On a class of one-parameter operator semigroups with state space Rn x Zm generated by pseudo-differential operators.

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On a Class of One-parameter Operator Semigroups with State Space $\mathbb{R}^n \times \mathbb{Z}^m$ Generated by Pseudo-differential Operators

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Abstract

The thesis shows that, under suitable conditions, a pseudo-differential operator, defined on some "nice" set of functions on $\mathbb{R}^n \times \mathbb{Z}^m$, with continuous negative definite symbol $q(x, \xi, \omega)$ extends to a generator of a Feller semigroup. Sections 1-5 are the preliminary sections, these sections discuss some harmonic analysis concerning locally compact Abelian groups. The essence of this thesis are Sections 6-13, which deals with obtaining the estimates required for the fulfilment of the conditions of the Hille-Yosida-Ray theorem.

Contents

Notation

In the following H denotes a locally compact Abelian group and $G \subset \mathbb{R}^n$ denotes an open set.

$\mathbb{N}$ natural numbers
$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$
$\mathbb{N}_0^n$ set of all multiindices
$\mathbb{Q}$ rational numbers
$\mathbb{R}$ real numbers
$\mathbb{R}^n$ euclidean vector space
$\mathbb{Z}$ integers
$\mathbb{Z}^m$ lattice points in $\mathbb{R}^m$
$\mathbb{Z}/N\mathbb{Z}$ integers of modulo N
$\mathbb{Z}(N)$ N^{th} roots of unity
$\mathbb{C}$ complex numbers
$\mathbb{C}^m$ unitary vector space
S^1 circle group
$\mathbb{T}^m$ m-dimensional torus group
$[0, 2\pi)^m = \underbrace{[0, 2\pi) \times \ldots \times [0, 2\pi)}_{\text{m times}}$
$a \wedge b = \min(a, b)$
$A \backslash B$ set theoretical difference of two sets
$\overline{A}^{\|\cdot\|_X}$ closure of the set A with respect to the norm $\|\cdot\|_X$
$|\alpha| = \alpha_1 + \ldots + \alpha_n$ for $\alpha \in \mathbb{N}_0^n$
$x^\alpha = x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}$, $\alpha \in \mathbb{N}_0^n$ and $x \in \mathbb{R}^n$
$\partial_x^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$
$D_x^\alpha u = (-i\partial_x)^\alpha u$
τ topology

$(H, \cdot)$ group
(X, τ) topological space
$(H, \cdot, \tau)$ topological group

$\operatorname{Re} f$ real part of a function
$(u_n)_{n \in \mathbb{N}}$ sequence in $\mathbb{C}$
$(u_k)_{k \in A}$ sequence in $\mathbb{C}$ with index A
$f_\nu \rightharpoonup f$ sequence of functions converging weakly to f
$u \cdot v$ product of functions
$u \circ v$ composition of functions
$u \otimes v$ tensor product of functions

$u \oplus v$ direct sum of functions
$u * v$ convolution of functions
$u *_{\mathbb{R}^n} v$ convolution of functions $u, v : \mathbb{R}^n \times \mathbb{Z}^m \to \mathbb{C}$ with respect to the variable $x \in \mathbb{R}^n$
$\chi : H \to S^1$ character on H
$\widehat{\chi_x} : \widehat{H} \to S^1$ character on $\widehat{H}$
j_ε, $J_\varepsilon(u)$ Friedrichs mollifier
$\tau_a(x) = x + a$ translation mapping on H
$\tau_a u(x) = u(x - a)$
$Sx = -x$ reflection at the origin
$H_\lambda x = \lambda x$ homothetic map
$\check{u}(x) = u(-x)$
$\tilde{\psi}(x) = \overline{\psi(-x)}$
$\hat{u}$, $F(u)$ Fourier transform of a function
$F_{x\mapsto\xi}(u)$ Fourier transform of a function $u : \mathbb{R}^n \to \mathbb{C}$
$F_{k\mapsto\omega}(u)$ Fourier transform of a function $u : \mathbb{Z}^m \to \mathbb{C}$
$F_{\omega\mapsto k}(u)$ Fourier transform of a function $u : \mathbb{T}^m \to \mathbb{C}$
$\hat{u}(\xi, k)$ Fourier transform of a function $u(\cdot, k) : \mathbb{R}^n \to \mathbb{C}$, $k \in \mathbb{Z}^m$ fixed
$\hat{u}(\eta, \xi, \omega)$ Fourier transform of a function $u(\cdot, \xi, \omega) : \mathbb{R}^n \to \mathbb{C}$, $\xi \in \mathbb{R}^n$ and $\omega \in \mathbb{T}^m$ both fixed
$\tilde{u}(\xi, \omega)$ Fourier transform of a function $u : \mathbb{R}^n \times \mathbb{Z}^m \to \mathbb{C}$
$F^{-1}(u)$ inverse Fourier transform of a function
$F^{-1}_{\xi\mapsto x}(u)$ inverse Fourier transform of a function $u : \mathbb{R}^n \to \mathbb{C}$
$F^{-1}_{\omega\mapsto k}(u)$ inverse Fourier transform of a function $u : \mathbb{T}^m \to \mathbb{C}$
$\operatorname{supp} u$ support of a function
$\operatorname{ess\,sup} u$ essential supremum of a function

$\mathcal{A}$ σ-algebra
$\mathcal{B}(H)$ Borel σ-algebra of H
μ measure
$(H, \mathcal{A})$ measurable space
$(H, \mathcal{A}, \mu)$ measure space
μ_H Haar measure on H
$\lambda^{(n)}$ Lebesgue measure on $\mathbb{R}^n$
ε_a Dirac measure at $a \in H$
$\mu_{\mathbb{R}^n} = \lambda^{(n)}$
$\mu_{\mathbb{Z}^m} = \sum_{k\in\mathbb{Z}^m} \varepsilon_k$ Haar measure on $\mathbb{Z}^m$
$\mu_{\mathbb{Z}/N\mathbb{Z}} = \frac{1}{N} \sum_{k=0}^{N-1} \varepsilon_k$ Haar measure on $\mathbb{Z}/N\mathbb{Z}$
$\mu_{\mathbb{T}^m} = \frac{1}{(2\pi)^m} \lambda^{(m)}$ Haar measure on $\mathbb{T}^m$
$\mu_1 \otimes \mu_2$ product of the measures μ_1 and μ_2

$\mu_1 * \mu_2$ convolution of the measures μ_1 and μ_2
$\widehat{\mu}_{\widehat{H}}$ dual Haar measure
$\hat{\mu}$ Fourier transform of a measure
$\|\mu\|$ total mass of a measure μ
$I_H(u) = \int_H u(x)\, \mu_H(\mathrm{d}x)$ Haar integral on H
$(\mu_t)_{t\geq 0}$ convolution semigroup of subprobabilities

$H^* = \mathrm{Hom}(H, S^1)$ group-homomorphisms on H
$V(H)$ functions on H
$\mathrm{span}\{e_k \mid e_k(x) = e^{ik\cdot x},\ k \in \mathbb{Z}^m\},\ x \in \mathbb{T}^m = [0, 2\pi)^m,$ trigonometric polynomials on $\mathbb{T}^m$
$V \otimes W = \{\sum_{l=1}^k v_l \otimes w_l \mid v_l \in V \text{ and } w_l \in W,\ k \in \mathbb{N}\}$
$C(H)$ continuous functions on H
$C_0(H)$ continuous functions on H with compact support
$C_0(\mathbb{Z}^m) = l_{\text{finite}}(\mathbb{Z}^m)$ finite sequences on $\mathbb{Z}^m$
$C_\infty(H)$ continuous functions on H vanishing at infinity
$C_b(H)$ bounded continuous functions on H
$C^m(G)$ m-times continuously differentiable functions on G
$C_0^m(G) = C^m(G) \cap C_0(G)$
$C^\infty(G) = \cap_{m\in\mathbb{N}} C^m(G)$
$C_0^\infty(G) = \cap_{m\in\mathbb{N}} C_0^m(G)$
$L^p(H, \mu_H) = \{u \in V(H) \text{ measurable} \mid \|u\|_{L^p(H,\mu_H)} < \infty\}$
$l^2(\mathbb{Z}^m) = L^2(\mathbb{Z}^m)$
$L^p_{per}(\mathbb{R}^m, \frac{1}{(2\pi)^m}\lambda^{(m)}) = L^p(\mathbb{T}^m),\ 1 \leq p < \infty$
$L^1_+(H, \mu_H) = \{u \in L^1(H, \mu_H) \mid u \geq 0\}$
$L^\infty(H, \mu_H) = \{u \in V(H) \text{ measurable} \mid \|u\|_{\infty,H} < \infty\}$
$H^{\psi,s}(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) \mid \|u\|_{H^{\psi,s}(\mathbb{R}^n)} < \infty\}$
$H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m) = \{u \in L^2(\mathbb{R}^n \times \mathbb{Z}^m) \mid \|u\|_{H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)} < \infty\}$
$S(\mathbb{R}^n) = \{u \in C^\infty(\mathbb{R}^n) \mid p_{m_1,m_2}(u) < \infty \text{ for all } m_1, m_2 \in \mathbb{N}_0\}$ Schwartz space of functions
$S(\mathbb{R}^n \times \mathbb{Z}^m) = \{u : \mathbb{R}^n \times \mathbb{Z}^m \to \mathbb{C},\ u(\cdot, k) \in C^\infty(\mathbb{R}^n) \mid |\partial_x^\alpha u(x, k)| \leq c_{r,s,\alpha,u}(1 + |x|^2)^{-\frac{r}{2}}(1 + |k|^2)^{-\frac{s}{2}} \text{ for all } \alpha \in \mathbb{N}_0^n, r, s \in \mathbb{N}_0\}$
$\mathcal{M}^+(H)$ Borel measures on H
$\mathcal{M}_b^+(H)$ bounded Borel measures on H
$\mathcal{M}(H)$ signed Borel measures on H
$\mathcal{M}_b(H)$ bounded signed Borel measures on H
$\mathcal{M}_{\mathbb{C}}^+(H)$ complex-valued Borel measures on H
$\mathcal{M}_{b,\mathbb{C}}^+(H)$ bounded complex-valued Borel measures on H
$\mathcal{M}_{\mathbb{C}}(H)$ complex-valued signed Borel measures on H
$\mathcal{M}_{b,\mathbb{C}}(H)$ bounded complex-valued signed Borel measures on H

$\widehat{H}$ dual group of H
$\widehat{\widehat{H}}$ double dual group of H
$\overline{H} = H \cup \{\infty\}$ one-point compactification of H
$CN(H)$ continuous negative definite functions on H
$CP(H)$ continuous positive definite functions on H
$N(H)$ negative definite functions on H
$P(H)$ positive definite functions on H

$|x|$ euclidean distance in $\mathbb{R}^n$
$|k| = (k_1^2 + \ldots + k_m^2)^{\frac{1}{2}},\ k \in \mathbb{Z}^m$
$|k|_1 = |k_1| + \ldots + |k_m|,\ k \in \mathbb{Z}^m$
$|k|_\infty = \max_{1 \le j \le m} |k_j|,\ k \in \mathbb{Z}^m$
$|z|$ euclidean distance in $\mathbb{C}^m$
$\|u\|_X$ norm of u in the space X
$\|u\|_{L^p(H,\mu_H)}$ norm in the space $L^p(H, \mu_H)$
$\|u\|_{L^p_{per}(\mathbb{R}^m, \frac{1}{(2\pi)^m}\lambda^{(m)})} = \|u\|_{L^p(\mathbb{T}^m)},\ 1 \le p < \infty$
$\|u\|_0 = \|u\|_{L^2(\mathbb{R}^n \times \mathbb{Z}^m)}$
$\|u\|_{\infty,H} = \|u\|_\infty = \max|u(x)|$, $\sup|u(x)|$ or $\operatorname{ess\,sup}|u(x)|$
$\|u\|_{H^{\psi,s}(\mathbb{R}^n)}$ norm in the space $H^{\psi,s}(\mathbb{R}^n)$
$\|u\|_{H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m)}$ norm in the space $H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m)$
$(u,v)_{L^2(H,\mu_H)}$ inner product on $L^2(H, \mu_H)$
$(u,v)_{L^2_{per}(\mathbb{R}^m, \frac{1}{(2\pi)^m}\lambda^{(m)})} = (u,v)_{L^2(\mathbb{T}^m)}$
$(u,v)_0 = (u,v)_{L^2(\mathbb{R}^n \times \mathbb{Z}^m)}$
$p_{m_1,m_2}(u) = \sup_{x \in \mathbb{R}^n}((1+|x|^2)^{\frac{m_1}{2}} \sum_{|\alpha| \le m_2} |\partial_x^\alpha u(x)|)$
$p_{\alpha,\beta}(u) = \sup_{x \in \mathbb{R}^n} |x^\alpha \partial_x^\beta u(x)|$

$(A, D(A))$ linear operator with domain $D(A)$
$D(A)$ domain of an operator
$R(A)$ range of an operator
$[A,B] = AB - BA$ commutator of operators
$B(u,v) = (q(x,D_x,D_k)u,v)_{L^2(\mathbb{R}^n \times \mathbb{Z}^m)}$ sesquilinear form associated with $q(x, D_x, D_k)$
$B^{(1)}(u,v) = (q_1(D_x,D_k)u,v)_{L^2(\mathbb{R}^n \times \mathbb{Z}^m)}$ sesquilinear form associated with $q_1(D_x, D_k)$
$B^{(2)}(u,v) = (q_2(x,D_x,D_k)u,v)_{L^2(\mathbb{R}^n \times \mathbb{Z}^m)}$ sesquilinear form associated with $q_2(x, D_x, D_k)$
$B_\lambda(u,v) = B(u,v) + \lambda(u,v)_{L^2(\mathbb{R}^n \times \mathbb{Z}^m)}$ for a sesquilinear form B
$q(x, D_x)$ pseudo-differential operator with symbol $q(x,\xi)$

$q(x, D_x, D_k)$ pseudo-differential operator with symbol $q(x, \xi, \omega)$
$q(D_x, D_k)$ pseudo-differential operator with symbol $q(\xi, \omega)$
$\psi(D_x)$ pseudo-differential operator with symbol $\psi(\xi)$
$(T_t)_{t\geq 0}$ one parameter semigroup of operators

Introduction

The famous **Hille-Yosida theorem** (in a variant due to Lumer and Phillips) states that a densely defined linear operator $(A, D(A))$ in a Banach space X is closable and its closure generates a strongly continuous contraction semigroup on X if and only if $(A, D(A))$ is **dissipative**, i.e.

$$|\lambda| \|u\|_X \leq \|(\lambda - A)u\|_X \quad \text{for all } \lambda > 0 \text{ and } u \in D(A),$$

and if for one, and hence for all, $\lambda > 0$ the range of $\lambda - A$ is dense in X, i.e. $\overline{(\lambda - A)(D(A))} = X$. Note that the second condition is a statement about the solvability of the equation $\lambda u - Au = f$, f in a dense set of X. In the case that X is a Banach space of real-valued functions defined on some Hausdorff space Y, say $X = C_\infty(Y) := \{u : Y \to \mathbb{R} \mid u \text{ is continuous and } \lim_{y\to\infty} u(y) = 0\}$, or $L^2(Y, \nu)$, where ν is a suitable measure on the Borel sets of Y, it makes sense to consider semigroups of operators which preserve positivity, i.e. $u \geq 0$ (or $u \geq 0$ ν-a.e.) implies $T_t u \geq 0$ (or $T_t u \geq 0$ ν-a.e.). Positivity preserving strongly continuous contraction semigroups on $C_\infty(Y)$ are called **Feller semigroups**. Recall a one-parameter semigroup $(T_t)_{t\geq 0}$ of bounded linear operators $T_t : X \to X$, $t \geq 0$, on a Banach space X is called a strongly continuous contraction semigroup if $T_t \circ T_s = T_{t+s}$ for all $s, t \geq 0$ and $T_0 = \mathrm{id}$, $\lim_{t\to 0} \|T_t u - u\|_X = 0$ for all $u \in X$ and $\|T_t\| \leq 1$ for all $t \geq 0$. It is well known that their generator must satisfy the **positive maximum principle**, i.e. if $u \in D(A)$ and $y_0 \in Y$ such that $u(y_0) = \sup_{y\in Y} u(y) \geq 0$, then $(Au)(y_0) \leq 0$, and in the Hille-Yosida theorem the dissipativity can be replaced by the positive maximum principle. For details on this paragraph we refer to [19].

If $Y = \mathbb{R}^n$, and the domain $D(A)$ of a generator A of a Feller semigroup contains the test functions $C_0^\infty(\mathbb{R}^n)$ and A maps $C_0^\infty(\mathbb{R}^n)$ into bounded continuous functions, then on $C_0^\infty(\mathbb{R}^n)$ (and larger sets such as the Schwartz space $S(\mathbb{R}^n)$) the operator A is a **pseudo-differential operator**

$$Au(x) := -q(x, D_x)u(x) := -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} q(x, \xi) \hat{u}(\xi) \,\mathrm{d}\xi$$

where $\hat{u}$ denotes the Fourier transform of u and the symbol $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ is a continuous function such that for every $x \in \mathbb{R}^n$ fixed $\xi \mapsto q(x, \xi)$ is continuous negative definite, i.e. $q(x, 0) \geq 0$ and $\xi \mapsto e^{-tq(x,\xi)}$ is for all $t > 0$ a continuous positive definite in the sense of Bochner. We call $q(x, \xi)$ the **symbol** of $q(x, D_x)$. Such an operator we call a pseudo-differential operator with negative definite symbol. Note that we always assume functions in $C_\infty(\mathbb{R}^n)$ (or $C_\infty(\mathbb{R}^n \times \mathbb{Z}^m)$) to be real-valued when dealing with the positive maximum principle or Feller semigroups. In [18] and further papers Jacob started to approach the converse problem: given a pseudo-differential operator with a symbol as above, when does it have an extension generating a Feller semigroup. Meanwhile, a lot of such results are known, many due to W. Hoh, e.g. [14]-[16].

Recently Evans and Jacob [12] made the observation that the Q-matrices, i.e. generators of Markov chains with state space $\mathbb{Z}^m$, have (under certain conditions) a representation of a pseudo-differential operator, the symbol of which is now defined on $\mathbb{Z}^m \times \mathbb{T}^m$.

The natural question is, given a pseudo-differential operator acting on "nice" functions defined on $\mathbb{R}^n \times \mathbb{Z}^m$, when does such an operator extend to a generator of a strongly continuous contraction semigroup on $C_\infty(\mathbb{R}^n \times \mathbb{Z}^m)$ and when is this semigroup even positivity preserving.

In this thesis we investigate pseudo-differential operators, with symbols $q(x, \xi, \omega)$ (where $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$ and $\omega \in \mathbb{T}^m$), defined by

$$-q(x, D_x, D_k)u(x, k) := -(2\pi)^{-\frac{n}{2}-m} \int_{\mathbb{R}^n} \int_{\mathbb{T}^m} e^{ix\cdot\xi} e^{ik\cdot\omega} q(x, \xi, \omega) \tilde{u}(\xi, \omega) \, \mathrm{d}\omega \mathrm{d}\xi,$$

where $\tilde{u}(\xi, \omega)$ denotes the Fourier transform of the function $u : \mathbb{R}^n \times \mathbb{Z}^m \to \mathbb{C}$. Further we take $q(x, \xi, \omega)$ to have the decomposition

$$q(x, \xi, \omega) = q_1(\xi, \omega) + q_2(x, \xi, \omega).$$

Note that the corresponding operators are translation-invariant with respect to the k-dependence.

We want to be able to apply the Hille-Yosida theorem (or its variant) to $-q(x, D_x, D_k)$. Out of the two conditions the first condition is the easiest to obtain, the dissipativity is added as a condition, for example in the form of the positive maximum principle on some nice set of functions defined on $\mathbb{R}^n \times \mathbb{Z}^m$. However, extensive work is required to arrive at the second condition, i.e. the solvability of the equation $\lambda u - (-q(x, D_x, D_k))u = f$. The purpose of this thesis is to get the estimates for $q(x, D_x, D_k)$ which will guarantee this equation to be solvable. This involves having to make a smallness condition on q_2 with respect to q_1, similar to the ideas in [20].

Many of our ideas extend to more general locally compact Abelian groups. For this reason we develop in Sections 1-5 more background material than actually needed for our concrete case. The contents of the sections are the following:

In Section 1 we discuss some concepts of topology and give the notion of a locally compact Abelian group H along with four important examples, referred to as the elementary locally compact Abelian groups, that will be considered throughout this thesis. The dual group of a locally compact Abelian group is defined and we identify the dual group of each of our examples of elementary locally compact Abelian groups. It turns out that these dual groups are again in the class of these examples. We further discuss spaces of continuous functions defined on a locally compact Abelian group. We end this section with the definition and some properties of a Haar measure.

In Section 2 some spaces of integrable functions (with respect to the Haar measure μ_H) are introduced. In particular, we discuss the spaces $L^p(H, \mu_H)$, $1 \leq p < \infty$, as Banach spaces and in the case $p = 2$ as Hilbert spaces. We investigate their properties in some detail for each of the elementary locally compact Abelian groups. Examples of dense subsets of $L^p(H, \mu_H)$ are then provided.

Section 3 concerns the Fourier transform over locally compact Abelian groups H. The definition of the Fourier transform on $L^1(H, \mu_H)$ is given and some of its properties are established including a few properties that hold for the case $H = \mathbb{R}^n$. The Fourier inversion theorem is stated which leads to the concept of a dual Haar measure $\widehat{\mu}_{\widehat{H}}$ on the dual group $\widehat{H}$. The notion of the Fourier transform on $L^2(H, \mu_H)$ is also considered. A few mapping properties of the Fourier transform are then given including Plancherel's theorem which claims the equality $\|u\|_{L^2(H,\mu_H)} = \|\hat{u}\|_{L^2(\widehat{H},\mu_{\widehat{H}})}$. Plancherel's theorem is a crucial application in this thesis that will be applied many times. This section finishes with a discussion of the inverse Fourier transform.

Section 4 introduces the extension of the Fourier transform from $L^1_+(H, \mu_H)$ to the set $\mathcal{M}_b^+(H)$ of bounded Borel measures on H, i.e. the Fourier (Fourier-Stieltjes) transform on $\mathcal{M}_b^+(H)$. It is mentioned that many of its properties carries over from the Fourier transform on $L^1_+(H, \mu_H)$. These properties are also listed. The notions of a positive definite and a negative definite function are introduced. Convolution semigroups on H are defined and their association with continuous negative definite functions on $\widehat{H}$ is illustrated. Finally, some important properties of negative definite functions are given which are then followed by the Lévy-Khinchin formula.

Section 5 deals with some spaces of functions of $u : \mathbb{R}^n \times \mathbb{Z}^m \to \mathbb{C}$. In particular, the Sobolev space $H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m)$, where $\psi : \mathbb{R}^n \to \mathbb{R}$ is a fixed continuous negative definite function, is defined and its Sobolev norm is investigated. There are, furthermore, some examples of dense sets of $H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m)$. Finally, the Friedrichs mollifier is studied.

Section 6 to Section 12 prepares our main result, namely that a certain class of pseudo-differential operators acting initially on $H^{\psi,2}(\mathbb{R}^n \times \mathbb{Z}^m)$ give rise to generators of strongly continuous contraction one parameter operator semigroups, on $C_\infty(\mathbb{R}^n \times \mathbb{Z}^m)$ or $L^2(\mathbb{R}^n \times \mathbb{Z}^m)$, and under additional conditions these semigroups are even Feller semigroups, i.e. positivity preserving strongly continuous contraction semigroups on $C_\infty(\mathbb{R}^n \times \mathbb{Z}^m)$. Hence the induced stochastic processes are with respect to $x \in \mathbb{R}^n$ typical Lévy-type processes with state space $\mathbb{R}^n$, while with respect to $k \in \mathbb{Z}^m$ they are Markov chains. However, we do not discuss the probabilistic aspects in this thesis.

Section 6 introduces the symbol classes needed to handle operators of type

$$q(x, D_x, D_k) = q_1(D_x, D_k) + q_2(x, D_x, D_k)$$

with $q_2(x, D_x, D_k)$ being a perturbation of the translation-invariant operator $q_1(D_x, D_k)$. In Section 7 we collect all estimates needed for $q_1(D_x, D_k)$ and Section 8 prepares with some auxiliary results of technical nature the more involved estimates for $q_2(x, D_x, D_k)$ which are given in Section 9. In Section 10 we combine these estimates for $q_1(D_x, D_k)$ and $q_2(x, D_x, D_k)$ to get estimates for $q(x, D_x, D_k)$.

In order to satisfy the range condition for $q(x, D_x, D_k) + \lambda$ as required by the Hille-Yosida theorem we introduce a sesquilinear form B associated with $q(x, D_x, D_k)$ and the notion of a variational solution to $q(x, D_x, D_k)u + \lambda u = f$. We establish first estimates for B and derive the existence of variational solutions. In Section 12 we prove that these solutions, of course under some conditions, are solutions of $q(x, D_x, D_k)u + \lambda u = f$ in a classical sense, i.e. for $f \in H^{\psi,s+2}(\mathbb{R}^n \times \mathbb{Z}^m)$ the solution belongs to $H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m)$, hence we have solvability in $C_\infty(\mathbb{R}^n \times \mathbb{Z}^m)$ for a dense set of right-hand sides f provided $H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m)$ is continuously embedded into $C_\infty(\mathbb{R}^n \times \mathbb{Z}^m)$. Eventually we add and discuss the dissipativity condition and prove our main result.

The basic structure is close to the approach in [18], where the case of $\mathbb{R}^n$ only was handled. The extension to $\mathbb{R}^n \times \mathbb{Z}^m$ with all its required additional changes is new.

1 Introductory Material

1.1 Some Basic Material on Locally Compact Abelian Groups

In this section we collect some auxiliary results on locally compact Abelian groups. We begin with the definition of a (Abelian) group:

Definition 1.1. *Let H be a non-empty set and $\cdot$ be a binary operation on H. We call $(H, \cdot)$ a* ***group*** *if the binary operation $\cdot : H \times H \to H$ satisfies the following group axioms:*

- *H is associative, i.e. $(f \cdot g) \cdot h = f \cdot (g \cdot h)$ for all f, g, $h \in H$.*
- *H contains an identity element e, i.e. there exists $e \in H$ such that $h \cdot e = e \cdot h = h$ for all $h \in H$.*
- *Every element in H has an inverse element also in H, i.e. for every $h \in H$ there exists $h^{-1} \in H$ such that $h \cdot h^{-1} = h^{-1} \cdot h = e$.*

If in addition $f \cdot g = g \cdot f$ for all $f, g \in H$, then we call $(H, \cdot)$ an ***Abelian group****.*

It is customary to denote the binary operation of a group by the multiplicative notation $\cdot$. We sometimes, under this binary operation, label the identity element as 1. At times the additive notation $+$ is used to denote the binary operation of a group in which case we write, in Definition 1.1, $f + g$ instead of $f \cdot g$, 0 instead of 1 for the identity element, and $-h$ instead of h^{-1} for the inverse element.

References Page 226 in [29].

