.6.9. $\mathbf{K}(\mathcal{A})$ *is a triangulated category.*

Definition 2.6.10. The *derived category* $\mathbf{D}(\mathcal{A})$ is defined to be the localization $Q^{-1}\mathbf{K}(\mathcal{A})$ of the category $\mathbf{K}(\mathcal{A})$ at the collection Q of quasi-isomorphisms.

Remark 2.6.11. In what follows we shall not discuss set theoretical issues related to the existence of $S^{-1}\mathcal{C}$. In all our results such issues will not occur.

In order to describe morphisms of $\mathbf{D}(\mathcal{A})$ explicitly, we need some facts from Gabriel–Zisman localization theory [13]

Definition 2.6.12. Let S be a collection of morphisms in a category $\mathcal{C}$. A localization of $\mathcal{C}$ with respect to S is a category $S^{-1}\mathcal{C}$, together with a functor $q : \mathcal{C} \to S^{-1}\mathcal{C}$ such that

1. $q(s)$ is a isomorphism in $S^{-1}\mathcal{C}$ for every $s \in S$.
2. Any functor $F : \mathcal{C} \to \mathcal{D}$ such that $F(s)$ is an isomorphism for all $s \in S$ factors in a unique way through q. (It follows that $S^{-1}\mathcal{C}$ is unique up to equivalence).

Example 2.6.13. 1. Let S be the collection of chain homotopy equivalences in $\mathbf{Ch}(\mathcal{A})$. The universal property for $\mathbf{Ch}(\mathcal{A}) \to \mathbf{K}(\mathcal{A})$ shows that $\mathbf{K}(\mathcal{A})$ is the localization $S^{-1}\,\mathbf{Ch}(\mathcal{A})$.

2. Let $\widetilde{Q}$ be the collection of all quasi-isomorphism in $\mathbf{Ch}(\mathcal{A})$. Since $\widetilde{Q}$ contains S of part (1), it follows that

$$\widetilde{Q}^{-1}\,\mathbf{Ch}(\mathcal{A}) = Q^{-1}(S^{-1}\,\mathbf{Ch}(\mathcal{A})) = Q^{-1}\mathbf{K}(\mathcal{A}) = \mathbf{D}(\mathcal{A}).$$

Therefore we could have defined the derived category to be the localization $\widetilde{Q}^{-1}\,\mathbf{Ch}(\mathcal{A})$. However, in order to prove that $\widetilde{Q}^{-1}\,\mathbf{Ch}(\mathcal{A})$ exists we must first prove that $Q^{-1}\mathbf{K}(\mathcal{A})$ exists, by giving an explicit description of morphisms.

Definition 2.6.14. A collection S of morphisms in a category $\mathcal{C}$ is called a *multiplicative system* in $\mathcal{C}$ if it satisfies the following three self-dual axioms:

1. S is closed under composition (if $s, t \in S$ are composable, then $st \in S$) and contains all identity morphisms ($\mathrm{id}_X \in S$ for all objects X in $\mathcal{C}$).
2. (Ore condition) If $t : Z \to Y$ is in S, then for every $g : X \to Y$ in $\mathcal{C}$ there is a commutative diagram "$gs = tf$" in $\mathcal{C}$ with s in S.

$$\begin{array}{ccc} W & \xrightarrow{f} & Z \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} \\ X & \xrightarrow{g} & Y \end{array}.$$

(The slogan is "$t^{-1}g = fs^{-1}$ for some f and s".) Moreover, the symmetric statement (whose slogan is "$fs^{-1} = t^{-1}g$ for some t and g") is also valid.

3. (Cancellation) If $f, g : X \to Y$ are parallel morphisms in $\mathcal{C}$, then the following two conditions are equivalent:

   (a) $sf = sg$ for some $s \in S$ with source Y.

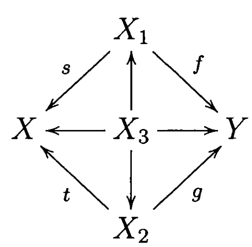
   (b) $ft = gt$ for some $t \in S$ with target X.

Example 2.6.15. The collection Q of quasi-isomorphisms is a multiplicative system in $\mathbf{K}(\mathcal{A})$.

Definition 2.6.16. Let $\mathcal{A}$ be a category and let S be a multiplicative system in $\mathcal{A}$. We call a chain in $\mathcal{A}$ of the form

$$fs^{-1} : X \xleftarrow{s} X_1 \xrightarrow{f} Y$$

a *(left) fraction* if s is in S. Call fs^{-1} *equivalent* to $X \xleftarrow{t} X_2 \xrightarrow{g} Y$ if there is a fraction $X \leftarrow X_3 \rightarrow Y$ fitting into a commutative diagram in $\mathcal{A}$:



Definition 2.6.17. A multiplicative system S is called *locally small* (on the left) if for each X there exists a set S_X of morphisms in S, all having target X, such that for every $X_1 \to X$ in S there is a map $X_2 \to X_1$ in $\mathcal{A}$ so that the composite $X_2 \to X_1 \to X$ in S_X.

Theorem 2.6.18 (Gabriel–Zisman [13]). *Let S be a locally small multiplicative system of morphisms in a category $\mathcal{C}$. Then the category $S^{-1}\mathcal{C}$ constructed above exists and is a localization of $\mathcal{C}$ with respect to S. The universal functor $q : \mathcal{C} \to S^{-1}\mathcal{C}$ sends $f : X \to Y$ to the sequence*

$$X \xleftarrow{1_X} X \xrightarrow{f} Y.$$

Now let $\mathcal{B} \subset \mathcal{C}$ be a full subcategory. Denote by $S \cap \mathcal{B}$ the class of morphisms of $\mathcal{B}$ lying in S. We say that $\mathcal{B}$ is *right cofinal* in $\mathcal{C}$ with respect to S, if for each morphism $s : X' \to X$ of S with $X' \in \mathcal{B}$, there is a morphism $m : X \to X''$ such that the composition ms belongs to $S \cap \mathcal{B}$. The *left* variant of this property is defined dually.

Lemma 2.6.19 ([33]). *Suppose $\mathcal{B}$ is right (respectively left) cofinal in $\mathcal{C}$ with respect to S. Then the class $S \cap \mathcal{B}$ admits a calculus of left (respectively right) fractions. If $\mathcal{B}$ is right (respectively left) cofinal in $\mathcal{C}$ with respect to S, then the canonical functor*

$$(S \cap \mathcal{B})^{-1}\mathcal{B} \to S^{-1}\mathcal{C}$$

is fully faithful.

Definition 2.6.20. A chain complex X is called *bounded* if almost all X_n are zero; if $X_n = 0$ unless $a \leqslant n \leqslant b$. A chain complex X called *bounded above* (respectively *bounded below*) if there is a bound b (respectively a) such that $X_n = 0$ for all $n > b$ (respectively $n < a$). The bounded (respectively bounded above, respectively bounded below) chain complexes form full subcategories of $\mathbf{Ch}(\mathcal{A})$ that are denoted $\mathbf{Ch}^b(\mathcal{A}), \mathbf{Ch}^-(\mathcal{A})$ and $\mathbf{Ch}^+(\mathcal{A})$ respectively.

We write $\mathbf{K}^b(\mathcal{A}), \mathbf{K}^-(\mathcal{A})$ and $\mathbf{K}^+(\mathcal{A})$ for the full subcategories of $\mathbf{K}(\mathcal{A})$ corresponding to the full subcategories $\mathbf{Ch}^b(\mathcal{A}), \mathbf{Ch}^-(\mathcal{A})$ and $\mathbf{Ch}^+(\mathcal{A})$ of bounded, bounded above, and bounded below chain complexes.

Example 2.6.21. By using Lemma 2.6.19, localizations of the full subcategories $\mathbf{K}^b(\mathcal{A}), \mathbf{K}^+(\mathcal{A})$ and $\mathbf{K}^-(\mathcal{A})$ of $\mathbf{K}(\mathcal{A})$ with respect to Q exist and are the full subcategories $\mathbf{D}^b(\mathcal{A}), \mathbf{D}^+(\mathcal{A})$, $\mathbf{D}^-(\mathcal{A})$ of $\mathbf{D}(\mathcal{A})$ whose objects are the chain complexes which are bounded, bounded below, and bounded above respectively (also see [52, 10.3.15]).

Theorem 2.6.22. $\mathbf{D}(\mathcal{A}), \mathbf{D}^b(\mathcal{A}), \mathbf{D}^+(\mathcal{A})$ *and* $\mathbf{D}^-(\mathcal{A})$ *are all triangulated categories, where triangles are those isomorphic to strict triangles (see 2.6.8).*

Corollary 2.6.23. *If I is a bounded below chain complex of injectives, then*

$$\mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(X, I) \cong \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(X, I)$$

for every X. Dually, if P is a bounded above cochain complex of projectives,

$$\mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(P, X) \cong \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(P, X).$$

Below we shall need the following useful fact:

Proposition 2.6.24. *The category of unbounded chain complexes* $\mathbf{Ch}(\mathcal{A})$ *of a Grothendieck category* $\mathcal{A}$ *is again a Grothendieck category.*

Sketch of proof. Colimits and limits are taken dimensionwise, filtered colimits are obviously exact. Following notation of Hovey [26], denote by $D^n X$, $n \in \mathbb{Z}$ and $X \in \mathcal{A}$, the complex which is X in degree n and $n-1$ and 0 elsewhere, with interesting differential being the identity map. If U is a generator of $\mathcal{A}$, then $\{D^n U\}_{n\in\mathbb{Z}}$ are generators of $\mathbf{Ch}(\mathcal{A})$. To see that these generate $\mathbf{Ch}(\mathcal{A})$, use the adjunction relation $\mathrm{Hom}_{\mathbf{Ch}(\mathcal{A})}(D^n U, X) \cong \mathrm{Hom}_{\mathcal{A}}(U, X_n)$.

Remark 2.6.25. This remark is to warn the reader that one should not confuse generators in abelian and triangulated categories. Precisely, we shall also work with the derived category $\mathbf{D}(\mathcal{A})$ of unbounded complexes of a Grothendieck category $\mathcal{A}$. Then generators for $\mathbf{D}(\mathcal{A})$ cannot be generators for $\mathbf{Ch}(\mathcal{A})$ and vice versa in general. Indeed, the generators $\{D^n U\}_{n\in\mathbb{Z}}$ of $\mathbf{Ch}(\mathcal{A})$ are contractible complexes, and hence zero in $\mathbf{D}(\mathcal{A})$.

On the other hand, suppose U is a generator for $\mathcal{A}$. Denote by $S^n U$, $n \in \mathbb{Z}$, the complex which is U in degree n and 0 elsewhere. Then $\{S^n U\}_{n\in\mathbb{Z}}$ is a family of generators for the derived category $\mathbf{D}(\mathcal{A})$ in the sense that for every non-zero object $X \in \mathbf{D}(\mathcal{A})$ there is a non-zero morphism in $\mathbf{D}(\mathcal{A})$ from some $S^n U$ to X. But these cannot generate $\mathbf{Ch}(A)$ as the following example shows.

Suppose K is a field, and

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & 0 & \longrightarrow & K & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \cdots \\
& \downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & 0 & \longrightarrow & K \oplus K & \xrightarrow{d_0} & K & \longrightarrow & 0 & \longrightarrow \cdots \\
& \downarrow & & \downarrow{\scriptstyle f} & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & 0 & \longrightarrow & K & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \cdots
\end{array}$$

is a commutative diagram in $\mathbf{Ch}(\mathrm{Mod}\,K)$, where $d_0(x,y) = f(x,y) = y$. We suppose the middle complex is concentrated in degrees 0 and -1. Clearly $\alpha(1) \in \mathrm{Ker}\, d_0$ implies $\alpha(1) = (x', 0)$ for some $x' \in K$. But $f\alpha(1) = f(x', 0) = 0$, so $f\alpha = 0$. Thus there is no non-zero map from $S^0 K$ to the middle complex such that the composite $f\alpha \neq 0$. Since there is no non-zero morphism from $S^n K$ to the middle complex for any $n \neq 0$, we see that $\{S^n K\}_{n\in\mathbb{Z}}$ are not generators for $\mathbf{Ch}(\mathrm{Mod}\, K)$.

Chapter 3

Enriched Category Theory

In an enriched category, the set of morphisms (the Hom-set) associated with every pair of objects is replaced by an object in some fixed monoidal category of "Hom-objects" in such a way that Hom-objects can be composed in the same fashion as Hom-sets in the usual category. Since enriched categories are of great utility in this project, we should collect some basic facts about them. We refer the reader to [4, 39] for details.

3.1 Enriched categories

Definition 3.1.1. A *monoidal* category $\mathcal{V}$ consists of the following data:

- ◇ a category $\mathcal{V}$;
- ◇ a bifunctor $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$, called the *tensor product*. We write $a \otimes b$ for the image under $\otimes$ of the pair (a, b);
- ◇ an object $e \in \mathcal{V}$, called the *unit*;
- ◇ for every triple a, b, c of objects, an associativity isomorphism

$$a_{abc} : (a \otimes b) \otimes c \to a \otimes (b \otimes c);$$

- ◇ for every object a, a left unit isomorphism

$$l_a : e \otimes a \to a;$$

◇ for every object a, a right unit isomorphism

$$r_a : a \otimes e \to a.$$

These data must satisfy the following requirements:

◇ the morphisms a_{abc} are natural in a, b, c;

◇ the morphisms l_a are natural in a;

◇ the morphisms r_a are natural in a;

◇ diagram (3.1) below is commutative for every quadruple of objects a, b, c, d (associativity coherence)

$$\begin{array}{ccc} ((a \otimes b) \otimes c) \otimes d & \xrightarrow{a_{a\otimes b,c,d}} & (a \otimes b) \otimes (c \otimes d) \\ {\scriptstyle a_{abc}\otimes 1}\downarrow & & \\ (a \otimes (b \otimes c)) \otimes d & & \downarrow{\scriptstyle a_{a,b,c\otimes d}} \\ {\scriptstyle a_{a,b\otimes c,d}}\downarrow & & \\ a \otimes ((b \otimes c) \otimes d) & \xrightarrow[1\otimes a_{bcd}]{} & a \otimes (b \otimes (c \otimes d)) \end{array} \tag{3.1}$$

◇ diagram (3.2) below is commutative for every pair a, b of objects (unit coherence)

$$\begin{array}{ccc} (a \otimes e) \otimes b & \xrightarrow{a_{aeb}} & a \otimes (e \otimes b) \\ & {\scriptstyle r_a\otimes 1}\searrow & \downarrow{\scriptstyle 1\otimes l_b} \\ & & a \otimes b \end{array} \tag{3.2}$$

Definition 3.1.2. With the notation of 3.1.1, a monoidal category is *symmetric* when, moreover, an isomorphism

$$s_{ab} : a \otimes b \to b \otimes a$$

is given for every pair a, b of objects. These isomorphisms must be such that:

◇ the morphisms s_{ab} are natural in a, b;

- ◇ diagram (3.3) below is commutative for every triple a, b, c of objects (associativity coherence);
- ◇ diagram (3.4) below is commutative for every object a (unit coherence);
- ◇ diagram (3.5) below is commutative (symmetry axiom) for every pair a, b of objects.

$$\begin{array}{ccc} (a \otimes b) \otimes c & \xrightarrow{s_{ab} \otimes 1} & (b \otimes a) \otimes c \\ {\scriptstyle a_{abc}} \downarrow & & \downarrow {\scriptstyle a_{bac}} \\ a \otimes (b \otimes c) & & b \otimes (a \otimes c) \\ {\scriptstyle s_{ab \otimes c}} \downarrow & & \downarrow {\scriptstyle 1 \otimes s_{ac}} \\ (b \otimes c) \otimes a & \xrightarrow[a_{bca}]{} & b \otimes (c \otimes a) \end{array} \tag{3.3}$$

$$\begin{array}{ccc} a \otimes e & \xrightarrow{s_{ae}} & e \otimes a \\ & {\scriptstyle r_a} \searrow & \downarrow {\scriptstyle 1 \otimes l_a} \\ & & a \end{array} \tag{3.4}$$

$$\begin{array}{ccc} a \otimes b & \xrightarrow{s_{ab}} & b \otimes a \\ & {\scriptstyle =} \searrow & \downarrow {\scriptstyle s_{ba}} \\ & & a \otimes b \end{array} \tag{3.5}$$

Definition 3.1.3. With the notation of 3.1.1, a monoidal category $\mathcal{V}$ is *biclosed* when, for each object $b \in \mathcal{V}$, both functors

$$- \otimes b : \mathcal{V} \to \mathcal{V}, \quad b \otimes - : \mathcal{V} \to \mathcal{V}$$

have a right adjoint. A biclosed symmetric monoidal category is called a *symmetric monoidal closed category*. The adjoint to the functor $- \otimes b$ will be denoted by $\underline{\mathrm{Hom}}(b, -)$ or $[b, -]$.

Definition 3.1.4. Let $\mathcal{V}$ be a closed symmetric monoidal category. A $\mathcal{V}$*-category* $\mathcal{C}$, or *a category enriched over* $\mathcal{V}$, consists of the following data:

1. a class $\mathrm{Ob}\,(\mathcal{C})$ of objects;
2. for every pair $a, b \in \mathrm{Ob}\,(\mathcal{C})$ of objects, an object $\mathcal{V}_{\mathcal{C}}(a, b)$ of $\mathcal{V}$;

3. for every triple $a, b, c \in \mathrm{Ob}\,(\mathcal{C})$ of objects, a composition morphism in $\mathcal{V}$,

$$c_{abc} : \mathcal{V}_{\mathcal{C}}(a,b) \otimes \mathcal{V}_{\mathcal{C}}(b,c) \to \mathcal{V}_{\mathcal{C}}(a,c);$$

4. for every object $a \in \mathcal{C}$, a unit morphism in $\mathcal{V}$,

$$u_a : e \to \mathcal{V}_{\mathcal{C}}(a,a).$$

These data must satisfy the following conditions:

- ◇ given objects $a, b, c, d \in \mathcal{C}$, diagram (3.6) below is commutative (associativity axiom);
- ◇ given objects $a, b \in \mathcal{C}$, diagram (3.7) below is commutative (unit axiom).

$$\begin{array}{ccc}
(\mathcal{V}_{\mathcal{C}}(a,b) \otimes \mathcal{V}_{\mathcal{C}}(b,c)) \otimes \mathcal{V}_{\mathcal{C}}(c,d) & \xrightarrow{c_{abc}\otimes 1} & \mathcal{V}_{\mathcal{C}}(a,c) \otimes \mathcal{V}_{\mathcal{C}}(c,d) \\
{\scriptstyle a_{\mathcal{V}_{\mathcal{C}}(a,b)\mathcal{V}_{\mathcal{C}}(b,c)\mathcal{V}_{\mathcal{C}}(c,d)}} \downarrow & & \\
\mathcal{V}_{\mathcal{C}}(a,b) \otimes (\mathcal{V}_{\mathcal{C}}(b,c) \otimes \mathcal{V}_{\mathcal{C}}(c,d)) & & \downarrow {\scriptstyle c_{acd}} \\
{\scriptstyle 1\otimes c_{bcd}} \downarrow & & \\
\mathcal{V}_{\mathcal{C}}(a,b) \otimes \mathcal{V}_{\mathcal{C}}(b,d) & \xrightarrow[c_{abd}]{} & \mathcal{V}_{\mathcal{C}}(a,d)
\end{array} \tag{3.6}$$

$$\begin{array}{ccccc}
e \otimes \mathcal{V}_{\mathcal{C}}(a,b) & \xrightarrow{l_{\mathcal{V}_{\mathcal{C}}(a,b)}} & \mathcal{V}_{\mathcal{C}}(a,b) & \xleftarrow{r_{\mathcal{V}_{\mathcal{C}}(a,b)}} & \mathcal{V}_{\mathcal{C}}(a,b) \otimes e \\
{\scriptstyle u_a\otimes 1} \downarrow & & \downarrow {\scriptstyle 1_{\mathcal{V}_{\mathcal{C}}(a,b)}} & & \downarrow {\scriptstyle 1\otimes u_b} \\
\mathcal{V}_{\mathcal{C}}(a,a) \otimes \mathcal{V}_{\mathcal{C}}(a,b) & \xrightarrow{c_{aab}} & \mathcal{V}_{\mathcal{C}}(a,b) & \xleftarrow{c_{abb}} & \mathcal{V}_{\mathcal{C}}(a,b) \otimes \mathcal{V}_{\mathcal{C}}(b,b)
\end{array} \tag{3.7}$$

When $\mathrm{Ob}\,\mathcal{C}$ is a set, the $\mathcal{V}$-category $\mathcal{C}$ is called a *small $\mathcal{V}$-category.*

Definition 3.1.5. Let $\mathcal{V}$ be a monoidal category. Given $\mathcal{V}$-categories $\mathcal{A}, \mathcal{B}$, a *$\mathcal{V}$-functor* or an *enriched functor* $F : \mathcal{A} \to \mathcal{B}$ consists in giving:

1. for every object $a \in \mathcal{A}$, an object $F(a) \in \mathcal{B}$;

2. for every pair $a, b \in \mathcal{A}$ of objects, a morphism in $\mathcal{V}$,

$$F_{ab} : \mathcal{V}_{\mathcal{A}}(a, b) \to \mathcal{V}_{\mathcal{B}}(F(a), F(b))$$

in such a way that the following axioms hold:

- ◇ for all objects $a, a', a'' \in \mathcal{A}$, diagram (3.8) below commutes (composition axiom);
- ◇ for every object $a \in \mathcal{A}$, diagram (3.9) below commutes (unit axiom).

