Hopf algebra and noncommutative differential structures.

Masmali, Ibtisam Ali

Hopf algebra and noncommutative Differential Structures

A thesis submitted to the University of Wales for the degree of

Doctor of Philosophy

By

Ibtisam Ali Masmali

School of Physical Sciences

Department of Mathematics

Swansea University

2010

ABSTRACT

In this thesis I will study noncommutative differential geometry, after the style of Connes and Woronowicz. In particular two examples of differential calculi on Hopf algebras are considered, and their associated covariant derivatives and Riemannian geometry. These are on the Heisenberg group, and on the finite group A_4. I consider bimodule connections after the work of Madore. In the last chapter noncommutative fibrations are considerd, with an application to the Leray spectral sequence.

NOTATION.

In this thesis equations are numbered as round brackets ( ), where (a.b) denotes equation b in chapter a, and references are indicated by square brackets [ ].

This thesis has been typeset using Latex, and some figures using the Visio program.

Declaration

This work has not previously been accepted in substance for any degree and is not beging concurrently submitted in candidature for any degree.

Ibtisam Ali Masmali. (Candidate)

Signed ....

Date ..27-.1.-..2011..........

Statement 1

This thesis is the result of my own investigation, except where otherwise stated. When correction services have been used, the extent and nature of the correction is clearly marked in a footnote(s).

Other sources are acknowledged by giving explicit references. A bibliography is appended.

Ibtisam Ali Masmali. (Candidate)

Signed ....

Date .....27.-..1.....-..2011....

Statement 2

I hereby give consent for this thesis, if accepted, to be available for photocopying and for inter-library loan, and for the title and summary to be made available to outside organisations.

Ibtisam Ali Masmali. (Candidate)

Signed ....

Date 27-1.-..2011................

This thesis is sincerely dedicated to
my great loving parents
my loving husband
and my daughters

Contents

ACKNOWLEDGEMENTS

Firstly, all my full gratitude is due to Allah who guided and helped me to complete this thesis.

My great appreciation and sincere gratitude to my supervisor Dr. Edwin Beggs for his support and great supervision during my PhD study, and for his constant direction to the knowledge and encouragment

I am also very grateful for the support I have received from Ministry of Higher Education in Saudi Arabia, and to Saudi Cultural Bureau in London. My deep thanks also to the Mathematics Department in Swansea University, especially to Prof. Aubrey Truman, Prof. Niels Jacob, and to secretaries, and to all the group of algebra, especially to Prof. Tomasz Brzezinski and Dr. Martin Crossley. My thanks also to Dr. Mohammed Alshomrani (King AbdulAziz University, Saudi Arabia) for the nice communications and support at the beginning of my study.

I wish to express my deep thanks to my great parents, who have backed me throughout my life, for their guidance, patience and wishing me all the best, and to my brothers and sisters. Also to all my friends in Swansea, especially to my best friend Mrs. Basmah Alhudithi for her great and nice friendship and kindly support.

Finally, I am deeply indebted and very grateful to my husband Dr. Ali Masmali for his constant support, help and encouragement to get all the success in my life and for his patience throughout my studies, and to my lovely daughters Arwa, Anwar and Raneem.

Introduction

Differential geometry may be dated to the work of Riemann [42], though much of the formalism was used previously in "flat space". This was later used by Einstein to write the general theory of relativity [19], and then in gauge theories of particle physics. Noncommutative differential geometry dates back to the work of Alain Connes on the Dirac operator [13], and was given a major boost with the work of Woronowicz [46] on differential calculi on quantum groups.

This led to considerable interest in noncommutative differential geometry from the physics community, based on the idea that combining quantum theory and gravity should lead to noncommutative space time. For example:

Moyal product[39][38] [47]this corresponds to one of the simplest possible noncommutative structure on space time, which is often used by physicists: it is similar to a noncommutative torus, but in more dimensions, eg.

$$f * g = fg + \frac{i\hbar}{2}\sum_{i,j}\Pi^{ij}(\partial_i f)(\partial_j g) - \frac{\hbar^2}{8}\sum_{i.j.k,m}\Pi^{ij}\Pi^{km}(\partial_i\partial_k f)(\partial_j\partial_m g) + \cdots$$

where Π_{ij} is a number valued matrix. Here f and g are smooth functions on $\mathbb{R}^n$, and $\hbar$ is a parameter.

Fuzzy spheres: this is a deformation of the algebra of functions on $\mathbb{R}^3$ given by

$$[X_i, X_j] = i\alpha\epsilon_{ijk}X_k$$

where α is a parameter, and

$$\epsilon_{ijk} = \begin{cases} 0 & \text{ijk has a repeat} \\ 1 & \text{ijk no repeat and is 123 in cyclic order} \\ -1 & \text{otherwise.} \end{cases}$$

eg. $\epsilon_{122} = 0 \quad \epsilon_{123} = \epsilon_{312} = 1 \quad \epsilon_{132} = -1.$

This, and the physical motivations behind it, is discussed in [45] [29] and [7].

Connes Standard Model: Connes and Marcoli gave an application of noncommutative differential geometry to give an alternative derivation of the standard model of particle physics. [14] [15]

Cosmology which in recently some predictions of a possible noncommutative structure of space time have become testable on a possible dependence of the velocity of light on frequency- so for measurements have been negative [32].

From the point of view of differential forms, the principle of noncommutative geometry is quite simple: make the forms into bimodules over a noncommutative algebra. However, this is not so simple in practice. But some examples which do work well are the calculi on quantum groups [46] (and their quotients, eg the quantum sphere) and on finite groups [34].

In this thesis we shall take two examples of differential calculi, one on the finite group A_4 and one on the Heisenberg group. We shall then apply methods of noncommutative differential geometry to these examples, and see how similar the results are to those of "classical" differential geometry. We shall see that there are substantial differences to the classical case, where there is a unique Levi-Civita connection.

In chapters 1 and 2 we will review some background material, and in chapters 3 and 4 we deal with the A_4 and Heisenberg example, respectively. In chapter 5 we look at an application of the Leray spectral sequence to a noncommutative fibration. In chapter 6 we give some general comments and possible directions for future work.

Chapter 1

Algebras, Hopf algebras and categories

Here we will give a brief introduction to the existing material on algebras, Hopf algebras and categories which we shall use later on.

1.1 Hopf algebra

Definition 1 *[36] An algebra (with unit)over a field* $\mathbb{K}$ *is a vector space,* A*, together with two linear maps, a multiplication* $\mu : A \otimes A \to A$*, and a unit map* $\eta : \mathbb{K} \to A$ *such that the following diagrams commute:*

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mu\otimes id"] \arrow[d, "id\otimes\mu"'] & A \otimes A \arrow[d, "\mu"] \\ A \otimes A \arrow[r, "\mu"'] & A \end{tikzcd} \tag{1.1}$$

$$
\begin{tikzcd}
\mathbb{K} \otimes A \arrow[r, "\eta\otimes id"] \arrow[dr] & A \otimes A \arrow[d, "\eta"] & A \otimes \mathbb{K} \arrow[l, "id\otimes\eta"'] \arrow[dl] \\
 & A &
\end{tikzcd}
\tag{1.2}
$$

where the lower left and right maps are simply scalar multiplication.

An algebra A is a star or $*$-algebra if there is a conjugate linear operation $a \mapsto a^*$ from A to A so that $(ab)^* = b^*a^*$, $1^* = 1$.

Definition 2 *[36] A* ***coalgebra*** *is a vector space C together with two linear maps, comultiplication $\Delta : C \to C \otimes C$ and counit $\epsilon : C \to \mathbb{K}$, such that the following two diagrams commute.*

The two upper maps in 1.4 are given by $c \to 1 \otimes c$ and $c \to c \otimes 1$ for any $c \in C$. C is cocomutative if $\tau \circ \triangle = \triangle$, where τ is the twist map $\tau(x \otimes y) = y \otimes x$.

$$
\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\Delta\otimes id"] \\
C \otimes C \arrow[r, "id\otimes\Delta"'] & C \otimes C \otimes C
\end{tikzcd}
\tag{1.3}
$$

$$
\begin{tikzcd}
\mathbb{K} \otimes C & C \arrow[l, "1\otimes"'] \arrow[r, "\otimes 1"] \arrow[d, "\Delta"'] & C \otimes \mathbb{K} \\
 & C \otimes C \arrow[ul, "\epsilon\otimes id"] \arrow[ur, "id\otimes\epsilon"'] &
\end{tikzcd}
\tag{1.4}
$$

Definition 3 *[36] A $\mathbb{K}$-vector space B is a* ***bialgebra*** *if (B, μ, η) is an algebra, (B, Δ, ϵ) is a coalgebra and either of the equivalent conditions holds :*

1) Δ and ϵ are algebra morphisms .

2) μ and η are coalgebra morphisms .

This bialgebra structure is often denoted by $(B, \triangle, \epsilon, \mu, \eta)$.

Definition 4 *[31] A* ***Hopf algebra*** *H is*

1) A bialgebra H , Δ , ϵ , μ , η .

2) A map $S : H \to H$ (the antipode) such that $\sum(Sh_{(1)})h_{(2)} = \epsilon(h) = \sum h_{(1)}Sh_{(2)}$ for all $h \in H$.

$$\begin{tikzcd} & H \otimes H \arrow[r, "S \otimes id"] & H \otimes H \arrow[dr, "\mu"] & \\ H \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] \arrow[rr, "\epsilon"] & & \mathbb{K} \arrow[r, "\eta"] & H \\ & H \otimes H \arrow[r, "id \otimes S"'] & H \otimes H \arrow[ur, "\mu"'] & \end{tikzcd} \tag{1.5}$$

Here Δ is the comultiplication of the bialgebra, μ its multiplication, η its unit and ϵ its counit.

Proposition 5 *[33] Let H be a Hopf algebra with antipode S . Then*

1) S is an anti-algebra morphism, that is $S(hK) = S(K)S(h)$, all $h, K \in H$ and $S(1) = 1$.

2) S is an anti-coalgebra morphism, that is $\Delta \circ S = \tau \circ (S \otimes S) \circ \Delta$ and $\epsilon \circ S = \epsilon$.

Two Hopf algebras H, H' are **dually paired** by a map $\langle , \rangle : H' \otimes H \to \mathbb{K}$ if

$$\langle \phi\psi, h \rangle = \langle \phi \otimes \psi, \Delta h \rangle \, , \, \langle 1, h \rangle = \epsilon(h)$$

$$\langle \Delta\phi, h \otimes g \rangle = \langle \phi, hg \rangle \, , \, \epsilon(\phi) = \langle \phi, 1 \rangle$$

$$\langle S\phi, h \rangle = \langle \phi, Sh \rangle$$

for all ϕ , $\psi \in H'$ and $h, g \in H$. Here $\langle , \rangle$ extends to tensor product pairwise, i.e.

$$\langle \phi \otimes \psi, a \otimes b \rangle = \langle \phi, a \rangle \, \langle \psi, b \rangle$$

1.1.1 Actions and coactions

The definition of an action of an algebra extends the idea of matrices acting on a vector space. The idea of coaction is then dual to an action.

Definition 6 *[31] A left action (or representation) of an algebra H is a pair (α, V), where V is a vector space and α is a linear map $H \otimes V \to V$, say $\alpha(h \otimes v) = \alpha_h(v)$, such that $\alpha_{gh}(v) = \alpha_h(\alpha_g(v))$, $\alpha(1 \otimes v) = v$. Instead of constantly writing α, we often simply denote it by $\triangleright$ (or simply by a period).Thus, $h \triangleright v = \alpha_n(v) \in V$, $(hg) \triangleright v = h \triangleright (g \triangleright v), \quad 1 \triangleright v = v$.*

[31] An algebra A is an H-module algebra if A is a left H-module (i.e. H acts on it form the left) and

$$h \triangleright (ab) = \sum (h_{(1)} \triangleright a)(h_{(2)} \triangleright b), \quad h \triangleright 1 = \epsilon(h)1.$$

and coalgebra C is a left H-module coalgebra if

$$\Delta(h \triangleright c) = \sum h_{(1)} \triangleright c_{(2)} \otimes h_{(2)} \triangleright c_{(2)}, \quad \epsilon(h \triangleright c) = \epsilon(h)\epsilon(c).$$

Example 7 *[31] The left regular action L of a bialgebra or Hopf algebra H on itself is $L_h(g) = hg$, and makes H into an H-module coalgebra.*

Proof: *For the complete proof of this example see [31].* □

Definition 8 *[31] A right coaction (or corepresentation) of a coalgebra H is a pair (β, V), where V is a vector space and β is a linear map $V \to V \otimes H$, such that $(\beta \otimes id) \circ \beta = (id \otimes \Delta) \circ \beta$ and $id = (id \otimes \epsilon) \circ \beta$.*

We shall now define modules and comodules for an algebra A and a coalgebra C respectively. We shall only consider the case where the modules and comodules are vector spaces over a field $\mathbb{K}$, not some more general picture.

Definition 9 *V is a right module over the algebra A if there is a linear map*

$$\triangleleft : V \otimes A \longrightarrow V$$

which is an action, i.e. $(v \triangleleft a) \triangleleft b = v \triangleleft (ab)$.
Similarly V is a left module over the algebra A if there is a linear map

$$\triangleright : A \otimes V \longrightarrow V$$

which is an action, i.e. $a \triangleright (b \triangleright v) = (ab) \triangleright v$.

For a unital algebra we shall assume that $1 \triangleright v = v$ and $v \triangleleft 1 = v$.

Definition 10 *V is a right comodule over the coalgebra C if there is a linear map*

$$\varrho : V \longrightarrow V \otimes C$$

which is a coaction, i.e. $(\varrho \otimes id)\varrho = (id \otimes \Delta)\varrho$.
V is a left comodule over the coalgebra C if there is a linear map

$$\lambda : V \longrightarrow C \otimes V$$

which is a coaction, i.e. $(id \otimes \lambda)\lambda = (\Delta \otimes id)\lambda$.

We shall assume that the counit coactions as the identity, i.e. $(id \otimes \epsilon)\varrho = id$ and $(\epsilon \otimes id)\lambda = id$

Example 11 *[31] The right regular coaction of a bialgebra or Hopf algebra H on itself is given by the coproduct of $R = \Delta : H \to H \otimes H$, and makes H into H -comodule algebra.*

Proof *: For the complete proof of this example see [31].* $\square$

The following definition comes in different left-right forms, we only give the one we will use later.

Definition 12 *A (right- right) Yetter-Drinfeld module V for a Hopf algebra H has a right coaction $\varrho : V \to V \otimes H$ (written $v \mapsto v_{[0]} \otimes v_{[1]}$) and a right action $\triangleleft : V \otimes H \to V$ for which*

$$\rho(v \triangleleft h) = \eta_{[0]} \triangleleft h_{(2)} \otimes S(h_{(1)})v_{[1]}h_{(3)},\ \forall\, v \in V,\ \forall\, h \in H$$

Proposition 13 *In the category of Yetter-Drinfeld modules see ([31], with module and comodule maps as morphism) there is a braiding*

$$\Psi : V \otimes W \to W \otimes V$$

$$\Psi(v \otimes w) = w_{[0]} \otimes v \triangleleft w_{[1]}$$

If S invertible, there is an inverse $\Psi^{-1}(w \otimes v) = v \triangleleft S^{-1}(w_{[1]}) \otimes w_{[0]}$

1.1.2 Star algebras and coalgebras

If A is a $*$ algebra , then there is a conjugate-linar operator $a \mapsto a^*$ and $(ab)^* = b^* a^*$

.

Definition 14 *[31] A Hopf $*$-algebra is a $*$-algebra H which also a Hopf algebra such that*

$$\Delta H^* = (\Delta h)^{*\otimes *}, \quad \epsilon(h^*) = \overline{\epsilon(h)}, \quad (S \circ *)^2 = id.$$

If A, H are two $*$-Hopf algebra, they are dually paired if they are dually paired as Hopf algebra and, in addition,

$$\langle \phi^*, h \rangle = \overline{\langle \phi, (Sh)^* \rangle}$$

for all $h \in H$ and $\phi \in A$. If we take nodules over a star algebra, it makes sense to consider their conjugate modules. We begin with the case of a vector space.
If E is vector space, then its conjugate $\overline{E}$ is defined to be E as a set, but with vector space operators (using $\overline{e} \in \overline{E}$ to denote the element e)

$$\overline{e} + \overline{f} = \overline{e + f}$$

$$\alpha \overline{e} = \overline{\alpha^* e} \quad , \alpha \in \mathbb{C}$$

If E is an A-bimodule, then the conjugate bimodule $\overline{E}$ is $\overline{E}$ as a vector space, and has actions

$$a.\overline{e} = \overline{e.a^*}, \quad \overline{e}.a = \overline{a^*.e}$$

There is a bimodule map

$$\Upsilon : \overline{E \otimes_A F} \longrightarrow \overline{F} \otimes_A \overline{E}$$

given by

$$\Upsilon(\overline{e \otimes f}) = \overline{f} \otimes \overline{e}.$$

1.2 Categories

Definition 15 *[24] A* ***Category*** *C consists*

1) of a class Ob (C) whose elements are called the objects of the category,

2) of a class $Hom(C)$ whose elements are called the morphisms of the category , and

3) of maps

identity $id : Ob(C) \to Hom(C)$,

source $s : Hom(C) \to Ob(C)$,

target $b : Hom(C) \to Ob(C)$,

composition $\circ : Hom(C) \times_{ob} Hom(C) \to Ob(C)$,

such that

a) for any object $V \in Ob(C)$, we have $s(id_v) = b(id_v) = V$,

b) for any morphism $f \in Hom(C)$, we have $id_{b(f)} \circ f = f \circ id_{s(f)} = f$

c) for any morphisms f, g, h satisfying $b(f) = s(g)$ and $b(g) = s(h)$, we have $(h \circ g) \circ f = h \circ (g \circ f)$.

Here $Hom(C) \times_{ob} Hom(C)$ is $\{(g, f) \in Hom(C) \times Hom(C) | b(S) = s(T)\}$

Example 16 *Vector spaces (object) and linear maps (morphism) $V \xrightarrow{\alpha} W$ where α is a linear map from V to W*

A subcategory C of a category D consists of a subclass $Ob(C)$ of $Ob(D)$ and of a subclass $Hom(C)$ of $Hom(D)$ which form a category with the identity, source, target and composition map in D.

Definition 17 *[24] A functor $F : C \to C'$ from the category C to the category C' consists of map $F : Ob(C) \to Ob(C')$ and of a map $F : Hom(C) \to Hom(C')$ such that*

a) for any object $V \in Ob(C)$, we have $F(id_v) = id_{f(v)}$,

b) for any morphism $f \in Hom(C)$, we have

$$s(F(f)) = F(s(f)) \quad \textit{and} \quad b(F(f)) = F(b(f)),$$

c) if f, g are composable morphisms in the category C, we have

$$F(g \circ f) = F(g) \circ F(f).$$

Example 18 *As an example of functor, consider a functor V from the category of finite sets (with functions as morphisms) to finite 2.dimensional vector space (with linear maps). Let $V(x)$ be the vector space with basis labelled by elements of x, so $V(\{a,b\})$ is the vector space with basis a and b. So $4a - 3b$ is an element of $V(\{a,b\})$. A functor $f : x \to y$ in finite sets gives a linear map*

$$V(f) : V(x) \to V(y)$$

$$V(f)(\sum_{x \in X} \alpha_x x) = \sum \alpha_x f(x)$$

e.g. $f : \{a,b\} \longrightarrow \{p,q,r\}$ and $f(a) = p$, $f(b) = q$.
Then $V(f)(4a - 3b) = 4p - 3q$.

Definition 19 *[24] Let F , G be functors from the category C to the category C'. A natural transformation η from F to G we write $\eta : F \to G$ is a family of morphisms $\eta(V) : F(V) \to G(V)$ in C' indexed by the objects V of C such that, for any morphism*

$F : V \to W$ in C, the square

$$\begin{array}{ccc} F(V) & \xrightarrow{\eta(V)} & G(V) \\ \Big\downarrow{\scriptstyle F(f)} & & \Big\downarrow{\scriptstyle G(f)} \\ F(W) & \xrightarrow{\eta(W)} & G(W) \end{array} \tag{1.6}$$

commutes.

$\eta(V)$ is an isomorphism of C' for any object V in C, we say That $\eta : F \to G$ is a natural isomorphism.

Let C be a category and $\otimes$ be a functor from $C \times C$ to C. This means that

a) we have an object $V \otimes W$ associated to any pair (V, W) of object of the category,

b) we have a morphism $f \otimes g$ associated to any pair (f, g) of morphisms of C such that $s(f \otimes g) = s(f) \otimes s(g)$ and $b(f \otimes g) = b(f) \otimes b(g)$,

c) if f' and g' are morphisms such that $s(f') = b(f)$ and $s(g') = b(g)$, then

$$(f' \otimes g')(f \otimes g) = (f' \circ f) \otimes (g' \circ g), \tag{1.7}$$

d) and $id_{V \otimes W} = id_V \otimes id_W$.

Relation (1.7) implies that $f \otimes g = (f \otimes id_{b(g)}) \circ (id_{s(f)} \otimes g) = (id_{b(f)} \otimes g) \circ (f \otimes id_s(g))$.

Any functor $\otimes : C \times C \to C$ obeying these conditions will be called a **tensor product** .

Definition 20 *[31] A monoidal category (or tensor category) is $(C, \otimes, \underline{1}, \Phi, l, r)$, where C is a category and $\otimes : C \times C \to C$ is a functor which is associative in the sense that there is a natural equivalence $\Phi : (\otimes)\otimes \to \otimes(\otimes)$, i.e. there are given* *functorial isomorphisms*

$$\Phi_{V,W,Z} : (V \otimes W) \otimes Z \cong V \otimes (W \otimes Z), \quad \forall V, W, Z \in C$$

obeying the pentagon condition in (1.8). In many cases $\Phi((V \otimes W) \otimes Z) = V \otimes (W \otimes Z)$*, and in these cases the category can be called " trivially associated" or "strictly monoidal", and* Φ *is often ommitted. We shall only be concered which such cases. We also require a unit object* $\underline{1}$ *and natural equivalences between the functors* $(\,) \times \underline{1}$*,* $\underline{1} \otimes (\,)$ *and the identity functor* $C \to C$*, i.e. there should be given functorial isomorphisms* $l_V : V \cong V \otimes \underline{1}$ *and* $r_V : V \cong \underline{1} \otimes V$ *obeying (1.9).*

$$\begin{tikzcd}
& (V \otimes W) \otimes (Z \otimes U) \arrow[dr, "\Phi"] & \\
((V \otimes W) \otimes Z) \otimes U \arrow[ur, "\Phi"] \arrow[d, "\Phi\otimes id"'] & & V \otimes (W \otimes (Z \otimes U)) \\
(V \otimes (W \otimes Z)) \otimes U \arrow[rr, "\Phi"'] & & V((\otimes W \otimes Z) \otimes U) \arrow[u, "id\otimes\Phi"']
\end{tikzcd} \tag{1.8}$$

$$\begin{tikzcd}
(V \otimes \underline{1}) \otimes W \arrow[rr, "\Phi"] & & V \otimes (\underline{1} \otimes W) \\
& V \otimes W \arrow[ul, "l\otimes id"] \arrow[ur, "id\otimes r"'] &
\end{tikzcd} \tag{1.9}$$

1.3 A braided category

The function $V \otimes^{op} W$ is defined in terms of $\otimes$ by $V \otimes^{op} W = W \otimes V$.

Definition 21 *[31] A* ***braided monoidal*** *or quasitensor* ***category*** $(C, \otimes, \Psi)$ *is a* *monoidal category $(C, \otimes)$ which is commutative in the sense that is a natural equivalence between the two functors $\otimes, \otimes^{op} : C \times C \to C$, i.e. there are given functorial isomorphisms*

$$\Psi_{V,W} : V \otimes W \to W \otimes V, \forall V, W \in C,$$

obeying the hexagon conditions in the following diagram :

$$\begin{tikzcd}
 & V \otimes (W \otimes Z) \arrow[dl, "id\otimes\Psi"'] \arrow[dr, "\Phi^{-1}"] & \\
V \otimes (Z \otimes W) \arrow[d, "\Phi^{-1}"'] & & (V \otimes W) \otimes Z \arrow[d, "\Psi"] \\
(V \otimes Z) \otimes W \arrow[dr, "\Psi\otimes id"'] & & Z \otimes (V \otimes W) \arrow[dl, "\Phi^{-1}"] \\
 & (Z \otimes V) \otimes W &
\end{tikzcd} \tag{1.10}$$

$$\begin{tikzcd}
 & (V \otimes W) \otimes Z \arrow[dl, "\Phi"'] \arrow[dr, "\Psi\otimes id"] & \\
V \otimes (W \otimes Z) \arrow[d, "\Psi"'] & & (W \otimes V) \otimes Z \arrow[d, "\Phi"] \\
(W \otimes Z) \otimes V \arrow[dr, "\Phi"'] & & W \otimes (V \otimes Z) \arrow[dl, "id\otimes\Psi"] \\
 & W \otimes (Z \otimes V) &
\end{tikzcd} \tag{1.11}$$

If we omit Φ (we have already stated that we are only interested in the trivially associative case)

$$\Psi_{V\otimes W,Z} = (\Psi_{V,Z} \otimes id)(id \otimes \Psi_{W,Z}),\ \Psi_{V,W\otimes,Z} = (id \otimes \Psi_{V,Z})(\Psi_{V,W} \otimes id)$$

It is not necessarily true that. $\Psi = \Psi^{-1}$, but if it does we call the monoidal category symmetric.

Simple example: vector spaces form a braided category in which the braiding is just transposition, i.e.

$$\Psi(V \otimes W) = W \otimes V.$$

There is a diagrammatic notation often used for monoidal category, for example see [31] ,[24]. The tensor product is written by placing names next to each other, and the identity map by unbroken lines. So

$$\begin{tikzcd} V \arrow[d] & W \arrow[d] \\ V & W \end{tikzcd} \tag{1.12}$$

denotes the identity from $V \otimes W$ to $V \otimes W$.

If we have a map $T : V \to U$, then $T \otimes id : V \otimes W \to U \otimes W$ would be writing

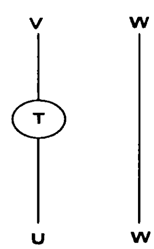

Figure 1.1

Now for a pair (V, W) we can denote $\Psi_{V,W}$ and its inverse $\Psi^{-1}_{W,V}$ respectively by figure 1.2.

One of the hexagon conditions can be represented by the following diagram (see figure 1.3)

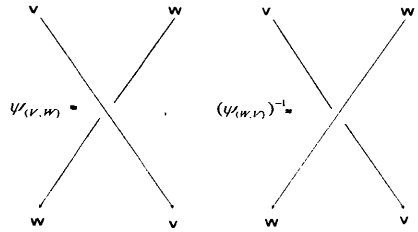

Figure 1.2

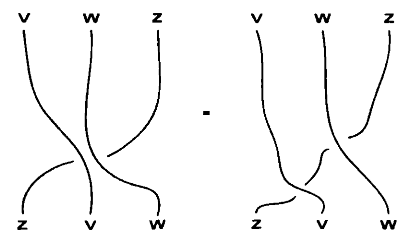

Figure 1.3

As a consequence we have the braid relation, represented by the following diagram

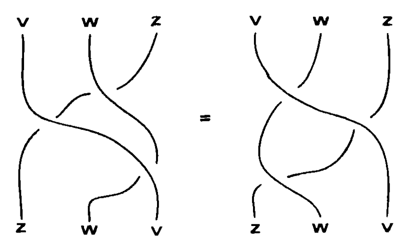

Figure 1.4

1.3.1 Vector bundles

We begin by a general definition. A submersion $f : E \to B$ is a differentiable map so that, for all $e \in E$, and all vector x at e, the set $f'(e; x)$ at $f(x)$ span all of the tangent space at $f(e)$.

Definition 22 *[27] Let B be a smooth manifold. A manifold E together with a smooth submersion $\pi : E \to B$, onto B, is called a **vector bundle of rank** k over B if the following holds:*
1) there is a k-dimensional vector space V , called typical fibre of E, such that for any point $p \in B$ the fibre $E_p = \pi^{-1}(p)$ of π over p is a vector space isomorphic to V
2)any point $p \in B$ has a neighbourhood U, such that there is a diffeomorphism

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\Phi_U"] \arrow[dr, "\pi"'] & & U \times V \arrow[dl, "pr_1"] \\ & U & \end{tikzcd} \tag{1.13}$$

and the diagram commutes, which means that every fibre E_p is mapped to $p \times V$. Φ_U is called a local trivialization of E over U and U is a trivializing neighbourhood for E.
3) $\Phi_U|_{E_p} : E_p \to V$ is an isomorphism of vector spaces.
B is called the base and E the total space of this vector bundle. $\pi : E \to B$ is a real or complex vector bundle corresponding to the typical fibre being a real or complex vector space.

Example 23 *[22] (1) The product or trivial bundle $E = B \times \mathbb{R}^n$ with p the projection onto the first factor.*
(2) The tangent bundle of the unit sphere S^n in $\mathbb{R}^{n+1}$, a vector bundle $p : E \to S^n$ where $E = \{(x, v) \in S^n \times \mathbb{R}^{n+1} | x \perp v\}$ and we think of v as a tangent vector to S^n by translating it so that its tail is at the head of x, on S^n. The map $p : E \to S^n$ sends (x, v) to x.

Definition 24 *[27] Any smooth map $s : B \to E$ such that $\pi \circ s = id_B$ is called a* *section of E. If s is only defined over a neighbourhood in B it is called a local section*

We denote the sections of the vector bundle $\pi : E \to B$ by ΓE, then ΓE is a bimodule over $C(B)$, the continuous functors on B, given by the

$$(f.s)(b) = f(b)s(b)$$

for $f \in C(B)$ and $s \in \Gamma E$. (In the commutative case we can set $f.s = s.f$, but in the noncommutative case we will need separate left and right actions.)
The K-theory of a topological space X is formed by taking an abelian group $K_0(X)$ generated from the vector bundles on X, up to equivalence [8], with group operation direct sum.
An element of $K_0(X)$ is given by $E - F$, where E and F are vector bundles, and $E - F$ is the "formal" difference of bundles. This is done so that we have inverses for the abelian group.

1.3.2 Tensor product of vector bundles

Let V and W be any two vector spaces over a field $\mathbb{K}$.Then $V \otimes W$ is the space of objects

$$v_1 \otimes w_1 + v_2 \otimes w_2 + \dots\dots\dots + v_k \otimes w_k$$

where

$$v_i \in V,\ w_i \in W$$

and with the following bilinear relations

$$a_1(v_1 \otimes w) + a_2(v_2 \otimes w) = (a_1 v_1 + a_2 v_2) \otimes w$$

and

$$a_1(v \otimes w_1) + a_2(v \otimes w_2) = v \otimes (a_1 w_1 + a_2 w_2)$$

where $a_i \in \mathbb{K}$, $v, v_i \in V$ and $w, w_i \in W$ [?].

If we have two vector bundles over X, their tensor product is given locally by (for an open subset of X, U and V, W are vector spaces)

$$(U \times V) \otimes (U \times W) = U \times (V \otimes W),$$

we have the picture in figure 1.5

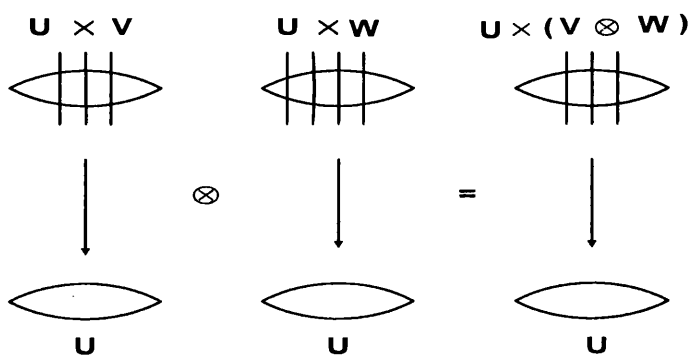

Figure 1.5

so if $s : U \to V$ and $t : U \to W$ give sections of the two bundles, then

$$(s \otimes t)(u) = s(u) \otimes t(u)$$

gives a section of the tensor product bundle.

If $f : U \to \mathbb{R}$ is any function, then

$$\begin{aligned} ((s.f)\otimes t)(u) &= s(u)f(u)\otimes t(u) \\ &= s(u)\otimes f(u)t(u) \\ &= (s\otimes f.t)(u), \end{aligned}$$

so

$$s.f\otimes t = s\otimes f.t \tag{1.14}$$

for any real function f.

1.3.3 Tensor product of modules

The tensor product $\otimes_A$ over an algebra A is defined as follows. If R is a right A-module, and L is a lift A-module, then $R\otimes_A L$ is the usual vector space. Tensor product $R\otimes L$, with the additional relation

$$r \triangleleft a\otimes l = r\otimes a \triangleright l$$

For a vector bundle E over X, we have seen that the sections ΓE of E is a $C(X)$ module.

Then we want $\Gamma(E\otimes F)$ to be given in terms of $\Gamma(E)$ and $\Gamma(F)$. But $\Gamma(E)\otimes\Gamma(F)$ is too big. We use equation 1.14 to see that we should have

$$\Gamma(E)\otimes_{C(X)}\Gamma(F).$$

This is designed to follow the equation 1.14, i.e. so that for vector bundles E and F over X

$$\Gamma(E \otimes F) = \Gamma(E) \otimes_{C(X)} \Gamma(F)$$

If R and L are bimodules, then $R \otimes_A L$ is also a bimodule, with

$$a \triangleright (r \otimes l) = (a \triangleright r) \otimes l$$

$$(r \otimes l) \triangleleft a = r \otimes (l \triangleleft a).$$

1.4 The Hopf fibration

The group SU_2 acts on $\mathbb{C}^2$ by matrix multiplication :

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} au + bv \\ cu + dv \end{pmatrix} \tag{1.15}$$

If we consider non zero vectors in $\mathbb{C}^2$, there is a map to the Riemann sphere $\mathbb{C}_\infty = \mathbb{C}U\{\infty\}$ given by

$$\begin{pmatrix} u \\ v \end{pmatrix} \longmapsto \frac{u}{v}$$

Where if $z \neq 0$ we get $\frac{w}{z} \in \mathbb{C}$, and if $z = 0$ we set $\frac{w}{z} = \infty$ (see figure 1.6).

This is the same as the construction of the projective space $\mathbb{P}^1\mathbb{C}$. Then SU_2 acts on projective space using 1.15 by putting $u = z$ and $v = 1$ to get $z \in \mathbb{C}_\infty$ mapping to

$$z \longmapsto \frac{au + bv}{cu + dv} = \frac{az + b}{cz + d}.$$

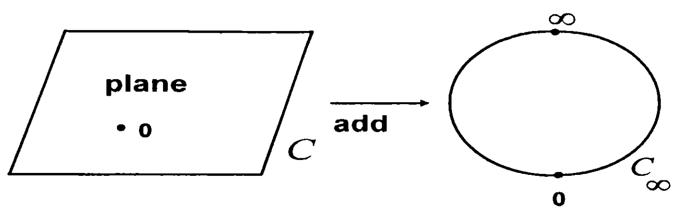

Figure 1.6

This is a Möbius transformation.

Now consider all the matrices $A \in SU_2$ for which $A(0) = 0 \in \mathbb{C}_\infty$. (the stabiliser of 0)

$$0 \longmapsto \frac{0a+b}{0c+d} = \frac{b}{d}$$

then we have $b = 0$, so

$$A = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$$

But $A \in SU_2$, so $AA^* = I_2$

$$\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}\begin{pmatrix} a^* & c^* \\ 0 & d^* \end{pmatrix} = \begin{pmatrix} aa^* & ac^* \\ ca^* & cc^* + dd^* \end{pmatrix}$$

so, $aa^* = 1$, and $c = 0$, and $dd^* = 1$ since $detA = ad = 1$, $d = \frac{1}{a}$.

So the subgroup of points which fix $0 \in \mathbb{C}_\infty$ is

$$T = \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix} \quad where \quad |a| = 1.$$

We have a fibration $SU_2 \longrightarrow \mathbb{C}_\infty$ sending A to $A(0)$ with fiber T. This is the Hopf fibration.

Note that topologically T is a circle S^1, SU_2 is S^3 and $\mathbb{C}_\infty$ is S^2, so we get a fibration $S^3 \longrightarrow S^2$, with fiber S^1.

1.5 Fiber bundles

Definition 25 *[27] A fibre bundle is a collection (E, B, F, π), where E, B, F are topological spaces and $\pi : E \to B$ is continuous surjection. E called total space, B called base space and F is fibre and π is the projection map (or bundle projection). The fibers of the map are the part of $\mathbb{R}^2$ which are mapped to the same point of $\mathbb{R}$.*

Example 26 *We can have $E = \mathbb{R}^2$ and $B = \mathbb{R}$, with $\pi(x, y) = x$. Then the fiber at a point $x \in \mathbb{R}$ is $\{(x, y)|y \in \mathbb{R}\}$, so $F = \mathbb{R}$. See figure 1.7.*

Example 27 *$S^1 \times S^1 \longmapsto S^1$ given by $(x, y) \longmapsto x$. Note that $S^1 \times S^1$ is torus.*

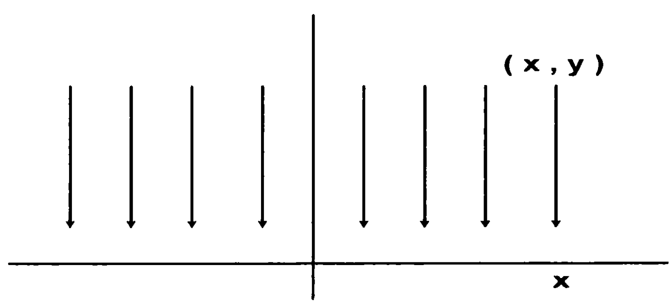

Figure 1.7

These example 26 and 27 are both trivial fiber bundles, where $E = B \times F$. We will now give some non trivial examples.

Example 28 *The Möbius bundle.*

Take a strip of paper, and glue the ends together two different ways, see figuer 1.8.

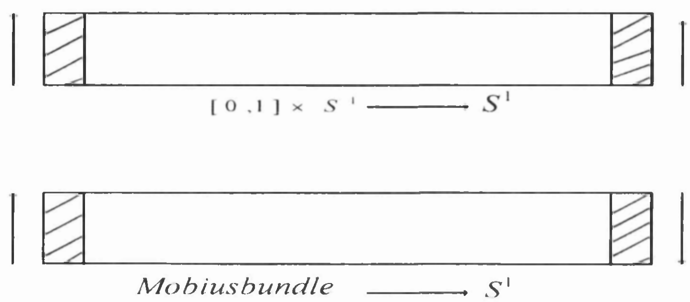

Figure 1.8

Now map the resulting spaces to the S^1 factor. The first glueing gives the trivial bundle, $[0,1] \times S^1 \longrightarrow S^1$ (see figure 1.9), The second gives a non-trivial bundle, the

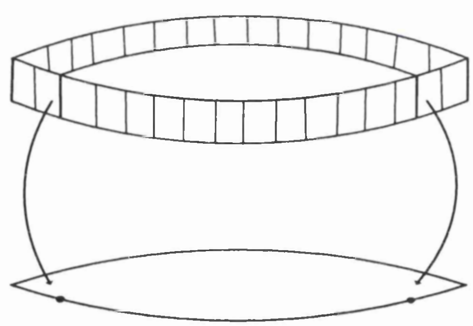

Figure 1.9

Mobius bundle, which is pictured in figure1.10

Example 29 *The Hopf fibration. This is a bundle $S^3 \to S^2$, with fiber S^1.*

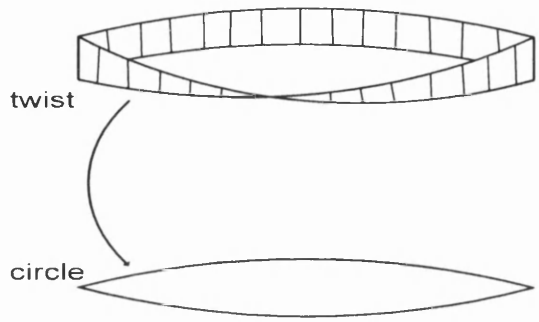

Figure 1.10

For these last two examples, the fibration is locally trivial, i.e. $B = U \cup V$ a union of two open sets. (in general we can have more than two), where the part of E mapping to U is of the form $U \times F$, and the part of E mapping to V is of the form $V \times F$. Figure 1.11 shows the two subsets which can be glued together to from either $[0, 1] \times S^1]$ on the Möbius band.

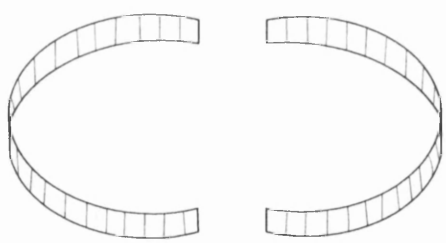

Figure 1.11

1.6 Exact sequences and flat modules

The maps of vector spaces

$$V \xrightarrow{S} W \xrightarrow{T} U$$

is "exact at W " if the kernel of $T : W \longrightarrow U$ is equal to the image of $S : V \longrightarrow W$. For example

$$\mathbb{R}^2 \xrightarrow{S} \mathbb{R}^3 \xrightarrow{T} \mathbb{R}$$

given by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x - y$$

$$S\begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} x \\ x \\ z \end{pmatrix}$$

is exact.

A sequence of linear maps

$$V_1 \xrightarrow{T_1} V_2 \xrightarrow{T_2} V_3 \ldots\ldots V_{n-1} \xrightarrow{T_{n-1}} V_n$$

is exact if it is exact at every entry with an incoming and an outcoming arrow i.e. exact at V_2 and V_3 and . . . . and V_{n-1}.

Note this sequnce may be infinite "A short exact sequence" is one of the form

$$0 \longrightarrow V_1 \xrightarrow{T_1} V_2 \xrightarrow{T_2} V_3 \longrightarrow 0$$

More general, we can make exactly the same definition for modules over an algebra A, and A-module maps.

We have the following definition (modified to include left and right) see [6] [12] [25].

Definition 30 *A right A-module E is flat if every short exact sequence of left A-modules*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

gives an exact sequence

$$0 \longrightarrow E \otimes_A L \longrightarrow E \otimes_A M \longrightarrow E \otimes_A N \longrightarrow 0.$$

Likewise, if E is a left A-module, it is called flat if for every short exact sequnce of right A-modules

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow\longrightarrow 0$$

we have a short exact sequence

$$0 \longrightarrow L \otimes_A E \longrightarrow M \otimes_A E \longrightarrow N \otimes_A E \longrightarrow 0.$$

Lemma 31 *Give two exact sequences*

$$0 \longrightarrow U \xrightarrow{t} V \xrightarrow{f} W \longrightarrow 0$$

$$0 \longrightarrow U \xrightarrow{t} V \xrightarrow{g} X \longrightarrow 0$$

There is an isomorphism $h : W \longrightarrow X$ given by $h(w) = g(v)$, where $f(v) = w$.

Proof: *We need to check that two choices $v, v' \in V$ with $f(v) = f(v') = w$ give $g(v) = g(v')$. If $f(v) = f(v')$, then $v - v' \in kerf = im\ t$. But $ker\ g = im\ t$ also, so $g(v) = g(v')$.*

To see that h is 1-1 (injective), suppose $h(w) = 0$. Then $g(v) = 0$, so $v \in kerg = kerf$, so $f(v) = 0 = w$, so $w = 0$.

To see that h is onto, for every $x \in X$ there is a $v \in V$ with $g(v) = x$. Now set $w = f(v)$, and $h(w) = x$. Then h is an isomorphism. $\square$

1.7 Finitely generated projective modules

For a left A module E, define $E^\circ =_A Hom(E, A)$ i.e. the left module maps from E to A. If E is also a bimodule, then E° is a bimodule, with the following actions of A ($a \in A, \quad e \in E, \quad \alpha \in E^\circ$)

$$(a.\alpha)(e) = \alpha(e.a),$$

$$(\alpha.a)(e) = \alpha(e).a$$

Definition 32 *E is finitely generated projective as a left A module if there are $e^1 \cdots e^n \in E$ and $e_1 \cdots e_n \in E^\circ$ (this is called a dual basis), so that for all $e \in E$,*

$$e = \sum_i e_i(e).e^i.$$

If E is also a bimodule, we can write the evaluation and coevaluation maps

$$ev : E \otimes_A E^\circ \longrightarrow A, \qquad e \otimes \alpha \longmapsto \alpha(e)$$

$$coev : A \longrightarrow E^\circ \otimes_A E, \qquad a \longmapsto a.e_i \otimes e^i$$

Proposition 33 *The matrix $P_{ij} = ev(e^i \otimes e_j) = e_j(e^i)$ obeys $P^2 = P$ (it is an* *idempotent).*

Proof:

$$\sum_j P_{ij}P_{jk} = \sum_j e_j(e^i)e_k(e^j)$$

using the fact that e_k *is a left module map* $[a.\alpha(e) = \alpha(a.e)]$, *this is*

$$\sum_j P_{ij}P_{jk} = \sum_j e_k(e_j(e^i)e^j).$$

and by the dual basis property,

$$\sum_j P_{ij}P_{jk} = e_k(e^i) = P_{ik}.$$

Chapter 2

Differential calculi and covariant derivatives

Begining with the work by Connes [13] and Woronwicz [46], there has been considerable interest in applying the methods of differential geometry to algebras. There has also been interest from the point of view of mathematical physics[34], as differential geometry is used to describe space - time, but quantum theory seems to force non commutativity on space time, at least at very small distances (the "Planck length").

2.1 Differential calculi on algebras

First consider the differential calculis on $\mathbb{R}^n$. Given coordinate functions $x_1, \ldots\ldots, x_n$, we have 1-forms $\sum f_i dx_i$, where $f_i : \mathbb{R}^n \to \mathbb{R}$ is a smooth function.
The smooth functions are those which can be differentiated arbitrarily many times . For example $f(x) = x|x|$ from $\mathbb{R}$ to $\mathbb{R}$ can be differentiated once, but not twice, so it is not smooth.

A 2-form is of type $\sum f_{ij} dx_i \wedge dx_j$, where f_{ij} is a smooth function, etc.

The n forms Ω^n form a differential graded algebra, i.e. we have

$$\wedge : \Omega^n \otimes \Omega^m \longrightarrow \Omega^{n+m}$$

$$d : \Omega^n \longrightarrow \Omega^{n+1}$$

with the following properties.

1) For $\xi \in \Omega^r$ and $\eta \in \Omega^m$, then $\xi \wedge \eta = (-1)^{rm} \eta \wedge \xi$ (graded commutativity)

2)$\Omega^0 = C^\infty(\mathbb{R}^n)$.

3)$d(\xi \wedge \eta) = d\xi \wedge \eta + (-1)^r \xi \wedge d\eta$ (signed derivation property) .

4) $d^2 = 0$ 5) For $\mathbb{R}^n$ we define $\wedge$ to be bilinear and d to be linear, and

$$\begin{aligned}(f\, dx_{i_1} \wedge ..... \wedge dx_{i_r}) \wedge (g\, dx_{j_1} \wedge ...... \wedge dx_{j_m}) &= f\, g\, dx_{i_1} \wedge .... \wedge dx_{i_r} \wedge dx_{j_1} \wedge ...... \wedge dx_{j_m} \\ d\,(f\, dx_{i_1} \wedge .... \wedge dx_{i_r}) &= \frac{\partial f}{\partial x_k} dx_k \wedge dx_{i_1} \wedge ..... \wedge dx_{i_r}.\end{aligned}$$

If we replace $C^\infty(\mathbb{R}^n)$, the smooth functions on $\mathbb{R}^n$, by a noncommutative algebra A, we slightly modify the definition of $\Omega^n A$.

Definition 34 *A differential graded algebra consists vector spaces $\Omega^n A$ with operators $\wedge$ and d so that*

1) $\wedge : \Omega^r A \otimes \Omega^m A \longrightarrow \Omega^{r+m} A$ is associative (we do not assume any graded commutative property)

2) $\Omega^0 A = A$ (this is really just notation)

3) $d : \Omega^n A \rightarrow \Omega^{n+1} A$ with $d^2 = 0$

4) $d(\xi \wedge \eta) = d\xi \wedge \eta + (-1)^r \xi \wedge d\eta$ for $\xi \in \Omega^r A$

5) $\Omega^1 A \wedge \Omega^n A = \Omega^{n+1} A$.