Remark 1.2. *$(H, \cdot)$ is written to emphasize the binary operation used. When no confusion can arise one may write H instead of $(H, \cdot)$*

An example of an Abelian group is $(\mathbb{Z}/N\mathbb{Z}, +)$ which is defined to be the set of integers modulo N, $N \in \mathbb{N}$, with the binary operation being addition modulo N. More important examples include $(\mathbb{Z}^m, +)$, $(\mathbb{R}^n, +)$ both with the usual addition, and $(\mathbb{T}^m, \cdot)$, $m, n \in \mathbb{N}$, with coordinatewise multiplication. $\mathbb{T}^m$ is given by $\underbrace{S^1 \times S^1 \times \ldots \times S^1}_{\text{m times}}$ with each $(S^1, \cdot) := (\{z \in \mathbb{C} \mid |z| = 1\}, \cdot)$ representing, under standard multiplication, the circle group. $(\mathbb{T}^m, \cdot)$ is called the m-dimensional torus group. Further, consider the set of N^{th} roots of unity $\mathbb{Z}(N) := \{1, e^{2\pi i/N}, e^{2\pi i 2/N}, \ldots, e^{2\pi i(N-1)/N}\}$. This set is, with respect to the standard multiplication, a group and it can be shown that $(\mathbb{Z}/N\mathbb{Z}, +)$ is isomorphic as a group to $(\mathbb{Z}(N), \cdot)$ meaning we may view $(\mathbb{Z}/N\mathbb{Z}, +)$ as $(\mathbb{Z}(N), \cdot)$. This will be clarified when isomorphisms are discussed later on in this section. However, from this point on we shall refer to $\mathbb{Z}/N\mathbb{Z}$ as the group of integers modulo N as it is the simplest group to work with. Of course we assume the reader to be familiar with basic concepts from group theory such as subgroups, quotient groups and product groups.

References Pages 219, 221 and 227 in [29]; Page 96 in [23].

Remark 1.3. *$(\mathbb{Z}/N\mathbb{Z}, +)$, $(\mathbb{Z}^m, +)$ and $(\mathbb{R}^n, +)$ are called additive groups whereas $(\mathbb{T}^m, \cdot)$ is a multiplicative group. Moreover, all these groups are Abelian. From now on we write these Abelian groups respectively as $\mathbb{Z}/N\mathbb{Z}$, $\mathbb{Z}^m$, $\mathbb{R}^n$ and $\mathbb{T}^m$. Note that $\mathbb{T}^1 = S^1$.*

Let $(H_1, \cdot_1)$ and $(H_2, \cdot_2)$ be two groups. A mapping $h : H_1 \to H_2$ is said to be a **group-homomorphism** if for all $x, y \in H_1$ it holds

$$h(x \cdot_1 y) = h(x) \cdot_2 h(y). \tag{1.1}$$

Direct consequences of (1.1) and the group axioms are

$$h(e_1) = e_2 \tag{1.2}$$

and

$$h(x^{-1}) = (h(x))^{-1}, \tag{1.3}$$

where e_1 and e_2 are the identity elements of the groups H_1 and H_2 respectively, and $x \in H_1$. If in addition the groups H_1 and H_2 are Abelian, then it also follows

$$h(x) \cdot_2 h(y) = h(y) \cdot_2 h(x) \tag{1.4}$$

for all $x, y \in H_1$.

References Pages 21-23 in [24].

Suppose $(H, +)$ is an Abelian group. Let us focus on group-homomorphisms $\gamma : H \to S^1$. Denote by $H^* := \mathrm{Hom}(H, S^1)$ the set of all such group-homomorphisms on H. Following equations (1.1)-(1.4) one immediately finds that a group-homomorphism $h \in H^*$ satisfies for all $x, y \in H$ the properties below:

- $\gamma(x + y) = \gamma(x) \cdot \gamma(y)$;
- $\gamma(0) = 1$;

- $\gamma(-x) = (\gamma(x))^{-1}$;
- $\gamma(x) \cdot \gamma(y) = \gamma(y) \cdot \gamma(x)$.

With multiplication defined pointwisely by

$$(\gamma_1 \cdot \gamma_2)(x) := \gamma_1(x) \cdot \gamma_2(x)$$

for all $x \in H$, it turns out $(H^*, \cdot)$ is an Abelian group with respect to pointwise multiplication. Indeed, the associativity of complex numbers under standard multiplication yields for all $x \in H$

$$\begin{aligned}((\gamma_1 \cdot \gamma_2)) \cdot \gamma_3)(x) &= (\gamma_1 \cdot \gamma_2)(x) \cdot \gamma_3(x) \\ &= (\gamma_1(x) \cdot \gamma_2(x)) \cdot \gamma_3(x) \\ &= \gamma_1(x) \cdot (\gamma_2(x) \cdot \gamma_2(x)) \\ &= \gamma_1(x) \cdot (\gamma_2 \cdot \gamma_3)(x) \\ &= (\gamma_1 \cdot (\gamma_2 \cdot \gamma_3))(x),\end{aligned}$$

implying the associativity of H^*.

Define the mapping $e : H \to S^1$ by $e(x) := 1$ for all $x \in H$. Obviously $e \in H^*$. Moreover, given $x \in H$, it holds

$$(\gamma \cdot e)(x) = \gamma(x) \cdot e(x) = \gamma(x) \cdot 1 = \gamma(x)$$

and

$$(e \cdot \gamma)(x) = e(x) \cdot \gamma(x) = 1 \cdot \gamma(x) = \gamma(x),$$

which shows e is the identity element in H^*.

For any $\gamma \in H^*$ define a mapping $\gamma^{-1} : H \to S^1$ by $\gamma^{-1}(x) := (\gamma(x))^{-1}$. Again, like for the identity element e, it is trivial to observe that $\gamma^{-1} \in H^*$. Moreover,

$$(\gamma \cdot \gamma^{-1})(x) = \gamma(x) \cdot \gamma^{-1}(x) = 1 = e(x)$$

and

$$(\gamma^{-1} \cdot \gamma)(x) = \gamma^{-1}(x) \cdot \gamma(x) = 1 = e(x)$$

for all $x \in H$, implying $\gamma^{-1} \in H^*$ is the inverse of $\gamma \in H^*$.

Finally, for any $\gamma_1, \gamma_2 \in H^*$ and $x \in H$, observe

$$(\gamma_1 \cdot \gamma_2)(x) = \gamma_1(x) \cdot \gamma_2(x) = \gamma_2(x) \cdot \gamma_1(x) = (\gamma_2 \cdot \gamma_1)(x).$$

Hence $(H^*, \cdot)$ is an Abelian group.

Our aim now is to introduce the notion of a topological group and then introduce locally compact (Abelian) groups, we are primarily interested in locally compact Abelian groups. We shall collect some facts from topology.

Definition 1.4. *Let X be a set. A* ***topology*** *τ on X is a collection of subsets (called* ***open sets****) of X such that*

- *$\emptyset \in \tau$ and $X \in \tau$;*
- *any arbitrary union of open sets is itself open, i.e. given $\{U_j \mid j \in \Gamma\} \subset \tau$, where Γ is any set, it follows $\bigcup_{j\in\Gamma} U_j \in \tau$;*
- *any finite intersection of open sets is itself open, i.e. given $\{U_1, \ldots U_n\} \subset \tau$, $n \in \mathbb{N}$, it follows $\bigcap_{j=1}^{n} U_j \in \tau$.*

The set X equipped with the topology τ, i.e. (X, τ), is called a ***topological space****.*

References Page 19 in [5].

Remark 1.5. *We may write H instead of (H, τ) if it is clear what topology is equipped to H or if the knowledge of the topology is not necessary.*

We assume the reader to have some understanding of the basic concepts of topology such as closed and compact sets, neighbourhoods and a basis of a topology. Given a topological space (X, τ), X is said to be locally compact if every point in X possesses a compact neighbourhood. We refer to the topology τ that makes (X, τ) into a locally compact topological space as the **locally compact topology** on X. For our purposes any topological space we work with will assumed to be Hausdorff. A topological space X is Hausdorff if for any two distinct points x and y in X there exists a neighbourhood U_x of x and a neighbourhood U_y of y such that they are disjoint, that is, $U_x \cap U_y = \emptyset$.

Let (X_1, τ_1) and (X_2, τ_2) be topological spaces. A mapping $u : X_1 \to X_2$ is continuous if the preimage of every open set is itself open, i.e. $u^{-1}(V) \in \tau_1$ whenever $V \in \tau_2$. We assume the reader to be familiar with basic properties of continuous mappings.

References Pages 20-21 and page 91 in [5].

Definition 1.6. *A group $(H, \cdot)$ is said to be a* ***topological group*** *if it is endowed with a topology τ such that (H, τ) is a topological space where, with respect to τ, the group operations $(x, y) \mapsto x \cdot y$ from $H \times H$ into H and*

$x \mapsto x^{-1}$ from H into H are continuous. We label the topological group by $(H, \cdot, \tau)$.

*Further, if $(H, \cdot)$ is Abelian and (H, τ) is locally compact, then $(H, \cdot, \tau)$ becomes a **locally compact Abelian group**.*

References Pages 57-58 in [27]; Page 186 in [22].

Remark 1.7. *In Definiton 1.6 we assume $H \times H$ is endowed with the product topology and take for granted that the reader is used to the concept of product topologies. In fact, in this thesis every product group is assumed to be equipped with the product topology. We write $(H, \cdot, \tau)$ to emphasize the binary operation and topology endowed to make H into a locally compact Abelian group, but we may write H instead of $(H, \cdot, \tau)$ if it is clear what binary operation and topology are endowed to H, or if the knowledge of the binary operation and topology is not required. For two distinct topologies, say τ_1 and τ_2, the topological groups $(H, \cdot, \tau_1)$ and $(H, \cdot, \tau_2)$ are different. For example, when taking $\mathbb{Q}$ with the usual addition as binary operation, the two topological groups obtained by taking on the one hand side the topology induced by the Euclidean distance and on the other hand the discrete topology are different. From now on H will only denote a locally compact Abelian group.*

Consider any discrete Abelian group $(G_1, \cdot)$ equipped with the discrete topology τ_1, namely $(G_1, \cdot, \tau_1)$. Further, suppose (G_2, τ_2) is an arbitrary topological space. As the topology τ_1 is discrete it turns out that any mapping $u : G_1 \to H$ is continuous. This result guarantees, for any group $(G_1, \cdot)$ equipped with the discrete topology τ_1, the continuity of the mappings $(x, y) \mapsto x \cdot y$ from $G_1 \times G_1$ into G_1 and $x \mapsto x^{-1}$ from G_1 into G_1. Consequently $(G_1, \cdot, \tau_1)$ is a topological group. Note that the (product) topology on $G_1 \times G_1$ is also the discrete topology, thus confirming the continuity of $\cdot : G_1 \times G_1 \to G_1$. Moreover, by definition of the discrete topology one immediately observes τ_1 is also a locally compact topology, implying $(G_1, \cdot, \tau_1)$ as a topological space (G_1, τ_1) is locally compact. In other words the discrete topology τ_1 is a locally compact topology and hence $(G_1, \cdot, \tau_1)$ is a locally compact Abelian group.

References Proposition 3.9 in [8]; Remark 3.14 on page 59 in [27]; Example (a) on page 186 in [22].

Let us turn our attention back to the four Abelian groups that we mentioned earlier. We now transform each of these Abelian groups into a locally compact Abelian group and list them in the example below along with their appropriate topologies. The first two examples follow from the paragraph above.

Example 1.8.

- *The additive Abelian group* $(\mathbb{Z}/N\mathbb{Z}, +)$ *endowed with the discrete topology.*
- *The additive Abelian group* $(\mathbb{Z}^m, +)$ *endowed with the discrete topology.*
- *The additive Abelian group* $(\mathbb{R}^n, +)$ *endowed with the usual topology, i.e. the topology whose basis consists of the open balls in* $(\mathbb{R}^n, +)$.
- *The multiplicative Abelian group* $(\mathbb{T}^m, \cdot) \subset (\mathbb{C}^m, \cdot)$ *endowed with the usual topology, i.e. the topology being the subspace topology.*
- *For finitely many arbitrary locally compact Abelian groups* $(H_1, \cdot_1), \ldots,$ $(H_n, \cdot_n)$*, the product group* $(H_1 \times \ldots \times H_n, \circ)$ *endowed with the product topology is a locally compact Abelian group.*

References Pages 6-7 in [1]; Examples (v) on page 16 in [13]; Page 31 in [8].

Remark 1.9. *In Example 1.8 we could choose the topology on* $\mathbb{T}^m$ *to be the natural topology induced by* $\mathbb{R}^{2m}$ *since* $\mathbb{C}^m$ *is homeomorphic to* $\mathbb{R}^{2m}$*. On the groups* $\mathbb{Z}^m$*,* $\mathbb{Z}/N\mathbb{Z}$ *and their subsets we always take the discrete topology and therefore refer to these sets, endowed with this topology, as discrete sets.*

$\mathbb{Z}/N\mathbb{Z}$ *and all its subsets are finite (and hence discrete) and all such spaces can easily be seen to be compact, therefore* $\mathbb{Z}/N\mathbb{Z}$ *is in fact a compact Abelian group. Any finite subset of* $\mathbb{Z}^m$ *is of course compact but not* $\mathbb{Z}^m$ *itself. From the Heine-Borel theorem, a set in* $\mathbb{R}^n$ *or* $\mathbb{T}^m$ *is compact if and only if the set is bounded and closed.*

The locally compact Abelian groups $\mathbb{Z}/N\mathbb{Z}$*,* $\mathbb{Z}^m$*,* $\mathbb{R}^n$ *and* $\mathbb{T}^m$ *together will be known as the* ***elementary locally compact Abelian groups****.*

References P.18 (a) on page 7 in [1]; Theorem 4.29 on page 48 in [8].

Let H be a locally compact Abelian group. In our thesis we are interested in a particular subset of H^*, namely the space of all continuous group-homomorphisms $\chi : H \to S^1$ which shall be designated by $\widehat{H}$. In fact, it turns out this subset is a subgroup of H^* under pointwise multiplication inherited from $(H^*, \cdot)$, and is therefore a group itself. The group $(\widehat{H}, \cdot)$ is called the dual group of H.

Definition 1.10. *Let $(H, \cdot)$ be a locally compact Abelian group. We call a complex-valued function χ a* ***character*** *on H if $\chi : H \to S^1$ is a continuous group-homomorphism. We denote the set of all such characters by $\widehat{H}$ and call it the* ***dual group*** *of H.*

References Page 230 in [29]; Pages 188-189 in [22].

Remark 1.11. *Every locally compact Abelian group has a dual group.*

It is easy to see that $\widehat{H}$ is a subgroup of H^* and as a result the mapping $e : H \to S^1$ defined by $e(x) := 1$ for all $x \in H$ is the identity element in $\widehat{H}$. We refer to e as the **unit character**. Observe also that $\widehat{H}$ becomes a locally compact Abelian group when endowed with the compact convergence topology (the topology of uniform convergence on compact sets). One can prove that the collection of all sets of the form

$$U(K, \varepsilon) := \{\chi \in \widehat{H} \mid |\chi(x) - 1| < \varepsilon \text{ for all } x \in K\},$$

where $\varepsilon > 0$ and $K \subset H$ is compact, produces a neighbourhood basis of the unit character and hence generates the locally compact topology (compact convergence topology) which turns $\widehat{H}$ into a locally compact Abelian group.

References Page 230 in [29]; Page 8 in [1]; Page 8 in [3].

Example 1.12.

- $(\widehat{\mathbb{Z}/N\mathbb{Z}}, \cdot) = \{e_\omega \mid \omega = 0, 1, \ldots, N-1\}$, *where*

$$e_\omega(k) := e^{2\pi i k\omega/N} \text{ for all } \omega = 0, 1, \ldots, N-1.$$

- $(\widehat{\mathbb{Z}^m}, \cdot) = \{e_\omega \mid \omega \in [0, 2\pi)^m\}$, *where*

$$e_\omega(k) := e^{ik\cdot\omega} \text{ for all } \omega \in [0, 2\pi)^m.$$

- $(\widehat{\mathbb{R}^n}, \cdot) = \{e_\xi \mid \xi \in \mathbb{R}^n\}$, *where*

$$e_\xi(x) := e^{ix\cdot\xi} \text{ for all } \xi \in \mathbb{R}^n.$$

- $(\widehat{\mathbb{T}^m}, \cdot) = \{e_k \mid k \in \mathbb{Z}^m\}$, *where e_k is taken to be a function on $[0, 2\pi)^m$ given by*

$$e_k(\omega) := e^{ik\cdot\omega} \text{ for all } k \in \mathbb{Z}^m.$$

References Page 231 in [29]; Pages 12-13 in [28].

Below we identify each of these dual groups as a locally compact Abelian group that will be easier to deal with. We will discuss in detail the dual group $\widehat{\mathbb{R}^n}$.

The dual group $\widehat{\mathbb{R}^n}$ is isomorphic as a group and homeomorphic as a topological space to $\mathbb{R}^n$, in other words, $\widehat{\mathbb{R}^n}$ is topologically isomorphic to $\mathbb{R}^n$, or $\widehat{\mathbb{R}^n} \cong \mathbb{R}^n$ for short with $\cong$ signifying the topological isomorphism. Indeed, recall from Example 1.12 that $\widehat{\mathbb{R}^n} = \{e_\xi \mid \xi \in \mathbb{R}^n\}$, where $e_\xi(x) = e^{ix\cdot\xi}$ for all $\xi \in \mathbb{R}^n$. This lead us to consider the mapping $i : \widehat{\mathbb{R}^n} \to \mathbb{R}^n$ defined by $i(e_\xi) := \xi$ which we now aim to prove is a group-isomorphism. Suppose $i(e_{\xi_1}) = i(e_{\xi_2})$, then $\xi_1 = \xi_2$, implying $e^{ix\cdot\xi_1} = e^{ix\cdot\xi_2}$ for all $x \in \mathbb{R}^n$, i.e. $e_{\xi_1} = e_{\xi_2}$. This shows i is injective. Now, given $\xi \in \mathbb{R}^n$, we find naturally e_ξ is an element in $\widehat{\mathbb{R}^n}$ such that $i(e_\xi) = \xi$. Finally, let $e_{\xi_1}, e_{\xi_2} \in \widehat{\mathbb{R}^n}$, then a simple computation yields

$$i(e_{\xi_1} \cdot e_{\xi_2}) = i(e_{\xi_1+\xi_2}) = \xi_1 + \xi_2 = i(e_{\xi_1}) + i(e_{\xi_2}),$$

i.e. i is a group-homomorphism. Thus i is a group-isomorphism, which concludes $\widehat{\mathbb{R}^n}$ is isomorphic as a group to $\mathbb{R}^n$. It also turns out that i is a homeomorphism, but we will not prove this.

References P.2 on page 1 in [1].

Analogue arguments yields

$$\widehat{\mathbb{Z}/N\mathbb{Z}} \text{ is topologically isomorphic to } \mathbb{Z}/N\mathbb{Z},$$

$$\widehat{\mathbb{Z}^m} \text{ is topologically isomorphic to } \mathbb{T}^m, \tag{1.5}$$

$$\widehat{\mathbb{T}^m} \text{ is topologically isomorphic to } \mathbb{Z}^m, \tag{1.6}$$

and of course we have just shown

$$\widehat{\mathbb{R}^n} \text{ is topologically isomorphic to } \mathbb{R}^n. \tag{1.7}$$

As the above examples show, each of these dual groups can be identified as either one of the elementary locally compact Abelian groups we are already familiar with. More importantly, given any finite number of locally compact Abelian groups $H_1, \dots, H_n$ having respectively the dual groups $\widehat{H_1}, \dots, \widehat{H_n}$, it holds

$$H_1 \widehat{\times \dots \times} H_n \text{ is topologically isomorphic to } \widehat{H_1} \times \dots \times \widehat{H_n}. \tag{1.8}$$

References P.19 (b) on page 8 and P.20 (a)-(c) on page 9 in [1].

Example 1.13. *Applying* (1.8) *with* (1.5) *and* (1.7) *yields* $\widehat{\mathbb{R}^n \times \mathbb{Z}^m} = \mathbb{R}^n \times \mathbb{T}^m$.

Notice that the dual group of the finite Abelian group $\mathbb{Z}/N\mathbb{Z}$ is isomorphic to the same finite Abelian group that we took the dual group of, namely $\mathbb{Z}/N\mathbb{Z}$. This follows from the fact that the dual group of any finite Abelian group H is topologically isomorphic to the same finite Abelian group H. Furthermore, the dual group of a compact group is discrete while the dual group of a discrete group is compact. Examples (1.6) and (1.5) respectively gives examples of these two group properties.

References P.19 on page 8 and P.20 (a) on page 9 in [1].

Recall from Remark 1.11 that every locally compact Abelian group has a dual group. We have already looked at the dual group of each of the elementary locally compact Abelian groups, and with a dual group $\widehat{H}$ of a locally compact Abelian group being a locally compact Abelian group itself it makes sense to investigate the dual group of a dual group, that is, the dual group of $\widehat{H}$. We denote this dual group by $\widehat{\widehat{H}}$ and refer to it as the **double dual group** of H.

Let the dual group of H be given by $\widehat{H} := \{\chi : H \to S^1 \mid \chi \text{ is a character}\}$. Then, for each $x \in H$, $\widehat{\chi_x} : \widehat{H} \to S^1$ defined by $\widehat{\chi_x}(\chi) := \chi(x)$ is a character on $\widehat{H}$ and moreover the theorem below confirms that $\widehat{\widehat{H}}$ consists only of characters of that form.

Theorem 1.14 (Pontryagin Duality Theorem). *Let H be a locally compact Abelian group and let $\widehat{\widehat{H}}$ be its double dual group. Then H is topologically isomorphic to $\widehat{\widehat{H}}$ where the topological isomorphism α is given by $\alpha(x) := \widehat{\chi_x}$.*

Hence $\widehat{\widehat{H}}$ may be regarded as H.

References P.19 (d) on page 8 in [1]; Theorem 4.2.11 (last sentence) on page 134 in [27].

Operators of particular interest to us are those whose domain consists of functions defined on a locally compact Abelian group. A few examples of such domains will now be discussed, and in what follows H is a locally compact Abelian group. We denote by $C(H)$ the vector space of all complex-valued continuous functions on H. $C_0(H)$ shall denote the vector space of all complex-valued continuous functions on H with compact support, that is, $C_0(H) := \{u \in C(H) \mid \text{supp } u \text{ is compact}\}$, where supp $u :=$

$\overline{\{x \in H \mid u(x) \neq 0\}}$. Let ∞ be an ideal point not belonging to H, then $\overline{H} := H \cup \{\infty\}$ is defined to be the one-point compactification of H endowed with the topology consisting of all open sets in the topology of H, and sets of the form $\overline{H} \backslash K$, $K \subset H$ compact. In the case when H is compact the point ∞ is an isolated point in $\overline{H}$. We say $u : H \to \mathbb{C}$ is a function vanishing at ∞ (infinity) if for every $\varepsilon > 0$ there exists $K \subset H$, K compact, such that $|u(x)| < \varepsilon$ for all $x \in H \backslash K$. The vector space of all complex-valued continuous functions on H vanishing at infinity is given by $C_\infty(H)$, i.e. $C_\infty(H) := \{u \in C(H) \mid u \text{ vanishes at infinity}\}$.

It holds in general

$$C_0(H) \subset C_\infty(H) \subset C(H)$$

and immediately we observe $C_0(H)$ and $C_\infty(H)$ are subspaces of $C(H)$. However, in the case when H is compact the subspaces coincide with $C(H)$, namely

$$C_0(H) = C_\infty(H) = C(H),$$

and hence $C_0(\mathbb{Z}/N\mathbb{Z}) = C_\infty(\mathbb{Z}/N\mathbb{Z}) = C(\mathbb{Z}/N\mathbb{Z})$ as well as $C_0(\mathbb{T}^m) = C_\infty(\mathbb{T}^m) = C(\mathbb{T}^m)$. As already stated, if H is equipped with the discrete topology, then any function defined on H is continuous and so $V(H) = C(H)$ with $V(H)$ signifying the vector space of all complex-valued functions on H. In particular, $V(\mathbb{Z}/N\mathbb{Z}) = C(\mathbb{Z}/N\mathbb{Z})$ and $V(\mathbb{Z}^m) = C(\mathbb{Z}^m)$. Most of these spaces along with $C_0(\mathbb{Z}^m)$ and $C_\infty(\mathbb{Z}^m)$ will be examined further in Section 2.

References Page 16 in [19]; Pages 166-167 in [2]; Page 92 in [5].

In Section 3 functions whose domain is the dual group of a locally compact Abelian group will be considered, namely the Fourier transform of a function. Recall that we were able to identify each dual group in Example 1.12 as one of the more well-known elementary locally compact Abelian groups. In addition, each of these identifications carries over to their corresponding vector spaces of functions as illustrated in the theorem below.

Theorem 1.15. *Let H_1 and H_2 be two locally compact Abelian groups such that H_1 and H_2 are isomorphic as groups. Moreover, let $V(H_1)$ and $V(H_2)$ be their corresponding vector spaces of complex-valued functions on H_1 and H_2 respectively. Then $V(H_1)$ and $V(H_2)$ are isomorphic as vector spaces.*

References Page 221 in [29].

Proof. Since H_1 and H_2 are isomorphic as groups, there exists a group-isomorphism from H_1 to H_2 which we denote by $J : H_1 \to H_2$. Our aim is to deduce that the mapping $T : V(H_1) \to V(H_2)$ defined by $v \mapsto v \circ J^{-1}$ is a vector space isomorphism.

Let $v_1, v_2 \in V(H_1)$ and $\lambda_1, \lambda_2 \in \mathbb{C}$. Then

$$\begin{aligned} T(\lambda_1 v_1 + \lambda_2 v_2) &= (\lambda_1 v_1 + \lambda_2 v_2) \circ J^{-1} \\ &= \lambda_1 v_1 \circ J^{-1} + \lambda_2 v_2 \circ J^{-1} \end{aligned}$$

$$\begin{aligned} &= \lambda_1 (v_1 \circ J^{-1}) + \lambda_2 (v_2 \circ J^{-1}) \\ &= \lambda_1 T(v_1) + \lambda_2 T(v_2), \end{aligned}$$

implying T is linear.

Suppose $v_1, v_2 \in V(H_1)$ such that $T(v_1) = T(v_2)$, i.e. $T(v_1)(y) = T(v_2)(y)$ for all $y \in H_2$. J being a group-isomorphism gives the existence, for each $y \in H_2$, of a unique element $x \in H_1$ such that $J(x) = y$ and therefore $J^{-1}(y) = x$. Now, for all $y \in H_2$

$$\begin{aligned} (v_1 \circ J^{-1})(y) &= (v_2 \circ J^{-1})(y) \\ v_1(x) &= v_2(x), \end{aligned}$$

i.e. $v_1 = v_2$, proving T is injective. Finally, given $w \in V(H_2)$, it is obvious $v := w \circ J \in V(H_1)$. Further it holds

$$T(v) = v \circ J^{-1} = (w \circ J) \circ J^{-1} = w,$$

giving the surjectivity of T.