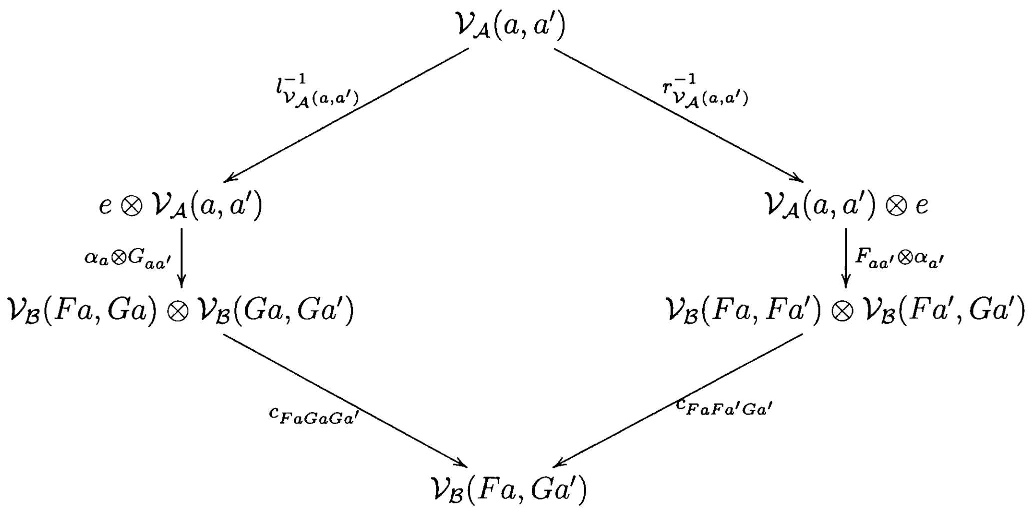
$$\begin{array}{ccc} \mathcal{V}_{\mathcal{A}}(a,a') \otimes \mathcal{V}_{\mathcal{A}}(a',a'') & \xrightarrow{c_{aa'a''}} & \mathcal{V}_{\mathcal{A}}(a,a'') \\ {\scriptstyle F_{aa'} \otimes F_{a'a''}} \downarrow & & \downarrow {\scriptstyle F_{aa''}} \\ \mathcal{V}_{\mathcal{B}}(Fa,Fa') \otimes \mathcal{V}_{\mathcal{B}}(Fa',Fa'') & \xrightarrow[c_{Fa,Fa',Fa''}]{} & \mathcal{V}_{\mathcal{B}}(Fa,Fa'') \end{array} \tag{3.8}$$

$$\begin{array}{ccc} e & \xrightarrow{u_a} & \mathcal{V}_{\mathcal{A}}(a,a) \\ & {\scriptstyle u_{Fa}} \searrow & \downarrow {\scriptstyle F_{aa}} \\ & & \mathcal{V}_{\mathcal{B}}(Fa,Fa) \end{array} \tag{3.9}$$

Definition 3.1.6. Let $\mathcal{V}$ be a monoidal category. Let $\mathcal{A}, \mathcal{B}$ be two $\mathcal{V}$-categories and $F, G : \mathcal{A} \to \mathcal{B}$ two $\mathcal{V}$-functors. A *$\mathcal{V}$-natural transformation* $\alpha : F \Rightarrow G$ consists in giving, for every object $a \in \mathcal{A}$, a morphism

$$\alpha_a : e \to \mathcal{V}_{\mathcal{B}}(F(a), G(a))$$

in $\mathcal{V}$ such that diagram (3.10) below commutes, for all objects $a, a' \in \mathcal{A}$.



(3.10)

We also observe that we can define the closed symmetric monoidal category **Set** as the category consisting of sets with arrows, the maps between them, and that categories in the usual sense are **Set**-categories (categories enriched over **Set**).

If $\mathcal{C}$ is a category, let $\mathbf{Set}_{\mathcal{C}}(a,b)$ denote the set of maps in $\mathcal{C}$ from a to b. A closed symmetric monoidal category $\mathcal{V}$ is a $\mathcal{V}$-category due to its internal Hom-objects. Let $\mathcal{V}(a,b)$ denote the $\mathcal{V}$-object $\underline{\mathrm{Hom}}_{\mathcal{V}}(a,b)$ of maps in $\mathcal{V}$. Any $\mathcal{V}$-category $\mathcal{C}$ defines a **Set**-category $\mathcal{UC}$. Its class of objects is $\mathrm{Ob}\,\mathcal{C}$, the morphism sets are $\mathbf{Set}_{\mathcal{UC}}(a,b) = \mathbf{Set}_{\mathcal{V}}(e, \mathcal{V}_{\mathcal{C}}(a,b))$ (see [4, p. 316]).

Definition 3.1.7. A $\mathcal{V}$-category $\mathcal{C}$ is a *right $\mathcal{V}$-module* if there is a $\mathcal{V}$-functor $\mathrm{act} : \mathcal{C} \otimes \mathcal{V} \to \mathcal{C}$, denoted $(c, A) \mapsto c \oslash A$ and a $\mathcal{V}$-natural unit isomorphism $r_c : \mathrm{act}(c,e) \to c$ subject to the following conditions:

1. there are natural associativity isomorphisms $c \oslash (A \otimes B) \to (c \oslash A) \otimes B$;

2. the isomorphisms $c \oslash (e \otimes A) \rightrightarrows c \oslash A$ coincide.

A right $\mathcal{V}$-module is *closed* if there is a $\mathcal{V}$-functor

$$\mathrm{coact} : \mathcal{V}^{\mathrm{op}} \otimes \mathcal{C} \to \mathcal{C}$$

such that for all $A \in \mathrm{Ob}\,\mathcal{V}$, and $c \in \mathrm{Ob}\,\mathcal{C}$, the $\mathcal{V}$-functor $\mathrm{act}(-, A) : \mathcal{C} \to \mathcal{C}$ is left $\mathcal{V}$-adjoint to $\mathrm{coact}(A, -)$ and $\mathrm{act}(c, -) : \mathcal{V} \to \mathcal{C}$ is left $\mathcal{V}$-adjoint to $\mathcal{V}_{\mathcal{C}}(c, -)$.

3.2 Ends and Coends

In this section we introduce the notions of ends and coends. They play an important role in constructing a closed symmetric monoidal structure for the category of enriched functors. We follow the work of Mac Lane [39] in this section.

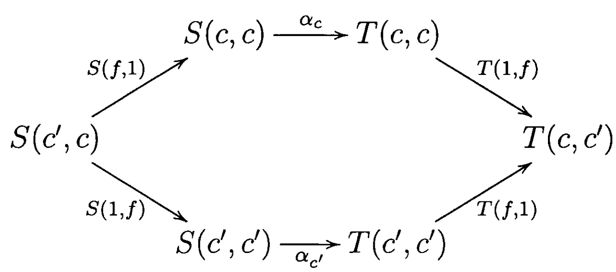
Definition 3.2.1. Let X, C be two categories. We define a *dinatural transformation*, $\alpha : S \to T$, between two functors S and T

$$S, T : C^{\mathrm{op}} \times C \to X$$

as a function which associates to every object $c \in C$ an arrow

$$\alpha_c : S(c, c) \to T(c, c)$$

of X and satisfies the coherence axiom, i.e. that the following diagram commutes for all arrows $f : c \to c'$ in C



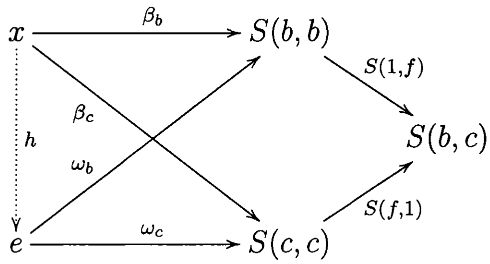
An *end* is a special type of limit, defined by the universal wedges in place of universal cones.

Definition 3.2.2. An *end* of a functor

$$S : C^{\mathrm{op}} \times C \to X$$

is a universal dinatural transformaton from a constant e to S. In other words, an end of S is a pair $< e, \omega >$, where e is an object of X and $\omega : e \to S$ is a dinatural transformation with the property that to every dinatural transformation $\beta : x \to S$ there is a unique arrow $h : x \to e$ of X with $\beta_a = \omega_a h$ for all $a \in C$.

Thus for each arrow $f : b \to c$ of C there is a diagram



(3.11)

such that both quadrilaterals commute (these are the dinatural conditions); the universal property of ω states that there is a unique h such that both triangles (at the left) commute.

The uniqueness property which applies to any universal states in this case that if $< e, \omega >$ and $< e', \omega' >$ are two ends for S, then there exists a unique isomorphism $u : e \to e'$ with $\omega' \cdot u = \omega$ (i.e., with $\omega'_c \cdot u = \omega_c$ for each $c \in C$). We call ω the *ending wedge* or the *universal wedge*, with components ω_c, while the object e itself, by abuse of language, is called the "end" of S and is written with the integral notation as

$$e = \int_c S(c, c) = \text{End of } S.$$

Note that the variable of integration c appears twice under the integral sign (once covariant and once contravariant) and is bound by the integral sign, in that the result no longer depends on c and so is unchanged if c is replaced by any other letter standing for an object of the category C. These properties are like those of the letter x is the usual integral

$$\int f(x)dx$$

of the calculus.

Natural transformations provide an example of ends. Two functors $U, V : C \to X$ define a functor $\mathrm{Hom}_X(U-, V-) : C^{\mathrm{op}} \times C \to \mathbf{Set}$, and if Y is any set, a wedge (=dinatural transformation) $\tau : Y \to \mathrm{Hom}_X(U-, V-)$, with components

$$\tau_c : Y \to \mathrm{Hom}_X(Uc, Vc), \quad c \in C,$$

assigns to each $y \in Y$ and to each $c \in C$ an arrow $\tau_{c,y} : Uc \to Vc$ of X such that for every arrow $f : b \to c$ one has the wedge coondition $Vf \cdot \tau_{b,y} = \tau_{c,y} \cdot Uf$. But this condition is just the commutativity of the square

$$\begin{array}{ccc} Ub & \xrightarrow{\tau} & Vb \\ {\scriptstyle Uf}\downarrow & & \downarrow{\scriptstyle Vf} \\ Uc & \xrightarrow[\tau]{} & Vc \end{array} \tag{3.12}$$

which asserts that $\tau_{-,y}$, for fixed y, is a natural transformation $\tau_{-,y} : U \to V$. Thus, if we write $Nat(U, V)$ for the set of all such natural transformations, the assignment $y \mapsto \tau_{-,y}$ is the unique function $Y \to Nat(U, V)$ which makes the following diagram commute.

$$\begin{array}{ccc} Y & \xrightarrow{\tau_c} & \mathrm{Hom}(Uc, Vc) \\ \vdots & & \downarrow{\scriptstyle 1} \\ Uc & \xrightarrow[\omega_c]{} & \mathrm{Hom}(Uc, Vc), \end{array}$$

where ω_c assigns to each natural $\lambda : U \to V$ its component $\lambda_c : Uc \to Vc$. This states exactly that ω is a universal wedge. Hence

$$Nat(U, V) = \int_c \mathrm{Hom}(Uc, Vc), \quad U, V : C \to X.$$

The definition of the coend of a functor

$$S : C^{\mathrm{op}} \times C \to X$$

is dual to that of an end.

Definition 3.2.3. A *coend* of S is a pair, $< d, \zeta : S \to d >$, consisting of an object $d \in X$ and a dinatural transformation ζ (a wedge), universal among dinatural transformations from S to a constant. The object d (when it exists, unique up to isomorphism) will usually be written with an integral sign and with the bound variable c as superscript. Thus,

$$S(c, c) \xrightarrow{\zeta_c} \int^c S(c, c) = d.$$

The formal properties of coends are dual to those of ends.

Coends are familiar under other names. For example, the tensor product of modules over a ring R is a coend. Specifically, a ring R is a preadditive category with one object (which we call R again) and with arrows the elements $r \in R$, composition of arrows being their product in R. A left R-module B is an additive functor $R \to \mathsf{Ab}$ which sends the (one) object R to the abelian group B and each arrow r in R to the scalar multiplication $r_* : b \mapsto rb$ in B. Similarly, a right R-module A is an additive functor $R^{\mathrm{op}} \to \mathsf{Ab}$ (contravariant on R to Ab). If $\otimes$ is the usual tensor product in Ab, then $R \mapsto A \otimes B$ is a bifunctor $R^{\mathrm{op}} \times R \to \mathsf{Ab}$. Moreover, the coend

$$\int^R A \otimes B = A \otimes_R B$$

is exactly the usual tensor product over R. Indeed, a wedge ζ from the bifunctor $A \otimes B$ to an abelian group M is precisely a (single) morphism $\varrho : A \otimes B \to M$ of abelian groups such that the diagram

$$\begin{array}{ccc} A \otimes B & \xrightarrow{1_A \otimes r_*} & A \otimes B \\ {\scriptstyle r_* \otimes 1_B}\downarrow & & \downarrow{\scriptstyle \varrho} \\ A \otimes B & \xrightarrow[\varrho]{} & M \end{array}$$

commutes for every arrow $r \in R$. With the above interpretation of modules as functors, this means for elements $a \in A$ and $b \in B$ that

$$\varrho(ar \otimes b) = \varrho(a \otimes rb).$$

We see that M is a coend precisely when M is $A \otimes B$ modulo all $ar \otimes b - a \otimes rb$, and this is precisely the usual despcription of the tensor product $M = A \otimes_R B$.

The point of these observations is not the reduction of the familiar to the unfamiliar (tensor products to coends) but the extension of the familiar to recover many more cases. If B is any monoidal category with monoidal product $\square$, then any two functors $T : P^{\mathrm{op}} \to B$ and $S : P \to B$ have a "tensor product"

$$T \square_P S = \int^P (Tp) \square (Sp),$$

an object of B.

By abusing language, we can often refer to the end of the functor S by the object e alone, and write

$$e = \int_c S(c,c) \text{ or just } \int_c S$$

We dually define the coend using the same approach by

$$e = \int^c S(c,c) \text{ or just } \int^c S.$$

If we consider a functor $F : C^{\mathrm{op}} \times C \to X$, where C is a category and assume our target category X for the functor F is complete, we can more formally describe this end as the equalizer in the diagram

$$\int_c F(c,c) \to \prod_{c \in C} F(c,c) \rightrightarrows \prod_{c \to c'} F(c',c).$$

There is again a dual definition for coends as a coequalizer of the dual diagram.

3.3 Categories of enriched functors

In this section we mostly follow Borceux [4] and Dundas–Röndigs–Østvær [10].

If $\mathcal{C}$ is a small $\mathcal{V}$-category, $\mathcal{V}$-functors from $\mathcal{C}$ to $\mathcal{V}$ and their $\mathcal{V}$-natural transformations form the category $[\mathcal{C}, \mathcal{V}]$ of $\mathcal{V}$-functors from $\mathcal{C}$ to $\mathcal{V}$. If $\mathcal{V}$ is complete, then $[\mathcal{C}, \mathcal{V}]$ is also a $\mathcal{V}$-category. We also denote this $\mathcal{V}$-category by $\mathcal{F}(\mathcal{C})$, or $\mathcal{F}$ if no confusion can arise. The morphism $\mathcal{V}$-object $\mathcal{V}_{\mathcal{F}}(X,Y)$ is the end

$$\int_{\operatorname{Ob}\mathcal{C}} \mathcal{V}(X(c), Y(c)). \tag{3.13}$$

Note that the underlying category $\mathcal{U}\mathcal{F}$ of the $\mathcal{V}$-category $\mathcal{F}$ is $[\mathcal{C}, \mathcal{V}]$.

One can compare $\mathcal{F}$ with $\mathcal{C}$ and $\mathcal{V}$ as follows. Given $c \in \operatorname{Ob}\mathcal{C}$, $X \mapsto X(c)$ defines the $\mathcal{V}$-functor $\mathrm{Ev}_c : \mathcal{F} \to \mathcal{V}$ called *evaluation at c*. The assignment $c \mapsto \mathcal{V}_{\mathcal{C}}(c,-)$ from $\mathcal{C}$ to $\mathcal{F}$ is again a $\mathcal{V}$-functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{F}$, called the *$\mathcal{V}$-Yoneda embedding*. $\mathcal{V}_{\mathcal{C}}(c,-)$ is a representable functor, represented by c.

Lemma 3.3.1 (Enriched Yoneda Lemma). *Let $\mathcal{V}$ be a complete closed symmetric monoidal category and $\mathcal{C}$ a small $\mathcal{V}$-category. For every $\mathcal{V}$-functor $X : \mathcal{C} \to \mathcal{V}$ and every $c \in \operatorname{Ob}\mathcal{C}$, there is a $\mathcal{V}$-natural isomorphism $X(c) \cong \mathcal{V}_{\mathcal{F}}(\mathcal{V}_{\mathcal{C}}(c,-), X)$.*

The isomorphism above is called the *Yoneda isomorphism.* It follows that every $\mathcal{V}$-functor can be expressed as a colimit of representable functors.

Lemma 3.3.2. *If $\mathcal{V}$ is a bicomplete closed symmetric monoidal category and $\mathcal{C}$ is a small $\mathcal{V}$-category, then $[\mathcal{C},\mathcal{V}]$ is bicomplete. (Co)limits are formed pointwise.*

Corollary 3.3.3. *Assume $\mathcal{V}$ is bicomplete, and let $\mathcal{C}$ be a small $\mathcal{V}$-category. Then any $\mathcal{V}$-functor $X : \mathcal{C} \to \mathcal{V}$ is $\mathcal{V}$-naturally isomorphic to the coend*

$$X \cong \int^{\operatorname{Ob}\mathcal{C}} \mathcal{V}_{\mathcal{C}}(c,-) \otimes X(c).$$

Let $(\mathcal{C},\diamond,u)$ be a small symmetric monoidal $\mathcal{V}$-category, where $\mathcal{V}$ is bicomplete. The *monoidal product* $\mathcal{C}\otimes\mathcal{C}$ is the $\mathcal{V}$-category, where

$$\operatorname{Ob}(\mathcal{C}\otimes\mathcal{C}) := \operatorname{Ob}\mathcal{C} \times \operatorname{Ob}\mathcal{C}$$

and

$$\mathcal{V}_{\mathcal{C}\otimes\mathcal{C}}((a,x),(b,y)) := \mathcal{V}_{\mathcal{C}}(a,b) \otimes \mathcal{V}_{\mathcal{C}}(x,y).$$

In [9], a closed symmetric monoidal product was constructed on the category $[\mathcal{C},\mathcal{V}]$ of $\mathcal{V}$-functors from $\mathcal{C}$ to $\mathcal{V}$. For $X, Y \in \operatorname{Ob}[\mathcal{C},\mathcal{V}]$, the monoidal product $X \odot Y \in \operatorname{Ob}[\mathcal{C},\mathcal{V}]$ is the coend

$$X \odot Y := \int^{\operatorname{Ob}(\mathcal{C}\otimes\mathcal{C})} \mathcal{V}_{\mathcal{C}}(c\diamond d,-) \otimes (X(c)\otimes Y(d)) : \mathcal{C} \to \mathcal{V}. \tag{3.14}$$

The following theorem is due to Day [9] and plays an important role in our analysis.

Theorem 3.3.4 (Day [9]). *Let $(\mathcal{V},\otimes,e)$ be a bicomplete closed symmetric monoidal category and $(\mathcal{C},\diamond,u)$ a small symmetric monoidal $\mathcal{V}$-category. Then the category $([\mathcal{C},\mathcal{V}],\odot,\mathcal{V}_{\mathcal{C}}(u,-))$ is closed symmetric monoidal with respect to monoidal product* (3.14). *The internal* Hom*-functor in $[\mathcal{C},\mathcal{V}]$ is given by the end*

$$\mathcal{F}(X,Y)(c) = \mathcal{V}_{\mathcal{F}}(X, Y(c\diamond,-)) = \int_{d\in\operatorname{Ob}\mathcal{C}} \mathcal{V}(X(d), Y(c\diamond d)). \tag{3.15}$$

The next lemma computes the tensor product of representable $\mathcal{V}$-functors.