6) $A.dA = \Omega^1 A$

Note that many differential graded algebras do not obey (5), but those in classical differential geometry do, and it will be true in all our examples. Also aspecial case of $\wedge : \Omega^0 A \otimes \Omega^n A = A \otimes \Omega^n A \to \Omega^n A$

If A is a star algebra, we can suppose that the star operation extends to $\Omega^n A$, so that

$$(a.db)^* = db^*.a^*$$

$$(d\xi)^* = d(\xi^*)$$

$$(\xi \wedge \eta)^* = (-1)^{|\xi||\eta|}\eta^* \wedge \xi^*$$

Note: We will often use $|\xi|$ for the degree of ξ, if $\xi \in \Omega^n A$, then $|\xi| = n$.

2.2 Differential calculi on Hopf algebras

The Hopf algebra H has a left coaction on itself by $\Delta : H \longrightarrow H \otimes H$. A 1-form in $\Omega^1 H$ should be written as a sum of $h.dg$ for $h, g \in H$. We would like this to give a left coaction of H on $\Omega^1 H$ by

$$h.dg \longmapsto h_{(1)}g_{(1)} \otimes h_{(2)}.dg_{(2)} \tag{2.1}$$

However there are, in general, relations between the $h.dg$ s, i.e. sums of these which give zero. It is necessary that these sums get sent to zero under $\otimes$, and this is a non-trivial condition. We shall assume that this left coaction is well defined, i.e. that we have a left covariant calculus. In this case, we can look at the left invariant 1-forms $L^1 H$, ie for those $\eta \in \Omega^1 H$ for which

$$\eta \longmapsto 1_H \otimes \eta.$$

We use $\xi \longmapsto \xi_{[-1]} \otimes \xi_{[0]}$ for the left coaction

There are lots of forms in L^1H, as for any $\xi \in \Omega^1(H)$ we have

$$S(\xi_{[-1]})\xi_{[0]} \in L^1H.$$

to see this, we do the following calculation : Applying the left coaction to $S(\xi_{[-1]})\xi_{[0]}$ gives

$$S(\xi_{[-1]})_{(1)}\xi_{[0][-1]} \otimes S(\xi_{[-1]})_{(2)}\xi_{[0][0]} \tag{2.2}$$

as S reverses the coproduct, this gives

$$S(\xi_{[-1](2)})\xi_{[0][-1]} \otimes S(\xi_{[-1](1)})\xi_{[0][0]}$$

as we have a left coaction, we write $P = Q$ in figure 2.1, or

$$\xi_{[-1](1)} \otimes \xi_{[-1](2)} \otimes \xi_{[0][-1]} \otimes \xi_{[0][0]} = \xi_{[-1](1)} \otimes \xi_{[-1](2)(1)} \otimes \xi_{[-1](2)(2)} \otimes \xi_{[0]} \tag{2.3}$$

Then, applying S to the second term of 2.3 and multiplying the second and third terms together,

$$\xi_{[-1](1)} \otimes S(\xi_{[-1](2)})\xi_{[0][-1]} \otimes \xi_{[0][0]} = \xi_{[-1](1)} \otimes S(\xi_{[-1](2)(1)})\xi_{[-1](2)(2)} \otimes \xi_{[0]}$$

but as $S(h_{(1)})h_{(2)} = \epsilon(h).1_H$, we get

$$\xi_{[-1](1)} \otimes S(\xi_{[-1](2)})\xi_{[0][-1]} \otimes \xi_{[0][0]} = \xi_{[-1](1)} \otimes \epsilon(\xi_{[-1](2)})1_H \otimes \xi_{[0]}.$$

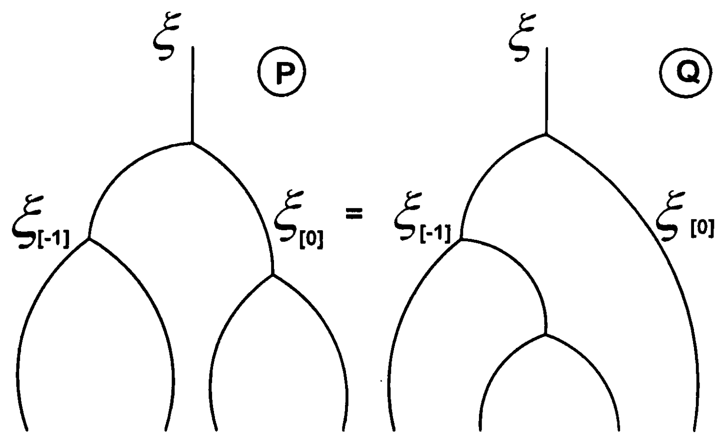

Figure 2.1

But now, using $h_{(1)}\epsilon(h_{(2)}) = h$, we get

$$\xi_{[-1](1)} \otimes S(\xi_{[-1](2)})\xi_{[0][-1]} \otimes \xi_{[0][0]} = \xi_{[-1]} \otimes 1_H \otimes \xi_{[0]}.$$

Now 2.2 gives $1_H \otimes S(\xi_{[-1]})\xi_{[0]}$ as required.

Note that $\xi_{[-2]}S(\xi_{[-1]})\xi_{[0]} = \xi$, as we have

$$\begin{aligned}
\xi_{[-2]}S(\xi_{[-1]})\xi_{[0]} &= \xi_{[-1](1)}S(\xi_{[-1](2))}\xi_{[0]} \\
&= \epsilon(\xi_{[-1]})\xi_{[0]} \\
&= \xi
\end{aligned}$$

and also $\xi_{[-2]}S(\xi_{[-1]})\xi_{[0]} = \xi_{[-1]}S(\xi_{[0][-1]})\xi_{[0][0]} \in H.L^1H$ as $S(\xi_{[0][-1]})\xi_{[0][0]} \in L^1H$. This proves the following proposition :

Proposition 35 $\Omega^1 H = H.L^1H$.

If we take L^1H to be the left invariant 1-form on the Hopf algebra H, we can give a right coaction by

$$\xi \triangleleft h = S(h_{(1)}).\xi.h_{(2)}$$

If the differential calculus is also right covariant, this right action and coaction make a Yetter-Drinfeld module. There is map $\varpi : H \longrightarrow L^1H$ given by

$$\varpi(h) = S(h_{(1)})dh_{(2)}. \tag{2.4}$$

By right covariant, we mean that

$$h.dg \longmapsto h_{(1)}.dg_{(1)} \otimes h_{(2)}g_{(2)}$$

extends to a well defined coaction on Ω^1H. In Woronowicz's paper on differential calculus on Hopf algebra [46] proposition(3.1) defines a braiding (we use Ψ instead of σ) with $\Psi(\omega \otimes \eta) = \eta \otimes \omega$, if η is right invariant and ω is left invariant
This is just the braiding in proposition 13 .
Woronowicz then defines the symmetric tensor product $S^2 = ker(id - \Psi) : \Omega^1 \otimes \Omega^1 \to \Omega^1 \otimes \Omega^1$ and the 2-forms as

$$\Omega^2 H = \frac{\Omega^1 H \otimes_A \Omega^1 H}{S^2}$$

2.3 Example: The function algebra of a finite group

To represent G, a finite group, we need to use the Hopf algebra $C(G)$, using δ_x as a basis for $x \in G$ (the function taking value 1 at x and zero elsewhere). This has the following operations which make it into a Hopf algebra.

$$\delta_x.\delta_y = \delta_{x,y}.\delta_x, \quad \Delta_x = \sum_{y,z\in G:yz=x} \delta_y \otimes \delta_z, \quad 1 = \sum_{x\in G} \delta_x, \quad \epsilon(\delta_x) = \delta_{x,e}, \quad S(\delta_x) = \delta_{x^{-1}}.$$

Here $e \in G$ represents the identity element, and $\delta_{x,y}$ represents the Kroneker delta. For $f \in C(G)$, it will be convenient to define the right translation $R_g(f) \in C(G)$ by $R_g(f)(x) = f(xg)$ so that $R_g(\delta_x) = \delta_{xg^{-1}}$. The star operation on $C(G)$ is given by

$$\star\delta_x = \overline{\delta_x}$$

We give $C(G)$ a differential calculus as the following [34]: Here C is a subset of G, but does not include the identity. Then take the left invariant 1-forms to have basis ξ^c for $c \in C$. The bimodule commutation relations and the exterior derivative are

$$\xi^c.f = (R_cf).\xi^c, \quad df = \sum_{c\in C}(R_cf - f).\xi^c \tag{2.5}$$

We can invert this to give

$$\xi^c = \sum_{u\in G} \delta_{uc^{-1}}.d\delta_u. \tag{2.6}$$

The calculus is bicovariant only when C is Ad-stable (i.e. $g \in C \Rightarrow xgx^{-1} \in C$ for all $x \in G$). The right action and the right coaction induced braiding(in the bicovariant case) (see section 2.2) are given by

$$\xi^a \triangleleft \delta_g = \delta_{a,g}\xi^a \quad \Delta_R\xi^c = \sum_{y\in G} \xi^{ycy^{-1}} \otimes \delta_y\,, \quad \Psi(\xi^a \otimes \xi^b) = \xi^{aba^{-1}} \otimes \xi^a \tag{2.7}$$

We also have see (2.4)

$$\varpi(\delta_g) = \sum_{xy=g} S(\delta_x)d(\delta_y) = \sum_{c\in C}(\delta_{g,c} - \delta_{g,e}).\xi^x$$

Thus ϖ has kernel with basis consisting of the elements $\sum_{c\in C}\delta_c + \delta_e$ and δ_g for $g \in G \setminus (C \cup \{e\})$.

If C is closed under inverse, we define $\xi^{a*} = -\xi^{a^{-1}}$. Then we have $(df)^* = df^*$, as

$$(d\delta_x)^* = \sum_{c\in C}((\delta_{xc^{-1}} - \delta_x).\xi^c)^* = -\sum_{c\in C}\xi^{c^{-1}}.(\delta_{xc^{-1}} - \delta_x) = \sum_{c\in C}(\delta_{xc} - \delta_x).\xi^{c^{-1}} = d\delta_x = d\delta_x^*.$$

A basis of $\Lambda^1 C(G))^\circ$ is given by ξ_c for $c \in C$, where we define $ev(\xi^a \otimes \xi_c) = \delta_{c,a}$. The action and coaction are represented by standard results on the dual Yetter-Drinfeld modules as

$$\xi_a \triangleleft \delta_b = \delta_{a.b^{-1}}\xi_a, \quad \Delta_R(\xi_a) = \sum_g \xi_{gag^{-1}} \otimes \delta_g.$$

Further were the calculus is inner when $\theta = \sum_{a\in C}\xi^a$ in the sense that d is given by a graded commutator $d = [\theta, -]$. Then the exterior derivative on 1-forms is given by

$$d\xi^c = \sum_{b,a\in C}(\xi^a \wedge \xi^c + \xi^c \wedge \xi^a) - \sum_{b,a\in C}\delta_{c,ab}\xi^a \wedge \xi^b. \tag{2.8}$$

2.4 Left covariant derivative

Historically covariant derivatives arose from trying to differential vector fields in differential geometry. E.g. suppose that we have the vector field of wind velocity on the earth. How to differentiate $V(X)$ in a direction at x? If we take a coordinate patch, a subset of $\mathbb{R}^n$, then we could take the partial derivative with respect to these

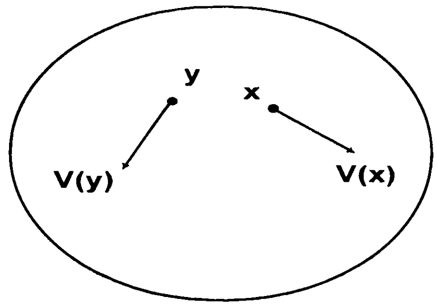

Figure 2.2

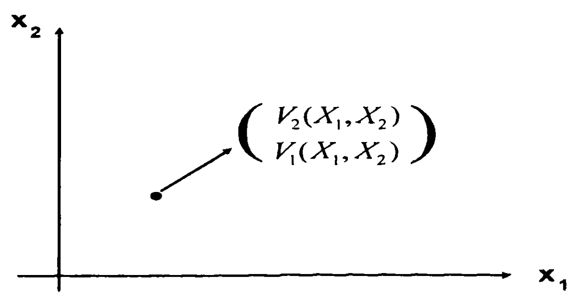

Figure 2.3

coordinates (as figure 2.2 and 2.3) to get

$$\frac{\partial v_i}{\partial x_j}. \tag{2.9}$$

However this is not well behaved on changing to different coordinates! The fix was to add Christoffel symbols to the formula 2.9, giving the idea of covariant derivative. We write $\nabla_W V$ to be the derivative of V in the direction of the vector W.

Since we are dealing mostly with forms rather than vector fields, it will be convenient to write the covariant derivative using forms. We write

$$\nabla_W V = (W \otimes id)\nabla V$$

where $\nabla V = \xi \otimes K \in \Omega X^1 \otimes_{C^\infty(X)}$ "Vector fields" and

$$(W \otimes id)(\xi \otimes K) = W(\xi).K.$$

Here we use the fact that vector fields are dual to forms to define $W(\xi)$.
Using coordinates $\nabla e = dx^i \otimes \nabla_i e$ where $\nabla_i e$ in the covariant derivative in the x^i direction.
In more generality we get the following definition for a left A-module E.

Definition 36 *[4] Given a left A-module E , a left A-covariant derivative is a map $\nabla : E \to \Omega^1 A \otimes_A E$ which obeys the condition $\nabla(a.e) = \mathrm{d}a \otimes e + a.\nabla e$ for all $e \in E$ and $a \in A$*

This is called the left Liebnitz rule.

Definition 37 *[4] The torsion of a left A-covariant derivative ∇ on $\Omega^1 A$ is the left A-module map $Tor = \wedge\nabla - d : \Omega^1 A \to \Omega^2 A$.*
That it is a left module map follows easily from the definition of a covariant derivative

$$Tor(a.\xi) = \wedge(a.\nabla\xi) + \wedge(da \otimes \xi) - a.d\xi + da \wedge \xi = a.Tor(\xi)$$

for all $\xi \in \Omega^1 A$ and $a \in A$.

2.5 The curvature

Curvature measure how "bent" the surface or manifold is purely by making measurements within it.

The standard measure of curvature. This can be given in terms of forms by the following calculation. Let A be an algebra and E be an A-module. A covariant derivative on E is

$$\nabla : E \longrightarrow \Omega^1 A \otimes_A E$$

obeying $\nabla(a.e) = da \otimes e + a.\nabla e$ for all $a \in A$ and $e \in E$.

Define the curvature

$$R : E \longrightarrow \Omega^2 A \otimes_A E$$

by using

$$\nabla^{[1]} : \Omega^1 A \otimes_A E \longrightarrow \Omega^2 A \otimes_A E$$

$$\nabla^{[1]}(\xi \otimes e) = d\xi \otimes e - \xi \wedge \nabla e.$$

Proposition 38 *[4] The curvature of a left A-covariant derivative ∇ is defined by*

$$R = (d \otimes id - id \wedge \nabla)\nabla : E \longrightarrow \Omega^2 A \otimes_A E$$

and is a left A-module map .

proof: To proof that this is well defined on $\xi.a \otimes e = \xi \otimes a.e$,

$$\begin{aligned}
\nabla^{[1]}(\xi.a \otimes e) &= d(\xi.a) \otimes e - \xi \wedge a\nabla e \\
&= d\xi.a \otimes e - \xi \wedge da \otimes e - \xi \wedge a\nabla e \\
&= d\xi \otimes a.e - \xi \wedge (da \otimes e + a.\nabla e) \\
&= d\xi \otimes a.e - \xi \wedge \nabla(a.e) \\
&= \nabla^{[1]}(\xi \otimes a.e).
\end{aligned}$$

Thus $R = \nabla^{[1]}\nabla : E \longrightarrow \Omega^2 A \otimes_A E.$ is a left module map.

2.6 Bimodule covariant derivative

In classical differential geometry much use is mode of tensoring bundles together. To have a working ideal of covariant derivative, we use the following idea, which was introduced by [16][17][20][30][37]

Definition 39 *A bimodule covariant derivative on an A-bimodule E is a triple (E, ∇, σ), where $\nabla : E \to \Omega^1 A \otimes_A E$ is a left A-covariant derivative, and $\sigma : E \otimes_A \Omega^1 A \to \Omega^1 A \otimes_A E$ is a bimodule map obeying*

$$\nabla(e.a) = \nabla(e).a + \sigma(e \otimes \mathrm{d}a), \quad \forall e \in E, a \in A$$

The reason for making this definition that we can apply ∇ to tensor products .

Proposition 40 *[10][18] Given (E, ∇_E, σ_E) a bimodule covariant derivative on the bimodule E and ∇_F a left covariant derivative on the left module F, there is a left*

A-covariant derivative on $E \otimes_A F$ given by

$$\nabla_{E\otimes F} = \nabla_E \otimes id_F + (\sigma_E \otimes id_F)(id_E \otimes \nabla_F)$$

Further if F is also an A-bimodule with a bimodule covariant derivative (∇_F, σ_F), then there is a bimodule covariant derivative $(\nabla_{E\otimes_A F}, \sigma_{E\otimes_A F})$ on $E \otimes_A F$ with

$$\sigma_{E\otimes F} = (\sigma_E \otimes id)(id \otimes \sigma_F)$$

Proposition 41 *[4] The torsion of a bimodule covariant derivative $(\Omega^1 A, \nabla, \sigma)$ is a bimodule map if and only if*

$$image(id + \sigma) \subset Ker(\wedge : \Omega^1 A \otimes_A \Omega^1 A \to \Omega^2 A).$$

We say in this case that ∇ is torsion-compatible .

Definition 42 *[4] The category ${}_A\mathcal{E}_A$ consists of objects A-bimodule covariant derivatives (E, ∇, σ) where $\sigma : E \otimes_A \Omega^1 A \to \Omega^1 A \otimes_A E$ is invertible . The morphisms are bimodule maps $\theta : E \to F$ which are preserved by the covariant derivatives , i.e.*

$$\nabla \circ \theta = (id \otimes \theta)\nabla : E \to \Omega^1 A \otimes_A F.$$

Then proposition(40) makes ${}_A\mathcal{E}_A$ into a monoidal category . The identity for the tensor product is the bimodule A with $\nabla_A = d : A \to \Omega^1 A \otimes_A A = \Omega^1 A$,and σ_A the is identity map $A \otimes_A \Omega^1 A$ to $\Omega^1 A \otimes_A A$ where both sides are identified with $\Omega^1 A$

Theorem 43 *[4] Suppose that A is a star algebra which has a differential structure $(\Omega^1 A, d)$ so that $\Omega^1 A$ is a star object and $\star d = \overline{d}\star : A \to \overline{\Omega^1 A}$. Then ${}_A\mathcal{E}_A$ described* *in definition (42) is a bar category (see [3]) with* $(\overline{E, \nabla_E, \sigma_E}) = (\overline{E}, \nabla_{\overline{E}}, \sigma_{\overline{E}})$ *given by*

$$\nabla_{\overline{E}}(\overline{e}) = (\star^{-1} \otimes id)\Upsilon\overline{\sigma_E^{-1}\nabla_E(e)},$$

$$\sigma_{\overline{E}} = (\star^{-1} \otimes id)\Upsilon\overline{\sigma_E^{-1}}\Upsilon^{-1}(id \otimes \star).$$

If E itself has a star operation $* : E \to \overline{E}$, then the covariant derivative is said to be star compatible if (see [4])

$$\sigma_E = (*^{-1} \otimes *^{-1})\Upsilon\overline{\sigma_E^{-1}}\Upsilon^{-1}(* \otimes *).$$

2.7 Hermitian structures and Riemannian geometry

On the surface of the earth there is an idea of length. But this idea of length seems to depend on the coordinates taken. For example sailors measure position in latitude and longitude.

One degree of longitude at the equator has a greater length than near the poles. To allow for this variation in length with coordinates we define a Riemannian metric to be an inner product

$$\langle\, , \rangle : Tangent\, space \otimes Tangent\, space \longrightarrow \mathbb{R}$$

where the coordinate directions $X_a = \dfrac{\partial}{\partial x^a}$ have $\langle X_a, X_b \rangle = g_{ab}$ and g_{ab} is a function on the space.

1) This is non degenerate, i.e. for every $X \neq 0$ in the Tangent space there is a Y so that $\langle X, Y \rangle \neq 0$.

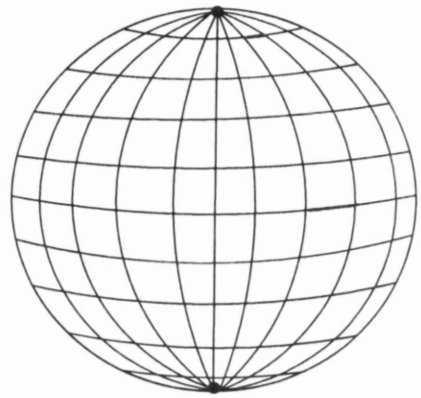

Figure 2.4

2) It is positive, for every $X \neq 0$ we have $\langle X, X\rangle \geq 0$.

3) It is symmetric, ie $\langle X, Y\rangle = \langle Y, X\rangle$.

We can use the nondegeneracy rule to get an inner product on the 1-forms, and this is what we will use in noncommutative geometry. As our forms will be complex valued, we replace the symmetry rule by

$$\langle \xi, \overline{\eta}\rangle = \langle \eta, \overline{\xi}\rangle^* .$$

Here $\overline{\xi} \in \overline{\Omega^1 A}$, see subsection 1.1.2.

Finally, the inner product of two 1-forms gives a function as we vary the point where the product takes place.

We get

$$\langle\, ,\, \rangle : \Omega^1 A \otimes_A \overline{\Omega^1 A} \longrightarrow A,$$

where A is the smooth functions on the space.

This is similar to the definition of a Hilbert C^* module [26], but we not require the completeness or the norms.

In general a Hermitian structure on a bimodule E will be taken to be

$$\langle\, , \rangle : E \otimes_A \overline{E} \longrightarrow A$$

with the properties

1) $\langle e, \overline{f} \rangle = \langle f, \overline{e} \rangle^*$.

2) $\langle\, , \rangle$ *is nondegenerate*$^\intercal$.

In the case where A is a c^* algebra, we normally have a positivity condition

3)$e \neq 0 \Longrightarrow \langle e, \overline{e} \rangle \geq 0$.

$\intercal$ The easiest way to explain non-degeneracy is to say that there is an invertible bimodule map $G : \overline{E} \longrightarrow E^\circ$, where $E^\circ = {}_A Hom(E, A)$, i.e. the left module maps from E to A, and that

$$\langle\, , \rangle = evaluation\,(id \otimes G) : E \otimes_A \overline{E} \longrightarrow A.$$

Here $\Theta \in {}_A Hom\ (E, F)$ (the left module maps) between E and F obey

$$\Theta(a \triangleright e) = a \triangleright \Theta(e)$$

A is a A-module by using multiplication, $a \triangleright b = ab$. The pairing between E° and E can be written

$$evaluation : E \otimes E^\circ \longrightarrow A \quad e \otimes \alpha = \alpha(e).$$

Note that we can correspondingly define $E' = Hom_A(E, A)$ (the right module maps from E to A) and in that case we get an evaluation

$$evaluation : E' \otimes E \longrightarrow A.$$

E° is always a right A-module, with

$$eval(e \otimes \alpha \triangleleft a) = eval(e \otimes \alpha).a$$

Also, if E is a bimodule, then E° also has a left A-action by

$$eval(e \otimes a \triangleright \alpha) = eval(e \triangleleft a \otimes \alpha)$$

and then we can write

$$eval : E \otimes_A E^\circ \longrightarrow A.$$

For finitely generated projective modules (see section 1.7) we also have

$$coevaluation : A \longrightarrow E^\circ \otimes_A E$$

and

$$coevaluation : A \longrightarrow E \otimes_A E'$$

The dual property can be written as a digram figure 2.5

Proposition 44 *[4] In the case of a finite group (see section 2.3), a left invariant Hermitian structure can be written as* $G : \overline{\Lambda^1 C(G)} \to (\Lambda^1 C(G))^\circ$ *given by* $G(\overline{\xi^a}) = \xi_b.g^{b,a}$ *, where* $g^{b,a} \in \mathbb{C}$ *. Then :*

1) If G *is a right module map , then* $g^{a,b} \in 0$ *only if* $a = b$ *, i.e. the metric is diagonal*

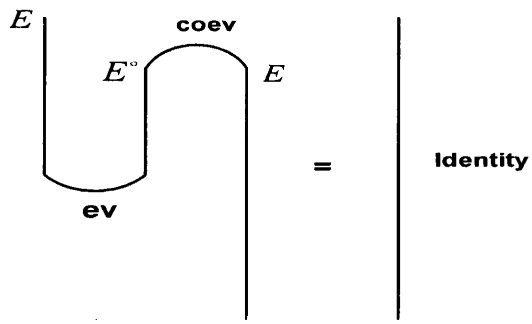

Figure 2.5

in our basis .

2) If G is a right comodule map , then for every $a \in C$ and $x \in G$, $g^{xax^{-1},xax^{-1}} = g^{a,a}$

.

Proof : For the complete proof of this proposition see [4]

Proposition 45 *[4] Suppose that E is finitely generated projective as a left module, with dual basis $e_i \otimes e^i \in E^\circ \otimes E$, and let G be a non-degenerate Hermitian structure on E. Suppose that we set $g^{ji} = \left\langle e^i, \overline{e^j} \right\rangle$, so it is automatic that $g^{ij*} = g^{ji}$. Then we have $G(\overline{e^i}) = e_j.g^{ji}$ (summation convention applies). We define $G^{-1}(e_i) = \overline{g_{ij}.e^j}$, where without loss of generality we can assume that $g_{ij}.ev(e^j \otimes e^k) = g_{ik}$. Then:*

a)$g^{ij}g_{jk} = ev(e^i \otimes e_k)$.

b)$g_{ij}g^{jk} = ev(e^k \otimes e_i)$.

c) ${}^*_{iq} = g_{qi}$.

To give a definition to the Christoffel symbols we begin with a left covariant derivative ∇ on a right A-module E. We suppose that E is finitely generated projective as a left A-module, with dual basis $e^i \in E$ and $e_i \in E^\circ$. Then we define the Christoffel symbol

$$\Gamma^j_i = -(id \otimes ev)(\nabla e^j \otimes e_i) \in \Omega^1 A. \tag{2.10}$$

(We choose the minus sign to fit with the standard convention for the covariant derivative of 1-forms, and the reader should remember that the basis of the 1-forms written with upper indices if the coefficients of a 1-form have lower indices, as is standard.[4]) We make the Christoffel symbols into a matrix by defining

$$(\Gamma)_{ji} = \Gamma^j_i$$

We use $g^\bullet$ as shorthand for the matrix g^{ij} and $g_\bullet$ as shorthand for the matrix g_{ij}.

Proposition 46 *[4] The condition for a connection to preserve the Hermitian metric is*

$$g_\bullet.\Gamma = \frac{1}{2}P^*.dg_\bullet.P + \phi,$$

where $\phi \in M_n(\Omega^1 A)$ with $\phi^* = -\phi$, $P^*\phi = \phi$ and $\phi P = \phi$. Form this we can deduce that

$$\Gamma = \frac{1}{2}g^\bullet \cdot dg_\bullet \cdot P + g^\bullet \cdot \phi - dP \cdot P\,.$$

Proposition 47 *[4] Contuning with our finite group example, the left invariant covariant derivative on $C(G)$ given by*

$$\nabla^L(\xi^a) = -\hat{\Gamma}^a_{bc}\xi^b \otimes \xi^c,$$

is a bimodule covariant derivative if and only if

$$a^{-1}bc \notin C \cup e \Rightarrow \hat{\Gamma}^a_{bc} = 0.$$

In this case σ is given by (summing over $b, c \in C$)

$$\sigma(\xi^d \otimes \xi^k) = \delta_{bc,dk}(\hat{\Gamma}^a_{bc} + \delta_{d,c})\xi^b \otimes \xi^c$$

Proposition 48 *[4] The condition for ∇ to preserve the metric is that the matrix $g_{a,b}\Gamma^b_c$ (summation over b) is antiHermitian . If $g_\bullet$ is diagonal , with all enteries on the diagonal equal (and necessarily real) , then this reduces to $\hat{\Gamma}^a_{d,c} = (\hat{\Gamma}^c_{d^{-1},a})^*$.*

Proposition 49 *[4] For (∇, σ) a bimodule covariant derivative as in proposition 47, ∇ is torsion comatible if and only if for all $b, c, d \in C$*

$$d^{-1}bc \in C \Longrightarrow \hat{\Gamma}^d_{b,c} - \hat{\Gamma}^d_{c,c^{-1}bc} = \delta_{cd,bc} - \delta_{b,d}$$

.

Definition 50 *[4] If E is a star-object in${}_A M_A$, we say that ∇ is star compatible if*

$$(id \otimes \star)\sigma_E = \sigma_{\overline{E}}(\star \otimes id) : E \otimes_A \Omega^1 A \longrightarrow \Omega^1 A \otimes_A \overline{E}$$

Proposition 51 *[4] The condition for star compatibility see definition (50) to hold is , suming over b' ,*

$$c^{-1}ab \in C \Longrightarrow (\hat{\Gamma}^a_{abb'^{-1},b'} + \delta_{a,b'})((\hat{\Gamma}^{b'^{-1}}_{b^{-1}a^{-1}c,c^{-1}})^* + \delta_{b'^{-1},c^{-1}}) = \delta_{a,c}.$$

2.8 Symplectic forms

The study of symplectic forms began with Hamiltonian dynamics. A symplectic form ω on a smooth mainfold X is a closed non-degenerate 2-form, i.e. $\omega \in \Omega^2 X$ with

$d\omega = 0$. And for all $x \neq 0$ on the tangent space to x, there is a y in the tangent space, so that $\omega(x, y) \neq 0$.

In noncommutative differential geometry, it is standard to see what $\omega \in \Omega^2 A$ with $d\omega = 0$ means.

The non-degeneracy condition is more of a problem, as there is no general way known to pair 2-forms and vector fields. However, in the cases we consider, taking invariant forms to Hopf algebra coactions results in finite dimensional vector spaces, and here we just ask for ω to be non-degenerate on the vector space, i.e. it has non-degenerate matrix.

Additionally, in classical geometry, ω is introduced at various stages in calculations. To just make ω "appear" in the non commutative case requiers that the map $A \longrightarrow \Omega^2 A$ sending $1 \in A$ to $\omega \in \Omega^2 A$ is a bimodule map, i.e that ω is central, $a.\omega = \omega.a$ all $a \in A$.

2.9 Cohomology

Definition 52 *A cochain complex is a sequence of objects*

$$C^0 \xrightarrow{d} C^1 \xrightarrow{d} C^2 \xrightarrow{d} C^3 \xrightarrow{d} C^4 \xrightarrow{d}$$

where the composition of any two maps is zero, i.e.

$$C^{n-1} \xrightarrow{d} C^n \xrightarrow{d} C^{n+1}$$

gives zero.

(We take abelian groups or vector spaces as examples of the objects here.) Then image($d : C^{n-1} \longrightarrow C^n$) is subset of kernel ($d : C^n \longrightarrow C^{n+1}$)

Then we defined the cohomology of the cochain complex as

$$H^n(C;d) = \frac{kernel(d : C^n \longrightarrow C^{n+1})}{image(d : C^{n-1} \longrightarrow C^n)}.$$

2.10 De Rham cohomology

A sheaf is another case of $\pi : Y \to B$, where B in a base space, which is more general than a tiber bundle. It also has an algebraic structure - each fiber is an abelian group, but that does not concern us here .

Each $e \in Y$ has an open neighbourhood U so that $\pi : U \longrightarrow imageU$ is homeo-

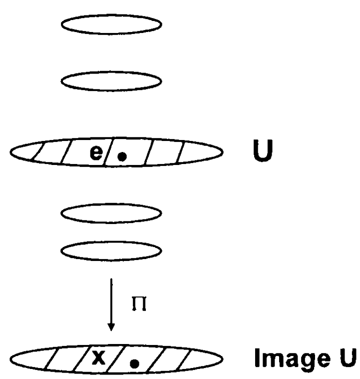

Figure 2.6

mophism of U onto a neighbourhood of $x = \pi(e)$.

This allows us - by assuming a little differentialilty (we do not make this precise) to lift a vector at x uniquely to a vector at e .

Note that a sheaf is often defined in terms of a presheaf, which can be thought of assigning to an open set U in B the set of continuous section $s : U \longrightarrow Y$ (with $\pi \circ s$ being the identity). This is not quite precise, but we are not going to construct an exact correspondence with sheaf theory in non commutative geometry, just an analogue.

If we take the functions on Y to be E, then E is an $A = C^\infty(X)$ bimodule, and the lifting of the vectors gives a flat covariant derivative

$$\nabla : E \longrightarrow \Omega^1 X \otimes_{C^\infty(X)} E.$$

We can now make this into a definition of a noncommutative sheaf [2]

In the differential graded $(\Omega^n A, \wedge, d)$, we have $d^2 = 0$.This means that

$$image\, d : \Omega^{n-1} A \longrightarrow \Omega^n A \subset ker\, d : \Omega^n A \longrightarrow \Omega^{n+1} A$$

. Then we define

$$H^n_{dR} = \frac{ker\, d : \Omega^n A \longrightarrow \Omega^{n+1} A}{image\, d : \Omega^{n-1} A \longrightarrow \Omega^n A}$$

(a quotient of vector spaces).This is the de Rham cohomology , first defined for smooth manifolds.

This can be generalised to give a version of sheaf cohomology in the non-commutative case [2]

Definition 53 *[2] Given an algebra A with differential calculus $(d, \Omega^\star A)$, we define the category ${}_A\mathcal{E}$ to consist of left A-modules E with connection $\nabla : E \to \Omega^1 A \otimes_A E$. A morphism $\phi : (E, \nabla) \to (F, \nabla)$ in the category is a left A-module map $\phi : E \to F$ which preserves the covariant derivative , i.e. $\nabla \circ \phi = (id \otimes \phi) \circ \nabla : E \longrightarrow \Omega^1 A \otimes_A F$.*

Definition 54 *[2] Given $(E, \nabla) \in {}_A\mathcal{E}$, define*

$$\nabla^{[n]} : \Omega^n A \otimes_A E \to \Omega^{n+1} A \otimes_A E, \quad \omega \otimes e \mapsto d\omega \otimes e + (-1)^n \omega \wedge \nabla e.$$

Then the curvature is defined as $R = \nabla^{[1]}\nabla : E \to \Omega^2 A \otimes E$, and is a left A-module map The covariant derivative is called flat if the curvature is zero . We write ${}_A\mathcal{F}$ for the full subcategory of ${}_A\mathcal{E}$ consisting of left A-modules with flat connections .

Proposition 55 *[2] For all $n \geq 0$, $\nabla^{[n+1]} \circ \nabla^{[n]} = id \wedge R : \Omega^n A \otimes_A E \to \Omega^{n+2} A \otimes_A E$.*

Proof : By explicit calculation,

$$\nabla^{[n+1]}(\nabla^{[n]}(\omega \otimes e)) = \nabla^{[n+1]}(d\omega \otimes e + (-1)^n \omega \wedge \nabla e).$$

Put $\nabla e = \xi_i \otimes e_i$ (summation implicit), and then

$$\begin{aligned} \nabla^{[n+1]}(\nabla^{[n]}(\omega \otimes e)) &= \nabla^{[n+1]}(d\omega \otimes e + (-1)^n \omega \wedge \xi_i \otimes e_i) \\ &= (-1)^{n+1} d\omega \wedge \nabla e + (-1)^n d\omega \wedge \xi_i \otimes e_i + \omega \wedge d\xi_i \otimes e_i \\ &= -\omega \wedge \xi_i \wedge \nabla e_i \\ &= \omega \wedge (d\xi_i \otimes e_i - \xi_i \wedge \nabla e_i) = \omega \wedge R(e). \square \end{aligned}$$

Definition 56 *[2] Given $(E, \nabla) \in {}_A\mathcal{F}$, define $H^\star(A; E, \nabla)$ to be the cohomology of the cochain complex*

$$E \xrightarrow{\nabla} \Omega^1 A \otimes_A E \xrightarrow{\nabla^{[1]}} \Omega^2 A \otimes_A E \xrightarrow{\nabla^{[2]}} \ldots\ldots$$

Note that $H^0(E, \nabla) = \Gamma E = \{e \in E : \nabla e = 0\}$, the flat section of E. We will often write $H^\star(A; E)$ where there is no danger of confusing the covariant derivative .

2.11 Spectral sequences

We use [35] as a basis reference for spectral sequences.

A spectral sequences consists of series of pages (indexed by r) and objects $E_r^{p,q}$ (e.g. vector spaces). We take $r \geq 1$ and $p, q \geq 0$, and set $E_r^{p,q} = 0$ if $p < 0$ or $q < 0$. There is a differential

$$d_r : E_r^{p,q} \longrightarrow E_r^{p+r,q+1-r}$$

such that $d_r d_r = 0$

e.g. when $r = 1$ (page 1) we have the picture in figure2.7

when $r = 2$ we have the picture in figure2.8

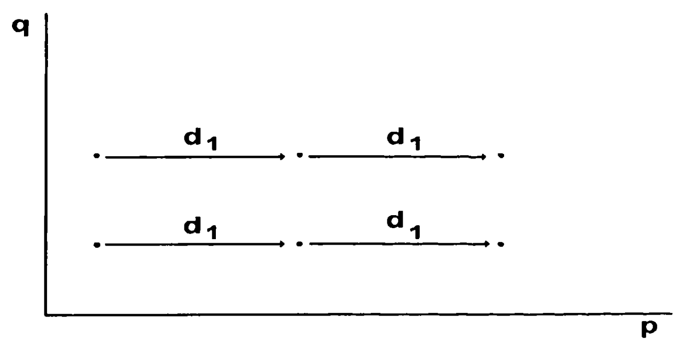

Figure 2.7: page 1

As $d_r d_r = 0$, we have $image\, d_r : E_r^{p-r,q+r-1} \to E_r^{p,q}$ is contained in

$$kernel\, d_r : E_r^{p,q} \to E_r^{p+r,q+1-r}.$$

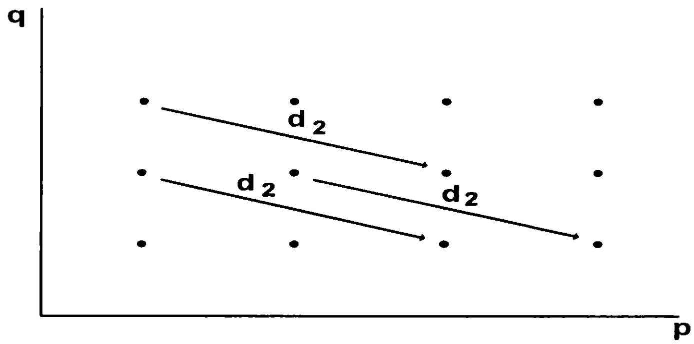

Figure 2.8: page 2

Now take the quotient (in our case,quotient of vector spaces)

$$\frac{kernel\, d_r : E_r^{p,q} \to E_r^{p+r,q+1-r}}{image\, d_r : E_r^{p-r,q+r-1} \to E_r^{p,q}} = H_r^{p,q}$$

Then the rule for going from page r to page $r+1$ is $E_{r+1}^{p,q} = H_r^{p,q}$

The maps d_{r+1} are given by a detailed formula on $H_r^{p,q}$.

The idea in that eventually, the $E_r^{p,q}$ will become fixed for r large enough. The spectral sequences is said to converge to these limiting cases as r increases. Spectral sequences are frequently used in algebraic topology and algebraic geometry. the use of a spectral sequences is summed up

input data $\to$ first or second page of spectral sequence $\to$ work out limit of the spectral sequence $\to$ read off results.

As an example of working out the limit, when $r = 2$ (page 2) we have the picture in figure 2.9

The are only two possible maps d_2 which are non zero, which are θ, ϕ. (because all maps $0 \to 0, 0 \to v$ and $v \to 0$ must be zero).

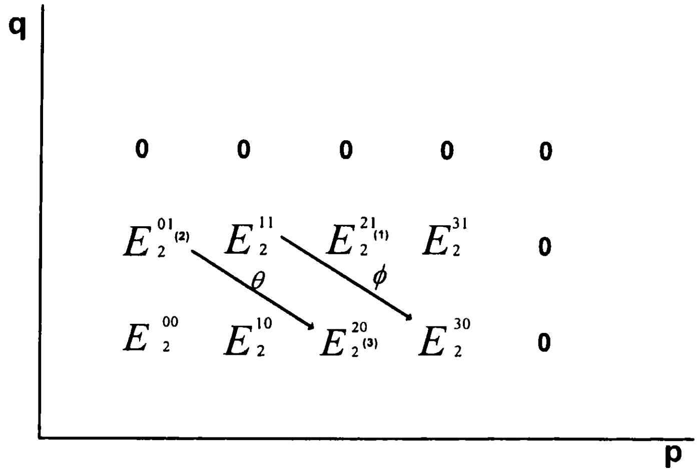

Figure 2.9: page 2

Construct next page

(1) $0 \xrightarrow{d_2} E_2^{21}$ $image = 0$

$E_2^{21} \xrightarrow{d_2} 0$ $kerel = E_2^{21}$

$E_3^{21} = \frac{ker}{image} = \frac{E_2^{21}}{0} = E_2^{21}$.

(2) $d_2 : 0 \to E_2^{01}$ $image = 0$

$d_2 : E_2^{01} \xrightarrow{\theta} E_2^{20}$ $kerel\theta$

$E_3^{01} = \frac{kerd_2}{imaged_2} = \frac{ker\theta}{0} = ker\theta$.

(3) $d_2 : E_2^{01} \xrightarrow{\theta} E_2^{20}$ $image\theta$

$d_2 : E_2^{20} \to 0$ $kerel = E_2^{20}$

$E_3^{20} = \frac{kerd_2}{imaged_2} = \frac{E_2^{20}}{im\theta} = \frac{E_2^{20}}{im\theta}$.

Every $d_3 : E_3^{p,q} \to E_3^{p+3,q-2}$

Now every page after this is the same. The spectral sequences has converged at page 3 in figure 2.10 .

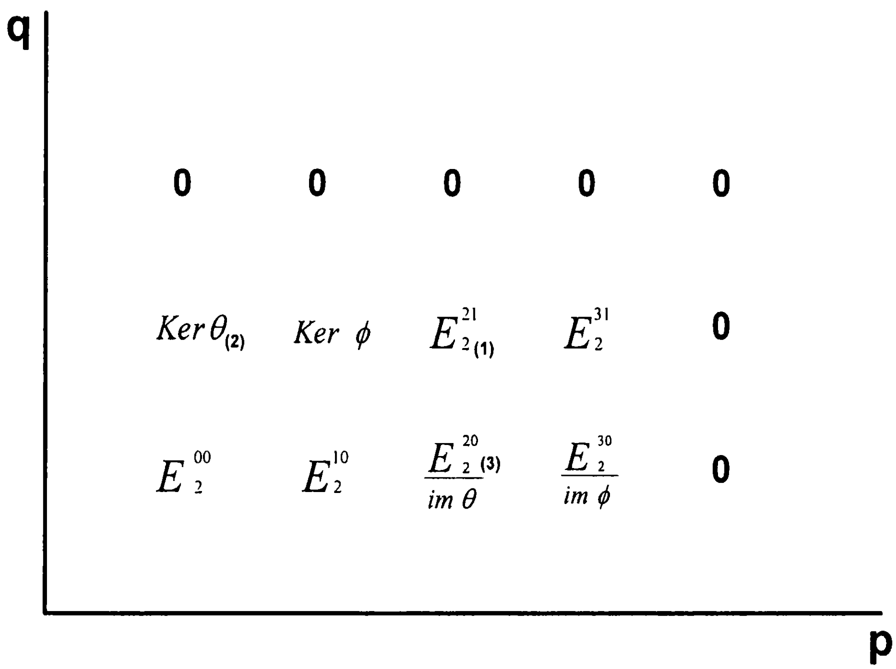

Figure 2.10: page 3

2.11.1 The Serre spectral sequence

The Serre spectral sequence is a machine for relating the cohomology of the total space E of the fiber bundle $E \to B$ to the cohomology of the base B with coefficients in the cohomology of the fiber F.

There is a spectral sequence with second page $H^p(B; H^q(F))$ which has limit $H^*(E)$. We should note that the cohomology $H^q(F)$ may be twisted, in the same way that the fibration itself may be non trivial. The best way to describe this is by sheaf theory.

About the simplest example is the torus $S^1 \times S^1 \to S^1$. We use complex coefficients.

In this case the bundle is trivial, and there is no problem with twisting.
$B = S^1$, $F = S^1$, $E = S^1 \times S^1$.

$$H^q(F, \mathbb{C}) = \begin{cases} \mathbb{C} & q = 0, 1 \\ 0 & otherwise \end{cases}$$

so

$$H^p(B, H^q(F, \mathbb{C})) = \begin{cases} H^p(S^1, \mathbb{C}) & q = 0, 1 \\ 0 & otherwise \end{cases} = \begin{cases} \mathbb{C} & (p = 0, 1) and (q = 0, 1) \\ 0 & otherwise \end{cases}$$

The second page is

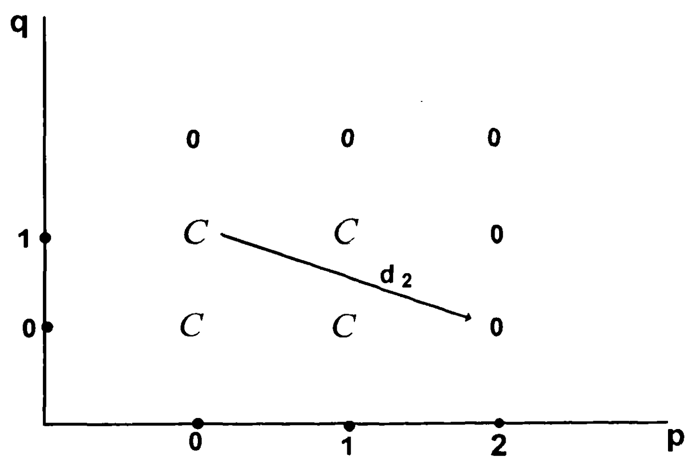

Figure 2.11

Every d_2 either maps to 0 on form 0, so the third page is the same as the second. In fact all the pages after are the same, the spectral sequences has converged.
We get $H^n(S^1 \times S^1) = \bigoplus_{p+q=n} E_\infty^{p,q}$

(E_∞ is the limit page)

$H^0(S^1 \times S^1, \mathbb{C}) = \mathbb{C}$

$H^1(S^1 \times S^1, \mathbb{C}) = \mathbb{C} \oplus \mathbb{C} = \mathbb{C}$

$H^2(S^1 \times S^1, \mathbb{C}) = \mathbb{C}$

$H^n(S^1 \times S^1, \mathbb{C}) = 0$ all $n \geq 3$

Note that the direct sum given is specific for vector spaces, and that a more complicated procedure might have to be used for other coefficients (e.g. $\mathbb{Z}$).

In [2] a noncommutative generalisation of the Serre spectral sequence was given for de Rham cohomology. To state it we need the idea of a non commutative fibration, $\iota : B \to E$. We now take B and E to be algebras, and the map between them has been reversed.

Definition 57 *[2]Define the cochain complexes*

$$\Xi^0_m X = \iota_* \Omega^m B.X \quad , \quad \Xi^n_m X = \frac{\iota_* \Omega^m B \wedge \Omega^n X}{\iota_* \Omega^{m+1} B \wedge \Omega^{n-1} X} \quad (n > 0),$$

with differential $d : \Xi^n_m X \to \Xi^{n+1}_m X$ defined by $d[\omega]_m = [d\omega]_m$, where $\omega \in \iota_* \Omega^m B \wedge \Omega^n X$ and $[\,]_m$ is the corresponding quotient map .

The maps $\Theta_m : \Omega^m B \otimes_B \Xi^n_0 X \to \Xi^n_m X$ defined by $\Theta_m(\omega \otimes [\xi]_0) = [\iota_* \omega \wedge \xi]_m$ are cochain maps if $\Omega^m B \otimes_B \Xi^*_0 X$ is given the differential $(-1)^m id \otimes d$.