□

Remark 1.16. *An important aspect of this proof was obtaining the definition of the vector space isomorphism T which we claimed to be $T : V(H_1) \to V(H_2)$ defined by $T(v) := v \circ J^{-1}$ with $J : H_1 \to H_2$ being the group-isomorphism with inverse $J^{-1} : H_2 \to H_1$.*

Example 1.17. *As $\widehat{\mathbb{R}^n}$ is isomorphic as a group to $\mathbb{R}^n$ we can deduce via Theorem 1.15 and Remark 1.16 (where $H_1 = \widehat{\mathbb{R}^n}$ and $H_2 = \mathbb{R}^n$) that $V(\widehat{\mathbb{R}^n})$ and $V(\mathbb{R}^n)$ are isomorphic as vector spaces with the vector space isomorphism T defined by $T(v) := v \circ J^{-1}$. Thus any function $v \in V(\widehat{\mathbb{R}^n})$, can be identified as $v \circ J^{-1} \in V(\mathbb{R}^n)$. Further, by defining $v \circ J^{-1} := v$ we may identify any function $v : \widehat{\mathbb{R}^n} \to \mathbb{C}$ as $v : \mathbb{R}^n \to \mathbb{C}$.*

Using the same argument, any functions $v_1 : \widehat{\mathbb{Z}/N\mathbb{Z}} \to \mathbb{C}$, $v_2 : \widehat{\mathbb{Z}^m} \to \mathbb{C}$ and $v_3 : \widehat{\mathbb{T}^m} \to \mathbb{C}$ can be identified as $v_1 : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$, $v_2 : \mathbb{T}^m \to \mathbb{C}$ and $v_3 : \mathbb{Z}^m \to \mathbb{C}$ respectively.

The corollary below follows directly from Theorem 1.15.

Corollary 1.18. *Let H_1 and H_2 be two locally compact Abelian groups such that H_1 and H_2 are topologically isomorphic. Then $C(H_1)$ and $C(H_2)$ are isomorphic as vector spaces.*

Therefore, one may choose to work with $C(\mathbb{Z}/N\mathbb{Z})$ instead of $C(\widehat{\mathbb{Z}/N\mathbb{Z}})$, or $C(\mathbb{T}^m)$ instead of $C(\widehat{\mathbb{Z}^m})$, or $C(\mathbb{Z}^m)$ instead of $C(\widehat{\mathbb{T}^m})$, or $C(\mathbb{R}^n)$ instead of $C(\widehat{\mathbb{R}^n})$.

We are familiar with integration on $\mathbb{R}^n$. However, integration can be generalized to locally compact Abelian groups and therefore one can define the integral of a function on a locally compact Abelian group $(H, +)$ (where the additive notation is used for the binary operation in H). To do so a measure space $(H, \mathcal{A}, \mu)$ needs to be constructed. We shall take the σ-algebra $\mathcal{A}$ of H to be the σ-algebra of H generated by the open sets of the topology of H, namely the Borel σ-algebra, labeled $\mathcal{B}(H)$. Thus $(H, \mathcal{B}(H))$, which we may simply write as H, is our measurable space and will always be throughout this thesis. We assume the reader to be familiar with concepts of measure and integration theory.

Fix $a \in H$ and let $\tau_a : H \to H$ be the translation mapping on H given by $\tau_a(x) := x + a$. τ_a is in fact a homeomorphism and consequently $\tau_a(A) \in \mathcal{B}(H)$ whenever $A \in \mathcal{B}(H)$. We know the Lebesgue measure $\lambda^{(n)}$ on $\mathbb{R}^n$ (unique up to a positive normalization factor) is translation-invariant which poses the question whether or not there exist a measure μ on an arbitrary H (by a measure on H we mean a measure on $\mathcal{B}(H)$) that is translation-invariant, i.e. $\mu(\tau_a(A)) = \mu(A)$ for all $A \in \mathcal{B}(H)$ and $a \in H$. Fortunately, the answer to this question is yes and the theorem stated below confirms this.

References Example 3 on page 36, Examples 2 on page 34 and 8.2 Corollary on page 39 in [2]; Page 58 (18 lines up) in [27].

Theorem 1.19. *Let H be a locally compact Abelian group. There exists a positive regular Borel measure μ on H, which is unique up to a positive normalization factor, that is translation-invariant and finite on compact sets. We call this measure (once a suitable normalization is chosen) the* ***Haar measure*** *on H and we denote it by μ_H.*

Theorem 1.19 claims that such a measure exists for any locally compact Abelian group. Hence for every locally compact Abelian group H we can construct the measure space $(H, \mathcal{B}(H), \mu_H)$. All measure spaces in this thesis will be of this form and when dealing with measure spaces we write H instead of $(H, \mathcal{B}(H), \mu_H)$.

References Page 187 in [22].

We now discuss some sets of measures on a locally compact Abelian group H that we will be working with throughout this thesis. Let $\mathcal{M}^+(H)$ and $\mathcal{M}_b^+(H)$ stand respectively for the set of all Borel measures on H and the set of all bounded Borel measures on H. Denote by $\mathcal{M}(H)$ and $\mathcal{M}_b(H)$ the set of all signed Borel measures on H (Borel measures on H that can have negative values) and the set of all bounded signed Borel measures on H respectively. In cases where measures are complex-valued we replace $\mathcal{M}^+(H)$ with $\mathcal{M}_{\mathbb{C}}^+(H)$, the set of all complex-valued Borel measures on H. The sets $\mathcal{M}_{b,\mathbb{C}}^+(H)$, $\mathcal{M}_{\mathbb{C}}(H)$ and $\mathcal{M}_{b,\mathbb{C}}(H)$ are defined analogously. For any Borel measure μ on H we define the **total mass of** μ, denoted $\|\mu\|$, to be the measure of H, that is, $\|\mu\| := \mu(H)$. By definition we have $\|\mu\| < \infty$ for $\mu \in \mathcal{M}_b(H)$ (or $\mathcal{M}_{b,\mathbb{C}}(H)$).

Before listing examples of Haar measures for some locally compact Abelian groups it is convenient to have a look at a few properties of a Haar measure on particular types of locally compact Abelian groups. If $H_1, \ldots, H_n$ are locally compact Abelian groups with corresponding Haar measures $\mu_{H_1}, \ldots \mu_{H_n}$ respectively, then the locally compact Abelian group $H_1 \times \ldots \times H_n$ has the product measure $\mu_{H_1} \otimes \ldots \otimes \mu_{H_n}$ as its Haar measure. The Haar measure μ_{H_d} on a discrete (topological) group H_d is usually normalized in such a way that the measure of each element x in H_d is 1, that is, $\mu_{H_d}(x) := 1$ for all $x \in H_d$. A measure with this property is called a counting measure. For a compact group H_c the Haar measure μ_{H_c} is usually normalized in a way that it has total mass 1, i.e. $\mu_{H_c}(H_c) := 1$. However if $H_{f,d}$ is a finite (and hence compact) and discrete (topological) group, then its Haar measure $\mu_{f,d}$ cannot be a counting measure of total mass one. Instead we take $\mu_{f,d}$ to be the non-counting Haar measure of total mass one.

References Pages 24-25 in [19]; 2.12 on page 11 in [3]; 1.1.3 on page 2 in [28]; Remark 4.4.7 on page 144 in [27].

Example 1.20.

- *On* $\mathbb{R}^n$, *with the exception of Section 3 we choose the Haar measure to be the Lebesgue measure* $\mu_{\mathbb{R}^n} := \lambda^{(n)}$ *with the normalization* $\lambda^{(n)}([0,1]^n) = 1$, *where* $[0,1]^n := \underbrace{[0,1] \times \ldots \times [0,1]}_{n \text{ times}}$. *In Section 3* $\mu_{\mathbb{R}^n}$ *will be taken as* $(2\pi)^{-\frac{n}{2}}\lambda^{(n)}$ *as this normalization will be of an advantage to us when dealing with the Fourier transform.*

- *We choose the Haar measure on* $\mathbb{Z}^m$ *to be the counting measure given by* $\mu_{\mathbb{Z}^m} := \sum_{k\in\mathbb{Z}^m} \varepsilon_k$, *where* ε_k *is the Dirac measure at* $k \in \mathbb{Z}^m$.

- *We know* $\mathbb{Z}/N\mathbb{Z}$ *is a finite and discrete group and therefore the associated Haar measure is taken to be the countable additive measure* $\mu_{\mathbb{Z}/N\mathbb{Z}} := \frac{1}{N}\sum_{k=0}^{N-1}\varepsilon_k$ *because only then do we obtain the normalization* $\mu_{\mathbb{Z}/N\mathbb{Z}}(\mathbb{Z}/N\mathbb{Z}) = 1$.

- *By referring to* $\mathbb{T}^m$ *as* $[0, 2\pi)^m$ *we identify the Haar measure* $\mu_{\mathbb{T}^m}$ *on* $\mathbb{T}^m$ *to be the restriction of* $\frac{1}{(2\pi)^m}\lambda^{(m)}$ *to* $[0, 2\pi)^m \subset \mathbb{R}^m$*. That way the normalization* $\mu_{\mathbb{T}^m}(\mathbb{T}^m) = 1$ *is obtained.*

References 1.5.3 on pages 24-26 in [28]; (3) in 2.3 on page 232 in [29]; Page 1 in [22].

2 Some Function Spaces on Locally Compact Abelian Groups

Before we make a start with this section it is worth considering the notion of a sequence. A sequence in $\mathbb{C}$, denoted by $(u_n)_{n\in\mathbb{N}}$, is a function $u : \mathbb{N} \to \mathbb{C}$, with $u_n := u(n)$. Therefore the space of all functions $u : \mathbb{N} \to \mathbb{C}$ may be viewed as the space of all sequences $(u_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$. One may however replace the index set $\mathbb{N}$ by a finite or discrete set A and instead speak of a sequence in $\mathbb{C}$, with index A, denoted by $(u_k)_{k\in A}$, which is identified as a function $u : A \to \mathbb{C}$, with $u_k := u(k)$. For the case $A = \mathbb{Z}/N\mathbb{Z}$ it follows that $\mathbb{C}^N$ (or equivalently the space of all sequences $(u_k)_{k\in 0,\dots,N-1}$, indexed by $\mathbb{Z}/N\mathbb{Z}$, in $\mathbb{C}$) is the space of all functions $u : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$. Here, $u : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ corresponds to $(u(0), \cdots, u(N-1)) \in \mathbb{C}^N$. Analogously, the space of all sequences $(u_k)_{k\in Z^m}$ in $\mathbb{C}$, indexed by $\mathbb{Z}^m$, is the space of all functions $u : \mathbb{Z}^m \to \mathbb{C}$. Here, we associate $u : \mathbb{Z}^m \to \mathbb{C}$ with the sequence $(u_k)_{k\in\mathbb{Z}^m}$, where $u(k) := u_k$ for all $k \in \mathbb{Z}^m$. From now on a sequence $(u_k)_{k\in A}$ will be said to be 'a sequence on A' rather than 'a sequence in $\mathbb{C}$ with index A'.

Earlier we came across the function spaces $C(H), C_0(H)$ and $C_\infty(H)$, H being a locally compact Abelian group. We wish to discuss further these spaces for the cases $\mathbb{Z}/N\mathbb{Z}$ and $\mathbb{Z}^m$. As already mentioned, $C(\mathbb{Z}/N\mathbb{Z})$ is exactly the space of all functions $u : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$, or the space $\mathbb{C}^N$. The space $C(\mathbb{Z}^m)$ consists of all functions $u : \mathbb{Z}^m \to \mathbb{C}$, or using the sequence notation, consists of all sequences $(u_k)_{k\in\mathbb{Z}^m}$ on $\mathbb{Z}^m$. A sequence in $C(\mathbb{Z}^m)$ is called finitely non-zero (or just finite) if there exists $N \in \mathbb{N}$ such that $u_k = 0$ for all $|k| \geq N$, or to put it another way, there exists only a finite number of non-zero elements in the sequence. We designate by $C_0(\mathbb{Z}^m)$ or $l_{\text{finite}}(\mathbb{Z}^m)$ the space consisting of all finite sequences on $\mathbb{Z}^m$. Finally, the space $C_\infty(\mathbb{Z}^m)$ consists of all sequences with the property that $\lim_{|k|\to\infty} u_k = 0$, or using the function notation, $\lim_{|k|\to\infty} f(k) = 0$. A function satisfying this property is said to be a function vanishing at infinity.

References Page xii in [2]; Problems 5F on page 76 in [26].

In the previous section we found for every locally compact Abelian group H the existence of a unique (up to a positive normalization factor) Haar measure μ_H. Remember that for the cases $H = \mathbb{Z}/N\mathbb{Z}$, $\mathbb{Z}^m$, $\mathbb{R}^n$ or $\mathbb{T}^m$, μ_H is taken to be the normalized Haar measure given in Example 1.20. For a moment let us assume μ_H to be arbitrary, then for every such Haar measure μ_H we may associate a non-zero positive linear functional I_H on $C_0(H)$ given by

$$I_H(u) := \int_H u(x)\, \mu_H(\mathrm{d}x) \tag{2.1}$$

that is μ_H-translation invariant. That is, a linear functional I_H on $C_0(H)$ not identically equal to zero with the following properties: $u \geq 0$ implies $I_H(u) \geq 0$, and for all $u \in C_0(H)$ we have $I_H(\tau_a u) = I_H(u)$ for $a \in H$ fixed, where $\tau_a u(x) := u(x-a)$ for all $x \in (H,+)$. This linear functional I_H on $C_0(H)$ is regarded as the **Haar integral** on H. Such an integral exists for every locally compact Abelian group and is unique up to a positive normalization factor. In the situation when H is discrete the integral in (2.1) possesses the form of a sum, this notion will be illustrated when the Haar integrals on $\mathbb{Z}/N\mathbb{Z}$ and $\mathbb{Z}^m$ are introduced in just a moment.

References Definitions on pages 63-64. Page 65 (first sentence), Theorem 6 on page 77 and Notation on page 78 in [13].

Given any Haar integral I_H on a locally compact Abelian group H, one may introduce the $L^p(H,\mu_H)$-norm $\|\cdot\|_{L^p(H,\mu_H)}$, $1 \leq p < \infty$, and the inner product $(\cdot,\cdot)_{L^2(H,\mu_H)}$ which, respectively, are defined formally by

$$\|u\|_{L^p(H,\mu_H)} := \left(\int_H |u(x)|^p\, \mu_H(\mathrm{d}x)\right)^{\frac{1}{p}}$$

and

$$(u,v)_{L^2(H,\mu_H)} := \int_H u(x)\overline{v(x)}\, \mu_H(\mathrm{d}x).$$

References 5.1 on page 66, and Examples 5 and 6 on page 164 in [26].

Consequently we can introduce the $L^p(H,\mu_H)$ space, this space is described to be the closure of $C_0(H)$ with respect to the $L^p(H,\mu_H)$-norm, i.e. $L^p(H,\mu_H) := \overline{C_0(H)}^{\|\cdot\|_{L^p(H,\mu_H)}}$. This definition immediately tells us that $C_0(H) \subset L^p(H,\mu_H)$ as a dense subspace. Equivalently $L^p(H,\mu_H)$ is the space of all (equivalence classes of) measurable functions $u : H \to \mathbb{C}$ for which $\|u\|_{L^p(H,\mu_H)} < \infty$. It is easy to verify $\|u\|_{L^p(H,\mu_H)}$ and $(u,v)_{L^2(H,\mu_H)}$ are well defined for all $u, v \in C_0(H)$. The notation $L^p(H,\mu_H)$ is used to emphasize the Haar measure being applied. Moreover, one may speak of the space $L^\infty(H,\mu_H)$, which by definition is the space of all measurable functions $u : H \to \mathbb{C}$ with finite essential supremum norm, designated $\|u\|_{\infty,H}$ (or $\|u\|_\infty$ if it is clear what H is). The essential supremum norm is defined as

$$\|u\|_{\infty,H} := \operatorname{ess\,sup} |u(x)| := \inf\{a \geq 0 \mid \mu_H\{x : |u(x)| > a\} = 0\}. \tag{2.2}$$

In the situation when u is continuous on H or when $\mu_H(A) = 0$, $A \subset H$, implies $A = \emptyset$, (2.2) can be reformulated as

$$\|u\|_{\infty,H} := \sup_{x \in H} |u(x)|. \tag{2.3}$$

This norm is called the supremum norm. If in addition H is finite, we arrive at

$$\|u\|_{\infty,H} := \max_{x \in H} |u(x)|.$$

References Page 188 (18 lines up) in [22]; 5.1 on page 66, page 74 (paragraph before 5.15 Theorem) in [26]; Examples (xi) and (xii) on page 25 in [5]; Page 86 (top of page) in [27].

When no confusion can arise we may adopt, for $1 \leq p \leq \infty$, the notations $L^p(H)$, $\|u\|_{L^p(H)}$ and $(u, v)_{L^2(H)}$ instead of $L^p(H, \mu_H)$, $\|u\|_{L^p(H,\mu_H)}$ and $(u, v)_{L^2(H,\mu_H)}$ respectively only when μ_H is normalized as in Example 1.20. The notations $L^p(H, \mu_H)$, $\|u\|_{L^p(H,\mu_H)}$ and $(u, v)_{L^2(H,\mu_H)}$ on the other hand will only be used when μ_H is taken to be arbitrary. From this point onwards we shall always assume any Haar measure on H to be normalized unless otherwise specified. It should also be noted that every continuous function is Borel measurable, hence all functions $u : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ and $u : \mathbb{Z}^m \to \mathbb{C}$ are by definition Borel measurable.

References Examples 2 on page 34 in [2].

The Haar integral of a function $u : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ (with respect to the Haar measure $\mu_{\mathbb{Z}/N\mathbb{Z}} = \frac{1}{N} \sum_{k=0}^{N-1} \varepsilon_k$) is given by

$$\int_{\mathbb{Z}/N\mathbb{Z}} u(x)\, \mu_{\mathbb{Z}/N\mathbb{Z}}(\mathrm{d}x) = \frac{1}{N} \int_{\mathbb{Z}/N\mathbb{Z}} u(x) \sum_{k=0}^{N-1} \varepsilon_k(\mathrm{d}x) = \frac{1}{N} \sum_{k=0}^{N-1} u(k).$$

Immediately we find for $1 \leq p < \infty$

$$\|u\|_{L^p(\mathbb{Z}/N\mathbb{Z})} = \left(\frac{1}{N} \sum_{k=0}^{N-1} |u(k)|^p \right)^{\frac{1}{p}}$$

and

$$(u, v)_{L^2(\mathbb{Z}/N\mathbb{Z})} = \frac{1}{N} \sum_{k=0}^{N-1} u(k)\overline{v(k)}.$$

Of course, every function $u : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ has a finite $L^p(\mathbb{Z}/N\mathbb{Z})$-norm, i.e. $\|u\|_{L^p(\mathbb{Z}/N\mathbb{Z})} < \infty$, meaning $L^p(\mathbb{Z}/N\mathbb{Z})$ coincides with $\mathbb{C}^N$. In other words, $L^p(\mathbb{Z}/N\mathbb{Z}) = C(\mathbb{Z}/N\mathbb{Z})$.

Consider now an arbitrary Haar measure η on $\mathbb{Z}/N\mathbb{Z}$. By definition of η, one finds that for $A \subset \mathbb{Z}/N\mathbb{Z}$ satisfying $\eta(A) = 0$ it follows $A = \emptyset$. This says that a property of $u \in L^p(\mathbb{Z}/N\mathbb{Z}, \eta)$ that holds almost everywhere actually turns out to hold everywhere. Hence, with $\mathbb{Z}/N\mathbb{Z}$ being finite, we find

$$\|u\|_{\infty,\mathbb{Z}/N\mathbb{Z}} = \max_{0 \leq k \leq N-1} |u(k)|,$$

which is obviously well defined for all $u : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$. Thus, like $L^p(\mathbb{Z}/N\mathbb{Z})$, $L^\infty(\mathbb{Z}/N\mathbb{Z})$ coincides with $\mathbb{C}^N$.

References Example 4 on page 85 in [13]; Examples 1 on page 66 in [2]; Page 222 in [29].

Next we turn to the case $H = \mathbb{Z}^m$. The Haar integral of $u : \mathbb{Z}^m \to \mathbb{C}$ (with respect to the Haar measure $\mu_{\mathbb{Z}^m} = \sum_{k \in \mathbb{Z}^m} \varepsilon_k$) is of the form

$$\int_{\mathbb{Z}^m} u(x)\, \mu_{\mathbb{Z}^m}(\mathrm{d}x) = \int_{\mathbb{Z}^m} u(x) \sum_{k \in \mathbb{Z}^m} \varepsilon_k(\mathrm{d}x) = \sum_{k \in \mathbb{Z}^m} u(k).$$

Let us look a bit more closely at the series $\sum_{k \in \mathbb{Z}^m} u(k)$ for a moment. Notice that the definition of this series is a bit obscure in the sense that we have no knowledge of the order of its terms. Let us assume the series $\sum_{k \in \mathbb{Z}^m} u(k)$ is absolutely convergent. Then the series $\sum_{k \in \mathbb{Z}^m} u(k)$ will always have the same sum no matter what the order of the terms are. For this reason $\int_{\mathbb{Z}^m} u(x)\, \mu_{\mathbb{Z}^m}(\mathrm{d}x)$ may be obtained as the limit of any sequence of partial sums. It is customary to choose $(S_N)_{N \in \mathbb{N}}$, $S_N := \sum_{|k| \leq N} u(k)$, as the sequence of (symmetric) partial sums to represent $\int_{\mathbb{Z}^m} u(x)\, \mu_{\mathbb{Z}^m}(\mathrm{d}x)$, that is, we set

$$\int_{\mathbb{Z}^m} u(x)\, \mu_{\mathbb{Z}^m}(\mathrm{d}x) = \lim_{N \to \infty} \sum_{|k| \leq N} u(k),$$

where $|k| := (k_1^2 + \ldots + k_m^2)^{\frac{1}{2}}$. However, we may replace $|k|$ with $|k|_1 := |k_1| + \ldots |k_m|$ or $|k|_\infty := \max_{1 \leq j \leq m} |k_j|$.

References Examples 5 on pages 86-87 in [13]; 1.4.6 Lemma on page 20 in [17]; Page 36 in [5].

Suppose $\sum_{k \in \mathbb{Z}^m} |u(k)|^p$ converges. Then using the argument above one may set

$$\sum_{k\in\mathbb{Z}^m}|u(k)|^p = \lim_{N\to\infty}\sum_{|k|\le N}|u(k)|^p.$$

In parallel with the case $H=\mathbb{Z}/N\mathbb{Z}$, we find for $1\le p<\infty$

$$\|u\|_{L^p(\mathbb{Z}^m)} = \left(\sum_{k\in\mathbb{Z}^m}|u(k)|^p\right)^{\frac{1}{p}}$$

and

$$(u,v)_{L^2(\mathbb{Z}^m)} = \sum_{k\in\mathbb{Z}^m} u(k)\overline{v(k)}.$$

Thus the space $L^p(\mathbb{Z}^m)$ consists off all functions $u:\mathbb{Z}^m\to\mathbb{C}$ with the property that $\sum_{k\in\mathbb{Z}^m}|u(k)|^p = \lim_{N\to\infty}\sum_{|k|\le N}|u(k)|^p$ is convergent. This convergence yields $\lim_{|k|\to\infty}u_k=0$ and therefore $L^p(\mathbb{Z}^m)\subset C_\infty(\mathbb{Z}^m)$ for $1\le p<\infty$. Moreover, one can show this inclusion is strict. Under this observation we understand there exist functions $u:\mathbb{Z}^m\to\mathbb{C}$ for which $\sum_{k\in\mathbb{Z}^m}|u(k)|^p$ diverges and as a result obtain the strict inclusion $L^p(\mathbb{Z}^m)\subset C(\mathbb{Z}^m)$. Now let η be an arbitrary measure on $\mathbb{Z}^m$. As in the case of $\mathbb{Z}/N\mathbb{Z}$, a property that holds η-almost everywhere actually turns out to hold everywhere, i.e. $\eta(A)=0$, $A\subset\mathbb{Z}^m$, implies $A=\emptyset$. This implies the essential supremum norm on $\mathbb{Z}^m$ is exactly the supremum norm

$$\|u\|_{\infty,\mathbb{Z}^m} = \sup_{k\in\mathbb{Z}^m}|u(k)|.$$

The space of all functions $u:\mathbb{Z}^m\to\mathbb{C}$ for which $\|u\|_{\infty,\mathbb{Z}^m}$ is finite is designated $L^\infty(\mathbb{Z}^m)$.

References Example 4 on page 67 and Example 5 on page 164 in [26]; Page 48 (first paragraph after (ii)) in [6].

For $1\le p<q<\infty$ it can be shown that

$$\|u\|_{L^q(\mathbb{Z}^m)} \le \|u\|_{L^p(\mathbb{Z}^m)},$$

which then implies

$$L^p(\mathbb{Z}^m)\subset L^q(\mathbb{Z}^m),$$

and this inclusion is strict.

References Lemma 3.1 and Corollary 3.2 on page 17 in [21].

Collecting all the inclusions together we arrive at

$$C_0(\mathbb{Z}^m) \subset L^p(\mathbb{Z}^m) \subset L^q(\mathbb{Z}^m) \subset C_\infty(\mathbb{Z}^m) \subset C(\mathbb{Z}^m).$$

Next we discuss the cases $\mathbb{R}^n$ and $\mathbb{T}^m$. Given a measurable function $u : \mathbb{R}^n \to \mathbb{C}$, its Haar integral (with respect to the Haar measure $\mu_{\mathbb{R}^n} = \lambda^{(n)}$) is written as

$$\int_{\mathbb{R}^n} u(x)\,\mu_{\mathbb{R}^n}(\mathrm{d}x) = \int_{\mathbb{R}^n} u(x)\,\lambda^{(n)}(\mathrm{d}x) := \int_{\mathbb{R}^n} u(x)\,\mathrm{d}x.$$

By $L^p(\mathbb{R}^n)$, $1 \le p < q < \infty$, we denote the space consisting of all measurable functions $u : \mathbb{R}^n \to \mathbb{C}$ for which $|u|^p$ is integrable, the $L^p(\mathbb{R}^n)$-norm and inner product respectively possess the forms

$$\|u\|_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |u(x)|^p\,\mathrm{d}x\right)^{\frac{1}{p}}$$

and

$$(u, v)_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} u(x)\overline{v(x)}\,\mathrm{d}x.$$

Meanwhile, in contrast with the cases $\mathbb{Z}/N\mathbb{Z}$ and $\mathbb{Z}^m$, the property $\mu_{\mathbb{R}^n}(A) = 0$ (for an arbitrary Haar measure $\mu_{\mathbb{R}^n}$ on $\mathbb{R}^n$) does not necessarily imply $A = \emptyset$. For instance, A may be any countable set in $\mathbb{R}^n$, and for this reason the essential supremum norm of a function $u : \mathbb{R}^n \to \mathbb{C}$ in $L^\infty(\mathbb{R}^n)$ is in general only of the form

$$\|u\|_{\infty,\mathbb{R}^n} = \inf\{a \ge 0 \mid \lambda^{(n)}\{x : |u(x)| > a\} = 0\}.$$

Only when u is continuous does its essential supremum norm adopt the same form as (2.3), that is, we get the supremum norm

$$\|u\|_{\infty,\mathbb{R}^n} = \sup_{x\in\mathbb{R}^n} |u(x)|.$$

References Example 1 on page 84 in [13]; Examples 2 on page 29 in [2].