Lemma 3.3.5. *The tensor product of representable functors is again representable. Precisely, there is a natural isomorphism*

$$\mathcal{V}_{\mathcal{C}}(c,-) \odot \mathcal{V}_{\mathcal{C}}(d,-) \cong \mathcal{V}_{\mathcal{C}}(c \odot d, -).$$

Below we shall need the following

Proposition 3.3.6. *Let $\mathcal{V}$ be a symmetric monoidal closed category. If $\mathcal{A}$ is a $\mathcal{V}$-category and $F, G : \mathcal{A} \rightrightarrows \mathcal{V}$ are $\mathcal{V}$-functors, giving a $\mathcal{V}$-natural transformation $\alpha : F \Rightarrow G$ is equivalent to giving a family of morphism $\alpha_a : F(a) \to G(a)$ in $\mathcal{V}$, for $a \in \mathcal{A}$, in such a way that the following diagram commutes for all $a, a' \in \mathcal{A}$*

$$\begin{array}{ccc} \mathcal{V}_{\mathcal{A}}(a,a') & \xrightarrow{F_{aa'}} & [F(a),F(a')] \\ {\scriptstyle G_{aa'}}\downarrow & & \downarrow{\scriptstyle [1,\alpha_{a'}]} \\ [G(a),G(a')] & \xrightarrow[{[\alpha_a,1]}]{} & [F(a),G(a')] \end{array}$$

Proof. See [4, 6.2.8]. □

Corollary 3.3.7. *Let $\mathcal{V}$ be a symmetric monoidal closed category. If $\mathcal{A}$ is a $\mathcal{V}$-category and $F, G : \mathcal{A} \rightrightarrows \mathcal{V}$ are $\mathcal{V}$-functors, giving a $\mathcal{V}$-natural transformation $\alpha : F \Rightarrow G$ is equivalent to giving a family of morphism $\alpha_a : F(a) \to G(a)$ in $\mathcal{V}$, for $a \in \mathcal{A}$, in such a way that the following diagram commutes for all $a, a' \in \mathcal{A}$*

$$\begin{array}{ccc} \mathcal{V}_{\mathcal{A}}(a,a') \otimes F(a) & \xrightarrow{\eta_F} & F(a') \\ {\scriptstyle 1\otimes\alpha_a}\downarrow & & \downarrow{\scriptstyle \alpha_{a'}} \\ \mathcal{V}_{\mathcal{A}}(a,a') \otimes G(a) & \xrightarrow[\eta_G]{} & G(a'), \end{array}$$

where η_F, η_G are the maps corresponding to the structure maps $F_{aa'}$ and $G_{aa'}$ respectively.

Corollary 3.3.8. *Let $\mathcal{V}$ be a symmetric monoidal closed category, $\mathcal{A}$ a small $\mathcal{V}$-category and $F, G : \mathcal{A} \rightrightarrows \mathcal{V}$ be $\mathcal{V}$-functors. Suppose $\alpha : F \Rightarrow G$ is a $\mathcal{V}$-natural transformation such that each $\alpha_a : F(a) \to G(a)$, $a \in \operatorname{Ob}\mathcal{A}$, is an isomorphism in $\mathcal{V}$. Then α is an isomorphism in $[\mathcal{A}, \mathcal{V}]$.*

Proof. This follows from the preceding corollary if we define $\alpha^{-1} : G \Rightarrow F$ by the collection of arrows α_a^{-1}, $a \in \operatorname{Ob}\mathcal{A}$. □

Chapter 4

Model Categories

A model category (sometimes called a Quillen model category or a closed model category) is a context for doing homotopy theory. Quillen [47] developed the definition of a model category to formalize the similarities between homotopy theory and homological algebra: the key examples which motivated his definition were the category of topological spaces, the category of simplicial sets, and the category of chain complexes. In recent decades, the language of model categories has been used in some parts of algebraic K-theory and algebraic geometry, where homotopy-theoretic approaches led to deep results (see, for example, [32, 41, 51]).

In this project we deal with categories enriched over a closed symmetric monoidal Grothendieck category $\mathcal{V}$. More precisely, we shall prove that the category of enriched functors

$$[\mathcal{C}, \mathcal{V}],$$

where $\mathcal{C}$ is a $\mathcal{V}$-category, is Grothendieck. Moreover, if $\mathcal{V}$ is a reasonable model category, then so is $[\mathcal{C}, \mathcal{V}]$. The latter is a monoidal model category whenever $\mathcal{V}$ is. So Grothendieck categories of enriched functors can have a rich homotopy theory in the sense of Quillen. To show this, we need to collect basic facts about model categories.

We mostly follow Dundas–Röndigs–Østvær [10], Hovey [26] and Schwede–Shipley [49] . All model structures we shall deal with in this project are cofibrantly generated.

4.1 Model structures

Definition 4.1.1. Suppose $\mathcal{C}$ is a category.

1. A map f in $\mathcal{C}$ is a *retract* of a map $g \in \mathcal{C}$ if f is a retract of g as objects of Map $\mathcal{C}$. That is, f is a retract of g if and only if there is a commutative diagram of the following form,

$$\begin{array}{ccccc} A & \longrightarrow & C & \longrightarrow & A \\ {\scriptstyle f}\big\downarrow & & {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ B & \longrightarrow & D & \longrightarrow & B \end{array}$$

   where the horizontal composites are identities.

2. A *functorial factorization* is an ordered pair (α, β) of functors Map $\mathcal{C} \to$ Map $\mathcal{C}$ such that $f = \beta(f) \circ \alpha(f)$ for all $f \in$ Map $\mathcal{C}$. In particular, the domain of $\alpha(f)$ is the domain of f, the codomain of $\alpha(f)$ is the domain of $\beta(f)$, and the codomain of $\beta(f)$ is the codomain of f.

Definition 4.1.2. Suppose $i : A \to B$ and $p : X \to Y$ are maps in a category $\mathcal{C}$. Then i has the *left lifting property with respect to* p and p has the *right lifting property with respect to* i if, for every commutative diagram of the following form,

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ B & \xrightarrow[g]{} & Y \end{array}$$

there is a lift $h : B \to X$ such that $hi = f$ and $ph = g$.

Definition 4.1.3. A model structure on a category $\mathcal{C}$ is three subcategories of $\mathcal{C}$ called *weak equivalence, cofibratons*, and *fibrations*, and two factorizations (α, β) and (γ, δ) satisfying the following properties:

1. (Two-out-of-three) If f and g are morphisms of $\mathcal{C}$ such that $g \circ f$ is defined and two of f,g and $g \circ f$ are weak equivalence, then so is the third.

2. (Retracts) If f and g are morphisms of $\mathcal{C}$ such that f is retract of g and g is a weak equivalence, cofibration, or fibration, then so is f.

3. (Lifting) Define a map to be a *trivial cofibration* if it is both a cofibration and a weak equivalence. Similarly, define a map to be a *trivial fibration* if it is both a fibration and a weak equivalence. Then trivial cofibrations have the left lifting property with respect to fibrations, and cofibrations have the left lifting property with respect to trivial fibrations.

4. (Factorization) For any morphism f, $\alpha(f)$ is a cofibrations, $\beta(f)$ is a trivial fibration $\gamma(f)$ is a trivial cofibration, and $\delta(f)$ is a fibration.

Definition 4.1.4. A *model category* is a category $\mathcal{C}$ with all limits and colimits together with a model structure on $\mathcal{C}$.

Example 4.1.5. Suppose $\mathcal{C}$ is a category with all small colimits and limits. We can put three different model structures on $\mathcal{C}$ by choosing one of the distinguished subcategories to be the isomorphisms and the other two to be all maps of $\mathcal{C}$. There are obvious choices for the functorial factorizations, and this gives a model structure on $\mathcal{C}$. For example, we could define a map to be a weak equivalence if and only if it is an isomorphism, and define every map to be both a cofibration and a fibration. In this case, we define the functors α and δ to be the identity functor, and define $\beta(f)$ to be the identity of codomain of f and $\gamma(f)$ to be the identity of the domain of f.

Example 4.1.6. Suppose $\mathcal{A}$ is a Grothendieck category. Let $\mathbf{Ch}\,\mathcal{A}$ be the category of unbounded chain complexes on $\mathcal{A}$. The injective model structure on $\mathbf{Ch}\,\mathcal{A}$ was first constructed by Joyal [31], and written down by Beck [2]. Its homotopy category is the derived category of $\mathcal{A}$. In this model structure the cofibrations are the monomorphisms, the weak equivalences are the quasi-isomorphisms, and the fibrations are certain epimorphims.

4.2 The homotopy category

Definition 4.2.1. Suppose $\mathcal{C}$ is a category with a subcategory of weak equivalences $\mathcal{W}$. Define the *homotopy* category $\operatorname{Ho}\mathcal{C}$ as follows. Form the free category

$F(\mathcal{C}, \mathcal{W}^{-1})$ on the arrows of $\mathcal{C}$ and the reversals of the arrows of $\mathcal{W}$. An object of $F(\mathcal{C}, \mathcal{W}^{-1})$ is an object of $\mathcal{C}$, and a morphism is a finite string of composable arrows $(f_1, f_2, ..., f_n)$ where f_i is either an arrow of $\mathcal{C}$ or the reversal w_i^{-1} of an arrow w_i of $\mathcal{W}$. The empty string at a particular object is the identity at that object, and composition is defined by concatenation of strings. Now, define $\mathrm{Ho}\,\mathcal{C}$ to be quotient category of $F(\mathcal{C}, \mathcal{W}^{-1})$ by the relations $1_A = (1_A)$ for all objects A, $(f, g) = (f \circ g)$ for all composable arrows f, g of $\mathcal{C}$, and $1_{\text{dom } w} = (w, w^{-1})$ and $1_{\text{codom } w} = (w^{-1}, w)$ for all $w \in \mathcal{W}$. Here dom w is the domain of w and codom w is the codomain of w.

Lemma 4.2.2. *Suppose $\mathcal{C}$ is a category with a subcategory $\mathcal{W}$.*

1. *If $F : \mathcal{C} \to \mathcal{D}$ is a functor that sends maps of $\mathcal{W}$ to isomorphisms, then there is a unique functor $\mathrm{Ho}\,\mathcal{C} \to \mathcal{D}$ such that $(\mathrm{Ho}\,F) \circ \gamma = F$.*

2. *Suppose $\delta : \mathcal{C} \to \mathcal{E}$ is a functor that takes maps of $\mathcal{W}$ to isomorphisms and enjoys the universal property of part (1). Then there is a unique isomorphism $\mathrm{Ho}\,\mathcal{C} \xrightarrow{F} \mathcal{E}$ such that $F\gamma = \delta$.*

3. *The correspondence of part (1) induces an isomorphism of categories between the category of functors $\mathrm{Ho}\,\mathcal{C} \to \mathcal{D}$ and natural transformations and the category of functors $\mathcal{C} \to \mathcal{D}$ that take maps of $\mathcal{W}$ to isomorphisms and natural transformations.*

Definition 4.2.3. Suppose $\mathcal{C}$ is a model category, and $f, g : B \to X$ are two maps in $\mathcal{C}$.

1. A *cylinder object* for B is a factorization of the fold map $\nabla : B \coprod B \to B$ into a cofibration $B \coprod B \xrightarrow{i_0 + i_1} B'$ followed by a weak equivalence $B' \xrightarrow{s} B$.

2. A *path object* for X is a factorization of the diagonal map $X \to X \times X$ into a weak equivalence $X \xrightarrow{r} X'$ followed by a fibration $X' \xrightarrow{(p_0, p_1)} X \times X$.

3. A *left homotopy* from f to g is a map $H : B' \to X$ for some cylinder object B' for B such that $Hi_0 = f$ and $Hi_1 = g$. We say that f and g are *left homotopic*, written $f \overset{l}{\sim} g$, if there is a left homotopy from f to g.

4. A *right homotopy* from f to g is a map $K : B \to X'$ for some path object X' for X such that $p_0 K = f$ and $p_1 K = g$. We say that f and g are *right homotopic*, written $f \overset{r}{\sim} g$, if there is a right homotopy from f to g.

5. We say that f and g are *homotopic*, written $f \sim g$, if they are both left and right homotopic.

6. f is a *homotopy equivalence* if there is a map $h : X \to B$ such that $hf \sim 1_B$ and $fh \sim 1_X$.

Proposition 4.2.4. *Suppose $\mathcal{C}$ is a model category, and $f, g : B \to X$ are two maps in $\mathcal{C}$.*

1. *If $f \overset{l}{\sim} g$ and $h : X \to Y$, then $hf \overset{l}{\sim} hg$. Dually, if $f \overset{r}{\sim} g$ and $h : A \to B$, then $fh \overset{r}{\sim} gh$.*

2. *If X is fibrant, $f \overset{l}{\sim} g$, and $h : A \to B$, then $fh \overset{l}{\sim} gh$. Dually, if B is cofibrant, $f \overset{r}{\sim} g$, and $h : X \to Y$, then $hf \overset{r}{\sim} hg$.*

3. *If B is cofibrant, then left homotopy is an equivalence relation on $\mathcal{C}(B, X)$. Dually, if X is fibrant, then right homotopy is an equivalence relation in $\mathcal{C}(B, X)$.*

4. *If B is cofibrant and $h : X \to Y$ is a trivial fibration or a weak equivalence of fibrant objects, then h induces an isomorphism*
$$\mathcal{C}(B, X)/ \overset{l}{\sim} \xrightarrow{\cong} \mathcal{C}(B, Y)/ \overset{l}{\sim} .$$
Dually, if X is fibrant and $h : A \to B$ is a trivial cofibration or weak equivalence of cofibrant object, then h induces an isomorphism
$$\mathcal{C}(B, X)/ \overset{r}{\sim} \xrightarrow{\cong} \mathcal{C}(A, X)/ \overset{r}{\sim} .$$

5. *If B is cofibrant, then $f \overset{l}{\sim} g$ implies $f \overset{r}{\sim} g$. Furthermore, if X' is any path object for X, there is a right homotopy $K : B \to X'$ from f to g. Dually, if X is fibrant, then $f \overset{r}{\sim} g$ implies $f \overset{l}{\sim} g$, and there is a left homotopy from f to g using any cylinder object for B.*

Proposition 4.2.5. *Suppose $\mathcal{C}$ is a model category. Let $\mathcal{C}_c$ (respectively $\mathcal{C}_f, \mathcal{C}_{cf}$) denote the full subcategory of cofibrant (respectively fibrant, cofibrant and fibrant) objects of $\mathcal{C}$. Then the inclusion functors induce an equivalence of categories*

$$\operatorname{Ho}\mathcal{C}_{cf} \to \operatorname{Ho}\mathcal{C}_c \to \operatorname{Ho}\mathcal{C}$$

and

$$\operatorname{Ho}\mathcal{C}_{cf} \to \operatorname{Ho}\mathcal{C}_f \to \operatorname{Ho}\mathcal{C}.$$

Theorem 4.2.6. *Suppose $\mathcal{C}$ is a model category. Let $\gamma : \mathcal{C} \to \operatorname{Ho}\mathcal{C}$ denote the canonical functor, and let Q denote the cofibrant replacement functor of $\mathcal{C}$ and R denote the fibrant replacement functor.*

1. *The inclusion $\mathcal{C}_{cf} \to \mathcal{C}$ induces an equivalence of categories*

$$\mathcal{C}_{cf}/\sim \xrightarrow{\cong} \operatorname{Ho}\mathcal{C}_{cf} \longrightarrow \operatorname{Ho}\mathcal{C}\ .$$

2. *There are natural isomorphisms*

$$\mathcal{C}(QRX, QRY)/\sim \cong \operatorname{Ho}\mathcal{C}(\gamma X, \gamma Y) \cong \mathcal{C}(RQX, RQY)/\sim\ .$$

   In addition, there is a natural isomorphism $\operatorname{Ho}\mathcal{C}(\gamma X, \gamma Y) \cong \mathcal{C}(QX, RY)/\sim$, and, if X is cofibrant and Y is fibrant, there is a natural isomorphism $\operatorname{Ho}\mathcal{C}(\gamma X, \gamma Y) \cong \mathcal{C}(X, Y)/\sim$. in particular, $\operatorname{Ho}\mathcal{C}$ is a category without moving to a higher universe.

3. *The functor $\gamma : \mathcal{C} \to \operatorname{Ho}\mathcal{C}$ identifies left or right homotopic maps.*

4. *If $f : A \to B$ is a map in $\mathcal{C}$ such that γf is isomorphic in $\operatorname{Ho}\mathcal{C}$, then f is a weak equivalence.*

Example 4.2.7. Suppose R is a ring, and let $\mathbf{Ch}(\operatorname{Mod} R)$ denote the category of unbounded chain complexes of R-modules. The projective model structure on $\mathbf{Ch}(\operatorname{Mod} R)$ is written down in [26, Section 2.3] (see also Definition 4.3.15). Its homotopy category is also the derived category of R-modules.

4.3 Cofibrantly generated model categories

Definition 4.3.1. Suppose $\mathcal{C}$ is a category with all small colimits, and λ is an ordinal. A *λ-sequence* in $\mathcal{C}$ is a colimit-preserving functor $X : \lambda \to \mathcal{C}$, commonly written as

$$X_0 \to X_1 \to \cdots \to X_\beta \to \cdots$$

Since X preserves colimits, for all limit ordinals $\gamma < \lambda$, the induced map

$$\operatorname{colim}_{\beta<\gamma} X_\beta \to X_\gamma$$

is an isomorphism. We refer to the map $X_0 \to \operatorname{colim}_{\beta<\lambda} X_\beta$ as the *composition* of the λ-sequence, though actually the composition is not unique, but only unique up to isomorphism under X, since the colimit is not unique. If $\mathcal{D}$ is a collection of morphisms of $\mathcal{C}$ and every map $X_\beta \to X_{\beta+1}$ for $\beta + 1 < \lambda$ is in $\mathcal{D}$, we refer to the composition $X_0 \to \operatorname{colim}_{\beta<\lambda} X_\beta$ as a *transfinite composition* of maps of $\mathcal{D}$.

Definition 4.3.2. Let γ be a cardinal. An ordinal α is *γ-filtered* if it is a limit ordinal and, if $A \subseteq \alpha$ and $|A| \leqslant \gamma$, then $\sup A < \alpha$.

Definition 4.3.3. Suppose $\mathcal{C}$ is a category with all small colimits, $\mathcal{D}$ is a collection of morphisms of $\mathcal{C}$, A is an object of $\mathcal{C}$ and $\varkappa$ is a cardinal. We say that A is *$\varkappa$-small relative to $\mathcal{D}$* if, for all $\varkappa$-filtered ordinals λ and all λ-sequences

$$X_0 \to X_1 \to \cdots \to X_\beta \to \cdots$$

such that each map $X_\beta \to X_{\beta+1}$ is in $\mathcal{D}$ for $\beta + 1 < \lambda$, the map of sets

$$\operatorname{colim}_{\beta<\lambda} \mathcal{C}(A, X_\beta) \to \mathcal{C}(A, \operatorname{colim}_{\beta<\lambda} X_\beta)$$

is an isomorphism. We say that A is *small relative to $\mathcal{D}$* if it is $\varkappa$-small relative to $\mathcal{D}$ for some $\varkappa$. We say that A is *small* if it is small relative to $\mathcal{C}$ itself.

Definition 4.3.4. Suppose $\mathcal{C}$ is a category with all small colimits, $\mathcal{D}$ is a collection of morphisms of $\mathcal{C}$, and A is an object of $\mathcal{C}$. We say that A is *finite relative to $\mathcal{D}$* if A is $\varkappa$-small relative to $\mathcal{D}$ for finite cardinals $\varkappa$. We say A is *finite* if it is finite relative to $\mathcal{C}$ itself. In this case, maps from A commute with colimits of arbitrary λ-sequences, as long as λ is a limit ordinal.

Definition 4.3.5. Let I be a class of maps in a category $\mathcal{C}$.

1. A map is *I-injective* if it has the right lifting property with respect to every map in I. The class of I-injective maps is denoted I-inj.

2. A map is *I-projective* if it has the left lifting property with respect to every map in I. The class of I-projective maps is denoted I-proj.

3. A map is an *I-cofibration* if it has the left lifting property with respect to every I-injective map. The class of I-cofibrations is the class (I-inj)-proj and is denoted I-cof.

4. A map is an *I-fibration* if it has the right lifting property with respect to every I-projective map. The class of I-fibrations is the class (I-proj)-inj and is denoted I-fib.

Definition 4.3.6. Let I be a set of maps in a category $\mathcal{C}$ containing all small colimits. A *relative I-cell complex* is a transfinite composition of pushouts of elements of I. That is, if $f : A \to B$ is a relative I-cell complex, then there is an ordinal λ and λ-sequence $X : \lambda \to \mathcal{C}$ such that f is the composition of X and such that, for each β such that $\beta + 1 < \lambda$, there is a pushout square as follows,

$$\begin{array}{ccc} C_\beta & \longrightarrow & X_\beta \\ {\scriptstyle g_\beta}\downarrow & & \downarrow \\ D_\beta & \longrightarrow & X_{\beta+1} \end{array}$$

such that $g_\beta \in I$. We denote the collection of relative I-cell complexes by I-cell. We say that $A \in \mathcal{C}$ is an *I-cell complex* if the map $0 \to A$ is a relative I-cell complex.