Proposition 58 *[2]Suppose that $\Theta_1 : \Omega^1 B \otimes_B \Xi^*_0 X \to \Xi^*_1 X$ (as defined in Definition 57) is invertible. Then is a left B-covariant derivative $\nabla : H^n(\Xi^*_0 X) \to \Omega^1 B \otimes_B H^n(\Xi^*_0 X)$ defined by $[\omega] \mapsto (id \otimes [\,])\Theta_1^{-1}[d\omega]_1$.*

Definition 59 *[2] The differential algebra map $\iota : B \to X$ is called a differential fibration if $\Theta_m : \Omega^m B \otimes_B \Xi^*_0 X \to \Xi^*_m X$(as given in Definition 57) is invertible for all $m \geq 0$.*

Theorem 60 *[2] Suppose that $\iota : B \to X$ is a differential fibration. Then there is a sepctral sequence converging to $H^*_{dR}(X)$ with*

$$E_2^{p,q} \cong H^p(B; H^q(\Xi_0^* X), \nabla)$$

Chapter 3

The group A_4

3.1 Introduction

Classically the discrete group A_4 does not have a non-trivial differential structure. However it is well known that there are non-trivial noncommutative differential structure on finite groups [34], these have no "analytic" content as it is usually thought of, but are of interest algebraically.

3.2 The group A_4

This consists of all even permutations of 4 objects. It has 12 elements which we can write as disjointed cycles. If we permute the objects 0,1,2,3, then the following are elements of A_4 (123), (12)(03). The cycle (123) sends the elements $0 \to 0$, $1 \to 2$, $2 \to 3$, $3 \to 1$.

There is an adjoint action of S_4 (the set of all permutations of 0, 1, 2, 3) on A_4 given by $a \mapsto gag^{-1}$ for $g \in S_4$

There is a representation of A_4 on $\mathbb{C}^4$ given in terms of a standard basis

$$e_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

by $T_g(e_i) = e_{g(i)}$ when $g \in A_4$.

For example $T_{(123)}(e_0) = e_0$, $T_{(123)}(e_1) = e_2$, $T_{(123)}(e_2) = e_3$, $T_{(123)}(e_3) = e_1$ giving the follwing matrix for $T_{(123)}$

$$T_{(123)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{3.1}$$

3.3 A differential calculus

A_4 has conjugacy classes

$$C = \left\{\pi_0, \pi_0^{-1}, \pi_1, \pi_1^{-1}, \pi_2, \pi_2^{-1}, \pi_3, \pi_3^{-1}\right\} \tag{3.2}$$

where

$$\begin{aligned} &\pi_0 = (123) \quad \pi_0^{-1} = (132) \quad \pi_1 = (023) \quad \pi_1^{-1} = (032) \\ &\pi_2 = (013) \quad \pi_2^{-1} = (031) \quad \pi_3 = (012) \quad \pi_3^{-1} = (021)\,. \end{aligned} \tag{3.3}$$

We use C to construct a differential calculus for A_4, as in section (2.3) the left invariant forms will have basis ξ^a for $a \in C$. To make it easier to read, instead of $\xi^{\pi_n^\pm}$ we write $\xi^{n^\pm}$, $(n \in 0,1,2,3)$ or use $\xi^{n'} = \xi^{n^{-1}}$.

The Woronowicz braiding Ψ is given by the formula from (2.3)

$$\Psi(\xi^a \otimes \xi^b) = \xi^{aba^{-1}} \otimes \xi^a,$$

and so Ψ acts separately on the subspace with basis $\xi^a \otimes \xi^b$ where $ab = x$, a fixed element of A_4. We can list these basis elements in the following table

Result x	*Cases* for ab
e	$\pi_i^{+1}\pi_i^{-1}, \pi_i^{-1}\pi_i^{+1}$
π_i^{-1}	$\pi_i\pi_i, \pi_{i+2}\pi_{i+1}^{-1}, \pi_{i+3}^{-1}\pi_{i+2}, \pi_{i+1}^{-1}\pi_{i+3}^{-1}$
π_i	$\pi_i^{-1}\pi_i^{-1}, \pi_{i+1}\pi_{i+2}^{-1}, \pi_{i+2}^{-1}\pi_{i+3}, \pi_{i+3}\pi_{i+1}$
$(03)(12)$	$\pi_0\pi_1, \pi_1\pi_3^{-1}, \pi_2\pi_3, \pi_3\pi_1^{-1}, \pi_3^{-1}\pi_2^{-1}, \pi_2^{-1}\pi_0, \pi_1^{-1}\pi_0^{-1}, \pi_0^{-1}\pi_2$
$(02)(13)$	$\pi_3\pi_2, \pi_3^{-1}\pi_0^{-1}, \pi_2\pi_1, \pi_2^{-1}\pi_3^{-1}, \pi_0\pi_3, \pi_1^{-1}\pi_2^{-1}, \pi_0^{-1}\pi_1^{-1}, \pi_1\pi_0$
$(01)(23)$	$\pi_0\pi_2^{-1}, \pi_0^{-1}\pi_3^{-1}, \pi_1\pi_2, \pi_1^{-1}\pi_3, \pi_2\pi_0^{-1}, \pi_2^{-1}\pi_1^{-1}, \pi_3\pi_0, \pi_3^{-1}\pi_1$
Total	$8+16+16+8+8+8=64$

(3.4)

It is helpful to note the following special case :

$\Psi(\xi^a \otimes \xi^a) = \xi^a \otimes \xi^a$ eigenvalue $+1$

and we include these in the cases below

Case 1 $x = e$

$\xi^a \otimes \xi^{a^{-1}} + \xi^{a^{-1}} \otimes \xi^a$ eigenvalue $+1$

$\xi^a \otimes \xi^{a^{-1}} - \xi^{a^{-1}} \otimes \xi^a$ eigenvalue -1.

Case 2 $x = \pi_i^{-1}$

$\Psi(\xi^{\pi_{i+2}} \otimes \xi^{\pi_{i+1}^{-1}}) = \xi^{\pi_{i+3}^{-1}} \otimes \xi^{\pi_{i+2}}$

$\Psi(\xi^{\pi_{i+3}^{-1}} \otimes \xi^{\pi_{i+2}}) = \xi^{\pi_{i+1}^{-1}} \otimes \xi^{\pi_{i+3}^{-1}}$

$\Psi(\xi^{\pi_{i+1}^{-1}} \otimes \xi^{\pi_{i+3}^{-1}}) = \xi^{\pi_{i+2}} \otimes \xi^{\pi_{i+1}^{-1}}.$

We get eigenvectors

$\xi^{\pi_{i+2}} \otimes \xi^{\pi_{i+1}^{-1}} + \omega \xi^{\pi_{i+3}^{-1}} \otimes \xi^{\pi_{i+2}} + \omega^2 \xi^{\pi_{i+1}^{-1}} \otimes \xi^{\pi_{i+3}^{-1}}$

with eigenvalue w^2, where $w^3 = 1$ (three complex roots)

Case 3 $x = \pi_i$

$\Psi(\xi^{\pi_{i+1}} \otimes \xi^{\pi_{i+2}^{-1}}) = \xi^{\pi_{i+3}} \otimes \xi^{\pi_{i+1}}$

$\Psi(\xi^{\pi_{i+3}} \otimes \xi^{\pi_{i+1}}) = \xi^{\pi_{i+2}^{-1}} \otimes \xi^{\pi_{i+3}}$

$\Psi(\xi^{\pi_{i+2}^{-1}} \otimes \xi^{\pi_{i+3}}) = \xi^{\pi_{i+1}} \otimes \xi^{\pi_{i+2}^{-1}}$

We get eigenvectors

$\xi^{\pi_{i+1}} \otimes \xi^{\pi_{i+2}^{-1}} + \omega \xi^{\pi_{i+3}} \otimes \xi^{\pi_{i+1}} + \omega^2 \xi^{\pi_{i+2}^{-1}} \otimes \xi^{\pi_{i+3}}$

with eigenvalue w^2 , where $w^3 = 1$ (three complex roots)

Case 4 $x = (03)(12)$

We split it into two parts :

First part :

$\Psi(\xi^{\pi_0} \otimes \xi^{\pi_1}) = \xi^{\pi_2^{-1}} \otimes \xi^{\pi_0}$

$\Psi(\xi^{\pi_2^{-1}} \otimes \xi^{\pi_0}) = \xi^{\pi_3^{-1}} \otimes \xi^{\pi_2^{-1}}$

$\Psi(\xi^{\pi_3^{-1}} \otimes \xi^{\pi_2^{-1}}) = \xi^{\pi_1} \otimes \xi^{\pi_3^{-1}}$

$\Psi(\xi^{\pi_1} \otimes \xi^{\pi_3^{-1}}) = \xi^{\pi_0} \otimes \xi^{\pi_1}.$

We get eigenvectors

$\xi^{\pi_0} \otimes \xi^{\pi_1} + \omega \xi^{\pi_2^{-1}} \otimes \xi^{\pi_0} + \omega^2 \xi^{\pi_3^{-1}} \otimes \xi^{\pi_2^{-1}} + \omega^3 \xi^{\pi_1} \otimes \xi^{\pi_3^{-1}},$

with eigenvalue w^3, where $w^4 = 1$ (four complex roots).

Second part :

$\Psi(\xi^{\pi_2} \otimes \xi^{\pi_3}) = \xi^{\pi_0^{-1}} \otimes \xi^{\pi_2}$

$\Psi(\xi^{\pi_0^{-1}} \otimes \xi^{\pi_2}) = \xi^{\pi_1^{-1}} \otimes \xi^{\pi_0^{-1}}$

$\Psi(\xi^{\pi_1^{-1}} \otimes \xi^{\pi_0^{-1}}) = \xi^{\pi_3} \otimes \xi^{\pi_1^{-1}}$

$\Psi(\xi^{\pi_3} \otimes \xi^{\pi_1^{-1}}) = \xi^{\pi_2} \otimes \xi^{\pi_3}.$

We get eigenvectors

$\xi^{\pi_2} \otimes \xi^{\pi_3} + \omega\xi^{\pi_0^{-1}} \otimes \xi^{\pi_2} + \omega^2\xi^{\pi_1^{-1}} \otimes \xi^{\pi_0^{-1}} + \omega^3\xi^{\pi_3} \otimes \xi^{\pi_1^{-1}},$

with eigenvalue w^3 , where $w^4 = 1$ (four complex roots).

Case 5 $x = (02)(13)$

We split it into two parts :

First part :

$\Psi(\xi^{\pi_3} \otimes \xi^{\pi_2}) = \xi^{\pi_0} \otimes \xi^{\pi_3}$

$\Psi(\xi^{\pi_0} \otimes \xi^{\pi_3}) = \xi^{\pi_1} \otimes \xi^{\pi_0}$

$\Psi(\xi^{\pi_1} \otimes \xi^{\pi_0}) = \xi^{\pi_2} \otimes \xi^{\pi_1}$

$\Psi(\xi^{\pi_2} \otimes \xi^{\pi_1}) = \xi^{\pi_3} \otimes \xi^{\pi_2}.$

We get eigenvectors

$\xi^{\pi_3} \otimes \xi^{\pi_2} + \omega\xi^{\pi_0} \otimes \xi^{\pi_3} + \omega^2\xi^{\pi_1} \otimes \xi^{\pi_0} + \omega^3\xi^{\pi_2} \otimes \xi^{\pi_1},$

with eigenvalue w^3 , where $w^4 = 1$ (four complex roots).

Second part :

$\Psi(\xi^{\pi_3^{-1}} \otimes \xi^{\pi_0^{-1}}) = \xi^{\pi_2^{-1}} \otimes \xi^{\pi_3^{-1}}$

$\Psi(\xi^{\pi_2^{-1}} \otimes \xi^{\pi_3^{-1}}) = \xi^{\pi_1^{-1}} \otimes \xi^{\pi_2^{-1}}$

$\Psi(\xi^{\pi_1^{-1}} \otimes \xi^{\pi_2^{-1}}) = \xi^{\pi_0^{-1}} \otimes \xi^{\pi_1^{-1}}$

$\Psi(\xi^{\pi_0^{-1}} \otimes \xi^{\pi_1^{-1}}) = \xi^{\pi_3^{-1}} \otimes \xi^{\pi_0^{-1}}.$

We get eigenvectors

$\xi^{\pi_3^{-1}} \otimes \xi^{\pi_0^{-1}} + \omega\xi^{\pi_2^{-1}} \otimes \xi^{\pi_3^{-1}} + \omega^2\xi^{\pi_1^{-1}} \otimes \xi^{\pi_2^{-1}} + \omega^3\xi^{\pi_0^{-1}} \otimes \xi^{\pi_1^{-1}},$

with eigenvalue w^3, where $w^4 = 1$ (four complex roots).

Case 6 $x = (01)(23)$

We split it into two parts :

First part :

$\Psi(\xi^{\pi_0} \otimes \xi^{\pi_2^{-1}}) = \xi^{\pi_3} \otimes \xi^{\pi_0}$

$\Psi(\xi^{\pi_3} \otimes \xi^{\pi_0}) = \xi^{\pi_1^{-1}} \otimes \xi^{\pi_3}$

$\Psi(\xi^{\pi_1^{-1}} \otimes \xi^{\pi_3}) = \xi^{\pi_2^{-1}} \otimes \xi^{\pi_1^{-1}}$

$\Psi(\xi^{\pi_2^{-1}} \otimes \xi^{\pi_1^{-1}}) = \xi^{\pi_0} \otimes \xi^{\pi_2^{-1}}$

We get eigenvectors

$\xi^{\pi_0} \otimes \xi^{\pi_2^{-1}} + \omega\xi^{\pi_3} \otimes \xi^{\pi_0} + \omega^2\xi^{\pi_1^{-1}} \otimes \xi^{\pi_3} + \omega^3\xi^{\pi_2^{-1}} \otimes \xi^{\pi_1^{-1}}$

with eigenvalue w^3, where $w^4 = 1$ (four complex roots)

Second part :

$\Psi(\xi^{\pi_0^{-1}} \otimes \xi^{\pi_3^{-1}}) = \xi^{\pi_2} \otimes \xi^{\pi_0^{-1}}$

$\Psi(\xi^{\pi_2} \otimes \xi^{\pi_0^{-1}}) = \xi^{\pi_1} \otimes \xi^{\pi_2}$

$\Psi(\xi^{\pi_1} \otimes \xi^{\pi_2}) = \xi^{\pi_3^{-1}} \otimes \xi^{\pi_1}$

$\Psi(\xi^{\pi_3^{-1}} \otimes \xi^{\pi 1_1}) = \xi^{\pi_0^{-1}} \otimes \xi^{\pi_3^{-1}}$

We get eigenvectors

$\xi^{\pi_0^{-1}} \otimes \xi^{\pi_3^{-1}} + \omega\xi^{\pi_2} \otimes \xi^{\pi_0^{-1}} + \omega^2\xi^{\pi_1} \otimes \xi^{\pi_2} + \omega^3\xi^{\pi_3^{-1}} \otimes \xi^{\pi_1}$

with eigenvalue w^3, where $w^4 = 1$ (four complex roots).

According to the paper by Woronowicz [46], the left invariant 2 forms on a Hopf algebra with differential structure are given in terms of Yetter-Drinfeld braiding Ψ by the following, where Γ is the left invariant 1-forms:

$$\Gamma \wedge \Gamma = \frac{\Gamma \otimes \Gamma}{S^2}$$

$$S^2 = kernel(id - \Psi) : \Gamma \otimes \Gamma \longrightarrow \Gamma \otimes \Gamma$$

i.e. S^2 is the $+1$ eigenspace of Ψ.

This gives the following relations on the wedge product:

1) $\xi^a \wedge \xi^a = 0$

2) $\xi^a \wedge \xi^{a^{-1}} + \xi^{a^{-1}} \wedge \xi^a = 0$

3) $\xi^{\pi_{i+2}} \wedge \xi^{\pi_{i+1}^{-1}} + \xi^{\pi_{i+3}^{-1}} \wedge \xi^{\pi_{i+2}} + \xi^{\pi_{i+1}^{-1}} \wedge \xi^{\pi_{i+3}^{-1}} = 0$

4) $\xi^{\pi_{i+1}} \wedge \xi^{\pi_{i+2}^{-1}} + \xi^{\pi_{i+3}} \wedge \xi^{\pi_{i+1}} + \xi^{\pi_{i+2}^{-1}} \wedge \xi^{\pi_{i+3}} = 0$

5) $\xi^{\pi_0} \wedge \xi^{\pi_1} + \xi^{\pi_2^{-1}} \wedge \xi^{\pi_0} + \xi^{\pi_3^{-1}} \wedge \xi^{\pi_2^{-1}} + \xi^{\pi_1} \wedge \xi^{\pi_3^{-1}} = 0$

6) $\xi^{\pi_2} \wedge \xi^{\pi_3} + \xi^{\pi_0^{-1}} \wedge \xi^{\pi_2} + \xi^{\pi_1^{-1}} \wedge \xi^{\pi_0^{-1}} + \xi^{\pi_3} \wedge \xi^{\pi_1^{-1}} = 0$

7) $\xi^{\pi_3} \wedge \xi^{\pi_2} + \xi^{\pi_0} \wedge \xi^{\pi_3} + \xi^{\pi_1} \wedge \xi^{\pi_0} + \xi^{\pi_2} \wedge \xi^{\pi_1} = 0$

8) $\xi^{\pi_3^{-1}} \wedge \xi^{\pi_0^{-1}} + \xi^{\pi_2^{-1}} \wedge \xi^{\pi_3^{-1}} + \xi^{\pi_1^{-1}} \wedge \xi^{\pi_2^{-1}} + \xi^{\pi_0^{-1}} \wedge \xi^{\pi_1^{-1}} = 0$

9) $\xi^{\pi_0} \wedge \xi^{\pi_2^{-1}} + \xi^{\pi_3} \wedge \xi^{\pi_0} + \xi^{\pi_1^{-1}} \wedge \xi^{\pi_3} + \xi^{\pi_2^{-1}} \wedge \xi^{\pi_1^{-1}} = 0$

10) $\xi^{\pi_0^{-1}} \wedge \xi^{\pi_3^{-1}} + \xi^{\pi_2} \wedge \xi^{\pi_0^{-1}} + \xi^{\pi_1} \wedge \xi^{\pi_2} + \xi^{\pi_3^{-1}} \wedge \xi^{\pi_1} = 0$

3.4 Covariant derivative on A_4

For connections on this calculus, we refer to section 2.4. These are given by Christoffel symbols Γ^a_{bc} for $a, b, c \in C$.

We write $\Gamma^{i^{\pm1}}_{j^{\pm2}k^{\pm3}}$ for $\Gamma^{\pi_i^{\pm1}}_{\pi_j^{\pm1}\pi_k^{\pm1}}$ for short. We suppose that the connection is invariant to the S_4 action, that is

$$\Gamma^{gxg^{-1}}_{gyg^{-1},gzg^{-1}} = \Gamma^x_{yz}, \quad for \quad all \quad g \in S_4 \tag{3.5}$$

We can use this symmetry to reduce the number of possibilities for the Christoffel symbols.

We consider all possible cases of $\Gamma^{i^{\pm1}}_{j^{\pm1}\,k^{\pm1}}$ below. There are two useful values of $g \in S_4$ to use in (3.5): We have $\theta_i \in S_4$,$i = 1, 2, 3$ defined by $\theta_i(j) = i + j$ mod 4. Then

$$\theta_i \pi_j^{\pm} \theta_i^{\pm1} = \pi_{i+j}^{\pm1} \tag{3.6}$$

For $i \neq j \in \mathbb{Z}_4$, take the transposition (or 2 cycle) $\sigma_{ij} = (i, j)$. Then

$$\begin{aligned}
\sigma_{ij}\, \pi_i\, \sigma_{ij}^{-1} &= \begin{cases} \pi_j^{-1} & i - j = 2 \\ \pi_j & \text{otherwise} \end{cases} \\
\sigma_{ij}\, \pi_k\, \sigma_{ij}^{-1} &= \pi_k^{-1} \quad i, j, k \text{ all different}
\end{aligned} \tag{3.7}$$

Now consider the various cases $\Gamma^{i^{\pm_1}}_{j^{\pm_2}\, k^{\pm_3}}$.

Case 1 If $j = k = i$, using θ_{-i},

$$\Gamma^{i^{\pm_1}}_{i^{\pm_2}\, i^{\pm_3}} = \Gamma^{0^{\pm_1}}_{0^{\pm_2}\, 0^{\pm_3}} \ , \tag{3.8}$$

and using σ_{12} if necessary,

$$\Gamma^{i^{\pm_1}}_{i^{\pm_2}\, i^{\pm_3}} = \Gamma^{0}_{0^{\pm_2 \pm_1}\, 0^{\pm_3 \pm_1}} \ . \tag{3.9}$$

We reduce to 4 values,

$$\Gamma^0_{0\,0} \quad , \Gamma^0_{0'\,0} \quad , \Gamma^0_{0\,0'} \quad , \Gamma^0_{0'\,0'} .$$

Case 2 If $j = k \neq i$, using θ_{-i},

$$\Gamma^{i^{\pm_1}}_{j^{\pm_2}\, j^{\pm_3}} = \Gamma^{0^{\pm_1}}_{(j-i)^{\pm_2}\, (j-i)^{\pm_3}} \tag{3.10}$$

Using σ_{pq} with $\{p, q\} = \mathbb{Z}_4 \setminus \{0, j - i\}$, if necessary,

$$\Gamma^{i^{\pm_1}}_{j^{\pm_2}\, j^{\pm_3}} = \Gamma^{0}_{(j-i)^{\pm_2 \pm_1}\, (j-i)^{\pm_3 \pm_1}} \tag{3.11}$$

We will show that these all reduce to

$$\Gamma^{0}_{2\,2} \quad , \Gamma^{0}_{2'\,2} \quad , \Gamma^{0}_{2\,2'} \quad , \Gamma^{0}_{2'\,2'}$$

If $j - i = 2$ we already have the result. However if $j - i \neq 2$ we apply $\sigma_{2,j-i}$ to get

$$\Gamma^{i^{\pm_1}}_{j^{\pm_2}\, j^{\pm_3}} = \Gamma^{0^{-1}}_{2^{\pm_2 \pm_1}\, 2^{\pm_3 \pm_1}} \ , \tag{3.12}$$

and applying σ_{13} gives

$$\Gamma^{i^{\pm_1}}_{j^{\pm_2}\, j^{\pm_3}} = \Gamma^{0}_{2^{- \pm_2 \pm_1}\, 2^{- \pm_3 \pm_1}} \ . \tag{3.13}$$

We summarise this by

$$\Gamma^{i^{\pm_1}}_{j^{\pm_2}\, j^{\pm_3}} = \begin{cases} \Gamma^{0}_{2^{\pm_2 \pm_1}\, 2^{\pm_3 \pm_1}} & j - i = 2 \\ \Gamma^{0}_{2^{- \pm_2 \pm_1}\, 2^{- \pm_3 \pm_1}} & j - i \neq 2. \end{cases} \tag{3.14}$$

Case 3 If $i = j \neq k$, using θ_{-i},

$$\Gamma^{i^{\pm_1}}_{i^{\pm_2}\, k^{\pm_3}} = \Gamma^{0^{\pm_1}}_{0^{\pm_2}\, (k-i)^{\pm_3}} \, . \tag{3.15}$$

Using σ_{pq} with $\{p, q\} = \mathbb{Z}_4 \setminus \{0, j - i\}$, if necessary,

$$\Gamma^{i^{\pm_1}}_{i^{\pm_2}\, k^{\pm_3}} = \Gamma^{0}_{0^{\pm_2 \pm_1}\, (k-i)^{\pm_3 \pm_1}} \, . \tag{3.16}$$

We will show that these all reduce to

$$\Gamma^{0}_{0\,2} \quad , \Gamma^{0}_{0'\,2} \quad , \Gamma^{0}_{0\,2'} \quad , \Gamma^{0}_{0'\,2'}$$

If $k-i=2$ we already have the result. However if $k-i\neq 2$ we apply $\sigma_{2,k-i}$ to get

$$\Gamma^{i\pm_1}_{i\pm_2\,k\pm_3}=\Gamma^{0^{-1}}_{0^-\pm_2\pm_1\,2\pm_3\pm_1}\ , \tag{3.17}$$

and applying σ_{13} gives

$$\Gamma^{i\pm_1}_{i\pm_2\,k\pm_3}=\Gamma^{0}_{0\pm_2\pm_1\,2^-\pm_3\pm_1}\ . \tag{3.18}$$

We summarise this by

$$\Gamma^{i\pm_1}_{i\pm_2\,k\pm_3}=\begin{cases}\Gamma^{0}_{0\pm_2\pm_1\,2\pm_3\pm_1} & k-i=2\\ \Gamma^{0}_{0\pm_2\pm_1\,2^-\pm_3\pm_1} & k-i\neq 2.\end{cases} \tag{3.19}$$

Case 4 If $j\neq i=k$, using θ_{-i},

$$\Gamma^{i\pm_1}_{j\pm_2\,i\pm_3}=\Gamma^{0\pm_1}_{(j-i)\pm_2\,(0)\pm_3}. \tag{3.20}$$

Using σ_{pq} with $\{p,q\}=\mathbb{Z}_4\setminus\{0,j-i\}$, if necessary,

$$\Gamma^{i\pm_1}_{j\pm_2\,i\pm_3}=\Gamma^{0}_{(j-i)\pm_2\pm_1\,(0)\pm_3\pm_1}. \tag{3.21}$$

We will show that these all reduce to

$$\Gamma^{0}_{2\,0}\ ,\Gamma^{0}_{2'\,0}\ ,\Gamma^{0}_{2\,0'}\ ,\Gamma^{0}_{2'\,0'}$$

If $j-i=2$ we already have the result. However if $j-i \neq 2$ we apply $\sigma_{2,j-i}$ to get

$$\Gamma^{i\pm_1}_{j\pm_2\, i\pm_3} = \Gamma^{0^{-1}\pm_1}_{2\pm_2\, 0^-\pm_3} , \tag{3.22}$$

and applying σ_{13} give

$$\Gamma^{i\pm_1}_{j\pm_2\, i\pm_3} = \Gamma^{0}_{2^-\pm_2\pm_1\, 0\pm_3\pm_1} . \tag{3.23}$$

We summarise this by

$$\Gamma^{i\pm_1}_{j\pm_2\, i\pm_3} = \begin{cases} \Gamma^{0}_{2\pm_2\pm_1\, 0\pm_3\pm_1} & j-i=2 \\ \Gamma^{0}_{2^-\pm_2\pm_1\, 0\pm_3\pm_1} & j-i \neq 2 \end{cases} \tag{3.24}$$

Case 5 If i, j, k all have different values, using θ_{-i}

$$\Gamma^{i\pm_1}_{j\pm_2\, k\pm_3} = \Gamma^{0\pm_1}_{(j-i)\pm_2\, (k-i)\pm_3} \tag{3.25}$$

We reduce these to represented by 8 cases

$$\Gamma^{0\pm}_{1\pm\, 3\pm} \tag{3.26}$$

If we have $j-i \neq 2$ and $k-i \neq 2$, then we have either $j-i=1$ and $k-i=3$, which is the case we want, or we have $j-i=3$ and $k-i=1$, in the case we use σ_{13} to get

$$\Gamma^{i\pm_1}_{j\pm_2\, k\pm_3} = \Gamma^{0\pm_1}_{3\pm_2\, 1\pm_3} = \Gamma^{0^-\pm_1}_{1^-\pm_2\, 3^-\pm_3} . \tag{3.27}$$

If $j-i=2$ and $k-i=1$, we use σ_{23} to get

$$\Gamma^{i\pm_1}_{j\pm_2\,k\pm_3} = \Gamma^{0\pm_1}_{2\pm_2\,1\pm_3} = \Gamma^{0-\pm_1}_{3\pm_2\,1-\pm_3} = \Gamma^{0\pm_1}_{1-\pm_2\,3\pm_3}. \tag{3.28}$$

If $j-i=1$ and $k-i=2$, we use σ_{23} to get

$$\Gamma^{i\pm_1}_{j\pm_2\,k\pm_3} = \Gamma^{0\pm_1}_{1\pm_2\,2\pm_3} = \Gamma^{0-\pm_1}_{1-\pm_2\,3\pm_3} \tag{3.29}$$

If $j-i=2$ and $k-i=3$, we use σ_{21} to get

$$\Gamma^{i\pm_1}_{j\pm_2\,k\pm_3} = \Gamma^{0\pm_1}_{2\pm_2\,3\pm_3} = \Gamma^{0-\pm_1}_{1\pm_2\,3-\pm_3} \tag{3.30}$$

If $j-i=3$ and $k-i=2$, we use σ_{21} to get

$$\Gamma^{i\pm_1}_{j\pm_2\,k\pm_3} = \Gamma^{0\pm_1}_{3\pm_2\,2\pm_3} = \Gamma^{0-\pm_1}_{3-\pm_2\,1\pm_3} = \Gamma^{0\pm_1}_{1\pm_2\,3-\pm_3} \tag{3.31}$$

3.4.1 Bimodule connections

The condition to have a bimodule connection is (see proposition 47 in section 2.7)

$$a^{-1}bc \notin C \cup \{e\} \Rightarrow \hat{\Gamma}^{a}_{b\,c} = 0 \tag{3.32}$$

We consider the various cases in turn ,and assign the letters to the Christoffel symbols that we will use later.

Case 1

$$a = \Gamma^{0}_{0\,0} \quad gives \quad a^{-1}bc = \pi_0$$
$$b = \Gamma^{0}_{0'\,0} \quad gives \quad a^{-1}bc = \pi_0^{-1}$$

$$c = \Gamma^0_{0\,0'} \quad gives \quad a^{-1}bc = \pi_0^{-1}$$
$$d = \Gamma^0_{0'\,0'} \quad gives \quad a^{-1}bc = e. \tag{3.33}$$

Case 2

$$e = \Gamma^0_{2\,2} \quad gives \quad a^{-1}bc = \pi_3^{-1}$$
$$f = \Gamma^0_{2'\,2} \quad gives \quad a^{-1}bc = \pi_0^{-1}$$
$$g = \Gamma^0_{2\,2'} \quad gives \quad a^{-1}bc = \pi_0^{-1}$$
$$\Gamma^0_{2'\,2'} \quad gives \quad a^{-1}bc = (03)\,(12) \notin C \cup \{e\} \;\;, so\; \hat{\Gamma}^0_{2'\,2'} = 0. \tag{3.34}$$

Case 3

$$h = \Gamma^0_{0\,2} \quad gives \quad a^{-1}bc = \pi_2$$
$$i = \Gamma^0_{0'\,2} \quad gives \quad a^{-1}bc = \pi_1$$
$$j = \Gamma^0_{0\,2'} \quad gives \quad a^{-1}bc = \pi_2^{-1}$$
$$\Gamma^0_{0'\,2'} \quad gives \quad a^{-1}bc = (01)\,(23) \notin C \cup \{e\} \;\;, so\, \hat{\Gamma}^0_{0'\,2'} = 0. \tag{3.35}$$

Case 4

$$k = \Gamma^0_{2\,0} \quad gives \quad a^{-1}bc = \pi_1^{-1}$$
$$m = \Gamma^0_{2'\,0} \quad gives \quad a^{-1}bc = \pi_1$$
$$n = \Gamma^0_{2\,0'} \quad gives \quad a^{-1}bc = \pi_2^{-1}$$
$$\Gamma^0_{2'\,0'} \quad gives \quad a^{-1}bc = (02)\,(13) \notin C \cup \{e\} \;\;, so\, \hat{\Gamma}^0_{2'\,0'} = 0. \tag{3.36}$$

Case5

$$\Gamma^0_{1\,3} \quad gives \quad a^{-1}bc = (03)\,(12) \notin C \cup \{e\} \;\;, so\; \hat{\Gamma}^0_{1'\,3'} = 0$$

$$p = \Gamma^{0}_{1'\,3} \quad gives \quad a^{-1}bc = \pi_2^{-1}$$
$$q = \Gamma^{0}_{1\,3'} \quad gives \quad a^{-1}bc = \pi_1$$
$$r = \Gamma^{0}_{1'\,3'} \quad gives \quad a^{-1}bc = \pi_0^{-1}$$
$$s = \Gamma^{0'}_{1\,3} \quad gives \quad a^{-1}bc = \phi_1$$
$$t = \Gamma^{0'}_{1'\,3} \quad gives \quad a^{-1}bc = \pi_3^{-1}$$
$$u = \Gamma^{0'}_{1\,3'} \quad gives \quad a^{-1}bc = \pi_2$$
$$v = \Gamma^{0'}_{1'\,3'} \quad gives \quad a^{-1}bc = e. \tag{3.37}$$

in the rest of this chapter, we always assume that ∇ is a bimodule connection, and that these letters and used.

3.4.2 Torsion compatible

We have given the Christoffel symbols short variable names, which we will use later. The condition for the connection to be torsion compatible is (see proposition (49) in section (2.7)):

$$d^{-1}bc \in C \implies \hat{\Gamma}^{d}_{b,c} - \hat{\Gamma}^{d}_{c,c^{-1}bc} = \delta_{cd,bc} - \delta_{b,d}. \tag{3.38}$$

We now apply this to our specific S_4 invariant connection on A_4.

For example

$\hat{\Gamma}^{0}_{0,0} - \hat{\Gamma}^{0}_{0,0'00} = \delta_{00,00} - \delta_{0,0}$

gives $a - a = 0$

$\hat{\Gamma}^{0'}_{1',3'} - \hat{\Gamma}^{0'}_{3',31'3'} = \delta_{3'0',1'3'} - \delta_{1',0'}$

$\hat{\Gamma}^{0'}_{1',3'} - \hat{\Gamma}^{0'}_{3',2} = 0$

$\hat{\Gamma}^{0'}_{1',3'} - \hat{\Gamma}^{0'}_{1',3'} = 0$

gives $p - p = 0$

and the same cacaulation with the other cases. Then with a small calculation we get:

$s = r = i = n = 0 \quad b = c \quad f = g$

$h = v - 1 = k \quad j = t - 1 = m = q - 1$

3.4.3 Implementing the S_4 symmetry on Christoffel symbols using Mathematica

This differential calculus was sufficiently large that we used Mathematica to do calculations with it.

We use the following Mathematica notation

$gam[\{i,r\},\{j,s\},\{k,t\}] = \Gamma^{i^r}_{j^s k^t}$, where $i,j,k \in 0,1,2,3$ and $r,s,t \in +1,-1$

and $gam0[r,j,s,k,t] = \Gamma^{o^r}_{j^s k^t}$

Using $\theta_i j^{\pm}\theta_i^{\pm 1} = (i+j)^{\pm 1}$, we can convert every Christoffel symbol, so that there is a zero in the top position.

gam[{i_, s1_}, {j_, s2_}, {k_, s3_}]:=gam0[s1, Mod[$j - i$, 4], s2, Mod[$k - i$, 4], s3].

We will now implement the symmetry in section 3.4, cases 1- 4. We use $endgam0[r,j,s,k,t] = \Gamma^{o^r}_{j^s k^t}$ to be one of the representative cases given . These statement are used to make the assignment:

gam0[s1_,j_,s2_,k_,s3_] := Which[

(j==0) && (k==0) , endgam0[+1,0,s1 s2 ,0,s1 s3], (* case 1 *)

j==k , Which[j==2, endgam0[+1,2,s1 s2 ,2,s1 s3], True , endgam0[+1,2,-s1 s2 ,2,-s1 s3]] , (* case 2 *)

j==0, Which[k==2, endgam0[+1,0,s1 s2 ,2,s1 s3], True , endgam0[+1,0,s1 s2 ,2,-s1 s3]] , (* case 3*)

k==0, Which[j==2, endgam0[+1,2,s1 s2 ,0,s1 s3], True , endgam0[+1,2,-s1 s2 ,0,s1

```
s3] ] , (* case 4*)
(* case 5 *) True, Which[
 (j==2) && (k==1) , endgam0[s1,1,-s2,3,s3],
(j==1) && (k==2) , endgam0[-s1,1,-s2,3,s3],
(j==2) && (k==3) , endgam0[-s1,1,s2,3,-s3],
(j==3) && (k==2) , endgam0[s1,1,s2,3,-s3],
(j==1) && (k==3) , endgam0[s1,1,s2,3,s3],
(j==3) && (k==1) , endgam0[-s1,1,-s2,3,-s3]]]
```

3.5 The Metric

The condition for the covariant dervative to preserve a diagonal metric (see proposition (48) in section (2.7)) is

$$\Gamma^{a}_{d\,c} = (\Gamma^{c}_{d'\,a})^* \tag{3.39}$$

Separating this into cases gives **Case 1** :

$$\begin{aligned}
a &= \Gamma^{0}_{0\,0} = (\Gamma^{0}_{0'\,0})^* = b^* \\
b &= \Gamma^{0}_{0'\,0} = (\Gamma^{0}_{0\,0})^* = a^* \\
c &= \Gamma^{0}_{0\,0'} = (\Gamma^{0'}_{0'\,0})^* = (\Gamma^{0}_{0\,0'})^*, \quad \text{so } \Gamma^{0}_{0\,0} \text{ is real} \\
d &= \Gamma^{0}_{0'\,0'} = (\Gamma^{0'}_{0\,0})^* = (\Gamma^{0}_{0'\,0'})^*, \quad \text{so } \Gamma^{0}_{0'\,0'} \text{ is real}
\end{aligned} \tag{3.40}$$

Case 2 :

$$\begin{aligned}
e &= \Gamma^{0}_{2\,2} = (\Gamma^{2}_{2'\,0})^* = (\Gamma^{0'}_{0\,2'})^* = (\Gamma^{0}_{0'\,2})^* = i^* \\
f &= \Gamma^{0}_{2'\,2} = (\Gamma^{2}_{2\,0})^* = (\Gamma^{0'}_{0'\,2'})^* = (\Gamma^{0}_{0\,2})^* = h^*
\end{aligned}$$

$$g = \Gamma^{0}_{2\,2'} = (\Gamma^{2'}_{2'\,0})^* = (\Gamma^{0}_{0\,2'})^* = j^*$$
$$\Gamma^{0}_{2'\,2'} = (\Gamma^{2'}_{2\,0})^* = (\Gamma^{0}_{0'\,2'})^* = 0, \quad \text{no more information} \tag{3.41}$$

Case 3 :

$$h = \Gamma^{0}_{0\,2} = (\Gamma^{0}_{2'\,2})^* = f^*$$
$$i = \Gamma^{0}_{0'\,2} = (\Gamma^{0}_{2\,2})^* = e^*$$
$$j = \Gamma^{0}_{0\,2'} = (\Gamma^{0}_{2\,2\prime})^* = g^*$$
$$\Gamma^{0}_{0'\,2'} = (\Gamma^{0}_{2'\,2\prime})^* = 0, \text{no more information} \tag{3.42}$$

Case 4 :

$$k = \Gamma^{0}_{2\,0} = (\Gamma^{0}_{2'\,0})^* = m^*$$
$$m = \Gamma^{0}_{2'\,0} = (\Gamma^{0}_{2\,0})^* = k^*$$
$$n = \Gamma^{0}_{2\,0'} = (\Gamma^{0'}_{2'\,0})^* = (\Gamma^{0}_{2\,0'})^* \quad \text{, so } \Gamma^{0}_{2\,0'} \text{ is real} \tag{3.43}$$
$$\Gamma^{0}_{2'\,0'} = (\Gamma^{0'}_{2\,0})^* = (\Gamma^{0}_{2'\,0\prime})^* = 0, \text{no more information} \tag{3.44}$$

Case 5 :

$$\Gamma^{0}_{1\,3} = (\Gamma^{3}_{1'\,0})^* = (\Gamma^{0}_{1\,3})^* = 0, \quad \text{no more information}$$
$$p = \Gamma^{0'}_{1'\,3'} = (\Gamma^{3'}_{1\,0'})^* = (\Gamma^{0'}_{1'\,3'})^* \quad \text{, so } \Gamma^{0'}_{1'\,3'}, \text{ is real}$$
$$q = \Gamma^{0}_{1'\,3} = (\Gamma^{3}_{1\,0})^* = (\Gamma^{0}_{1'\,3})^* \quad \text{, so } \Gamma^{0}_{1'\,3} \text{ is real}$$
$$r = \Gamma^{0'}_{1\,3'} = (\Gamma^{3'}_{1'\,0'})^* = (\Gamma^{0'}_{1\,3'})^*, \quad \text{so } \Gamma^{0'}_{1\,3'} \text{ is real}$$
$$s = \Gamma^{0'}_{1'\,3} = (\Gamma^{3}_{1\,0'})^* = (\Gamma^{0}_{1'\,3'})^* = u^*$$
$$t = \Gamma^{0}_{1\,3'} = (\Gamma^{3'}_{1'\,0})^* = (\Gamma^{0'}_{1\,3})^* = v^*$$
$$u = \Gamma^{0}_{1'\,3'} = (\Gamma^{3'}_{1\,0})^* = (\Gamma^{0'}_{1'\,3})^* = s^*$$

$$v = \Gamma^{0'}_{1\,3} = (\Gamma^{3}_{1'\,0'})^* = (\Gamma^{0}_{1\,3'})^* = t^* \tag{3.45}$$

This can be summarised as (where $*$ denotes a complex conjugate)

$$\begin{array}{lllll} a = b^* & b = a^* & c = c^* & d = d^* & e = i^* \\ f = h^* & g = j^* & h = f^* & i = e^* & j = g^* \\ k = m^* & m = k^* & n = n^* & p = p^* & q = q^* \\ r = r^* & s = u^* & t = v^* & u = s^* & v = t^*. \end{array} \tag{3.46}$$

We state these results as a proposition.

Proposition 61 *The condition for the bimodule covariant derivative ∇ to preserve a diagonal metric is:*

c, d, n, p, q and r are real and $a = b^*$ $b = a^*$ $e = i^*$

$f = h^*$ $g = j^*$ $h = f^*$ $i = e^*$ $j = g^*$

$k = m^*$ $m = k^*$ $s = u^*$ $t = v^*$ $u = s^*$ $v = t^*$

Proposition 62 *There are torsion covariant derivatives which satisfy the diagonal metric with $u = e = s = r = i = n = 0$, $a = b = c$, $f = h = v - 1 = k = g = j = t - 1 = m = q - 1$ and all them are real and also d, p are real.*

Proof: *Using section 3.4.2 and proposition 61.* $\square$

3.6 The generalised braiding σ

We give a convention for writing tensor products of matrices as single matrices, consistent with the Mathematica Kronecker product:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes M = \begin{pmatrix} aM & bM \\ cM & dM \end{pmatrix}$$

We have $\sigma : \Omega^1 \otimes \Omega^1 \longrightarrow \Omega^1 \otimes \Omega^1$, and we have 8 generators of Ω^1 , so we need a 64×64 matrix for σ.

We use the formula (proposition (47) in section (2.7))

$$\sigma(\xi^d \otimes \xi^k) = \delta_{bc,dk}(\Gamma^d_{bc} + \delta_{d,c})\xi^b \otimes \xi^c$$

To calculate this in Mathematica, we use the following functions

We consider the conjugacy class C, and enumerate it from 1 to 8 as

f[1]={0,1} $= \pi_0^{+1}$

f[2]={0,-1} $= \pi_0^{-1}$

f[3]={1,1} $= \pi_1^{+1}$

f[4]={1,-1} $= \pi_1^{-1}$

f[5]={2,1} $= \pi_2^{+1}$

f[6]={2,-1} $= \pi_2^{-1}$

f[7]={3,1} $= \pi_3^{+1}$

f[8]={3,-1} $= \pi_3^{-1}$

This corresponds to the representing matrices (as 3.1))

rep[1] ={{1,0,0,0},{0,0,0,1},{0,1,0,0},{0,0,1,0}}

rep[2]={{1,0,0,0},{0,0,1,0},{0,0,0,1},{0,1,0,0}}

```
rep[3]={{0,0,0,1},{0,1,0,0},{1,0,0,0},{0,0,1,0}}
rep[4]={{0,0,1,0},{0,1,0,0},{0,0,0,1},{1,0,0,0}}
rep[5]={{0,0,0,1},{1,0,0,0},{0,0,1,0},{0,1,0,0}}
rep[6]={{0,1,0,0},{0,0,0,1},{0,0,1,0},{1,0,0,0}}
rep[7]={{0,0,1,0},{1,0,0,0},{0,1,0,0},{0,0,0,1}}
rep[8]={{0,1,0,0},{0,0,1,0},{1,0,0,0},{0,0,0,1}}
```

The Kroneker $\delta_{i,j}$ is implemeted as

```
delta[ppp_,qqq_] := Which[ ppp==qqq ,1,True, 0]
```

and the Chistoffel symbols are enumerated by the

```
symbol[i_,j_,k_]:=gam[f[i],f[j],f[k]]
```

The tensor product is converted to a single matrix (σ is a 64 by 64 matrix) by using

```
sizeclass=8
pairtonumber[{i_,j_}] := j+(i-1) sizeclass
numbertopair[n_] := { IntegerPart[(n-1)/sizeclass]+1,n- sizeclass IntegerPart[(n-1)/sizeclass]
```

Now the following formula give the entries of σ

```
newsigmaentry[n_,m_] := delta[rep[numbertopair[m][[1]]].rep[numbertopair[m][[2]]],
rep[numbertopair[n][[1]]].rep[numbertopair[n][[2]]] ] (
symbol[numbertopair[m][[1]],numbertopair[n][[1]] , numbertopair[n][[2]] ] +
delta[numbertopair[m][[1]], numbertopair[n][[2]] ])
```

Now we make this into a matrix using the table command

```
MatrixForm[sigmamatrix=Table[newsigmaentry[n,m],{n,1,64},{m,1,64}]]
```

and convert to our single letter names of the Christoffel symbols using

MatrixForm[newsigma = sigmamatrix//.{endgam0[1, 0, 1, 0, 1] $\to a$, endgam0[1, 0, −1, 0, 1] $\to$ b, endgam0[1, 0, 1, 0, −1] $\to$ c, endgam0[1, 0, −1, 0, −1] $\to$ d, endgam0[1, 2, 1, 2, 1] $\to$ e, endgam0[1, 2, −1, 2, 1] $\to$ f, endgam0[1, 2, 1, 2, −1] $\to$ g, endgam0[1, 2, −1, 2, −1] $\to$ 0, endgam0[1, 0, 1, 2, 1] $\to$ h, endgam0[1, 0, −1, 2, 1] $\to$ i, endgam0[1, 0, 1, 2, −1] $\to$ j, endgam0[1, 0, −1, 2, −1] $\to$ 0, endgam0[1, 2, 1, 0, 1] $\to$ k, endgam0[1, 2, −1, 0, 1] $\to$ m, endgam0[1, 2, 1, 0, −1] $\to$ n, endgam0[1, 2, −1, 0, −1] $\to$ 0, endgam0[1, 1, 1, 3, 1] $\to$ 0, endgam0[−1, 1, −1, 3, −1] $\to$ p, endgam0[1, 1, −1, 3, 1] $\to$ q, endgam0[−1, 1, 1, 3, −1] $\to$ r, endgam0[−1, 1, −1, 3, 1] $\to$ s, endgam0[1, 1, 1, 3, −1] $\to$ t, endgam0[1, 1, −1, 3, −1] $\to$ u, endgam0[−1, 1, 1, 3, 1] $\to$ v}]

Note that σ does not depend on p or d.

3.7 The Braid Relations

The braid relations for σ are defined using

$\sigma_{12} = \sigma \otimes \mathbf{I}_8 : \Omega^1 \otimes \Omega^1 \otimes \Omega^1 \longrightarrow \Omega^1 \otimes \Omega^1 \otimes \Omega^1$

$\sigma_{23} = \mathbf{I}_8 \otimes \sigma : \Omega^1 \otimes \Omega^1 \otimes \Omega^1 \longrightarrow \Omega^1 \otimes \Omega^1 \otimes \Omega^1$

σ satisfies the braid relation if $\sigma_{12}\sigma_{23}\sigma_{12} - \sigma_{23}\sigma_{12}\sigma_{23} = 0$

We implement this on mathematica by

sigma23=KroneckerProduct[IdentityMatrix[8],newsigma];

sigma12=KroneckerProduct[newsigma,IdentityMatrix[8]]

MatrixForm[test=sigma12.sigma23.sigma12-sigma23.sigma12.sigma23]

We look at the entries of "test" above, which are formulae in the values a ,b , c ,...., v.

Note : that σ does not depend on p or d.

The following 71 cases list all the values of a, ...,v which satisfy the braid relations.