Eventually we turn to the case $\mathbb{T}^m$. For the Haar integral of a measurable function $u : \mathbb{T}^m \to \mathbb{C}$ (with respect to the Haar measure $\mu_{\mathbb{T}^m} = \frac{1}{(2\pi)^m}\lambda^{(m)}$) we write

$$\frac{1}{(2\pi)^m}\int_0^{2\pi} \cdots \int_0^{2\pi} u(e^{it})\,\lambda^{(m)}(\mathrm{d}t) = \frac{1}{(2\pi)^m}\int_0^{2\pi} \cdots \int_0^{2\pi} u(e^{it})\,\mathrm{d}t.$$

Here an abuse of notation has been used, by $u(e^{it})$ we mean $u(e^{it_1}, \dots, e^{it_m})$, where $t_k \in [0, 2\pi)$, $1 \le k \le m$. On the other hand however, each function $u : \mathbb{T}^m \to \mathbb{C}$ can be identified as a 2π-periodic function on $\mathbb{R}^m$ which shall also be denoted by u, and moreover

$$\frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} u(e^{it}) \, \mathrm{d}t = \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} u(x) \, \mathrm{d}x.$$

Immediately we find for $1 \le p < \infty$

$$\begin{aligned} \|u\|_{L^p(\mathbb{T}^m)} &= \left(\frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} |u(e^{it})|^p \, \mathrm{d}t \right)^{\frac{1}{p}} \\ &= \left(\frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} |u(x)|^p \, \mathrm{d}x \right)^{\frac{1}{p}} := \|u\|_{L^p_{per}(\mathbb{R}^m, \frac{1}{(2\pi)^m}\lambda^{(m)})}, \end{aligned}$$

as well as

$$\begin{aligned} (u, v)_{L^2(\mathbb{T}^m)} &= \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} u(e^{it})\overline{v(e^{it})} \, \mathrm{d}t \\ &= \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} u(x)\overline{v(x)} \, \mathrm{d}x := (u, v)_{L^2_{per}(\mathbb{R}^m, \frac{1}{(2\pi)^m}\lambda^{(m)})}. \end{aligned}$$

References Examples 2 on page 84 in [13]; Basic Examples (3) on pages 96-97 in [23]; Page 128 (last sentence before Remark 4.1.1) in [27]; Pages 1-2 and 27 in [22].

Of course, we define $L^p(\mathbb{T}^m)$ (respectively $L^p_{per}(\mathbb{R}^m, \frac{1}{(2\pi)^m}\lambda^{(m)})$) to be the space of all measurable functions $u : \mathbb{T}^m \to \mathbb{C}$ (respectively 2π-periodic functions $u : \mathbb{R}^m \to \mathbb{C}$) such that $\|u\|_{L^p(\mathbb{T}^m)} < \infty$ (respectively $\|u\|_{L^p_{per}(\mathbb{R}^m, \frac{1}{(2\pi)^m}\lambda^{(m)})}$ $< \infty$). It follows from what we have just discussed above that $L^p(\mathbb{T}^m)$ is isometrically isomorphic to $L^p_{per}(\mathbb{R}^m, \frac{1}{(2\pi)^m}\lambda^{(m)})$, or roughly speaking $L^p(\mathbb{T}^m)$ is the space $L^p_{per}(\mathbb{R}^m, \frac{1}{(2\pi)^m}\lambda^{(m)})$. Meanwhile, the space $L^\infty(\mathbb{T}^m)$ consists of all measurable functions $u : \mathbb{T}^m \to \mathbb{C}$ satisfying $\|u\|_{\infty, \mathbb{T}^m} < \infty$, where

$$\|u\|_{\infty, \mathbb{T}^m} = \inf\{a \ge 0 \mid \frac{1}{(2\pi)^m}\lambda^{(m)}\{e^{it} : |u(e^{it})| > a\} = 0\}.$$

Like in the case $\mathbb{R}^n$, if u is in addition continuous, then

$$\|u\|_{\infty, \mathbb{T}^m} = \sup_{\omega \in \mathbb{T}^m} |u(\omega)|.$$

Since any continuous function on a compact set is bounded, we have $C(\mathbb{T}^m) \subset L^p(\mathbb{T}^m)$, hence $C_0(\mathbb{T}^m) = C_\infty(\mathbb{T}^m) = C(\mathbb{T}^m)$ is a subspace of $L^p(\mathbb{T}^m)$.

References 4.16 on pages 55-56 in [26].

To comprehend inclusion relations for the spaces $L^p(\mathbb{T}^m, \mu_{\mathbb{T}^m})$, $1 \le p < \infty$, the following observation is required: Let $(H, \mathcal{A}, \mu_H)$ be a measure space where H is a locally compact Abelian group and μ_H is an arbitrary Haar measure on H with finite total mass, i.e. $\mu_H(H) < \infty$. Then for $u \in L^q(H, \mu_H)$, $1 \le p < q < \infty$, we get the estimate

$$\|u\|_{L^p(H,\mu_H)} \le \mu_H(H)^{\frac{1}{p}-\frac{1}{q}} \|u\|_{L^q(H,\mu_H)} \tag{2.4}$$

and therefore $L^q(H, \mu_H) \subset L^p(H, \mu_H)$. However, if μ_H is not of finite total mass, then it is obvious from estimate (2.4) that in general there will be no inclusion relation between $L^p(H, \mu_H)$ and $L^q(H, \mu_H)$ for $p \neq q$. Without identifying $\mathbb{T}^m$ as $[0, 2\pi)^m$ and $\mu_{\mathbb{T}^m}$ with $\frac{1}{(2\pi)^m}\lambda^{(m)}$ we have in the case $H = \mathbb{T}^m$

$$\|u\|_{L^p(\mathbb{T}^m)} \le \mu_{\mathbb{T}^m}(\mathbb{T}^m)^{\frac{1}{p}-\frac{1}{q}} \|u\|_{L^q(\mathbb{T}^m)}, \tag{2.5}$$

and since $\mathbb{T}^m$ is compact, we know from Theorem 1.19 that $\mu_{\mathbb{T}^m}(\mathbb{T}^m) < \infty$ and hence $L^q(\mathbb{T}^m) \subset L^p(\mathbb{T}^m)$ (or $L^q_{per}(\mathbb{R}^m, \frac{1}{(2\pi)^m}\mu_{\mathbb{R}^m}) \subset L^p_{per}(\mathbb{R}^m, \frac{1}{(2\pi)^m}\mu_{\mathbb{R}^m})$). Notice that this inclusion relation is the reverse of the inclusion relation $L^p(\mathbb{Z}^m) \subset L^q(\mathbb{Z}^m)$ established earlier.

By contrast, given any Haar measure $\mu_{\mathbb{R}^n}$ on $\mathbb{R}^n$, there is in general no inclusion relation between $L^p(\mathbb{R}^n, \mu_{\mathbb{R}^n})$ and $L^q(\mathbb{R}^n, \mu_{\mathbb{R}^n})$ for $p \neq q$. This is because, analogously to estimate (2.5), we find

$$\|u\|_{L^q(\mathbb{R}^n,\mu_{\mathbb{R}^n})} \le \mu_{\mathbb{R}^n}(\mathbb{R}^n)^{\frac{1}{p}-\frac{1}{q}} \|u\|_{L^p(\mathbb{R}^n,\mu_{\mathbb{R}^n})},$$

which is of no sense as $\mu_{\mathbb{R}^n}(\mathbb{R}^n) = \infty$.

References Page 30 in [19].

For each of these spaces of integrable functions we now give examples of dense subsets which will prove useful for proving estimates later on. A dense subspace of $L^p(\mathbb{Z}^m)$, $1 \le p < \infty$, is given by the space of all finite sequences on $\mathbb{Z}^m$, designated $C_0(\mathbb{Z}^m)$ or $l_{\text{finite}}(\mathbb{Z}^m)$, in fact we can restrict ourselves to finite sequences with rational value. Recall that a sequence is said to be finite if only a finite number of the terms in the sequence are non-zero. Since in addition this space is countable, it follows $L^p(\mathbb{Z}^m)$ is separable.

A common example of a dense subset of $L^p(\mathbb{R}^n)$, $1 \le p < \infty$, is the Schwartz space. The Schwartz space, denoted by $S(\mathbb{R}^n)$, consists of all functions $u \in C^\infty(\mathbb{R}^n)$ for which the semi-norm

$$p_{m_1,m_2}(u) := \sup_{x\in\mathbb{R}^n}((1+|x|^2)^{\frac{m_1}{2}} \sum_{|\alpha|\leq m_2} |\partial_x^\alpha u(x)|)$$

or equivalently

$$p_{\alpha,\beta}(u) := \sup_{x\in\mathbb{R}^n} |x^\alpha \partial_x^\beta u(x)|$$

are for all $m_1, m_2 \in \mathbb{N}_0$ and $\alpha, \beta \in \mathbb{N}_0^n$ finite. Other examples of dense subsets of $L^p(\mathbb{R}^n)$ include $C_0(\mathbb{R}^n)$ and the test functions $C_0^\infty(\mathbb{R}^n)$, but $S(\mathbb{R}^n)$ is preferable when dealing with the Fourier transform later on. Note that $C_\infty(\mathbb{R}^n)$ is neither a subspace of $L^p(\mathbb{R}^n)$ nor is $L^p(\mathbb{R}^n)$ a subspace of $C_\infty(\mathbb{R}^n)$.

Finally, consider the set of all trigonometric polynomials on $\mathbb{T}^m = [0, 2\pi)^m$, defined by $\text{span}\{e_k \mid e_k(x) = e^{ik\cdot x},\ k \in \mathbb{Z}^m\}$. It turns out $\text{span}\{e_k \mid e_k(x) = e^{ik\cdot x},\ k \in \mathbb{Z}^m\}$ is a dense subset of $L^p(\mathbb{T}^m)$ for $1 \leq p < \infty$. The space $C(\mathbb{T}^m)$ provides a further example of a dense subset of $L^p(\mathbb{T}^m)$.

References Problems 5G on page 76 in [26]; Pages 31 and 43, and Corollary 2.6.1 on page 44 in [19]; Page 24 and E8 on page 268 in [28].

3 The Fourier Transform of Functions

Recollecting what we have discussed about dual groups and $L^p(H, \mu_H)$ spaces, we can now combine these ideas together to introduce the notion of the Fourier transform over a locally compact Abelian group. In this section we take $(2\pi)^{-\frac{n}{2}}\lambda^{(n)}$ instead of $\lambda^{(n)}$ as the normalized Haar measure on $\mathbb{R}^n$. Given a locally compact Abelian group H endowed with the Haar measure μ_H, we define the **Fourier transform** of a function $u \in L^1(H, \mu_H)$ by

$$\hat{u}(\chi) := F(u)(\chi) := (u, \chi)_{L^2(H,\mu_H)} = \int_H u(x)\overline{\chi(x)}\,\mu_H(\mathrm{d}x), \qquad \chi \in \widehat{H}. \quad (3.1)$$

Definition (3.1) clearly illustrates the Fourier transform F assigns a function on H to a function on $\widehat{H}$. Compare page 107 in [23]. Examples of the Fourier transforms over each of the elementary locally compact Abelian groups are given below.

Example 3.1.

- *Let* $\widehat{\mathbb{Z}/N\mathbb{Z}} = \{e_\omega \mid \omega = 0, 1, \ldots, N-1\}$ *be the dual group of* $\mathbb{Z}/N\mathbb{Z}$ *as defined in Example 1.12. Then the Fourier transform of a function* $u \in C(\mathbb{Z}/N\mathbb{Z}) = L^1(\mathbb{Z}/N\mathbb{Z})$ *is given by*

$$\hat{u}(e_\omega) = (u, e_\omega)_{L^2(\mathbb{Z}/N\mathbb{Z})} = \frac{1}{N}\sum_{k=0}^{N-1} u(k)e^{-2\pi i k\omega/N}, \qquad e_\omega \in \widehat{\mathbb{Z}/N\mathbb{Z}},$$

or due to the correspondence $\omega \longleftrightarrow e_\omega$ *we may, by applying Theorem 1.15, restate* (3.1) *as*

$$\hat{u}(\omega) = \frac{1}{N}\sum_{k=0}^{N-1} u(k)e^{-2\pi i k\omega/N}, \qquad \omega \in \mathbb{Z}/N\mathbb{Z}. \quad (3.2)$$

It is more convenient to choose (3.2) *as the representation of the Fourier transform of u, in other words, we will view* $\hat{u}$ *of having domain* $\mathbb{Z}/N\mathbb{Z}$ *instead of* $\widehat{\mathbb{Z}/N\mathbb{Z}}$. *We will apply the same argument to define the Fourier transforms over* $\mathbb{Z}^m$, $\mathbb{T}^m$ *and* $\mathbb{R}^n$.

- *The Fourier transform of a function* $u \in L^1(\mathbb{Z}^m)$ *is given by*

$$F_{k\mapsto\omega}(u)(\omega) := \hat{u}(\omega) = \sum_{k\in\mathbb{Z}^m} u(k)e^{-ik\cdot\omega}, \qquad \omega \in [0, 2\pi)^m. \quad (3.3)$$

- *The Fourier transform of a function* $u \in L^1(\mathbb{T}^m)$ *is given by*

$$F_{\omega\mapsto k}(u)(k) := \hat{u}(k) = \frac{1}{(2\pi)^m}\int_0^{2\pi}\cdots\int_0^{2\pi} u(\omega)e^{-ik\cdot\omega}\,\mathrm{d}\omega, \qquad k \in \mathbb{Z}^m.$$

- *The Fourier transform of a function* $u \in L^1(\mathbb{R}^n)$ *is given by*

$$F_{x\mapsto\xi}(u)(\xi) := \hat{u}(\xi) = (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} u(x)e^{-ix\cdot\xi}\,\mathrm{d}x, \qquad \xi \in \mathbb{R}^n. \tag{3.4}$$

References Pages 223 and 235 in [29]; Page 13 in [28].

Without proof we now state some important properties of all members of the Fourier transform family. Some properties hold for all Fourier transforms while there are properties that are not valid for all of these Fourier transforms.

Lemma 3.2. *1. Let* $u, v \in L^1(H, \mu_H)$*, where* $(H, +, \tau)$ *is a locally compact Abelian group equipped with the Haar measure* μ_H*. Further let* $\tau_a u(x) = u(x-a)$ *and* $\check{u}(x) := u(-x)$ *for all* $x \in H$*. Then for* $\lambda, \mu \in \mathbb{C}$*,* $a \in H$ *and* $\chi_1 \in \widehat{H}$ *it follows*

(a) $(\lambda u + \mu v)^\wedge(\chi) = \lambda\hat{u}(\chi) + \mu\hat{v}(\chi)$ *(linearity);*

(b) $(\tau_a u)^\wedge(\chi) = \overline{\chi(a)}\hat{u}(\chi)$ *(shifting);*

(c) $(\chi_1 \cdot u)^\wedge(\chi) = \hat{u}(\chi \cdot \chi_1^{-1})$*;*

(d) $(\check{u})^\wedge(\chi) = \hat{u}(\chi^{-1})$*;*

(e) $(\overline{\check{u}})^\wedge(\chi) = \overline{\hat{u}(\chi)}$ *(conjugate);*

(f) $(u * v)^\wedge(\chi) = \hat{u}(\chi) \cdot \hat{v}(\chi)$ *(convolution theorem).*

2. Let $H_\lambda(x) := \lambda x$*. Then for* $u \in S(\mathbb{R}^n)$ *we have*

$$(u \circ H_\lambda)^\wedge(\xi) = \lambda^{-n}\hat{u}\left(\frac{\xi}{\lambda}\right), \quad \lambda > 0 \quad \textit{(scaling)}.$$

3. Let $\alpha, \beta \in \mathbb{N}_0^n$ *and set* $D_\xi^\alpha := (-i\partial_\xi)^\alpha$*,* $D_x^\beta := (-i\partial_x)^\beta$*. For* $u \in S(\mathbb{R}^n)$ *we have*

$$(x^\alpha u(\cdot))^\wedge(\xi) = (-1)^{|\alpha|}D_\xi^\alpha\hat{u}(\xi)$$

and

$$(D_x^\beta u)^\wedge(\xi) = \xi^\beta \hat{u}(\xi).$$

References 2.3 Proposition on page 9 in [3]; Theorem 4.11.5 on page 197 in [9]; Corollary 3.1.3 on page 77, Lemma 3.1.9.B on page 80 and Theorem 3.2.4 on page 84 in [19].

For the rest of this section H will still represent a locally compact Abelian group.

The theorem below claims that we can recover any function $u \in L^1(H, \mu_H)$ for which $\hat{u} \in L^1(\widehat{H}, \mu_{\widehat{H}})$ from its Fourier transform.

Theorem 3.3 (Fourier Inversion Theorem). *Suppose $u \in L^1(H, \mu_H)$. Then there exists a unique Haar measure on $\widehat{H}$, denoted $\widehat{\mu}_{\widehat{H}}$, such that if $\hat{u} \in L^1(\widehat{H}, \widehat{\mu}_{\widehat{H}})$, the formula*

$$u(x) = \int_{\widehat{H}} \hat{u}(\chi)\chi(x)\, \widehat{\mu}_{\widehat{H}}(\mathrm{d}\chi) \tag{3.5}$$

holds almost everywhere in H. If in addition u is continuous, then (3.5) *holds pointwisely.*

The unique Haar measure $\widehat{\mu}_{\widehat{H}}$ satisfying (3.5) is known as the **dual Haar measure** (with respect to μ_H). From now on we write $L^1(\widehat{H})$ and $L^2(\widehat{H})$ instead of $L^1(\widehat{H}, \widehat{\mu}_{\widehat{H}})$ and $L^2(\widehat{H}, \widehat{\mu}_{\widehat{H}})$ while the notations $L^1(\widehat{H}, \mu_{\widehat{H}})$ and $L^2(\widehat{H}, \mu_{\widehat{H}})$ will be used only when $\mu_{\widehat{H}}$ is an arbitrary Haar measure on $\widehat{H}$.

Example 3.4. *For any $u \in S(\mathbb{R}^n)$*

$$u(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \hat{u}(\xi) e^{ix\cdot\xi}\, \mathrm{d}\xi, \qquad x \in \mathbb{R}^n,$$

holds pointwisely. Of course, $\hat{u} \in S(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$ and by definition u is continuous. This example shows the dual Haar measure $\widehat{\mu}_{\widehat{\mathbb{R}^n}}$ (with respect to $\mu_{\mathbb{R}^n} = (2\pi)^{-\frac{n}{2}}\lambda^{(n)}$) is $(2\pi)^{-\frac{n}{2}}\lambda^{(n)}$, implying $\mu_{\mathbb{R}^n} = \widehat{\mu}_{\widehat{\mathbb{R}^n}}$.

References Theorem 4.4.5, Definition 4.4.6 and Remark 4.4.8 on page 144 in [27]; Corollary 3.2.12 on page 88 in [19].

We now go on to define the Fourier transform on $L^2(H, \mu_H)$. In general it does not hold that $L^2(H, \mu_H) \subset L^1(H, \mu_H)$. Earlier we found $L^1(\mathbb{Z}^m) \subset$

$L^2(\mathbb{Z}^m)$ and that in general there is no inclusion relation between $L^1(\mathbb{R}^n, \mu_{\mathbb{R}^n})$ and $L^2(\mathbb{R}^n, \mu_{\mathbb{R}^n})$. Thus $L^2(\mathbb{Z}^m)$ and $L^2(\mathbb{R}^n)$ are examples for which $L^2(H, \mu_H) \subset L^1(H, \mu_H)$ is not valid as oppose to $L^2(\mathbb{Z}/N\mathbb{Z})$ and $L^2(\mathbb{T}^m)$ in which the inclusion relation does hold. Consequently we cannot respectively take definitions (3.3) and (3.4) to represent the Fourier transforms on $L^2(\mathbb{Z}^m)$ and $L^2(\mathbb{R}^n)$. That being the case, a different notion of a Fourier transform is required on these spaces. But before this notion of the Fourier transform on $L^2(H, \mu_H)$ can be introduced we require a theorem.

Theorem 3.5. *Let $u \in L^1(H, \mu_H) \cap L^2(H, \mu_H) \subset L^2(H, \mu_H)$. Then $\hat{u} \in L^2(\widehat{H}, \mu_{\widehat{H}})$ and*

$$\|u\|_{L^2(H,\mu_H)} = c_{\mu_{\widehat{H}}} \|\hat{u}\|_{L^2(\widehat{H},\mu_{\widehat{H}})}, \tag{3.6}$$

where $c_{\mu_{\widehat{H}}}$ is a constant depending on the normalization of the Haar measure $\mu_{\widehat{H}}$. Moreover, $c_{\mu_{\widehat{H}}} = 1$ only in the case $\mu_{\widehat{H}} = \widehat{\mu}_{\widehat{H}}$ and hence the Fourier transform is an isometry from $L^1(H, \mu_H) \cap L^2(H, \mu_H)$ into $L^2(\widehat{H})$.

References Theorem 4.4.13 and Remark 4.4.14 on page 147 in [27]; 2.5 Theorem on page 9 in [3].

The fact that $L^1(H, \mu_H) \cap L^2(H, \mu_H)$ is dense in $L^2(H, \mu_H)$ and $F : L^1(H, \mu_H) \cap L^2(H, \mu_H) \to L^2(\widehat{H})$ is an isometry immediately implies that the isometry extends uniquely by continuity to an isometry, denoted again by F, with domain $L^2(H, \mu_H)$. We call $F : L^2(H, \mu_H) \to L^2(\widehat{H})$ also the Fourier transform. We now give the definition of the Fourier transform of a function $u \in L^2(H, \mu_H)$.

Definition 3.6. *Let $u \in L^2(H, \mu_H)$ and $(u_n)_{n \in \mathbb{N}}$, $u_n \in L^1(H, \mu_H) \cap L^2(H, \mu_H)$, be a sequence such that $\lim_{n\to\infty} u_n = u$ in the sense of the space $L^2(H, \mu_H)$, i.e. $\lim_{n\to\infty} \|u - u_n\|_{L^2(H,\mu_H)} = 0$. We define the Fourier transform of $u \in L^2(H, \mu_H)$ by*

$$\hat{u} := \lim_{n\to\infty} \hat{u}_n,$$

where the limit is understood in the $L^2(\widehat{H})$-sense.

References (1.6.6) on page 33 in [4]; Definition 4.11.2 on pages 199-200 in [9].

Furthermore, equality (3.6) also holds for $u \in L^2(H, \mu_H)$ giving rise to the **Plancherel's theorem** established in Theorem 3.9.

Let us summarize some mapping properties of the Fourier transform which we will need later.

Theorem 3.7. *1. The Fourier transform is a continuous linear operator from $S(\mathbb{R}^n)$ onto itself.*

2. The Fourier transform is a continuous linear operator from $L^1(\mathbb{R}^n)$ into $C_\infty(\mathbb{R}^n)$ satisfying

$$\|\hat{u}\|_\infty \leq \|u\|_{L^1(\mathbb{R}^n)} \tag{3.7}$$

for all $u \in L^1(\mathbb{R}^n)$. Hence $\hat{u}$ vanishes at infinity.

3. Let $u \in L^2(\mathbb{R}^n)$. Then

$$\|u\|_{L^2(\mathbb{R}^n)} = \|\hat{u}\|_{L^2(\mathbb{R}^n)}.$$

Furthermore the Fourier transform is an isometric isomorphism (and hence continuous) from $L^2(\mathbb{R}^n)$ onto $L^2(\mathbb{R}^n)$.

References Theorem 3.1.7 on page 78 and Theorem 3.2.1 on page 83 in [19]; (1.6.7) Theorem on page 33 in [4]; 2.5 Theorem on page 9 in [3].

2. and 3. can be generalized to any locally compact Abelian group, i.e.

Lemma 3.8. *The Fourier transform is a continuous linear operator from $L^1(H, \mu_H)$ into $C_\infty(\widehat{H})$ satisfying*

$$\|\hat{u}\|_\infty \leq \|u\|_{L^1(H,\mu_H)}$$

for all $u \in L^1(H, \mu_H)$. Hence $\hat{u}$ vanishes at infinity.

and

Theorem 3.9. *Let $u \in L^2(H, \mu_H)$. Then $\hat{u} \in L^2(\widehat{H}, \mu_{\widehat{H}})$ and*

$$\|u\|_{L^2(H,\mu_H)} = c_{\mu_{\widehat{H}}} \|\hat{u}\|_{L^2(\widehat{H},\mu_{\widehat{H}})},$$

where $c_{\mu_{\widehat{H}}}$ is a constant depending on the normalization of the Haar measure $\mu_{\widehat{H}}$. Furthermore, in the case $\mu_{\widehat{H}} = \widehat{\mu}_{\widehat{H}}$ the Fourier transform is an isometric isomorphism from $L^2(H, \mu_H)$ onto $L^2(\widehat{H})$.

Note that 2. is the Riemann-Lebesgue lemma while 3. is Plancherel's theorem.

References Theorem 3.2.1 on page 83 in [19]; (1.6.7) Theorem on page 33 in [4]; 2.5 Theorem on page 9 in [3]; 1.2.4 Theorem (d) on page 9 in [28].

We cannot use (3.5) for the inverse Fourier transform say in $L^2(H, \mu_H)$. However, using (3.5) and the fact that elements $u \in L^2(H, \mu_H)$ may be approximated by functions $u_n \in L^1(H, \mu_H) \cap L^2(H, \mu_H)$ for which $\hat{u}_n \in L^1(\widehat{H}) \cap L^2(\widehat{H})$ we find

$$u(x) = \lim_{n\to\infty} \int_{\widehat{H}} \hat{u}_n(\chi)\chi(x)\, \widehat{\mu}_{\widehat{H}}(\mathrm{d}\chi), \tag{3.8}$$

where the limit is understood in the $L^2(H, \mu_H)$-sense. Remember that $\lim_{n\to\infty} \hat{u}_n = \hat{u}$ in the $L^2(\widehat{H})$-sense. By an abuse of notation one may interpret (3.8) as (3.9) stated in Theorem 3.10.

Theorem 3.10 (Fourier Inversion Theorem on $L^2(H, \mu_H)$). *Let $u \in L^2(H, \mu_H)$. Then*

$$u(x) = \int_{\widehat{H}} \hat{u}(\chi)\chi(x)\, \widehat{\mu}_{\widehat{H}}(\mathrm{d}\chi) \tag{3.9}$$

in the sense of the space $L^2(H, \mu_H)$.

This claims any function $u \in L^2(H, \mu_H)$ can be recovered from its Fourier transform.

References 10.4.9 and 10.4.10 Remarks on pages 729-730 in [10].

We now turn our attention for a moment to examining the Fourier transform of a function on $\widehat{H}$ rather than the Fourier transform of a function on H. By definition the Fourier transform maps a function on $\widehat{H}$ into a function on $\widehat{\widehat{H}}$, but in view of the Pontryagin Duality theorem the Fourier transform may be regarded as an operator, denote it by $\check{F}$, mapping a function on $\widehat{H}$ into a function on H which, for a function $U \in L^1(\widehat{H}, \mu_{\widehat{H}})$, takes the form

$$\check{U}(x) := \check{F}(U)(x) := \int_{\widehat{H}} U(\chi)\overline{\chi(x)}\, \mu_{\widehat{H}}(\mathrm{d}\chi), \qquad x \in H. \tag{3.10}$$

By comparison of (3.5) in Theorem 3.3 with (3.10) we define the **inverse Fourier transform** of a function $U \in L^1(\widehat{H}, \mu_{\widehat{H}})$ by $F^{-1}(U)(x) := \check{F}(U)(-x)$, that is,

$$F^{-1}(U)(x) := \int_{\widehat{H}} U(\chi)\chi(x)\, \mu_{\widehat{H}}(\mathrm{d}\chi), \qquad x \in H.$$

From this definition it follows F^{-1} has the same mapping properties as F and consequently F^{-1} is an isometry from $L^1(\widehat{H}) \cap L^2(\widehat{H})$ into $L^2(H, \mu_H)$ that extends uniquely by continuity to an isometric isomorphism, denoted as well by F^{-1}, with domain $L^2(\widehat{H})$. We call $F^{-1} : L^2(\widehat{H}) \to L^2(H, \mu_H)$ also the inverse Fourier transform. It also turns out $F^{-1}: L^2(\widehat{H}) \to L^2(H, \mu_H)$ is the inverse of $F: L^2(H, \mu_H) \to L^2(\widehat{H})$.