Definition 4.3.7. Suppose $\mathcal{C}$ is a model category. We say that $\mathcal{C}$ is *cofibrantly generated* if there are sets I and J of maps such that:

1. the domains of the maps of I are small relative to I-cell;

2. the domains of the maps of J are small relative to J-cell;

3. the class of fibrations is J-inj; and

4. the class of trivial fibrations is I-inj.

We refer to I as the set of *generating cofibrations*, and to J as the set of *generating trivial cofibrations*. A cofibrantly generated model category $\mathcal{C}$ is called *finitely generated* if we can choose the sets I and J so that the domains and codomains of I and J are finite relative to I-cell.

Theorem 4.3.8. *Suppose $\mathcal{C}$ is a category with all small colimits and limits. Suppose $\mathcal{W}$ is a subcategory of $\mathcal{C}$, and I and J are sets of maps of $\mathcal{C}$. Then there is a cofibrantly generated model structure on $\mathcal{C}$ with I as the set of generating cofibrations, J as the set of generating acyclic cofibrations, and $\mathcal{W}$ as the subcategory of weak equivalences if and only if the following conditions are satisfied.*

1. *The subcategory of $\mathcal{W}$ has the two out of three property and is closed under retracts.*
2. *The domains of I are small relative to I-cell.*
3. *The domains of J are small relative to J-cell.*
4. *J-cell $\subseteq \mathcal{W} \cap I$-cof.*
5. *I-inj $\subseteq \mathcal{W} \cap J$-inj .*
6. *Either $\mathcal{W} \cap I$-cof $\subseteq J$-cof or $\mathcal{W} \cap J$-inj $\subseteq I$-inj .*

Definition 4.3.9. A ring R is a *Frobenius ring* if the projective and injective left or right R-modules coincide.

Example 4.3.10. Every group ring of a finite group over a field is a Frobenius ring.

Definition 4.3.11. Suppose R is a ring. Given maps $f, g : M \to N$ of R-modules, define f to be *stably equivalent* to g, written $f \sim g$, if $f - g$ factors through a projective module.

Definition 4.3.12. Let R be a ring. The *stable category of R-modules* is the category whose objects are left R-modules and whose morphisms are stable equivalence classes of R-module maps. A map f of R-modules is a *stable equivalence* if it is an isomorphism in this category.

Definition 4.3.13. Suppose R is a Frobenius ring. Let I denote the set of inclusions $\mathfrak{a} \to R$, where $\mathfrak{a}$ is a left ideal in R. Let J denote the set consisting of the inclusion $0 \to R$. Define a map of R-modules to be a *fibration* if it has the right lifting property with respect ot J, and define f to be a *cofibration* if $f \in I$-cof.

Theorem 4.3.14. *Suppose R is a Frobenius ring. Then there is a cofibrantly generated model structure on* $\mathrm{Mod}\, R$ *where the cofibrations are injections, the fibrations are the surjections, and the weak equivalences are the stable equivalences. The model structure is finitely generated.*

Definition 4.3.15. Let R be a ring. The projective model category structure on chain complexes is defined as follows. Given an R-module M, define $S^n(M) \in \mathbf{Ch}(\mathrm{Mod}\, R)$ by $S^n(M)_n = M$ and $S^n(M)_k = 0$ if $k \neq n$. Similarly, define $D^n(M)$ by $D^n(M)_k = M$ if $k = n$ or $k = n-1$, and 0 otherwise. The differential d_n in $D^n(M)$ is the identity. We often denote $S^n(R)$ by simply S^n, and $D^n(R)$ by D^n. There is an evident injection $S^{n-1}(M) \to D^n(M)$. Now define the set I to consist of the maps $S^{n-1} \to D^n$, and define the set J to consist of the maps $0 \to D^n$. Define a map to be *fibration* if it is in J-inj, and define a map to be a *cofibration* if it is in I-cof . Define a map f to be a *weak equivalence* if the induced map $H_n f$ on homology is an isomorphism for all n.

Lemma 4.3.16. *Suppose R is a ring. If A is a cofibrant chain complex, then A_n is a projective R-module for all n. Conversely, any bounded below complex of projective R-modules is cofibrant.*

Proposition 4.3.17. *Suppose R is a ring. Then a map $i : A \to B$ in* $\mathbf{Ch}(\mathrm{Mod}\, R)$ *is a cofibration if and only if i is a dimensionwise split inclusion with cofibrant cokernel.*

Theorem 4.3.18. *The projective model structure on* $\mathbf{Ch}(\mathrm{Mod}\, R)$ *is a finitely generated model category with I as its generating set of cofibrations, J as its generating set of trivial cofibrations, and quasi-isomorphisms as its weak equivalences. The fibrations are surjections.*

4.4 Monoidal model categories

Definition 4.4.1. Given model categories $\mathcal{C}, \mathcal{D}$ and $\mathcal{E}$, an adjunction of two variables $(\otimes, \mathrm{Hom}_r\, \mathrm{Hom}_l, \varphi_r, \varphi_l) : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ is called a *Quillen adjunction of two variables*, if, given a cofibration $f : U \to V$ in $\mathcal{C}$ and a cofibration $g : W \to X$ in $\mathcal{D}$, the induced map

$$f \square g : P(f,g) = (V \otimes W) \amalg_{U \otimes W} (U \otimes X) \to V \otimes X$$

is a cofibration in $\mathcal{E}$ that is trivial if either f or g is. We refer to the left adjoint F of a Quillen adjunction of two variables as a *Quillen bifunctor*, and often abuse notation by using the term "Quillen bifunctor $\otimes$" when we really mean "Quillen adjunction of two variables $(\otimes, \mathrm{Hom}_r, \mathrm{Hom}_l, \varphi_r, \varphi_l)$."

The map $f \square g$ occurring in Definition 4.4.1 is sometimes called the *pushout product* of f and g.

Proposition 4.4.2. *Suppose $\otimes : \mathcal{C} \otimes \mathcal{D} \to \mathcal{E}$ is an adjunction of two variables between model categories. Suppose as well that $\mathcal{C}$ and $\mathcal{D}$ are cofibrantly generated, with generating cofibratons I and I' respectively, and generating trivial cofibrations J and J' respectively. Then $\otimes$ is Quillen bifunctor if and only if $I \square I'$ consists of cofibrations and both $I \square J'$ and $J \square I'$ consists of trivial cofibrations.*

Definition 4.4.3. A *monoidal model category* is a closed category $\mathcal{C}$ with a model structure making $\mathcal{C}$ into a model category, such that the following conditions hold.

1. The monoidal structure $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a Quillen bifunctor.

2. Let $QS \xrightarrow{q} S$ be the cofibrant replacement for the unit S, obtained by using the functorial factorizations to factor $0 \to S$ into a cofibration followed by a trivial fibration. Then the natural map $QS \otimes X \xrightarrow{q \otimes 1} S \otimes X$ is a weak equivalence for all cofibrant X. Similarly, the natural map $X \otimes QS \xrightarrow{1 \otimes q} X \otimes S$ is a weak equivalence for all cofibrant X.

Note that this second condition is automatic if S is cofibrant.

Lemma 4.4.4. *Suppose $\otimes : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ is an adjunction of two variables, I is a set of maps in $\mathcal{C}$, I' is a set of maps in $\mathcal{D}$, and K is a set of maps in $\mathcal{E}$. Suppose as well that $I \square I' \subseteq K$. Then $(I\text{-cof}) \square (I'\text{-cof}) \subseteq K\text{-cof}$.*

Example 4.4.5. Let R be a commutative ring. Then $\mathbf{Ch}(\mathrm{Mod}\, R)$, the category of unbounded chain complexes of R-modules, given the projective model structure of Definition 4.3.15, is a symmetric monoidal model category.

First we recall that $\mathbf{Ch}(\mathrm{Mod}\, R)$ is indeed a closed symmetric monoidal category. Given chain complexes X and Y, we define

$$(X \otimes Y)_n = \bigoplus_k X_k \otimes_R Y_{n+k}, \tag{4.1}$$

where $d(x \otimes y) = dx \otimes y + (-1)^{|x|} x \otimes dy$. The unit is the complex S^0 consisting of R in the degree 0. The commutativity isomorphism is defined by $T(x \otimes y) = (-1)^{|x||y|} y \otimes x$ for the homogeneous elements x and y. To see that $\mathbf{Ch}(\mathrm{Mod}\, R)$ is in fact a closed symmetric monoidal category, we define

$$\mathrm{Hom}(X, Y)_n = \prod_k \mathrm{Hom}_R(X_k, Y_{n+k}), \tag{4.2}$$

with $(df)(x) = df(x) + (-1)^{n+1} f(dx)$ for $f \in \mathrm{Hom}_R(X_k, Y_{n+k})$.

As the unit S^0 is cofibrant, it suffices to verify that the tensor product is a Quillen bifunctor. Recall that the generating cofibrations are the maps $S^{n-1} \to D^n$, and the generating trivial cofibrations are the maps $0 \to D^n$. The pushout product of two generating cofibrations is an injection with bounded below dimensionwise projective cokernel. Hence by Lemma 4.3.16, the cokernel is cofibrant. Proposition 4.3.17 then implies that the pushout product of two generating cofibrations is a cofibration. Lemma 4.4.4 implies that the pushout product of any two cofibrations is cofibration. The pushout product of $S^{n-1} \to D^n$ and $0 \to D^m$ is the map $D^{m+n+1} \to D^m \otimes D^n$, which is a weak equivalence as required.

Example 4.4.6. Another class of examples arises from modular representation theory. We let k be a field and G a finite group; the interesting cases will be those where the characteristic of k does divide the order of G. The group algebra kG is a Frobenius ring, that is, the classes of its projective and injective modules coincide. The stable module category $\mathrm{Stmod}\, kG$ has as objects all (left, say) kG-modules, and the group of morphisms in $\mathrm{Stmod}\, kG$ is defined to be the quotient of the group of module homomorphisms by the subgroup of those homomorphisms which factor through a projective (equivalently, an injective) module; see for example [5, 7]. The stable module category is in fact the homotopy category associated to a model category structure on the category of all kG-modules; see Theorem 4.3.14. The cofibrations are the monomorphisms, the fibrations are the epimorphisms, and the weak equivalences are maps which become isomorphisms in the stable module category. This model category is quite special because every object is both fibrant and cofibrant. The above model category structure exists over any Frobenius ring, but for the group algebra kG there is a compatible monoidal structure. For two kG-modules M and N, the tensor product over the ground field $M \otimes_k N$ becomes a kG-module when endowed with the diagonal G-action. Similarly, the group $\mathrm{Hom}_k(M, N)$ of k-linear maps supports a G-action by conjugation. This data makes the category of kG-modules into a symmetric monoidal closed category with unit object the trivial module k.

Hovey [27] generalizes this model structure to kG-modules, where k is a principle ideal domain and G is finite. The model category is cofibrantly generated and is called the *projective model structure* of kG-modules.

Proposition 4.4.7. *Suppose $\mathcal{C}, \mathcal{D}$ and $\mathcal{E}$ are model categories, and*

$$(\otimes, \mathrm{Hom}_r, \mathrm{Hom}_l, \varphi_r, \varphi_l) : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$$

is a Quillen adjunction of two variables. Then the total derived functor defines an adjunction of two variables

$$(\otimes^L, R\,\mathrm{Hom}_r, R\,\mathrm{Hom}_l, R\varphi_r, R\varphi_l) : \mathrm{Ho}\,\mathcal{C} \times \mathrm{Ho}\,\mathcal{D} \to \mathrm{Ho}\,\mathcal{E}$$

Proof. See [26, Proposition 4.3.1]. □

Theorem 4.4.8. *Suppose $\mathcal{C}$ is a (symmetric) monoidal model category. Then* $\mathrm{Ho}\,\mathcal{C}$ *can be given the structure of a closed (symmetric) monoidal category. The adjunction of two variables $(\otimes^L, R\,\mathrm{Hom}_r, R\,\mathrm{Hom}_l)$ that is part of the closed structure of* $\mathrm{Ho}\,\mathcal{C}$ *is the total derived adjunction of $(\otimes, \mathrm{Hom}_r, \mathrm{Hom}_l)$. The associativity and unit isomorphisms (and the commutativity isomorphism in case $\mathcal{C}$ is symmetric on* $\mathrm{Ho}\,\mathcal{C}$*) are derived from the corresponding isomorphisms of $\mathcal{C}$.*

Proof. See [26, Theorem 4.3.2]. □

4.5 The monoid axiom

Definition 4.5.1. A monoidal model category $\mathcal{C}$ satisfies the *monoid axiom* if every map in

$$(\{\text{trivial cofibrations}\} \otimes \mathcal{C})\text{- cof}$$

is a weak equivalence.

Lemma 4.5.2. *Let $\mathcal{C}$ be a cofibrantly generated model category endowed with a closed symmetric monoidal structure.*

1. *If the pushout product axiom holds for a set of generating cofibrations and a set of generating trivial cofibrations, then it holds in general.*

2. *Let J be a set of generating trivial cofibration. If every map in $(J \otimes \mathcal{C})$-cof is a weak equivalence, then the monoid axiom holds.*

Proof. See [49]. □

Theorem 4.5.3 (Hovey [27]). *Suppose $R = k[G]$, where k is a principal ideal domain and G is a finite group. Then the projective model structure on R-modules satisfies the monoid axiom, and so there is an induced model structure on the category of monoids and the category of modules over a given monoid. Furthermore, a weak equivalence of monoids induces a Quillen equivalence of the corresponding module categories.*

4.6 Weakly finitely generated model categories

Definition 4.6.1. An object $a \in \mathrm{Ob}(\mathcal{C})$ is *finitely presentable* if the Hom-functor $\mathbf{Set}_{\mathcal{C}}(a, -)$ commutes with all filtered colimits. If $\mathcal{C}$ is a $\mathcal{V}$-category, $a \in \mathrm{Ob}(\mathcal{C})$ is *$\mathcal{V}$-finitely presentable* if the functor $\mathcal{V}_{\mathcal{C}}(a, -)$ commutes with all filtered colimits.

Definition 4.6.2. A cofibrantly generated model category $\mathcal{V}$ is *weakly finitely generated* if I and J can be chosen such that the following conditions hold:

1. the domains and codomains of the maps in I are finitely presentable;

2. domains of the maps J are small;

3. there exists a subset J' of J of maps with finitely presentable domains and codomains, such that map $f : A \to B$ in $\mathcal{V}$ with fibrant codomain B is a fibration if and only if it is contained in J'-inj.

Our pointwise notions of weak equivalence, fibrations and cofibrations are as follows.

Definition 4.6.3. A morphism $f \in [\mathcal{C}, \mathcal{V}]$ is a

- *pointwise weak equivalence* if $f(c)$ is a weak equivalence in $\mathcal{V}$ for all $c \in \mathcal{C}$;
- *pointwise fibration* if $f(c)$ is fibration in $\mathcal{V}$;
- *cofibration* if f has the left lifting property with respect to all pointwise trivial fibrations.

Theorem 4.6.4. *Let $\mathcal{V}$ be a weakly finitely generated monoidal model category, and let $\mathcal{C}$ be a small $\mathcal{V}$-category. Suppose the monoid axiom holds in $\mathcal{V}$. Then $[\mathcal{C}, \mathcal{V}]$, with the classes of maps in 4.6.3, is a weakly finitely generated model category.*

Proof. See [10, Theorem 4.2]. □

Theorem 4.6.5. *Consider $\mathcal{V}$ and $\mathcal{C}$ as in 4.6.4. Then the pointwise model structure gives $[\mathcal{C}, \mathcal{V}]$ the structure of a $\mathcal{V}$-model category. Likewise, $[\mathcal{C}, \mathcal{V}]$ is a monoidal $\mathcal{V}$-category provided $\mathcal{C}$ is a symmetric monoidal $\mathcal{V}$-category and the monoid axiom in the sense of Schwede–Shipley [49] holds.*

Proof. See [10, Theorem 4.4]. □

Chapter 5

Grothendieck categories of enriched functors and localizations

In this chapter we prove that the category of enriched functors $[\mathcal{C}, \mathcal{V}]$ is a Grothendieck category whenever $\mathcal{V}$ is a closed symmetric monoidal Grothendieck category, giving us new Grothendieck categories in practice. An advantage of this result is that we can recover some well-known theorems for Grothendieck categories in the case $\mathcal{V} = \mathsf{Ab}$. Another advantage is that $\mathcal{V}$ can also contain some rich homological or homotopical information, which is extended to the category of enriched functors $[\mathcal{C}, \mathcal{V}]$. This homotopical information will be of great utility in the next chapters to study monoidal structures for the derived category of generalized modules. Also, this result implies that localization theory of Grothendieck categories becomes available for $[\mathcal{C}, \mathcal{V}]$. Moreover, $[\mathcal{C}, \mathcal{V}]$ is closed symmetric monoidal whenever $\mathcal{C}$ is a symmetric monoidal $\mathcal{V}$-category.

5.1 Grothendieck categories of enriched functors

Before proving the main result we want to collect some important examples of closed symmetric monoidal Grothendieck categories.

Example 5.1.1. 1. Given any commutative ring R, the triple $(\mathrm{Mod}\, R, \otimes_R, R)$

is a closed symmetric monoidal Grothendieck category.

2. More generally, let X be a quasi-compact quasi-separated scheme. Consider the category $\mathrm{Qcoh}(\mathcal{O}_X)$ of quasi-coherent $\mathcal{O}_X$-modules. By [30, 3.1] $\mathrm{Qcoh}(\mathcal{O}_X)$ is a locally finitely presented Grothendieck category, where quasi-coherent $\mathcal{O}_X$-modules of finite type form a family of finitely presented generators. The tensor product on $\mathcal{O}_X$-modules preserves quasi-coherence, and induces a closed symmetric monoidal structure on $\mathrm{Qcoh}(\mathcal{O}_X)$.

3. Let R be any commutative ring. Let $C' = \{C'_n, \partial'_n\}$ and $C'' = \{C''_n, \partial''_n\}$ be two chain complexes of R-modules. Their tensor product $C' \otimes_R C'' = \{(C' \otimes_R C'')_n, \partial_n\}$ is the chain complex defined by

$$(C' \otimes_R C'')_n = \bigoplus_{i+j=n} (C'_i \otimes_R C''_j),$$

and

$$\partial_n(t'_i \otimes s''_j) = \partial'_i(t'_i) \otimes s''_j + (-1)^i t'_i \otimes \partial''_j(s''_j), \quad \text{for all } t'_i \in C'_i,\ s''_j \in C''_j,\ (i+j=n),$$

where $C'_i \otimes_R C''_j$ denotes the tensor product of R-modules C'_i and C''_j. Then the triple $(\mathbf{Ch}(\mathrm{Mod}\, R), \otimes_R, R)$ is a closed symmetric monoidal Grothendieck category (see Proposition 2.6.24). Here R is regarded as a complex concentrated in zeroth degree.

4. $(\mathrm{Mod}\, kG, \otimes_k, k)$ is a closed symmetric monoidal Grothendieck category, where k is a field and G is a finite group.

The main result of this Chapter is as follows.

Theorem 5.1.2. *Let $\mathcal{V}$ be a closed symmetric monoidal Grothendieck category with a set of generators $\{g_i\}_I$. If $\mathcal{C}$ is a small $\mathcal{V}$-category, then the category of enriched functors $[\mathcal{C}, \mathcal{V}]$ is a Grothendieck $\mathcal{V}$-category with the set of generators $\{\mathcal{V}(c, -) \oslash g_i \mid c \in \mathrm{Ob}\,\mathcal{C}, i \in I\}$. Moreover, if $\mathcal{C}$ is a small symmetric monoidal $\mathcal{V}$-category, then $[\mathcal{C}, \mathcal{V}]$ is closed symmetric monoidal with monoidal product and internal* Hom*-object computed by formulas of Day* (3.14) *and* (3.15).

Proof. If $\mathcal{C}$ is a small $\mathcal{V}$-category, then $[\mathcal{C},\mathcal{V}]$ is a $\mathcal{V}$-category by section 3.3. The internal Hom-object is given by (3.13). Let show that $[\mathcal{C},\mathcal{V}]$ is a preadditive category. Given $\mathcal{V}$-functors $X, Y \in [\mathcal{C},\mathcal{V}]$, we have that

$$\mathrm{Hom}_{[\mathcal{C},\mathcal{V}]}(X,Y) = \mathrm{Hom}_{\mathcal{V}}(e, \int_{c\in \mathrm{Ob}\,\mathcal{C}} \mathcal{V}(X(c),Y(c))$$

is an abelian group, because $\mathcal{V}$ is preadditive. We can also describe explicitly the abelian group structure as follows. The morphisms of $[\mathcal{C},\mathcal{V}]$ are, by definition, the $\mathcal{V}$-functors from $\mathcal{C}$ to $\mathcal{V}$. Using Corollary 3.3.7, for any $\mathcal{V}$-natural transformations

$$\alpha, \alpha' : X \to Y$$

its sum $\alpha + \alpha'$ is determined by the arrows

$$\alpha_c + \alpha'_c : e \to \mathcal{V}(X(c),Y(c)).$$

Recall that $\mathrm{Hom}_{\mathcal{V}}(X(c),Y(c)) = \mathrm{Hom}_{\mathcal{V}}(e, \mathcal{V}(X(c),Y(c)))$ and $\alpha_c + \alpha'_c$ is addition of α_c and α'_c in the abelian group $\mathrm{Hom}_{\mathcal{V}}(X(c),Y(c))$.