This list was produced by taking the matrix $\sigma_{12}\sigma_{23}\sigma_{12} - \sigma_{23}\sigma_{12}\sigma_{23}$ in Mathematica (called test above), and examining the enteries to see when they were zero. Solving these equations using mathematical software. Seemed not to be an option - it was certainly beyond my (and my supervisor's) knowledge of how to get the software to work on such a big problem. We resorted to a partially manual and partially computer assisted approach.

This involved taking the simplest entries in the matrix, factorising them, and producing a tree as assumptions were made at various stages. The tree is written in Appendix A.

Note : Here $x = \pm 1$, $y = \pm 1$ and $z = \pm 1$

1) a=-1, b=b, c=0, e=0, f=f, g=0, h=0, i=0 , j=0,k=-1, m=m, n=0,
q=0, r=0, s=0, t=t, u=0, v=0

2) a=a, b=-1-f, c=0, e=0, f=f, g=0, h=1+a, i=0, j=0 , k=-1, m=m,
n=0, q=0, r=0, s=0, t=t, u=0, v=0

3) a=a, b=b, c=0, e=0, f=0, g=0, h=1+a, i=0, j=0 , k=-1, m=m, n=0,
q=0 , r=0, s=0, t=t, u=0, v=0

4) a=k, b=m, c=0, e=0, f=0, g=0, h=0, i=0, j=0, k=k, m=m, n=0,
q=0, r=0, s=0, t=0, u=0, v=0

5) a=-1, b=b, c=0, e=0, f=f, g=0, h=0, i=0, j=0, k=-1, m=m, n=0,
q=0, r=0, s=0, t=0, u=0, v=0

6) a=-1, b=b, c=0, e=0, f=0, g=0, h=0, i=0, j=0, k=-1, m=-1, n=0,
q=0, r=0, s=0, t=t, u=0, v=0

7) a=a, b=b, c=0, e=0, f=0, g=0, h=0, i=0, j=0, k=-1, m=-1, n=0,

q=0, r=0, s=0, t=1+b, u=0, v=1+a

8) a=-1, b=b, c=0, e=0, f=f, g=0, h=0, i=0, j=0, k=-1, m=-1, n=0,

q=0, r=0, s=0, t=t, u=0, v=0

9) a=-1, b=-1, c=c, e=0, f=0, g=0, h=0, i=0, j=0, k=-1, m=-1, n=n,

q=q, r=r, s=0 , t=0, u=0, v=0 , at least two of n , q , r =0

10) a=-1, b=-1, c=0, e=0, f=0, g=g, h=0, i=0, j=0, k=-1, m=-1,

n=0, q=0, r=0, s=0, t=0, u=0, v=0

11)a=-1+h, b=-1, c=h, e=0, f=0, g=0, h=h, i=0, j=h, k=-1, m=-1,

n=0, q=0, r=0, s=0, t=0, u=0, v=0

12) a=-1+j, b=-1, c=j, e=0, f=0, g=0, h=j, i=0, j=j, k=-1, m=-1,

n=0, q=0, r=0, s=0, t=0, u=0, v=0

13)a=-1+c, b=-1, c=c, e=0, f=0, g=0, h=c, i=0, j=c, k=-1, m=-1,

n=0, q=0, r=0, s=0, t=0, u=0, v=0

14)a=a, b=-1, c=1+a, e=0, f=0, g=0, h=1+a, i=0, j=1+a, k=-1,

m=-1, n=0, q=0, r=0, s=0, t=0, u=0, v=0

15) a=-1, b=-1, c=0, e=0, f=0, g=j, h=0, i=0, j=j, k=-1, m=-1,

n=n, q=0, r=r, s=0, t=0, u=0, v=0

16) a=-1, b=-1, c=0, e=0, f=0, g=j, h=0, i=0, j=j, k=-1, m=-1,

n=0, q=q, r=r, s=0, t=0, u=0, v=0

17) a=-1, b=-1, c=0, e=0, f=0, g=j, h=0, i=0, j=j, k=-1, m=-1,

n=n, q=q, r=0, s=0, t=0, u=0, v=0

18) a=-1, b=-1, c=$\frac{nqr}{j^2}$, e=0, f=0, g=j, h=0, i=0, $j \neq 0$, k=-1,

m=-1, n=n, q=q, r=r, s=0, t=0, u=0, v=0

19)a=-1, b=-1, c=$\frac{nqr}{j^2}$, e=0, f=0, g=j, h=0, i=0, $j \neq 0$, k=-1,

m=-1, n=n, q=q, r=r, s=0, t=0, u=0, v=0

20)a=-1, b=-1, c=$\frac{nqr}{j^2}$, e=0, f=0, g=j, h=0, i=0, $j \neq 0$, k=-1, m$= -1$, n$=n$, q$=q$, r$=r$, s=0, t=0, u=0, v=0

21)a$= -1$, b$= -1$, c$=x$, e=0, f=0, g$=x$, h=0, i=0, j$=x$, k$= -1$,

m$= -1$, n$= -x$, q$=x$, r$= -x$, s=0, t=0, u=0, v=0

22)a$= -1$, b$= -1$, c$=x$, e=0, f=0, g$=x$, h=0, i=0, j$=x$, k$= -1$,

m$= -1$, n$=x$, q$=x$, r$=x$, s=0, t=0, u=0, v=0

23)a$= -1$, b$= -1$, c$=x$, e=0, f=0, g$=x$, h=0, i=0, j$=x$, k$= -1$, m$= -1$,

n$= -x$, q$= -x$, r$=x$, s=0, t=0, u=0, v=0

24)a$= -1$, b$= -1$, c$=x$, e=0, f=0, g$=x$, h=0, i=0, j$=x$, k$= -1$, m$= -1$,

n$=x$, q$= -x$, r$= -x$, s=0, t=0, u=0, v=0

25)a$= -1-4g$, b$= -1$, c$= -3g$, e=0, f=0, g$=g$, h$= -4g$, i=0, j$= -3g$,

k$= -1$, m$= -1$, n$=gxy$, q$=gy$, r$=gx$, s=0, t=0, u=0, v=0

26)a$= -1+4g$, b$= -1$, c$=g$, e=0, f=0, g$=g$, h$=4g$, i=0, j$=g$, k$= -1$,

m$= -1$, n$=gxy$, q$=gy$, r$=gx$, s=0, t=0, u=0, v=0

27)a$=k$, b$= -1$, c=0, e=0, f=0, g=0, h=0, i=0, j=0, k$=k$, m$= -1$,

n=0, q=0, r=0, s=0, t=0, u=0, v=0

28)a$= -1+v$, b$= -1$, c=0, e=0, f=0, g=0, h=0, i=0, j=0, k$= -1$,

m$= -1$, n=0, q=0, r=0, s=0, t=0, u=0, v$=v$

29)a$= -1$, b$= -1$, c$=c$, e=0, f=0, g=0, h=0, i=0, j=0, k$= -1$,

m$= -1$, n=0, q=0, r=0, s=0, t=0, u=0, v=0

30)a$= -1+c$, b$= -1$, c$=c$, e=0, f=0, g=0, h=0, i=0, j=0, k$= -1$,

m$= -1$, n=0, q=0, r=0, s=0, t=0, u=0, v=0

31)a$=a$, b$= -1$, c=0, e=0, f=0, g=0, h=0, i=0, j=0, k$= -1$, m$= -1$,

n=0, q=0, r=0, s=0, t=0, u=0, v=0

32)a$= -1$, b$= -1$, c=0, e=0, f=0, g=0, h$=j$, i=0, j$=j$, k$= -1$, m$= -1$,

n=0, q=0, r=0, s=0, t=0, u=0, v=0

33)a$=a$, b$= -1$, c=0, e=0, f=0, g=0, h$=1+a$, i=0, j$=1+a$, k$= -1$, m= -1, n=0, q=0, r=0, s=0, t=0, u=0, v=0

34)a= -1, b= -1, c=j, e=0, f=0, g=0, h=j, i=0, j=j, k= -1, m= -1, n=0, q=0, r=0, s=0, t=0, u=0, v=0

35)a=a, b= -1, c=$1+a$, e=0, f=0, g=0, h=$1+a$, i=0, j=$1+a$, k= -1, m= -1, n=0, q=0, r=0, s=0, t=0, u=0, v=0

36)a=a, b=b, c=0, e=0, f=0, g=0, h=0, i=0, j=0, k= -1, m= -1, n=0, q=0, r=0, s=0, t=0, u=0, v=$1+a$

37)a= -1, b=b, c=0, e=0, f=f, g=g, h=0, i=0, j=0, k= -1, m= -1, n=0, q=0, r=0, s=0, t=0, u=0, v=0

38)a= -1, b=b, c=0, e=0, f=f, g=0, h=0, i=i, j=0, k= -1, m=m, n=0, q=0, r=0, s=s, t=t, u=0, v=0

39)a=a, b= $-1-f$, c=0, e=0, f=f, g=0, h=$1+a$, i=i, j=0, k= -1, m= $-1-t$, n=0, q=0, r=0, s=s, t=t, u=0, v=0

40)a= $-1-3uy$, b= -1, c=0, e=u, f=0, g=0, h= $-3uy$, i=0, j=0, k= $-1+uy$, m= -1, n=0, q=0, r=0, s=0, t=0, u=u, v=uy

41)a= $-1+uy$, b= -1, c=0, e=u, f=0, g=0, h=uy, i=0, j=0, k= $-1+uy$, m= -1, n=0, q=0, r=0, s=0, t=0, u=u, v=uy

42)a= $-1+u$, b= -1, c=u, e=u, f=0, g=u, h=u, i=0, j=u, k= $-1+u$, m= -1, n=u, q=u, r=u, s=0, t=0, u=u, v=u

43)a= $-1-3uy$, b= -1, c=$uy\left(-5+2y^2\right)$, e=u, f=0, g=uy, h= $-3uy$, i=0, j= $-3uy$, k= $-1+uy$, m= -1, n=u, q=uy, r=u, s=0, t=0, u=u, v=uy

44)a= $-1+uy$, b= $-1+f$, c=0, e=u, f=f, g=0, h=uy, i=fy, j=0, k= $-1+uy$, m= $-1+f$, n=0, q=0, r=0, s=fy, t=f, u=u, v=uy

45)a= $-1+v$, b= $-1-f$, c=0, e= $-u$, f=f, g=0, h= $-v$, i= $-s$, j=0, k= $-1+v$, m= $-1-f$, n=0, q=0, r=0, s=s, t= $-f$, u=u, v=v

46)a= $-1+h$, b= -1, c=0, e= $-u$, f=0, g=0, h=h, i=0, j=0, k= $-1-v$, m= -1, n=0, q=0, r=0, s=0, t=0, u=u, v=v

47)a= $-1+h$, b= $-1-f$, c=0, e= $-u$, f=f, g=0, h=h, i= $-s$, j=0, k= $-1-v$, m= $-1-f$, n=0, q=0, r=0, s=s, t= $-f$, u=u, v=v

48)a=-1, b=b, c=0, e=0, f=f, g=0, h=0, i=i, j=0, k=-1, m=m, n=0, q=0, r=0, s=iy, t=t, u=0, v=0

49)a= $-1+h$, b=b, c=0, e=0, f=0, g=0, h=h, i=i, j=0, k= -1, m=$b-t$, n=0, q=0, r=0, s= $-i$, t=t, u=0, v=0

50)a= $-1+h$, b=b, c=0, e=0, f=0, g=0, h=h, i=i, j=0, k= -1, m= $-2-b-t$, n=0, q=0, r=0, s= $-i$, t=t, u=0, v=0

51)a= $-1+h$, b= $-1+2i$, c=0, e=0, f=0, g=0, h=h, i=i, j=0, k= -1, m= $-1-t$, n=0, q=0, r=0, s=i, t=t, u=0, v=0

52)a= $-1+h$, b= $-1-2i$, c=0, e=0, f=0, g=0, h=h, i=i, j=0, k= -1, m= $-1-t$, n=0, q=0, r=0, s=i, t=t, u=0, v=0

53)a= $-1+h$, b= $-1-f$, c=0, e=0, f=f, g=0, h=h, i=i, j=0, k= -1, m= $-1+2f-t$, n=0, q=0, r=0, s= $-i$, t=t, u=0, v=0

54)a= $-1+h$, b= $-1-f$, c=0, e=0, f=f, g=0, h=h, i=i, j=0, k= -1, m= $-1-2f-t$, n=0, q=0, r=0, s= $-i$, t=t, u=0, v=0

55)a= -1, b= -1, c= $-3q$, e=0, f=0, g= $-3q$, h=0, i=i, j=q, k= -1, m= $-1+iz$, n=qz, q=q, r=qz, s=i, t=iz, u=0, v=0

56)a= -1, b= $-1+f$, c= $-3rz$, e=0, f=f, g= $-3rz$, h=0, i=i, j=rz, k= -1, m= $-1+iz$, n=r, q=rz, r=r, s=i, t=iz, u=0, v=0

57)a=k, b= $-1+iyz$, c=0, e=0, f=iz, g=0, h=0, i=i, j=0, k=k, m= $-1+iyz$, n=0, q=0, r=0, s=iy, t=iyz, u=0, v=0

58)a=k, b= $-1+iyz$, c=0, e=0, f=iz, g=0, h=0, i=i, j=0, k=k, m= $-1+iyz$, n=0, q=0, r=0, s=iy, t=iyz, u=0, v=0

59)a= $-1+v$, b= $-1+iz$, c=0, e=0, f=iyz, g=0, h=0, i=i, j=0, k= -1, m$= -1 + iz$, n=0, q=0, r=0, s=iy, t=iz, u=0, v=v

60)a=a, b=b, c=0, e=0, f=0, g=0, h=$1 + a$, i=$(1 + b)x$, j=0, k=-1, m=-1, n=0, q=0, r=0, s=0, t=0, u=0, v=0

61)a=a, b=$-1 - \frac{(1+m)^2 x}{s} + sx$, c=0, e=0, f=0, g=0, h=$1 + a$, i=$-\frac{(1+m)^2}{s}$, j=0, k=$-1$, m=$m$, n=0, q=0, r=0, s=$s$, t=$-1 - m$, u=0, v=0

62)a=$-1 + u$, b=-1, c=$4u$, e=u, f=0, g=0, h=u, i=0, j=$4u$, k=$-1 + u$, m=-1, n=0, q=0, r=0, s=0, t=0, u=u, v=u

63)a=$-1 - 3uy$, b=-1, c=$-4uy$, e=u, f=0, g=0, h=$-3uy$, i=0, j=$-4uy$, k=$-1 + uy$, m=-1, n=0, q=0, r=0, s=0, t=0, u=u, v=uy

64)a=$-1 - \left(-4 + \sqrt{15}\right) f$, b=$-1 + f$, c=$-\left(-3 + \sqrt{15}\right) f$, e=0, f=$f$, g=$-f$, h=$-\left(-4 + \sqrt{15}\right) f$, i=0, j=$-\left(-4 + \sqrt{15}\right) f$, k=$-1$, m=$-1$, n=0, q=0, r=0, s=0, t=0, u=0, v=0

65)a=$-1+\left(4 + \sqrt{15}\right) f$, b=$-1+f$, c=$\left(3 + \sqrt{15}\right) f$, e=0, f=$f$, g=$-f$, h=$\left(4 + \sqrt{15}\right) f$, i=0, j=$\left(4 + \sqrt{15}\right) f$, k=$-1$, m=$-1$, n=0, q=0, r=0, s=0, t=0, u=0, v=0

66)a=$-1+\left(-4 + \sqrt{15}\right) f$, b=$-1+f$, c=$\left(-3 + \sqrt{15}\right) f$, e=0, f=$f$, g=$f$, h=$\left(-4 + \sqrt{15}\right) f$, i=0, j=$\left(-4 + \sqrt{15}\right) f$, k=$-1$, m=$-1$, n=0, q=0, r=0, s=0, t=0, u=0, v=0

67)a=$-1-\left(4 + \sqrt{15}\right) f$, b=$-1+f$, c=$-\left(3 + \sqrt{15}\right) f$, e=0, f=$f$, g=$f$, h=$-\left(4 + \sqrt{15}\right) f$, i=0, j=$-\left(4 + \sqrt{15}\right) f$, k=$-1$, m=$-1$, n=0, q=0, r=0, s=0, t=0, u=0, v=0

68)a=$-1 + uy$, b=-1, c=uy^3, e=u, f=0, g=uy, h=uy, i=0, j=uy, k=$-1 + uy$, m=-1, n=u, q=uy, r=u, s=0, t=0, u=u, v=uy

69)a=$-1 - 3uy$, b=-1, c=$-uy^3$, e=u, f=0, g=$-uy$, h=$-3uy$, i=0, j=$-uy$, k=$-1 + uy$, m=-1, n=$-u$, q=$-uy$, r=$-u$, s=0, t=0, u=u, v=uy

70)a=-1, b=-1, c=$-\left(1 + \sqrt{3}\right) g$, e=0, f=0, g=$g$, h=0, i=0, j=$-\left(2 + \sqrt{3}\right) g$, k=$-1$, m=$-1$, n=0, q=0, r=0, s=0, t=0, u=0, v=0

71)a=-1, b=-1, c=$\left(-1 + \sqrt{3}\right) g$, e=0, f=0, g=$g$, h=0, i=0, j=$\left(-2 + \sqrt{3}\right) g$, k=$-1$, m=$-1$, n=0, q=0, r=0, s=0, t=0, u=0, v=0

Proposition 63 *The only cases where σ satisfies the braid relations and is torsion compatible are :*

11)$a=-1+h,\ b=-1,\ c=h,\ e=0,\ f=0,\ g=0,\ h=h,\ i=0,\ j=h,\ k=-1,\ m=-1,\ n=0,$ $q=0,\ r=0,\ s=0,\ t=0,\ u=0,\ v=0$

12) $a=-1+j,\ b=-1,\ c=j,\ e=0,\ f=0,\ g=0,\ h=j,\ i=0,\ j=j,\ k=-1,\ m=-1,\ n=0,$ $q=0,\ r=0,\ s=0,\ t=0,\ u=0,\ v=0$

13)$a=-1+c,\ b=-1,\ c=c,\ e=0,\ f=0,\ g=0,\ h=c,\ i=0,\ j=c,\ k=-1,\ m=-1,\ n=0,$ $q=0,\ r=0,\ s=0,\ t=0,\ u=0,\ v=0$

14)$a=a,\ b=-1,\ c=1+a,\ e=0,\ f=0,\ g=0,\ h=1+a,\ i=0,\ j=1+a,\ k=-1,\ m=-1,\ n=0,$ $q=0,\ r=0,\ s=0,\ t=0,\ u=0,\ v=0$

34) $a=-2,\ b=-1,\ c=-1,\ e=0,\ f=0,\ g=0,\ h=-1,\ i=0,\ j=-1,\ k=-1,$ $m=-1,\ n=0,\ q=0,\ r=0,\ s=0,\ t=0,\ u=0,\ v=0$

35) $a=-1,\ b=-1,\ c=-1,\ e=0,\ f=0,\ g=0,\ h=1+a,\ i=0,\ j=-1,\ k=-1,\ m=-1,\ n=0,$ $q=0,\ r=0,\ s=0,\ t=0,\ u=0,\ v=0$

Proposition 64 *The only cases where σ satisfies the braid relations and preserves a diagonal metric are :*

All cases work except 2 , 11 ,12 , 13 , 14 , 15 , 39 , 47

The others are listed with additional restriction if necessary

1) $m=b=-1$

3)$m=a=b=-1$ *and* $t=0$

4)$b=m=a=k$

5) $f=0$ *and* $m=b=-1$

6)$b=-1$ *and* $t=0$

7) $a=b$

8) $b=-1$ *and* $f=t=0$

9) Ok

10) $g = 0$

11) $h = 0$

12) $j = 0$

13) $c = 0$

14 $a = -1$

16) Ok

17) Ok

18) Ok

19) Ok

20) Ok

21) Ok

22) Ok

23) Ok

24) Ok

25) $g = 0$

26) $g = 0$

27) $k = -1$

28) $v = 0$

29) Ok

30) $c = 0$

31) $j = 0$

32) $j = 0$

33) $a = -1$

34) $j = 0$

35) $a = -1$

36) $a = b = -1$

37) $j = f = 0$ and $b = -1$

38) $s = f = 0$ and $m = b = -1$

40) $u = 0$

41) $u = 0$

42) $u = 0$

43) $u = 0$

44) $u = s = fy$

45) $t = -f$

46) $h = v = u = 0$

48) $f = i = t = 0$ and $b = m = -1$

49) $s = t = t = 0$ and $b = -1$

50) $i = h = t = 0$ and $b = -1$

51) $i = h = t = s = 0$

52) $i = h = t = s = 0$

53) Ok

54) Ok

55) $i = q = 0$

56) $i = r = 0$

57) $i = 0$ and $k = -1$

58) $i = 0$ and $k = -1$

59) $i = v = 0$

60) $a = b = -1$

61) $s = 0$ and $s = a = -1$

62) $u = 0$

63) $u = 0$

64) $f = 0$

65) $f = 0$

66) $f = 0$

67) $f = 0$

68) $u = 0$

69) $u = 0$

70) $g = 0$

71) $g = 0$

3.8 Star compatibility

The condition for the covariant derivatives to be star compatibile is that (see proposition(51 in section 2.7)

$$c^{-1}ab \in C \Longrightarrow \sum_d (\hat{\Gamma}^a_{abd^{-1},d} + \delta_{a,d})((\hat{\Gamma}^{d^{-1}}_{b^{-1}a^{-1}c,c^{-1}})^* + \delta_{d^{-1},c^{-1}}) = \delta_{a,c}. \tag{3.47}$$

To calculate this in Mathematica, we use the following statements:

The Kronecker delta is

delta[ppp_, qqq_]:=Which[ppp == qqq, 1, True, 0]

The following function assigns the number (see section 3.6) in the class C to the representing matrix, with 0 returned if the matrix is not a representive of C.

numberfrommatrix[m_]:=Which[m == rep[1], 1, m == rep[2], 2,
m == rep[3], 3, m == rep[4], 4, m == rep[5], 5, m == rep[6], 6, m == rep[7], 7, m ==
rep[8], 8, True, 0]

The next function takes these numbers, and finds the corresponding Christoffel symbol which will be 0 if we use Γ^a_{bc} with a, b, c not in C.

newgam[a_, b_, c_]:=Which[$a == 0, 0, b == 0, 0, c == 0, 0$, True, symbol[$a, b, c$]]

Now we implent (3.47), using $\gamma[-]$ as conjugate

```
Sum[
(newgam[aaa, numberfrommatrix[rep[aaa].rep[bbb].Inverse[rep[ddd]]
], ddd] + delta[aaa, ddd])
(γ[newgam[
numberfrommatrix[Inverse[rep[ddd]]],
numberfrommatrix[Inverse[rep[bbb]].Inverse[rep[aaa]].rep[ccc]],
numberfrommatrix[Inverse[rep[ccc]]]
]] + delta[ddd, ccc]
)
− delta[aaa, ccc], {ddd, 1, 8}] //.{γ[0] → 0}
```

Now we use the letters for the Christoffel symbols

```
formula[aaa_, bbb_, ccc_]:=Sum[
(newgam[aaa, numberfrommatrix[rep[aaa].rep[bbb].Inverse[rep[ddd]]
], ddd] + delta[aaa, ddd])
(γ[newgam[
numberfrommatrix[Inverse[rep[ddd]]],
numberfrommatrix[Inverse[rep[bbb]].Inverse[rep[aaa]].rep[ccc]],
numberfrommatrix[Inverse[rep[ccc]]]
]] + delta[ddd, ccc]
), {ddd, 1, 8}]−delta[aaa, ccc]//. {endgam0[1, 0, 1, 0, 1] → a, endgam0[1, 0, −1, 0, 1] →
b, endgam0[1, 0, 1, 0, −1] →  c, endgam0[1, 0, −1, 0, −1] → d,
endgam0[1, 2, 1, 2, 1] → e, endgam0[1, 2, −1, 2, 1] → f, endgam0[1, 2, 1, 2, −1] → g, endgam0
```

$0, \mathrm{endgam0}[1,0,1,2,1] \to h,$

$\mathrm{endgam0}[1,0,-1,2,1] \to i, \mathrm{endgam0}[1,0,1,2,-1] \to j, \mathrm{endgam0}[1,0,-1,2,-1] \to$

$0, \mathrm{endgam0}[1,2,1,0,1] \to k, \mathrm{endgam0}[1,2,-1,0,1] \to m,$

$\mathrm{endgam0}[1,2,1,0,-1] \to n, \mathrm{endgam0}[1,2,-1,0,-1] \to 0, \mathrm{endgam0}[1,1,1,3,1] \to$

$0, \mathrm{endgam0}[-1,1,-1,3,-1] \to p, \mathrm{endgam0}[1,1,-1,3,1] \to q,$

$\mathrm{endgam0}[-1,1,1,3,-1] \to r, \mathrm{endgam0}[-1,1,-1,3,1] \to s, \mathrm{endgam0}[1,1,1,3,-1] \to$

$t, \mathrm{endgam0}[1,1,-1,3,-1] \to u, \mathrm{endgam0}[-1,1,1,3,1] \to v,$

$\gamma[0] \to 0\}$

And finally implement the conjugates, according to (section 3.5). This means that we restrict to the case of preserving a diagonal metric .

$\mathrm{MatrixForm}[\mathrm{Table}[\mathrm{formula}[3, y, x], \{x, 1, 8\}, \{y, 1, 8\}]] //.\{\gamma[a] \to b,$

$\gamma[b] \to a, \gamma[c] \to c, \gamma[d] \to d, \gamma[e] \to i, \gamma[f] \to h, \gamma[g] \to j, \gamma[h] \to f,$

$\gamma[i] \to e, \gamma[j] \to g, \gamma[k] \to m, \gamma[m] \to k, \gamma[n] \to n, \gamma[p] \to p, \gamma[q] \to q,$

$\gamma[r] \to r, \gamma[s] \to u. \gamma[t] \to v, \gamma[u] \to s, \gamma[v] \to t\}$

This comes from the summary of section(3.5), given in equation (3.46)

c, d, n, p, q and r are real and $a = b^*$ $\quad b = a^*$ $\quad e = i^*$

$f = h^*$ $\quad g = j^*$ $\quad h = f^*$ $\quad i = e^*$ $\quad j = g^*$

$k = m^*$ $\quad m = k^*$ $\quad s = u^*$ $\quad t = v^*$ $\quad u = s^*$ $\quad v = t^*$

In equation (3.47) we have three parameters each taking values 1, ... ,8. In the Nathematica code formula [x, y,z] we also have three parameters. The easiest way to look at this is to print 8 matrices, each 8 by 8. For example, here is formula [3, y, x] (as in the Mathematica above).

The result is that the following matrix needs to vanish :

$$
\left(\begin{array}{cc}
j(1+k)+(1+m)q+er+ir+ns+gt+nu+qv & 0 \\
e(1+m)+gn+jn+2qr+(1+k)s+tu+iv & h(1+m)+es+(1+k)t+f \\
-1+gj+(1+k)(1+m)+n^2+q^2+r^2+es+iu+tv & -1+fh+(1+k)(1+m)+es \\
gi+(1+k)n+(1+m)n+eq+qs+rt+ju+rv & 0 \\
g(1+m)+en+in+(1+k)q+rs+qt+ru+jv & 0 \\
i(1+k)+2nq+gr+jr+et+(1+m)u+sv & f(1+k)+es+ht+(1+m) \\
ei+gq+jq+2nr+(1+k)t+su+(1+m)v & (1+b)e+hi+i(1+k)+iu \\
ej+iq+(1+k)r+(1+m)r+gs+nt+qu+nv & 0
\end{array}\right.
$$

$$
\begin{array}{cc}
0 & cf+(1+a)g+ch+2gh+(1+b)j+2fj \\
(1+a)s+fu+(1+m)u+tu & (1+a)f+cg+(1+b)h+2fh+cj+2gj \\
-1+(1+a)(1+b)+3iu & -1+(1+a)(1+b)+c^2+3fh+3gj \\
0 & (1+a)c+(1+b)c+3gh+3fj \\
0 & cf+(1+a)g+ch+2gh+(1+b)j+2fj \\
(1+a)s+fu+(1+m)u+tu & (1+a)f+cg+(1+b)h+2fh+cj+2gj \\
(1+a)s+fu+(1+m)u+tu & (1+a)f+cg+(1+b)h+2fh+cj+2gj \\
0 & cf+(1+a)g+ch+2gh+(1+b)j+2fj
\end{array}
$$

$$
\begin{array}{cc}
ej+iq+(1+k)r+(1+m)r+gs+nt+qu+nv & 0 \\
ei+gq+jq+2nr+(1+k)t+su+(1+m)v & (1+b)e+hi+i(1+k)+iv \\
-1+gj+(1+k)(1+m)+n^2+q^2+r^2+es+iu+tv & -1+fh+(1+k)(1+m)+es+tv \\
gi+(1+k)n+(1+m)n+eq+qs+rt+ju+rv & 0 \\
j(1+k)+(1+m)q+er+ir+ns+gt+nu+qv & 0 \\
e(1+m)+gn+jn+2qr+(1+k)s+tu+iv & h(1+m)+es+(1+k)t+fv \\
i(1+k)+2nq+gr+jr+et+(1+m)u+sv & f(1+k)+es+ht+(1+m)v \\
g(1+m)+en+in+(1+k)q+rs+qt+ru+jv & 0
\end{array}
$$

$$
\left.
\begin{array}{cc}
0 & g(1+m)+en+in+(1+k)q+rs+qt+ru+jv \\
f(1+k)+es+ht+(1+m)v & i(1+k)+2nq+gr+jr+et+(1+m)u+sv \\
-1+fh+(1+k)(1+m)+es+tv & -1+gj+(1+k)(1+m)+n^2+q^2+r^2+es+iu+tv \\
0 & gi+(1+k)n+(1+m)n+eq+qs+rt+ju+rv \\
0 & ej+iq+(1+k)r+(1+m)r+gs+nt+qu+nv \\
(1+b)e+hi+i(1+k)+iv & ei+gq+jq+2nr+(1+k)t+su+(1+m)v \\
h(1+m)+es+(1+k)t+fv & e(1+m)+gn+jn+2qr+(1+k)s+tu+iv \\
0 & j(1+k)+(1+m)q+er+ir+ns+gt+nu+qv
\end{array}
\right)
$$

The tree for solving these equations is given in Appendix B. For the note on the approach taken to solve the problem, see the note in section 3.7. The result of these methods using Mathematica is given in the following proposition.

Proposition 65 *The covariant derivative ∇ in proposition 61 (∇ assumed to preserve a diagonal metric) is is star compatible if and only if one of the following cases apply :*

1) $a=-1+x$, $b=-1+\frac{1}{x}$, $c=0$, $e=0$, $f=0$, $g=0$, $h=0$, $i=0$, $j=0$,
$k=-1$, $m=-1$, $n=0$, $q=0$, $r=0$, $s=0$, $t=\frac{1}{v}$, $u=0$, $v\neq 0$,
$w=0$, $x\neq 0$, *—x—=1, —v—=1 and d , p are real.*

2) $a = -1 + x,\ b = -1 + \frac{1}{x},\ c{=}0,\ e{=}0,\ f{=}0,\ g{=}0,\ h{=}0,\ i{=}0,\ j{=}0,$
$k = -1 + w,\ m = -1 + \frac{1}{w},\ n{=}0,\ q{=}0,\ r{=}0,\ s{=}0,\ t{=}0,\ u{=}0, v{=}0,$
$w \neq 0, x \neq 0,\ -x-=1,\ -w-=1$ *and* d , p *are real.*

Combining this with the long list in section(3.7) gives

Proposition 66 *The covariant derivative* ∇ *in proposition 65 additionally gives* σ *satisfying the braid relations in the following cases :*
1) case 1 in proposition (65), is the same as case 7 in the list in section(3.7), with an extra condition $t = \frac{1}{x}$, $v = x$ *, giving:*
$a = -1 + x,\ b = -1 + \frac{1}{x},\ c{=}0,\ e{=}0,\ f{=}0,\ g{=}0,\ h{=}0,\ i{=}0,\ j{=}0,$
$k = -1,\ m = -1,\ n{=}0,\ q{=}0,\ r{=}0,\ s{=}0,\ t{=}\frac{1}{x},\ u{=}0, v = x,$
$w{=}0, x \neq 0, y{=}\frac{1}{x},\ z{=}0$
2) case 2 in proposition (65), is the same as case 4 in the list in section(3.7), with an extra condition $x = w$ *giving :*
$a = -1 + x,\ b = -1 + \frac{1}{x},\ c{=}0,\ e{=}0,\ f{=}0,\ g{=}0,\ h{=}0,\ i{=}0,\ j{=}0,$
$k = -1 + w,\ m = -1 + \frac{1}{w},\ n{=}0,\ q{=}0,\ r{=}0,\ s{=}0,\ t{=}0,\ u{=}0, v{=}0,$
$w \neq 0, y{=}\frac{1}{w},\ z{=}\frac{1}{w}$

Proposition 67 *There are no cases which satisfy the conditions of proposition 65, and are additionally star compatible and torsion compatible.*

proof : Use proposition 65, and section 3.4.2. □

Classically for a Riemannian metric there is a unique torsion zero covariant derivative which preserves the metric, called the Levi-Civita connection. We see that this is not the case here. There are two ways of looking at this result. The first is that in a noncommutative context the existence of such covariant derivatives is just not expected. The second would be to find weaker conditions for which a weaker idea of "Levi-Civita connection" existed.

3.9 Calculating the Torsion

Using the single letter labels for the Christoffel symbols (3.4), the condition for torsion compatibility (3.4.2) is

$s = r = i = n = 0 \quad b = c \quad f = g$

$h = v - 1 = k \quad j = t - 1 = m = q - 1$

Assuming that we have torsion compatibility, we have

$-\nabla\xi^0 = a\xi^0 \otimes \xi^0 + b(\xi^0 \otimes \xi^{0'} + \xi^{0'} \otimes \xi^0) + d\xi^{0'} \otimes \xi^{0'} + e(\xi^{1'} \otimes \xi^{1'} + \xi^2 \otimes \xi^2 + \xi^{3'} \otimes \xi^{3'}) + g(\xi^1 \otimes \xi^{1'} + \xi^{1'} \otimes \xi^1 + \xi^2 \otimes \xi^{2'} + \xi^{2'} \otimes \xi^2 + \xi^3 \otimes \xi^{3'} + \xi^{3'} \otimes \xi^3) + k(\xi^0 \otimes \xi^{1'} + \xi^{1'} \otimes \xi^0 + \xi^0 \otimes \xi^2 + \xi^2 \otimes \xi^0 + \xi^0 \otimes \xi^{3'} + \xi^{3'} \otimes \xi^0) + m(\xi^0 \otimes \xi^1 + \xi^0 \otimes \xi^{2'} + \xi^0 \otimes \xi^3 + \xi^1 \otimes \xi^0 + \xi^{2'} \otimes \xi^0 + \xi^3 \otimes \xi^0) + p(\xi^{1'} \otimes \xi^3 + \xi^2 \otimes \xi^1 + \xi^{3'} \otimes \xi^{2'}) + (m+1)(\xi^1 \otimes \xi^{3'} + \xi^3 \otimes \xi^{1'} + \xi^{2'} \otimes \xi^{1'} + \xi^1 \otimes \xi^2 + \xi^{2'} \otimes \xi^{3'} + \xi^3 \otimes \xi^2) + (k+1)(\xi^3 \otimes \xi^1 + \xi^1 \otimes \xi^{2'} + \xi^{2'} \otimes \xi^3) + u(\xi^{3'} \otimes \xi^1 + \xi^{1'} \otimes \xi^{2'} + \xi^2 \otimes \xi^3)$

Now apply $\wedge$ to $-\nabla\xi^0$ to get

$-\wedge\nabla\xi^0 = a\xi^0 \wedge \xi^0 + b(\xi^0 \wedge \xi^{0'} + \xi^{0'} \wedge \xi^0) + d\xi^{0'} \wedge \xi^{0'} + e(\xi^{1'} \wedge \xi^{1'} + \xi^2 \wedge \xi^2 + \xi^{3'} \wedge \xi^{3'}) + g(\xi^1 \wedge \xi^{1'} + \xi^{1'} \wedge \xi^1 + \xi^2 \wedge \xi^{2'} + \xi^{2'} \wedge \xi^2 + \xi^3 \wedge \xi^{3'} + \xi^{3'} \wedge \xi^3) + k(\xi^0 \wedge \xi^{1'} + \xi^{1'} \wedge \xi^0 + \xi^0 \wedge \xi^2 + \xi^2 \wedge \xi^0 + \xi^0 \wedge \xi^{3'} + \xi^{3'} \wedge \xi^0) + m(\xi^0 \wedge \xi^1 + \xi^0 \wedge \xi^{2'} + \xi^0 \wedge \xi^3 + \xi^1 \wedge \xi^0 + \xi^{2'} \wedge \xi^0 + \xi^3 \wedge \xi^0) + p(\xi^{1'} \wedge \xi^3 + \xi^2 \wedge \xi^1 + \xi^{3'} \wedge \xi^{2'}) + (m+1)(\xi^1 \wedge \xi^{3'} + \xi^3 \wedge \xi^{1'} + \xi^{2'} \wedge \xi^{1'} + \xi^1 \wedge \xi^2 + \xi^{2'} \wedge \xi^{3'} + \xi^3 \wedge \xi^2) + (k+1)(\xi^3 \wedge \xi^1 + \xi^1 \wedge \xi^{2'} + \xi^{2'} \wedge \xi^3) + u(\xi^{3'} \wedge \xi^1 + \xi^{1'} \wedge \xi^{2'} + \xi^2 \wedge \xi^3)$

From the list in section(3.3), we have

1) $\xi^a \wedge \xi^a = 0$

2) $\xi^a \wedge \xi^{a^{-1}} + \xi^{a^{-1}} \wedge \xi^a = 0$, so we get

$-\wedge\nabla\xi^0 = k(\xi^0 \wedge \xi^{1'} + \xi^{1'} \wedge \xi^0 + \xi^0 \wedge \xi^2 + \xi^2 \wedge \xi^0 + \xi^0 \wedge \xi^{3'} + \xi^{3'} \wedge \xi^0) + m(\xi^0 \wedge \xi^1 + \xi^0 \wedge \xi^{2'} + \xi^0 \wedge \xi^3 + \xi^1 \wedge \xi^0 + \xi^{2'} \wedge \xi^0 + \xi^3 \wedge \xi^0) + p(\xi^{1'} \wedge \xi^3 + \xi^2 \wedge \xi^1 + \xi^{3'} \wedge \xi^{2'}) + (m+1)(\xi^1 \wedge \xi^{3'} + \xi^3 \wedge \xi^{1'} + \xi^{2'} \wedge \xi^{1'} + \xi^1 \wedge \xi^2 + \xi^{2'} \wedge \xi^{3'} + \xi^3 \wedge \xi^2) + (k+1)(\xi^3 \wedge \xi^1 + \xi^1 \wedge \xi^{2'} + \xi^{2'} \wedge \xi^3) + u(\xi^{3'} \wedge \xi^1 + \xi^{1'} \wedge \xi^{2'} + \xi^2 \wedge \xi^3)$

From (2.8)

$$d\xi^c = \sum_{a\in C}(\xi^a \wedge \xi^c + \xi^c \wedge \xi^a) - \sum_{b,a\in C} \delta_{c,ab}\xi^a \wedge \xi^b.$$

We get

$d\xi^0 = (\xi^1 \wedge \xi^0 + \xi^0 \wedge \xi^1 + \xi^{2'} \wedge \xi^0 + \xi^0 \wedge \xi^{2'} + \xi^3 \wedge \xi^0 + \xi^0 \wedge \xi^3) - (\xi^1 \wedge \xi^{2'} + \xi^{2'} \wedge \xi^3 + \xi^3 \wedge \xi^1)$

hence

$d\xi^0 - \wedge\nabla\xi^0 = k(\xi^0 \wedge \xi^{1'} + \xi^{1'} \wedge \xi^0 + \xi^0 \wedge \xi^2 + \xi^2 \wedge \xi^0 + \xi^0 \wedge \xi^{3'} + \xi^{3'} \wedge \xi^0) + (m + 1)(\xi^0 \wedge \xi^1 + \xi^0 \wedge \xi^{2'} + \xi^0 \wedge \xi^3 + \xi^1 \wedge \xi^0 + \xi^{2'} \wedge \xi^0 + \xi^3 \wedge \xi^0) + p(\xi^{1'} \wedge \xi^3 + \xi^2 \wedge \xi^1 + \xi^{3'} \wedge \xi^{2'}) + (m+1)(\xi^1 \wedge \xi^{3'} + \xi^3 \wedge \xi^{1'} + \xi^{2'} \wedge \xi^{1'} + \xi^1 \wedge \xi^2 + \xi^{2'} \wedge \xi^{3'} + \xi^3 \wedge \xi^2) + k(\xi^3 \wedge \xi^1 + \xi^1 \wedge \xi^{2'} + \xi^{2'} \wedge \xi^3) + u(\xi^{3'} \wedge \xi^1 + \xi^{1'} \wedge \xi^{2'} + \xi^2 \wedge \xi^3)$

Use relations (3-10) in the list in the section (3.3) to substitute

3) $\xi^{\pi'_{i+1}} \wedge \xi^{\pi'_{i+3}} = -\xi^{\pi_{i+2}} \wedge \xi^{\pi'_{i+1}} - \xi^{\pi'_{i+3}} \wedge \xi^{\pi_{i+2}}$(not used)

4) When $i = 0$ we get $\xi^3 \wedge \xi^1 = -\xi^1 \wedge \xi^{2'} - \xi^{2'} \wedge \xi^3$

When $i = 1$ we get $\xi^0 \wedge \xi^2 = -\xi^2 \wedge \xi^{3'} - \xi^{3'} \wedge \xi^0$

When $i = 3$ we get $\xi^2 \wedge \xi^0 = -\xi^0 \wedge \xi^{1'} - \xi^{1'} \wedge \xi^2$

5) $\xi^{3'} \wedge \xi^{2'} = -\xi^0 \wedge \xi^1 - \xi^{2'} \wedge \xi^0 - \xi^1 \wedge \xi^{3'}$

6) $\xi^{1'} \wedge \xi^{0'} = -\xi^2 \wedge \xi^3 + \xi^{0'} \wedge \xi^2 - \xi^3 \wedge \xi^{-1'}$(not used)

7) $\xi^3 \wedge \xi^2 = -\xi^0 \wedge \xi^3 - \xi^1 \wedge \xi^0 - \xi^2 \wedge \xi^1$

8) $\xi^{0'} \wedge \xi^{1'} = -\xi^{3'} \wedge \xi^{0'} - \xi^{2'} \wedge \xi^{3'} - \xi^{1'} \wedge \xi^{2'}$(not used)

9) $\xi^{2'} \wedge \xi^{1'} = -\xi^0 \wedge \xi^{2'} - \xi^3 \wedge \xi^0 - \xi^{1'} \wedge \xi^3$

10) $\xi^{0'} \wedge \xi^{3'} = -\xi^2 \wedge \xi^{0'} - \xi^1 \wedge \xi^2 - \xi^{3'} \wedge \xi^1$(not used)

Now use these relations to get rid of all cases

$d\xi^0 - \wedge\nabla\xi^0 = k(\xi^{1'} \wedge \xi^0 - \xi^2 \wedge \xi^{3'} - \xi^{1'} \wedge \xi^2 + \xi^0 \wedge \xi^{3'}) + (m+1)(\xi^0 \wedge \xi^1 + \xi^{2'} \wedge \xi^0 + \xi^3 \wedge \xi^{1'} + \xi^1 \wedge \xi^{3'} - \xi^{1'} \wedge \xi^{3'} + \xi^1 \wedge \xi^2 + \xi^{2'} \wedge \xi^{3'} - \xi^2 \wedge \xi^1) + p(\xi^{1'} \wedge \xi^3 + \xi^2 \wedge \xi^1 - \xi^0 \wedge \xi^1 - \xi^{2'} \wedge \xi^0 - \xi^1 \wedge \xi^{3'}) + u(\xi^{3'} \wedge \xi^1 + \xi^{1'} \wedge \xi^{2'} + \xi^2 \wedge \xi^3)$

The condition for the torsion to vanish is that $k = p = u = 0$ and $m = -1$.

From section 3.5 we have the condition for ∇ to preserve a diagonal metric, which includes $k = m^*$, so we can not have torsion zero for such a metric preserving ∇.

Proposition 68 *Torsion zero is given by covariant derivatives with* $h = s = r = i = n = k = p = u = q = t = 0$, $j = m = -1$, $v = 1$, $b = c$, $f = g$.

Corollary 69 *There are no zero torsion covariant derivatives which satisfy the braid relations.*

proof : Use proposition 63, and proposition 68. □

Proposition 70 *There are no zero torsion covariant derivatives which preserve a diagonal metric.*

proof : Use proposition 61, and proposition 68. □

3.10 Summary for chapter 3

We summarise the properties of "covariant derivatives" on A_4 for the given calculus which are S_4 invariant:

M	Preserves a diagonal metric	from Proposition 61
S	Preserves the star operation	note $*$
T	Torsion compatible	from section 3.4.2
$T0$	Zero torsion	from Proposition 68
B	Braid relations	from section 3.7

$*$ **Note** : We did not consider S by itself, but only $M \cap S$, because of the complexity of the equations.

Property	*result*
$M \cap S$	Proposition 65
$M \cap S \cap B$	Proposition 66
$T0 \cap B$	$\emptyset$ (see corollary 69)
$B \cap T$	Proposition 63
$B \cap M$	Proposition 64
$M \cap S \cap T$	$\emptyset$ (see proposition 67)
$M \cap T$	Proposition 62
$M \cap T0$	$\emptyset$ (see proposition 70)

(3.48)

Chapter 4

The Heisenberg group

Here we will study a rather different case, to the previus chapter, we now take an infinite discrete group. This was recently taken as an example of a noncommutative fibering with a classical base space by [43] and [44]

4.1 The Heisenberg group

The Heisenberg group H is defined to be following subgroup of $M_3(\mathbb{R})$ under multiplication.

$$\left\{ \begin{pmatrix} 1 & n & k \\ 0 & 1 & m \\ 0 & 0 & 1 \end{pmatrix} : n, m, k \in Z \right\}$$

We can take generators for the group u, v , w , where w is central and there is one more relation $uv = wvu$. The generators correspond to the matrices

$$u = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad v = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad w = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

There is an isomorophism $\theta : H \longrightarrow H$, for every matrix

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \in SL(2, Z),$$

given by the formla $\theta(u) = u^a v^b$, $\theta(v) = u^c v^d$, $\theta(w) = w$.

To check this,

$$\begin{aligned} \theta(u)\theta(v) &= u^a v^b u^c v^d \\ &= w^{-bc} u^{a+c} v^{b+d} \\ \theta(w)\theta(v)\theta(u) &= w u^c v^d u^a v^b \\ &= w^{1-ad} u^{c+a} v^{d+b} \end{aligned} \tag{4.1}$$

so the relation $uv = wvu$ implies $1 = ad - bc$.

From $vu = w^{-1}uv$ and w central we can prove by induction that $v^n u^m = w^{-nm} u^m v^n$.

4.2 Differential calculus on the Heisenberg group

We assume that there is a differential calculus on the group algebra $\mathbb{K}H$ of H . For $x \in \{u, v, w\}$, we write $e^x = x^{-1} \cdot dx$, a left invariant element of $\Omega^1 \mathbb{K}H$ (see section

2.2)

We suppose that $\Omega^1\mathbb{K}H$ is free as left $\mathbb{K}H$ module, with generators $\{e^u, e^v, e^w\}$. This means that every element of $\Omega^1\mathbb{K}H$ can be written uniquely as $a^u.e^u + a^v.e^v + a^w.e^w$, for $a^u, a^v, a^w \in \mathbb{K}H$.