Example 3.11. *The inverse Fourier transform of a function $U \in L^1(\mathbb{R}^n)$ is defined by*

$$F^{-1}_{\xi\mapsto x}(U)(x) := F^{-1}(U)(x) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} U(\xi) e^{ix\cdot\xi}\, \mathrm{d}\xi, \qquad x \in \mathbb{R}^n. \quad (3.11)$$

Moreover, $F^{-1}: S(\mathbb{R}^n) \to S(\mathbb{R}^n)$ is the inverse of $F: S(\mathbb{R}^n) \to S(\mathbb{R}^n)$.

References 10.4.9 on pages 729-730 in [10]; (1.6.8) on page 34 in [4]; Definition 3.1.5 and Theorem 3.1.7 on page 78 in [19].

For the rest of this thesis the Haar measure on $\mathbb{R}^n$ will be taken to be $\lambda^{(n)}$ but we will still use (3.4) and (3.11) for the definitions of the Fourier transform and inverse Fourier transform respectively. Accordingly there will be changes regarding constants to Lemma 3.2.(f) and (3.7) in Theorem 3.7, i.e. Lemma 3.2.(f) becomes $(u * v)^\wedge(\xi) = (2\pi)^{\frac{n}{2}} \hat{u}(\xi) \cdot \hat{v}(\xi)$ as oppose to $(u * v)^\wedge(\xi) = \hat{u}(\xi) \cdot \hat{v}(\xi)$, and (3.7) in Theorem 3.7 becomes $\|\hat{u}\|_\infty \leq (2\pi)^{-\frac{n}{2}} \|u\|_{L^1(\mathbb{R}^n)}$.

References Theorem 3.2.1 on page 83 and Theorem 3.2.4 on page 84 in [19].

4 The Fourier Transform of Measures

We now want to extend the Fourier transform to measures. In this section H represents a locally compact Abelian group.

Definition 4.1. *Let $\mu \in \mathcal{M}_b^+(H)$. The* ***Fourier (Fourier-Stieltjes) transform*** *of μ is given by*

$$\hat{\mu}(\chi) := \int_H \overline{\chi(x)}\, \mu(\mathrm{d}x), \qquad \chi \in \widehat{H}. \tag{4.1}$$

It can easily be shown that $L^1_+(H, \mu_H) := \{u \in L^1(H, \mu_H) \mid u \geq 0\}$ is injectively embedded into $\mathcal{M}_b^+(H)$ with u being associated with the measure, labeled $u\,\mu_H$, having density u with respect to μ_H (see 17.1 Theorem on page 96 in [2] for the definition of this measure). Hence (4.1) may be interpreted as an extension of the Fourier transform from $L^1_+(H, \mu_H)$ to $\mathcal{M}_b^+(H)$ with the Fourier transform on $\mathcal{M}_b^+(H)$ coinciding with the Fourier transform on $L^1_+(H, \mu_H)$ in the case $\mu = u\,\mu_H$. Moreover, switching from positive measures to bounded signed or even bounded complex-valued measures $\mu \in \mathcal{M}_{b,\mathbb{C}}(H)$ we see that the Fourier transform extends from $L^1(H, \mu_H)$ to $\mathcal{M}_{b,\mathbb{C}}(H)$.

References 2.2 on page 8 in [3]; Page 191 in [22]; 17.2 Definition on page 96 in [2].

Theorem 4.2. *Let $\mu \in \mathcal{M}_b^+(H)$. Then its Fourier transform $\hat{\mu}$ is a uniformly continuous and bounded function on $\widehat{H}$. In addition we have the estimate*

$$\|\hat{\mu}\|_\infty \leq \|\mu\|.$$

References 2.3 Proposition on page 9 in [3].

It should be noted that there is in general no analogue to Lemma 3.8 (Riemann-Lebesgue lemma), namely $\hat{\mu}$ in general does not vanish at infinity. Nevertheless there are properties of the Fourier transform on $L^1(H, \mu_H)$ which do carry over to the Fourier transform on $\mathcal{M}_b^+(H)$. In particular, the analogues of (a)-(f) in Lemma 3.2 hold for $\hat{\mu}$ which we now exhibit in the following lemma:

Lemma 4.3. *Let $\mu, \nu \in \mathcal{M}_b^+(H)$, where $(H, +, \tau)$ is a locally compact Abelian group with Haar measure μ_H. Further let $Sx := -x$ for all $x \in H$. Then for $\alpha, \beta \geq 0$, $a \in H$ and $\chi_1 \in \widehat{H}$ it follows*

1. $(\alpha\mu + \beta\nu)^\wedge(\chi) = \alpha\hat{\mu}(\chi) + \beta\hat{\nu}(\chi)$ *(linearity);*
2. $(\tau_a(\mu))^\wedge(\chi) = \overline{\chi(a)}\hat{\mu}(\chi)$ *(shifting the argument);*

3. $(\chi_1\mu)^\wedge(\chi) = \hat{\mu}(\chi \cdot \chi_1^{-1})$;

4. $[S(\mu)]^\wedge(\chi) = \hat{\mu}(\chi^{-1})$;

5. $(\overline{\mu})^\wedge(\chi) = \overline{\hat{\mu}(\chi^{-1})}$ *(conjugate);*

6. $(\mu * \nu)^\wedge(\chi) = \hat{\mu}(\chi) \cdot \hat{\nu}(\chi)$ *(convolution theorem).*

References Page 146 in [27]; 2.3 Proposition on page 9 in [3].

We want to characterize the Fourier transform of $\mu \in \mathcal{M}_b^+(H)$, but first a definition of a positive definite function is required.

Definition 4.4. *Let $(H, \cdot, \tau)$ be a locally compact Abelian group. A function $u : H \to \mathbb{C}$ is said to be* ***positive definite*** *if for all $k \in \mathbb{N}$ and elements $x_1, \dots, x_k \in H$ the matrix $(u(x_j x_l^{-1}))_{j,l=1,\dots,k}$ is positive Hermitian, i.e. for all $\lambda_1, \dots, \lambda_k \in \mathbb{C}$ we have*

$$\sum_{j,l=1}^{k} u(x_j x_l^{-1}) \lambda_j \overline{\lambda_l} \geq 0.$$

We denote by $P(H)$ the set of all positive definite functions on H while $CP(H)$ denotes the set of all **continuous positive definite functions** on H. Obviously $CP(H) \subset P(H)$.

References Definition 3.5.3 on page 106 in [19]; 3.3 Definition on page 12 in [3].

Lemma 4.5. *$\hat{\mu}$ is a positive definite function on $\widehat{H}$ whenever $\mu \in \mathcal{M}_b^+(H)$.*

References Lemma 3.5.4 on page 106 in [19].

Lemma 4.5 together with Theorem 4.2 immediately gives us that $\hat{\mu}$ is a continuous and positive definite function on $\widehat{H}$. One can easily show that any character on H is also a continuous and positive definite function.

References 3.5 on page 13 in [3].

We can now introduce a version of **Bochner's theorem** which characterizes the Fourier transform of a measure $\mu \in \mathcal{M}_b^+(H)$.

Theorem 4.6. *A function $u : \widehat{H} \to \mathbb{C}$ is the Fourier transform of a measure $\mu \in \mathcal{M}_b^+(H)$ if and only if u is continuous and positive definite.*

References Page 191 (last sentence before Section 5) in [22].

Theorem 4.7. *The Fourier transform is a homeomorphism from $\mathcal{M}_b^+(H)$ endowed with the Bernoulli topology onto $CP(\widehat{H})$ endowed with the topology of uniform convergence on compact sets.*

References 3.13 Theorem on page 15 in [3].

Following the definition of a positive definite function on H we are now in a position to introduce a notion of a negative definite function on a locally compact Abelian group. In view of Theorem 4.10 it is convenient to define a negative definite function on the dual group $\widehat{H}$ rather than H, remember that $\widehat{H}$ itself is a locally compact Abelian group. In what follows, e represents the unit character in $\widehat{H}$.

Definition 4.8. *We refer to a function $\psi : \widehat{H} \to \mathbb{C}$ as* ***negative definite*** *if*

$$\psi(e) \geq 0$$

and

$$\chi \mapsto e^{-t\psi(\chi)} \text{ is positive definite for } t \geq 0.$$

In the case when $\chi \mapsto e^{-t\psi(\chi)}$ is also continuous, $\psi : \widehat{H} \to \mathbb{C}$ becomes a ***continuous negative definite function****.*

The sets of negative definite functions and continuous negative definite functions on a locally compact Abelian group H are labeled respectively by $N(H)$ and $CN(H)$.

References Definition 3.6.5 on page 122 in [19]; 8.4 Corollary on page 50 and 7.1 Definition on page 39 in [3].

Next we go on to define a family of measures on a locally compact Abelian group H which will be of significant importance to us in the sense that such a family associates with a continuous negative definite function on the dual group $\widehat{H}$.

References 8.3 Theorem on page 49 in [3].

Definition 4.9. *A family $(\mu_t)_{t \geq 0}$, $\mu_t \in \mathcal{M}_b^+(H)$, of bounded Borel measures on H which satisfies the following properties*

- $\|\mu_t\| \leq 1$ *for all* $t \geq 0$;
- $\mu_s * \mu_t = \mu_{s+t}$ *for all* $s, t \geq 0$;
- $\lim_{t \to 0} \mu_t = \mu_0$ *vaguely where* $\mu_0 = \varepsilon_0$;

constitutes a ***convolution semigroup*** *on* H*.*

References 8.1 Definition on page 48 in [3]; Definition 3.6.1 on page 121 in [19].

Theorem 4.10. *For every convolution semigroup* $(\mu_t)_{t\geq 0}$ *on* H *there exists* $\psi \in CN(\widehat{H})$*, unique to* $(\mu_t)_{t\geq 0}$*, such that*

$$\hat{\mu}_t(\chi) = e^{-t\psi(\chi)} \quad \text{for all } t \geq 0 \text{ and } \chi \in \widehat{H}. \tag{4.2}$$

The converse also holds: For every $\psi \in CN(\widehat{H})$ *there exists a convolution semigroup* $(\mu_t)_{t\geq 0}$ *on* H*, unique to* ψ*, such that* (4.2) *holds.*

Hence there is a one-to-one correspondence between the family of all convolution semigroups on H *and* $CN(\widehat{H})$*.*

References 8.3 Theorem on page 49 in [3]; Theorem 3.6.16 on pages 127-128 in [19].

This next lemma states some important properties of negative definite functions which are required later on.

Lemma 4.11. *Let* $\chi_1, \chi_2 \in \widehat{H}$ *and let* $\tilde{\psi}$ *be a function defined by* $\tilde{\psi}(\chi) := \overline{\psi(\chi^{-1})}$ *for all* $\chi \in \widehat{H}$*. For any* $\psi \in N(\widehat{H})$ *we have*

$$\psi(e) \geq 0 \text{ and } \psi = \tilde{\psi};$$

$$\sqrt{|\psi(\chi_1 \cdot \chi_2)|} \leq \sqrt{|\psi(\chi_1)|} + \sqrt{|\psi(\chi_2)|}$$

and

$$\left|\sqrt{|\psi(\chi_1)|} - \sqrt{|\psi(\chi_2)|}\right| \leq \sqrt{|\psi(\chi_1 \cdot \chi_2^{-1})|};$$

$$\frac{1 + |\psi(\chi_1)|}{1 + |\psi(\chi_2)|} \leq 2(1 + |\psi(\chi_1 \cdot \chi_2^{-1})|).$$

Furthermore, if $\psi \in CN(\mathbb{R}^n)$*, then there exists a constant* $c_\psi > 0$ *such that*

$$|\psi(\xi)| \leq c_\psi(1 + |\xi|^2) \quad \text{for all } \xi \in \mathbb{R}^n.$$

Note that a similar result holds for $\widehat{H} = \mathbb{Z}^m$*.*

References 7.5 Proposition (i) and (ii) on page 40, and 7.15 Proposition on page 45 in [3]; Lemma 3.6.21 on page 133, Lemma 3.6.22 and Lemma 3.6.23 on page 134 in [19].

Finally, the formula below establishes the representation that every continuous negative definite function on $\mathbb{R}^n$ possesses.

Lemma 4.12. *Given $\psi \in CN(\mathbb{R}^n)$, we can find $c \geq 0$, $d \in \mathbb{R}^n$ and a symmetric positive semidefinite continuous quadratic form q on $\mathbb{R}^n$ such that*

$$\psi(\xi) = c + i(d \cdot \xi) + q(\xi) + \int_{\mathbb{R}^n \setminus \{0\}} \left(1 - e^{-iy\cdot\xi} - \frac{iy \cdot \xi}{1 + |y|^2}\right) \nu(\mathrm{d}y), \qquad (4.3)$$

where ν is a Borel measure on $\mathbb{R}^n \setminus \{0\}$ that integrates $y \mapsto (|y|^2 \wedge 1)$.

(4.3) is known as the **Lévy-Khinchin formula**.

References Pages 138 and 153 in [19].

5 Some more Function Spaces

In this section we introduce and study some spaces of functions $u : \mathbb{R}^n \times \mathbb{Z}^m \to \mathbb{C}$. On $\mathbb{R}^n$ we consider the standard Euclidean topology while $\mathbb{Z}^m$ is equipped with the discrete topology.

By $C(\mathbb{R}^n \times \mathbb{Z}^m)$ we denote all continuous functions defined on $\mathbb{R}^n \times \mathbb{Z}^m$, $C_b(\mathbb{R}^n \times \mathbb{Z}^m)$ consists of all bounded continuous functions, and $C_\infty(\mathbb{R}^n \times \mathbb{Z}^m)$ is the space of all continuous functions vanishing at infinity. For $u \in C(\mathbb{R}^n \times \mathbb{Z}^m)$ we write $u(\cdot, k) \in C^m(\mathbb{R}^n)$ if for $k \in \mathbb{Z}^m$ fixed the function $x \mapsto u(x,k)$ is m-times continuously differentiable, similar notations are $u(\cdot,k) \in C_b^m(\mathbb{R}^n)$ or $u(\cdot, k) \in C_\infty(\mathbb{R}^n)$. The norm on $C_b(\mathbb{R}^n \times \mathbb{Z}^m)$ and $C_\infty(\mathbb{R}^n \times \mathbb{Z}^m)$ is the supremum norm, i.e.

$$\|u\|_{C_b(\mathbb{R}^n \times \mathbb{Z}^m)} = \|u\|_{C_\infty(\mathbb{R}^n \times \mathbb{Z}^m)} := \sup_{(x,k) \in \mathbb{R}^n \times \mathbb{Z}^m} |u(x,k)|.$$

Both $C_b(\mathbb{R}^n \times \mathbb{Z}^m)$ and $C_\infty(\mathbb{R}^n \times \mathbb{Z}^m)$ are with respect to the supremum norm Banach spaces.

The space $L^2(\mathbb{R}^n \times \mathbb{Z}^m)$ consists of all (equivalence classes of) measurable functions $u : \mathbb{R}^n \times \mathbb{Z}^m \to \mathbb{C}$ with the finite norm

$$\|u\|^2_{L^2(\mathbb{R}^n \times \mathbb{Z}^m)} = \int_{\mathbb{R}^n} \sum_{k \in \mathbb{Z}^m} |u(x,k)|^2 \,\mathrm{d}x.$$

With this norm the space $L^2(\mathbb{R}^n \times \mathbb{Z}^m)$ is a Hilbert space. Using Plancherel's theorem we find

$$\begin{aligned}
\|u\|^2_{L^2(\mathbb{R}^n \times \mathbb{Z}^m)} &= \int_{\mathbb{R}^n} \sum_{k \in \mathbb{Z}^m} |u(x,k)|^2 \,\mathrm{d}x \\
&= \int_{\mathbb{R}^n} \sum_{k \in \mathbb{Z}^m} |\hat{u}(\xi,k)|^2 \,\mathrm{d}\xi \\
&= (2\pi)^{-m} \int_{\mathbb{R}^n} \int_{\mathbb{T}^m} |\tilde{u}(\xi,\omega)|^2 \,\mathrm{d}\omega \,\mathrm{d}\xi \\
&= (2\pi)^{-m} \int_{\mathbb{T}^m} \int_{\mathbb{R}^n} |\tilde{u}(\xi,\omega)|^2 \,\mathrm{d}\xi \,\mathrm{d}\omega,
\end{aligned}$$

where $\hat{u}(\xi,k)$, for each k fixed, denotes the Fourier transform of $u(\cdot,k) : \mathbb{R}^n \to \mathbb{C}$ and $\tilde{u}(\xi,\omega)$ denotes the Fourier transform of $u : \mathbb{R}^n \times \mathbb{Z}^m \to \mathbb{C}$, i.e.

$$\tilde{u}(\xi,\omega) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \sum_{k \in \mathbb{Z}^m} u(x,k) e^{-ix\cdot\xi} e^{-ik\cdot\omega} \,\mathrm{d}x, \qquad (\xi,\omega) \in \mathbb{R}^n \times [0,2\pi)^m,$$

for $u \in L^1(\mathbb{R}^n \times \mathbb{Z}^m)$.

Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a fixed continuous negative definite function. By c_ψ we denote the smallest constant such that

$$\psi(\xi) \leq c_\psi(1 + |\xi|^2).$$

Later on we will use **Peetre's inequality**, compare Lemma 3.6.23 in [19], stating

$$\frac{1 + \psi(\xi)}{1 + \psi(\eta)} \leq 2(1 + \psi(\xi - \eta)).$$

Moreover, we will require that for some $r_0 > 0$ and $c_{0,\psi} > 0$ it holds

$$(1 + \psi(\xi))^{\frac{1}{2}} \geq c_{0,\psi}(1 + |\xi|^2)^{\frac{r_0}{2}}. \tag{5.1}$$

Given ψ as above we define for $s \geq 0$, $s \in \mathbb{R}$, the norm

$$\begin{aligned}
\|u\|^2_{H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m)} &:= \int_{\mathbb{R}^n} \sum_{k \in \mathbb{Z}^m} (1 + \psi(\xi))^s |\hat{u}(\xi, k)|^2 \,\mathrm{d}\xi \\
&= (2\pi)^{-m} \int_{\mathbb{R}^n} \int_{\mathbb{T}^m} (1 + \psi(\xi))^s |\tilde{u}(\xi, \omega)|^2 \,\mathrm{d}\omega \,\mathrm{d}\xi \\
&= (2\pi)^{-m} \int_{\mathbb{T}^m} \int_{\mathbb{R}^n} (1 + \psi(\xi))^s |\tilde{u}(\xi, \omega)|^2 \,\mathrm{d}\xi \,\mathrm{d}\omega.
\end{aligned}$$

The space of all $u \in L^2(\mathbb{R}^n \times \mathbb{Z}^m)$ for which $\|u\|_{H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m)}$ is finite is denoted by $H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m)$, and equipped with $\|\cdot\|_{H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m)}$ the space $H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m)$ is a Hilbert space where the inner product is given by

$$\begin{aligned}
&\int_{\mathbb{R}^n} \sum_{k \in \mathbb{Z}^m} (1 + \psi(\xi))^s \hat{u}(\xi, k) \overline{\hat{v}(\xi, k)} \,\mathrm{d}\xi \\
&= (2\pi)^{-m} \int_{\mathbb{R}^n} \int_{\mathbb{T}^m} (1 + \psi(\xi))^s \tilde{u}(\xi, \omega) \overline{\tilde{v}(\xi, \omega)} \,\mathrm{d}\omega \,\mathrm{d}\xi.
\end{aligned}$$

Obviously we have the estimate

$$\|u\|_{H^{\psi,t}(\mathbb{R}^n \times \mathbb{Z}^m)} \leq \|u\|_{H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m)}$$

for $s > t$, implying that $H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m)$ is continuously embedded into $H^{\psi,t}(\mathbb{R}^n \times \mathbb{Z}^m)$. For $s = 0$ we have $H^{\psi,0}(\mathbb{R}^n \times \mathbb{Z}^m) = L^2(\mathbb{R}^n \times \mathbb{Z}^m)$.

An easy consequence of the definition is the following **Ehrling-type** or interpolation estimate.

Lemma 5.1. *Suppose that* $\lim_{|\xi|\to\infty}\psi(\xi)=\infty$*, then for every* $\varepsilon>0$ *and for every* $0\le t<s$ *there exists a constant* $\kappa_{s,t,\epsilon}\ge 0$ *such that for all* $u\in H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)$ *it holds*

$$\|u\|^2_{H^{\psi,t}(\mathbb{R}^n\times\mathbb{Z}^m)}\le\varepsilon\|u\|^2_{H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)}+\kappa_{s,t,\varepsilon}\|u\|^2_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)}.$$

Proof. Since $\lim_{|\xi|\to\infty}\psi(\xi)=\infty$, it follows that for $\varepsilon>0$ there exists $\kappa_{s,t,\varepsilon}\ge 0$ such that

$$(1+\psi(\xi))^t\le\varepsilon(1+\psi(\xi))^s+\kappa_{s,t,\varepsilon},$$

implying that

$$\begin{aligned}\|u\|^2_{H^{\psi,t}(\mathbb{R}^n\times\mathbb{Z}^m)}&=(2\pi)^{-m}\int_{\mathbb{R}^n}\int_{\mathbb{T}^m}(1+\psi(\xi))^t|\tilde{u}(\xi,\omega)|^2\,\mathrm{d}\omega\,\mathrm{d}\xi\\&\le\varepsilon(2\pi)^{-m}\int_{\mathbb{R}^n}\int_{\mathbb{T}^m}(1+\psi(\xi))^s|\tilde{u}(\xi,\omega)|^2\,\mathrm{d}\omega\,\mathrm{d}\xi\\&\quad+\kappa_{s,t,\varepsilon}(2\pi)^{-m}\int_{\mathbb{R}^n}\int_{\mathbb{T}^m}|\tilde{u}(\xi,\omega)|^2\,\mathrm{d}\omega\,\mathrm{d}\xi\\&=\varepsilon\|u\|^2_{H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)}+\kappa_{s,t,\varepsilon}\|u\|^2_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)},\end{aligned}$$

proving the lemma.

□

We may define the **pseudo-differential operator**

$$(1+\psi(D_x))^{\frac{s}{2}}u(x,k):=(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}e^{ix\cdot\xi}(1+\psi(\xi))^{\frac{s}{2}}\hat{u}(\xi,k)\,\mathrm{d}\xi$$

to find again by Plancherel's theorem that

$$\begin{aligned}\|u\|^2_{H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)}&=\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}^m}|(1+\psi(D_x))^{\frac{s}{2}}u(x,k)|^2\,\mathrm{d}x\\&=(2\pi)^{-m}\int_{\mathbb{R}^n}\int_{\mathbb{T}^m}|(1+\psi(D_x))^{\frac{s}{2}}F_{k\mapsto\omega}(u)(x,\omega)|^2\,\mathrm{d}\omega\,\mathrm{d}x\\&=\|(1+\psi(D_x))^{\frac{s}{2}}u\|^2_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)}.\end{aligned}$$

Using the standard theory of topological tensor products of Hilbert spaces, compare [31], we may deduce that each of the following sets are dense in $H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)$, $s\ge 0$,

$$H^{\psi,s}(\mathbb{R}^n)\otimes l^2(\mathbb{Z}^m), \qquad S(\mathbb{R}^n)\otimes l^2(\mathbb{Z}^m),$$
$$H^{\psi,s}(\mathbb{R}^n)\otimes l_{\text{finite}}(\mathbb{Z}^m), \qquad S(\mathbb{R}^n)\otimes l_{\text{finite}}(\mathbb{Z}^m),$$

where $S(\mathbb{R}^n)$ denotes the Schwartz space, $l_{\text{finite}}(\mathbb{Z}^m)$ the space of all finite sequences on $\mathbb{Z}^m$, and $H^{\psi,s}(\mathbb{R}^n)$ is the space of all $u\in L^2(\mathbb{R}^n)$ such that

$$\|u\|^2_{H^{\psi,s}(\mathbb{R}^n)} := \int_{\mathbb{R}^n}(1+\psi(\xi))^s|\hat{u}(\xi)|^2\,\mathrm{d}\xi < \infty.$$

Note that in order to prove that a linear operator A maps $H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)$ continuously into $H^{\psi,t}(\mathbb{R}^n\times\mathbb{Z}^m)$ it is sufficient to prove

$$\|Au\|_{H^{\psi,t}(\mathbb{R}^n\times\mathbb{Z}^m)} \le c\|u\|_{H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)}$$

for a dense subset of $H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)$. In most cases to follow we will work with the set of all $u:\mathbb{R}^n\times\mathbb{Z}^m\to\mathbb{C}$ with the property that $u(x,\cdot)\in l^2(\mathbb{Z}^m)$ and $u(\cdot,k)\in S(\mathbb{R}^n)$ and $\|u\|_{H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)}$ is finite. This set contains $S(\mathbb{R}^n)\otimes l^2(\mathbb{Z}^m)$ and hence is dense in $H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)$ for every $s\ge 0$. We denote this set by $\mathcal{D}(\mathbb{R}^n\times\mathbb{Z}^m)$. Alternatively we may use $S(\mathbb{R}^n\times\mathbb{Z}^m)$, defined as $S(\mathbb{R}^n\times\mathbb{Z}^m) := \{u:\mathbb{R}^n\times\mathbb{Z}^m\to\mathbb{C},\ u(\cdot,k)\in C^\infty(\mathbb{R}^n)\mid |\partial_x^\alpha u(x,k)|\le c_{r,s,\alpha,u}(1+|x|^2)^{-\frac{r}{2}}(1+|k|^2)^{-\frac{s}{2}} \text{ for all } \alpha\in\mathbb{N}_0^n, r,s\in\mathbb{N}_0\}$.

Let ψ satisfy (5.1) and take $u\in S(\mathbb{R}^n\times\mathbb{Z}^m)$ or $\mathcal{D}(\mathbb{R}^n\times\mathbb{Z}^m)$. By the Fourier inversion theorem we find for $s>\frac{n}{2r_0}$

$$\begin{aligned}|u(x,k)| &= |F^{-1}_{\xi\mapsto x}F_{y\mapsto\xi}(u)(x,k)|\\ &= \left|(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat{u}(\xi,k)\,\mathrm{d}\xi\right|\\ &\le (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\frac{1}{(1+\psi(\xi))^{\frac{s}{2}}}(1+\psi(\xi))^{\frac{s}{2}}|\hat{u}(\xi,k)|\,\mathrm{d}\xi\\ &\le (2\pi)^{-\frac{n}{2}}\left(\int_{\mathbb{R}^n}\frac{1}{(1+\psi(\xi))^s}\,\mathrm{d}\xi\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^n}(1+\psi(\xi))^s|\hat{u}(\xi,k)|^2\,\mathrm{d}\xi\right)^{\frac{1}{2}},\end{aligned}$$

and since by assumption $(1+\psi(\xi))^{\frac{1}{2}}\ge c_{0,\psi}(1+|\xi|^2)^{\frac{r_0}{2}}$, $r_0>0$, we have

$$\frac{1}{(1+\psi(\xi))^s}\le\frac{1}{c_{0,\psi}^{2s}}\frac{1}{(1+|\xi|^2)^{r_0 s}}$$

and for $r_0 s>\frac{n}{2}$ it follows that $\frac{1}{(1+\psi(\cdot))^s}\in L^1(\mathbb{R}^n)$. Thus we arrive at

$$|u(x,k)|^2 \leq \sum_{k\in\mathbb{Z}^m} |u(x,k)|^2 \leq c \sum_{k\in\mathbb{Z}^m} \int_{\mathbb{R}^n} (1+\psi(\xi))^s |\hat{u}(\xi,k)|^2 \,\mathrm{d}\xi.$$

This implies

Proposition 5.2. *Under the assumption of* (5.1), *if* $s > \frac{n}{2r_0}$, *then*

$$|u(x,k)| \leq c\|u\|_{H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)}$$

and $\lim_{|x|+|k|\to\infty} u(x,k) = 0$. *In addition* u *has a modification which is continuous on* $\mathbb{R}^n$ *for* $k \in \mathbb{Z}^m$ *fixed and on* $\mathbb{R}^n \times \mathbb{Z}^m$.