To show that the addition is bilinear, let

$$\begin{aligned}\beta \in \mathrm{Hom}_{\mathcal{V}}(e, \int_c \mathcal{V}(Y(c),Z(c)) &= \int_c \mathrm{Hom}_{\mathcal{V}}(e, \mathcal{V}(Y(c),Z(c)) \\ &= \int_c \mathrm{Hom}_{\mathcal{V}}(Y(c),Z(c)).\end{aligned}$$

Using Corollary 3.3.7, we set

$$(\beta\alpha)_c := \beta_c \circ \alpha_c.$$

Using the fact that $\mathcal{V}$ is preadditive, we have

$$\begin{aligned}(\beta(\alpha+\alpha'))_c &= \beta_c(\alpha_c + \alpha'_c) \\ &= \beta_c \circ \alpha_c + \beta_c \circ \alpha'_c \\ &= (\beta\alpha + \beta\alpha')_c.\end{aligned}$$

Similarly, $(\alpha+\alpha')\gamma = \alpha\gamma + \alpha'\gamma$. We see that $[\mathcal{C},\mathcal{V}]$ is preadditive.

Since $\mathcal{V}$ is a bicomplete closed symmetric monoidal category and $\mathcal{C}$ is a small $\mathcal{V}$-category, then by Lemma 3.3.2 the category $[\mathcal{C},\mathcal{V}]$ is bicomplete. Moreover, limits and colimits are formed objectwise. In particular, $[\mathcal{C},\mathcal{V}]$ has finite products. It follows from [39, VIII.2.2] that $[\mathcal{C},\mathcal{V}]$ is an additive category. Furthermore, $[\mathcal{C},\mathcal{V}]$ has kernels and cokernels which are defined objectwise.

Given a morphism α in $[\mathcal{C},\mathcal{V}]$, the canonical map

$$\bar{\alpha} : \operatorname{Coker}(\ker\alpha) \to \operatorname{Ker}(\operatorname{coker}\alpha)$$

is an isomorphism objectwise. Corollary 3.3.8 implies $\bar{\alpha}$ is an isomorphism. It follows that $[\mathcal{C},\mathcal{V}]$ is an abelian category.

Next, direct limits exist in $[\mathcal{C},\mathcal{V}]$ and are defined objectwise. They are exact in $[\mathcal{C},\mathcal{V}]$, because so are direct limits in $\mathcal{V}$ (by assumption, $\mathcal{V}$ is a Grothendieck category). So, $[\mathcal{C},\mathcal{V}]$ is an Ab5-category.

It remains to find generators for $[\mathcal{C},\mathcal{V}]$. By [10, 2.4] $[\mathcal{C},\mathcal{V}]$ is a closed $\mathcal{V}$-module, and hence there is an action

$$\oslash : [\mathcal{C},\mathcal{V}]\otimes\mathcal{V} \to [\mathcal{C},\mathcal{V}].$$

Now for any non zero functor $X \in [\mathcal{C},\mathcal{V}]$ we have natural isomorphisms

$$\begin{aligned}\operatorname{Hom}_{[\mathcal{C},\mathcal{V}]}(\mathcal{V}(c,-)\oslash g_i, X)) &\cong \operatorname{Hom}_{\mathcal{V}}(g_i, \mathcal{V}_{[\mathcal{C},\mathcal{V}]}(\mathcal{V}(c,-),X)) \\ &\cong \operatorname{Hom}_{\mathcal{V}}(g_i, X(c)).\end{aligned}$$

Let $\alpha : X \to Y$ be a non-zero map in $[\mathcal{C},\mathcal{V}]$. We want to show that there are $i \in I$, a map $\beta : \mathcal{V}(c,-)\oslash g_i \to X$ such that $\alpha\beta \neq 0$.

Since α is non-zero, then $\alpha_c : X(c) \to Y(c)$ is a non-zero map in $\mathcal{V}$ for some $c \in \operatorname{Ob}\mathcal{C}$. By assumption $\{g_i\}_I$ are generators of $\mathcal{V}$, and so there is a map $\bar{\beta} : g_i \to X(c)$ such that $\alpha_c\bar{\beta} \neq 0$. By the above isomorphism we can find a unique map $\beta : \mathcal{V}(c,-)\oslash g_i \to X$ corresponding to $\bar{\beta}$. Now $\alpha_c\bar{\beta} \neq 0$ implies $\alpha\beta \neq 0$ as required.

If $\mathcal{C}$ is a small symmetric monoidal $\mathcal{V}$-category, then $[\mathcal{C},\mathcal{V}]$ is closed symmetric monoidal by Day's theorem 3.3.4. Monoidal product and internal Hom-object are computed by formulas of Day (3.14) and (3.15). □

Below we give a couple of examples illustrating the preceding theorem.

Example 5.1.3. Let R be a commutative ring with unit. Consider the closed symmetric monoidal category $\mathcal{V} = (\operatorname{Mod} R, \otimes_R, R)$. Consider a $\mathcal{V}$-category $\mathcal{C}$

defined as follows. Its objects are integers $\operatorname{Ob}\mathcal{C} = \mathbb{Z}$. Given two integers $m, n \in \operatorname{Ob}\mathcal{C}$, we define a Hom-object as

$$\mathcal{V}_\mathcal{C}(m,n) = \begin{cases} 0 & \text{if } m \neq n; \\ R & \text{if } m = n. \end{cases}$$

Clearly, $\mathcal{C}$ is a $\mathcal{V}$-category and the category $[\mathcal{C},\mathcal{V}]$ of $\mathcal{V}$-functors from $\mathcal{C}$ to $\mathcal{V}$ is the product category $\prod_\mathbb{Z} \operatorname{Mod} R$. By definition, $\operatorname{Ob}(\prod_\mathbb{Z} \operatorname{Mod} R)$ are tuples $(M_i)_{i\in\mathbb{Z}}$ and morphisms are tuples of R-homomorphisms $(f_i : M_i \to N_i)_{i\in\mathbb{Z}}$.

Let $\operatorname{Gr} R$ be the category of $\mathbb{Z}$-graded R-modules and graded homomorphisms. It is easy to see that the functor

$$\prod_\mathbb{Z} \operatorname{Mod} R \to \operatorname{Gr} R, \quad (M_i)_{i\in\mathbb{Z}} \mapsto \oplus_\mathbb{Z} M_i,$$

is an isomorphism of categories. It is well known that $\operatorname{Gr} R$ is a closed symmetric monoidal Grothendieck category with tensor product

$$(M \otimes_R N)_k := \bigoplus_{i+j=k} M_i \otimes_R N_j \text{ for all } M, N \in \operatorname{Gr} R. \tag{5.1}$$

We want to show that this tensor product is recovered from Day's theorem for $[\mathcal{C},\mathcal{V}]$. Indeed, we define a symmetric monoidal product on $\mathcal{C}$ as follows. For every $m, n \in \operatorname{Ob}\mathcal{C}$

$$m \boxtimes n := m + n.$$

Given $a \in \mathcal{V}_\mathcal{C}(m,m) = R$ and $b \in \mathcal{V}_\mathcal{C}(n,n) = R$, we set

$$a \boxtimes b = a \cdot b \in \mathcal{V}_\mathcal{C}(m+n, m+n) = R.$$

Clearly, $m \boxtimes n = n \boxtimes m$. Since R is commutative, it follows that $\boxtimes$ defines a strictly symmetric monoidal tensor product on $\mathcal{C}$.

Now for every $M, N \in [\mathcal{C},\mathcal{V}] \cong \operatorname{Gr} R$, Day's theorem implies

$$\begin{aligned} M \odot N &= \int^{(m,n)\in\mathbb{Z}\boxtimes\mathbb{Z}} M(m) \otimes_R N(n) \otimes_R \mathcal{V}_\mathcal{C}(m \boxtimes n, -) \\ &= \int^{(m,n)\in\mathbb{Z}\boxtimes\mathbb{Z}} M_m \otimes_R N_n \otimes_R \mathcal{V}_\mathcal{C}(m+n, -). \end{aligned}$$

Thus,

$$(M \odot N)_k = \bigoplus_{m+n=k} M_m \otimes_R N_n$$

and

$$\begin{aligned}\underline{\mathrm{Hom}}(M,N)(n) &= \int_{m\in \mathrm{Ob}\,\mathcal{C}=\mathbb{Z}} \mathrm{Hom}_R(M(m), N(m \boxtimes n)) \\ &= \int_m \mathrm{Hom}_R(M_m, N_{m+n}) = \mathrm{Hom}_{\mathrm{Gr}\,R}(M, N(n)).\end{aligned}$$

So tensor product (5.1) as well as internal Hom-functor for graded modules are recovered from Day's theorem.

By Theorem 5.1.2 $[\mathcal{C}, \mathcal{V}]$ is Grothendieck with $\{R(n)\}_{n\in\mathbb{Z}}$ generators, where

$$R(n)_m = \begin{cases} 0, & \text{if } m \neq n; \\ R, & \text{if } m = n. \end{cases}$$

It follows that $\mathrm{Gr}\,R$ is Grothendieck. Moreover, $[\mathcal{C}, \mathcal{V}] \cong \mathrm{Gr}\,R$ is closed symmetric monoidal.

Example 5.1.4. Let R be a commutative ring. Recall that an R-linear DG-category $\mathcal{C}$ is just a category enriched over $\mathcal{V} = \mathbf{Ch}(\mathrm{Mod}\,R)$. Right DG-modules over the DG-category $\mathcal{C}$ are just contravariant $\mathcal{V}$-functors from $\mathcal{C}$ to $\mathcal{V}$. In turn, DG-morphisms of DG-modules are nothing but $\mathcal{V}$-natural transformations. The category of right DG-modules and DG-morphisms is denoted by $\mathrm{Mod}\,\mathcal{C}$. Using our notation, one has, by definition, $\mathrm{Mod}\,\mathcal{C} = [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$.

Theorem 5.1.2 and Proposition 2.6.24 imply $\mathrm{Mod}\,\mathcal{C}$ is a Grothendieck category with the family of generators

$$D^n(\mathcal{V}_{\mathcal{C}}(-,c)) := \mathcal{V}_{\mathcal{C}}(-,c) \oslash D^n R, \quad c \in \mathrm{Ob}\,\mathcal{C}, n \in \mathbb{Z}.$$

Here $D^n R$ stands for the complex which is R in degrees n and $n-1$ and zero elsewhere, with interesting differential being the identity.

Example 5.1.5. Any preadditive category $\mathcal{B}$ is nothing but a category enriched over abelian groups $\mathcal{V} = \mathsf{Ab}$. $\mathcal{V}$-functors from $\mathcal{B}$ to $\mathcal{V}$ are the same as additive functors. Theorem 5.1.2 says that the category of additive functors $(\mathcal{B}, \mathsf{Ab})$ is Grothendieck with representable functors $\{h^B = \mathcal{V}_{\mathcal{B}}(B,-) \oslash \mathbb{Z}\}_{B\in\mathcal{B}}$ being a family of generators. Thus the fact that $(\mathcal{B}, \mathsf{Ab})$ is Grothendieck (see Example 2.3.5) follows from Theorem 5.1.2.

5.2 Localizations

We say that a full subcategory $\mathcal{S}$ of an abelian category $\mathcal{C}$ is a *Serre subcategory* if for any short exact sequence

$$0 \to X \to Y \to Z \to 0$$

in $\mathcal{C}$ an object $Y \in \mathcal{S}$ if and only if X, $Z \in \mathcal{S}$. A Serre subcategory $\mathcal{S}$ of a Grothendieck category $\mathcal{C}$ is *localizing* if it is closed under taking direct limits. Equivalently, the inclusion functor $i : \mathcal{S} \to \mathcal{C}$ admits the right adjoint $t = t_{\mathcal{S}} : \mathcal{C} \to \mathcal{S}$ which takes every object $X \in \mathcal{C}$ to the maximal subobject $t(X)$ of X belonging to $\mathcal{S}$. The functor t we call the *torsion functor*. An object C of $\mathcal{C}$ is said to be $\mathcal{S}$*-torsionfree* if $t(C) = 0$. Given a localizing subcategory $\mathcal{S}$ of $\mathcal{C}$ the *quotient category* $\mathcal{C}/\mathcal{S}$ consists of $C \in \mathcal{C}$ such that $t(C) = t^1(C) = 0$. The objects from $\mathcal{C}/\mathcal{S}$ we call $\mathcal{S}$*-closed objects*. Given $C \in \mathcal{C}$ there exists a canonical exact sequence

$$0 \to A' \to C \xrightarrow{\lambda_C} C_{\mathcal{S}} \to A'' \to 0$$

with $A' = t(C)$, $A'' \in \mathcal{S}$, and where $C_{\mathcal{S}} \in \mathcal{C}/\mathcal{S}$ is the maximal essential extension of $\widetilde{C} = C/t(C)$ such that $C_{\mathcal{S}}/\widetilde{C} \in \mathcal{S}$. The object $C_{\mathcal{S}}$ is uniquely defined up to a canonical isomorphism and is called the $\mathcal{S}$*-envelope* of C. Moreover, the inclusion functor $i : \mathcal{C}/\mathcal{S} \to \mathcal{C}$ has the left adjoint *localizing functor* $(-)_{\mathcal{S}} : \mathcal{C} \to \mathcal{C}/\mathcal{S}$, which is also exact. It takes each $C \in \mathcal{C}$ to $C_{\mathcal{S}} \in \mathcal{C}/\mathcal{S}$. Then,

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \cong \mathrm{Hom}_{\mathcal{C}/\mathcal{S}}(X_{\mathcal{S}}, Y)$$

for all $X \in \mathcal{C}$ and $Y \in \mathcal{C}/\mathcal{S}$.

If $\mathcal{C}$ and $\mathcal{D}$ are Grothendieck categories, $q : \mathcal{C} \to \mathcal{D}$ is an exact functor, and a functor $s : \mathcal{D} \to \mathcal{C}$ is fully faithful and right adjoint to q, then $\mathcal{S} := \operatorname{Ker} q$ is a localizing subcategory and there exists an equivalence $\mathcal{C}/\mathcal{S} \overset{H}{\cong} \mathcal{D}$ such that $H \circ (-)_{\mathcal{S}} = q$. We shall refer to the pair (q, s) as the *localization pair*.

Example 5.2.1. Let $\mathcal{A}$ be a small preadditive category. Consider the category $(\mathcal{A}, \mathsf{Ab})$ of additive functors from $\mathcal{A}$ to Ab. Let $p \in \operatorname{Ob} \mathcal{A}$, then we have

$$\mathrm{Hom}_{(\mathcal{A},\mathsf{Ab})}((p, -), (p, -)) = \mathrm{End}_{\mathcal{A}}\, p.$$

Let

$$\mathcal{S}_p = \{F \in (\mathcal{A}, \mathsf{Ab}) \mid F(p) = 0\}.$$

Then $\mathcal{S}_p$ is a localizing subcategory of $(\mathcal{A}, \mathsf{Ab})$. By [14] and [18] there is an equivalence of categories

$$(\mathcal{A}, \mathsf{Ab})/\mathcal{S}_p \cong \mathrm{Mod}(\mathrm{End}_{\mathcal{A}}\, p)^{\mathrm{op}}.$$

This result has some important applications in ring and module theory (see [16, 17]).

More generally, given a collection of objects $\mathcal{P}$ in $\mathcal{A}$, we can consider the localizing subcategory

$$\mathcal{S}_{\mathcal{P}} = \{F \in (\mathcal{A}, \mathsf{Ab}) \mid F(p) = 0 \text{ for all } p \in \mathcal{P}\}$$

of $(\mathcal{A}, \mathsf{Ab})$ and then

$$(\mathcal{A}, \mathsf{Ab})/\mathcal{S}_{\mathcal{P}} \cong (\mathcal{P}, \mathsf{Ab}),$$

where $\mathcal{P}$ on the right hand side is regarded as a full subcategory in $\mathcal{A}$ (see [14] for details).

Our next goal is to obtain an enriched analog of this result.

Suppose $\mathcal{V}$ is a closed symmetric monoidal Grothendieck category. Let $\mathcal{C}$ be a $\mathcal{V}$-category. By Theorem 5.1.2 $[\mathcal{C}, \mathcal{V}]$ is a Grothendieck category. Suppose $\mathcal{P}$ is a collection of objects in $\mathcal{C}$. We shall also regard $\mathcal{P}$ as a natural $\mathcal{V}$-subcategory. Then

$$\mathcal{S}_{\mathcal{P}} = \{F \in [\mathcal{C}, \mathcal{V}] \mid F(p) = 0 \text{ for all } p \in \mathcal{P}\}$$

is localizing in $[\mathcal{C}, \mathcal{V}]$.

We shall prove below that there is a natural equivalence of Grothendieck categories

$$[\mathcal{C}, \mathcal{V}]/\mathcal{S}_{\mathcal{P}} \cong [\mathcal{P}, \mathcal{V}].$$

Thus the same result of [14, 18] for the category of additive functors $(\mathcal{A}, \mathsf{Ab})$ with $\mathcal{A}$ preaddtive (see Example 5.2.1 above) is recovered from the case when $\mathcal{V} = \mathsf{Ab}$. But first we prove the following

Proposition 5.2.2. *Suppose $\mathcal{V}$ is a closed symmetric monoidal Grothendieck category. Let $\mathcal{C}$ be a $\mathcal{V}$-category and let $\mathcal{P}$ consist of a collection of objects of $\mathcal{C}$. Then the inclusion map $i : \mathcal{P} \to \mathcal{C}$ is a $\mathcal{V}$-functor. It induces two adjoint functors*

$$i_* : [\mathcal{P}, \mathcal{V}] \rightleftarrows [\mathcal{C}, \mathcal{V}] : i^*$$

where i_* is the enriched left Kan extension and i^* is just restriction to $\mathcal{P}$.

Proof. Although this fact is a consequence of [4, 6.7.7], we give a proof here for the convenience of the reader.

If $F \in [\mathcal{P}, \mathcal{V}]$ then by Corollary 3.3.3 we have

$$F \cong \int^{\mathrm{Ob}\,\mathcal{P}} \mathcal{V}(p, -) \oslash F(p).$$

By definition of the left Kan extension, we have

$$i_* F \cong \int^{\mathrm{Ob}\,\mathcal{P}} \mathcal{V}(i(p), -) \oslash F(p).$$

We want to show that

$$\mathrm{Hom}_{[\mathcal{C},\mathcal{V}]}(i_* F, G) \cong \mathrm{Hom}_{[\mathcal{P},\mathcal{V}]}(F, i^* G).$$

Using (3.13), one has isomorphisms of $\mathcal{V}$-objects

$$\mathcal{V}_{\mathcal{F}}(F, i^* G) = \mathcal{V}_{\mathcal{F}}(F, G \circ i) = \int_{\mathrm{Ob}\,\mathcal{P}} \mathcal{V}(F(p), G(i(p))). \tag{5.2}$$

On the other hand,

$$\begin{aligned} \mathcal{V}_{\mathcal{F}}(i_* F, G) &= \mathcal{V}_{\mathcal{F}}\Big(\int^{\mathrm{Ob}\,\mathcal{P}} \mathcal{V}(i(p), -) \oslash F(p), G\Big) \\ &= \int_{\mathrm{Ob}\,\mathcal{P}} \mathcal{V}_{\mathcal{F}}(F(p), \mathcal{V}_{[\mathcal{C},\mathcal{V}]}(\mathcal{V}((i(p), -), G))) \\ &= \int_{\mathrm{Ob}\,\mathcal{P}} \mathcal{V}(F(p), G(i(p))). \end{aligned} \tag{5.3}$$

We have used here the fact that the functor $\mathcal{V}_{\mathcal{F}}(-, G)$ takes $\mathcal{V}$-coends to $\mathcal{V}$-ends [4, 6.6.11] as well as the fact that $[\mathcal{C}, \mathcal{V}]$ is a closed $\mathcal{V}$-module. Now (5.2) and (5.3) imply i_* and i^* are adjoint functors. □

Theorem 5.2.3. *Let $\mathcal{S}_\mathcal{P} = \{G \in [\mathcal{C},\mathcal{V}] \mid G(p) = 0$ for all $p \in \mathcal{P}\}$. Then $\mathcal{S}_\mathcal{P}$ is a localizing subcategory of $[\mathcal{C},\mathcal{V}]$ and $[\mathcal{P},\mathcal{V}]$ is equivalent to the quotient category $[\mathcal{C},\mathcal{V}]/\mathcal{S}_\mathcal{P}$.*

Proof. Obviously, $\mathcal{S}_\mathcal{P}$ is localizing. Let

$$\varkappa : [\mathcal{P},\mathcal{V}] \to [\mathcal{C},\mathcal{V}]/\mathcal{S}_\mathcal{P}$$

be the composition of the left Kan extension functor $i_* : [\mathcal{P},\mathcal{V}] \to [\mathcal{C},\mathcal{V}]$ and the localization functor $(-)_{\mathcal{S}_\mathcal{P}} : [\mathcal{C},\mathcal{V}] \to [\mathcal{C},\mathcal{V}]/\mathcal{S}_\mathcal{P}$. We want to prove that $\varkappa$ is an equivalence of categories.