We assume that each x commutes with e^x, and that w commutes with all of them. We assume that $e^v \cdot u = u \cdot (e^v + A)$, and $e^u \cdot v = v \cdot (e^u + B)$, and furthermore that A commutes with u and B commutes with v. By induction

$$u^{-n}e^v u^n = e^v + nA \quad , \quad v^{-n}e^u v^n = e^u + nB \tag{4.2}$$

As w is central, we get

$$uw = wu$$

$$du.w + u.dw = w.du + dw.u.$$

We assumed that w commutes with each e^x. So,

$$u.dw = dw.u,$$

i.e. u commutes with e^w. Likewise we see that v commutes with e^w, so e^w is central. From the relation on the group $uv = wvu$, we apply d to get

$$\begin{aligned} v^{-1} \cdot e^u \cdot v + e^v &= e^w + u^{-1}e^v u + e^u && (4.3) \\ e^u + B + e^v &= e^w + e^v + A + e^u && (4.4) \end{aligned}$$

So, the relation on the group implies

$$B - A = e^w. \tag{4.5}$$

We want θ to preserve the relations of the differential calculus, so
$\theta(e^u) = \theta(u)^{-1}d\theta(u)$,
$\theta(e^u) = v^{-b}u^{-a}(d(u^a) \cdot v^b + u^a \cdot d(v^b))$.
As we assume that u commutes with du and v commutes with dv,
$u^{-n}d(u^n) = ne^u$, $v^{-n}d(v^n) = ne^v$. So,

$$\begin{aligned}
\theta(e^u) &= av^{-b}e^u v^b + be^v \\
&= ae^u + abB + be^v \quad \textit{Similarly,} \\
\theta(e^v) &= \theta(v)^{-1}d\theta(v) \\
&= v^{-d}u^{-c}(d(u^c) \cdot v^d + u^c \cdot d(v^d)) \\
&= cv^{-d}e^u v^d + de^v \\
&= ce^u + cdB + de^v \quad \textit{and} \\
\theta(e^w) &= e^w.
\end{aligned}$$

In the calculus $ue^u u^{-1} = e^u$ so if we apply θ to both sides we get $\theta(u)\theta(e^u)\theta(u^{-1}) = \theta(e^u)$, and this must be true, and gives another condition on θ.

$$\begin{aligned}
\theta(u)\theta(e^u)\theta(u^{-1}) &= \theta(e^u) \\
u^a v^b \theta(e^u) v^{-b} u^{-a} &= \theta(e^u) \\
u^a v^b (ae^u + abB + be^v) v^{-b} u^{-a} &= ae^u + abB + be^v \\
v^b (ae^u + abB + be^v) v^{-b} &= u^{-a}(ae^u + abB + be^v) u^a \\
v^b ae^u v^{-b} + abB + be^v &= ae^u + u^{-a} abBu^a + u^{-a} be^v u^a
\end{aligned}$$

If we also assume that B commutes with u, then we get

$$\begin{aligned}
ae^u - abB + abB + be^v &= ae^u + abB + be^v + abA \\
ae^u + be^v &= ae^u + be^v + ab(B + A).
\end{aligned}$$

So $ab(A + B) = 0$. But this should be true for all matrices in $SL_2(\mathbb{Z})$, so $A + B = 0$. We now apply θ to $ve^v v^{-1} = e^v$

$$\begin{aligned}
\theta(v)\theta(e^v)\theta(v^{-1}) &= \theta(e^v) \\
u^c v^d \theta(e^v) v^{-d} u^{-c} &= \theta(e^v) \\
v^d (ce^u + cdB + de^v) v^{-d} &= u^{-c}(ce^u + cdB + de^v) u^c \\
v^d ce^u v^{-b} + cdB + de^v &= ce^u + u^{-c} cdBu^c + u^{-c} de^v u^c \\
ce^u - cdB + cdB + de^v &= ce^u + cdB + de^v + cdA \\
ce^u + de^v &= ce^u + de^v + cd(B + A)
\end{aligned}$$

Then $cd(A+B)=0$, so $B+A=0$, and we find $B=-A$.

So $A=\frac{-1}{2}e^w$, $B=\frac{1}{2}e^w$ from (4.5), and e^w commutes with u, and v and w.

We summarise this in the following proposition

Proposition 71 *There is a differential calculus on* $\mathbb{K}H$ *with (left invariant) generators* $e^x = x^{-1}dx$ *for* $x \in \{u, v, w\}$, *and relations.*

$x \cdot e^x = e^x \cdot x$ *for all* $x \in \{u, v, w\}$

$x \cdot e^w = e^w \cdot x$

$w \cdot e^x = e^x \cdot w$

$u^{-n}e^v u^n = e^v - \frac{n}{2}e^w$

$v^{-n}e^u v^n = e^v + \frac{n}{2}e^w$.

Further the map θ in section 4.1 induced by the matrix

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \in SL(2, Z),$$

extends to a map of 1-forms given by

$$\theta(e^w) = e^w,$$

$$\theta(e^u) = a.e^u + be^v + \frac{ab}{2}e^w,$$

$$\theta(e^v) = ce^u + de^v + \frac{cd}{2}e^w.$$

Proposition 72 *The left invariant 1-forms are just sums of numbers times* e^x .

Proof : *Suppose* $a.e^x$ *is invariant and apply* λ, *to get*

$$a_{(1)} \otimes a_{(2)} e^x = e \otimes a e^x,$$

so, as we have free generators, $a_{(1)} \otimes a_{(2)} = e \otimes a$. *Then apply* ϵ *to the second factor* $a_{(1)}\epsilon(a_{(2)}) = e.\epsilon(a)$, *so* $a = e.\epsilon(a)$ *is a multiple of the identity.* $\square$

This proposition means that where we do calculations on left invariant forms, we do not have to worry about $y.e^x$ where y is an algebra element, we just have a numerical coefficient. I.e. the forms e^x are a vector space basis for the left invariant forms.

4.2.1 Star operation

The group algebra $\mathbb{K}H$ has a star operator $x^* = x^{-1}$, for all $x \in H$. For this to extend to the 1-forms we need $(e^x)^* = (x^{-1}dx)^* = dx^* \cdot x^{*-1}$, so

$$\begin{aligned} (e^x)^* &= dx^{-1} \cdot x \\ &= -x^{-1} dx \\ &= -e^x \end{aligned}$$

Here we have used $d(xx^{-1}) = 0$, using the product rule

$$dx \cdot x^{-1} + x \cdot d(x^{-1}) = 0$$

and rearranging to get

$$d(x^{-1}) = -x^{-1} \cdot dx \cdot x^{-1}.$$

4.2.2 The higher forms

On a Hopf algebra H we have a coproduct which is compatible with the product,

$$\Delta h = h_{(1)} \otimes h_{(2)},$$

$$\Delta(hg) = h_{(1)}g_{(1)} \otimes h_{(2)}g_{(2)}.$$

This coproduct may extend to a left coaction on the one forms on H (see section 2.2): This coaction has the formula

$$\lambda(h \cdot dg) = h_{(1)}g_{(1)} \otimes h_{(2)} \cdot dg_{(2)} \in H \otimes \Omega^1 H$$

and there may be a right coaction

$$\varrho(h \cdot dg) = h_{(1)} \cdot dg_{(1)} \otimes h_{(2)}g_{(2)} \in \Omega^1 H \otimes H$$

for x in the group, $\Delta x = x \otimes x$, $e^x = x^{-1} \cdot dx$, and then

$$\lambda(e^x) = x^{-1}x \otimes x^{-1} \cdot dx$$

$$\varrho(e^x) = x^{-1} \cdot dx \otimes x^{-1}x$$

so $\lambda(e^x) = e \otimes e^x$ and $\varrho(e^x) = e^x \otimes e$, and so e^x are both left and right invariant. This means that Woronowicz braiding (see proposition 13 and section 2.2) is just transposition.

$$\Psi(e^x \otimes e^y) = e^y \otimes e^x.$$

This means that the kernel of wedge is the symmetric tensors. i.e. $e^x \otimes e^x$ and $e^x \otimes e^y + e^y \otimes e^x$, this is because $\Psi(e^x \otimes e^y + e^y \otimes e^x) = e^y \otimes e^x + e^x \otimes e^y$.

Proposition 73 $de^x = 0$

Proof :

$$\begin{aligned} de^x &= d(x^{-1}dx) \\ &= dx^{-1} \wedge dx \\ &= -x^{-1}dx \wedge x^{-1}dx \\ &= -e^x \wedge e^x = 0, \end{aligned} \tag{4.6}$$

as $e^x \otimes e^x$ *is in the kernel of* $\wedge$. $\square$.

4.3 Covariant derivatives on the Heisenberg group

We consider a bimodule covariant derivative ∇ on $E = \Omega^1 A$ (see section 2.6), where A is the group algebra of the Heisenberg group. We suppose that ∇ is left invariant to the coaction of A, so each $\nabla(e^x)$ is a sum of numbers times $e^y \otimes e^z$.

We suppose ∇ to be a bimodule connection, i.e. that there is a bimodule map, $\sigma : \Omega^1 A \otimes_A \Omega^1 A \longrightarrow \Omega^1 A \otimes_A \Omega^1 A$, defined by $\sigma(e^x \otimes dy) = \nabla(e^x \cdot y) - \nabla(e^x) \cdot y$ (see definition 39).

The definition of a left covariant derivative is

$$\nabla(y \cdot e^x) = dy \otimes e^x + y \cdot \nabla(e^x) \tag{4.7}$$

Suppose $y = w$, and remember that w is central and use 4.7

$$\begin{aligned}
\sigma(e^x \otimes dw) &= \nabla(e^x \cdot w) - \nabla(e^x) \cdot w \\
&= \nabla(w.e^x) - w \cdot \nabla(e^x) \\
&= dw \otimes e^x + w.\nabla(e^x) - w.\nabla(e^x) \\
&= dw \otimes e^x.
\end{aligned}$$

Suppose $x = w$, remember that w commutes with A,

$\sigma(e^w \otimes dy) = dy \otimes e^w + y \cdot \nabla(e^w) - \nabla(e^w) \cdot y$

and the other cases are :

$\sigma(e^u \otimes du) = du \otimes e^u + u \cdot \nabla(e^u) - \nabla(e^u) \cdot u$

$\sigma(e^v \otimes dv) = dv \otimes e^v + v \cdot \nabla(e^v) - \nabla(e^v) \cdot v$

$\sigma(e^v \otimes du) = du \otimes e^v - \frac{1}{2}u\nabla(e^w) - \frac{1}{2}du \otimes e^w + u \cdot \nabla(e^v) - \nabla(e^v) \cdot u$

$\sigma(e^u \otimes dv) = dv \otimes e^u + v \cdot \nabla(e^u) - \nabla(e^u) \cdot v + \frac{1}{2}v\nabla(e^w) + \frac{1}{2}dv \otimes e^w.$

As σ is a right module map

$\sigma(e^x \otimes dw)w^{-1} = \sigma(e^x \otimes e^w) = e^w \otimes e^x$

$\sigma(e^w \otimes e^y) = e^y \otimes e^w + y\nabla(e^w)y^{-1} - \nabla(e^w)$

$\sigma(e^u \otimes e^u) = e^u \otimes e^u + u\nabla(e^u)u^{-1} - \nabla(e^u)$

$\sigma(e^v \otimes e^v) = e^v \otimes e^v + v\nabla(e^v)v^{-1} - \nabla(e^v)$

$\sigma(e^v \otimes e^u) = e^u \otimes e^v - \frac{1}{2}u\nabla(e^w)u^{-1} + u\nabla(e^v)u^{-1} - \nabla(e^v)$

$\sigma(e^u \otimes e^v) = e^v \otimes e^u + \frac{1}{2}v\nabla(e^w)v^{-1} + v\nabla(e^u)v^{-1} - \nabla(e^u).$

Lemma 74 *For*

$$g = \sum_{x,y \in \{u,v,w\}} g_{xy} e^x \otimes e^y \tag{4.8}$$

we have

$$\begin{aligned} u^{-1}gu-g &= e_u\otimes e_w(\frac{-1}{2}g_{uv})+e_w\otimes e_u(\frac{-1}{2}g_{vu})+e_w\otimes e_v(\frac{-1}{2}g_{vv}) \\ &\quad +e_v\otimes e_w(\frac{-1}{2}g_{vv})+e_w\otimes e_w(\frac{1}{4}g_{vv}-\frac{1}{2}g_{vw}-\frac{1}{2}g_{wv}) \\ v^{-1}gv-g &= e_w\otimes e_u(\frac{1}{2}g_{uu})+e_u\otimes e_w(\frac{1}{2}g_{uu})+e_w\otimes e_v(\frac{1}{2}g_{uv}) \\ &\quad +e_v\otimes e_w(\frac{1}{2}g_{vu})+e_w\otimes e_w(\frac{1}{4}g_{uu}+\frac{1}{2}g_{uw}+\frac{1}{2}g_{wu}). \end{aligned}$$

Proof: *Use the commutaion relation in proposition 71.*

As a consequence of this result, if we assume that $\nabla(e^w)$ commutes with all elements of A, then for some numbers a, b , c we have (putting $g=\nabla(e^w)$ in lemma 74)

$$\nabla(e^w)=ae^w\otimes e^w+b(e^v\otimes e^w-e^w\otimes e^v)+c(e^u\otimes e^w-e^w\otimes e^u) \tag{4.9}$$

We write the covariant derivatives of e^u and e^v as

$$\nabla e^u=\sum\phi_{xy}e^x\otimes e^y \quad \textit{and} \quad \nabla e^v=\sum\psi_{xy}e^x\otimes e^y. \tag{4.10}$$

Proposition 75 *To be torsion compatible (see proposition 41) the following conditions on ∇ in 4.9 and 4.10 must be satisfied, $b=c=0$, $\phi_{vu}=\phi_{uv}$, $\psi_{vu}=\psi_{uv}$.*

Proof: The condition for ∇ to be torsion compatible is Image $(\sigma+id)\subset$ kernel $\wedge$. Now calculate

$(\sigma+id)(e^w\otimes e^u)=e^u\otimes e^w+e^w\otimes e^u$

$(\sigma+id)(e^w\otimes e^w)=2e^w\otimes e^w$

$(\sigma+id)(e^u\otimes e^w)=e^w\otimes e^u+e^u\otimes e^w$

$$(\sigma + id)(e^v \otimes e^w) = e^w \otimes e^v + e^v \otimes e^w$$
$$(\sigma + id)(e^w \otimes e^v) = e^v \otimes e^w + e^w \otimes e^v$$
$$(\sigma + id)(e^u \otimes e^u) = 2e^u \otimes e^u + u(\nabla e^u)u^{-1} - \nabla e^u$$
$$(\sigma + id)(e^v \otimes e^v) = 2e^v \otimes e^v + v(\nabla e^v)v^{-1} - \nabla e^v$$
$$(\sigma + id)(e^u \otimes e^v) = e^v \otimes e^u + e^u \otimes e^v + v(\nabla e^u)v^{-1} - \nabla e^u + \tfrac{1}{2}\nabla e^w$$
$$(\sigma + id)(e^v \otimes e^u) = e^u \otimes e^v + e^v \otimes e^u + u(\nabla e^v)u^{-1} - \nabla e^v - \tfrac{1}{2}\nabla e^w$$

Then to be torsion compatible, we have the restriction that the following things are symmetric

$$\begin{aligned}
&1) u(\nabla e^u)u^{-1} - \nabla e^u \\
&2) v(\nabla e^v)v^{-1} - \nabla e^v \\
&3) v(\nabla e^u)v^{-1} - \nabla e^u + \frac{1}{2}\nabla e^w \\
&4) u(\nabla e^v)u^{-1} - \nabla e^v - \frac{1}{2}\nabla e^w.
\end{aligned} \tag{4.11}$$

Now we get

$$\begin{aligned}
u(\nabla e^u)u^{-1} - \nabla e^u \;=\; & -\phi_{uv} e^u \otimes (-\frac{1}{2}e^w) - \phi_{vu}(-\frac{1}{2}e^w) \otimes e^u + \phi_{vv}(-(-\frac{1}{2}e^w) \otimes e^v \\
& - e^v \otimes (-\frac{1}{2}e^w) + (-\frac{1}{2}e^w) \otimes (-\frac{1}{2}e^w)) - \phi_{vw}(-\frac{1}{2}e^w) \otimes e^w \\
& - \phi_{wv} e^w \otimes (-\frac{1}{2}e^w)
\end{aligned}$$

For the following to be zero we get $b = \frac{\phi_{vu} - \phi_{uv}}{2}$ to be symmetric as follows

$$\begin{aligned}
v(\nabla e^u)v^{-1} - \nabla e^u + \frac{1}{2}\nabla e^w \;=\; & \phi_{uu}(-e^u \otimes (\frac{1}{2}e^w) - (\frac{1}{2}e^w) \otimes e^u + (\frac{1}{2}e^w) \otimes (\frac{1}{2}e^w)) \\
& - \phi_{uv}(\frac{1}{2}e^w) \otimes e^v - \phi_{uw}(\frac{1}{2}e^w) \otimes e^w - \phi_{vu} e^v \otimes (\frac{1}{2}e^w) \\
& - \phi_{wu} e^w \otimes (\frac{1}{2}e^w) + \frac{1}{2} a e^w \otimes e^w + \frac{b}{2}(e^v \otimes e^w - e^w \otimes e^v)
\end{aligned}$$

$$+\frac{c}{2}(e^u \otimes e^w - e^w \otimes e^u)$$

Now consider

$$\nabla e^v = \sum \psi_{xy} e^x \otimes e^y$$

giving the following which is symmetri, if $\psi_{uv} = \psi_{vu}$,

$$\begin{aligned} v(\nabla e^v)v^{-1} - \nabla e^v &= -\psi_{vu} e^v \otimes (\frac{1}{2}e^w) - \psi_{uv}(\frac{1}{2}e^w) \otimes e^v + \psi_{uu}(-(\frac{1}{2}e^w) \otimes e^u \\ &\quad -e^u \otimes (\frac{1}{2}e^w) + (\frac{1}{2}e^w) \otimes (\frac{1}{2}e^w)) - \psi_{uw}(\frac{1}{2}e^w) \otimes e^w \\ &\quad -\psi_{wu} e^w \otimes (\frac{1}{2}e^w) \end{aligned}$$

Finally,

$$\begin{aligned} u(\nabla e^v)u^{-1} - \nabla e^v - \frac{1}{2}\nabla e^w &= \psi_{vv}(-e^v \otimes (\frac{-1}{2}e^w) - (\frac{-1}{2}e^w) \otimes e^v + (\frac{-1}{2}e^w) \otimes (\frac{-1}{2}e^w)) \\ &\quad -\psi_{vw}(\frac{-1}{2}e^w) \otimes e^w - \psi_{vu}(\frac{-1}{2}e^w) \otimes e^u - \psi_{uv} e^u \otimes (\frac{-1}{2}e^w) \\ &\quad -\psi_{wv} e^w \otimes (\frac{-1}{2}e^w) - \frac{1}{2}a e^w \otimes e^w - \frac{b}{2}(e^v \otimes e^w - e^w \otimes e^v) \\ &\quad -\frac{c}{2}(e^u \otimes e^w - e^w \otimes e^u) \end{aligned}$$

For this to be symmetric, we need $b = 0$ $\square$.

4.4 The matrix for σ

Write σ as a matrix $\sigma(e^x \otimes e^y) = \sum_{p,q} \Sigma^{xy}_{pq} e^p \otimes e^q$ and we write Σ^{xy}_{pq} as a 9 by 9 matrix, using the conventions

$e^u \otimes e^u \mapsto x^1, e^u \otimes e^v \mapsto x^2, e^u \otimes e^w \mapsto x^3$

$e^v \otimes e^u \mapsto x^4, e^v \otimes e^v \mapsto x^5, e^v \otimes e^w \mapsto x^6$

$e^w \otimes e^u \mapsto x^7, e^w \otimes e^v \mapsto x^8, e^w \otimes e^w \mapsto x^9$

We calculate the entries of Σ using the convention from 4.9 and 4.10. We consider the torsion compatible case only (see proposition 75)

For example $\sigma(x^1) = \sigma(e^u \otimes e^u) = x^1 + u\nabla(e^u)u^{-1} - \nabla(e^u) = x^1 + \frac{\phi_{uv}}{2}(e^u \otimes e^w + e^w \otimes e^u) + \frac{\phi_{vv}}{2}(e^w \otimes e^v + e^v \otimes e^w + \frac{1}{2}e^w \otimes e^w) + (\frac{\phi_{wv}}{2} + \frac{\phi_{vw}}{2})e^u \otimes e^w$

We summarise the results as

$\sigma(x^1) = x^1 + \frac{\phi_{uv}}{2}x^3 + \frac{\phi_{uv}}{2}x^7 + \frac{\phi_{vv}}{2}x^6 + \frac{\phi_{vv}}{2}x^8 + (\frac{\phi_{vv}}{4} + \frac{\phi_{wv}}{2} + \frac{\phi_{vw}}{2})x^9$

$\sigma(x^2) = x^4 - \frac{\phi_{uu}}{2}x^3 - \frac{\phi_{uu}}{2}x^7 - \frac{\phi_{vu}}{2}x^6 - \frac{\phi_{uv}}{2}x^8 + (\frac{\phi_{uu}}{4} - \frac{\phi_{uw}}{2} + \frac{\phi_{wu}}{2} + \frac{a}{2})x^9$

$\sigma(x^3) = x^7$

$\sigma(x^4) = x^2 + \frac{\psi_{uv}}{2}x^3 + \frac{\psi_{vu}}{2}x^7 + \frac{\psi_{vv}}{2}x^6 + \frac{\psi_{vv}}{2}x^8 + (\frac{\psi_{vv}}{4} + \frac{\psi_{vw}}{2} + \frac{\psi_{wv}}{2} - \frac{a}{2})x^9$

$\sigma(x^5) = x^5 - \frac{\psi_{uu}}{2}x^3 - \frac{\psi_{uu}}{2}x^7 - \frac{\psi_{vu}}{2}x^6 - \frac{\psi_{uv}}{2}x^8 + (\frac{\psi_{uu}}{4} - \frac{\psi_{uw}}{2} - \frac{\psi_{wu}}{2})x^9$

$\sigma(x^6) = x^8$

$\sigma(x^7) = x^3$

$\sigma(x^8) = x^6$

$\sigma(x^9) = x^9$

Assuming that σ is torsion compatible, we set

$$\begin{aligned}
&\phi_{uv} = 2d = \phi_{vu}, \quad \phi_{vv} = 4e, \quad \phi_{uu} = 4f \\
&\phi_{wv} + \phi_{vw} = 2g, \quad \phi_{uw} + \phi_{wu} = 2h \\
&\psi_{vv} = 4i, \quad \psi_{uu} = 4j, \quad \psi_{uv} = 2k = \psi_{vu}, \\
&\psi_{vw} + \psi_{wv} = 2m, \quad \psi_{uw} + \psi_{wu} = 2n.
\end{aligned} \tag{4.12}$$

As a result we can build the matrix as follows

$$\Sigma = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ d & -2f & 0 & k & -2j & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2e & -d & 0 & 2i & -k & 0 & 0 & 1 & 0 \\ d & -2f & 1 & k & -2j & 0 & 0 & 0 & 0 \\ 2e & -d & 0 & 2i & -k & 1 & 0 & 0 & 0 \\ e+g & \frac{a}{2}+f-h & 0 & -\frac{a}{2}+i+m & j-n & 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.13}$$

Note det $\Sigma = -1$, so σ is always invertible.

4.5 Star compatibility

The condition for the covariant derivatives to be * compatible (see section 2.6) is that

$$\overline{\sigma}\Upsilon^{-1}(*\otimes *)\sigma_E = \Upsilon^{-1}(*\otimes *). \tag{4.14}$$

If we set

$$\sigma(e^x \otimes e^y) = \sum_{p,q} \Sigma^{xy}_{pq} e^p \otimes e^q,$$

then

$$(*\otimes *)\sigma(e^x \otimes e^y) = \sum \Sigma^{xy}_{pq} \overline{e^p} \otimes \overline{e^q}$$

Now, using Υ as in subsection 1.1.2

$$\begin{aligned}
\Upsilon^{-1}(*\otimes *)\sigma(e^x\otimes e^y) &= \sum \Sigma^{xy}_{pq}\overline{e^p\otimes e^q} \\
\overline{\sigma}\Upsilon^{-1}(*\otimes *)\sigma(e^x\otimes e^y) &= \sum \Sigma^{xy}_{pq}\overline{e^q\otimes e^p} \\
&= \sum \Sigma^{xy}_{pq}\overline{\sigma(e^q\otimes e^p)} \\
&= \sum_{p,q,j,k} \Sigma^{xy}_{pq}\overline{\Sigma^{qp}_{jk}(e^j\otimes e^k)} \\
&= \sum_{pqjk} \Sigma^{xy}_{pq}(\Sigma^{qp}_{jk})^*\overline{e^j\otimes e^k}
\end{aligned}$$

This must be equal to the RHS of (4.14) applied to $e^x\otimes e^y$ which is R H S $(e^x\otimes e^y) = \Upsilon^{-1}(*\otimes *)(e^x\otimes e^y) = \Upsilon^{-1}(\overline{e^x}\otimes\overline{e^y}) = \overline{e^y\otimes e^x}$.

So the condition for star compatibility becomes

$$\sum_{pq} \Sigma^{xy}_{pq}(\Sigma^{qp}_{jk})^* = \delta_{y,j}\delta_{x,k} \tag{4.15}$$

However the summation in this equation is not quite matrix multiplication. To turn it into matrix multiplication, we use a matrix T for which

$T(e^x\otimes e^y) = e^y\otimes e^x$ which is 9 by 9, a matrix given by $T^{xy}_{yx} = 1$, i.e. $T^{3x+y-3}_{3y+x-3} = 1$, and zeros elsewhere. i.e.

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Proposition 76 *The condition for ∇ to be torsion compatible and to preserve star is that all of a , d , e , f , g , h , i , j , k , m , n are imaginary.*

Proof : *If we solve $\Sigma T\overline{\Sigma}T - I_9 = 0$, this is equivalent to following matrix vanishing.*

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ d+\overline{d} & k+\overline{k} & 0 & -2f-2\overline{f} & -2j-2\overline{j} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2e+2\overline{e} & 2i+2\overline{i} & 0 & -d-\overline{d} & -k-\overline{k} & 0 & 0 & 0 & 0 \\ d+\overline{d} & k+\overline{k} & 0 & -2f-2\overline{f} & -2j-2\overline{j} & 0 & 0 & 0 & 0 \\ 2e+2\overline{e} & 2i+2\overline{i} & 0 & -d-\overline{d} & -k-\overline{k} & 0 & 0 & 0 & 0 \\ e+\overline{e}+g+\overline{g} & -\frac{a}{2}-\frac{\overline{a}}{2}+i+\overline{i}+m+\overline{m} & 0 & \frac{a}{2}+\frac{\overline{a}}{2}+f+\overline{f}-h-\overline{h} & j+\overline{j}-n-\overline{n} & 0 & 0 & 0 & 0 \end{pmatrix}$$

4.6 The Braid relations

Proposition 77 : *The condition for ∇ (from section 4.4) to be torsion compatible and to have σ obeying the braid relations is one of the following 3 cases :*

1) $d = e = f = i = j = k = 0$

2) $a = h + m, d = f = i = j = k = n = 0$

3) $a = h + i - \frac{i^2}{e} + m, d = -2i, f = \frac{i^2}{e}, g = \frac{e(-hi+em+in)}{i^2}, j = \frac{i^3}{e^2}, k = \frac{-2i^2}{e}$

Proof: *We use Mathematica to find the matrix, using (4.13)*

$(I_4 \otimes \Sigma)(\Sigma \otimes I_3)(I_3 \otimes \Sigma) - (\Sigma \otimes I_3)(I_3 \otimes \Sigma)(\Sigma \otimes I_3)$

MatrixForm[test22=z2.z1.z2-z1.z2.z1]

$$\begin{pmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
2d^2 + 4ek & -4df - 8ej \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
4de + 8ei & -2d^2 - 4ek \\
0 & 0 \\
0 & 0 \\
0 & -4ej + dk \\
0 & 0 \\
0 & 0 \\
0 & -d^2 + 4ef + 2di - 2ek \\
-2d^2 + 8ef & -4ej + dk \\
0 & -3d^2 + 12ef + 2di - 2ek \\
-2e\left(\frac{a}{2} + f - h\right) + 2e\left(-\frac{a}{2} + i + m\right) & 4f(e+g) + 3d\left(\frac{a}{2} + f - h\right) + d\left(-\frac{a}{2} + i + m\right) + 4e
\end{pmatrix}$$

$$
\begin{array}{cc}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & -4df + 4fi - 4ej - 2dk \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & -8ef - 4ek \\
0 & 0 \\
0 & 0 \\
0 & 4fi - dk \\
0 & 0 \\
0 & 0 \\
0 & d^2 - 4ef + 2di - 2ek \\
0 & 8fi + 4ej - 3dk \\
0 & d^2 - 4ef + 2di - 2ek \\
0 & -4f(e+g) - d\left(\frac{a}{2} + f - h\right) - 4\left(\frac{a}{2} + f - h\right) i - 2(e+g)k - d\left(-\frac{a}{2} + i + m\right)
\end{array}
$$

$$
\begin{array}{ccccc}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
8f^2 + 8dj - 4fk & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
4df - 4fi + 4ej + 2dk & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
4dj - 4fk & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-4fi + 4ej & 0 & 0 & 0 & 0 \\
4dj - 4fk & 0 & 0 & 0 & 0 \\
8ej - 2dk & 0 & 0 & 0 & 0 \\
-2f\left(\frac{a}{2} + f - h\right) + 4(e+g)j + 2\left(\frac{a}{2} + f - h\right)k - 2f\left(-\frac{a}{2} + i + m\right) - 2d(j-n) & 0 & 0 & 0 & 0
\end{array}
$$

$$
\begin{array}{c}
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\
-4fi + 4ej + 2dk + 4ik \\
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\
-4di + 8i^2 + 8ek \\
0 \\ 0 \\
-4fi + 4ej \\
0 \\ 0 \\
-4di + 4ek \\
8ej - 2dk \\
-4di + 4ek \\
2\left(\frac{a}{2} + f - h\right)i + 2(e+g)k - 2d\left(-\frac{a}{2} + i + m\right) + 2i\left(-\frac{a}{2} + i + m\right) - 4e(j-n)
\end{array}
$$

$$
\begin{array}{cc}
0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\
-4dj - 8ij & 0 \\
0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\
4fi - 4ej - 2dk - 4ik & 0 \\
0 & 0 \\ 0 & 0 \\
-2dj - 4ij + 2fk + k^2 & 0 \\
0 & 0 \\ 0 & 0 \\
4fi - dk & 0 \\
-2dj - 4ij + 2fk + k^2 & 0 \\
8fi + 4ej - 3dk & 0 \\
\left(\frac{a}{2} + f - h\right)k + 4f\left(-\frac{a}{2} + i + m\right) + k\left(-\frac{a}{2} + i + m\right) + 2d(j-n) + 4i(j-n) & 0
\end{array}
$$

$$
\begin{array}{cc}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
-4dj - 2k^2 & 8fj + 4jk \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
-8ej - 4ik & 4dj + 2k^2 \\
0 & 0 \\
0 & 0 \\
-2dj + 4ij + 2fk - k^2 & 0 \\
0 & 0 \\
0 & 0 \\
-4ej + dk & 0 \\
-2dj + 12ij + 2fk - 3k^2 & 0 \\
-4ej + dk & 8ij - 2k^2 \\
-4(e+g)j - \left(\frac{a}{2} + f - h\right)k - 3k\left(-\frac{a}{2} + i + m\right) - 4i(j-n) & -2\left(\frac{a}{2} + f - h\right)j + 2j\left(-\frac{a}{2} + i + m\right)
\end{array}
$$

$$
\left.\begin{array}{ccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
$$

4.7 The matrix of Christoffel symbols

Here we assume that ∇ is torsion compatible, and use proposition 75. refer to (4.12) for the notation We use

$$\nabla e^j = -\sum_y \Gamma^j_y \otimes e^y$$

from (2.10), where Γ is a matrix of 1-forms. The minus corresponds to the convention that differentiating forms gives minus Christoffel symbols. From(4.10) we have $\Gamma^u_y = -\sum_x \phi_{xy} e^x$

When $y = u$, $\Gamma^u_u = -\sum_x \phi_{xu} e^x = -4fe^u - 2de^v - \phi_{wu} e^w$.

When $y = v$, $\Gamma^u_v = -\sum_x \phi_{xv} e^x = -2de^u - 4ee^v - \phi_{wv} e^w$.

When $y = w$, $\Gamma^u_w = -\sum_x \phi_{xw} e^x = -\phi_{uw} e^u - \phi_{vw} e^v - \phi_{ww} e^w$.

From (4.10) we have $\nabla e^v = -\sum_{x,y} \psi_{xy} e^x \otimes e^y$, so $\Gamma^v_y = \sum_x \psi_{xy} e^x$.

When $y = u$,$-\Gamma^v_u = \sum_x \psi_{xu} e^x = 4je^u + 2ke^v + \psi_{wu} e^w$.

When $y = v$, $-\Gamma^v_v = \sum_x \psi_{xv} e^x = 2ke^u + 4ie^v + \phi_{wv} e^w$.

When $y = w$, $-\Gamma^v_w = \sum_x \psi_{xw} e^x = \psi_{uw} e^u + \psi_{vw} e^v + \psi_{ww} e^w$.

From (4.9) $\nabla(e^w) = ae^w \otimes e^w$, so

$$\Gamma^w_y = \begin{cases} 0 & y \neq w \\ -ae^w & y = w \end{cases}$$

When $y = u$, $-\Gamma^w_u = 0$, when $y = v$, $-\Gamma^w_v = 0$, when $y = w$, $-\Gamma^w_w = ae^w$

The end result is the matrix of Christoffel symbols for the torsion compatible case

$$\Gamma = -\begin{pmatrix} 4fe^u + 2de^v + \phi_{wu} e^w & 2de^u + 4ee^v + \phi_{wv} e^w & \phi_{uw} e^u + \phi_{vw} e^v + \phi_{ww} e^w \\ 4je^u + 2ke^v + \psi_{wu} e^w & 2ke^u + 4ie^v + \psi_{wv} e^w & \psi_{uw} e^u + \psi_{vw} e^v + \psi_{ww} e^w \\ 0 & 0 & ae^w \end{pmatrix} \tag{4.16}$$

$$\wedge \nabla e^j = -\sum_y \Gamma^j_y \wedge e^y =$$

$$\begin{pmatrix} \phi_{wu}e^w \wedge e^u + \phi_{wv}e^w \wedge e^v + \phi_{uw}e^u \wedge e^w + \phi_{vw}e^v \wedge e^w \\ \psi_{wu}e^w \wedge e^u + \psi_{wv}e^w \wedge e^v + \psi_{uw}e^u \wedge e^w + \psi_{vw}e^v \wedge e^w \\ 0 \end{pmatrix} \quad (4.17)$$

Proposition 78 *The condition for the torsion vanishing are :* $\phi_{wu} = \phi_{uw}$, $\phi_{wv} = \phi_{vw}$, $\psi_{wu} = \psi_{uw}$ *and* $\psi_{wv} = \psi_{vw}$

Proof : *From 4.17 and the fact that* $de^x = 0$ *(see proposition 73).*

4.8 The curvature

Given that $d\Gamma = 0$,the matrix for the curvature is given by $-R = \Gamma \wedge \Gamma =$

$$e^u \wedge e^v \begin{pmatrix} 4dk - 16ej & 16fe - 4d^2 + 8di - 8ek & 4f\phi_{vw} - 2d\phi_{uw} 2d\psi_{vw} - 4e\psi_{uw} \\ 8jd - 8kf - 8di - 8ek & 16je - 4kd & 4j\phi_{vw} - 2k\phi_{uw} + 2k\psi_{vw} - 4i\psi_{uw} \\ 0 & 0 & 0 \end{pmatrix}$$

$+e^u \wedge e^w$

$$\begin{pmatrix} 2d\psi_{wu} - 4j\phi_{wv} & 16fe - 4d^2 + 8di - 8ek & 4f\phi_{vw} - 2d\phi_{uw} 2d\psi_{vw} - 4e\psi_{uw} \\ 4j\phi_{ww} - 4f\psi_{wu} + 2d\psi_{wv} - 2k\phi_{wv} & 16je - 4kd & 4j\phi_{vw} - 2k\phi_{uw} + 2k\psi_{vw} - 4i\psi_{uw} \\ 0 & 0 & 0 \end{pmatrix}$$

$+e^v \wedge e^w$

$$\left(\begin{array}{ccc} 4e\psi_{wu}2k\phi_{wv} & 4f\phi_{wv} - 2d\phi_{wv} + 2d\psi_{wv} - 2k\phi_{wv} & 4f\phi_{ww} - \phi_{wu}\phi_{uw} + 2d\psi_{ww} - \phi_{ww}\psi_{uw} \\ 2k\phi_{wu} - 2d\psi_{wu} - 2d\psi_{wu} - 4e\psi_{wv} - 4i\phi_{wv} & 2k\phi_{wv} - 4e\psi_{wu} & 2k\phi_{ww} - \psi_{wu}\phi_{vw} - 4i\psi_{ww} - \psi_{wv}\psi_{vw} \\ 0 & 0 & 0 \end{array} \right)$$

The trace of the R matrix is zero, this is expected as the bundle $\Omega^1 A$ is trivial

4.9 Connections which are invariant to the automorphism Θ

The covariant derivative, for H a Hopf algebra,

$$\nabla : \Omega^1 H \longrightarrow \Omega^1 H \otimes \Omega^1 H.$$

by our assumption of left invariance reduces to a linear map on the left invariant 1-forms

$$\nabla^L : L^1 H \longrightarrow L^1 H \otimes L^1 H \tag{4.18}$$

To see this, remember that H is a Hopf algebra and that

$$\nabla : \Omega^1 H \longrightarrow \Omega^1 H \otimes \Omega^1 H$$

is defined in terms of $\nabla^L : L^1H \longrightarrow L^1H$ by the Liebnitz rule for $a \in H$, $\eta \in L^1H$ by

$$\nabla(a.\eta) = da \otimes \eta + a.\nabla(\eta).$$

We use proposition 35 to see that this defines ∇ on all of Ω^1H. We use the order of basis (e^u, e^v, e^w) as $(1, 2, 3)$ for Ω^1A, then can write 4.18 using the matrix

$$\kappa = \begin{pmatrix} 4f & 4j & 0 \\ 2d & 2k & 0 \\ \phi_{\rm uw} & \psi_{\rm uw} & 0 \\ 2d & 2k & 0 \\ 4e & 4i & 0 \\ \phi_{\rm vw} & \psi_{\rm vw} & 0 \\ \phi_{\rm wu} & \psi_{\rm wu} & 0 \\ \phi_{\rm wv} & \psi_{\rm wv} & 0 \\ \phi_{\rm ww} & \psi_{\rm ww} & a \end{pmatrix} \tag{4.19}$$

Where we use the labelling for the tensor product $e^i \otimes e^j$ in the $3(i-1)+j$ position. For the matrix

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL_2(2, Z), (ps - rq = 1) \tag{4.20}$$

We have met the map $\Theta : A \longrightarrow A$ given by

$$\Theta(u) = u^p v^r$$

$$\Theta(v) = u^q p v^s$$

$$\Theta(w) = w$$

In proposition 71 the matrix giving θ in terms of the basis (e^u, e^v, e^w) of $\Omega^1 A$ is

$$\Theta = \begin{pmatrix} p & q & 0 \\ r & s & 0 \\ \frac{pr}{2} & \frac{qs}{2} & 1 \end{pmatrix}$$

The connections which are invariant to Θ are given by

$$[\theta \otimes \theta].\kappa - (\kappa.\theta) = 0 \tag{4.21}$$

This matrix, from 4.21 , is

$$\left(\begin{array}{c} -4fp + 4fp^2 + 4dpq + 4eq^2 - 4jr \\ -2dp - 2kr + 4fpr + 2dqr + 2dps + 4eqs \\ 2fp^2r + dpqr + dpqs + 2eq^2s + q\phi_{\mathrm{vw}} - r\psi_{\mathrm{uw}} \\ -2dp - 2kr + 4fpr + 2dqr + 2dps + 4eqs \\ -4ep - 4ir + 4fr^2 + 4drs + 4es^2 \\ 2fpr^2 + dprs + dqrs + 2eqs^2 + r\phi_{\mathrm{uw}} - p\phi_{\mathrm{vw}} + s\phi_{\mathrm{vw}} - r\psi_{\mathrm{vw}} \\ 2fp^2r + dpqr + dpqs + 2eq^2s + q\phi_{\mathrm{wv}} - r\psi_{\mathrm{wu}} \\ 2fpr^2 + dprs + dqrs + 2eqs^2 + r\phi_{\mathrm{wu}} - p\phi_{\mathrm{wv}} + s\phi_{\mathrm{wv}} - r\psi_{\mathrm{wv}} \\ -\frac{1}{2}apr + fp^2r^2 + dpqrs + eq^2s^2 + \frac{1}{2}pr\phi_{\mathrm{uw}} + \frac{1}{2}qs\phi_{\mathrm{vw}} + \frac{1}{2}pr\phi_{\mathrm{wu}} + \frac{1}{2}qs\phi_{\mathrm{wv}} + \phi_{\mathrm{ww}} - p\phi_{\mathrm{ww}} - \end{array} \right.$$

$$
\begin{matrix}
4jp^2 - 4fq + 4kpq + 4iq^2 - 4js & 0 \\
-2dq + 4jpr + 2kqr - 2ks + 2kps + 4iqs & 0 \\
2jp^2r + kpqr + kpqs + 2iq^2s - q\phi_{\mathrm{uw}} + p\psi_{\mathrm{uw}} - s\psi_{\mathrm{uw}} + q\psi_{\mathrm{vw}} & 0 \\
-2dq + 4jpr + 2kqr - 2ks + 2kps + 4iqs & 0 \\
-4eq + 4jr^2 - 4is + 4krs + 4is^2 & 0 \\
2jpr^2 + kprs + kqrs + 2iqs^2 - q\phi_{\mathrm{vw}} + r\psi_{\mathrm{uw}} & 0 \\
2jp^2r + kpqr + kpqs + 2iq^2s - q\phi_{\mathrm{wu}} + p\psi_{\mathrm{wu}} - s\psi_{\mathrm{wu}} + q\psi_{\mathrm{wv}} & 0 \\
2jpr^2 + kprs + kqrs + 2iqs^2 - q\phi_{\mathrm{wv}} + r\psi_{\mathrm{wu}} & 0 \\
jp^2r^2 - \frac{aqs}{2} + kpqrs + iq^2s^2 - q\phi_{\mathrm{ww}} + \frac{1}{2}pr\psi_{\mathrm{uw}} + \frac{1}{2}qs\psi_{\mathrm{vw}} + \frac{1}{2}pr\psi_{\mathrm{wu}} + \frac{1}{2}qs\psi_{\mathrm{wv}} + \psi_{\mathrm{ww}} - s\psi_{\mathrm{ww}} & 0
\end{matrix}\right)
$$

We require this to be true for all matrices in 4.20 . On the assumption $r \neq 0$, we find

$j \to \frac{-fp+fp^2+dpq+eq^2}{r}$,
$i \to \frac{-ep+fr^2+drs+es^2}{r}$,
$k \to \frac{-dp+2fpr+dqr+dps+2eqs}{r}$
, $\psi_{\mathrm{uw}} \to \frac{2fp^2r+dpqr+dpqs+2eq^2s+q\phi_{\mathrm{vw}}}{r}$
$\psi_{\mathrm{wu}} \to \frac{2fp^2r+dpqr+dpqs+2eq^2s+q\phi_{\mathrm{wv}}}{r}$,
$\psi_{\mathrm{vw}} \to \frac{2fpr^2+dprs+dqrs+2eqs^2+r\phi_{\mathrm{uw}}-p\phi_{\mathrm{vw}}+s\phi_{\mathrm{vw}}}{r}$,
$\psi_{\mathrm{wv}} \to \frac{2fpr^2+dprs+dqrs+2eqs^2+r\phi_{\mathrm{wu}}-p\phi_{\mathrm{wv}}+s\phi_{\mathrm{wv}}}{r}$,
$\psi_{\mathrm{ww}} \to \frac{-apr+2fp^2r^2+2dpqrs+2eq^2s^2+pr\phi_{\mathrm{uw}}+qs\phi_{\mathrm{vw}}+pr\phi_{\mathrm{wu}}+qs\phi_{\mathrm{wv}}+2\phi_{\mathrm{ww}}-2p\phi_{\mathrm{ww}}}{2r}$;

These conditions make the whole first column of the matrix from (4.21) vanish . The top 8 entries of the second column are linear equations in f, e, d whose cofficients are long polynomials in p, q, r, s, and these equations must be true for matrices in $SL_2\mathbb{Z}$.

We substitute in the numeral value of the matrices

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} and \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$$

and find that $f = e = d = 0$.

Using these substitutions gives all entries of (4.21), except the last entry of the second column zero, and κ itself is (for $r \neq 0$) .

$$\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
\phi_{\mathrm{uw}} & \frac{(-1+ps)\phi_{\mathrm{vw}}}{r^2} & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\phi_{\mathrm{vw}} & \frac{r\phi_{\mathrm{uw}} - p\phi_{\mathrm{vw}} + s\phi_{\mathrm{vw}}}{r} & 0 \\
\phi_{\mathrm{wu}} & \frac{(-1+ps)\phi_{\mathrm{wv}}}{r^2} & 0 \\
\phi_{\mathrm{wv}} & \frac{r\phi_{\mathrm{wu}} - p\phi_{\mathrm{wv}} + s\phi_{\mathrm{wv}}}{r} & 0 \\
\phi_{\mathrm{ww}} & -\frac{apr^2 - pr^2\phi_{\mathrm{uw}} + s\phi_{\mathrm{vw}} - ps^2\phi_{\mathrm{vw}} - pr^2\phi_{\mathrm{wu}} + s\phi_{\mathrm{wv}} - ps^2\phi_{\mathrm{wv}} - 2r\phi_{\mathrm{ww}} + 2pr\phi_{\mathrm{ww}}}{2r^2} & a
\end{pmatrix}$$

Since κ must be independent of p, q, s, r , this gives $\phi_{vw} = 0$ and $\phi_{wv} = 0$ and κ is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \phi_{\mathrm{uw}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \phi_{\mathrm{uw}} & 0 \\ \phi_{\mathrm{wu}} & 0 & 0 \\ 0 & \phi_{\mathrm{wu}} & 0 \\ \phi_{\mathrm{ww}} & -\frac{apr - pr\phi_{\mathrm{uw}} - pr\phi_{\mathrm{wu}} - 2\phi_{\mathrm{ww}} + 2p\phi_{\mathrm{ww}}}{2r} & a \end{pmatrix}$$

For this to be independent of $\{p, q, r, s\}$ we read $\phi_{ww} = 0$,$a = \phi_{uw} + \phi_{wu}$ to summarise this

Proposition 79 *The torsion compatible connections invariant to the automorphism $\in$ are given by the matrix (see 4.19)*

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \phi_{\mathrm{uw}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \phi_{\mathrm{uw}} & 0 \\ \phi_{\mathrm{wu}} & 0 & 0 \\ 0 & \phi_{\mathrm{wu}} & 0 \\ 0 & 0 & \phi_{\mathrm{uw}} + \phi_{\mathrm{wu}} \end{pmatrix}$$

4.10 Riemannian metrics and 2-forms

We need to decide what we mean by a Riemannian metric on an algebra A. As we shall see there is no single obvious choice. We only consider left invariant metrics.
A Riemannian metric ought to be a non degenerate symmetric inner product on $\Omega^1 A$. However the condition that the inner product

$$\langle \quad \rangle : \Omega^1 A \otimes_A \overline{\Omega^1 A} \longrightarrow A$$

is a bimodule map is very strong, and as we shall see in our case, no non degenerate such metrics exist .
It will be convenient to look for central elements $g \in \Omega^1 A \otimes \Omega^1 A$, rather than looking at the inner product. (If $\langle \quad \rangle$ was non-degenerate, this would be equivalent.)
We consider lemma (74) to find symmetric elements of $\Omega^1 A \otimes_A \Omega^1 A$ which commute with A, and elements of $\Omega^2 A$ which commute with A. If we assume that they are all left invariant there is only one symmetric tensor which commutes with A, $e^w \otimes e^w$
If we allow g to be central up to a multiple of $e^w \otimes e^w$ in which case
$g_{uu} = g_{vv} = g_{uv} = g_{vu} = 0$, and we have
$u^{-1} g u - g = -\frac{e_w \otimes e_w}{2}(g_{vw} + g_{wv})$
$v^{-1} g v - g = -\frac{e_w \otimes e_w}{2}(g_{uw} + g_{wu})$. Thus

$$g = \begin{pmatrix} 0 & 0 & g_{uw} \\ 0 & 0 & g_{vw} \\ g_{wu} & g_{wv} & g_{uw} \end{pmatrix} \tag{4.22}$$

This is degenerate (i.e. the matrix for g_{ij} is not invertible).
We come to the conclusion that A does not have a standard Riemannian structure .