Later on we will need to work with the Friedrichs mollifier for elements in $H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m)$ where the mollifier acts only on the x-variable. Following page 31 in [19] we set

$$j(x) := \begin{cases} c_0 \exp((|x|^2-1)^{-1}), & |x| < 1 \\ 0, & |x| \geq 1 \end{cases},$$

where $c_0^{-1} = \int_{|x|<1} \exp((|x|^2-1)^{-1})\,\mathrm{d}x$. For $\varepsilon > 0$ we set $j_\varepsilon(x) := \varepsilon^{-n} j\left(\frac{x}{\varepsilon}\right)$. The operator J_ε is defined, say for $u \in L^2(\mathbb{R}^n)$, by

$$J_\varepsilon(u)(x) := (j_\varepsilon * u)(x) = \int_{\mathbb{R}^n} j_\varepsilon(x-y)u(y)\,\mathrm{d}y, \tag{5.2}$$

and the operator is called the **Friedrichs mollifier**. Now we extend the definition to $L^2(\mathbb{R}^n \times \mathbb{Z}^m)$ by

$$J_\varepsilon(u)(x,k) := \int_{\mathbb{R}^n} j_\varepsilon(x-y)u(y,k)\,\mathrm{d}y. \tag{5.3}$$

By inspection of the classical proofs, compare Proposition 2.3.17 in [19], we get immediately for $u, v \in L^2(\mathbb{R}^n \times \mathbb{Z}^m)$

$$\int_{\mathbb{R}^n} |J_\varepsilon(u)(x,k)|^2\,\mathrm{d}x \leq \int_{\mathbb{R}^n} |u(x,k)|^2\,\mathrm{d}x,$$

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} |J_\varepsilon(u)(x,k) - u(x,k)|^2\,\mathrm{d}x = 0,$$

and

$$\int_{\mathbb{R}^n} J_\varepsilon(u)(x,k)\overline{v(x,k)}\,\mathrm{d}x = \int_{\mathbb{R}^n} u(x,k)\overline{J_\varepsilon(v)(x,k)}\,\mathrm{d}x.$$

These results imply now

$$\begin{aligned}\|J_\varepsilon(u)\|^2_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)} &= \sum_{k\in\mathbb{Z}^m}\int_{\mathbb{R}^n}|J_\varepsilon(u)(x,k)|^2\,\mathrm{d}x\\ &= \int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}^m}|J_\varepsilon(u)(x,k)|^2\,\mathrm{d}x\\ &\le \int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}^m}|u(x,k)|^2\,\mathrm{d}x\\ &= \|u\|^2_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)},\end{aligned}$$

and

$$\lim_{\varepsilon\to 0}\|J_\varepsilon(u)-u\|_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)} = 0, \tag{5.4}$$

as well as

$$(J_\varepsilon(u),v)_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)} = (u, J_\varepsilon(v))_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)}.$$

In addition we can easily see that $u \in H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)$ implies $J_\varepsilon(u) \in H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)$. Following Proposition 2.3.15 in [20] we can prove

Lemma 5.3. *For any $u \in H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)$, $s\ge 0$, it follows that*

$$J_\varepsilon(u) \in \bigcap_{t\ge 0} H^{\psi,t}(\mathbb{R}^n\times\mathbb{Z}^m). \tag{5.5}$$

Proof. Since $j \in C_0^\infty(\mathbb{R}^n)$, hence $\hat{j}_\varepsilon \in S(\mathbb{R}^n)$, we have the estimate

$$(1+\psi(\xi))^{\frac{t}{2}}|\hat{j}_\varepsilon(\xi)| \le c_{t,s,\varepsilon}(1+\psi(\xi))^{\frac{s}{2}},$$

which gives

$$\begin{aligned}\|J_\varepsilon(u)\|^2_{H^{\psi,t}(\mathbb{R}^n\times\mathbb{Z}^m)} &= \sum_{k\in\mathbb{Z}^m}\int_{\mathbb{R}^n}(1+\psi(\xi))^t|(j_\varepsilon * u)^\wedge(\xi,k)|^2\,\mathrm{d}\xi\\ &= (2\pi)^n\sum_{k\in\mathbb{Z}^m}\int_{\mathbb{R}^n}(1+\psi(\xi))^t|\hat{j}_\varepsilon(\xi)|^2|\hat{u}(\xi,k)|^2\,\mathrm{d}\xi\\ &\le (2\pi)^n c_{t,s,\varepsilon}\sum_{k\in\mathbb{Z}^m}\int_{\mathbb{R}^n}(1+\psi(\xi))^s|\hat{u}(\xi,k)|^2\,\mathrm{d}\xi\\ &= (2\pi)^n c_{t,s,\varepsilon}\|u\|^2_{H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)},\end{aligned} \tag{5.6}$$

which implies $J_\varepsilon(u) \in H^{\psi,t}(\mathbb{R}^n \times \mathbb{Z}^m)$ for any $t \geq 0$.

□

Note that for $t = 0$ (5.6) implies in particular

$$\int_{\mathbb{R}^n} |J_\varepsilon(u)(x,k)|^2 \, \mathrm{d}x \leq (2\pi)^n c_{0,s,\varepsilon} \|u\|^2_{H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m)}.$$

It is easy to see that for $k \in \mathbb{Z}^m$ fixed the function $x \mapsto J_\varepsilon(u)(x,k)$ belongs to $C^\infty(\mathbb{R}^n)$. The next result extends (2.167) and (2.168) in Proposition 2.3.15 in [20].

Lemma 5.4. *For $u \in H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m)$, $s \geq 0$, it holds*

$$\|J_\varepsilon(u)\|_{H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m)} \leq \|u\|_{H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m)} \tag{5.7}$$

and

$$\lim_{\varepsilon \to 0} \|J_\varepsilon(u) - u\|_{H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m)} = 0.$$

Proof. For $u \in H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m)$ we find

$$\begin{aligned}
\|J_\varepsilon(u)\|^2_{H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m)} &= (2\pi)^n \sum_{k \in \mathbb{Z}^m} \int_{\mathbb{R}^n} (1 + \psi(\xi))^s |\hat{j}_\varepsilon(\xi)|^2 |\hat{u}(\xi,k)|^2 \, \mathrm{d}\xi \\
&\leq \sum_{k \in \mathbb{Z}^m} \int_{\mathbb{R}^n} (1 + \psi(\xi))^s |\hat{u}(\xi,k)|^2 \, \mathrm{d}\xi \\
&= \|u\|^2_{H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m)},
\end{aligned}$$

which proves (5.7). Furthermore we have

$$\begin{aligned}
&\|J_\varepsilon(u) - u\|^2_{H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m)} \\
&\quad = \sum_{k \in \mathbb{Z}^m} \int_{\mathbb{R}^n} (1 + \psi(\xi))^s |(j_\varepsilon * u)^\wedge(\xi,k) - \hat{u}(\xi,k)|^2 \, \mathrm{d}\xi \\
&\quad = \sum_{k \in \mathbb{Z}^m} (2\pi)^n \int_{\mathbb{R}^n} (1 + \psi(\xi))^s |\hat{j}_\varepsilon(\xi) - (2\pi)^{-\frac{n}{2}}|^2 |\hat{u}(\xi,k)|^2 \, \mathrm{d}\xi.
\end{aligned}$$

Since $\hat{j}_\varepsilon(\xi) = \hat{j}(\varepsilon\xi) \to \hat{j}(0) = (2\pi)^{-\frac{n}{2}}$ as $\varepsilon \to 0$ and in addition $|\hat{j}_\varepsilon(\xi)| \leq (2\pi)^{-\frac{n}{2}}$ as well as

$$\sum_{k \in \mathbb{Z}^m} \int_{\mathbb{R}^n} (1 + \psi(\xi))^s |\hat{u}(\xi,k)|^2 \, \mathrm{d}\xi < \infty,$$

the dominated convergence theorem implies

$$\lim_{\varepsilon\to 0}\|J_\varepsilon(u)-u\|_{H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)}=0.$$

□

Most important is the following result extending the final assertion of Proposition 2.3.15 in [20].

Proposition 5.5. *Let $s\geq 0$, $\varepsilon\in(0,\rho)$ and assume for $u\in L^2(\mathbb{R}^n\times\mathbb{Z}^m)$ that*

$$\|J_\varepsilon(u)\|_{H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)}\leq c_{u,s} \tag{5.8}$$

for all $\varepsilon\in(0,\rho)$ with $c_{u,s}$ independent of ε. Then $u\in H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)$.

Proof. Since $H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)$ is weakly compact, from (5.8) it follows that there exists a subsequence $(J_{\frac{1}{n_l}}(u))_{n_l>\frac{1}{\rho}}$ converging weakly in $H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)$ to some $v\in H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)$. The embedding of $H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)$ into $L^2(\mathbb{R}^n\times\mathbb{Z}^m)$ is linear and continuous, hence $(J_{\frac{1}{n_l}}(u))_{n_l>\frac{1}{\rho}}$ converges also in $L^2(\mathbb{R}^n\times\mathbb{Z}^m)$ weakly to v. However, in addition, by (5.4) we know that $(J_{\frac{1}{n_l}}(u))_{n_l>\frac{1}{\rho}}$ converges in $L^2(\mathbb{R}^n\times\mathbb{Z}^m)$ to u, hence $v=u\in H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)$.

□

6 Symbol Classes

We want to study pseudo-differential operators acting on functions defined on $\mathbb{R}^n \times \mathbb{Z}^m$. We want to extend these operators from some nice domain to a Banach space such that the extension generates a Feller semigroup, hence a Feller process. In case of functions defined on $\mathbb{R}^n$ only, the appropriate condition would be that the symbol of the generator is with respect to the co-variable on $\mathbb{R}^n$ a continuous negative definitive function, compare Courrège [7] or [19]. For functions defined on $\mathbb{Z}^m$ alone we would expect the generator to be a Q-matrix, see [25] and [30]. In [12] it was pointed out that under certain general conditions Q-matrices are indeed also pseudo-differential operators whose symbols with respect to the co-variable are continuous negative definite functions, of course now on the torus $\mathbb{T}^m$. Here we want to look at symbols $q : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{T}^m \to \mathbb{R}$. While we assume that $\xi \mapsto q(x, \xi, \omega)$ is a continuous negative definite function, we are more flexible with respect to the function $\omega \mapsto q(x, \xi, \omega)$, mainly due to a lack of a proper understanding of the general relations of Q-matrices and harmonic analysis. The x-dependence of q is seen as a perturbation, i.e. we intend to look at the translation-invariant operator $q(x_0, D_x, D_k)$ where the "coefficient" is frozen and the operator $q(x, D_x, D_k) - q(x_0, D_x, D_k)$ is supposed to be a (small) perturbation of $q(x_0, D_x, D_k)$. Having in mind the constructions of W. Hoh [14], [15] and [16] using the martingale problem and stopping time techniques, this approach is more convenient than to look at operators of type $\psi(D_x, D_k) + \tilde{q}(x, D_x, D_k)$, also in the basic estimates there is no difference.

In the following we fix a continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$\psi(\xi) \leq c_\psi(1 + |\xi|^2), \tag{6.1}$$

and with some $\rho_0 > 0$ and $c_{0,\psi} > 0$

$$(1 + \psi(\xi))^{\frac{1}{2}} \geq c_{0,\psi}(1 + |\xi|^2)^{\frac{\rho_0}{2}}.$$

Given

$$q : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{T}^m \to \mathbb{R},$$

we assume that we can decompose q according to

$$q(x, \xi, \omega) = q_1(\xi, \omega) + q_2(x, \xi, \omega).$$

Hence we have a decomposition into an operator $q_1(D_x, D_k)$ which is translation invariant, i.e. has constant coefficients, and an operator $q_2(x, D_x, D_k)$

with variable coefficients. The latter will be considered as a perturbation of the first one. Note that such a decomposition is always possible by freezing a coefficient, i.e.

$$q(x,\xi,\omega) = q(x_0,\xi,\omega) + (q(x,\xi,\omega) - q(x_0,\xi,\omega)) = q_1(\xi,\omega) + q_2(x,\xi,\omega).$$

We require q, q_1 and q_2 to be continuous and that both $\xi \mapsto q(x,\xi,\omega)$ and $\xi \mapsto q_1(\xi,\omega)$ are continuous negative definite functions for all $x \in \mathbb{R}^n$ and $\omega \in \mathbb{T}^m$. Moreover, it is assumed

A.1.

$$|q_1(\xi,\omega)| \leq \gamma_1(1+\psi(\xi))$$

and

$$1 + |q_1(\xi,\omega)| \geq \gamma_0(1+\psi(\xi)),$$

where $\gamma_0 > 0$ and $\gamma_1 \geq 0$ are independent of ω. For the symbol q_2 we assume

A.2.M. $x \mapsto q_2(x,\xi,\omega)$ belongs to $C^M(\mathbb{R}^n)$, $M \in \mathbb{N}_0$, and for all $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leq M$, it holds

$$|\partial_x^\alpha q_2(x,\xi,\omega)| \leq \varphi_\alpha(x)(1+\psi(\xi)), \tag{6.2}$$

for some functions $\varphi_\alpha \in L^1(\mathbb{R}^n)$.

Condition (6.2) has an important consequence.

Proposition 6.1. *Suppose that q_2 satisfies A.2.M and denote by $\hat{q}_2(\eta,\xi,\omega)$ the Fourier transform of q_2 with respect to x, i.e.*

$$\hat{q}_2(\eta,\xi,\omega) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\eta} q_2(x,\xi,\omega)\,\mathrm{d}x.$$

There exists a constant $\tilde{\gamma}_{M,n,m}$ such that

$$|\hat{q}_2(\eta,\xi,\omega)| \leq \tilde{\gamma}_{M,n,m} \sum_{|\alpha|\leq M} \|\varphi_\alpha\|_{L^1(\mathbb{R}^n)}(1+|\eta|^2)^{-\frac{M}{2}}(1+\psi(\xi)). \tag{6.3}$$

Proof. We follow closely the considerations in [20], p.68-69, to find for $|\beta| \leq M$

$$\left|\eta^\beta \int_{\mathbb{R}^n} e^{-ix\cdot\eta} q_2(x,\xi,\omega)\,\mathrm{d}x\right| = \left|\int_{\mathbb{R}^n} e^{-ix\cdot\eta}\partial_x^\beta q_2(x,\xi,\omega)\,\mathrm{d}x\right|$$
$$\leq \|\varphi_\beta\|_{L^1(\mathbb{R}^n)}(1+\psi(\xi)),$$

which implies (6.3).

□

Given $q(x,\xi,\omega) = q_1(\xi,\omega) + q_2(x,\xi,\omega)$, we can define on $\mathcal{D}(\mathbb{R}^n \times \mathbb{Z}^m)$ or $S(\mathbb{R}^n \times \mathbb{Z}^m)$ the following pseudo-differential operators

$$\begin{aligned} q(x,D_x,D_k)u(x,k) &:= F^{-1}_{\xi\mapsto x}F^{-1}_{\omega\mapsto k}(q(x,\xi,\omega)\tilde{u}(\xi,\omega))(x,k) \\ &= F^{-1}_{\omega\mapsto k}F^{-1}_{\xi\mapsto x}(q(x,\xi,\omega)\tilde{u}(\xi,\omega))(x,k), \end{aligned} \tag{6.4}$$

$$\begin{aligned} q_1(D_x,D_k)u(x,k) &:= F^{-1}_{\xi\mapsto x}F^{-1}_{\omega\mapsto k}(q_1(\xi,\omega)\tilde{u}(\xi,\omega))(x,k) \\ &= F^{-1}_{\omega\mapsto k}F^{-1}_{\xi\mapsto x}(q_1(\xi,\omega)\tilde{u}(\xi,\omega))(x,k) \end{aligned} \tag{6.5}$$

and $q_2(x,D_x,D_k)$ by replacing in (6.4) the symbol q by the symbol q_2.

(For clarity we prefer here and there as in (6.4) and (6.5) some abuse of notation, i.e writing $F^{-1}_{\xi\mapsto x}F^{-1}_{\omega\mapsto k}(q_1(\xi,\omega)\tilde{u}(\xi,\omega))(x,k)$ for $F^{-1}_{\xi\mapsto x}F^{-1}_{\omega\mapsto k}(q_1\tilde{u})(x,k)$).

Our aim is to establish for $q(x,D_x,D_k)$ estimates which will allow an application of the Hille-Yosida theorem to prove that $-q(x,D_x,D_k)$ generates a strongly continuous contraction semigroup on some suitable Banach spaces, for example $L^2(\mathbb{R}^n \times \mathbb{Z}^m)$. We adopt basic ideas from [18] where the case of operators acting on functions defined on $\mathbb{R}^n$ was discussed.

7 Estimates for $q_1(D_x, D_k)$

Estimates for $q_1(D_x, D_k)$ with respect to the norm in $H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m)$, $s \geq 0$, are straight forward once we observe that for $u \in \mathcal{D}(\mathbb{R}^n \times \mathbb{Z}^m)$ or $S(\mathbb{R}^n \times \mathbb{Z}^m)$ we have

$$(q_1(D_x, D_k)u)^\sim(\xi, \omega) = q_1(\xi, \omega)\tilde{u}(\xi, \omega). \tag{7.1}$$

Proposition 7.1. *Assume A.1. Then for $s \geq 0$ it holds*

$$\|q_1(D_x, D_k)u\|_{H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m)} \leq \gamma_1 \|u\|_{H^{\psi,s+2}(\mathbb{R}^n \times \mathbb{Z}^m)}$$

for all $u \in H^{\psi,s+2}(\mathbb{R}^n \times \mathbb{Z}^m)$.

Proof. We need only to observe for $u \in \mathcal{D}(\mathbb{R}^n \times \mathbb{Z}^m)$ or $S(\mathbb{R}^n \times \mathbb{Z}^m)$ that by $A.1$

$$\begin{aligned}
&\|q_1(D_x, D_k)u\|^2_{H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m)} \\
&\quad = \int_{\mathbb{R}^n} \sum_{k \in \mathbb{Z}^m} |(1 + \psi(D_x))^{\frac{s}{2}} q_1(D_x, D_k)u(x, k)|^2 \,\mathrm{d}x \\
&\quad = (2\pi)^{-m} \int_{\mathbb{R}^n} \int_{\mathbb{T}^m} |(1 + \psi(\xi))^{\frac{s}{2}} q_1(\xi, \omega)\tilde{u}(\xi, \omega)|^2 \,\mathrm{d}\omega \mathrm{d}\xi \\
&\quad \leq (2\pi)^{-m}\gamma_1^2 \int_{\mathbb{R}^n} \int_{\mathbb{T}^m} (1 + \psi(\xi))^s (1 + \psi(\xi))^2 |\tilde{u}(\xi, \omega)|^2 \,\mathrm{d}\omega \mathrm{d}\xi \\
&\quad = (2\pi)^{-m}\gamma_1^2 \int_{\mathbb{R}^n} \int_{\mathbb{T}^m} |(1 + \psi(\xi))^{\frac{s+2}{2}} \tilde{u}(\xi, \omega)|^2 \,\mathrm{d}\omega \mathrm{d}\xi \\
&\quad = \gamma_1^2 \|u\|^2_{H^{\psi,s+2}(\mathbb{R}^n \times \mathbb{Z}^m)}.
\end{aligned}$$

□

Further we have some lower bounds for $q_1(D_x, D_k)$, namely a type of **Gårding inequality**.

Proposition 7.2. *Suppose A.1 holds and $\lim_{|\xi| \to \infty} \psi(\xi) = \infty$. Then there exists $\tilde{\kappa}_0 > 0$ and $\tilde{\kappa}_1 \geq 0$ such that*

$$\|q_1(D_x, D_k)u\|_{H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m)} \geq \tilde{\kappa}_0 \|u\|_{H^{\psi,s+2}(\mathbb{R}^n \times \mathbb{Z}^m)} - \tilde{\kappa}_1 \|u\|_{L^2(\mathbb{R}^n \times \mathbb{Z}^m)} \tag{7.2}$$

holds for all $u \in H^{\psi,s+2}(\mathbb{R}^n \times \mathbb{Z}^m)$. (Note that $\tilde{\kappa}_1$ depends on s).

Proof. We note for $u \in \mathcal{D}(\mathbb{R}^n \times \mathbb{Z}^m)$ or $S(\mathbb{R}^n \times \mathbb{Z}^m)$ that

$$
\begin{aligned}
\|u\|^2_{H^{\psi,s+2}(\mathbb{R}^n\times\mathbb{Z}^m)} &= (2\pi)^{-m}\int_{\mathbb{R}^n}\int_{\mathbb{T}^m}(1+\psi(\xi))^{s+2}|\tilde{u}(\xi,\omega)|^2\,\mathrm{d}\omega\mathrm{d}\xi \\
&\le \frac{1}{\gamma_0^2}(2\pi)^{-m}\int_{\mathbb{R}^n}\int_{\mathbb{T}^m}(1+\psi(\xi))^{s}(1+q_1(\xi,\omega))^2|\tilde{u}(\xi,\omega)|^2\,\mathrm{d}\omega\mathrm{d}\xi \\
&\le \frac{2}{\gamma_0^2}(2\pi)^{-m}\int_{\mathbb{R}^n}\int_{\mathbb{T}^m}((1+\psi(\xi))^{\frac{s}{2}}q_1(\xi,\omega)|\tilde{u}(\xi,\omega)|)^2\,\mathrm{d}\omega\mathrm{d}\xi \\
&\quad + \frac{2}{\gamma_0^2}(2\pi)^{-m}\int_{\mathbb{R}^n}\int_{\mathbb{T}^m}((1+\psi(\xi))^{\frac{s}{2}}|\tilde{u}(\xi,\omega)|)^2\,\mathrm{d}\omega\mathrm{d}\xi \\
&= \frac{2}{\gamma_0^2}\|q_1(D_x,D_k)u\|^2_{H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)} \\
&\quad + \frac{2}{\gamma_0^2}(2\pi)^{-m}\int_{\mathbb{R}^n}\int_{\mathbb{T}^m}((1+\psi(\xi))^{\frac{s}{2}}|\tilde{u}(\xi,\omega)|)^2\,\mathrm{d}\omega\mathrm{d}\xi \\
&\le \frac{2}{\gamma_0^2}\|q_1(D_x,D_k)u\|^2_{H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)} \\
&\quad + \frac{1}{2}(2\pi)^{-m}\int_{\mathbb{R}^n}\int_{\mathbb{T}^m}(1+\psi(\xi))^{s+2}|\tilde{u}(\xi,\omega)|^2\,\mathrm{d}\omega\mathrm{d}\xi \\
&\quad + \kappa_1(2\pi)^{-m}\int_{\mathbb{R}^n}\int_{\mathbb{T}^m}|\tilde{u}(\xi,\omega)|^2\,\mathrm{d}\omega\mathrm{d}\xi, \qquad (7.3)
\end{aligned}
$$

where we applied for the last estimate Lemma 5.1. Thus we arrive at

$$
\begin{aligned}
\|u\|^2_{H^{\psi,s+2}(\mathbb{R}^n\times\mathbb{Z}^m)} \le \frac{2}{\gamma_0^2}&\|q_1(D_x,D_k)u\|^2_{H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)} \\
&+ \frac{1}{2}\|u\|^2_{H^{\psi,s+2}(\mathbb{R}^n\times\mathbb{Z}^m)} + \kappa_1\|u\|^2_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)},
\end{aligned}
$$

implying (7.2).

□

Remark 7.3. *Note that in* (7.3) *we may obtain for every* $\varepsilon > 0$

$$
\begin{aligned}
\|u\|^2_{H^{\psi,s+2}(\mathbb{R}^n\times\mathbb{Z}^m)} \le{} & \frac{2}{\gamma_0^2}\|q_1(D_x,D_k)u\|^2_{H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)} \\
& + \frac{\varepsilon}{2}(2\pi)^{-m}\int_{\mathbb{R}^n}\int_{\mathbb{T}^m}(1+\psi(\xi))^{s+2}|\tilde{u}(\xi,\omega)|^2\,\mathrm{d}\omega\mathrm{d}\xi \\
& + \kappa_1(\varepsilon)(2\pi)^{-m}\int_{\mathbb{R}^n}\int_{\mathbb{T}^m}|\tilde{u}(\xi,\omega)|^2\,\mathrm{d}\omega\mathrm{d}\xi,
\end{aligned}
$$

which allows us to replace (7.2) *by*

$$
\begin{aligned}
&\|q_1(D_x,D_k)u\|^2_{H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)} \\
&\qquad \ge \gamma_0(1-\varepsilon)\|u\|^2_{H^{\psi,s+2}(\mathbb{R}^n\times\mathbb{Z}^m)} - \kappa_\varepsilon\|u\|^2_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)}.
\end{aligned}
$$

Finally we would like to remark that since $q_1(D_x,D_k)$ is translation-invariant, it commutes with convolution. Especially we have for all $\varepsilon>0$

$$
[J_\varepsilon, q_1(D_x,D_k)] = 0, \tag{7.4}
$$

where J_ε is defined as in (5.2).

8 Some Auxiliary Results

In order to handle $q_2(x, D_x, D_k)$ it is convenient to handle in advance some commutator estimates. First we want to estimate $[J_\varepsilon, q_2(x, D_x, D_k)]$ where J_ε is the Friedrichs mollifier acting only on $x \in \mathbb{R}^n$. We follow closely the consideration made in [20], p 79-81.