First observe that

$$i^*i_*F \cong F$$

for all $F \in [\mathcal{P},\mathcal{V}]$. Second, given $G \in [\mathcal{C},\mathcal{V}]$ the adjunction map $\beta : i_*i^*G \to G$ is such that $\operatorname{Ker}\beta, \operatorname{Coker}\beta \in \mathcal{S}_\mathcal{P}$. Indeed, applying the exact functor i^* to the exact sequence

$$\operatorname{Ker}\beta \rightarrowtail i_*i^*G \xrightarrow{\beta} G \twoheadrightarrow \operatorname{Coker}\beta,$$

we get an exact sequence in $[\mathcal{P},\mathcal{V}]$

$$i^*(\operatorname{Ker}\beta) \rightarrowtail i^*i_*i^*G \xrightarrow{i^*(\beta)} i^*G \twoheadrightarrow i^*(\operatorname{Coker}\beta).$$

Since the composite map

$$i^*G \to i^*i_*i^*G \xrightarrow{i^*(\beta)} i^*G$$

is the identity map and the left arrow is an isomorphism, then so is the right arrow. Thus

$$i^*(\operatorname{Ker}\beta) = i^*(\operatorname{Coker}\beta) = 0,$$

and hence $\operatorname{Ker}\beta, \operatorname{Coker}\beta \in \mathcal{S}_\mathcal{P}$. It also follows that

$$(i_*i^*G)_{\mathcal{S}_\mathcal{P}} \cong G_{\mathcal{S}_\mathcal{P}}. \tag{5.4}$$

We have for all $F, F' \in [\mathcal{P},\mathcal{V}]$

$$\begin{aligned}\operatorname{Hom}_{[\mathcal{C},\mathcal{V}]/\mathcal{S}_\mathcal{P}}(\varkappa(F),\varkappa(F')) &\cong \operatorname{Hom}_{[\mathcal{C},\mathcal{V}]/\mathcal{S}_\mathcal{P}}((i_*(F)_{\mathcal{S}_\mathcal{P}}, i_*(F')_{\mathcal{S}_\mathcal{P}})\\ &\cong \operatorname{Hom}_{[\mathcal{C},\mathcal{V}]}(i_*F,(i_*F')_{\mathcal{S}_\mathcal{P}})\\ &\cong \operatorname{Hom}_{[\mathcal{P},\mathcal{V}]}(F, i^*((i_*F')_{\mathcal{S}_\mathcal{P}}))\\ &\cong \operatorname{Hom}_{[\mathcal{P},\mathcal{V}]}(F, i^*i_*F')\\ &\cong \operatorname{Hom}_{[\mathcal{P},\mathcal{V}]}(F, F').\end{aligned}$$

We use here an isomorphism $i^*G \cong i^*(G_{\mathcal{S}_\mathcal{P}})$ for any $G \in [\mathcal{C}, \mathcal{V}]$. The isomorphism is obtained by applying the exact functor i^* to the exact sequence

$$S \rightarrowtail G \xrightarrow{\lambda_G} G_{\mathcal{S}_\mathcal{P}} \twoheadrightarrow S', \qquad S, S' \in \mathcal{S}_\mathcal{P},$$

where λ_G is the $\mathcal{S}_\mathcal{P}$-envelope of G.

We see that $\varkappa$ is fully faithful. Now let $G \in [\mathcal{C}, \mathcal{V}]/\mathcal{S}_\mathcal{P}$ be any $\mathcal{S}_\mathcal{P}$-closed object, then isomorphism (5.4) implies

$$\varkappa(i^*G) = (i_*i^*G)_{\mathcal{S}_\mathcal{P}} \cong G_{\mathcal{S}_\mathcal{P}} \cong G.$$

If we set $F := i^*G$, then $\varkappa(F) \cong G$. This shows that $\varkappa$ is an equivalence of categories. □

Corollary 5.2.4 (Garkusha [14]). *Let $\mathcal{C}$ be a Grothendieck category with finitely generated projective generators $\mathcal{B} = \{p_i\}_{i \in I}$. Let $\mathcal{P} = \{p_j\}_{j \in J}$ be a subfamily in $\mathcal{B}$, where $J \subset I$. Then*

$$\mathcal{S}_\mathcal{P} = \{x \in \mathcal{C} \mid \mathrm{Hom}_\mathcal{C}(p_j, x) = 0 \textit{ for all } p_j \in \mathcal{P}\}$$

is localizing and $\mathcal{C}/\mathcal{S}_\mathcal{P}$ is a Grothendieck category with $\{(p_j)_{\mathcal{S}_\mathcal{P}}\}$ a family of finitely generated projective generators.

Proof. By Mitchell's Theorem $\mathcal{C}$ is equivalent to the category $(\mathcal{B}^{\mathrm{op}}, \mathsf{Ab})$ by means of the functor T sending $x \in \mathcal{C}$ to $(-, x)$. Now the latter category is the same as the category of enriched $\mathcal{V}$-functors from $\mathcal{B}^{\mathrm{op}}$ to $\mathcal{V}$ where $\mathcal{V} = \mathsf{Ab}$. We use as well the fact that a category is preadditive if and only if it is enriched in Ab.

It follows that T induces an equivalence of categories $\mathcal{C}/\mathcal{S}_\mathcal{P}$ and $(\mathcal{B}^{\mathrm{op}}, \mathsf{Ab})/\widetilde{\mathcal{S}}_\mathcal{P}$, where

$$\widetilde{\mathcal{S}}_\mathcal{P} = \{F \in (\mathcal{B}^{\mathrm{op}}, \mathsf{Ab}) | F(p_j) = 0 \text{ for all } p_j \in \mathcal{P}\}.$$

Theorem 5.2.3 implies the latter quotient category is equivalent to $(\mathcal{P}^{\mathrm{op}}, \mathsf{Ab})$. The proof of Theorem 5.2.3 shows that the functor

$$(\mathcal{P}^{\mathrm{op}}, \mathsf{Ab}) \to \mathcal{C}/\mathcal{S}_\mathcal{P}$$

which sends $F \in (\mathcal{P}^{\mathrm{op}}, \mathsf{Ab})$ to $(T^{-1}(i_*F))_{\mathcal{S}_\mathcal{P}}$ is an equivalence of categories. It follows that $\{(p_j)_{\mathcal{S}_\mathcal{P}}\}_{j \in J}$ is a family of finitely generated projective generators of $\mathcal{C}/\mathcal{S}_\mathcal{P}$. □

Example 5.2.5. Let $\mathcal{C}$ and $\mathcal{V}$ be as in Example 5.1.3 and let $n \in \operatorname{Ob}\mathcal{C} = \mathbb{Z}$. We set

$$\mathcal{S}_n = \{F \in [\mathcal{C}, \mathcal{V}] \cong \operatorname{Gr} R \mid F(n) = 0\}.$$

Then Theorem 5.2.3 implies

$$[\mathcal{C}, \mathcal{V}]/\mathcal{S}_n \cong \operatorname{Gr} R/\mathcal{S}_n \cong [\mathcal{P}, \operatorname{Mod} R],$$

where $\mathcal{P}$ has one object $n \in \mathbb{Z}$ and $\mathcal{V}_{\mathcal{P}}(n, n) = R$. But $[\mathcal{P}, \operatorname{Mod} R] = \operatorname{Mod} R$, hence $\operatorname{Gr} R/\mathcal{S}_n \cong \operatorname{Mod} R$.

Below we shall work with ring objects and modules over them.

Definition 5.2.6. Let $(\mathcal{C}, \otimes, e)$ be a monoidal category. A *monoid (ring object)* $\mathcal{R}$ in $\mathcal{C}$ is an object $\mathcal{R} \in \mathcal{C}$ together with two arrows $\mu : \mathcal{R} \otimes \mathcal{R} \to \mathcal{R}, \eta : e \to \mathcal{R}$ such that the diagrams

$$\begin{array}{ccccc}
\mathcal{R} \otimes (\mathcal{R} \otimes \mathcal{R}) & \xrightarrow{\cong} & (\mathcal{R} \otimes \mathcal{R}) \otimes \mathcal{R} & \xrightarrow{\mu\otimes 1} & \mathcal{R} \otimes \mathcal{R} \\
{\scriptstyle 1\otimes\mu}\downarrow & & & & \downarrow{\scriptstyle \mu} \\
\mathcal{R} \otimes \mathcal{R} & & \xrightarrow{\quad\quad\mu\quad\quad} & & \mathcal{R}
\end{array}$$

$$\begin{array}{ccccc}
e \otimes \mathcal{R} & \xrightarrow{\eta\otimes 1} & \mathcal{R} \otimes \mathcal{R} & \xleftarrow{1\otimes\eta} & \mathcal{R} \otimes e \\
& {\scriptstyle l_{\mathcal{R}}}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle r_{\mathcal{R}}} & \\
& & \mathcal{R} & &
\end{array}$$

are commutative.

Definition 5.2.7. We fix a ring object $\mathcal{R}$ in a monoidal category $\mathcal{C}$. A *left* $\mathcal{R}$-*module* M over $\mathcal{R}$ is an object equipped with an arrow $v : \mathcal{R} \otimes M \to M$ called the action, which is compatible with the ring object composition. More precisely, we require that the following diagrams commute

$$\begin{array}{ccccc}
\mathcal{R} \otimes (\mathcal{R} \otimes M) & \xrightarrow{\cong} & (\mathcal{R} \otimes \mathcal{R}) \otimes M & \xrightarrow{\mu\otimes 1} & \mathcal{R} \otimes M \\
{\scriptstyle 1\otimes v}\downarrow & & & & \downarrow{\scriptstyle v} \\
\mathcal{R} \otimes M & & \xrightarrow{\quad\quad v\quad\quad} & & M
\end{array}$$

$$\begin{array}{ccc}
\mathcal{R} \otimes e & \xrightarrow{1\otimes\eta} & \mathcal{R} \otimes M \\
& {\scriptstyle r_{\mathcal{R}}}\searrow & \downarrow{\scriptstyle v} \\
& & M
\end{array}$$

Lemma 5.2.8. *Let $\mathcal{R}$ be a ring object of a closed symmetric monoidal Grothendieck category $\mathcal{V}$. Then the category of left $\mathcal{R}$-modules $\mathcal{R}\,\mathrm{Mod}$ can naturally be identified with the Grothendieck category $[\mathcal{C},\mathcal{V}]$, where $\mathrm{Ob}\,\mathcal{C}=\{*\}$ and $\mathcal{V}_{\mathcal{C}}(*,*)=\mathcal{R}$.*

Proof. Given a left $\mathcal{R}$-module M, define a $\mathcal{V}$-functor $F:\mathcal{C}\to\mathcal{V}$ as follows. We set $F(*)=M$ and the structure map

$$F:\mathcal{V}(*,*)\to\mathcal{V}(F(*),F(*))$$

to be the map adjoint to the structure map $\mathcal{R}\otimes M\to M$. This rule clearly yields the desired identification. □

Example 5.2.9. To illustrate the previous lemma, let R be a commutative unital ring and let $A=A_0\oplus A_1\oplus\cdots$ be a graded R-algebra. Then A is a ring object in $[\mathcal{C},\mathcal{V}]=\mathrm{Gr}\,R$ with respect to tensor product (5.1). We can regard A as a one object $\mathcal{V}$-category, where $\mathcal{V}=\mathrm{Gr}\,R$. In our notation $A=\mathcal{V}_{\mathrm{Gr}\,R}(*,*)$. Lemma 5.2.8 implies $[\mathcal{C},\mathcal{V}]\cong A\,\mathrm{Mod}$ is a Grothendieck category. Moreover, $\{A(n)=A\oslash R(n)\}_{n\in\mathbb{Z}}$ are generators by Theorem 5.1.2 (see as well Example 5.1.3).

Corollary 5.2.10. *Let $\mathcal{C}$ be a small $\mathcal{V}$-category, and let c be any object of $\mathcal{C}$. Then there is a natural equivalence of Grothendieck categories*

$$R\,\mathrm{Mod}\cong[\mathcal{C},\mathcal{V}]/\mathcal{S}_c,$$

where $\mathcal{S}_c=\{G\in[\mathcal{C},\mathcal{V}]\mid G(c)=0\}$ and $R=\mathcal{V}(c,c)$.

Proof. This follows from Theorem 5.2.3 and Lemma 5.2.8. □

Chapter 6

Chain complexes of generalized modules

Recall that the category of generalized modules $\mathcal{C}_R$ is defined as

$$\mathcal{C}_R := (\operatorname{mod} R, \mathsf{Ab}),$$

whose objects are the additive functors $F : \operatorname{mod} R \to \mathsf{Ab}$ from the category of right finitely presented R-modules $\operatorname{mod} R$ to the category of abelian groups Ab. Its morphisms are the natural transformations of functors. Similarly, the category ${}_R\mathcal{C}$ consists of the additive functors from the category of left finitely presented R-modules to Ab.

In this chapter we prove that the category of chain complexes of generalized modules $\mathbf{Ch}\,\mathcal{C}_R$ over a commutative ring R can be identified with the category of enriched functors $[\operatorname{mod} R, \mathbf{Ch}(\operatorname{Mod} R)]$, where the category of finitely presented modules $\operatorname{mod} R$ is regarded as a full subcategory of complexes $\mathbf{Ch}(\operatorname{Mod} R)$ concentrated in zeroth degree and this single entry is finitely presented. As an application of this, we show that $\mathbf{Ch}\,\mathcal{C}_R$ is a closed symmetric monoidal category. We shall also establish that $\mathbf{Ch}\,\mathcal{C}_R$ is a closed symmetric monoidal model category with nice finiteness conditions.

6.1 Generalized modules as enriched functors

Below we shall need the following

Theorem 6.1.1 ([8]). *Suppose R is a commutative ring. Then the category of generalized R-modules $\mathcal{C}_R$ can naturally be identified with the category of enriched functors* $[\operatorname{mod} R, \operatorname{Mod} R]$.

Proof. We briefly recall the proof from [8] for the convenience of the reader.

I. We first associate to any object in $[\operatorname{mod} R, \operatorname{Mod} R]$ an object in $\mathcal{C}_R$.

Let $F \in \operatorname{Ob}[\operatorname{mod} R, \operatorname{Mod} R]$. By Definition 3.1.5 F takes $\operatorname{Ob}(\operatorname{mod} R)$ to $\operatorname{Ob}(\operatorname{Mod} R)$ and for all $M, M' \in \operatorname{mod} R$ there is a R-module homomorphism

$$F_{MM'} : \operatorname{Hom}_R(M, M') \to \operatorname{Hom}_R(F(M), F(M')). \qquad (6.1)$$

Let $f : M \to M'$ be a homomorphism in $\operatorname{mod} R$, then we set $F(f)$ to be the image of f in $\operatorname{Hom}_R(M, M')$ under (6.1). Observe that an R-module structure on $F(M)$ is given by

$$x \cdot r = F_{MM}(r)(x)$$

for all $x \in F(M)$ and $r \in R$. Here $F_{MM}(r)$ stands for the image of the right multiplication endomorphism $r : M \to M$.

II. Next, we want to show that the morphisms of $[\operatorname{mod} R, \operatorname{Mod} R]$ can naturally be regarded as morphisms of $\mathcal{C}_R$.

We have to verify that $\mathcal{V}$-natural transformations in $[\operatorname{mod} R, \operatorname{Mod} R]$ are morphisms in $\mathcal{C}_R$. Given $F, G \in [\operatorname{mod} R, \operatorname{Mod} R]$, the first step shows that $F, G \in \mathcal{C}_R$. So given a $\mathcal{V}$-natural transformation t in $[\operatorname{mod} R, \operatorname{Mod} R]$, we want to prove that t yields a morphism in $\mathcal{C}_R$ in a natural way. For t we have structure homomorphisms in $\operatorname{Mod} R$ (see Definition 3.1.6)

$$t_M : R \to \operatorname{Hom}_R(F(M), G(M)).$$

for all $M, M' \in \operatorname{mod} R$. For any $M \in \operatorname{Mod} R$, set

$$\tau_M := t_M(1) : F(M) \to G(M).$$

Therefore t yields a natural transformation in $\mathcal{C}_R$

$$\tau : F \to G.$$

We see that morphisms in $[\operatorname{mod} R, \operatorname{Mod} R]$ can naturally be regarded as morphisms of $\mathcal{C}_R$. So $F(M) \in \operatorname{Mod} R$.

III. In this step we shall show that any object F of $\mathcal{C}_R$ can be regarded as an enriched functor in $[\operatorname{mod} R, \operatorname{Mod} R]$.

For any $M \in \operatorname{mod} R$ we have that $F(M) \in \mathsf{Ab}$. Let us show that $F(M)$ is an R-module. For all $x \in F(M)$, we have to define $x \cdot r$, where $r \in R$. The element r defines an R-module homomorphism $r : M \to M$ sending $m \in M$ to $m \cdot r$. We have a morphism $F(r) : F(M) \to F(M)$ and we set

$$x \cdot r := F(r)(x).$$

So $F(M) \in \operatorname{Mod} R$. Now we define an enriched functor associated with F. We therefore define a morphism in $\operatorname{Mod} R$

$$F_{MM'} : \operatorname{Hom}_R(M, M') \to \operatorname{Hom}_R(F(M), F(M')), \quad F_{MM'}(f) := F(f),$$

for all M, M' in $\operatorname{mod} R$.

Next we construct diagram (3.9). We have that

$$u_M : R \to \operatorname{Hom}_R(M, M)$$

is given by the right multiplication homomorphism. One has

$$F_{MM}(u_M(r)) = F(r)$$

for all $r \in R$, and hence the diagram

$$\begin{array}{ccc} R & \xrightarrow{u_M} & \operatorname{Hom}_R(M, M) \\ & {\scriptstyle u_{F(M)}} \searrow & \downarrow {\scriptstyle F_{MM}} \\ & & \operatorname{Hom}_R(F(M), F(M)) \end{array}$$

is commutative. The structure of an enriched functor for F, denoted by the same letter, is completed.

So every object in $\mathcal{C}_R$ can naturally be regarded as an enriched functor in $[\operatorname{mod} R, \operatorname{Mod} R]$.

IV. In this step we shall show that morphisms in $\mathcal{C}_R$ (recall that these are natural transformations of additive functors) can naturally be regarded as $\mathcal{V}$-natural transformations in $[\operatorname{mod} R, \operatorname{Mod} R]$, i.e. as morphisms of $[\operatorname{mod} R, \operatorname{Mod} R]$.

Let $\tau : F \to G$ be any natural transformation in $\mathcal{C}_R$. Then for each object $M \in \operatorname{mod} R$, there exists a homomorphism $\tau_M : F(M) \to G(M)$ in Ab and for each homomorphism $f : M \to M'$ in $\operatorname{mod} R$ the diagram

$$\begin{array}{ccc} F(M) & \xrightarrow{\tau_M} & G(M) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(M') & \xrightarrow[\tau_M]{} & G(M') \end{array}$$

is commutative.

By above we can regard F, G as enriched functors. We want to show that τ yields a $\mathcal{V}$-natural transformation t between the $\mathcal{V}$-functors F and G. For any $M \in \operatorname{mod} R$ define a map

$$t_M : R \to \operatorname{Hom}_R(F(M), G(M))$$

as $t_M(r) := \tau_M \circ F(r) = G(r) \circ \tau_M$. By definition, $t_M(1) = \tau_M$. Then the maps t_M yield $\mathcal{V}$-natural transformations between the $\mathcal{V}$-functors F and G. □

6.2 Chain complexes of generalized modules as enriched functors

Let $\mathbf{Ch}(\operatorname{Mod} R)$ be the category of chain complexes of modules over a commutative ring R. It is a closed symmetric monoidal cofibrantly generated and weakly finitely generated model category by sections 4.4 and 4.6. Weak equivalences are the quasi-isomorphisms and fibrations are the surjective chain maps. The tensor product and internal Hom-object are defined as in (4.1) and (4.2). The monoidal unit is the chain complex with the ring R concentrated in zeroth degree and other degrees are zero.