A symplectic form ω is a 2 form which is closed (i.e. $d\omega = 0$). We restrict ourselves to the left invariant case, where we only consider the finite dimensional vector spaces $L^1 A$ and their wedge products.

In that case

$$\omega = \omega_{xy} e^x \wedge e^y$$

where ω_{xy} is taken to be antisymmetric, and we consider non-degenerate to mean that the antisymmetric matrix ω_{xy} invertible. To be able to introduce the symplectic form into calculations, we require that it is central. This allows us to just write ω in a formula, without having it in the original formula.

For example, consider introducing ω into the middle of the expression

$$e \otimes_A a.f \in E \otimes_A F$$

If we introduce ω here we get $e \otimes_A \omega \otimes_A a.f$.

However $e \otimes_A a.f = e.a \otimes_A f$, so introducing ω here gives

$$e.a \otimes_A \omega \otimes_A f \quad and \quad e \otimes_A \omega \otimes_A a.f$$

For these to be equal requires

$$e \otimes_A (a.\omega - \omega.a) \otimes_A f = 0.$$

To just introduce ω arbitrarily into a calculation, rather than having it at the beginning, we should have $a.\omega = \omega.a$ all $a \in A$.

For 2 forms , if $\omega = \sum_{x,y} \omega_{xy} e^x \wedge e^y$

$v^{-1}\omega v - \omega = \frac{1}{2} e^w \wedge e^v (\omega_{uv} - \omega_{vu})$

$u^{-1}\omega u - \omega = \frac{1}{2}e^u \wedge e^w(\omega_{vu} - \omega_{uv})$.

Then we have $\omega_{uv} = \omega_{vu}$ for ω to be central, in which case the term $e^u \wedge e^v$ does not appear in Ω, so

$$\omega_{uv}e^u \wedge e^v + w_{vu}e^v \wedge e^u = w_{uv}(e^u \wedge e^v + e^v \wedge e^u) = 0.$$

Now we check where such a left invariant central 2-form ω is closed, i.e. $d\omega = 0$. Form proposition (73) any such ω is closed .

Then $\omega_{x,y}$ is the matrix (with order u, v, w)

$$\begin{pmatrix} 0 & 0 & \omega_{uw} \\ 0 & 0 & \omega_{vw} \\ -\omega_{uw} & -\omega_{vw} & \omega_{ww} \end{pmatrix}$$

and the determinant is still zero !

Here we have used the antisymmetry of $\wedge$ in our example ($e^x \wedge e^y = -e^y \wedge e^x$) so

$$\begin{aligned} \omega &= \frac{1}{2}\sum \omega_{xy}e^x \wedge e^y + \frac{1}{2}\sum \omega_{xy}e^x \wedge e^y \\ &= \frac{1}{2}\sum \omega_{xy}e^x \wedge e^y - \frac{1}{2}\sum \omega_{xy}e^y \wedge e^x \\ &= \frac{1}{2}\sum \omega_{xy}e^x \wedge e^y - \frac{1}{2}\sum \omega_{yx}e^y \wedge e^x \\ &= \frac{1}{2}\sum(\omega_{xy} - \omega_{yx})e^x \wedge e^y \end{aligned} \tag{4.23}$$

so without loss of generality we can take the matrix to be antisymmetric. Thus we have not had much luck is getting either non-degenerate metrics or symplectic 2-forms on A. However this will change when we consider the fibration 4.26 later.

4.10.1 Covariant derivatives preserving the degenerate metric

Although the metric g_{ab} in (4.22) is degenerate (as the matrix for g is not invertible), we shall look for connections preserving it.

The entries of the matrix g_{ab} are just numbers, so $dg_{\bullet} = 0$. From proposition(46), to preserve this metric we require that $g\Gamma$ is antiHermitian

$$(g\Gamma)^* = -g\Gamma \tag{4.24}$$

[Note : A is Hermitian if $A^* = A$ and A is antihermitian it $A^* = -A$]. We use the form of Γ in (4.16), so we assume that the connection is torsion compatible. Then

$$-g_{\bullet}\Gamma = \begin{pmatrix} 0 & 0 & g_{uw}ae^w \\ 0 & 0 & g_{vw}ae^w \\ e^u(4fg_{wu}+4jg_{wv})+e^v(2dg_{wu}+2kg_{wv})+e^w(g_{wu}\phi_{wu}+g_{wv}\psi_{wu}) & e^u(2dg_{wu}+2kg_{wv})+e^v(4eg_{wu}+4ig_{wv})+e^w(g_{wu}\phi_{wv}+g_{wv}\psi_{wv}) & e^u(g_{wu}\phi_{uw}+g_{wv}\psi_{uw})+e^v(g_{wu}\phi_{vw}+g_{wv}\psi_{vw})+e^w(g_{wu}\phi_{ww}+g_{wv}\psi_{ww}+g_{ww}a) \end{pmatrix} \tag{4.25}$$

We use the fact that $(e^x)^* = -e^x$, and get the following conditions for $g_{\bullet}$ to be antihermitian

Proposition 80 *The conditions for a torsion compatible connection to preserve the degenerate metric (4.22) are, in the notation of (4.16),*

$f g_{wu} + j g_{wv} = 0$

$d g_{wu} + k g_{wv} = 0$

$e g_{wu} + i g_{wv} = 0$

$\phi_{wu} g_{wu} + \psi_{wu} g_{wv} = g_{wu} \overline{a}$

$\phi_{wv} g_{wu} + \psi_{wv} g_{wv} = g_{wv} \overline{a}$

and the following are real

$g_{wu}\phi_{uw} + g_{wv}\psi_{uw}$, $g_{wu}\phi_{vw} + g_{wv}\psi_{vw}$, $g_{wu}\phi_{ww} + g_{wv}\psi_{ww} + g_{ww}a$

Proof: *From (4.24) and (4.25).* □

Proposition 81 *Suppose both g_{wu} and g_{wv} are non zero and set $g_{wv} = x g_{wu}$. The condition that a star compatible covariant derivative is both torsion zero and preserves the metric is (in the notation of 4.12)*

Case1: *When x is real, and $f = -jx$, $d = -kx$, $e = -ix$, $a = -h - xn = -m - \frac{g}{x}$, and $g_{wu}(\phi_{ww} + x\psi_{ww}) + g_{wu}a$ is real, and if $a \neq 0$, we get g_{wu} is imaginary.*

Case2: *When x is not real, and $a = j = f = d = k = e = i = h = n = m = g = 0$, and $g_{wu}(\phi_{ww} + x\psi_{ww})$ is real.*

Proof:*From proposition 78, torsion zero implies $\phi_{wu} = \phi_{uw}$, $\phi_{wv} = \phi_{vw}$, $\psi_{wu} = \psi_{uw}$ and $\psi_{wv} = \psi_{vw}$. Using this in the definition of the letters in (4.12), we get $\phi_{wu} = \phi_{uw} = h$, $\phi_{wv} = \phi_{vw} = g$, $\psi_{wu} = \psi_{uw} = n$, $\psi_{wv} = \psi_{vw} = m$.*

Now we use proposition 80 to write, for $g_{wv} = x g_{wu}$, (so $x \neq 0$). Note that since both $g_{wu}\overline{a}$ and $g_{wu}x\overline{a}$ are real, if $a \neq 0$ we have x is real. We split into two cases:

Case1: *x is real, so $f = -jx$, $d = -kx$, $e = -ix$, $g_{wu}(h + xn) = g_{wu}\overline{a}$ and $g_{wu}(g + xm) = g_{wu}x\overline{a}$are real, and also $g_{wu}(\phi_{ww} + x\psi_{ww}) + g_{ww}a$ is real.*

Case2: *x is not real. From proposition 76, we have f and j both imaginary, so*

$f = -jx$ implies that $f = j = 0$. Likewise we have $a = j = f = d = k = e = i = h = n = m = g = 0$. $\square$

The reader should note that this is not the classical result of a unique Levi Civita connection (i.e. torsion zero and metric preserving). This is not surprising, since the metric we started with was degenerate, so the classical result would not apply anyway

Proposition 82 *The condition that a star compatible covariant derivative is both torsion zero and preserves the metric and, is the invariant to the automorphism Θ is (in the notation of 4.12) $\phi_{ww} = \psi_{ww} = a = j = f = d = k = e = i = h = n = g = m = 0$*
i.e. all Christoffel symbols vanish. In this the matrix for Σ becomes

$$\Sigma = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Proof: *As $\phi_{ww} = \psi_{ww} = 0$ from proposition 79, we see $g_{ww}a$ is real. Also $n = g = 0$ and $a = 2h$. Putting this into $g_{wu}(h + xn) = g_{wu}\overline{a}$, $g_{wu}(g + xm) = g_{wu}x\overline{a}$ and $g_{wu}(\phi_{ww} + x\psi_{ww}) + g_{ww}a$ to get $g_{wu}h = g_{wu}\overline{a}$, and $g_{wu}xm = g_{wu}x\overline{a}$, so $h = \overline{a}$, now $a = 2h = \overline{h}$ gives $h = a = 0$.*

Proposition 83 *The condition that a star compatible covariant derivative is torsion zero and preserves the metric and satisfies the braid relation are*
$d = e = f = i = j = k = 0$, $a = -h - xn = -m - \frac{g}{x}$ *and* $g_{wu}(\phi_{ww} + x\psi_{ww}) + a g_{ww}$ *is real, if* $a \neq 0$ *we get* g_{ww} *is imaginary.*

Proof: *Using proposition 77 and proposition 81, we can see all of case2 in proposition 81 is contaned in case1 in proposition 77. We now just have to look at case1 when* $x \neq 0$ *is real and* $i \neq 0$*, then with a small calculation we get* $e = -ix$, $d = -2i$,
$f = \frac{-i}{x}$, $j = \frac{i}{x^2}$, $k = \frac{2i}{x}$
$h = \frac{-1}{2}(1+x)^2(m + \frac{i}{x})$ *and*
$n = i + \frac{i}{x} + (2+x)m$
$g = -x(\frac{-1}{2}(1+x)^2(m + \frac{i}{x}) + i + \frac{i}{x} + 2m)$
$a = (1 - \frac{1}{2}(1+x)^2)(m + \frac{i}{x}) + i$
case 2 in proposition 81 gives case 2 in proposition 77, and so satisfies the braid relations. $\square$

4.11 The noncommutative torus $\mathbb{T}^2_q$

This has generators u, v and a complex number q of norm 1 and relation $uv = qvu$

$$u^* = u^{-1}, v^* = v^{-1} (i.e. u, v\, are\, unitary).$$

There is a map from $C(S^1)$, the functions on the unit circle, to the group algebra of the Heisenberg group given by $q \in S^1$(or the identity function $: S^1 \to \mathbb{C}$) maps to $w \in H$.

If we set $w = q$ in the relations for the Heisenberg group algebra, we get the noncommutative torus. We can consider that the noncommutative torus is the fiber of the map

$$C(S^1) \longrightarrow A.$$

Here we take w to be a complex number of unit norm, the coordinate function on S^1. Then $w^* = w^{-1}$. The map sends $w^n \in C(S^1)$ to $w^n \in A$. The differential structure of the fiber space is

$$\Omega^n F = \frac{\Omega^n A}{\Omega^1 C(S^1) \wedge \Omega^{n-1} A} \tag{4.26}$$

i.e. we put $dw = 0$ in $\Omega^n F$ (i.e. put $e^w = 0$). This is because in 4.26 we divide by everything of the form $e^w \wedge \xi$. To see that this gives a fibration, we note that a linear basis for the left invariant n-forms in as follows:

$\Omega^1 A \quad e^u, e^v, e^w$

$\Omega^2 A \quad e^u \wedge e^v,\ e^w \wedge e^u\ ,\ e^w \wedge e^v$

$\Omega^3 A \quad e^v \wedge e^u \wedge e^w$

Then the invariant forms is Ξ^n_m are :

$\Xi^0_0 = 1,\quad \Xi^0_1 = \langle e^w \rangle,\quad \Xi^0_m = 0\ ,\ m > 1$

$$\Xi^1_0 = \frac{\langle e^u, e^v, e^w \rangle}{\langle e^w \rangle} = \langle e^u, e^v \rangle$$

$$\Xi^2_0 = \frac{\langle e^u \wedge e^v, e^w \wedge e^u, e^w \wedge e^v \rangle}{\langle e^w \wedge e^u, e^w \wedge e^v \rangle} = \langle e^u \wedge e^v \rangle$$

$$\Xi^3_0 = \frac{\langle e^w \wedge e^u \wedge e^v \rangle}{\langle e^w \wedge e^u \wedge e^v \rangle} = 0$$

$\Xi^n_0 = 0 \quad n \geq 4$

$$\Xi^1_1 = \frac{e^w \wedge \langle e^w, e^u, e^v \rangle}{\langle 0 \rangle} = \langle e^w \wedge e^u, e^w \wedge e^v \rangle$$

$$\Xi^2_1 = \frac{e^w \wedge \langle e^w \wedge e^u, e^w \wedge e^v, e^u \wedge e^v \rangle}{\langle 0 \rangle} = \langle e^w \wedge e^u \wedge e^v \rangle$$

all others are zero. Then the map

$$\Omega^1 C(S^1) \otimes_{C(S^1)} \Xi^n_0 \longrightarrow \Xi^n_1$$

$$\Omega^n F = \frac{\Omega^n A}{dw \wedge \Omega^{n-1} A}$$

is one-to-one and onto. We have a basis of 1-forms e^u, u^v and relations

$$e^u \wedge e^u = e^v \wedge e^v = 0$$

$$e^u \wedge e^v = -e^v \wedge e^u.$$

This gives a fibration in the sense of [2], and a spectral sequence.

We now go back to (4.10) and observe that although the metrics there were degenerate, we do get non-degenerate metrics on the fibers. Now, the map Θ sends w to w, so we then have a map preserving the fibers.

We can restrict the covariant derivative in section 4.10 using basis order e^u, e^v, to get

$$\Gamma = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The torsion of a connection on $\Omega^1 A$ is given in definition 37. In the case of matrix (4.11) , we get $\nabla e^x = 0$, so

$$\begin{aligned} Tor(e^x) &= de^x \\ &= d(x^{-1}dx) \\ &= -x^{-1}dx \wedge x^{-1}dx \\ &= -e^x \wedge e^x \\ &= 0. \end{aligned}$$

4.11.1 Θ independent metrics and 2-forms on the torus

Now we see that our previous calculations of Riemannian metrice and symplectic form work better on the fiber space.

If we set $g_\bullet = e^u \otimes e^v + e^v \otimes e^u$, then $g_\bullet$ is central in 4.26.

If we set $\omega = e^u \wedge e^v$, then ω is central in 4.26.

We get e^u and e^v commuting with all algebra elements . Look for Θ independent Riemannian metric as follows

$$g = \sum g_{ab} e^a \otimes e^b$$

$$(\Theta \otimes \Theta)g = \sum g_{ab}\Theta(e^a) \otimes \Theta(e^b)$$

$$from \quad \Theta(e^x) = \sum_y \Theta_{yx} e^y$$

we get

$$
\begin{aligned}
(\Theta \otimes \Theta) g &= \sum_{ab,x,y} g_{ab} \Theta_{xa} e^x \otimes \Theta_{yb} e^y \\
&= \sum_{ab,x,y} \Theta_{xa} g_{ab} \Theta_{yb} e^x \otimes e^y
\end{aligned}
$$

so the matrix for $(\Theta \otimes \Theta) g$ is $\Theta g \Theta^T$ in terms of the matrix for g. and

$$
g = \begin{pmatrix} a & b \\ d & e \end{pmatrix}
$$

For the noncommutative torus

$$
\Theta = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \; (with\, ps - qr = 1),
$$

For invariant, we want $\Theta\, g\, \Theta^T - g = 0$, i.e.

$$
\begin{pmatrix} -a + p(ap + dq) + q(bp + eq) & -b + (ap + dq)r + (bp + eq)s \\ -d + p(ar + ds) + q(br + es) & -e + r(ar + ds) + s(br + es) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{4.27}
$$

to impose the determinant 1 rule we set $r \to (ps - 1)/q$. Then (4.27) requires
$e \to \frac{ak - ap^2 - bpq - dpq}{q^2}$
$b \to \frac{apqr + dq^2 r + aks - ap^2 s - dpqs}{kq}$
For this to be true for all Θ, we see that g is a multiple of

$$
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.28}
$$

If we multiply this matrix by i we get

$$\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

which is Hermitian (see proposition 45), and this would be the only (up to real multiple) inner product does not preserve reality, as the inner product of ie^u and $i\epsilon^v$(which are both real) is imaginary. but the antisymmetric matrix (4.28) does give a reality preseving symplectic form $\omega = e^u \wedge e^v$,which is preserved by Θ.

This is form as $d\omega = 0$ (see section(2.8).

For any left invariant Riemannian metric on the noncommutative torus, the torsion free connection with $\Gamma = 0$ preserves the metric .

Chapter 5

The Leray spectral sequence

Strictly speaking, we talk only about a specific case of the classical Leray spectral sequence [9], that related to a sheaf over a fiber bundle. However it is hoped that this will be suffciently general to give an interesting result. The material in this chapter is joint work with my supervisor. We give further comments on the Leray spectral sequence in section 5.7 and 6.3

5.1 Classical theory

The statement of the general Leray spectral sequence can be found in [9]. We shall omit the supports and the subsets as we are only currently interested in a non commutative analogue of the spectral sequence.

Then the statement reads that, given $f : X \to Y$ and $\mathcal{S}$ a sheaf on X, that there is a spectral sequence

$$E_2^{pq} = H^p(Y, H^q(f, f|\mathcal{S}))$$

converging to $H^{p+q}(X, \mathcal{S})$.
Here $H^q(f, f|\mathcal{S})$ is a sheaf on Y which is given by the presheaf for an open $U \subset Y$

$$U \to H^q(f^{-1}U; \mathcal{S}|_{f^{-1}U}).$$

Here $f^{-1}U$ is an open set of X, and $\mathcal{S}|_{f^{-1}U}$ is the sheaf $\mathcal{S}$ restricted to this open set. We shall consider the special case of a differential fibration. This is the background to the Serre spectral sequence, but we consider a sheaf on the total space.
The Leray spectral sequence of a fibration is a spectral sequence (see section 2.11) whose input is the cohomology of the base space B with coefficients in the cohomology of the fiber F, and converges to the cohomology of the total space E. Here

$$\pi : E \to B$$

is a fibration with fiber F. The difference of this from the Serre spectral sequence is that the cohomology above may have coefficients in a sheaf on E.
We shall apply noncommutative sheaf cohomology (see definition 56) to the Leray spectral sequence. To do this we use the same definition of fibration as that used for the noncommutative Serre spectral sequence in [2], and we discuss this in the next section.

5.2 Differential fibration

We have previously mentioned the idea of differential fibration (see definition 59), but now it may be useful of spend a little time justifying it. Take a trivial fibration

$$\mathbb{R}^n \times \mathbb{R}^m \xrightarrow{\pi} \mathbb{R}^n$$

$$(x_1, ..., x_n, y_1, ..., y_m) \longmapsto (x_1, ..., x_n).$$

Here the base space is $B = \mathbb{R}^n$, the fiber is $\mathbb{R}^m$, and the total space is $E = \mathbb{R}^{n+m}$. We can write a basis for the differential forms on the total space, putting the B terms (the dx_i) first. A form of degree p in the base and q in the fiber (total degree $p+q$) is

$$dx_{i_1} \wedge .... \wedge dx_{i_p} \wedge dy_{j_1} \wedge .... \wedge dy_{j_q}$$

e.g. $\quad dx_2 \wedge dx_4 \wedge dy_1 \wedge dy_7 \wedge dy_9$

If we have the projection map $\pi : E \longrightarrow B$, we can write this as

$$\pi^*(dx_2 \wedge dx_4) \wedge (dy_1 \wedge dy_7 \wedge dy_9)$$

so we have a form in $\pi^*\Omega^2 B \wedge \Omega^3 E$. Another element of $\pi^*\Omega^2 B \wedge \Omega^3 E$ might be

$$\pi^*(dx_2 \wedge dx_4) \wedge (dx_3 \wedge dy_1 \wedge dy_7).$$

Note, we now just look at $\Omega^3 E$, not the forms in the fiber direction, as in the noncommutative case we will not know (at least in the begining) what the fiber is. We need to describe the forms on the fiber space more indirectly.

Now look at the vector space quotient

$$\frac{\pi^*\Omega^2 B \wedge \Omega^3 E}{\pi^*\Omega^3 B \wedge \Omega^2 E}.$$

Consider our two elements of the top line,

$$\alpha = \pi^*(dx_2 \wedge dx_4) \wedge (dy_1 \wedge dy_7 \wedge dy_9)$$

$$\beta = \pi^*(dx_2 \wedge dx_4) \wedge (dx_3 \wedge dy_1 \wedge dy_7)$$

Here β is also an element of the bottom line, as we could write

$$\beta = \pi^*(dx_2 \wedge dx_4 \wedge dx_3) \wedge (dy_1 \wedge dy_7)$$

so, denoting the quotient by square brackets, $[\beta] = 0$. On the other hand, α is not in the bottom line, so $[\alpha] \neq 0$. We can now use

$$\frac{\pi^* \Omega^p B \wedge \Omega^q E}{\pi^* \Omega^{p+1} B \wedge \Omega^{q-1} E}$$

to denote the forms on the total space which are of degree p in the base and degree q in the fiber, without explicitly having any coordinates for the fiber.

5.3 The spectral sequence of filtration

We have already discussed spectral sequences in section 2.11, but it will be convenient to go into a little more detail here, and to quote the result from [35] again.

A decreasing filtration of vector space V is a sequence of subspaces $F^m V$ for which $F^{m+1} V \subset F^m V$.

For example, we could have (where we take all $x_i \in \mathbb{R}$)

$$F^m(\mathbb{R}^n) = \left\{ \begin{pmatrix} x_1 \\ . \\ . \\ x_{n-m} \\ 0 \\ . \\ . \\ 0 \end{pmatrix} \in \mathbb{R}^n \right\}$$

$$F^1(\mathbb{R}^3) = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}, \quad F^2(\mathbb{R}^3) = \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix}$$

$$F^3(\mathbb{R}^3) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad F^0(\mathbb{R}^3) = \mathbb{R}^3$$

The reader should refer to [35] for the details of the homological algebra used to construct the spectral sequence. We will merely quote the results.

Remark 84 *[2]Start with a differential graded module C^n (for $n \geq 0$) and $d : C^n \to C^{n+1}$ with $d^2 = 0$. Suppose that C has a filtration $F^mC \subset C = \oplus_{n\geq 0} C^n$ for $m \geq 0$ so that:*

(1) $dF^mC \subset F^mC$ for all $m \geq 0$ (i.e. the filtration is preserved by d);

(2) $F^{m+1}C \subset F^mC$ for all $m \geq 0$ (i.e. the filtration is decreasing);

(3) $F^0C = C$ and $F^mC^n = F^mC \cap C^n = \{0\}$ for all $m > n$ (a boundedness

condition).

Then there is a spectral sequence $(\mathcal{E}_r^{*,*}, d_r)$ for $r \geq 1$ (r counts the page of the spectral sequence) with d_r of bidegree $(r, 1-r)$ and

$$\mathcal{E}_1^{p,q} = H^{p+q}(F^pC/F^{p+1}C) = \frac{\ker d : F^pC^{p+q}/F^{p+1}C^{p+q} \to F^pC^{p+q+1}/F^{p+1}C^{p+q+1}}{\operatorname{im} d : F^pC^{p+q-1}/F^{p+1}C^{p+q-1} \to F^pC^{p+q}/F^{p+1}C^{p+q}} \quad (5.1)$$

In more detail, we define

$$\begin{aligned} Z_r^{p,q} &= F^pC^{p+q} \cap d^{-1}(F^{p+r}C^{p+q+1}) , \\ B_r^{p,q} &= F^pC^{p+q} \cap d(F^{p-r}C^{p+q-1}) , \\ \mathcal{E}_r^{p,q} &= Z_r^{p,q}/(Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q}) . \end{aligned}$$

The differential $d_r : \mathcal{E}_r^{p,q} \to \mathcal{E}_r^{p+r,q-r+1}$ is the map induced on quotienting $d : Z_r^{p,q} \to Z_r^{p+r,q-r+1}$.

The spectral sequence converges to $H^*(C,d)$ in the sense that

$$\mathcal{E}_\infty^{p,q} \cong \frac{F^pH^{p+q}(C,d)}{F^{p+1}H^{p+q}(C,d)} ,$$

where $F^pH^*(C,d)$ is the image of the map $H^*(F^pC,d) \to H^*(C,d)$ induced by inclusion $F^pC \to C$.

5.4 The filtration of the cochain complex

We suppose that E is a left A module, with a left covariant derivative

$$\nabla : E \longrightarrow \Omega^1 A \otimes_A E$$

and that this covariant derivative is flat, i.e. that its curvature vanishes (see proposition 38). Then $\nabla^{[n]} : \Omega^n A \otimes_A E \longrightarrow \Omega^{n+1} A \otimes_A E$ is a cochain complex (see definition 56). If $i : B \longrightarrow A$ is a fibration (see section 5.2) (we used π for a fibration of topological spaces, and we will use i for algebras). We can define a filtration of $\Omega^n A \otimes_A E$ by

$$F^m(\Omega^n A \otimes_A E) = \begin{cases} i_*\Omega^m B \wedge \Omega^{n-m} A \otimes_A E & 0 \le m \le n; \\ 0 & \textit{otherwise.} \end{cases} \tag{5.2}$$

Proposition 85 *The filtration in 5.2 satisfies the conditions of remark 84.*

Proof: *First* $\quad F^0(\Omega^n A \otimes_A E) = i_*\Omega^0 B \wedge \Omega^n A \otimes_A E,$

but $1 \in i_*\Omega^0 B = i_* B$, $\quad$ *so* $F^0(\Omega^n A \otimes_A E) = \Omega^n A \otimes_A E.$

To show it is decreasing, (using condition 5 from definition 34)

$$\begin{aligned} F^{m+1}(\Omega^n A \otimes_A E) &= i_*\Omega^{m+1} B \wedge \Omega^{n-m-1} A \otimes_A E \\ &= i_*\Omega^m B \wedge (i_*\Omega^1 B \wedge \Omega^{n-m-1} A) \otimes_A E \\ &\subset i_*\Omega^m B \wedge \Omega^{n-m} A \otimes_A E \\ &\subset F^m(\Omega^n A \otimes_A E). \end{aligned}$$

To show that the filtration is preserved by d, take $i_*\xi \wedge \eta \otimes e \in F^m(\Omega^n A \otimes_A E)$ where $\xi \in \Omega^m B$, and $\eta \in \Omega^{n-m} A$. Then

$$d(i_*\xi \wedge \eta \otimes e) = i_* d\xi \wedge \eta \otimes e + (-1)^m i_*\xi \wedge d\eta \otimes e + (-1)^n i_*\xi \wedge \eta \wedge \nabla e$$

This is in $F^m C$, as the first term is in $F^{m+1} C \subset F^m C$, and the other two are in $F^m C$. $\square$

Now we have a spectral sequence which converges to $H^*_{dR}(A; E)$. All we have to do is to find the first and second pages of the spectral sequence, though this is quite lengthy.

5.5 Calculation the first page of the spectral sequence

From section 5.3, to use the filtration in section 5.4 we need to work with

$$M_{p,q} = \frac{F^p C^{p+q}}{F^{p+1} C^{p+q}} = \frac{i_* \Omega^p B \wedge \Omega^q A \otimes_A E}{i_* \Omega^{p+1} B \wedge \Omega^{q-1} A \otimes_A E}$$

Then we look, for p fixed (following (5.1)), at the sequence

$$\cdots M_{p,q-1} \xrightarrow{d} M_{p,q} \xrightarrow{d} M_{p,q+1} \xrightarrow{d} \cdots \tag{5.3}$$

as the cohomology of this sequence gives the first page of the spectral sequence. Denote the quotient in $M_{p,q}$ by $[\ \ ]_{p,q}$, so if $x \in i_* \Omega^p B \wedge \Omega^q A \otimes_A E$, then $[x]_{p,q} \in M_{p,q}$. Then we have a map of left B modules

$$\Omega^p B \otimes_B M_{0,q} \longrightarrow M_{p,q}$$

$$\xi \otimes [y]_{0.q} \longrightarrow [i_* \xi \wedge y]_{pq}.$$

Here $y \in \Omega^q A \otimes_A E$ and the left action of $b \in B$ on y is $i(b)y$.

For notation, set $D_{p,q} = i^* \Omega^p B \wedge \Omega^q A$, and $N_{p,q} = \dfrac{D_{p,q}}{D_{p+1,q-1}}$.

Proposition 86 *If E is flat as a left A module, then $N_{p,q} \otimes_A E \cong M_{p,q}$ with isomorphism $[z] \otimes e \longmapsto [z \otimes e]_{p,q}$.*

Proof: *We have, by definition, a short exact sequence, where inc is inclusion and* $[\,]$ *is quotient*

$$0 \longrightarrow D_{p+1,q-1} \xrightarrow{inc} D_{p,q} \xrightarrow{[\,]} N_{p,q} \longrightarrow 0.$$

As E is flat, we get another short exact sequence,

$$0 \longrightarrow D_{p+1,q-1} \otimes_A E \xrightarrow{inc \otimes id} D_{p,q} \otimes_A E \xrightarrow{[\,]\otimes id} N_{p,q} \otimes_A E \longrightarrow 0$$

but by definition we also have

$$0 \longrightarrow D_{p+1,q-1} \otimes_A E \xrightarrow{inc \otimes id} D_{p,q} \otimes_A E \xrightarrow{[\,]_{p,q}} M_{p,q} \longrightarrow 0.$$

and the result follows from lemma 31. □

We can now restate definition 59 in terms of our current notation.

Definition 87 $i : B \longrightarrow A$ *is a differential fibration if the map*

$$\xi \otimes [x] \longrightarrow [i_* \xi \wedge x]$$

gives an isomorphism from $\Omega^p B \otimes_B N_{0,q}$ *to* $N_{p,q}$ *for all* p, q.

Proposition 88 *If E is flat a left A module, and* $i : B \longrightarrow A$ *is a fibering in the sense of definition 87, then*

$$\Omega^p B \otimes_B N_{0,q} \otimes_A E \cong M_{p,q}$$

via the map

$$\xi \otimes [x] \otimes e \longmapsto [i_* \xi \wedge x \otimes e]_{p,q}.$$

Proof: *Definition 87 says that we have an isomorphism*

$$\Omega^p B \otimes_B N_{0,q} \longrightarrow N_{p,q}$$

given by $\xi \otimes [x] \longmapsto [i_*\xi \wedge x]$. *Now use proposition 86.* $\square$

We now return to the problem of calculating the cohomology of the sequence 5.3. Take $\xi \otimes [x] \otimes e \in \Omega^p B \otimes_B N_{0,q} \otimes_A E$ which maps to $[i_*\xi \wedge x \otimes e] \in M_{p,q}$, and apply d (in this case $\nabla^{[p+q]}$ and $x \in \Omega^q A$) to it to get

$$d(i_*\xi\wedge x)\otimes e+(-1)^{p+q}i_*\xi\wedge x\wedge\nabla e = i_*d\xi\wedge x\otimes e+(-1)^p i_*\xi\wedge dx\otimes e+(-1)^{p+q}i_*\xi\wedge x\wedge\nabla e$$

But $d\xi \in \Omega^{p+1}B$, and

$$M_{p,q+1} = \frac{i_*\Omega^p B \wedge \Omega^{q+1} A \otimes_A E}{i_*\Omega^{p+1} B \wedge \Omega^q A \otimes_A E},$$

so the first term vanishes on applying $[\quad]_{p,q+1}$. Then

$$d[i_*\xi \wedge x \otimes e]_{p,q} = (-1)^p [i_*\xi \wedge (dx \otimes e + (-1)^q x \wedge \nabla e)]_{p.q+1} \tag{5.4}$$

Then, using proposition 88, we have an isomorphism

$$\Omega^p B \otimes_B M_{0,q} \cong M_{p,q} \tag{5.5}$$

$$\xi \otimes [y]_{0,q} \longrightarrow [i_*\xi \wedge y]_{p,q},$$

and using this isomorphism, d on $M_{p,q}$ can be writen as (see 5.4)

$$d(\xi \otimes [y]_{0,q}) = (-1)^p \xi \otimes [\nabla^{[q]} y]_{0,q+1} \tag{5.6}$$

where $y \in \Omega^q A \otimes_A E$. From 5.6 we see that we should study $[\nabla^{[q]}] : M_{0,q} \longrightarrow M_{0,q+1}$, defined by $[y]_{0,q} \longmapsto [\nabla^{[q]} y]_{0,q+1}$.

Proposition 89 *We show that*

$$[\nabla^{[q]}] : M_{0,q} \longrightarrow M_{0,q+1}$$

is a left B module map. Remember that $b \triangleright [\eta \otimes e] = [i(b)\eta \otimes e]$, for $b \in B$ and $\eta \otimes e \in \Omega^q A \otimes_A E$.

Proof: *First,*

$$\begin{aligned} [\nabla^{[q]}](b \triangleright [\eta \otimes e]_{0,q}) &= [d(i(b)\eta) \otimes e + (-1)^q i(b)\eta \wedge \nabla e]_{0,q+1} \\ &= [i_*(db) \wedge \eta \otimes e + i(b).d\eta \otimes e + (-1)^q i(b)\eta \wedge \nabla e]_{0,q+1} \end{aligned}$$

Now

$$i_*(db) \wedge \eta \otimes e \in i_*\Omega^1 B \wedge \Omega^q A \otimes_A E$$

so $[i_*(db) \wedge \eta \otimes e]_{0,q+1} = 0$ in $M_{0,q+1}$. Then

$$\begin{aligned} [\nabla^{[q]}](b \triangleright [\eta \otimes e]_{0,q}) &= [i(b).d\eta \otimes e + (-1)^q i(b)\eta \wedge \nabla e]_{0,q+1} \\ &= b \triangleright [d\eta \otimes e + (-1)^q \eta \wedge \nabla e]_{0,q+1}. \quad \square \end{aligned}$$

Proposition 90 *If $\Omega^p B$ is flat as aright B module, the cohomology of the cochain complex*

$$\cdots M_{p,q-1} \xrightarrow{d} M_{p,q} \xrightarrow{d} M_{p,q+1} \xrightarrow{d} \cdots$$

is given by $\Omega^p B \otimes_B \hat{H}_q$, where $\hat{H}_q$ is defined as the cohomology of the cochain complex

$$\cdots \xrightarrow{d} M_{0,q} \xrightarrow{d} M_{0,q+1} \xrightarrow{d} \cdots .$$

Proof: As we now know that $d = [\nabla^{[q]}] : M_{0,q} \longrightarrow M_{0,q+1}$ is a left B module map, we have an exact sequence of left B modules, where the first map is inclusion

$$0 \longrightarrow K_q \xrightarrow{inc} M_{0,q} \xrightarrow{d} Z_{q+1} \longrightarrow 0 \tag{5.7}$$

where

$$\begin{aligned} Z^q &= image\, d : M_{0,q-1} \longrightarrow M_{0,q}, \\ K^q &= kernel\, d : M_{0,q} \longrightarrow M_{0,q+1}. \end{aligned} \tag{5.8}$$

Now we define

$$\hat{H}_q = \frac{K_q}{Z_q} \tag{5.9}$$

to be the cohomology of

$$[\nabla^{[q]}] : M_{0,q} \longrightarrow M_{0,q+1},$$

so we have a short exact sequence

$$0 \longrightarrow Z_q \longrightarrow K_q \longrightarrow \hat{H}_q \longrightarrow 0. \tag{5.10}$$

To calculate the cohomology of 5.3, we need to calculate both of

$$\hat{Z}_{p,q} = image\, d : \Omega^p B \otimes_B M_{0,q-1} \longrightarrow \Omega^p B \otimes_B M_{0,q},$$

$$\hat{K}_{p,q} = kernel\, d : \Omega^p B \otimes_B M_{0,q} \longrightarrow \Omega^p B \otimes_B M_{0,q+1}.$$

If $\Omega^P B$ is flat as a right B module, then we have another exact sequence from 5.7

$$0 \longrightarrow \Omega^p B \otimes_B K_q \stackrel{id\otimes inc}{\longrightarrow} \Omega^p B \otimes_B M_{0,q} \stackrel{id\otimes d}{\longrightarrow} \Omega^p B \otimes_B Z_{q+1} \longrightarrow 0 \tag{5.11}$$

But by 5.6 the last map $id \otimes d$ is $(-1)^p d$ on $M_{p,q}$ so we have

$$\hat{Z}_{p,q} = \Omega^p B \otimes_B Z_q$$

and

$$\hat{K}_{p,q} = \Omega^p B \otimes_B K_q.$$

Now apply $\Omega^p B \otimes_B$ to (5.10) to get, using $\Omega^p B$ flat as a right B module again,

$$0 \longrightarrow \Omega^p B \otimes_B Z_q \longrightarrow \Omega^p B \otimes_B K_q \longrightarrow \Omega^p B \otimes_B \hat{H}_q \longrightarrow 0.$$

But by our previous result, this is

$$0 \longrightarrow \hat{Z}_{p,q} \longrightarrow \hat{K}_{p,q} \longrightarrow \Omega^p B \otimes_B \hat{H}_q \longrightarrow 0 \tag{5.12}$$

so the cohomology of $M_{p,q}$ is isomorphic to $\Omega^p B \otimes_B \hat{H}_q$. By definition we have

$$0 \longrightarrow \hat{Z}_{p,q} \longrightarrow \hat{K}_{p,q} \longrightarrow H(M_{p,q}) \longrightarrow 0$$

and this gives the isomophism by 31.
If we write $\langle \quad \rangle_{p,q}$ for the equivalence class in cohomology $(M_{p,q})$, this isomophism is given by

$$\langle i_* \xi \wedge x \rangle_{p,q} \longrightarrow \xi \otimes \langle x \rangle_{0,q} \tag{5.13}$$

for $\xi \in \Omega^p B$ and $x \in \Omega^q A \otimes_A E$. $\square$

5.6 Calculation the second page of the spectral sequence

Now we need to move to the second page of the spectral sequence, in which we take the cohomology of the previous cohomology, i.e. the cohomology of

$$d : cohomology\,(M_{p,q}) \longrightarrow cohomology\,(M_{p+1,q}).$$

By the isomophism discussed in the last section 5.5, we can view this as

$$d : \Omega^p B \otimes_B \hat{H}_q \longrightarrow \Omega^{p+1} B \otimes_B \hat{H}_q.$$

Proposition 91 *The differential d gives a left covariant derivative*

$$\nabla_q : \hat{H}_q \longrightarrow \Omega^1 B \otimes_B \hat{H}_q.$$

If $\langle \xi \otimes e \rangle_{0.q} \in \hat{H}_q$, this is given by using (5.13)

$$\langle \xi \otimes e \rangle_{0.q} \longmapsto \eta \otimes \langle \omega \otimes f \rangle_{0,q}$$

where

$$d\xi \otimes e + (-1)^q \xi \wedge \nabla e = i_* \eta \wedge \omega \otimes f.$$

Proof: *Take* $\langle x\rangle_{0,q} \in \hat{H}_q$, *where* $x \in K_q$ *(see (5.8)). Suppose* $x = \xi \otimes e$, *where* $\xi \in \Omega^q A$ *and* $e \in E$ *(summation implicit). As* $x \in K_q$ *we have*

$$[dx]_{0,q+1} = [d\xi \otimes e + (-1)^q \xi \wedge \nabla e]_{0,q+1} = 0$$

in $M_{0,q+1}$, *so*

$$d\xi \otimes +(-1)^q \xi \wedge \nabla e \in i_* \Omega^1 B \wedge \Omega^q A \otimes_A E.$$

We write (summation implicit), for $\eta \in \Omega^1 B$,

$$d\xi \otimes e + (-1)^q \xi \wedge \nabla e = i_* \eta \wedge \omega \otimes f \tag{5.14}$$

Under the isomorphism (5.5), this corresponds to $\eta \otimes [\omega \otimes f]_q \in \Omega^1 B \otimes_B M_{0,q}$. *As the curvature of* E *vanishes, we have from* $\nabla^{[q+1]}$ *to (5.14),*

$$i_* d\eta \wedge \omega \otimes f - i_* \eta \wedge d\omega \otimes f + (-1)^{q+1} i_* \eta \wedge \omega \wedge \nabla f = 0. \tag{5.15}$$

We take this as an element of $M_{1,q+1}$, *so we apply* $[\quad]_{1,q+1}$ *to (5.15). Then as the denominator of* $M_{1,q+1}$ *is*

$$i_* \Omega^2 B \wedge \Omega^q A \otimes_A E,$$

we see that the first term of 5.15 vanishes on taking the quotient, giving

$$-[i_* \eta \wedge (d\omega \otimes f + (-1)^q \omega \wedge \nabla f)]_{1,q+1} = 0.$$

Under the isomorphism (5.5) this corresponds to

$$-\eta \otimes_B [d\omega \otimes f + (-1)^q \omega \wedge \nabla f]_{0,q+1} = 0. \tag{5.16}$$

This means that

$$\eta \otimes [\omega \otimes f]_{0,q} \in \Omega^1 B \otimes_B M_{0,q}$$

is in the kernel of the map id $\otimes$ d in (5.11), and as (5.11) is an exact sequence we have

$$\eta \otimes [\omega \otimes f]_{0,q} \in \Omega^1 B \otimes_B K_q,$$

so we can see take the cohomology class to get

$$\eta \otimes \langle \omega \otimes f \rangle_{0,q} \in \Omega^1 B \otimes_B \hat{H}_q.$$

This completes showing that ∇_q exists, but we need to show that it is a left covariant derivative. For $b \in B$, we calculate $\nabla_q(b.\xi \otimes e)$ using the formula to get

$$d(b.\xi) \otimes e + (-1)^q b.\xi \wedge \nabla e = db \wedge \xi \otimes e + b.(d\xi \otimes e + (-1)^q \xi \wedge \nabla e),$$

so we get

$$\nabla_q \langle b.\xi \otimes e \rangle_{0,q} = db \otimes \langle \xi \otimes e \rangle_{0,q} + b.\nabla_q \langle \xi \otimes e \rangle_{0,q}. \quad \square$$

Proposition 92 *The curvature of the covariant derivative ∇_q in proposition 91 is zero.*

Proof: *Using the notation of proposition 91, equation (5.14)*

$$\nabla_q \langle \xi \otimes e \rangle_{0,q} = \eta \otimes \langle \omega \otimes f \rangle_{0,q}.$$

If we apply $\nabla_q^{[1]}$ (see proposition 38), we get

$$R_q \langle \xi \otimes e \rangle_{0,q} = d\eta \otimes \langle \omega \otimes f \rangle_{0,q} - \eta \wedge \nabla_q \langle \omega \otimes f \rangle_{0,q}. \tag{5.17}$$

To find $\nabla_q \langle w \otimes f \rangle_{0,q}$, *refering to the proof of proposition 91, formula (5.16), we have*

$$\eta \otimes_B (d\omega \otimes f + (-1)^q \omega \wedge \nabla f) \in \Omega^1 B \otimes_B (i_* \Omega^1 B \wedge \Omega^q A \otimes_A E).$$

Now we write (summation implicit). This comes for tensoring the exact sequence

$$0 \longrightarrow i_* \Omega^1 B \wedge \Omega^q A \otimes_A E \longrightarrow \Omega^{q+1} A \otimes_A E \stackrel{[\,]_{0,q+1}}{\longrightarrow} M_{0,q+1} \longrightarrow 0$$

on the left by $\Omega^1 B$, *and use* $\Omega^1 B$ *flat.*

$$\eta \otimes (d\omega \otimes f + (-1)^q \omega \wedge \nabla f) = \eta' \otimes (i_* \kappa \wedge \zeta \otimes g) \tag{5.18}$$

for $\eta', \kappa \in \Omega^1 B$, $\zeta \in \Omega^q A$ *and* $g \in E$. *Then, from proposition 91,*

$$\eta \wedge \nabla_q \langle \omega \otimes f \rangle_{0,q} = \eta' \wedge \kappa \otimes \langle \zeta \otimes g \rangle_{0,q}$$

so from (5.17),

$$R_q \langle \xi \otimes e \rangle_{0,q} = d\eta \otimes \langle \omega \otimes f \rangle_{0,q} - \eta' \wedge \kappa \otimes \langle \zeta \otimes g \rangle_{0,q}. \tag{5.19}$$

Now (5.18) implies that

$$i_* \eta \wedge (d\omega \otimes f + (-1)^q \omega \wedge \nabla f) = i_* \eta' \wedge i_* \kappa \wedge \zeta \otimes g,$$

and substituting this into (5.15) gives

$$i_* d\eta \wedge \omega \otimes f - i_* \eta' \wedge i_* \kappa \wedge \zeta \otimes g = 0,$$

so on taking equivalence classes in $M_{2,q}$ we find, using the isomophism (5.5),

$$d\eta \otimes [\omega \otimes f]_{0,q} - \eta' \wedge \kappa \otimes [\zeta \otimes g]_{0,q} = 0.$$

and this shows that $R = 0$ by 5.19. $\square$

Theorem 93 *Given*

1) a map $i : B \longrightarrow A$ which is a differential fibration (see definition 87)

2) A flat left A module E, with a zero-curvature left covariant derivative

$$\nabla_E : E \longrightarrow \Omega^1 A \otimes_A E$$

3) Each $\Omega^p B$ is flat as a right B module.

Then there is a spectral sequence converging to $H^*(A, E, \nabla_E)$ with second page $H^*(B, \hat{H}_q, \nabla_q)$ where $\hat{H}_q$ is defined as the cohomology of the cochain complex

$$\cdots \xrightarrow{d} M_{0,q} \xrightarrow{d} M_{0,q+1} \xrightarrow{d} \cdots$$

where

$$M_{0,q} = \frac{\Omega^q A \otimes_A E}{i_* \Omega^1 B \wedge \Omega^{q-1} A \otimes_A E}$$

and

$$d[x \otimes e]_{0,q} = [dx \otimes e + (-1)^q x \wedge \nabla_E e]_{0,q}.$$

The zero curvature left conariant derivative

$$\nabla_q : \hat{H}_q \longrightarrow \Omega^1 B \otimes_B \hat{H}_q$$

is as defined in proposition 91.

Proof: *The first part of the proof is given in proposition 90. Now we need to calculate the cohomology of*

$$d : \Omega^p B \otimes_B \hat{H}_q \longrightarrow \Omega^{p+1} B \otimes_B \hat{H}_q$$

This is given for $\xi \otimes \langle \eta \otimes e \rangle_{0,q}$ (for $\xi \in \Omega^p B$, $\eta \in \Omega^q A$ and $e \in E$) as follows : this element corresponds to $i_* \xi \wedge \eta \otimes e$, and applying d to this gives

$$i_* d\xi \wedge \eta \otimes e + (-1)^p i_* \xi \wedge d\eta \otimes e + (-1)^{p+q} i_* \xi \wedge \eta \wedge \nabla e.$$

But we have calculated the effect of d on $\hat{H}_q$ in proposition 91, so we get

$$d(\xi \otimes \langle \eta \otimes e \rangle_{0,q}) = d\xi \otimes \langle \eta \otimes e \rangle_{0,q} + (-1)^p \xi \wedge \nabla_q \langle \eta \otimes e \rangle_{0,q}.$$

The covariant derivative ∇_q has zero curvature by proposition 92. $\square$

We have the following examples of a noncommutative differential fibration:

Example 94 *(see section 8.5 of [2]) Given the left covariant calculus on the quantum group $SU_q(2)$ given by Woronowicz [46], the corresponding differential calculus on the quantum sphere S_q^2 gives a differential fibration*

$$i : S_q^2 \longrightarrow SU_q(2).$$

Here the algebra SU_q^2 is the invariants of $SU_q(2)$ under a circle action, and i is just the inclusion.

Example 95 *In section 4.11 it is shown that the noncommutative torus $\mathbb{T}_q^2$ is the fiber of the map*

$$C(S^1) \longrightarrow A,$$

where A is the group algebra of the Heisenberg group.
In the paper [43] the authors discuss noncommutative tours bundles over topological spaces. In fact this paper was the motivation behind checking that the idea of differential algebra fibration definition applied to the Heisenberg group algebra.