For the Fourier transform of $[J_\varepsilon, q_2(x, D_x, D_k)]u$ we find

$$\begin{aligned}
&([J_\varepsilon, q_2(x, D_x, D_k)]u)^\sim(\xi, \omega) \\
&\quad = (J_\varepsilon(q_2(x, D_x, D_k)u))^\sim(\xi, \omega) - (q_2(x, D_x, D_k)J_\varepsilon(u))^\sim(\xi, \omega) \\
&\quad = \int_{\mathbb{R}^n} \hat{q}_2(\xi - \eta, \eta, \omega)\hat{j}(\varepsilon\xi)\tilde{u}(\eta, \omega)\mathrm{d}\eta \\
&\qquad - \int_{\mathbb{R}^n} \hat{q}_2(\xi - \eta, \eta, \omega)\hat{j}(\varepsilon\eta)\tilde{u}(\eta, \omega)\,\mathrm{d}\eta \\
&\quad = \int_{\mathbb{R}^n} \hat{q}_2(\xi - \eta, \eta, \omega)(\hat{j}(\varepsilon\xi) - \hat{j}(\varepsilon\eta))\tilde{u}(\eta, \omega)\,\mathrm{d}\eta.
\end{aligned}$$

We use in addition, compare [20], (2.172), where the lengthy proof is given in detail, the estimate

$$|\hat{j}(\varepsilon\xi) - \hat{j}(\varepsilon\eta)|(1 + |\xi|^2)^{\frac{1}{2}} \leq c^*(1 + |\xi - \eta|^2)^{\frac{1}{2}},$$

where c^* is independent of $\varepsilon \in (0, 1]$. To proceed further we recall (6.1) and (6.3). Now we find by arguments analogous to those in [20] that

$$\begin{aligned}
&|(1 + \psi(\xi))^{\frac{s}{2}}([J_\varepsilon, q_2(x, D_x, D_k)]u)^\sim(\xi, \omega)| \\
&\quad = \left| \int_{\mathbb{R}^n} \hat{q}_2(\xi - \eta, \eta, \omega)(\hat{j}(\varepsilon\xi) - \hat{j}(\varepsilon\eta))(1 + \psi(\xi))^{\frac{s}{2}}\tilde{u}(\eta, \omega)\,\mathrm{d}\eta \right| \\
&\quad \leq c \int_{\mathbb{R}^n} (1 + |\xi - \eta|^2)^{-\frac{M}{2}}(1 + \psi(\eta))(1 + |\xi - \eta|^2)^{\frac{1}{2}} \times \\
&\qquad \times \frac{(1 + \psi(\xi))^{\frac{1}{2}}}{(1 + |\xi|^2)^{\frac{1}{2}}}(1 + \psi(\xi))^{\frac{s-1}{2}}|\tilde{u}(\eta, \omega)|\mathrm{d}\eta \\
&\quad \leq c' \int_{\mathbb{R}^n} (1 + |\xi - \eta|^2)^{-\frac{M-1}{2}}(1 + \psi(\xi - \eta))^{\frac{|s-1|}{2}}(1 + \psi(\eta))^{\frac{s+1}{2}}|\tilde{u}(\eta, \omega)|\,\mathrm{d}\eta \\
&\quad \leq c'' \int_{\mathbb{R}^n} (1 + |\xi - \eta|^2)^{\frac{-M+1+|s-1|}{2}}(1 + \psi(\eta))^{\frac{s+1}{2}}|\tilde{u}(\eta, \omega)|\,\mathrm{d}\eta.
\end{aligned}$$

Denoting by $v *_{\mathbb{R}^n} \omega$ the convolution of $v, w : \mathbb{R}^n \times \mathbb{Z}^m \to \mathbb{C}$ only with respect to the variable x, we can rewrite the above result as

$$
\begin{aligned}
&|(1+\psi(\xi))^{\frac{s}{2}}([J_\varepsilon, q_2(x, D_x, D_k)]u)^{\sim}(\xi,\omega)| \\
&\quad \le c'' \left((1+|\cdot|^2)^{\frac{-M+1+|s-1|}{2}} *_{\mathbb{R}^n} (1+\psi(\cdot))^{\frac{s+1}{2}} |\tilde{u}(\cdot,\omega)|\right)(\xi,\omega), \qquad (8.1)
\end{aligned}
$$

which implies now

Proposition 8.1. *For $s \ge 0$ let $M > 1 + |s-1| + n$ be such that A.2.M is fulfilled. Then it holds*

$$
\|[J_\varepsilon, q_2(x, D_x, D_k)]u\|_{H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)} \le \tilde{c}\|u\|_{H^{\psi,s+1}(\mathbb{R}^n\times\mathbb{Z}^m)} \qquad (8.2)
$$

for all $\varepsilon \in (0,1]$ and all $u \in H^{\psi,s+1}(\mathbb{R}^n \times \mathbb{Z}^m)$ with $\tilde{c}$ independent of $\varepsilon \in (0,1]$.

Proof. For $u \in H^{\psi,s+1}(\mathbb{R}^n \times \mathbb{Z}^m)$ we find using (8.1) and Young's inequality that

$$
\begin{aligned}
&\|[J_\varepsilon, q_2(x, D_x, D_k)]u\|_{H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)} \\
&\quad = \|(1+\psi(\cdot))^{\frac{s}{2}}([J_\varepsilon, q_2(x, D_x, D_k)]u)^{\sim}(\cdot,\cdot)\|_{L^2(\mathbb{R}^n\times\mathbb{T}^m)} \\
&\quad = (2\pi)^{-\frac{m}{2}} \left(\int_{\mathbb{T}^m} \|(1+\psi(\cdot))^{\frac{s}{2}}([J_\varepsilon, q_2(x, D_x, D_k)]u)^{\sim}(\cdot,\omega)\|^2_{L^2(\mathbb{R}^n)} \,\mathrm{d}\omega\right)^{\frac{1}{2}} \\
&\quad \le c''(2\pi)^{-\frac{m}{2}} \times \\
&\quad \times \left(\int_{\mathbb{T}^m} \|(1+|\cdot|^2)^{\frac{-M+1+|s-1|}{2}} *_{\mathbb{R}^n} (1+\psi(\cdot))^{\frac{s+1}{2}}|\tilde{u}(\cdot,\omega)|\|^2_{L^2(\mathbb{R}^n)} \,\mathrm{d}\omega\right)^{\frac{1}{2}} \\
&\quad \le c''(2\pi)^{-\frac{m}{2}} \times \\
&\qquad \times \left(\int_{\mathbb{T}^m} \|(1+|\cdot|^2)^{\frac{-M+1+|s-1|}{2}}\|^2_{L^1(\mathbb{R}^n)} \|(1+\psi(\cdot))^{\frac{s+1}{2}}|\tilde{u}(\cdot,\omega)|\|^2_{L^2(\mathbb{R}^n)} \,\mathrm{d}\omega\right)^{\frac{1}{2}} \\
&\quad = \tilde{c}(2\pi)^{-\frac{m}{2}} \left(\int_{\mathbb{T}^m} \|(1+\psi(\cdot))^{\frac{s+1}{2}}|\tilde{u}(\cdot,\omega)|\|^2_{L^2(\mathbb{R}^n)} \,\mathrm{d}\omega\right)^{\frac{1}{2}} \\
&\quad = \tilde{c}(2\pi)^{-\frac{m}{2}} \left(\int_{\mathbb{T}^m}\int_{\mathbb{R}^n} (1+\psi(\xi))^{s+1}|\tilde{u}(\xi,\omega)|^2 \,\mathrm{d}\xi\mathrm{d}\omega\right)^{\frac{1}{2}}
\end{aligned}
$$

$$= \tilde{c}\|u\|_{H^{\psi,s+1}(\mathbb{R}^n\times\mathbb{Z}^m)},$$

proving the proposition.

□

Using (7.4) we have

Corollary 8.2. *With* $q_2(x, D_x, D_k)$ *as in Proposition* 8.1 *and* $q_1(D_x, D_k)$, *where the symbol* q_1 *satisfies A.1, it holds for* $q(x, D_x, D_k) = q_1(D_x, D_k) + q_2(x, D_x, D_k)$ *and* $u \in H^{\psi,s+1}(\mathbb{R}^n \times \mathbb{Z}^m)$

$$\|[J_\varepsilon, q(x, D_x, D_k)]u\|_{H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)} \leq \tilde{c}\|u\|_{H^{\psi,s+1}(\mathbb{R}^n\times\mathbb{Z}^m)} \tag{8.3}$$

with $\tilde{c}$ *independent of* $\varepsilon \in (0, 1]$.

Next we want to study the commutator $[(1+\psi(D_x))^{\frac{s}{2}}, q_2(x, D_x, D_k)]$. For this we need a modification of Lemma 2.3.3 in [20].

Lemma 8.3. *Let* $k \in L^1(\mathbb{R}^n)$. *Then we have for all* $u, v \in L^2(\mathbb{R}^n \times \mathbb{T}^m)$

$$\left|(2\pi)^{-m}\int_{\mathbb{T}^m}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} k(\xi-\eta)u(\eta,\omega)v(\xi,\omega)\,\mathrm{d}\eta\,\mathrm{d}\xi\mathrm{d}\omega\right|$$
$$\leq \|k\|_{L^1(\mathbb{R}^n)}\|u\|_{L^2(\mathbb{R}^n\times\mathbb{T}^m)}\|v\|_{L^2(\mathbb{R}^n\times\mathbb{T}^m)}.$$

Proof. The proof of Lemma 2.3.3 in [20] yields

$$\begin{aligned}
&\left|(2\pi)^{-m}\int_{\mathbb{T}^m}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} k(\xi-\eta)u(\eta,\omega)v(\xi,\omega)\,\mathrm{d}\eta\,\mathrm{d}\xi\mathrm{d}\omega\right| \\
&\leq (2\pi)^{-m}\int_{\mathbb{T}^m}\|k\|_{L^1(\mathbb{R}^n)}\left(\int_{\mathbb{R}^n}|u(\eta,\omega)|^2\,\mathrm{d}\eta\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^n}|v(\xi,\omega)|^2\,\mathrm{d}\xi\right)^{\frac{1}{2}}\,\mathrm{d}\omega \\
&\leq \|k\|_{L^1(\mathbb{R}^n)}\left((2\pi)^{-m}\int_{\mathbb{T}^m}\int_{\mathbb{R}^n}|u(\eta,\omega)|^2\,\mathrm{d}\eta\mathrm{d}\omega\right)^{\frac{1}{2}}\times \\
&\quad\times\left((2\pi)^{-m}\int_{\mathbb{T}^m}\int_{\mathbb{R}^n}|v(\xi,\omega)|^2\,\mathrm{d}\xi\mathrm{d}\omega\right)^{\frac{1}{2}}.
\end{aligned}$$

□

To handle the commutator $[(1+\psi(D_x))^{\frac{s}{2}}, q_2(x,D_x,D_k)]$ we note first

$$([(1+\psi(D_x))^{\frac{s}{2}}, q_2(x,D_x,D_k)]u)^{\sim}(\xi,\omega)$$
$$= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \hat{q}_2(\xi-\eta,\eta,\omega)((1+\psi(\xi))^{\frac{s}{2}} - (1+\psi(\eta))^{\frac{s}{2}})\tilde{u}(\eta,\omega)\,\mathrm{d}\eta.$$

As in the proof of Theorem 2.3.9 in [20] we have

$$|(1+\psi(\xi))^{\frac{s}{2}} - (1+\psi(\eta))^{\frac{s}{2}}| \le \widehat{c}_{s,\psi}(1+|\xi-\eta|^2)^{\frac{s}{2}}(1+\psi(\eta))^{\frac{s-1}{2}},$$

and we will use also (6.3), i.e.

$$|\hat{q}_2(\xi-\eta,\eta,\omega)| \le \tilde{\gamma}_{M,n,m} \sum_{|\alpha|\le M} \|\varphi_\alpha\|_{L^1(\mathbb{R}^n)}(1+|\xi-\eta|^2)^{-\frac{M}{2}}(1+\psi(\eta)).$$

Proposition 8.4. *For $s>0$ let $M>n+s$ be such that A.2.M is fulfilled. Then with some constant $\gamma=\gamma_{M,n,m,s,\psi}$ it holds for all $u\in H^{\psi,s+1}(\mathbb{R}^n\times\mathbb{Z}^m)$*

$$\|[(1+\psi(D_x))^{\frac{s}{2}}, q_2(x,D_x,D_k)]u\|_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)}$$
$$\le \gamma \textstyle\sum_{|\alpha|\le M} \|\varphi_\alpha\|_{L^1(\mathbb{R}^n)}\|u\|_{H^{\psi,s+1}(\mathbb{R}^n\times\mathbb{Z}^m)}.$$

Proof. For $u\in H^{\psi,s+1}(\mathbb{R}^n\times\mathbb{Z}^m)$ and $v\in L^2(\mathbb{R}^n\times\mathbb{Z}^m)$ we find

$$\left|([(1+\psi(D_x))^{\frac{s}{2}}, q_2(x,D_x,D_k)]u,v)_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)}\right|$$
$$= \Bigg|(2\pi)^{-\frac{n}{2}-m}\int_{\mathbb{T}^m}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{q}_2(\xi-\eta,\eta,\omega)((1+\psi(\xi))^{\frac{s}{2}}-(1+\psi(\eta))^{\frac{s}{2}}) \times$$
$$\times \tilde{u}(\eta,\omega)\overline{\tilde{v}(\xi,\omega)}\,\mathrm{d}\eta\mathrm{d}\xi\mathrm{d}\omega\Bigg|$$
$$\le c_{M,n,m,s,\psi}\sum_{|\alpha|\le M}\|\varphi_\alpha\|_{L^1(\mathbb{R}^n)}(2\pi)^{-m}\times$$
$$\times\int_{\mathbb{T}^m}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(1+|\xi-\eta|^2)^{-\frac{M-s}{2}}(1+\psi(\eta))^{\frac{s+1}{2}}|\tilde{u}(\eta,\omega)||\tilde{v}(\xi,\omega)|\,\mathrm{d}\eta\mathrm{d}\xi\mathrm{d}\omega$$
$$\le c_{M,n,m,s,\psi}\sum_{|\alpha|\le M}\|\varphi_\alpha\|_{L^1(\mathbb{R}^n)}\|(1+|\cdot|^2)^{-\frac{M-s}{2}}\|_{L^1(\mathbb{R}^n)}\times$$
$$\times\|(1+\psi(\cdot))^{\frac{s+1}{2}}\tilde{u}(\cdot,\cdot)\|_{L^2(\mathbb{R}^n\times\mathbb{T}^m)}\|\tilde{v}\|_{L^2(\mathbb{R}^n\times\mathbb{T}^m)}$$

$$= \gamma \sum_{|\alpha| \le M} \|\varphi_\alpha\|_{L^1(\mathbb{R}^n)} \|u\|_{H^{\psi,s+1}(\mathbb{R}^n \times \mathbb{Z}^m)} \|v\|_{L^2(\mathbb{R}^n \times \mathbb{Z}^m)},$$

which implies

$$\|[(1+\psi(D_x))^{\frac{s}{2}}, q_2(x, D_x, D_k)]u\|_{L^2(\mathbb{R}^n \times \mathbb{Z}^m)}$$

$$= \sup_{\|v\|_{L^2(\mathbb{R}^n \times \mathbb{Z}^m)} \neq 0} \frac{\left|([(1+\psi(D_x))^{\frac{s}{2}}, q_2(x, D_x, D_k)]u, v)_{L^2(\mathbb{R}^n \times \mathbb{Z}^m)}\right|}{\|v\|_{L^2(\mathbb{R}^n \times \mathbb{Z}^m)}}$$

$$\le \gamma \sum_{|\alpha| \le M} \|\varphi_\alpha\|_{L^1(\mathbb{R}^n)} \|u\|_{H^{\psi,s+1}(\mathbb{R}^n \times \mathbb{Z}^m)},$$

and the proposition is proved.

□

Remark 8.5. *Note that for $s = 0$ the proposition is trivial.*

9 Estimates for $q_2(x, D_x, D_k)$

Assume that the symbol q_2 satisfies A.2.M with M sufficiently large. We want to prove that $q_2(x, D_x, D_k)$ maps the space $H^{\psi,s+2}(\mathbb{R}^n \times \mathbb{Z}^m)$ continuously into $H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m)$ with a bound being controlled by $\sum_{|\alpha|\leq M}\|\varphi_\alpha\|_{L^1(\mathbb{R}^n)}$. Let $u \in \mathcal{D}(\mathbb{R}^n \times \mathbb{Z}^m)$ or $u \in S(\mathbb{R}^n \times \mathbb{Z}^m)$ and take $v \in L^2(\mathbb{R}^n \times \mathbb{Z}^m)$. It follows that

$$\begin{aligned}
&(q_2(x, D_x, D_k)u, v)_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)} \\
&\quad = (2\pi)^{-\frac{n}{2}-m} \int_{\mathbb{T}^m}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{q}_2(\xi-\eta, \eta, \omega)\tilde{u}(\eta, \omega)\overline{\tilde{v}(\xi, \omega)}\,\mathrm{d}\eta\mathrm{d}\xi\mathrm{d}\omega
\end{aligned}$$

and therefore by using arguments as before, especially (6.3) and Lemma 8.3, with $\tilde{\gamma}_M = \tilde{\gamma}_{M,n,m}$ we find

$$\begin{aligned}
&|(q_2(x, D_x, D_k)u, v)_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)}| \\
&\quad \leq \tilde{\gamma}_M \sum_{|\alpha|\leq M} \|\varphi_\alpha\|_{L^1(\mathbb{R}^n)}(2\pi)^{-\frac{n}{2}-m} \times \\
&\qquad \times \int_{\mathbb{T}^m}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (1+|\xi-\eta|^2)^{-\frac{M}{2}}(1+\psi(\eta))|\tilde{u}(\eta,\omega)||\tilde{v}(\xi,\omega)|\,\mathrm{d}\eta\mathrm{d}\xi\mathrm{d}\omega \\
&\quad \leq \gamma'_M \sum_{|\alpha|\leq M} \|\varphi_\alpha\|_{L^1(\mathbb{R}^n)}\|u\|_{H^{\psi,2}(\mathbb{R}^n\times\mathbb{Z}^m)}\|v\|_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)},
\end{aligned}$$

which yields

$$\|q_2(x, D_x, D_k)u\|_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)} \leq \gamma'_M \sum_{|\alpha|\leq M} \|\varphi_\alpha\|_{L^1(\mathbb{R}^n)}\|u\|_{H^{\psi,2}(\mathbb{R}^n\times\mathbb{Z}^m)}. \tag{9.1}$$

It is easy to see that for (9.1) to hold $M = n+1$ is sufficient. Since

$$\|u\|_{H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)} = \|(1+\psi(D_x))^{\frac{s}{2}}u\|_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)},$$

we find further for $s \geq 0$ that

$$\begin{aligned}
&\|q_2(x, D_x, D_k)u\|_{H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)} \\
&\quad = \|(1+\psi(D_x))^{\frac{s}{2}}q_2(x, D_x, D_k)u\|_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)} \\
&\quad \leq \|q_2(x, D_x, D_k)(1+\psi(D_x))^{\frac{s}{2}}u\|_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)} \\
&\qquad + \|[(1+\psi(D_x))^{\frac{s}{2}}, q_2(x, D_x, D_k)]u\|_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)}
\end{aligned}$$

$$\leq \gamma'_M \sum_{|\alpha|\leq M} \|\varphi_\alpha\|_{L^1(\mathbb{R}^n)} \|(1+\psi(D_x))^{\frac{s}{2}} u\|_{H^{\psi,2}(\mathbb{R}^n\times\mathbb{Z}^m)}$$
$$+\gamma \sum_{|\alpha|\leq M} \|\varphi_\alpha\|_{L^1(\mathbb{R}^n)} \|u\|_{H^{\psi,s+2}(\mathbb{R}^n\times\mathbb{Z}^m)},$$

where we used (9.1) and Proposition 8.4 and we have assumed that M is large enough for both estimates to hold. Noting that

$$\|(1+\psi(D_x))^{\frac{s}{2}} u\|_{H^{\psi,2}(\mathbb{R}^n\times\mathbb{Z}^m)} = \|u\|_{H^{\psi,s+2}(\mathbb{R}^n\times\mathbb{Z}^m)}$$

we eventually arrive at

Proposition 9.1. *For $s \geq 0$ let M be such that A.2.M holds for the symbol q_2 as well as Proposition 8.4 and* (9.1)*. Then there exists a constant $\gamma_1 = \gamma_{M,n,m,s,\psi}$ such that*

$$\|q_2(x,D_x,D_k)u\|_{H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)} \leq \gamma_1 \sum_{|\alpha|\leq M} \|\varphi_\alpha\|_{L^1(\mathbb{R}^n)} \|u\|_{H^{\psi,s+2}(\mathbb{R}^n\times\mathbb{Z}^m)}.$$

10 Estimates for $q(x, D_x, D_k)$

Now we consider the operator $q(x, D_x, D_k) = q_1(D_x, D_k) + q_2(x, D_x, D_k)$. For the symbol q_1 we assume A.1 and for q_2 we assume A.2.M where M will in general depend on $s \geq 0$ and $n \in \mathbb{N}$ as needed in the estimates.

Theorem 10.1. *For $s \geq 0$ the operator $q(x, D_x, D_k)$ maps the space $H^{\psi,s+2}(\mathbb{R}^n \times \mathbb{Z}^m)$ continuously into $H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m)$ and it holds*

$$\|q(x, D_x, D_k)u\|_{H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)} \leq c\|u\|_{H^{\psi,s+2}(\mathbb{R}^n\times\mathbb{Z}^m)}.$$

Proof. This follows from Proposition 7.1, Proposition 9.1 and

$$\|q(x, D_x, D_k)u\|_{H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)} \leq \|q_1(D_x, D_k)u\|_{H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)} + \|q_2(x, D_x, D_k)u\|_{H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)}.$$

□

Theorem 10.2. *Suppose $\lim_{|\xi|\to\infty} \psi(\xi) = \infty$. Given γ_1 and M as in Proposition 9.1 and $\tilde{\kappa}_1 \geq 0$ as in (7.2). If with $\tilde{\kappa}_0$ as in (7.2) it holds*

$$\sum_{|\alpha|\leq M} \|\varphi_\alpha\|_{L^1(\mathbb{R}^n)} \leq \frac{\tilde{\kappa}_0}{2\gamma_1}, \tag{10.1}$$

then there exists a constant $\delta_0 > 0$ such that for $s \geq 0$ we have

$$\|q(x, D_x, D_k)u\|_{H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)} \geq \delta_0\|u\|_{H^{\psi,s+2}(\mathbb{R}^n\times\mathbb{Z}^m)} - \tilde{\kappa}_1\|u\|_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)}. \tag{10.2}$$

(Note that δ_0 and $\tilde{\kappa}_1$ depends on s).

Proof. We observe that

$$\begin{aligned}
&\|q(x, D_x, D_k)u\|_{H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)} \\
&\quad= \|q_1(D_x, D_k)u + q_2(x, D_x, D_k)u\|_{H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)} \\
&\quad\geq \|q_1(D_x, D_k)u\|_{H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)} - \|q_2(x, D_x, D_k)u\|_{H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)} \\
&\quad\geq \tilde{\kappa}_0\|u\|_{H^{\psi,s+2}(\mathbb{R}^n\times\mathbb{Z}^m)} - \tilde{\kappa}_1\|u\|_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)} \\
&\qquad -\gamma_1 \sum_{|\alpha|\leq M} \|\varphi_\alpha\|_{L^1(\mathbb{R}^n)}\|u\|_{H^{\psi,s+2}(\mathbb{R}^n\times\mathbb{Z}^m)} \\
&\quad= \left(\tilde{\kappa}_0 - \gamma_1 \sum_{|\alpha|\leq M} \|\varphi_\alpha\|_{L^1(\mathbb{R}^n)}\right)\|u\|_{H^{\psi,s+2}(\mathbb{R}^n\times\mathbb{Z}^m)} - \tilde{\kappa}_1\|u\|_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)},
\end{aligned}$$

where we have used Proposition 7.2 and Proposition 9.1. Hence, under condition (10.1) the result follows.

□

Remark 10.3. *Note that under the assumption that* $(q(x, D_x, D_k)u, u)_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)} \geq 0$, *the above result implies also for* $\lambda \geq 0$

$$\begin{aligned}
&\|q(x, D_x, D_k)u + \lambda u\|^2_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)} \\
&\quad = \|q(x, D_x, D_k)u\|^2_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)} + 2\lambda(q(x, D_x, D_k)u, u)_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)} \\
&\quad\quad + \lambda^2\|u\|^2_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)} \\
&\quad \geq \|q(x, D_x, D_k)u\|^2_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)} + \lambda^2\|u\|^2_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)},
\end{aligned}$$

or

$$\begin{aligned}
&\|q(x, D_x, D_k)u + \lambda u\|_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)} \\
&\quad \geq \frac{1}{\sqrt{2}}\|q(x, D_x, D_k)u\|_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)} + \frac{\lambda}{\sqrt{2}}\|u\|_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)},
\end{aligned}$$

which via (10.2) *yields for* $\lambda \geq \tilde{\kappa}_1$

$$\|q(x, D_x, D_k)u + \lambda u\|_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)} \geq \frac{\delta_0}{\sqrt{2}}\|u\|_{H^{\psi,2}(\mathbb{R}^n\times\mathbb{Z}^m)}.$$

The next theorem is a type of elliptic regularity result.

Theorem 10.4. *Suppose that* $q(x, D_x, D_k) = q_1(D_x, D_k) + q_2(x, D_x, D_k)$ *where the symbol* q_1 *satisfies A.1 and for* M *suitable, i.e. depending on* s *and* $n \in \mathbb{N}$, *we assume that the symbol* q_2 *satisfies A.2.M. If for some* $f \in H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m)$, $s \geq 0$, *and* $\lambda \in \mathbb{R}^n$ *we have a solution* $u \in H^{\psi,s+1}(\mathbb{R}^n \times \mathbb{Z}^m)$ *to the equation*

$$q_\lambda(x, D_x, D_k)u := q(x, D_x, D_k)u + \lambda u = f,$$

then it follows that $u \in H^{\psi,s+2}(\mathbb{R}^n \times \mathbb{Z}^m)$ *provided* (10.1) *holds.*

Proof. Using (10.2) and (7.4) we find for $\varepsilon \in (0, 1]$ that

$$\begin{aligned}
&\delta_0\|J_\varepsilon(u)\|_{H^{\psi,s+2}(\mathbb{R}^n\times\mathbb{Z}^m)} - \tilde{\kappa}_1\|J_\varepsilon(u)\|_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)} - \|\lambda J_\varepsilon(u)\|_{H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)} \\
&\quad \leq \|q(x, D_x, D_k)J_\varepsilon(u)\|_{H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)} - \|\lambda J_\varepsilon(u)\|_{H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)} \\
&\quad \leq \|q(x, D_x, D_k)J_\varepsilon(u) + \lambda J_\varepsilon(u)\|_{H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)}
\end{aligned}$$

$$
\begin{aligned}
&\leq \|J_\varepsilon(q(x,D_x,D_k)u+\lambda u)\|_{H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)} \\
&\quad +\|[J_\varepsilon, q_2(x,D_x,D_k)]u\|_{H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)} \\
&= \|J_\varepsilon(f)\|_{H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)} + \|[J_\varepsilon, q_2(x,D_x,D_k)]u\|_{H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)}.
\end{aligned}
$$

Using Proposition 8.1 we get further

$$
\begin{aligned}
&\delta_0\|J_\varepsilon(u)\|_{H^{\psi,s+2}(\mathbb{R}^n\times\mathbb{Z}^m)} \\
&\quad \leq \|J_\varepsilon(f)\|_{H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)} + \tilde{\kappa}_1\|J_\varepsilon(u)\|_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)} \\
&\qquad +|\lambda|\|J_\varepsilon(u)\|_{H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)} + \|[J_\varepsilon, q_2(x,D_x,D_k)]u\|_{H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)} \\
&\quad \leq \|f\|_{H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)} + \tilde{\kappa}_1\|u\|_{H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)} + |\lambda|\|u\|_{H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)} \\
&\qquad +\tilde{c}\|u\|_{H^{\psi,s+1}(\mathbb{R}^n\times\mathbb{Z}^m)} \\
&\quad = \|f\|_{H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)} + (\tilde{\kappa}_1+|\lambda|)\|u\|_{H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)} + \tilde{c}\|u\|_{H^{\psi,s+1}(\mathbb{R}^n\times\mathbb{Z}^m)}.
\end{aligned}
$$

Now an application of Proposition 5.5 yields the theorem.