The category of finitely presented modules $\operatorname{mod} R$ is a small symmetric monoidal category naturally enriched over $\operatorname{Mod} R$. Moreover, $\operatorname{mod} R$ can also be enriched over $\mathbf{Ch}\operatorname{Mod} R$ if we regard a module $M \in \operatorname{mod} R$ as the chain complex with M concentrated in zeroth degree and other degrees are zero. Given, $M, N \in \operatorname{mod} R$ the internal Hom-object is the chain complex

$$\cdots \to 0 \to \operatorname{Hom}_R(M, N) \to 0 \to \cdots$$

where $\mathrm{Hom}_R(M, N)$ is concentrated in zeroth degree. Observe that this complex equals $\underline{\mathrm{Hom}}(M, N)$, where M, N are regarded as complexes defined above.

If there is no likelihood of confusion, we shall also write the internal Hom-chain complex as $\mathcal{V}(A_\bullet, B_\bullet)$, where $A_\bullet, B_\bullet$ are two chain complexes. In other words,

$$\mathcal{V}(A_\bullet, B_\bullet) := \underline{\mathrm{Hom}}(A_\bullet, B_\bullet).$$

Theorem 6.2.1. *Suppose R is a commutative ring. Then the category of chain complexes of generalized R-modules $\mathbf{Ch}(\mathcal{C}_R)$ can naturally be identified with the category* $[\mathrm{mod}\, R, \mathbf{Ch}(\mathrm{Mod}\, R)]$.

Proof. Given $M \in \mathrm{mod}\, R$ we want to describe chain morphisms as follows

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & 0 & \xrightarrow{0} & M & \xrightarrow{0} & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle\alpha} & & \downarrow & & \\
\cdots & \longrightarrow & \mathcal{V}(A_\bullet, B_\bullet)_1 & \xrightarrow{\partial_1} & \mathcal{V}(A_\bullet, B_\bullet)_0 & \xrightarrow{\partial_0} & \mathcal{V}(A_\bullet, B_\bullet)_{-1} & \longrightarrow & \cdots
\end{array}$$

From the diagram we have $\partial_0 \alpha = 0$. This means $\partial_0 \alpha(m) = 0$, for all $m \in M$.

$$\alpha(m) \in \mathcal{V}(A_\bullet, B_\bullet)_0 = \prod_{p \in \mathbb{Z}} \mathrm{Hom}_R(A_p, B_p)$$
$$\alpha(m) = (\alpha_p(m) : A_p \to B_p)_{p \in \mathbb{Z}}$$
$$\partial \alpha_p(m) = 0 = \partial \alpha_p(m) - \alpha(m)_{p-1} \partial.$$

We get a commutative diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & A_{p+1} & \xrightarrow{\partial_{p+1}} & A_p & \xrightarrow{\partial_p} & A_{p-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle\alpha_{p+1}(m)} & & \downarrow{\scriptstyle\alpha_p(m)} & & \downarrow{\scriptstyle\alpha_{p-1}(m)} & & \\
\cdots & \longrightarrow & B_{p+1} & \xrightarrow[\partial_{p+1}]{} & B_p & \xrightarrow[\partial_p]{} & B_{p-1} & \longrightarrow & \cdots
\end{array}$$

This shows $\alpha(m) : A_\bullet \to B_\bullet$ is a chain map.

I. Consider a $\mathcal{V}$-functor $F \in [\mathrm{mod}\, R, \mathbf{Ch}(\mathrm{Mod}\, R)]$. By definition, we have $F(M) \in \mathbf{Ch}(\mathrm{Mod}\, R)$ for any M in $\mathrm{mod}\, R$. We also have a map

$$F_{MM'} : \mathcal{V}(M, M') \to \mathcal{V}(F(M), F(M')) \in \mathbf{Ch}(\mathrm{Mod}\, R).$$

This is the same as a chain map

$$F_{MM'} : (\cdots \to 0 \to \mathrm{Hom}_R(M, M') \to 0 \to \cdots) \longrightarrow \underline{\mathrm{Hom}}(F(M), F(M')).$$

By above we have a chain map

$$F_{MM'}(f) : F(M) \to F(M')$$

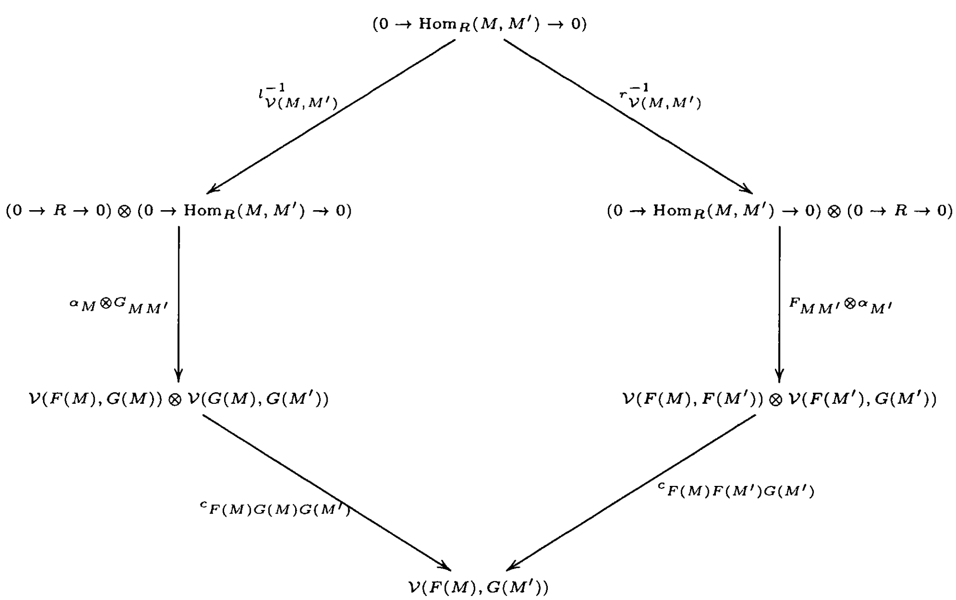
for all $f \in \mathrm{Hom}_R(M, M')$. It is directly verified that $F_{MM}(\mathrm{id}_M) = \mathrm{id}_{F(M)}$ and $F_{MM''}(gf) = F_{M'M''}(g)F_{MM'}(f)$ for any $g \in \mathrm{Hom}_R(M', M'')$ (see Definition 3.1.5).

If we observe that $\mathbf{Ch}\,\mathcal{C}_R$ is the same as the category $(\mathrm{mod}\,R, \mathbf{Ch}(\mathsf{Ab}))$ of additive functors from $\mathrm{mod}\,R$ to $\mathbf{Ch}(\mathsf{Ab})$, it follows that the enriched functor F gives rise to an object in $\mathbf{Ch}\,\mathcal{C}_R$. We denote this object by the same letter.

II. Now let $\alpha : F \Rightarrow G$ be a $\mathcal{V}$-map in $[\mathrm{mod}\,R, \mathbf{Ch}(\mathrm{Mod}\,R)]$. It consists of giving a chain map

$$\alpha_M : (0 \to R \to 0) \longrightarrow \mathcal{V}(F(M), G(M)) = \underline{\mathrm{Hom}}(F(M), G(M)),$$

which is equivalent to giving a chain map $\alpha_M(r) : F(M) \to G(M)$ for all $r \in R$, such that the following diagram commutes



The diagram implies the following:

(a) For all $r \in R$ and $f \in \mathrm{Hom}_R(M, M')$ the chain map $c(\alpha_M(r) \otimes G_{MM'}(f)) : F(M) \to G(M')$ equals the chain map $G_{MM'}(f) \circ \alpha_M(r)$.

(b) For all $r \in R$ and $f \in \mathrm{Hom}_R(M, M')$ the chain map $c(F_{MM'}(f) \otimes \alpha_M(r)) : F(M) \to G(M')$ equals $\alpha_{M'}(r) \circ F_{MM'}(f)$.

So we get a commutative diagram

$$\begin{array}{ccc} F(M) & \xrightarrow{\alpha_M(r)} & G(M) \\ {\scriptstyle F_{MM'}(f)}\downarrow & & \downarrow{\scriptstyle G_{MM'}(f)} \\ F(M') & \xrightarrow[\alpha_{M'}(r)]{} & G(M') \end{array}$$

Thus $\alpha_M(r) : F(M) \to G(M)$ gives rise to a morphism in $(\mathrm{mod}\, R, \mathbf{Ch}(\mathsf{Ab})) = \mathbf{Ch}\,\mathcal{C}_R$.

We shall associate to the enriched map $\alpha : F \Rightarrow G$ in $[\mathrm{mod}\, R, \mathbf{Ch}(\mathrm{Mod}\, R)]$ the map $F \to G$ in $\mathbf{Ch}\,\mathcal{C}_R$ given by the chain maps $\alpha_M(1) : F(M) \to G(M)$, $M \in \mathrm{mod}\, R$. We denote the associated chain map by the same letter α.

III. By Theorem 6.1.1 the category $\mathbf{Ch}[\mathrm{mod}\, R, \mathrm{Mod}\, R]$ is identified with $\mathbf{Ch}\,\mathcal{C}_R$. Let $F_\bullet \in \mathbf{Ch}[\mathrm{mod}\, R, \mathrm{Mod}\, R]$, so we have a chain complex

$$F_\bullet = \cdots \longrightarrow F_{n+1} \xrightarrow{\partial_{n+1}} F_n \xrightarrow{\partial_n} F_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

with each $F_n \in [\mathrm{mod}\, R, \mathrm{Mod}\, R]$ and each ∂_n being a $\mathcal{V}$-natural transformation from $\mathrm{mod}\, R$ to $\mathrm{Mod}\, R$. We have to associate a $\mathcal{V}$-functor in $[\mathrm{mod}\, R, \mathbf{Ch}(\mathrm{Mod}\, R)]$ to $F_\bullet$.

If $M \in \mathrm{mod}\, R$ then

$$F_\bullet(M) = \cdots \longrightarrow F(M)_{n+1} \xrightarrow{\partial_{n+1,M}(1)} F(M)_n \xrightarrow{\partial_{n,M}(1)} F(M)_{n-1} \xrightarrow{\partial_{n-1,M}(1)} \cdots \in \mathbf{Ch}(\mathrm{Mod}\, R)$$

Also, for any map $f : M \to M' \in \mathrm{mod}\, R$ we have that $F_\bullet(f) : F_\bullet(M) \to F_\bullet(M')$ is a chain map, because each square of the following diagram is commutative

$$\begin{array}{ccccccccc} \cdots \longrightarrow & F_{n+1}(M) & \xrightarrow{\partial_{n+1}} & F_n(M) & \xrightarrow{\partial_n} & F_{n-1}(M) & \longrightarrow \cdots \\ & {\scriptstyle F_{MM'}(f)}\downarrow & & {\scriptstyle F_{MM'}(f)}\downarrow & & \downarrow{\scriptstyle F_{MM'}(f)} & \\ \cdots \longrightarrow & F_{n+1}(M') & \xrightarrow[\partial_{n+1}]{} & F_n(M')_n & \xrightarrow[\partial_n]{} & F(M')_{n-1}, & \longrightarrow \cdots \end{array}$$

Thus $F_{MM'}(f)$ is a chain map, and hence one gets a chain map

$$F_{MM'} : (\cdots \to 0 \to \mathrm{Hom}_R(M, M') \to 0 \to \cdots) \longrightarrow \underline{\mathrm{Hom}}(F_\bullet(M), F_\bullet(M')).$$

Since $F_{MM}(\mathrm{id}_M) = \mathrm{id}_{F_\bullet(M)}$ and $F_{MM''}(gf) = F_{M'M''}(g)F_{MM'}(f)$ for all $g \in \mathrm{Hom}_R(M', M'')$, $F_\bullet$ yields a $\mathcal{V}$-functor in $[\mathrm{mod}\, R, \mathbf{Ch}(\mathrm{Mod}\, R)]$ denoted by the same letter.

IV. Next let $\beta : F_\bullet \to G_\bullet$ be a chain map in $\mathbf{Ch}[\mathrm{mod}\, R, \mathrm{Mod}\, R]$, then we have the following commutative diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & F_{n+1} & \xrightarrow{\partial_{n+1}} & F_n & \xrightarrow{\partial_n} & F_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \beta_{n+1}} & & \downarrow{\scriptstyle \beta_n} & & \downarrow{\scriptstyle \beta_{n-1}} & & \\
\cdots & \longrightarrow & G_{n+1} & \xrightarrow[\partial_{n+1}]{} & G_n & \xrightarrow[\partial_n]{} & G_{n-1} & \longrightarrow & \cdots
\end{array}$$

Note that each $\beta_{n,M} : R \to \mathrm{Hom}_R(F_n(M), G_n(M))$ is such that diagram (3.10) is commutative for it. So we are given maps $\beta_{n,M}(r) : F_n(M) \to G_n(M)$ for all $r \in R$.

One has a commutative diagram for all $r \in R$

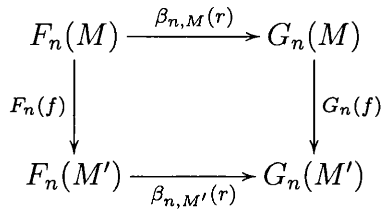
$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & F_{n+1}(M) & \xrightarrow{\partial_{n+1,M}(1)} & F_n(M) & \xrightarrow{\partial_{n,M}(1)} & F_{n-1}(M) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \beta_{n+1,M}(r)} & & \downarrow{\scriptstyle \beta_{n,M}(r)} & & \downarrow{\scriptstyle \beta n,M(r)} & & \\
\cdots & \longrightarrow & G_{n+1}(M) & \xrightarrow[\partial_{n+1,M}(1)]{} & G_n(M) & \xrightarrow[\partial_{n,M}(1)]{} & G_{n-1}(M)) & \longrightarrow & \cdots
\end{array}$$

In particular, $\beta_M(r) : F_\bullet(M) \to G_\bullet(M)$ is a chain map.

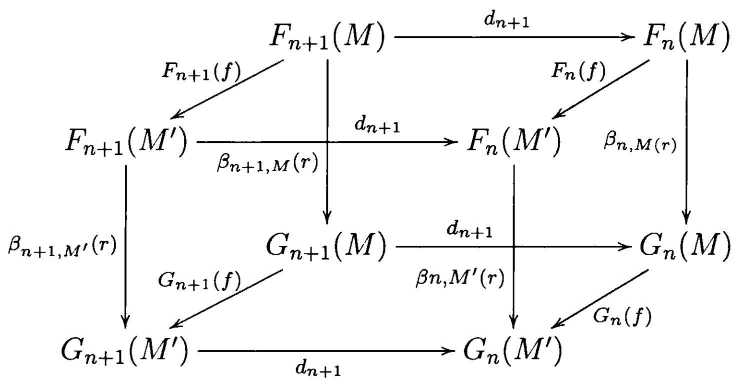
Now we want to show that the diagram

$$\begin{array}{ccc}
F_\bullet(M) & \xrightarrow{\beta_M(r)} & G_\bullet(M) \\
\downarrow{\scriptstyle F_\bullet(f)} & & \downarrow{\scriptstyle G_\bullet(f)} \\
F_\bullet(M') & \xrightarrow[\beta_{M'}(r)]{} & G_\bullet(M')
\end{array} \tag{6.2}$$

commutes. Commutativity of diagram (3.10) implies commutativity of the following diagram for all $n \in \mathbb{Z}$:



Commutativity of the latter together with the facts that $\beta_M(r), G_\bullet(f), F_\bullet(f)$ are all chain maps is enough to check commutativity of (6.2) as shown in the diagram below



Thus we have constructed a $\mathcal{V}$-natural map $\beta : F_\bullet \to G_\bullet$.

It is now easily verified that associations given in steps I–IV yield the desired isomorphisms of categories $[\mathrm{mod}\, R, \mathbf{Ch}(\mathrm{Mod}\, R)]$ and $\mathbf{Ch}\,\mathcal{C}_R$. □

Theorem 6.2.2. *Let R be a commutative ring, then $\mathbf{Ch}(\mathcal{C}_R)$ is a left and right proper closed symmetric monoidal $\mathcal{V}$-model category, where $\mathcal{V} = \mathbf{Ch}(\mathrm{Mod}\, R)$. The tensor product of two complexes $F_\bullet, G_\bullet \in \mathbf{Ch}(\mathcal{C}_R)$ is given by*

$$F_\bullet \odot G_\bullet = \int^{(M,N)\in \mathrm{mod}\, R \otimes \mathrm{mod}\, R} F_\bullet(M) \otimes_R G_\bullet(N) \otimes \mathrm{Hom}_R(M \otimes N, -). \tag{6.3}$$

Here $\mathrm{Hom}_R(M \otimes_R N, -)$ is regarded as a complex concentrated in zeroth degree. The internal Hom-object is defined as

$$\underline{\mathrm{Hom}}(F_\bullet, G_\bullet)(M) = \int_{N \in \mathrm{mod}\, R} \underline{\mathrm{Hom}}_{\mathbf{Ch}(\mathrm{Mod}\, R)}(F_\bullet(N), G_\bullet(M \otimes_R N)). \tag{6.4}$$

Moreover, $\mathbf{Ch}(\mathcal{C}_R)$ satisfies the monoid axiom in the sense of Schwede–Shipley [49].

Proof. By Theorem 6.2.1 we know that $\mathbf{Ch}\,\mathcal{C}_R$ can be identified with $[\operatorname{mod} R, \mathbf{Ch}(\operatorname{Mod} R)]$. Note that $\operatorname{mod} R$ is symmetric monoidal category enriched over $\mathbf{Ch}(\operatorname{Mod} R)$. Now formulas (6.3)-(6.4) as well as the fact that $\mathbf{Ch}\,\mathcal{C}_R$ is closed symmetric monoidal follow from Theorem 3.3.4.

It remains to show that $\mathbf{Ch}\,\mathcal{C}_R$ is a left and right proper closed symmetric monoidal $\mathcal{V}$-model category. Since the category $\mathbf{Ch}\,\mathcal{C}_R$ can be identified with $[\operatorname{mod} R, \mathbf{Ch}(\operatorname{Mod} R)]$ by Theorem 6.2.1, it is enough to verify this for the category $[\operatorname{mod} R, \mathbf{Ch}(\operatorname{Mod} R)]$. The category $\mathbf{Ch}(\operatorname{Mod} R)$ is a closed symmetric monoidal cofibrantly generated model category, where the weak equivalences are quasi-isomorphisms and the fibrations are surjective chain morphisms (see Theorem 4.3.18 and Example 4.4.5). Moreover, $\mathbf{Ch}(\operatorname{Mod} R)$ is weakly finitely generated in the sense of [10, Section 3.1]. Also, $\mathbf{Ch}(\operatorname{Mod} R)$ satisfies the monoid axiom in the sense of [49]. It follows from [10, 4.2] that $[\operatorname{mod} R, \mathbf{Ch}(\operatorname{Mod} R)]$ is a weakly finitely generated model category, where fibrations and weak equivalences are defined objectwise. Furthermore, the category $[\operatorname{mod} R, \mathbf{Ch}(\operatorname{Mod} R)]$ is a monoidal $\mathcal{V}$-enriched model category satisfying the monoid axiom by Theorem 4.6.5, because $\operatorname{mod} R$ is a symmetric monoidal category enriched over $\mathbf{Ch}(\operatorname{Mod} R)$. Finally, [10, 4.8] implies $[\operatorname{mod} R, \mathbf{Ch}(\operatorname{Mod} R)]$ is both left and right proper. □

Corollary 6.2.3. *Let $\mathcal{R}$ be a ring object in $\mathbf{Ch}(\mathcal{C}_R)$. Then the category of left $\mathcal{R}$-modules is a cofibrantly generated model Grothendieck category. If $\mathcal{R}$ is a commutative ring object, then the category of $\mathcal{R}$-modules is a cofibrantly generated, monoidal model category satisfying the monoid axiom, and the category of $\mathcal{R}$-algebras is a cofibrantly generated model category.*

Proof. This is a consequence of Lemma 5.2.8, Theorem 6.2.2 and [49, 4.1]. □

In order to construct some Grothendieck categories inside $\mathbf{Ch}(\mathcal{C}_R)$, we shall recall the notion of a coalgebra.

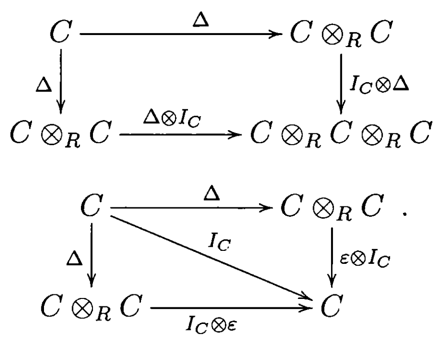
Definition 6.2.4. An *R-coalgebra* over a commutative ring R is an R-module C with R-linear maps

$$\Delta : C \to C \otimes_R C \quad \text{and } \varepsilon : C \to R,$$

called *(coassociative) coproduct* and *counit*, respectively, with the properties

$$(I_C \otimes \Delta) \circ \Delta = (\Delta \otimes I_C) \circ \Delta, \quad \text{and } (I_C \otimes \varepsilon) \circ \Delta = I_C = (\varepsilon \otimes I_C) \circ \Delta,$$

which can be expressed by commutativity of the diagrams



A coalgebra (C, Δ, ε) is said to be *cocommutative* if $\Delta = S \circ \Delta$, where

$$S : C \otimes_R C \to C \otimes_R C, \quad a \otimes b \to b \otimes a,$$

is the twist map.