5.7 Some comments on the Leray spectral sequence

In chapter 6, we discuss a possible application (as get very tentative) of the Leray spectral sequence (in the version we give here) to the representation theory of quantum groups.
In example 95 we have already mentioned the paper [43]. In a sequel to this, in [44], the authors discuss the idea of C^* algebra fibrations over topological spaces in more generality. This is rather different to our point of view. We can have noncommutative base algebras (example 94), but we also require the existence of a differential structure. However it is intersting that [44] includes a discussion of the Leray spectral sequence of the fibration with base a simplicial complex, and that this forms an important part of their theory.

Chapter 6

Conclusion

6.1 Summary

We have looked at noncommutative differential geometry, stated in terms of differential forms, and seen how it can be used in Riemannian geometry and sheaf cohomology. The example of differential calculi which we considered in detail were the Heisenberg group (chapter 4) and the group algebra of A_4 (chapter 3). What was found supports the general trend of example in noncommutative geometry: some ideas from classical geometry work in considerable generality (like K theory), and some work only in more special circumstances.

For example, in classical differential geometry, on a Riemannian manifold there is always a unique Levi-Civita connection (which preserves the metric and is torsion free). We have seen that this is not necessarily the case in noncommutative geometry.

Part of the problem with noncommutative geometry is that when a classical idea does not work, it is not certain whether to say simply that it does not work, or whether to look for how the noncommutative construction might differ. For example in the theory of geometric quantisation, it is not obvious to say that quantisation of differential form does not work in the case of curvature, or whether we should allow the possibility of non associative calculi (see [5] and [23]).

However some nonassociative structures arise as part of string theory (see [11]).

For example for the Riemannian metric on the Heisenberg group we found that the only metric to commute with the algebra was degenerate. For the Levi-civita metric we have tried to say that the metric is exactly preserved by the covariant derivative. Maybe, for many examples, this is not natural condition. We might have $\nabla G \neq 0$, but set equal to another interesting quantity. In the past, mathematical physics has been a good source of interesting generalisations.

6.2 Further work on differential geometry

To extend the work on general covariant derivatives to other cases would probably require extensive use of computer algebra (for example the noncommutative algebra packages for Mathematica or Sage). However, even then it is not at all obvious that many polynomials in many variable could be solved (for example, the braid relations). This problem of doing general calculations is well known in classical differential geometry, for example solving the general case of Einstein's equations in general relativity is viewed as extremely different. There is no solution to the two body problem in G. R.

Following the spirit of finding the black hole solution in general relatvity, it is likely that symmetry will be needed to reduce the complexity. It is likely that more can be said in general on quotients of Hopf algebra with differential calculi, such as the quantum sphere. There is some work being done on higher dimensional examples in this direction(see [1] on a quantum S^4).

If a group G acts on a topological space X transitively (i.e. every point can be moved to every other point), then we have a one to one correspondence $\frac{G}{H} \longrightarrow X$, given by $x \in X$ and $[g] \longmapsto g \triangleright x$. Here H is a subgroup (the stabiliser of $x \in X$) of G given by

$$H = \{h \in G \,|\, h \triangleright x = x\}.$$

An example of this is of the rotation group acting on a sphere. An example of a not transitive action is $\mathbb{R}$ acting $\mathbb{R}^2$

$$z \triangleright (x, y) = (x + z, y)$$

Here is transitive action of $\mathbb{R}^2$ on $\mathbb{R}$

$$(u, v) \triangleright (x, y) = (x + u, y + v)$$

These spaces on which a group acts transitively are important in pure mathematics and in physics (e.g. cosmology).

Those examples most studied in noncommutative geometry are given by a Hopf algebra H, and a surjective Hopf algebra map $\pi : H \longrightarrow K$ for another Hopf algebra K. Then the algebra is

$$A = H^{coK} = \{h \in H : (id \otimes \pi)\nabla h = h \otimes 1\}.$$

It might be possible to study more general quotient spaces than this. The fact that A above is not completely general can be seen by the fact that there is an algebra map given by the counit,

$$\epsilon : A \longrightarrow \mathbb{C},$$

so A contains at least are "classical point". It is not expected that a more general quotient would have such an algebra map. It would be interesting to consider the differential geometry of these more general quotients.

6.3 Further work on the Leray spectral sequence

One use for the Leray spectral sequence (in the fibration version we have given) is given in [40]. The Borel-Weil-Bott theorem is about representations of Lie groups [21] [28] [48]. We summarise a little detail for SU_2. Then we can take the Hopf fibration

$$SU_2 \longrightarrow S^2 = \frac{SU_2}{Diagonal\, matrices\, in\, SU_2}$$

We can take line bundles on S^2, and give an action of SU_2 on it. Then we can classify all irreducible representation of SU_2 as complex analytic sections of the line bundles (one dimensional vector bundles).

One way of proving the result for more general groups is to use the Leray spectral sequence, and this proof may well generalise to the noncommutative case. It would be interesting to compare our result to the results and definitions in [44] in more detail. The paper [44] is discussed in more detail in section 5.7.

Bibliography

[1] F. Andrea, L. Dabrowski, G. Landi, *The Isospectral Dirac Operator on the 4-dimensional Orthogonal Quantum Sphere.* Commun. Math. Phys. 279, 77-116 (2008).

[2] E.J. Beggs, T. Brzezinski, *The Serre spectral sequence of noncommutative fibration for de Rham cohomology.* Acta Mathematica 195 (2005), p155-196

[3] E. J. Beggs , S. Majid, *Bar categories and star operations,* Algebra. Represent. Theory 12 (2009),no.2-5,103-152

[4] E.J. Beggs , S. Majid, **-Compatible Connections in noncommutative Riemannian geometry.* Journal of Geometry and Physics 61 (2011) pp. 95-124.

[5] E.J. Beggs, S. Majid, *Quantisation by cochain twists and nonassociative differentials.*. J. Math. Phys. 51, 053522 (2010); doi:10.1063/1.3371677 (32 pages)

[6] A.J. Berrick, M.E. Keating, *An Introduction to Ring and Modules with K-theory in view.* C.U.P. 2000.

[7] F.A. Berezin, *General concept of quantization,* Commun. Math. Phys. 40, 153-174 (1975).

[8] B. Blackadar, *K-Theory for Operator algebra,* 1986 by Springer-Verlag New York Inc.

[9] G.E. Bredon, *Sheaf Theory,* 1967 by McGraw-Hill.

[10] K. Bresser, F. Muller-Hoissen, A. Dimakis, A. Sitarz, *Noncommutative geometry of finite group.* J. of Physics A (Math. and General), 29 :2705- 2735, 1996.

[11] P. Bouwknegt, K. Hannabuss, V. Mathai, *Nonassociative Tori and Applications to T-Duality.* Commun. Math. Phys. 264, 41-69 (2006).

[12] T. Brzezinski, *Lecture on rings, modules and categories.* Swansea University

[13] A. Connes, *Noncommutative geometry,* Academic Press. San Diego; London: 1994.

[14] A. Connes, *Gravity coupled with matter and the foundation of noncommutative geometry,* Comm. Math. Phys. 155: 109. 1996

[15] A. Connes, M. Marcolli, *Noncommutative Geometry: Quantum Fields and Motives,* American Mathematical Society (2007).

[16] M. Dubois-Violette, T. Masson, *On the first-order operators in bimodules.* Lett. Math. Phys. 37, 467-474, 1996.

[17] M. Dubois-Violette, P.W. Michor, *Connections on central bimodules in noncommutative differential geometry.* J.Geom. Phys. 20, 218-232, 1996

[18] M. Dubois-Violette, *Lectures on graded differential algebras and noncommutative geometry.* In Y. et al. Maeda, editor, Noncommutative Differential Geometry and its Applications to Physics, pages 245-306. Shonan, Japan, 1999, Kluwer Academic Publishers, 2001.

[19] A. Einstein, *Relativity: The Special and General Theory,* New York: H. Holt and Company, 1916

[20] G. Fiore ,J. Madore, *Leibniz rules and reality conditions.* Eur. Phys. J. C Part. Fields 17 (2000), no. 2, 359-366.

[21] W. Fulton, J. Harris, *Representation Theory.* A First Course, Springer-Verlag New York Inc, 1991.

[22] A. Hatcher, *Vector Bundles and K-theory.* Version 1.1, November 2000. (http://www.math.cornell.edu/ hatcher/VBKT/VB.pdf)

[23] E. Hawkins, *Noncommutative Rigidity.* Commun. Math. Phys. 246, 211-235 (2004).

[24] C. Kassel, *Quantum Groups.* vol. 155 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1995.

[25] W. Klingenberg, *A Course in Differential Geometry.* Springer-Verlag, New York Inc, 1978.

[26] E. C. Lance, *Hilbert C^*-modules, A toolkit for operator algebraists.* London Mathematical Society Lecture Note Series 210. Cambridge University Press, 1995.

[27] J. Lepotier, *Lectures on Vector Bundles.* Cambridge studies in advanced mathematics. Cambridge University Press 1997

[28] J. Lurie, *A Proof of the Borel-Weil-Bott Theorem,* (http://www-math.mit.edu/ lurie/papers/bwb.pdf), Retrieved on Dec. 14, 2007.

[29] J. Madore, *The fuzzy sphere,* Class. Quant. Grav. 9, 69-87 (1992).

[30] J. Madore, *An introduction to noncommutative differential geometry and its physical application.* London Mathematical Society Lecture Note Series, 257, CUP 1999.

[31] S. Majid, *Foundations of Quantum Group Theory.* Cambridge University Press, 1995.

[32] S. Majid, *Waves on noncommutative spacetime and gamma-ray bursts,* Int.Jour,Mod.Phys.A15 4301-4323 (2000).

[33] S. Majid, *Quantum Groups Primer.* Cambridge University Press, 2002

[34] S. Majid, *Riemannian geometry of quantum groups and finite groups with nonuniversal differentials.* Commun. Math. Phys 225, 131-170 (2002)

[35] J. McCleary, *A User's Guide to Spectral Sequences.* 2nd ed., Cambridge University Press, Cambridge (2001).

[36] S. Montgomery, *Hopf Algebras and Their Actions on Rings.* American Mathematical Society, 1993 .

[37] J. Mourad, *Linear connections in noncommutative geometry.* Class. Quantum Grav. 12, 965-974, 1995.

[38] A. Moyal, *Maverick Mathematician: The Life and Science of J.E. Moyal,* ANU E-press, 2006

[39] J.E. Moyal, *Quantum mechanics as a statistical theory,* Proceedings of the Cambridge Philosophical Society, 45 (1949) pp. 99-124.

[40] A. Pressley , G. Segal, *Loop Groups,* Oxford University Press, 1988

[41] P. Petersen, *Riemannian Geometry.* Springer-Verlag, New York Inc, 1998 .

[42] B. Riemann, *Ueber die Hypothesen, welche der Geometrie zu Grunde liegen.,* Aus dem dreizehnten Bande der Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen. (http://www.emis.de/classics/Riemann/Geom.pdf).

[43] Siegfried Echterhoff, Ryszard Nest, and Herve Oyono-Oyono, *Principal noncommutative torus bundles,* arXiv:0810.0111(October 2008)

[44] Siegfried Echterhoff, Ryszard Nest, and Herve Oyono-Oyono, *Fibration with noncommutative fibers,* J. Noncommut. Geom. 3 (2009), no. 3, 377-417

[45] Ursula Carow-Watamura , S. Watamura, *Noncommutative Geometry and Gauge Theory on Fuzzy Sphere,* Department of Physics Graduate School of Science Tohoku University Aoba-ku, Sendai 980-8577, JAPAN

[46] S. L. Woronoeicz, *Differential Calculus on Compact Matrix Pseudogroups (Quantum Groups).* Commun. Math. Phys. 122,125-170 (1989)

[47] $http://en.wikipedia.org/wiki/Moyal-product, on\, 10-8-2010.$

[48] $http://en.wikipedia.org/wiki/Borel-Wei-Bott-theorem, on\, 9-8-2010$

Appendix A

We have terms $3f(es-iu), -t(es-iu), -2(e-u)(e+u)v, -(1+b)(es-iu)$

Case1 If $es-iu \neq 0$, we must have $f=t=0$ and $b=-1$. Recalculate the matrix. Substituting this into the matrix, we find $i^2u=0$, so $iu=0$, and so $e \neq 0$, and $s \neq 0$.

The matrix contains $egs=0$, so $g=0$.

The matrix contains $es^2=0$, which is a contradiction.

Case2 $es-iu=0$.

The matrix contains $v(e^2-u^2)=0$ and $(1+k)(e^2-u^2)=0$. We split into two cases: **Case2a** $(e^2-u^2 \neq 0)$ and **Case2b** $(e^2-u^2=0)$.

Case2a $es-iu=0$ and $e^2-u^2 \neq 0$.

The matrix contains $v(e^2-u^2)=0$ and $(1+k)(e^2-u^2)=0$, so we deduce that $v=0$ and $k=-1$.

The matrix contains $u^3=0$, so $u=0$ and we deduce that $e \neq 0$.

The matrix contains $e^3=0$, so $e=0$ - Contradiction. End of case 2a

Case2b $es-iu=0$ and $e^2-u^2=0$.

It splits to two cases:

Case2ba when $u=e=0$ and **Case2bb** when $u \neq 0 \neq e$.

Case2ba We get a matrix entry $(i+a-h)(s^2-i^2)=(1+k)(s^2-i^2)=v(s^2-i^2)=0$.

Split into two cases

Case2baa when$s^2=i^2$ and **Case2bab** when$s^2 \neq i^2$.

Case2baa $u=e=0$ and $s^2=i^2$.

Split into two cases

Case2baaa when$s=i=0$ and **Case2baab** when $s=yi \neq 0$ where $y=\pm 1$.

Case2baaa $s=i=u=e=0$.

We get entry $t^2(1+k)-v(1+m)^2$, $v(1+k)(1+k+v)$, $v(1+k)(1+a-h-v)$, $v(1+k)(a-h-k)$.

We begin by assuming that $t^2(1+k)\neq 0$. Then all of $t, 1+k, v, 1+m$ are non zero. Then the terms 2-4 listed above give $1+k+v=0$, $1+a-h-v=0$, $a-h-k=0$, and from this we deduce $v=0$, a contradiction. So $t^2(1+k)=v(1+m)=0$.

Next we have terms $f((1+k)t+fv)$ and $f(f(1+k)+v(1+m))$, so we deduce $fv=f(1+k)=0$.

We split into 4 cases:

Case2baaaa when $(1+m)t\neq 0$ then from the entries $t(1+m)(1+k+v)$, $t(1+m)(1+a-h-v)$, $t(1+m)(a-h-k)$, so if $t(1+m)\neq 0$ we deduce $v=0$,$k=-1$and $h=a+1$.

For the next 3 cases we have $(1+m)t=v(1+m)0$, remembering that $v(1+k)=0$:

Case2baaab when $m\neq -1$, so $t=0$ and $v=0$ and $f(1+k)=0$.

Case2baaac where $t\neq 0$,so $m=k=-1$ and $fv=0$.

Case2baaad where $t=0$ and $m=-1$, so $fv=f(1+k)=0$.

Case2baaaa $s=i=u=e=v=0$ and$k=-1$ and $h=a+1$ and $t(1+m)\neq 0$

We get entries qt^2, nt^2, rt^2 and gt^2, so $g=q=n=r=0$.

We now get j^2t, so $j=0$.

We now get $c(1+m)t$, so $c=0$.

We now get $(1+a)f(1+b+f)$, $(1+a)((1+m)^2+t^2)$, $(1+a)((1+b)^2+3f^2-2t(1+m))$.

If $a=-1$ SOLUTION

If $a\neq -1$, then we split into two cases

$f\neq 0$ gives $b=-1-f$ and $f^2=t(1+m)/2$ and $t^2=-(1+m)^2$ SOLUTION.

$f=0$ and $(1+b)^2=2t(1+m)$ and $t^2=-(1+m)^2$ SOLUTION.

Case2baaab $s=i=u=e=v=t=0$ and $m\neq -1$.

We get $(1+m)r^2$, $(1+m)q^2$, $(1+m)j^2$, so $r=q=j=0$.

Next we get $(1+m)g^2$, so $g=0$.

Next we get $(1+m)h^2$ and $(a-k)(1+m)^2$, so $h=0$ and $a=k$.

Get terms fn^2, fc^2, $f^2(1+k)$.

Split into 2 cases:

Case2baaaba $s=i=u=e=v=t=r=q=j=g=h=f=0$ and $a=k$ and $m \neq -1$.

Case2baaabb $s=i=u=e=v=t=r=q=j=g=h=n=c=0$ and $a=k=-1$ and $f \neq 0$ and $m \neq -1$.

Case2baaaba $s=i=u=e=v=t=r=q=j=g=h=f=0$ and $a=k$ and $m \neq -1$.

We get $(1+m)^2c$ and $(1+m)^2n$, so $c=n=0$.

We get $(1+k)(b-m)(1+m)$.

We get 2 cases:

$b=m$ SOLUTION

$k=-1$ SOLUTION

Case2baaabb $s=i=u=e=v=t=r=q=j=g=h=n=c=0$ and $a=k=-1$ and $f \neq 0$ and $m \neq -1$ SOLUTION

Case2baaac $s=i=u=e=0$, $t \neq 0$ and $k=m=-1$.

We get n^2t, q^2t, j^2t, so $n=q=j=0$.

Then we get ht^2 and gt^2, so $h=g=0$.

Then rt^2, so $r=0$.

Then ct^2, so $c=0$.

We get $(1+a)f^2$ and f^2v, $f((1+a)(1+b)-tv)$.

Split into two cases:

Case2baaaca $s=i=u=e=n=q=j=h=g=r=c=f=0$, $t \neq 0$ and $k=m=-1$.

Case2baaacb $s = i = u = e = n = q = j = h = g = r = c = v = 0$, $f \neq 0$, $t \neq 0$ and $a = k = m = -1$.

Case2baaaca $s = i = u = e = n = q = j = h = g = r = c = f = 0$, $t \neq 0$ and $k = m = -1$.

We get $t^2(1 + a - v)$ and $v(1 + b - t)$, so $v = 1 + a$.

Split into two cases:

$a = -1$ SOLUTION $a \neq -1$ and $t = 1 + b$ SOLUTION.

Case2baaacb $s = i = u = e = n = q = j = h = g = r = c = v = 0$, $f \neq 0$, $t \neq 0$ and $a = k = m = -1$.

SOLUTION

Case2baaad $s = i = u = e = t = 0$ and $m = -1$.

We get $f^2(1 + k)$ and $f^2 v$.

Split into 2 cases:

Case2baaada when $f = 0$ and **Case2baaadb** when $f \neq 0$ so $k = -1$and $v = 0$.

Case2baaada $s = i = u = e = t = f = 0$ and $m = -1$.

We get $(1 + b)c^2$, $(1 + b)n^2$, $(1 + b)q^2$ and $(1 + b)r^2$

Split into two cases

Case2baaadaa when $b = -1$ and **Case2baaadab** when $b \neq -1$ so $c = n = q = r = 0$.

Case2baaadaa $s = i = u = e = t = f = 0$ and $b = m = -1$.

We have entries $g(r(1 + k) + nv)$, $g(n(1 + k) + rv)$, $q(r(1 + k) + nv)$, $q(n(1 + k) + rv)$, $(h - j)(r(1 + k) + nv)$, $(h - j)(n(1 + k) + rv)$, $r(1 + k)n + q^2 v$, $q^2(1 + k) + nrv$ and $v(1 + k)(1 + k + v)$.

Split into four cases :

Case2baaadaa4 at least one of $r(1 + k) + nv$ or $n(1 + k) + rv$ is non-zedro, so $g = 0$, $q = 0$ and $h = j$. **Cases 2baaadaa1+2baaadaa2+2baaadaa3** when $r(1+k)+nv = n(1+k)+rv = 0$, $v(q^2-n^2) = v(q^2-r^2) = 0$ and $(1+k)(q^2-n^2) = (1+k)(q^2-r^2) = 0$,so

Case 1 when $v = 1+k = 0$

Case 2+3at least one of $v, 1+k \neq 0$ from $r(1+k)+nv = n(1+k)+rv = 0$

Deduce that $n = 0$ if and only if $r = 0$ (oas $n = \frac{-r(1+k)}{v}$ or $n = \frac{-rv}{1+k}$)

Case2baaadaa2 when $q = n = r = 0$

Case2baaadaa3 when $v^2 = (1+k)^2 \neq 0$, so $k \to -1-v$ and $q^2 = n^2 = r^2 \neq 0$

Case2baaadaa1 $s = i = u = e = t = f = v = 0$ and $b = m = k = -1$ We have $g^2(1+a+2h)+c^2h-ch^2$, $g((c-h)(1+a)+ch+2gh)$ and $g(g(1+a+h)+2ch-h^2$

Solve gives four cases :

case 2baaadaa1A when $g = 0$, $ch(c-h) = 0$.

case 2baaadaa1B when $g \neq 0$, $h = 0$ and $a = -1$.

case 2baaadaa1C when $g \neq 0$, $h \neq 0$,$c = g+h$ and $a = \frac{-(g+3gh+h^2)}{g}$.

case 2baaadaa1D when $g \neq 0$, $c = g$, $h = 4g$ and $a = -1+4g$.

Case2baaadaa1A $s = i = u = e = t = f = v = g = 0$ and $b = m = k = -1$.

We have entries $3nqr$, $(j-c)qr+n(r^2+q^2)$, $(j-c)nr+q(r^2+n^2)$ and $(j-c)qn+r(n^2+q^2)$

We get at least one of $n, q, r = 0$ but then we deduce at least two of $n, q, r = 0$

terms $j(jn+2qr)$, $j(jq+2nr)$ and $j(jr+2qn)$.

Split into two cases : **case 2baaadaa1A1** when $j = 0$, at least two of $n, q, r = 0$ and **case 2baaadaa1A2** when $j \neq 0$ so $n = q = r = 0$.

Case2baaadaa1A1 $s = i = u = e = t = f = v = g = j = 0$ and $b = m = k = -1$, at least two of $n, q, r = 0$ from the term $h(n^2+r^2+q^2)$.

Deduce either $n = r = q = 0$ (Go to case 1A2) or $h = 0$

Set $h = 0$ we get $(1+a)n^2 = 0$, $(1+a)q^2 = 0$ and $(1+a)r^2 = 0$, so if $n = r = q = 0$

Go to case 1A2 or $a = -1$ SOLUTION.

Case2baaadaa1A2 $s = i = u = e = t = f = v = g = n = q = r = 0$ and $b = m = k = -1$ and $j \neq 0$.

We have entries $(1+a)(1+a-c)c$, $(1+a)(1+a-h)h$, $(1+a)(1+a-j)j$,$c(c-h)h$,$c(c-j)j$ and $h(h-j)j$.

Any non-zero available in the set $j, 1+a, h, c$ must be equal. SOLUTION.

Case2baaadaa1B $s = i = u = e = t = f = v = h = v = 0$ and $b = m = k = a = -1$ and $g \neq 0$.

We have entry $g(c^2 + 2gj - n^2 - q^2 - r^2)$

Split into 4 cases :

2baaadaa1B1 when $n = q = r = 0$ and **2baaadaa1B2+2baaadaa1B3+2baaadaa1B4** at least one of n, r, q are non zero (two or three of n, q, r are non zero).

2baaadaa1B1 $s = i = u = e = t = f = v = h = v = n = q = r = 0$ and $b = m = k = a = -1$ and $g \neq 0$.

We get $c^2 + 2gj$

So $j \rightarrow \frac{-c^2}{2g}$.

We get $c(c+2g)(c^2 + 2cg - 2g^2)$.

We split into three cases :

When $c = 0$ SOLUTION.

When $c = -2g$, we get g^3 CONTRADICTION.

When $c^2 + 2cg - 2g^2 = 0$, we get $c = -g - \sqrt{3}g$ or $c = -g + \sqrt{3}g$ SOLUTION.

2baaadaa1B2$s = i = u = e = t = f = v = h = v = 0$ and $b = m = k = a = -1$ and $g \neq 0$, if exacitly one of n, r, q are non zero (eg r)

We get $g^2 r$ so $r = 0$ CONTRADICTION.

2baaadaa1B3 $s = i = u = e = t = f = v = h = v = 0$ and $b = m = k = a = -1$ and $g \neq 0$, if exacitly two of n, r, q are non zero (eg n, r)

We get $2(g-j)nr$, so $g=j$ and also get $nr(c+g-j)$ so $c=0$

Then we get $r(g^2-n^2)$ and $n(g^2-r^2)$, so $g^2=n^2=r^2$

So we get three cases :

When $q=0$ we get $r^2=g^2$ and $n^2=g^2$ SOLUTION.

When $n=0$ we get $r^2=g^2$ and $q^2=g^2$ SOLUTION.

When $r=0$ we get $q^2=g^2$ and $n^2=g^2$ SOLUTION.

2baaadaa1B4$s=i=u=e=t=f=v=h=v=0$ and $b=m=k=a=-1$ and $g\neq 0$, when all n,q,r are non zero

We have entries $(g-j)(n(g+j)+2qr)$, $(g-j)(q(g+j)+2nr)$ and $(g-j)(r(g+j)+2qn)$

Split into two cases:

2baaadaa1B4a when $g=j$ and**2baaadaa1B4b** when $g\neq j$

2baaadaa1B4a $s=i=u=e=t=f=v=h=v=0$ and $b=m=k=a=-1$ and $g\neq 0$, and $g=j$.

We get $3(cg^2-nqr)$,so $c=\frac{nqr}{g^2}$

We get $(g^2-r^2)(g^2-n^2)$, $(g^2-r^2)(g^2-q^2)$ and $(g^2-q^2)(g^2-n^2)$.

So $g^2=r^2$, $g^2=n^2$ and $g^2=q^2$, at least two of them are equal, so we have 12 cases:

Suppose $g^2=n^2=r^2$ so $r\to g$ and $n\to g$ or $r\to -g$ and $n\to g$ or $r\to g$ and $n\to -g$ or $r\to -g$ and $n\to -g$

Suppose $g^2=n^2=q^2$ so $q\to g$ and $n\to g$ or $q\to -g$ and $n\to g$ or $q\to g$ and $n\to -g$ or $q\to -g$ and $n\to -g$

Suppose $g^2=q^2=r^2$ so $r\to g$ and $q\to g$ or $r\to -g$ and $q\to g$ or $r\to g$ and $q\to -g$ or $r\to -g$ and $q\to -g$ SOLUTION.

2baaadaa1B4b $s=i=u=e=t=f=v=h=v=0$ and $b=m=k=a=-1$ and $g\neq 0$ and $g\neq j$

Then $g+j\neq 0$, so $g+j=\frac{2nr}{q}=\frac{2qr}{n}=\frac{2qn}{r}=2x$, (define $x\neq 0$)

So $j\to 2x-g$ and $q\to\frac{xn}{r}$

Deduce $x^2 = r^2 = q^2 = n^2$

We get $8(g-x)x^3$, so $g = x$

We now get $3(-c+x)x^2$,so $c = x$

We have entries $r(n-x)(n+x)$ and $\frac{n(r-x)(r+x)(r^2+x^2)}{r^2}$, so we have 4 cases:

$n = -x$ and $r = -x$ or $n = x$ and $r = x$ or $n = -x$ and $r = x$ or $n = x$ and $r = -x$

SOLUTION.

Case2baaadaa1C $s = i = u = e = t = f = 0$ and $b = m = -1$ and $g \neq 0$ and $h \neq 0$ and $c = g + h$ and $a = \frac{-(g+3gh+h^2)}{g}$.

We get $(3g+h)(4g+h)$ and $(2g+h)^2(4g+h)$, can not have both $2g+h$ and $3g+h$ as $g \neq 0$, so must have $4g+h = 0$ and so $h = -4g$

We get $4gj + j^2 + n^2 + q^2 + r^2$ and $2gj + 9g^2 - n^2 - q^2 - r^2$ adding these gives $9g^2 + 6gj + j^2 = (3g+j)(3g+) = 0$, so $j = -3g$

We get $g^3 - nqr$, so all of n, q ,r are non zero

We get $3g^2 - n^2 - q^2 - r^2$, $gn = qr$, $gq = nr$ and $gr = nq$, so $n = qr/g$ and $g^2 = r^2 = q^2 \neq 0$, so $r = xg$ and $q = yg$,when $y^2 = x^2 = 1$ SOLUTION.

Case2baaadaa1d $s = i = u = e = t = f = v = 0$ and $b = m = k = -1$, $g \neq 0$, $c = g$, $h = 4g$ and $a = -1 + 4g$

We get $g(2gj + g^2 - n^2 - q^2 - r^2)$ and $4g(4gj - j^2 - n^2 - q^2 - r^2)$, subtracting these gives $g^2 - 2gj + j^2 = (g-j)^2 = 0$, so $j = g$

We get $g^3 - nqr$, so all of n, q ,r are non zero

We get $3g^2 - n^2 - q^2 - r^2$, $gn = qr$, $gq = nr$ and $gr = nq$, so $n = qr/g$ and $g^2 = r^2 = q^2 \neq 0$, so $r = xg$ and $q = yg$, when $y^2 = x^2 = 1$ SOLUTION.

Case2baaadaa2 $s = i = u = e = t = f = q = n = r = 0$, $b = m = -1$ and at least one of $v, 1+k \neq 0$.

We get gv^2 and $g(1+k)^2$, so $g = 0$

We get $h(1+k)^2 + h^2v$ and $h(1+k)^2 + h^2v - (1+k)v^2$, subtracting these gives

$(1+k)v^2 = 0$

Split into cases:

Case2baaadaa2a when $v = 0$ and $k \neq -1$ and **Case2baaadaa2b** when $v \neq 0$ and $k = -1$

Case2baaadaa2a $s = i = u = e = t = f = q = n = r = g = v = 0$, $b = m = -1$ and $k \neq -1$.

We get $h(1+k)^2$, $j(1+k)^2$ and $c(1+k)^2$, so $c = j = h = 0$

We get $(a-k)(1+k)^2$, so $a = k$ SOLUTION.

Case2baaadaa2b $s = i = u = e = t = f = q = n = r = g = 0$, $b = m = k = -1$ and $v \neq 0$.

We get h^2v, jv^2, cv^2 and $(1+a-v)v^2$, so $c = h = j = 0$ and $a = v - 1$ SOLUTION.

Case2baaadaa3 $s = i = u = e = t = f = 0$ and $b = m = -1$, $v^2 = (1+k)^2 \neq 0$, $k = -1 - v$ and $q^2 = n^2 = r^2 \neq 0$.

We have $q^2 = n^2 = r^2 \neq 0$, so $q = xn$ and $r = yn$ when $x^2 = y^2 = 1$

We get $n^2v(1-y)$, so $y = 1$

We get $v(2hj - j^2 + cv + n^2)$and$2hj - j^2 + jv + n^2$, subtracting these gives $v(c-j)$, so $c = j$

We get $-(1+a)v^2 + hv^2 - v^3$ and $-(1+a)v^2 + hv^2 + v^3$, subtracting these gives $-2v^3$ CONTRADICTION.

Case2baaadaa4 $s = i = u = e = t = f = g = q = 0$ and $b = m = -1$ and $h = j$.

We get j^2n, j^2r

Split into two cases:

Case2baaadaa4a when $j = 0$ and **Case2baaadaa4b** when $j \neq 0$, so $n = r = 0$

Case2baaadaa4a $s = i = u = e = t = f = g = q = j = 0$ and $b = m = -1$ and $h = j$.

We get $(1+k)r$, $(1+k)n$, $(1+k)q$ and $(1+k)^2(a-k)$

split into two cases :

Case2baaadaa4aa when $k = -1$ and **Case2baaadaa4ab** when $k \neq -1$, so $n = r = v = 0$ and $a = k$

Case2baaadaa4aa $s = i = u = e = t = f = g = q = j = 0$ and $b = m = k = -1$ and $h = j$.

We get r^2v and n^2r

split into two cases :

Case2baaadaa4aaa when $r = 0$ and **Case2baaadaa4aab** when $r \neq 0$, so $n = v = 0$

Case2baaadaa4aaa $s = i = u = e = t = f = g = q = j = r = 0$ and $b = m = k = -1$ and $h = j$.

We get nv^2 and $(1 + a)n^2$

split into two cases :

Case2baaadaa4aaaa when $n = 0$ and **Case2baaadaa4aaab** when $n \neq 0$, so $v = 0$ and $a = -1$

Case2baaadaa4aaaa $s = i = u = e = t = f = g = q = j = r = n = 0$ and $b = m = k = -1$ and $h = j$.

We get cv^2 and $(1 + a - v)v^2$

Split into two cases:

When $v = 0$ and when $v \neq 0$, when $v = 0$, we get $(1 + a)(1 + a - c)c$, so split into three cases:

When $a = -1$ SOLUTION.

When $a = c - 1$ SOLUTION.

When $c = 0$ SOLUTION.

And when $v \neq 0$ and $c = 0$ and $a = v - 1$ SOLUTION.

Case2baaadaa4aaab $s = i = u = e = t = f = g = q = j = r = v = 0$ and

$b = m = k = a = -1$, $n \neq 0$ and $h = j$. SOLUTION.

Case2baaadaa4aab $s = i = u = e = t = f = g = q = j = n = v = 0$ and $b = m = k = -1$, $r \neq 0$ and $h = j$.

We get $(1 + a)r^2$, so $a = -1$ SOLUTION.

Case2baaadaa4ab $s = i = u = e = t = f = g = q = j = n = r = v = 0$ and $b = m = -1$ and $h = j$ and $a = k$.

We get $c(1 + k)^2$, so $c = 0$ SOLUTION.

Case2baaadaa4b $s = i = u = e = t = f = g = q = n = r = 0$ and $b = m = -1$, $j \neq 0$ and $h = j$.

We get $2j(1 + k)v$, split into two cases: **Case2baaadaa4ba** when $v = 0$ and **Case2baaadaa4bb** when $v \neq 0$, so $k = -1$

Case2baaadaa4ba $s = i = u = e = t = f = g = q = n = r = v = 0$ and $b = m = -1$, $j \neq 0$ and $h = j$.

We get $h^2(1 + k)$, so $k = -1$ because $j \neq 0$ and $h = j$

We get $(1 + a)(1 + a - c)c$ $c(c - j)j$ and $(1 + a)(1 + a - j)j$

When $c = 0$, we get $(1 + a)(1 + a - j)j$, split into cases: when $a = -1$ SOLUTION and when $j = a + 1$ SOLUTION.

When $c \neq 0$, so $c = j$, we get $(1 + a)(1 + a - j)j$

Split into cases: when $a = -1$ SOLUTION and when $j = a + 1$ SOLUTION.

Case2baaadaa4bb $s = i = u = e = t = f = g = q = n = r = 0$ and $b = m = k = -1$,$j \neq 0$, $v \neq 0$ and $h = j$.

We get jv CONTRADICTION.

Case2baaadab $s = i = u = e = t = f = c = n = q = r = 0$ and $m = -1$ and $b \neq -1$.

We have entries g^2h , g^2j , v^2g and $g(1 + k)^2$

Split into two cases

Case2baaadaba when $g = 0$ and **Case2baaadabb** when $g \neq 0$ so $h = j = v = 0$ and $k = -1$.

Case2baaadaba $s = i = u = e = t = f = c = n = q = r = g = 0$ and $m = -1$ and $b \neq -1$.

We get $(1 + b)^2 h$ so $h = 0$

We now get $j(1 + k)^2$,$(1 + k)v^2$ and $(a - k)(1 + k)^2$ and $j^2 v$

Split into two cases:

Case2baaadabaa when $k = -1$ and **Case2baaadabab** when $k \neq -1$ so $j = v = 0$ and $a = k \neq -1$.

Case2baaadabaa $s = i = u = e = t = f = c = n = q = r = g = h = 0$ and $k = m = -1$ and $b \neq -1$.

We get $(1 + b)^2 j$ so $j = 0$

We now get $(1 + a - v)v^2$

Split into two cases:

$v = 0$ SOLUTION.

$v = 1 + a$ SOLUTION.

Case2baaadabab $s = i = u = e = t = f = c = n = q = r = g = h = j = v = 0$ and $m = -1$ and $b \neq -1$ and $a = k \neq -1$.

SOLUTION.

Case2baaadabb $s = i = u = e = t = f = c = n = q = r = h = j = v = 0$ and $m = k = -1$ and $b \neq -1$ and $g \neq 0$.

We get $(1 + a)g^2$ so $a = -1$

SOLUTION.

Case2baaadb $s = i = u = e = t = v = 0$ and $f \neq 0$ and $m = k = -1$.

We have entry $(1 + b)f(1 + a - h)$

plit into two cases

Case2baaadba when $b = -1$ and **Case2baaadbb** when $b \neq -1$ so $h = 1 + a$.

Case2baaadba $s = i = u = e = t = v = 0$ and $k = b = m = -1$ and $f \neq 0$.

We have entry $fg(h + j)$

split into two cases:

Case2baaadbaa $s = i = u = e = t = v = g = 0$ and $k = b = m = -1$ and $f \neq 0$.

We have entry $f^2 j$, $f^2 h$ and $c^2 f$, so $c = h = j = 0$

We get $(1 + a)f^2$, so $a = -1$

We get fnr and fqr and fqn

split into two cases

Case2baaadbaaa when $r = 0$ and **Case2baaadbaab** when $r \neq 0$, so $q = n = 0$.

Case2baaadbaaa $s = i = u = e = t = v = g = c = h = j = r = 0$ and $a = k = b = m = -1$ and $f \neq 0$.

We get fn^2 and fq^2, so $n = q = 0$

SOLUTION.

Case2baaadbaab $s = i = u = e = t = v = g = c = h = j = 0$ and $a = k = b = m = -1$ and $f \neq 0$ and $r \neq 0$.

We get fr^2 CONTRADICTION.

Case2baaadbab $s = i = u = e = t = v = 0$, $k = b = m = -1$, $f \neq 0$, $g \neq 0$ and $h = -j$.

We have entries $c^2 f$, $(1 + a + c)fg$, so $c = 0$ and $a = -1$

We get fj^2, so $j = 0$

We get fnr and fqr and fqn

split into two cases

Case2baaadbaba when $r = 0$ and **Case2baaadbabb** when $r \neq 0$ so $q = n = 0$.

Case2baaadbaba $s = i = u = e = t = v = c = j = r = 0$ and $a = k = b = m = -1$ and $f \neq 0$, $g \neq 0$ and $h = -j$.

We get fn^2 and fq^2, so $n = q = 0$ SOLUTION.

Case2baaadbabb $s = i = u = e = t = v = c = j = 0$ and $a = k = b = m = -1$ and $f \neq 0$ and $r \neq 0$, $g \neq 0$ and $h = -j$.

We get fr^2 CONTRADICTION.

Case2baaadbb $s = i = u = e = t = v = 0$, $f \neq 0$, $m = k = -1$, $b \neq -1$ and $h = 1 + a$.

We have entries $f(n(1+a-j) - 2qr) - g(1+b)n$, $f(q(1+a-j) - 2nr) - g(1+b)q$ and $f(r(1+a-j) - 2qn) - g(1+b)r$

If exacitly one of n , q , r is zero (suppose n=0), get $-2qrf = 0$, which is a contradiction, as $f \neq 0$

Terms $q(n^2 + r^2 - g^2) + nr(j - g - c)$, $n(r^2 + r^2 - g^2) + qr(j - g - c)$ and $r(n^2 + q^2 - g^2) + nq(j - g - c)$

If exacitly two of n , q , r are zero (suppose n=r=0), we get $-qg^2 = 0$, so $g = 0$

We have entry $g(1+b)(1+a-j) - f(n^2 + q^2 + r^2)$.

If $g = 0$ we deduce $fq^2 = 0$ CONTRADICTION.

Split into two cases

Case2baaadbb1 when $n = r = q = 0$ and **Case2baaadbb2** when n, q, r all non-zero.

Case Case2baaadbb1: when$n = r = q = 0$

We have entries $(1+b)g(1+a-j)$, $fg(1+a-j)$, $c^2(1+b) + 3fg(1+a+j)$, $(1+a)(1+a-j)j$ and $2f(1+a+b+ab+f+af) + g(c^2 + 2gj)$

Split into two cases

Case2baaadbb1a when$g = 0$ and **Case2baaadbb1b** when $g \neq 0$ and $j = 1 + a$.

Case2baaadbb1a when$g = 0$ We get $c^2 f$, so $c = 0$

We get now $f^2 j$, so $j = 0$

We get $(1+a)f(1+b+f)$, $(1+a)((1+b)^2 + 3f^2)$

split into two cases

When $a = -1$ SOLUTION.

And when $a \neq 0$ and $1 + b = -f$, so $4f = 0$ CONTRADICTION.

Case2baaadbb1b $s = i = u = e = t = v = n = r = q = 00$ and $f \neq 0$ and $m = k = -1$ and $b \neq -1$ and $h = 1 + a$, $g \neq 0$ and $j = 1 + a$.

We have entries $c^2(1+b) + 6fg(1+a)$, $2f(1+a)(1+b+f) + g(c^2 + 2g(1+a))$ and $c^2 f + 2g(1 + a + b + a + 2f + 2af)$

split into two cases:

When $c = 0$ and $a = -1$ SOLUTION.

When $c \neq 0$ and $a \neq -1$, we get $1 + a + 2b + 2ab + b^2 + ab^2 - c - 2ac - a^2c + c^2 + ac^2 + 2f^2 + 2af^2 + cf^2 + 3g^2 + 3ag^2$and $1 + a + 2b + 2ab + b^2 + ab^2 - c - 2ac - a^2c + c^2 + ac^2 + 3f^2 + 3af^2 + 2g^2 + 2ag^2 + cg^2$ subtract to get $(1 + a - c)(f - g)(f + g)$

Split into two cases:

Case2baaadbb1ba when$a = c - 1$ and **Case2baaadbb1bb** when $c \neq= 1 + a$, so $g = fx$, then $x^2 = 1$.

Case2baaadbb1ba $s = i = u = e = t = v = n = r = q = 00$ and $f \neq 0$ and $m = k = -1$ and $b \neq -1$ and $h = 1 + a$, $g \neq 0$ and $j = 1 + a$ and $a = c - 1$, $c \neq 0$ and $a \neq -1$.

We get $c(c + bc + 6fg)$, so $b = -1 - 6fg/c$

we get $cf(c - 2g)(c + 6g)$, split into two cases:

When $c = 2g$, we get $16g^2(f^2 - g^2)$ and $12g^2(4f^2 + g^2)$, subtracting these gives $-32g^2f^2$ CONTRADICTION.

When $c = -6g$, we get $-144g^2(f^2 - g^2)$ and $36g^2(4f^2 + 3g^2)$ CONTRADICTION.

Case2baaadbb1bb $s = i = u = e = t = v = n = r = q = 00$, $f \neq 0$, $m = k = -1$, $b \neq -1$, $h = 1 + a$, $g \neq 0$ $j = 1 + a$, $c \neq= 1 + a$ and $g = fx$, then $x^2 = 1$, $c \neq 0$ and $a \neq -1$.

We get $c^2 + bc^2 + 6f^2x + 6af^2x$, so $b = -1 - 6f^2x(1+a)/c^2$

We get $(2fx(1+a) - c^2)(6fx(1+a) + c^2)$, so $c^2 = 2\beta fx(1+a)$, then $\beta = 1 or -3$

And we also get $(1+a-c)f^2x(6(1+a)^2 - 6c(1+a) - c^2)$, so we get $2(1+a)(3(1+a) - 3c - \beta fx) = 0$, so $c = 1 + a - \beta fx/3$

We put $a = z - 1$ when $z = yf$.

So we have four cases:

case 2baaadbb1bb 1 when $x = 1$ and $\beta = 1$we can not find solution to $(1 - 24y + 9y^2)(1 + 48y + 9y^2) = 0$, so CONTRADICTION.

case 2baaadbb1bb 2 when $x = -1$ and $\beta = 1$ also no solution CONTRADICTION.

case 2baaadbb1bb 3 when $x = 1$ and $\beta = -3$ we get $y = 4 \pm \sqrt{15}$ SOLUTION.

case 2baaadbb1bb 4 when $x = -1$ and $\beta = -3$ we get $y = -4 \pm \sqrt{15}$ SOLUTION.

Case2baaadbb2: when all of n, q , r non zero

We have entries $g(1+b)(1+a-j) - f(n^2+q^2+r^2)$, $-fg(1+a-j) + f(n^2+q^2) + r^2(1+b)$, $-fg(1+a-j) + f(r^2+q^2) + n^2(1+b)$ and $-fg(1+a-j) + f(n^2+r^2) + q^2(1+b)$

If $n^2 + q^2 + r^2 = 0$, then $g(1+a-j) = 0$, so $r^2 = \frac{-f(n^2+q^2)}{1+b}$

put $x = \frac{-f}{1+b} = 0$, so $r^2 = -x(n^2 + q^2)$

Then $n^2 + q^2 + \frac{r^2}{x} = 0$ and $n^2 + q^2 + r^2 = 0$, so $x = 1$ and $f = 1 + b$

If $n^2 + q^2 + r^2 \neq 0$

We have $f(2(n^2 + q^2 + r^2)) + (n^2 + q^2 + r^2)(1 + b) = 3fg(1 + a - j)$

$(n^2 + q^2 + r^2)(2f + 1 + b) = \frac{3f^2(n^2+q^2+r^2)}{1+b}$

$2f + (1 + b) = \frac{3f^2}{1+b}$, so $3f^2 - 2f(1+b) - (1+b)^2 = 0$

$(3f + (1+b))(f - (1+b)) = 0$, so split into cases **Case Case2baaadbb2a** When $b = f - 1$

Case Case2baaadbb2b when $b = -3f - 1$ and $n^2 + q^2 + r^2 = 0$ and $g \neq 0$ and $1 + a - j \neq 0$.

Case2baaadbb2a When $b = f - 1$

We have entries $f(n(1+a-g-j)-2qr)$, $f(r(1+a-g-j)-2qn)$ and $f(q(1+a-g-j)-2nr)$

We deduce that there is $x \neq 0$, so $1+a-g-j = 2x$ and $q^2 = n^2 = r^2 = x^2$

Also we have $f(g(1+a-j)-(n^2+r^2+q^2))$

So we deduce $g(1+a-j) = 3x^2$ and from this and the defintion of x a have, $(g+3x)(g-x) = 0$

So $g \neq 0$ and $1+a-j \neq 0$ and $a \to 2x+g+j-1$

From this entry $nq(c+g-j)+r(g^2-n^2-q^2)$, get $x(c-j) = 2x^2-gx-g^2$

When $g = 0$ gives $c = j$

And when $g = -3x$ gives $c \to -4x+j$

Split into two cases:

Case2baaadbb2a1 when$g = x$ and **Case2baaadbb2a2** when$g = -3x$.

Case2baaadbb2a1 when$g = 0$ and $c = j$

We get $f(j+3x)^2$, so $j \to -3x$

We get $-12f^2x$ CONTRADICTION.

Case2baaadbb2a2 when$g = -3x$ and $c = -4x+j$

We get $f(n^2+q^2+r^2-15x^2)$ gives $-12fx^2$=0 CONTRADICTION

Case Case2baaadbb2b when $b = -3f-1$ and $n^2+q^2+r^2 = 0$, $g \neq 0$, $1+a-j \neq 0$

we have entries $f(3g(1+a-j)+n^2+q^2+r^2)$, $f(q(1+a-j+3g)-2nr)$, $f(n(1+a-j+3g)-2qr)$, $f(r(1+a-j+3g)-2nq)$ and $3f(c^2-g(1+a+j))$

So$c^2 = g(1+a+j)$ and $1+a-j+3g = \frac{2nr}{q} = \frac{2qr}{n} = \frac{2nq}{r} = 2x$

So $a \to 2x-1+j-3g$ and $x^2 = n^2 = r^2 = q^2$

We also have $f(g(1+a-j)+3n^2-q^2-r^2)$ and $g(1+a-j) = -x^2$ and $g(1+a-j+3g) = 2xg$

So $3g^2 = 2xg+x^2$ and get $(3g+x)(g-x) = 0$ so $g = x$ or $g = \frac{-x}{3}$

Also we have entry $q(g^2 - n^2 - r^2) + nr(c + g - j)$ and using $x = \frac{nr}{q}$, get $g^2 - 2x^2 + x(c + g - j) = 0$, so $x(c - j) = 2x^2 - g^2 - xg$

Split into two cases:

Case2baaadbb2b1 when$g = x$ and **Case2baaadbb2b2** when$g = \frac{-x}{3}$.