□

11 The Sesquilinear Form Associated with $q(x, D_x, D_k)$

We assume that $q(x, D_x, D_k) = q_1(D_x, D_k) + q_2(x, D_x, D_k)$ is given as before with symbols q_1 and q_2 satisfying A.1 and A.2.M respectively. Again we assume M to be sufficiently large so that the results, i.e. the estimates, for $q_2(x, D_x, D_k)$ hold, and whenever required for these estimates we assume $\sum_{|\alpha|\leq M} \|\varphi_\alpha\|_{L^1(\mathbb{R}^n)}$ to be small.

On $S(\mathbb{R}^n \times \mathbb{Z}^m)$ or $\mathcal{D}(\mathbb{R}^n \times \mathbb{Z}^m)$ we introduce the sesquilinear form B by

$$B(u, v) := (q(x, D_x, D_k)u, v)_{L^2(\mathbb{R}^n \times \mathbb{Z}^m)}.$$

Clearly, B splits according to

$$B(u, v) = B^{(1)}(u, v) + B^{(2)}(u, v),$$

where

$$B^{(1)}(u, v) := (q_1(D_x, D_k)u, v)_{L^2(\mathbb{R}^n \times \mathbb{Z}^m)}$$

and

$$B^{(2)}(u, v) := (q_2(x, D_x, D_k)u, v)_{L^2(\mathbb{R}^n \times \mathbb{Z}^m)}.$$

Proposition 11.1. *On $S(\mathbb{R}^n \times \mathbb{Z}^m)$ (or $\mathcal{D}(\mathbb{R}^n \times \mathbb{Z}^m)$) it holds*

$$|B^{(1)}(u, v)| \leq \gamma_1 \|u\|_{H^{\psi,1}(\mathbb{R}^n \times \mathbb{Z}^m)} \|v\|_{H^{\psi,1}(\mathbb{R}^n \times \mathbb{Z}^m)} \tag{11.1}$$

and

$$B^{(1)}(u, u) \geq \gamma_0 \|u\|^2_{H^{\psi,1}(\mathbb{R}^n \times \mathbb{Z}^m)} - \|u\|^2_{L^2(\mathbb{R}^n \times \mathbb{Z}^m)}, \tag{11.2}$$

where γ_0 and γ_1 are as in A.1.

Proof. For $u, v \in S(\mathbb{R}^n \times \mathbb{Z}^m)$ (or $\mathcal{D}(\mathbb{R}^n \times \mathbb{Z}^m)$) it follows that

$$\begin{aligned}
|B^{(1)}(u, v)| &= |(q_1(D_x, D_k)u, v)_{L^2(\mathbb{R}^n \times \mathbb{Z}^m)}| \\
&= |(2\pi)^{-m} \int_{\mathbb{R}^n} \int_{\mathbb{T}^m} q_1(\xi, \omega)\tilde{u}(\xi, \omega)\overline{\tilde{v}(\xi, \omega)} \,\mathrm{d}\omega \mathrm{d}\xi| \\
&\leq \gamma_1 (2\pi)^{-m} \int_{\mathbb{R}^n} \int_{\mathbb{T}^m} (1 + \psi(\xi))|\tilde{u}(\xi, \omega)||\tilde{v}(\xi, \omega)| \,\mathrm{d}\omega \mathrm{d}\xi \\
&\leq \gamma_1 \|u\|_{H^{\psi,1}(\mathbb{R}^n \times \mathbb{Z}^m)} \|v\|_{H^{\psi,1}(\mathbb{R}^n \times \mathbb{Z}^m)},
\end{aligned}$$

where in the last step we used the Cauchy-Schwarz inequality. Moreover we find

$$\begin{aligned}
B^{(1)}(u,u) &= (2\pi)^{-m}\int_{\mathbb{R}^n}\int_{\mathbb{T}^m} q_1(\xi,\omega)|\tilde{u}(\xi,\omega)|^2\,\mathrm{d}\omega\mathrm{d}\xi \\
&= (2\pi)^{-m}\int_{\mathbb{R}^n}\int_{\mathbb{T}^m}(1+q_1(\xi,\omega))|\tilde{u}(\xi,\omega)|^2\,\mathrm{d}\omega\mathrm{d}\xi \\
&\quad -(2\pi)^{-m}\int_{\mathbb{R}^n}\int_{\mathbb{T}^m}|\tilde{u}(\xi,\omega)|^2\,\mathrm{d}\omega\mathrm{d}\xi \\
&\geq (2\pi)^{-m}\int_{\mathbb{R}^n}\int_{\mathbb{T}^m}\gamma_0(1+\psi(\xi))|\tilde{u}(\xi,\omega)|^2\,\mathrm{d}\omega\mathrm{d}\xi \\
&\quad -(2\pi)^{-m}\int_{\mathbb{R}^n}\int_{\mathbb{T}^m}|\tilde{u}(\xi,\omega)|^2\,\mathrm{d}\omega\mathrm{d}\xi \\
&= \gamma_0\|u\|^2_{H^{\psi,1}(\mathbb{R}^n\times\mathbb{Z}^m)} - \|u\|^2_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)},
\end{aligned}$$

proving the proposition.

□

Corollary 11.2. *The sesquilinear form $B^{(1)}$ has a continuous extension to $H^{\psi,1}(\mathbb{R}^n\times\mathbb{Z}^m)$ and* (11.1), (11.2) *holds for all $u,v\in H^{\psi,1}(\mathbb{R}^n\times\mathbb{Z}^m)$.*

In the following we will denote the extension of $B^{(1)}$ to $H^{\psi,1}(\mathbb{R}^n\times\mathbb{Z}^m)$ again by $B^{(1)}$.

Proposition 11.3. *On $S(\mathbb{R}^n\times\mathbb{Z}^m)$ (or $\mathcal{D}(\mathbb{R}^n\times\mathbb{Z}^m)$) we have the estimate*

$$|B^{(2)}(u,v)| \leq \tilde{\gamma}_M\sum_{|\alpha|\leq M}\|\varphi_\alpha\|_{L^1(\mathbb{R}^n)}\|u\|_{H^{\psi,1}(\mathbb{R}^n\times\mathbb{Z}^m)}\|v\|_{H^{\psi,1}(\mathbb{R}^n\times\mathbb{Z}^m)}. \tag{11.3}$$

Proof. We note that

$$\begin{aligned}
&|B^{(2)}(u,v)| = |(q_2(x,D_x,D_k)u,v)_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)}| \\
&= \left|(2\pi)^{-\frac{n}{2}-m}\int_{\mathbb{R}^n}\int_{\mathbb{T}^m}\int_{\mathbb{R}^n}\hat{q}_2(\xi-\eta,\eta,\omega)\tilde{u}(\eta,\omega)\overline{\tilde{v}(\xi,\omega)}\,\mathrm{d}\eta\mathrm{d}\omega\mathrm{d}\xi\right| \\
&\leq \tilde{\gamma}_{M,n,m}\sum_{|\alpha|\leq M}\|\varphi_\alpha\|_{L^1(\mathbb{R}^n)}(2\pi)^{-m}\int_{\mathbb{R}^n}\int_{\mathbb{T}^m}\int_{\mathbb{R}^n}(1+|\xi-\eta|^2)^{-\frac{M}{2}}\times \\
&\quad\times(1+\psi(\eta))|\tilde{u}(\eta,\omega)||\tilde{v}(\xi,\omega)|\,\mathrm{d}\eta\mathrm{d}\omega\mathrm{d}\xi
\end{aligned}$$

$$\leq \gamma_M^* \sum_{|\alpha|\leq M} \|\varphi_\alpha\|_{L^1(\mathbb{R}^n)} (2\pi)^{-m} \int_{\mathbb{T}^n} \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} (1+|\xi-\eta|^2)^{-\frac{M-1}{2}} \times$$

$$\times (1+\psi(\eta))^{\frac{1}{2}} |\tilde{u}(\eta,\omega)| (1+\psi(\xi))^{\frac{1}{2}} |\tilde{v}(\xi,\omega)| \,\mathrm{d}\eta \mathrm{d}\xi \mathrm{d}\omega$$

$$\leq \tilde{\gamma}_M \sum_{|\alpha|\leq M} \|\varphi_\alpha\|_{L^1(\mathbb{R}^n)} \|u\|_{H^{\psi,1}(\mathbb{R}^n\times\mathbb{Z}^m)} \|v\|_{H^{\psi,1}(\mathbb{R}^n\times\mathbb{Z}^m)},$$

where we used A.2.M with $M \geq n+2$, Peetre's inequality, Proposition 6.1 and Lemma 8.3.

□

Corollary 11.4. *The sesquilinear form $B^{(2)}$ has a continuous extension to $H^{\psi,1}(\mathbb{R}^n \times \mathbb{Z}^m)$ which we denote again by $B^{(2)}$. For this extension estimate* (11.3) *holds.*

Theorem 11.5. *The sesquilinear form $B = B^{(1)} + B^{(2)}$ is defined on $H^{\psi,1}(\mathbb{R}^n \times \mathbb{Z}^m)$ and for $u, v \in H^{\psi,1}(\mathbb{R}^n \times \mathbb{Z}^m)$ we have*

$$|B(u,v)| \leq \tau \|u\|_{H^{\psi,1}(\mathbb{R}^n\times\mathbb{Z}^m)} \|v\|_{H^{\psi,1}(\mathbb{R}^n\times\mathbb{Z}^m)}.$$

In addition, for $\tilde{\gamma}_M \sum_{|\alpha|\leq M} \|\varphi_\alpha\|_{L^1(\mathbb{R}^n)} \leq \frac{\gamma_0}{2}$ the following Gårding inequality holds

$$|B(u,u)| \geq \operatorname{Re} B(u,u) \geq \frac{\gamma_0}{2} \|u\|^2_{H^{\psi,1}(\mathbb{R}^n\times\mathbb{Z}^m)} - \|u\|^2_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)} \tag{11.4}$$

with $\gamma_0 > 0$ as in A.1.

Proof. From Corollaries 11.2 and 11.4 we deduce that B is defined on $H^{\psi,1}(\mathbb{R}^n \times \mathbb{Z}^m)$ and it holds

$$\begin{aligned} |B(u,v)| &\leq |B^{(1)}(u,v)| + |B^{(2)}(u,v)| \\ &\leq \left(\gamma_1 + \tilde{\gamma}_M \sum_{|\alpha|\leq M} \|\varphi_\alpha\|_{L^1(\mathbb{R}^n)} \right) \|u\|_{H^{\psi,1}(\mathbb{R}^n\times\mathbb{Z}^m)} \|v\|_{H^{\psi,1}(\mathbb{R}^n\times\mathbb{Z}^m)}. \end{aligned}$$

Moreover we find

$$
\begin{aligned}
|B(u,u)| &\geq \operatorname{Re} B(u,u) \\
&\geq B^{(1)}(u,u) - |B^{(2)}(u,u)| \\
&\geq \gamma_0 \|u\|^2_{H^{\psi,1}(\mathbb{R}^n\times\mathbb{Z}^m)} - \|u\|^2_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)} - \tilde{\gamma}_M \sum_{|\alpha|\leq M} \|\varphi_\alpha\|_{L^1(\mathbb{R}^n)} \|u\|^2_{H^{\psi,1}(\mathbb{R}^n\times\mathbb{Z}^m)}
\end{aligned}
$$

$$
\begin{aligned}
&\geq \gamma_0 \|u\|^2_{H^{\psi,1}(\mathbb{R}^n\times\mathbb{Z}^m)} - \|u\|^2_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)} - \frac{\gamma_0}{2} \|u\|^2_{H^{\psi,1}(\mathbb{R}^n\times\mathbb{Z}^m)} \\
&= \frac{\gamma_0}{2} \|u\|^2_{H^{\psi,1}(\mathbb{R}^n\times\mathbb{Z}^m)} - \|u\|^2_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)},
\end{aligned}
$$

implying (11.4).

□

12 Solving the Equation $q(x, D_x, D_k)u + \lambda u = f$

As in the first paragraph of Section 11 we assume that $q(x, D_x, D_k) = q_1(D_x, D_k) + q_2(x, D_x, D_k)$ is given as before with symbols q_1 and q_2 satisfying A.1 and A.2.M respectively. Again we assume M to be sufficiently large so that the results, i.e. the estimates, for $q_2(x, D_x, D_k)$ hold, and whenever required for these estimates we assume $\sum_{|\alpha|\leq M} \|\varphi_\alpha\|_{L^1(\mathbb{R}^n)}$ to be small.

The aim of this section is to use the estimates established before to solve for $\lambda \geq 0$ the equation

$$q(x, D_x, D_k)u + \lambda u = f. \tag{12.1}$$

If (12.1) holds **pointwise**, i.e for all $x \in \mathbb{R}^n$ and all $k \in \mathbb{Z}^m$, we call u a **classical solution**. If $f \in H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m)$ and $u \in H^{\psi,s+2}(\mathbb{R}^n \times \mathbb{Z}^m)$ such that (12.1) holds for all $k \in \mathbb{Z}^m$ and almost every $x \in \mathbb{R}^n$, then we call u an **L^2-solution**.

We need a further, weaker notion of a solution. Since we are eventually interested to generate Feller semigroups, we may restrict our considerations to real-valued functions avoiding some problems with anti-linear functionals.

Definition 12.1. *Let $\lambda \geq 0$. We call $u \in H^{\psi,1}(\mathbb{R}^n \times \mathbb{Z}^m)$ a **variational solution** to the equation* (12.1) *if for all $\varphi \in H^{\psi,1}(\mathbb{R}^n \times \mathbb{Z}^m)$ it holds*

$$B_\lambda(u, \varphi) := B(u, \varphi) + \lambda(u, \varphi)_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)} = (f, \varphi)_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)}. \tag{12.2}$$

Using the Lax-Milgram theorem, compare [19], we can prove

Theorem 12.2. *Suppose that q_1 and q_2 satisfy the conditions stated in the first paragraph of this section. Then for every $\lambda \geq 1$ equation* (12.1) *has for every $f \in L^2(\mathbb{R}^n \times \mathbb{Z}^m)$ a unique variational solution $u \in H^{\psi,1}(\mathbb{R}^n \times \mathbb{Z}^m)$.*

Proof. First we note that

$$|(f, \varphi)_0| \leq \|f\|_0 \|\varphi\|_0 \leq \|f\|_0 \|\varphi\|_{H^{\psi,1}(\mathbb{R}^n\times\mathbb{Z}^m)},$$

where $\|\cdot\|_0$ denotes the norm on $L^2(\mathbb{R}^n \times \mathbb{Z}^m)$. Hence every $f \in L^2(\mathbb{R}^n \times \mathbb{Z}^m)$ defines a continuous linear functional on $H^{\psi,1}(\mathbb{R}^n\times\mathbb{Z}^m)$. Further, by Theorem 11.5 the sesquilinear form B_λ is continuous on $H^{\psi,1}(\mathbb{R}^n \times \mathbb{Z}^m)$ and for $\lambda \geq 1$ it also satisfies the estimate

$$|B_\lambda(u, u)| \geq \operatorname{Re} B_\lambda(u, u) \geq \frac{\gamma_0}{2} \|u\|^2_{H^{\psi,1}(\mathbb{R}^n\times\mathbb{Z}^m)}. \tag{12.3}$$

Thus, by the Lax-Milgram theorem exists a unique $u \in H^{\psi,1}(\mathbb{R}^n \times \mathbb{Z}^m)$ representing the continuous linear functional $l_f : H^{\psi,1}(\mathbb{R}^n \times \mathbb{Z}^m) \to \mathbb{C}$, $l_f(\varphi) := (f, \varphi)_0$ via B_λ, i.e.

$$l_f(\varphi) = (f, \varphi)_0 = B_\lambda(u, \varphi)$$

for all $\varphi \in H^{\psi,1}(\mathbb{R}^n \times \mathbb{Z}^m)$, proving the theorem.

□

Next we want to prove that a variational solution to (12.1) is for $f \in L^2(\mathbb{R}^n \times \mathbb{Z}^m)$ already an L^2-solution.

Theorem 12.3. *Let q_1 and q_2 be as in Theorem 12.2. In addition suppose that Proposition 8.1 and Theorem 10.2 hold for $s = 0$. Then a variational solution $u \in H^{\psi,1}(\mathbb{R}^n \times \mathbb{Z}^m)$ to (12.1) with $f \in L^2(\mathbb{R}^n \times \mathbb{Z}^m)$ is an L^2-solution, in particular, $u \in H^{\psi,2}(\mathbb{R}^n \times \mathbb{Z}^m)$.*

Proof. Let $u \in H^{\psi,1}(\mathbb{R}^n \times \mathbb{Z}^m)$ be a variational solution to (12.1) and $(u_l)_{l\in\mathbb{N}}$, $u_l \in S(\mathbb{R}^n \times \mathbb{Z}^m)$ (or $D(\mathbb{R}^n \times \mathbb{Z}^m)$), a sequence converging in $H^{\psi,1}(\mathbb{R}^n \times \mathbb{Z}^m)$ to u. Further let $\varepsilon \in (0, 1]$ and denote by J_ε the Friedrichs mollifier (with respect to the variable x) as in Section 5. Then

$$J_\varepsilon(u_l) \in \bigcap_{s\geq 0} H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m) \subset H^{\psi,2}(\mathbb{R}^n \times \mathbb{Z}^m)$$

and it holds

$$\begin{aligned} B_\lambda(J_\varepsilon(u_l), \varphi) &= (q(x, D_x, D_k)J_\varepsilon(u_l) + \lambda J_\varepsilon(u_l), \varphi)_0 \\ &= (J_\varepsilon((q(x, D_x, D_k) + \lambda)u_l), \varphi)_0 - ([J_\varepsilon, q(x, D_x, D_k)]u_l, \varphi)_0 \\ &= B_\lambda(u_l, J_\varepsilon(\varphi)) - ([J_\varepsilon, q_2(x, D_x, D_k)]u_l, \varphi)_0. \end{aligned}$$

From Proposition 8.1 we deduce (for l large)

$$\|[J_\varepsilon, q_2(x, D_x, D_k)]u_l\|_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)} \leq \tilde{c}\|u_l\|_{H^{\psi,1}(\mathbb{R}^n\times\mathbb{Z}^m)} \leq c'_u,$$

implying (for a subsequence of $([J_\varepsilon, q_2(x, D_x, D_k)]u_l)_{l\in\mathbb{N}}$, which we denote again by $([J_\varepsilon, q_2(x, D_x, D_k)]u_l)_{l\in\mathbb{N}}$) that $[J_\varepsilon, q_2(x, D_x, D_k)]u_l \rightharpoonup \omega_\varepsilon$ in $L^2(\mathbb{R}^n \times \mathbb{Z}^m)$ and that $\|\omega_\varepsilon\|_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)} \leq c'_u$ where c'_u depends on u but is independent of ε. For $l \to \infty$ we find now

$$\begin{aligned} B_\lambda(J_\varepsilon(u), \varphi) &= B_\lambda(u, J_\varepsilon(\varphi)) - (\omega_\varepsilon, \varphi)_0 \\ &= (f, J_\varepsilon(\varphi))_0 - (\omega_\varepsilon, \varphi)_0 \\ &= (J_\varepsilon(f), \varphi)_0 - (\omega_\varepsilon, \varphi)_0. \end{aligned}$$

It follows that

$$\begin{aligned}\|q_\lambda(x,D_x,D_k)J_\varepsilon(u)\|_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)} &\leq \|J_\varepsilon(f)\|_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)} + \|\omega_\varepsilon\|_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)}\\ &\leq \|f\|_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)} + c_u',\end{aligned}$$

or by Theorem 10.2 we find

$$\begin{aligned}&\|q_\lambda(x,D_x,D_k)J_\varepsilon(u)\|_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)}\\ &\quad\geq \|q(x,D_x,D_k)J_\varepsilon(u)\|_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)} - |\lambda|\|J_\varepsilon(u)\|_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)}\\ &\quad\geq \delta_0\|J_\varepsilon(u)\|_{H^{\psi,2}(\mathbb{R}^n\times\mathbb{Z}^m)} - \tilde{\kappa}_1\|J_\varepsilon(u)\|_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)} - |\lambda|\|J_\varepsilon(u)\|_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)},\end{aligned}$$

implying

$$\delta_0\|J_\varepsilon(u)\|_{H^{\psi,2}(\mathbb{R}^n\times\mathbb{Z}^m)} \leq \|f\|_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)} + (\tilde{\kappa}_1 + |\lambda|)\|u\|_{L^2(\mathbb{R}^n\times\mathbb{Z}^m)} + c_u'.$$

Now we may apply Proposition 5.5 to deduce that u belongs to $H^{\psi,2}(\mathbb{R}^n\times\mathbb{Z}^m)$ which in turn implies

$$B_\lambda(u,\varphi) = (q(x,D_x,D_k)u + \lambda u, \varphi)_0,$$

and the theorem is proved.

□

Corollary 12.4. *If in addition to the hypothesis of Theorem 12.3 those of Theorem 10.4 hold, then a variational solution to* (12.1) *with* $f \in H^{\psi,s}(\mathbb{R}^n\times\mathbb{Z}^m)$ *belongs to* $H^{\psi,s+2}(\mathbb{R}^n\times\mathbb{Z}^m)$.

13 The Operator $-q(x, D_x, D_k)$ as a Generator of a Semigroup

So far we have seen by combining the results of Sections 10-12 that under suitable assumptions we can consider the operator $-q(x, D_x, D_k)$ as well defined on $H^{\psi,s+2}(\mathbb{R}^n \times \mathbb{Z}^m)$ with image in $H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m)$. Moreover, for λ large enough $q(x, D_x, D_k) + \lambda$ is bijective from $H^{\psi,s+2}(\mathbb{R}^n \times \mathbb{Z}^m)$ to $H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m)$.

We change our point of view and assuming (5.1) and $s > \frac{n}{2r_0}$ we can consider $-q(x, D_x, D_k)$ as densely defined on

$$C_\infty(\mathbb{R}^n \times \mathbb{Z}^m) = \overline{S(\mathbb{R}^n \times \mathbb{Z}^m)}^{\|\cdot\|_\infty} = \overline{H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m)}^{\|\cdot\|_\infty}, \tag{13.1}$$

where $\|\cdot\|_\infty$ denotes the supremum norm

$$\|u\|_\infty := \sup_{(x,k) \in \mathbb{R}^n \times \mathbb{Z}^m} |u(x,k)|. \tag{13.2}$$

Thus $(-q(x, D_x, D_k), H^{\psi,s+2}(\mathbb{R}^n \times \mathbb{Z}^m))$ is a densely defined operator on $C_\infty(\mathbb{R}^n \times \mathbb{Z}^m)$ with image $H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m) \subset C_\infty(\mathbb{R}^n \times \mathbb{Z}^m)$, and for λ large enough we find for every $f \in H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m)$, a dense subset of $C_\infty(\mathbb{R}^n \times \mathbb{Z}^m)$, there exists a unique solution to

$$(\lambda - (-q(x, D_x, D_k)))u = f. \tag{13.3}$$

In view of the Hille-Yosida theorem, once we know that $-q(x, D_x, D_k)$ is dissipative in $C_\infty(\mathbb{R}^n \times \mathbb{Z}^m)$ we can conclude that the operator $(-q(x, D_x, D_k),$ $H^{\psi,s+2}(\mathbb{R}^n \times \mathbb{Z}^m))$ is closable and its closure generates a strongly continuous contraction semigroup on $C_\infty(\mathbb{R}^n \times \mathbb{Z}^m)$.

It is well known, compare [11], that if $-q(x, D_x, D_k)$ satisfies the positive maximum principle, then it is dissipative and moreover if $-q(x, D_x, D_k)$ satisfies the positive maximum principle and generates a strongly continuous contraction semigroup on $C_\infty(\mathbb{R}^n \times \mathbb{Z}^m)$, then this semigroup is a Feller semigroup. But thanks to the Lévy-Khinchin formula, when $(\xi, \omega) \mapsto q(x, \xi, \omega)$ is for every $x \in \mathbb{R}^n$ a continuous negative definite function, it follows that $-q(x, D_x, D_k)$ satisfies the positive maximum principle. We refer to [19] where the case of operators acting on functions defined on $\mathbb{R}^n$ is discussed and to [12] where the case of operators acting on sequences is investigated. The general case follows from both results by superposition. Hence by the Hille-Yosida-Ray theorem, i.e. the Hille-Yosida theorem with the dissipativity replaced by the positive maximum principle, we may state

Theorem 13.1. *Under the assumptions of Corollary* 12.4, *if the function* $(\xi,\omega)\mapsto q(x,\xi,\omega)$ *is for every* $x\in\mathbb{R}^n$ *a continuous negative definite function, then* $(-q(x,D_x,D_k),H^{\psi,s+2}(\mathbb{R}^n\times\mathbb{Z}^m))$, $s>\frac{n}{2r_0}$, *extends to a generator of a Feller semigroup* $(T_t)_{t\geq 0}$ *on* $C_\infty(\mathbb{R}^n\times\mathbb{Z}^m)$.

Let $\varphi:\mathbb{R}^n\to\mathbb{R}$ and $\sigma:\mathbb{T}^m\to\mathbb{R}$ be two continuous negative definite functions. Then $\varphi\oplus\sigma:\mathbb{R}^n\times\mathbb{T}^m\to\mathbb{R}$, $(\xi,\omega)\mapsto\varphi(\xi)+\sigma(\omega)$ is a continuous negative definite function on $\mathbb{R}^n\times\mathbb{T}^m$. This allows us to construct examples $q(x,\xi,\omega)=r(x,\xi)+s(x,\omega)$ with $(\xi,\omega)\mapsto q(x,\xi,\omega)$ being continuous negative definite.

The function

$$p(\omega)=2\sum_{k=1}^{\infty}\frac{1}{\pi k^2}(1-\cos\omega k)=\omega-\frac{\omega^2}{2\pi}$$

is a continuous negative definite function on $\mathbb{T}^1$, see [12], p.639. Moreover, on $\mathbb{R}^n$, for $0<\alpha<2$, the function $\tilde{q}_\alpha(\xi)=|\xi|^\alpha$ is a continuous negative definite function. For $j=1,2$ let $a_j:\mathbb{R}^n\to\mathbb{R}$, $a_j(x)=a_{0j}+\varphi(x)$, $0\leq\varphi\in S(\mathbb{R}^n)$, $a_{0j}>0$. Consider

$$\begin{aligned}q(x,\xi,\omega)&=a_1(x)\tilde{q}_\alpha(\xi)+a_2(x)p(\omega)\\&=a_{01}|\xi|^\alpha+(a_1(x)-a_{01})|\xi|^\alpha+a_{02}p(\omega)+(a_2(x)-a_{02})p(\omega)\\&=a_{01}|\xi|^\alpha+a_{02}p(\omega)+((a_1(x)-a_{01})|\xi|^\alpha+(a_2(x)-a_{02})p(\omega))\\&=q_1(\xi,\omega)+q_2(x,\xi,\omega).\end{aligned}$$

With $\psi(\xi)=(1+|\xi|^2)^{\frac{\alpha}{2}}$ it follows now that $A.1$ and $A.2.M$ hold.

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