Example 6.2.5. A typical example of a ring object in $\mathbf{Ch}(\mathcal{C}_R)$ is constructed as follows. Let C be a finitely presented R-module. We regard the representable functor $(C, -) \in \mathcal{C}_R$ as a complex in zeroth degree. Lemma 3.3.5 implies a natural isomorphism of representable functors

$$(C, -) \odot (C, -) \cong (C \otimes_R C, -).$$

It follows that $(C, -)$ is a ring object of $\mathbf{Ch}(\mathcal{C}_R)$ if and only if C is an R-coalgebra. It is a commutative ring object if and only if C is a cocommutative R-coalgebra. By the previous corollary we have a Grothendieck model category of $(C, -)$-modules inside $\mathbf{Ch}(\mathcal{C}_R)$.

Chapter 7

Almost stable homotopy category structure for $\mathcal{D}(\mathcal{C}_R)$

In this chapter we give another application of Theorem 6.2.2. Namely, we prove that the derived category $\mathcal{D}(\mathcal{C}_R)$ of the Grothendieck category of generalized modules is a closed symmetric monoidal compactly generated triangulated category with duality on compact objects. However, compact objects are not strongly dualizable as it will be shown below. Thus $\mathcal{D}(\mathcal{C}_R)$ is an example of a category which satisfies all the axioms of a unital algebraic stable homotopy theory in the sense of Hovey–Palmieri–Strickland [29] except the property that compact objects are strongly dualizable. This kind of categories is new. We refer to these as *unital algebraic almost stable homotopy categories.* A basic example of this class of categories is $\mathcal{D}(\mathcal{C}_R)$.

We start with the following

Theorem 7.0.6. *Let R be a commutative ring. Then the derived category $\mathcal{D}(\mathcal{C}_R)$ of the Grothendieck category $\mathcal{C}_R$ is a compactly generated triangulated closed symmetric monoidal category, where formulas* (6.3) *and* (6.4) *yield the derived tensor product $F_\bullet \odot^L G_\bullet$ and derived internal* Hom*-object $R\underline{\mathrm{Hom}}(F_\bullet, G_\bullet)$. The compact objects of $\mathcal{D}(\mathcal{C}_R)$ are the complexes isomorphic to bounded complexes of coherent functors in* $\mathrm{coh}\,\mathcal{C}_R$.

Proof. By Theorem 6.2.2 $\mathbf{Ch}(\mathcal{C}_R)$ is a left and right proper closed symmetric monoidal $\mathcal{V}$-model category, where $\mathcal{V} = \mathbf{Ch}(\mathrm{Mod}\,R)$. Hence the derived category $\mathcal{D}(\mathcal{C}_R)$ is identified with the homotopy category $\mathrm{Ho}(\mathbf{Ch}(\mathcal{C}_R))$. But the latter category is a closed symmetric monoidal category by [26, 4.3.2] with derived tensor product and internal Hom-functors induced by (6.3) and (6.4) from Theorem 6.2.2.

Since $\mathcal{C}_R$ is a Grothendieck category with finitely generated projective generators $\{(M,-)\}_{M\in \operatorname{mod} R}$, then its derived category $\mathcal{D}(\mathcal{C}_R)$ is a compactly generated triangulated category. The compact objects are those quasi-isomorphic to bounded complexes of representable functors. They are also called perfect complexes. Since every coherent functor $C \in \operatorname{coh}\mathcal{C}_R$ has a resolution (see, e.g., [25, 2.1])

$$0 \longrightarrow (L,-) \longrightarrow (K,-) \longrightarrow (M,-) \longrightarrow C \longrightarrow 0,$$

where $K, L, M \in \operatorname{mod} R$, then every bounded complex of coherent functors is quasi-isomorphic to a bounded complex of representable functors. It follows that every bounded complex of coherent functors is quasi-isomorphic to a perfect complex. This finishes the proof. □

Remark 7.0.7. A monoidal unit in $\mathbf{Ch}\mathcal{C}_R$ and $\mathcal{D}(\mathcal{C}_R)$ is $(R,-) \cong - \otimes_R R$ regarded as a complex concentrated in zeroth degree.

Definition 7.0.8. Let $\mathcal{V}$ be a closed symmetric monoidal additive category, with monoidal product $x \otimes y$, unit e, and internal function objects $\mathcal{V}(x,y)$. An object $x \in \mathcal{V}$ is *strongly dualizable* if the natural map $\mathcal{V}(x,e) \otimes y \to \mathcal{V}(x,y)$ is an isomorphism for all y. We shall also write $x^\vee$ to denote $\mathcal{V}(x,e)$.

It follows from [38, Theorem 7.1.6] that the functor

$$x \in \mathcal{V} \mapsto x^\vee \in \mathcal{V}$$

puts the full subcategory of strongly dualizable objects of $\mathcal{V}$ in duality.

We want to show below that $\mathcal{D}(\mathcal{C}_R)$ has a duality on the full subcategory of compact objects $\mathcal{D}(\mathcal{C}_R)^c$ but these are not strongly dualizable in general. To this end we shall need a categorical duality

$$D : (\operatorname{coh}\mathcal{C}_R)^{\mathrm{op}} \to \operatorname{coh}{}_R\mathcal{C}$$

of Auslander [1] and Gruson–Jensen [23] (see [25] as well) defined over any non-commutative ring R as follows. Given ${}_RN \in R\operatorname{mod}$, we have

$$(DC)({}_RN) := \operatorname{Hom}_{\mathcal{C}_R}(C, - \otimes_R N).$$

If $\eta : B \to C$ is a morphism in $\operatorname{coh}\mathcal{C}_R$, then

$$D(\eta)_N : D(C)({}_RN) \to D(B)({}_RN)$$

is defined to be $\operatorname{Hom}_{\mathcal{C}_R}(\eta, - \otimes_R N)$. For $M_R \in \operatorname{mod} R$ and ${}_RN \in R\operatorname{mod}$ we have that

$$D(M_R, -) \cong M \otimes_R - \quad \text{and} \quad D(- \otimes_R N) \cong ({}_RN, -).$$

We shall refer to this duality as the *Auslander–Gruson–Jensen Duality*.

Suppose now R is commutative. Then the category $\mathcal{C}_R$ is closed symmetric monoidal for the same reasons that $\mathbf{Ch}\,\mathcal{C}_R$ is. The monoidal product $C \odot C'$ and internal Hom-object $\underline{\operatorname{Hom}}(C, C')$ are computed by formulas, which are similar to (6.3)-(6.4). It can also be shown that the Auslander–Gruson–Jensen Duality D defined above is isomorphic to the internal Hom-functor

$$\underline{\operatorname{Hom}}(-, (R, -)) \cong \underline{\operatorname{Hom}}(-, - \otimes_R R)$$

(we refer the reader to [8] for further details).

The following example shows that compact objects of $\mathcal{D}(\mathcal{C}_R)$ are not strongly dualizable in general.

Example 7.0.9. There are objects $C \in \mathcal{D}(\mathcal{C}_R)^c$ and $X \in \mathcal{D}(\mathcal{C}_R)$ such that the natural arrow

$$C^\vee \odot^L X \to R\underline{\operatorname{Hom}}(C, X) \tag{7.1}$$

is not an isomorphism, where

$$C^\vee := R\underline{\operatorname{Hom}}(C, - \otimes_R R) = R\underline{\operatorname{Hom}}(C, (R, -)).$$

Let $R = \mathbb{Z}$ and $M = N = \mathbb{Z}_2 \in \operatorname{mod}\mathbb{Z}$. We have an exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow 0.$$

We want to compute $(- \otimes M)^\vee \odot^L - \otimes N = (- \otimes \mathbb{Z}_2)^\vee \odot^L - \otimes \mathbb{Z}_2$.

To compute $(- \otimes \mathbb{Z}_2)^\vee$, consider a projective resolution for $- \otimes \mathbb{Z}_2$ in $\mathcal{C}_\mathbb{Z}$

$$0 \longrightarrow (\mathbb{Z}_2, -) \longrightarrow (\mathbb{Z}, -) \xrightarrow{(2,-)} (\mathbb{Z}, -) \longrightarrow - \otimes \mathbb{Z}_2 \longrightarrow 0.$$

Then $(-\otimes\mathbb{Z}_2)^\vee$ is the value of $\underline{\mathrm{Hom}}(-,(\mathbb{Z},-))$ at the projective resolution of $-\otimes\mathbb{Z}_2$. Since $\underline{\mathrm{Hom}}(-,(\mathbb{Z},-))$ puts the category of coherent objects $\mathrm{coh}\,\mathcal{C}_\mathbb{Z}$ in duality and takes representable functors $(L,-)$ to $-\otimes L$, then $(-\otimes\mathbb{Z}_2)^\vee$ is the complex

$$\cdots\to 0\to -\otimes\mathbb{Z}\to -\otimes\mathbb{Z}\to -\otimes\mathbb{Z}_2\to 0\to\cdots$$

This complex has only one non-zero homology group $(\mathbb{Z}_2,-)$, which we place in zeroth degree as a complex. We see that $(-\otimes\mathbb{Z}_2)^\vee\cong(\mathbb{Z}_2,-)$.

We take a projective resolution for $-\otimes N$ in $\mathrm{coh}\,\mathcal{C}_\mathbb{Z}$ as above

$$0\longrightarrow(\mathbb{Z}_2,-)\longrightarrow(\mathbb{Z},-)\xrightarrow{(2,-)}(\mathbb{Z},-)\longrightarrow -\otimes\mathbb{Z}_2\longrightarrow 0.$$

Tensoring it with $(M,-)=(\mathbb{Z}_2,-)$ we get $(M,-)\odot^L-\otimes N$ which is the complex

$$\cdots\to 0\to(\mathbb{Z}_2,-)\odot(\mathbb{Z}_2,-)\to(\mathbb{Z}_2,-)\odot(\mathbb{Z},-)\to(\mathbb{Z}_2,-)\odot(\mathbb{Z},-)\to 0\to\cdots$$

By Lemma 3.3.5 it equals the complex

$$\cdots\longrightarrow 0\longrightarrow(\mathbb{Z}_2,-)\longrightarrow(\mathbb{Z}_2,-)\xrightarrow{0}(\mathbb{Z}_2,-)\longrightarrow 0\longrightarrow\cdots$$

Note that evaluation of this complex at $\mathbb{Z}$ is zero.

Now compute $R\underline{\mathrm{Hom}}(-\otimes M,-\otimes N)$. It is the value of $\underline{\mathrm{Hom}}(-,-\otimes N)$ at the projective resolution of $-\otimes M$

$$0\longrightarrow(\mathbb{Z}_2,-)\longrightarrow(\mathbb{Z},-)\xrightarrow{(2,-)}(\mathbb{Z},-)\longrightarrow -\otimes M\longrightarrow 0.$$

Applying $\underline{\mathrm{Hom}}(-,-\otimes N)$ to the complex

$$0\longrightarrow(\mathbb{Z}_2,-)\longrightarrow(\mathbb{Z},-)\xrightarrow{(2,-)}(\mathbb{Z},-)\longrightarrow 0$$

we get a complex

$$0\to\underline{\mathrm{Hom}}((\mathbb{Z},-),-\otimes\mathbb{Z}_2)\to\underline{\mathrm{Hom}}((\mathbb{Z},-),-\otimes\mathbb{Z}_2)\to\underline{\mathrm{Hom}}((\mathbb{Z}_2,-),-\otimes\mathbb{Z}_2)\to 0$$

Using enriched Yoneda Lemma 3.3.1, it is equal to

$$0\longrightarrow -\otimes\mathbb{Z}_2\xrightarrow{0}-\otimes\mathbb{Z}_2\xrightarrow{\cong}-\otimes\mathbb{Z}_2\longrightarrow 0.$$

The value of this complex at $\mathbb{Z}$ has non-trivial homology. We conclude that (7.1) cannot be an isomorphism in general. We conclude that compact objects of $\mathcal{D}(\mathcal{C}_R)$ are not strongly dualizable.

Lemma 7.0.10. *The triangulated category $\mathcal{D}(\mathcal{C}_R)^c$ of compact objects of $\mathcal{D}(\mathcal{C}_R)$ is triangle equivalent to the derived category $\mathcal{D}^b(\operatorname{coh}\mathcal{C}_R)$ of bounded complexes in* $\operatorname{coh}\mathcal{C}_R$.

Proof. By the proof of Theorem 7.0.6 $\mathcal{D}(\mathcal{C}_R)^c$ is triangle equivalent to the full subcategory $\widetilde{\mathcal{D}}(\mathcal{C}_R)^c$ of bounded complexes in $\operatorname{coh}\mathcal{C}_R$. Let

$$\begin{array}{ccc} & X & \\ {\scriptstyle\sim}\swarrow & & \searrow \\ M & & N \end{array}$$

be a morphism in $\mathcal{D}(\mathcal{C}_R)$ with $M, N \in \widetilde{\mathcal{D}}(\mathcal{C}_R)^c$. Let $P \to M$ be a projective resolution of M. Then P is in $\widetilde{\mathcal{D}}(\mathcal{C}_R)^c$ and P is isomorphic to X in $\mathcal{D}(\mathcal{C}_R)$. But P is a bounded complex of projectives in $\mathcal{C}_R$. Therefore there is a quasi-isomorphism $P \to X$. Then we get a diagram in $\widetilde{\mathcal{D}}(\mathcal{C}_R)^c$

$$\begin{array}{ccc} & P & \\ & {\scriptstyle\sim}\downarrow & \\ & X & \\ {\scriptstyle\sim}\swarrow & & \searrow \\ M & & N \end{array}$$

By [33, 9.1] the natural functor

$$\mathcal{D}^b(\operatorname{coh}\mathcal{C}_R) \to \widetilde{\mathcal{D}}(\mathcal{C}_R)^c$$

is fully faithful. But objects are the same, and therefore these subcategories of $\mathcal{D}(\mathcal{C}_R)$ coincide. □

Lemma 7.0.11. *There is a triangle equivalence of triangulated categories $\mathcal{D}^b(\operatorname{coh}\mathcal{C}_R)$ and $\mathcal{D}^b((\operatorname{coh}\mathcal{C}_R)^{\mathrm{op}})$ taking $X \in \mathcal{D}^b(\operatorname{coh}\mathcal{C}_R)$ to $\underline{\mathrm{Hom}}(X, - \otimes R)$.*

Proof. By [8] $\underline{\mathrm{Hom}}(-, - \otimes R) : \operatorname{coh}\mathcal{C}_R \to (\operatorname{coh}\mathcal{C}_R)^{\mathrm{op}}$ is an equivalence of abelian categories. Moreover, this functor is isomorphic to the Auslander–Gruson–Jensen duality (see [8, 4.6]). The fact that equivalent abelian categories have equivalent derived categories finishes the proof. □

Corollary 7.0.12. *$\mathcal{D}^b(\operatorname{coh}\mathcal{C}_R)$ is triangle equivalent to $(\mathcal{D}^b(\operatorname{coh}\mathcal{C}_R))^{\mathrm{op}}$.*

Proof. This follows from the previous lemma and the fact that $\mathcal{D}^b(\mathcal{A}^{\mathrm{op}})$ is triangle equivalent to $(\mathcal{D}^c(\mathcal{A}))^{\mathrm{op}}$ for any abelian category $\mathcal{A}$. □

Theorem 7.0.13 (Auslander–Gruson–Jensen Duality for compact objects). *Let $\mathcal{D}(\mathcal{C}_R)^c$ be the full triangulated subcategory of $\mathcal{D}(\mathcal{C}_R)$ of compact objects. Then there is a duality*

$$D : (\mathcal{D}(\mathcal{C}_R)^c)^{\mathrm{op}} \to \mathcal{D}(\mathcal{C}_R)^c$$

that takes a compact object $C_\bullet$ to

$$DC_\bullet := R\underline{\mathrm{Hom}}(C_\bullet, - \otimes_R R).$$

Proof. By Lemma 7.0.10 $\mathcal{D}(\mathcal{C}_R)^c \simeq \mathcal{D}^b(\operatorname{coh}\mathcal{C}_R)$. Let $\mathcal{D}^b(\operatorname{proj}\mathcal{C}_R)$ be a full subcategory of $\mathcal{D}(\mathcal{C}_R)^c$ consisting of bounded complexes of representable coherent functors. The composition

$$\mathcal{D}^b(\operatorname{proj}\mathcal{C}_R) \longrightarrow \mathcal{D}(\mathcal{C}_R)^c \xrightarrow{\simeq} \mathcal{D}^b(\operatorname{coh}\mathcal{C}_R)$$

is an equivalence of triangulated categories.

Lemma 7.0.11 and Corollary 7.0.12 imply $\underline{\mathrm{Hom}}(-, - \otimes R)$ is an equivalence of triangulated categories $\mathcal{D}^b(\operatorname{coh}\mathcal{C}_R) \simeq (\mathcal{D}^b(\operatorname{coh}\mathcal{C}_R))^{\mathrm{op}}$.

Now the composite of equivalences

$$\mathcal{D}^b(\operatorname{proj}\mathcal{C}_R) \to \mathcal{D}(\mathcal{C}_R)^c \to \mathcal{D}^b(\operatorname{coh}\mathcal{C}_R) \to (\mathcal{D}^b(\operatorname{coh}\mathcal{C}_R))^{\mathrm{op}} \to (\mathcal{D}(\mathcal{C}_R)^c)^{\mathrm{op}}$$

computes the desired equivalence $R\underline{\mathrm{Hom}}(-, - \otimes R)$ of triangulated categories. □

Definition 7.0.14 (Hovey–Palmieri–Strickland [29]). A *stable homotopy category* is a category $\mathcal{C}$ with the following extra structure:

1. A triangulation.
2. A closed symmetric monoidal structure, compatible with the triangulation.
3. A set $\mathcal{G}$ of strongly dualizable objects of $\mathcal{C}$, such that the only localizing subcategory of $\mathcal{C}$ containing $\mathcal{G}$ is $\mathcal{C}$ itself.
   We also assume that $\mathcal{C}$ satisfies the following:

4. Arbitrary coproducts of objects of $\mathcal{C}$ exist.

5. Every cohomology functor on $\mathcal{C}$ is representable.

We shall say that such a category $\mathcal{C}$ is *algebraic* if the objects of $\mathcal{G}$ are compact. If, in addition, the unit object e is compact, we say that $\mathcal{C}$ is *unital algebraic.*

Definition 7.0.15. An *almost stable homotopy category* is a category $\mathcal{C}$ which satisfies axioms (1)-(2) and (4)-(5) for a stable homotopy category and the following axiom:

(3′) There is a full small subcategory $\mathcal{G}$ of $\mathcal{C}$ with duality $D : \mathcal{G}^{\mathrm{op}} \to \mathcal{G}$, such that the only localizing subcategory of $\mathcal{C}$ containing $\mathcal{G}$ is $\mathcal{C}$ itself.

Algebraic and *unital algebraic* almost stable homotopy categories are defined as in Definition 7.0.14.

Remark 7.0.16. Every stable homotopy category $\mathcal{C}$ with generating set of strongly dualizable $\mathcal{G}$ is an almost stable homotopy category. Indeed, we can assume without loss of generality that $x^\vee \in \mathcal{G}$ for every $x \in \mathcal{G}$, and then the full subcategory of $\mathcal{C}$ whose objects are those of $\mathcal{G}$ has a duality $x \mapsto x^\vee$ and generates $\mathcal{C}$. The following theorem shows that there are algebraic almost stable homotopy categories which are not stable homotopy categories.

Theorem 7.0.17. *Let R be a commutative ring. Then $\mathcal{D}(\mathcal{C}_R)$ is a unital algebraic almost stable homotopy category, which is not an algebraic stable homotopy category in the sense of Definition 7.0.14.*

Proof. Let $\mathcal{G}$ be the full subcategory of compact objects of $\mathcal{D}(\mathcal{C}_R)$. The fact that $\mathcal{D}(\mathcal{C}_R)$ is a unital algebraic almost stable homotopy category follows from Theorems 7.0.6 and 7.0.13. We also use here the fact that every cohomology functor on a compactly generated triangulated category is representable by a theorem of Neeman [42, Theorem 3.1].

Suppose $\mathcal{D}(\mathcal{C}_R)$ is generated by compact strongly dualizable objects $\mathcal{G}$ as required for an algebraic stable homotopy category. By [29, Theorem A.2.5] we may assume without loss of generality that $\mathcal{G}$ is a thick subcategory in the triangulated category of compact objects. If $\mathcal{G}$ generated $\mathcal{D}(\mathcal{C}_R)$, then another theorem of

Neeman [42, Theorem 2.1] would imply that $\mathcal{G}$ contains all compact objects. But Example 7.0.9 shows that compact objects of $\mathcal{D}(\mathcal{C}_R)$ are not strongly dualizable in general. This contradiction finishes the proof. □

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