Case2baaadbb2b1 when$g = x$ and gives $c = j$

We get $f(j - x)^2$, so $j = x$

We get $-4f^2x$. CONTRADICTION.

Case2baaadbb2b2 when$g = \frac{-x}{3}$and gives$c = \frac{20x}{9} + j$

We get $\frac{2}{3}f(3nq + 13rx)$ and using $x = \frac{nq}{r}$, get $\frac{2}{3}f(3x + 13rx) = 0$. CONTRADICTION.

Case2bab $u = e = 0$ and $s^2 \neq i^2$.

We get a matrix entry $(i + a - h)(s^2 - i^2) = (1 + k)(s^2 - i^2) = v(s^2 - i^2) = 0$ and deduce $h = 1 + a$, $v = 0$ and $k = -1$

The matrix contains $g(n + mn + iq - s - as + cs + rt)$, $g(i + ai - ci - r - mr - qs - nt)$, $g(1 + a - c + m + am - cm - ir - ns - qt)$, $g(in + q + mq + rs - t - at + ct)$,

Split into two cases:

Case2baba when$g = 0$ and **Case2babb** when$g \neq 0$.

Case2baba $u = e = v = g = 0$ and $h = 1 + a$, and $k = -1$ and $s^2 \neq i^2$.

We have terms $(1 + b)c^2$ and $c^2 f$.

Split into two cases:

Case2babaa when $c = 0$ and **Case2babab** when$c \neq 0$ (implying that $f = 0$ and $b = -1$).

Case2babaa $u = e = v = g = c = 0$ and $h = 1 + a$, and $k = -1$ and $s^2 \neq i^2$.

We have terms $f^2 j$.

Split into two cases:

Case2babaaa when $f = 0$ and **Case2babaab** when$f \neq 0$ (implying that $j = 0$).

Case2babaaa $u = e = v = g = c = f = 0$ and $h = 1 + a$, and $k = -1$ and $s^2 \neq i^2$.

We have terms $(1+b)^2 j$, $(1+b)q^2$, $(1+b)n^2$ and $(1+b)r^2$.

Split into two cases:

Case2babaaaa when $b = -1$ and **Case2babaaab** when $b \neq -1$ (implying that $j = q = n = r = 0$).

Case2babaaaa $u = e = v = g = c = f = 0$ and $h = 1 + a$, $k = -1$, $b = -1$ and $s^2 \neq i^2$.

We get entrys nrq, $(1+a)rs$, $(1+a)ri$, $(1+a)ns$, $(1+a)ni$, $(1+a)qs$ and $(1+a)qi$. Subtracting these gives $n(a+1)(i-s) = 0$, $r(a+1)(i-s) = 0$ and $q(a+1)(i-s) = 0$, so $(1+a)n = 0$, $(1+a)r = 0$ and $(1+a)q = 0$

Split into two cases:

Case2baba5 when $a = -1$ and $nqr = 0$ and **Case2baba4b** when $a \neq -1$ (implying that $q = n = r = 0$).

Case2baba5 $u = e = v = g = c = f = 0$ and $h = 1 + a$, and $k = -1$ and $b = -1$ and $a = -1$ and $s^2 \neq i^2$.

we have entries nqr, $j(s^2 + 2(1+m)t)$ and $j(i^2 + 2(1+m)t)$. Subtracting these gives $j(s^2 - i^2) = 0$, so $j = 0$.

Now we have entries $ir^2 + nqt$ and $-r^2 s - nqt$, and on adding we have $r^2(i-s) = 0$, so $r = 0$.

Now we have entries iq^2 and sq^2, and on subtracting we have $q^2(i-s) = 0$, so $q = 0$.

Now we have entries ns^2 and $in^2 = 0$, so likewise $n = 0$ SOLUTION.

Case2baba4b $u = e = v = g = c = f = q = n = r = 0$ and $h = 1 + a$, and $k = -1$ and $b = -1$ and $s^2 \neq i^2$ and $a \neq -1$.

We have terms $(1+a)ij$ and $(1+a)sj$, so we deduce $(1+a)j = 0$, so $j = 0$.

We have terms $(1+a)(i+s)(1+m+t)$, so $t = -1 - m$.

Now we have terms $(1+m)^2 + is = 0$ and $i^2 + s^2 - 2(1+m)^2 = 0$, so we get $(i+s)^2 = 0$.

CONTRADICTION.

Case2babaaab $u = e = v = g = c = f = j = q = n = r = 0$ and $h = 1 + a$, and $k = -1$ and $s^2 \neq i^2$ and $b \neq -1$.

We have terms $(1 + a)(1 + m + t)$, $(1 + a)((1 + m)^2 + 2is + t^2)$, $(1 + a)((1 + b)^2 - i^2 - s^2 - 2(1 + m)t)$,

Split into two cases

Case2babaaaba when $a = -1$ and **Case2babaaabb** when $a \neq -1$.

Case2babaaaba $u = e = v = g = c = f = j = q = n = r = 0$ and $h = 1 + a$, and $a = -1$ and $k = -1$ and $s^2 \neq i^2$ and $b \neq -1$. SOLUTION.

Case2babaaabb $u = e = v = g = c = f = j = q = n = r = 0$ and $h = 1 + a$, and $a \neq -1$ and $t = -1 - m$ and $k = -1$ and $s^2 \neq i^2$ and $b \neq -1$.

We get $(1 + b)^2 = (i + s)^2$ and $(1 + m)^2 + is = 0$, split into two cases:

When $s = 0$, we get $i^2 = (1 + b)^2$ and $(1 + m)^2$, so $i = x(1 + b)$ and $m = -1$ when $x^2 = 1$ SOLUTION.

When $s \neq 0$, we get $i = -(1 + m)^2/s$

We get $b = x(i + s) - 1$ and $x^2 = 1$ SOLUTION.

Case2babaab $u = e = v = g = c = j = 0$, $h = 1 + a$, $k = -1$ and $s^2 \neq i^2$ and $f \neq 0$.

We have terms $(1 + a)qi$, $(1 + a)qs$, $(1 + a)ri$, $(1 + a)rs$, $(1 + a)ni$, $(1 + a)ns$.

We split into two cases: **Case2babaaba** $a = -1$ and **Case2babaabb** $a \neq -1$ and so $q = r = n = 0$.

Case2babaaba $u = e = v = g = c = j = 0$ and $h = 1 + a$, and $a = k = -1$ and $s^2 \neq i^2$ and $f \neq 0$.

We have terms fnr, frq, fnq, $f(n^2 + q^2 + r^2)$. As $f \neq 0$ we have $nr = nq = rq = 0 = n^2 + r^2 + q^2$. The only solution to this is if $q = r = n = 0$. SOLUTION

Case2babaabb $u = e = v = g = c = j = q = n = r = 0$, $h = 1 + a$, $k = -1$ and

$s^2 \neq i^2$ and $f \neq 0$ and $a \neq -1$.

We have entries $(1+a)f(1+b+f)$ and $(1+a)(i+s)(1+m+t)$. We deduce $b = -1-f$ and $m = -1-t$.

Now we get entries $(1+a)(is+t^2)$ and $(1+a)(4f^2 - i^2 - s^2 + 2t^2)$, and deduce $t^2 = -is$ and $4f^2 = (i+s)^2$. SOLUTION

Case2babab $u = e = v = g = f = 0$ and $h = 1+a$, and $b = k = -1$ and $s^2 \neq i^2$ and $c \neq 0$.

We have terms $(1+a-c)iq$, $(1+a-c)sq$, $(1+a-c)ir$, $(1+a-c)sr$, $(1+a-c)in$, $(1+a-c)sn$, $(1+a-c)(1+a)c$, nqr. We deduce that $(1+a-c)q = (1+a-c)n = (1+a-c)r = (1+a-c)(1+a) = 0$.

Split into two cases

Case2bababa when $a = c-1$ and **Case2bababb** when$a \neq c-1$ (implying that $n = r = q = 0$ and $a = -1$).

Case2bababb $u = e = v = g = f = n = r = q = h = 0$ a, and $a = b = k = -1$ and $s^2 \neq i^2$ and $c \neq 0$ and $a \neq c-1$.

We get j^2s and j^2i, so $j = 0$

We get cs^2 and ci^2, so $c = 0$. CONTRADICTION

Case2bababa $u = e = v = g = f = 0$ and $h = 1+a$, and $b = k = -1$ and $s^2 \neq i^2$ and $a = c-1$ and $c \neq 0$..

We have terms $inq - qrs - nrt$, $nqs + irq + nrt$, $qi^2 + (n+r)st$, $qs^2 + (n+r)it$, $i^2n + qs(m+1) + rt^2$, $i^2r + qs(m+1) + nt^2$.

Split into two cases:

Case2babababaa when $q = 0$ and **Case2babababab** when $q \neq 0$.

Case2babababaa $u = e = v = g = f = q = 0$ and $h = 1+a$, and $b = k = -1$ and $s^2 \neq i^2$ and $a = c-1$ and $c \neq 0$.

Get r^2s and r^2i, so $r = 0$.

Then get ns^2 and in^2, so $n = 0$.

Then get $j^2 s$ and ij^2, so $j = 0$.

Then get $c^2 s$ and $c^2 i$, so $c = 0$. CONTRADICTION.

Case2babababab $u = e = v = g = f = 0$ and $h = 1 + a$, and $b = k = -1$ and $s^2 \neq i^2$ and $a = c - 1$ and $c \neq 0$. and $q \neq 0$.

We have terms $inq - qrs - nrt$, $nqs + irq + nrt$, $qi^2 + (n + r)st$, $qs^2 + (n + r)it$, $i^2 n + qs(m + 1) + rt^2$, $i^2 r + qs(m + 1) + nt^2$, nqr.

On subtracting terms, we get $q(s^2 - i^2) + t(n + r)(i - s) = 0$, and on dividing by $s - i$ we get $q(s + i) = t(n + r)$. As $i + s \neq 0$ this means that $t \neq 0$ and $n + r \neq 0$. Now the equation $qi^2 + (n + r)st = 0$ gives, on substitution, $q(i^2 + is + s^2) = 0$, so $i^2 + is + s^2 = 0$. From this we must have both $s \neq 0$ and $i \neq 0$. We deduce that $i = sx$, where $x \neq 1$ is a solution of $x^3 = 1$. Now we get $nr = 0$, and then $q(in - rs) = 0$ and $q(ir + ns) = 0$. As q, s are nonzero and $i = sx$ we get $r = xn$ and $n = -x^2 n$, so $n = r = 0$.

On substituting this, we get entries $q^2 t$ and $(1 + m)^2 q$, so $t = 0$ and $m = -1$.

Then we get $i^2 q = 0$. CONTRADICTION

Case2babb $u = e = v = 0$ and $a = h - 1$, and $k = -1$ and $s^2 \neq i^2$ and $g \neq 0$.

We have entries $g((c - h)i + n(1 + m) + qs + rt)$, $g((c - h)i + r(1 + m) + qs + nt)$.

Subtract to get $(n - r)(1 + m - t)$.

Split into two cases:

2babba when $r = n$ and **2babbb** when $m = t - 1$ and $r \neq n$.

2babba $u = e = v = 0$ and $a = h - 1$ and $k = -1$ and $s^2 \neq i^2$ and $g \neq 0$and $r = n$

We get $i^2 q + 2ns(1 + m) + gt^2$ and $s^2 q + 2ni(1 + m) + gt^2$.

Subtract to get $(i^2 - s^2)q + 2n(s - i)(1 + m) = 0$ dividing by $(i - s)$, get $(i + s)q = 2n(1 + m)$

And we have also entries $gi^2 + 2ns(1 + m) + qt^2$, $gs^2 + 2ni(1 + m) + qt^2$ gives

$g(s+i) = 2n(1+m)$, as $s+i \neq 0$, get $g = q$

And from $g(s+i) = 2n(1+m)$, we deduce $n \neq 0$ and $m \neq -1$

We have $n(g(c+g-j) - n^2)$, so $g(c+g-j) = n^2$

And we have $n(3fg - (h-j)(1+b))$, so $3fg = (h-j)(1+b)$

And we have $n(g(1+b) + 2fg - f(h-j))$ so $g(1+b) + 2fg - f(h-j) = 0$

And we have $2gi(1+m) + n(s^2+t^2)$, $2gs(1+m) + n(i^2+t^2)$, so $2g(1+m) = n(s+i)$

times g gives $2g^2(1+m) = ng(s+i) = 2n^2(1+m)$ so $g^2 = n^2$.

From$g(c+g-j) = n^2$ get $g(c-j) = 0$, so $c = j$.

We get $3fg(h+j) + j^2(1+b)$, and we have $3fg = (h-j)(1+b)$, from them get $(1+b)h^2$

And we have $h(g(1+b) + f(2g+j))$, so if $1+b = 0$ then $f = 0$

split into two cases:

2babbaawhen $b = -1$ and $f = 0$ and **2babbab** when $b \neq -1$, so $h = 0$

2babbaa $u = e = v = f = 0$, $a = h-1$, $b = k = -1$, $s^2 \neq i^2$, $q = g \neq 0$, $r = n$, $n \neq 0$, $m \neq -1$, $c = j$ and $g^2 = n^2$, $2g(1+m) = n(s+i)$ and $g(s+i) = 2n(1+m)$

We have entries $g(2gj + j^2 - 3n^2)$, but $n^2 = g^2$ and $g \neq 0$, so $(j+3g)(j-g) = 0$, so $j = g$ or $(j = -3g)$.

We have $g(g(1+m) + n(i+s) + t(j-h))$, use $2g(1+m) = n(s+i)$ to get $3g(1+m) = t(h-j)$,so $t \neq 0$ and$h \neq 0$.

We have $(1+m)^2g + i^2 + 2nst$, $(1+m)^2g + s^2 + 2nit$

Deduce $g(i+s) = 2nt$, so $1+m = t$, and $m = t-1$

Now $3gt = t(h-j)$, so $h = 3g+j$

We get $g(gi - 3gs + 2nt)$ and $g(3gi - gs - 2nt)$ add to get $g(4gi - 4gs) = 0$, so $4g^2(i-s) = 0$ CONTRADICTION.

2babbab $u = e = v = h = 0$, $a = h-1$, $k = -1$, $s^2 \neq i^2$, $q = g \neq 0$, $r = n$, $n \neq 0$, $m \neq -1$, $c = j$, $g^2 = n^2$ and $b \neq -1$.

We have $g(3g^2-2gj-j^2)$, so $j=g$ or $j=-3g$,and so $j\neq 0$.

And we have $j(i+s)(1+m+t)$, deduce $m=-t-1$.

And we have $j(3fg+j(1+b))$ and $j((1+b)^2+3f^2)$ gives a cntradiction with either $j=g$ or $j=-3g$ CONTRADICTION.

Case2babbb $u=e=v=0$ and $a=h-1$, and $k=-1$ and $s^2\neq i^2$ and $g\neq 0$, $m=t-1$ and $r\neq n$.

We have $ns^2+git+iqt+rt^2$, $ins+git+qst+rt^2$, subtract to get $ns(s-i)+qt(i-s)=0$, dividing by $(s-i)$ to get $ns-qt=0$.

We also have $ins+git+qst+rt^2$, $irs+git+qst+nt^2$, subtract to get $is(n-r)+(r-n)t^2=0$, dividing by $(n-r)$ to get $is-t=0$.

We also have $ns^2+git+iqt+rt^2$, $rs^2+git+iqt+nt^2$, subtract to get $(i-s)rs+qt(s-i)=0$, dividing by $(i-s)$ to get $rs-qt=0$.

So $ns=qt=rs$ but $n\neq r$, so $s=0$ and $i\neq 0$

We get$t^2=0$, so $t=0$

We get $gi^2=0$ CONTRADICTION.

Case2bb When $u\neq 0\neq e$, $es-iu=0$ and $e^2=u^2/neq0$, put $e=xu$ and so $i=xs$, where $x=\pm 1$.

We have entry $(1+a-h)u(1+k-v)$

Split into two cases:

case 2bba when $k=v-1$ and **2bbb** when $k\neq v-1$, so $a=h-1$.

case2bba$e=xu$, $i=xs$ $k=v-1$, $u\neq 0\neq e$ and $x=\pm 1$.

We have entry $(1+m-t)(1+b-fx)u$

Split into two cases:

case 2bbaa when $b=fx-1$ and **2bbab** when $b\neq fx-1$, so $m=t-1$.

case2bbaa $e=xu$, $i=xs$ $k=v-1$, $u\neq 0\neq e$ and $x=\pm 1$ and $b=fx-1$.

We have entries $fgh-fgj-fn^2-fxq^2-fr^2-gsux$, $fgh-fgj-fxn^2-fq^2-fr^2-$

$gsux$, $fgh-fgj-fn^2-fq^2-fxr^2-gsux$. Subtracting these gives $f(1-x)(q^2+n^2)$, $f(1-x)(r^2+n^2)$, $f(1-x)(q^2+r^2)$.

Split into three cases : **case2bbaaa** when $f=0$ and **case2bbaab**when $x=1$ and $f\neq 0$ and**case2bbaac** when $f\neq 0$ and $x\neq 1$ so $x=-1$ and $q=n=r=0$.

case2bbaaa $e=xu$, $i=xs$ $k=v-1$, $u\neq 0\neq e$ and $x=\pm 1$ and $b=fx-1$, $f=0$.

We have entry gsu

Split into two cases:

case 2bbaaaa when $s=0$ and **2bbaaab** when $s\neq 0$, so $g=0$.

case2bbaaaa $e=xu$, $i=xs$ $k=v-1$, $u\neq 0\neq e$ and $x=\pm 1$and $b=fx-1$, $f=s=0$.

We get $(1+m)^2u$ and t^2u , so $t=0$ and $m=-1$

We get $-(1+a)^2+h(1+a)-h^2+u^2-2hv+2v^2$ and $-h(1+a)+u^2-2hv+2v^2$

Subtracting these gives $(1+a)^2-2h(1+a)+h^2$=$(1+a-h)^2=0$, so $a=h-1$

We get $2u^3(u-v)(u+v)$, so $v=yu$ and $y^2=1$

We also get $u(h-uy)(h+3uy)$, so $h=\alpha yu$ and $\alpha=1or-3$

We have entries $(g-j)((g+j)r+2nq)$, $(g-j)((g+j)n+2rq)$ and $(g-j)((g+j)q+2nr)$

Split into four cases:

case2bbaaaa1 when $g=j$.

case2bbaaaa2 when $g\neq j$, so $g=-j$ and at least two of n,q,r are zero .

case2bbaaaa3 when $g^2\neq j^2$ and all of n,q,r are zero .

case2bbaaaa4 when $g^2\neq j^2$ and all of n,q,r are non- zero and $\frac{(g+j)^2}{4}=r^2=q^2=n^2$.

case2bbaaaa1$e=xu$, $i=xs$ $k=v-1$, $u\neq 0\neq e$ and $x=\pm 1$and $b=fx-1$, $f=s=0$, $t=0$ and $m=-1$, $a=h-1$, $v=yu$ and $y^2=1$, $g=j$.

We $j(c^2+2j^2-n^2-q^2-r^2)$, $3(cj^2-nqr)$, $q(n^2+r^2-j^2)-cnr$, $n(q^2+r^2-j^2)-cqr$, $r(n^2+q^2-j^2)-cnq$ and $nq+qr-cu+j^2y+nry-ju-2ju\alpha$

Split into two cases :

case 2bbaaaa1a when $j = 0$ and **2bbaaaa1 b** when $j \neq 0$.

case2bbaaaa1a $e = xu$, $i = xs$ $k = v - 1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx - 1$, $f = s = 0$, $t = 0$ and $m = -1$, $a = h - 1$, $v = yu$ and $y^2 = 1$, $g = j$ and $j = 0$, $h = \alpha yu$ and $\alpha = 1 or - 3$, $x = 1$.

We get $cu\alpha$, so $c = 0$

And we get $3nqr$, $q(n^2 + r^2)$, $n(q^2 + r^2)$ and $r(n^2 + q^2)$, so must have at least two of n, q, r are zero.

But we get also $q + ry + ny\alpha$, so all n , q , r are zero SOLUTION.

case2bbaaaa1b $e = xu$, $i = xs$ $k = v - 1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx - 1$, $f = s = 0$, $t = 0$ and $m = -1$, $a = h - 1$, $v = yu$ and $y^2 = 1$, $g = j$ and $j \neq 0$.

We have entry $3(cj^2 - nqr)$, so $c = \frac{nqr}{j^2}$

Suppose $n = r = 0$, we get qj^2, so $q = 0$

Suppose $q = 0$, get $n(r^2 - j^2)$ and $r(n^2 - j^2)$, so $r^2 = n^2 = j^2 \neq 0$, so can not have all n , q , r are zero, and so only one or all them are non - zero.

Split into four cases:

case2bbaaaa1b1 when $n = 0$ and $q, r \neq 0$,$c = 0$.

case2bbaaaa1b2 when $r = 0$ and $q, n \neq 0$,$c = 0$.

case2bbaaaa1b3 when $q = 0$ and $n, r \neq 0$,$c = 0$.

case2bbaaaa1b4 when all n , q ,r are non - zero.

case2bbaaaa1b1 when $n = 0$ and $q, r \neq 0$,$c = 0$ and $r^2 = q^2 = j^2$.

We get $ju(-u + jy\alpha)$, so $u = jy\alpha$.

We get $j^2\alpha^2(\alpha^2 - 1)$, so $\alpha = 1$, we get $j^2 q$ CONTRADICTION.

case2bbaaaa1b2 when $r = 0$ and $n, q \neq 0$,$c = 0$ and $q^2 = n^2 = j^2$

We get $ju(-u + jy\alpha)$, so $u = jy\alpha$

We get $j^2\alpha^2(\alpha^2 - 1)$, so $\alpha = 1$, we get $j^4 qy$ CONTRADICTION.

case2bbaaaa1b3 when $q = 0$ and $n, r \neq 0$,$c = 0$.

We get $ju(-u + jy\alpha)$, so $u = jy\alpha$

We get $j^2\alpha^2(\alpha^2 - 1)$, so $\alpha = 1$, we get j^2ry CONTRADICTION.

case2bbaaaa1b4 when all n , q , r are non - zero and $e = xu$, $i = xs$ $k = v - 1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx - 1$, $f = s = 0$, $t = 0$ and $m = -1$, $a = h - 1$, $v = yu$ and $y^2 = 1$, $g = j$ and $j \neq 0$ and $h = \alpha yu$.

We have entries $q(j^2 - r^2)(j^2 - n^2)$, $r(j^2 - q^2)(j^2 - n^2)$ and $n(j^2 - r^2)(j^2 - q^2)$, so must have at least two of $j^2 = q^2$,$j^2 = r^2$ and $j^2 = n^2$

We get $u^3(x - 1)$ and $u(h - uy)(h + 3uy)$, so $x = 1$ and $h = \alpha uy$ then $\alpha = 1 or - 3$

We get $cn + jr - qu + j^2y + jqy - ruy - ju\alpha - nuy\alpha$ and $jn + cr - qu + j^2y + jqy - nuy - ju\alpha - ruy\alpha$ subtracting these gives $(n - r)(c - j + uy - uy\alpha)$

And we get $nu + juy + quy - 2nqy\alpha - 2jry\alpha + ru\alpha^2$ and $ru + juy + quy - 2jny\alpha - 2qry\alpha + nu\alpha^2$ subtracting these gives $(n - r)(-u - 2jy\alpha + 2qy\alpha + u\alpha^2)$

Split into two cases :

case 2bbaaaa1b4a when $n = r$ and **case 2bbaaaa1b4b** when $n \neq r$, so $c = j + uy(1 - \alpha)$ and $q = yu(1 - \alpha^2)/2\alpha + j$

case2bbaaaa1b4a when all n , q ,r are non - zero and $e = xu$, $i = xs$ $k = v - 1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx - 1$, $f = s = 0$, $t = 0$ and $m = -1$, $a = h - 1$, $v = yu$ and $y^2 = 1$, $g = j$ and $j \neq 0$ and $h = \alpha yu$ and $n = r$.

We must have at least two of $j^2 = r^2$, $j^2 = n^2$ and $j^2 = q^2$

When $j^2 = r^2$, we get $cj^2 = qr^2$, so $j^2(c - q)$ and $j \neq 0$, so $c = q$ and $r = zj$ then $z^2 = 1$

And we get $cu + 2ju - j^2y\alpha - n^2y\alpha - q^2y\alpha - r^2y\alpha + ju\alpha^2$, so $c = (-2ju + j^2y\alpha + q^2y\alpha + 2r^2y\alpha - ju\alpha^2)/u$

We get $1/2u^2(-1 + yz)(1 + \alpha)^2$

Split into cases :

When $z = -y$, we get $(j-q)uy$, $j = q$, and we get $-2juy(1+\alpha)$ CONTRADICTION.

When $z = y$, we get $2j^2+2jq-juy-quy-juy\alpha$ and $2j^2+2jq-2juy-juy\alpha-quy\alpha$, subtracting these gives $(j-q)uy(\alpha-1)$

Split into two cases:

When $\alpha = 1$ and when $\alpha = -3$ and $q = j$

When $\alpha = 1$, we get $2j^2 + 2jq - 3juy - quy$ and $3ju + qu - 3j^2y - q^2y$, subtracting these gives $(j-q)^2y$, so $q = j$, we can see that if $\alpha = 1or - 3$ we get $q = j$

We get $2u^3(-1+x)$, so $x = 1$

We get $zj - uy - uy\alpha$, so $j = uy(1+\alpha)/2$

We get $-(1/2)u^2y(-1+\alpha)(1+\alpha)(3+\alpha)$ we have two possiple $\alpha = 1$ and $\alpha = -3$ SOLUTION in the both cases.

case2bbaaaa1b4b when all n , q ,r are non - zero and $e = xu$, $i = xs$ $k = v - 1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx - 1$, $f = s = 0$, $t = 0$ and $m = -1$, $a = h - 1$, $v = yu$ and $y^2 = 1$, $g = j$ and $j \neq 0$ and $h = \alpha yu$, $n \neq r$, $c = j + uy(1-\alpha)$ and $q = yu(1-\alpha^2)/2\alpha + j$.

We get $2(n-r)u(\alpha-1)$, so $\alpha = 1$

We get $u(n+r-2jy)$, so $n = 2jy - r$

We get $4j(j-uy)$ and $(j-r)^2$, so $r = j$ and $j = uy$

We get $2u^2(y-1)$, so $y = 1$

We get $u^3(x-1)$, so $x = 1$ SOLUTION.

case2bbaaaa2 $e = xu$, $i = xs$ $k = v - 1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx - 1$, $f = s = 0$, $t = 0$ and $m = -1$, $a = h - 1$, $v = yu$ and $y^2 = 1$, $g \neq j$, $g = -j$ and at least two of n, q, r are zero .

When $q = n = 0$ and $r \neq 0$, we get $u^2(j - ry)$, so $j = ry$

We get r^2uy CONTRADICTION.

When $r = q = 0$ and $n \neq 0$, we get $u^2(j - ny)$, so $j = ny$.

We get n^2uy CONTRADICTION.

When $r = n = 0$ and $q \neq 0$, we get $(j-q)u^2y$, so $j = q$

We get q^2uy CONTRADICTION.

When $n = q = r = 0$, we get ju^2, so $j = 0$, but $g \neq j$ and $g = -j$ CONTRADICTION.

case2bbaaaa3 $e = xu$, $i = xs$ $k = v - 1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx - 1$, $f = s = 0$, $t = 0$ and $m = -1$, $a = h - 1$, $v = yu$ and $y^2 = 1$, $g^2 \neq j^2$ and all of n, q, r are zero.

We get gu^2, so $g = 0$

We get $u(h - uy)(h + 3uy)$, so $h = \alpha yu$ when $\alpha = 1 or - 3$

We get $u^3(x - 1)$, so $x = 1$

We get $c(c - j)j$

Split in two cases :

When $c = 0$, we get $3u^2j$ CONTRADICTION.

When $c = j$, we get $ju(j - 2uy - 2uy\alpha)$ and $ju(-3u + jy\alpha - u\alpha^2)$, so split into cases:

When $\alpha = 1$, we get $ju(j - 4uy)$, so $j = 4uy$, we get $16u^3(y - 1)$, so $y = 1$ SOLUTION.

When $\alpha = -3$, we get $ju(j + 4uy)$, so $j = -4uy$, SOLUTION when $y = 1 or - 1$

case2bbaaaa4 $e = xu$, $i = xs$ $k = v - 1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx - 1$, $f = s = 0$, $t = 0$ and $m = -1$, $a = h - 1$, $v = yu$ and $y^2 = 1$, $g^2 \neq j^2$ and all of n, q, r are non- zero and $\frac{(g+j)^2}{4} = r^2 = q^2 = n^2$.

We get $u(h - uy)(h + 3uy)$, so $h = \alpha yu$ and $\alpha = 1 or - 3$

We get $u^3(x - 1)$, so $x = 1$

We get $gn + jn + 2qr = 0$, so $j = -g - \frac{2qr}{n} \neq g$

We have $r^2 = q^2 = n^2$, we let $r = zn$ and $q = \beta n$ and $\beta^2 = z^2 = 1$, so $g \neq -z\beta n$

We get $(z - 1)(u - gy - u\alpha + 3ny\beta)$ and $(z - 1)(c - g + uy - uy\alpha)$

Split into two cases :

case 2bbaaaa4a when $z = 1$ and **case 2bbaaaa4b** when $z \neq 1$ so $z = -1$, $g = c + uy - uy\alpha$ and $g = uy - uy\alpha + 3n\beta$, when subtracting these gives $c = 3n\beta$

case2bbaaaa4a $e = xu$, $i = xs$ $k = v - 1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx - 1$, $f = s = 0$, $t = 0$ and $m = -1$, $a = h - 1$, $v = yu$ and $y^2 = 1$, $g^2 \neq j^2$ and all of n, q, r are non- zero and $\frac{(g+j)^2}{4} = r^2 = q^2 = n^2$, $h = \alpha yu$ and $\alpha = 1 or - 3$,$x = 1$, $j = -g - \frac{2qr}{n} \neq g$, $r = zn$ and $q = \beta n$ and $\beta^2 = z^2 = 1$ and $z = 1$.

We get$-cn - gn + g^2y + nuy + gu\alpha + nuy\alpha + nu\beta + gny\beta$, so $c = (-gn + g^2y + nuy + gu\alpha + nuy\alpha + nu\beta + gny\beta)/n$.

We get $u\alpha(\beta - y)(ny + g)$.

Split into two cases :

case 2bbaaaa4aa when $\beta = y$ and **case 2bbaaaa4ab** when $\beta \neq y$, so $\beta = -y$ and $g = -ny$

case2bbaaaa4aa $e = xu$, $i = xs$ $k = v - 1$, $u \neq 0 \neq e$ and $x = \pm 1$and $b = fx - 1$, $f = s = 0$, $t = 0$ and $m = -1$, $a = h - 1$, $v = yu$ and $y^2 = 1$, $g^2 \neq j^2$ and all of n, q, r are non- zero and $\frac{(g+j)^2}{4} = r^2 = q^2 = n^2$, $h = \alpha yu$ and $\alpha = 1 or - 3$,$x = 1$, $j = -g - \frac{2qr}{n} \neq g$, $r = zn$ and $q = \beta n$ and $\beta^2 = z^2 = 1$, $z = 1$, $c = (-gn + g^2y + nuy + gu\alpha + nuy\alpha + nu\beta + gny\beta)/n$ and $\beta = y$.

We get $n(\alpha - 1)(2g + 2ny - uy + uy\alpha)$

Split into two cases :

case 2bbaaaa4aaa when $\alpha = 1$ and **case 2bbaaaa4aab** when $\alpha \neq 1$, so $\alpha = -3$ and $g = y(2u - n)$

case2bbaaaa4aaa $e = xu$, $i = xs$ $k = v - 1$, $u \neq 0 \neq e$ and $x = \pm 1$and $b = fx - 1$, $f = s = 0$, $t = 0$ and $m = -1$, $a = h - 1$, $v = yu$ and $y^2 = 1$, $g^2 \neq j^2$ and all of n, q, r are non- zero and $\frac{(g+j)^2}{4} = r^2 = q^2 = n^2$, $h = \alpha yu$ and $\alpha = 1 or - 3$, $x = 1$, $j = -g - \frac{2qr}{n} \neq g$, $r = zn$ and $q = \beta n$ and $\beta^2 = z^2 = 1$, $z = 1$

$,c=(-gn+g^2y+nuy+gu\alpha+nuy\alpha+nu\beta+gny\beta)/n$ and $\beta=y$ and $\alpha=1$.

We get $2gn+gu+2n^2y+3nuy$ and $2n^2+3nu+2gny+guy$ subtracting these gives $2(g-n)(g+n)$, so $g=\gamma n$ and $\gamma^2=1$

We get $2ny+3uy+2n\gamma+u\gamma= 2n(y+\gamma)+u(3y+\gamma)$

When $\gamma=-y$, we get $2uy$ CONTRADICTION.

When $\gamma=y$, we get $2uy$ CONTRADICTION.

case2bbaaaa4aab$e=xu$, $i=xs$ $k=v-1$, $u\neq 0\neq e$ and $x=\pm 1$ and $b=fx-1$, $f=s=0$, $t=0$ and $m=-1$, $a=h-1$, $v=yu$ and $y^2=1$, $g^2\neq j^2$ and all of n,q,r are non- zero and $\frac{(g+j)^2}{4}=r^2=q^2=n^2$, $h=\alpha yu$ and $\alpha=1 or-3$,$x=1$, $j=-g-\frac{2qr}{n}\neq g$, $r=zn$ and $q=\beta n$ and $\beta^2=z^2=1$, $z=1$, $c=(-gn+g^2y+nuy+gu\alpha+nuy\alpha+nu\beta+gny\beta(/n$ and $\beta=y$, $\alpha=-3$ and $g=y(2u-n)$.

We get $n(n-u)u^2y$, so $n=u$ SOLUTION.

case2bbaaaa4ab $e=xu$, $i=xs$ $k=v-1$, $u\neq 0\neq e$ and $x=\pm 1$and $b=fx-1$, $f=s=0$, $t=0$ and $m=-1$, $a=h-1$, $v=yu$ and $y^2=1$, $g^2\neq j^2$ and all of n,q,r are non- zero and $\frac{(g+j)^2}{4}=r^2=q^2=n^2$,$h=\alpha yu$ and $\alpha=1 or-3$, $x=1$, $j=-g-\frac{2qr}{n}\neq g$, $r=zn$ and $q=\beta n$ and $\beta^2=z^2=1$ and $z=1$ $,c=(-gn+g^2y+nuy+gu\alpha+nuy\alpha+nu\beta+gny\beta)/n$, $\beta=-y$ and $g=-ny$.

We get $2n^2uy(4n+u(1-\alpha))$, so $n=-u(1-\alpha)/4$ and so $\alpha\neq -1$ because $n\neq 0$

We get $4u^4(\alpha-1)^2(2+\alpha)$ CONTRADICTION.

case2bbaaaa4a $e=xu$, $i=xs$ $k=v-1$, $u\neq 0\neq e$ and $x=\pm 1$ and $b=fx-1$, $f=s=0$, $t=0$ and $m=-1$, $a=h-1$, $v=yu$ and $y^2=1$, $g^2\neq j^2$ and all of n,q,r are non- zero and $\frac{(g+j)^2}{4}=r^2=q^2=n^2$, $h=\alpha yu$ and $\alpha=1 or-3$,$x=1$, $j=-g-\frac{2qr}{n}\neq g$, $r=zn$ and $q=\beta n$, $\beta^2=z^2=1$, $z=-1$ and $c=3n\beta$.

We get $uy(\alpha-1)(uy(1-\alpha)+4n\beta)$ and $u(u(\alpha-1)-4ny\beta)$, when $\alpha=1$, we get $-4ny\beta u$, so $\alpha=-3$

We get $4u(u + ny\beta)$, so $u = -ny\beta$

We get $4n^2y\beta$ CONTRADICTION.

case2bbaaab $e = xu$, $i = xs$ $k = v - 1$, $u \neq 0 \neq e$ and $x = \pm 1$and $b = fx - 1$, $f = g = 0$ and $s \neq 0$.

We get $t^2u + 2sv + 2msv$ and $t^2ux + 2sv + 2msv$. Subtracting these gives $t^2u(x - 1)$

Split into two cases :

case 2bbaaaba when $x = 1$ and **2bbaaabb** when $x \neq 1$, so $t = 0$ and $x = -1$.

case2bbaaaba $e = xu$, $i = xs$ $k = v - 1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx - 1$, $f = g = 0$ and $s \neq 0$ and $x = 1$.

We get $s(1 + m - t)u$ so $m = t - 1$.

We get stu, so $t = 0$

We get s^2u CONTRADICTION.

case2bbaaabb$e = xu$, $i = xs$ $k = v - 1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx - 1$, $f = g = t = 0$ and $s \neq 0$ and $x = -1$.

We get s^2u CONTRADICTION.

case2bbaab$e = xu$, $i = xs$ $k = v - 1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx - 1$, $f \neq 0$ and $x = 1$.

We get $(-1 + f - m)(tu - vs)$

Split into two cases :

case 2bbaaba when $m = f - 1$ and **2bbaabb** when $m \neq f - 1$ and $v = \frac{tu}{s}$.

case2bbaaba$e = xu$, $i = xs$ $k = v - 1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx - 1$, $f \neq 0$ and $x = 1$ and $m = f - 1$.

We have entries $fs + afs - fhs + 2f^2u + ftu - t^2u - 2fsv$ and $fs + afs - fhs + f^2u + s^2u - 2fsv$. Subtracting these gives $u(f^2 - s^2 + ft - t^2)$.

And we have also $(2f^2 - s^2 - ft)u$ Subtracting these gives $(f^2 - 2ft + t^2)u$, so $(f - t)^2u$ and so $t = f$

We get $(f^2-s^2)u$, so$s=yf$ and $y^2=1$

We get $f^2(1+a-h-2v+2uy)$, so $a=-1+h+2v-2uy$

We $u(u-yv)^2$, so $v=yu$

We get $u(h-uy)(h+3uy)=0$

Split into two cases :

case 2bbaabaa when $h=uy$ and **2bbaabab** when $h=-3uy$

case2bbaabaa$e=xu$, $i=xs$ $k=v-1$, $u\neq 0\neq e$ and $x=\pm 1$ and $b=fx-1$,$f\neq 0$ and $x=1$ and $m=f-1$, $t=f$,$s=yf$ and $y^2=1$,$a=-1+h+2v-2uy$,$v=yu$ and $h=uy$.

We get $f(cn+jr+j^2y+jqy)$ and $f(jn+cr+j^2y+jqy)$, subtracting these gives $(c-g)(n-r)=0$.

We get $f(jn+gr+q^2y+nqy)$ and $f(gn+jr+q^2y+nqy)$, subtracting these gives $(g-j)(n-r)=0$.

We also get $f(jn+nq+gqy+r^2y)$ and $f(jr+rq+gqy+n^2y)$, subtracting these gives $(n-r)(j+q-(n+r)y)=0$

Split into two cases :

case 2bbaabaaa when $n=r$ and **2bbaabaab** when $n\neq r$ so, $c=g=j=2ny-q$

case2bbaabaaa $e=xu$, $i=xs$ $k=v-1$, $u\neq 0\neq e$ and $x=\pm 1$ and $b=fx-1$, $f\neq 0$ and $x=1$ and $m=f-1$, $t=f$, $s=yf$ and $y^2=1$,$a=-1+h+2v-2uy$,$v=yu$ and $h=uy$ and $n=r$.

We have entries $q^2+r^2+gry+jry=0$ and $jq+r^2+gry+qry=0$, subtracting these gives $(q-j)(q-ry)=0$

Split into two cases :

case 2bbaabaaaa when $q=j$ and **2bbaabaaab** when $q\neq j$ so, $q=2ry$

case2bbaabaaaa$e=xu$, $i=xs$, $k=v-1$, $u\neq 0\neq e$ and $x=\pm 1$ and $b=fx-1$,$f\neq 0$ and $x=1$ and $m=f-1$, $t=f$,$s=yf$ and $y^2=1$,$a=-1+h+2v-2uy$,$v=$

yu and $h = uy$ and $n = r$ and $q = j$.

We have entries $f(gj + j^2 + 2r^)$ and $f(gj + r^2 + 2jry)$, subtracting these gives$(j^2 + r^2 - 2jry)$

And we have $(j^2+r^2+gry+jry)$, subtracting these gives $gry+3jry$, so $r(g+3j) = 0$

Split into two cases :

case 2bbaabaaaaa when $r = 0$ and **2bbaabaaaab** when $r \neq 0$ so, $g = 3g$

case2bbaabaaaaa$e = xu$, $i = xs$ $k = v - 1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx - 1$,$f \neq 0$ and $x = 1$ and $m = f - 1$, $t = f$,$s = yf$ and $y^2 = 1$,$a = -1 + h + 2v - 2uy$, $v = yu$ and $h = uy$ and $n = r$ and $q = j$ and $r = 0$.

We get $fj^2 = 0$, so $j = 0$

We get $fg^2y = 0$, so$g = 0$

We get $fc^2 = 0$, so $c = 0$ SOLUTION.

case2bbaabaaaab $e = xu$, $i = xs$ $k = v-1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx-1$, $f \neq 0$ and $x = 1$ and $m = f-1$, $t = f$, $s = yf$ and $y^2 = 1$,$a = -1+h+2v-2uy$,$v = yu$ and $h = uy$ and $n = r$ and $q = j$, $r \neq 0$ and $g = 3g$.

We get $f(2j^2 + r^2)$ so $j \neq 0$

And we get $(j^2 + r^2 + 2jry)$, subtracting these gives$j(j - 2ry)$, so $j = 2ry$

We get fr^2 CONTRADICTION.

case2bbaabaaab $e = xu$, $i = xs$ $k = v - 1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx - 1$,$f \neq 0$ and $x = 1$ and $m = f-1$, $t = f$, $s = yf$ and $y^2 = 1$,$a = -1+h+2v-2uy$,$v = yu$ and $h = uy$ and $n = r$, $q \neq j$ and $q = 2ry$.

We get$f(gj + 3r^2)$ so $gj = -3r^2$

And we get $f(c^2 + 3gj)$ so $(c^2 - 9r^2) = 0$ and get $c = 3zr$

We get$f(2jr+j^2y+3r^2z)$ and $f(2gr+g^2y+3r^2z)$ subtracting these gives $(j-g)(2r+yg+yj) = 0$.

Split into two cases :

case2bbaabaaaba when $j = g$ and **case2bbaabaaabb** when $j \neq g$ so, $g = -2ry - j$

case2bbaabaaaba $e = xu$, $i = xs$ $k = v - 1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx - 1$, $f \neq 0$ and $x = 1$ and $m = f - 1$, $t = f$, $s = yf$ and $y^2 = 1$,$a = -1 + h + 2v - 2uy$,$v = yu$ and $h = uy$ and $n = r$, $c = 3zr$,$q \neq j$ and $q = 2ry$ and $j = g$.

We get $2fr(r + gy)$

Split into two cases :

case2bbaabaaabaa when $r = 0$ and **case2bbaabaaabab** when $r \neq 0$ so, $r = -gy$ and $g \neq 0$

case2bbaabaaabaa $e = xu$, $i = xs$ $k = v - 1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx - 1$,$f \neq 0$ and $x = 1$ and $m = f - 1$, $t = f$,$s = yf$ and $y^2 = 1$,$a = -1 + h + 2v - 2uy$,$v = yu$ and $h = uy$ and $n = r$, $q \neq j$ and $q = 2ry$, $c = 3zr$ and $j = g$ and $r = 0$, so$q = 0$.

We get fg^2 so $g = 0$ and so $j = 0$ but $q \neq j$ CONTRADICTION.

case2bbaabaaabab $e = xu$, $i = xs$ $k = v - 1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx - 1$, $f \neq 0$ and $x = 1$ and $m = f - 1$, $t = f$,$s = yf$ and $y^2 = 1$,$a = -1 + h + 2v - 2uy$, $v = yu$ and $h = uy$ and $n = r$, $q \neq j$ and $q = 2ry$, $c = 3zr$ and $j = g$, $r \neq 0$, $g \neq 0$ and $r = -gy$.

We get fg^2 CONTRADICTION.

case2bbaabaaabb $e = xu$, $i = xs$ $k = v - 1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx - 1$,$f \neq 0$ and $x = 1$ and $m = f - 1$, $t = f$,$s = yf$ and $y^2 = 1$,$a = -1 + h + 2v - 2uy$,$v = yu$ and $h = uy$ and $n = r$, $q \neq j$ and $q = 2ry$,$c = 3zr$, $j \neq g$and , $g = -2ry - j$.

We get $f(-c + 4j + 9ry)$, so $c = 4j + 9ry$

We get $2f^2(j + 3ry)$, so $j = -3ry$

We get $3fr^2yz$, so $r = 0$ but we have $q \neq j$ and $q = 2ry$ and $j = -3ry$ CONTRADICTION.

case2bbaabaab $e = xu$, $i = xs$, $k = v - 1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx - 1$,$f \neq 0$ and $x = 1$ and $m = f - 1$, $t = f$,$s = yf$ and $y^2 = 1$,$a = -1 + h + 2v - 2uy$,$v =$

yu and $h = uy$, $n \neq r$ and $c = g = j = 2ny - q$.

We have entry $c^2 + 2gj = 0$ so $4c^2 = 0$ and so $c = g = j = 0$ and $q = 2ny$

We get $4g^2f$, so $g = 0$

We get $u^2y(n + r)$, so $n = -r$

We get $2f^2r$ so $r = 0$ but we have $n \neq r$ and $n = -r$ CONTRADICTION.

case2bbaaba $e = xu$, $i = xs$ $k = v-1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx-1$,$f \neq 0$ and $x = 1$ and $m = f - 1$, $t = f$,$s = yf$ and $y^2 = 1$,$a = -1 + h + 2v - 2uy$,$v = yu$ and $h = -3uy$.

We have entries $f(jn+gr+q^2+nry+4nuy)$, $f(gn+jr+q^2+nry+4ruy)$ subtracting these gives $(n - r)(j - g + 4uy)$ Split into two cases :

case2bbaabaa when $n = r$ and **case2bbaabab** when $n \neq r$ so , $j = g - 4uy$.

case2bbaabaa $e = xu$, $i = xs$ $k = v - 1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx - 1$,$f \neq 0$ and $x = 1$ and $m = f - 1$, $t = f$,$s = yf$ and $y^2 = 1$,$a = -1+h+2v-2uy$,$v = yu$ and $h = -3uy$,$n = r$.

We get $f(j^2 + cq - ur + 2jry - 4quy - 7ru)$, $f(j^2 + jq - ur + jry + cry - 4quy - 7ru)$ subtracting these gives$(c - j)(q - ry)$

Split into two cases :

case2bbaabaaa when $c = j$ and **case2bbaabaab** when $c \neq j$ so , $q = ry$.

case2bbaabaaa $e = xu$, $i = xs$ $k = v - 1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx - 1$,$f \neq 0$ and $x = 1$ and $m = f - 1$, $t = f$,$s = yf$ and $y^2 = 1$,$a = -1+h+2v-2uy$,$v = yu$ and $h = -3uy$,$n = r$ and $c = j$.

We get $f(jr + qr + gqy + r^2y + 4ruy)$ and $f(gr + jr + q^2y + r^2y + 4ruy)$ subtracting these gives$(q - g)(r - qy)$

Split into two cases :

case2bbaabaaaa when $q = g$ and **case2bbaabaaab** when $q \neq g$ so , $r = yq$

case2bbaabaaaa $e = xu$, $i = xs$ $k = v - 1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx - 1$

,$f \neq 0$ and $x = 1$ and $m = f - 1$, $t = f$,$s = yf$ and $y^2 = 1$,$a = -1 + h + 2v - 2uy$,$v = yu$ and $h = -3uy$,$n = r$, $c = j$ and $q = g$.

We get $f(2ru - 2guy)$, so $r = gy$

We get $fg(4u + 3gy + jy)$

When $g = 0$, we get $16f^2uy$ CONTRADICTION.

When $j = -4uy - 3g$, we get $16fu^2$ CONTRADICTION.

case2bbaabaaab $e = xu$, $i = xs$, $k = v - 1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx - 1$,$f \neq 0$ and $x = 1$ and $m = f - 1$, $t = f$,$s = yf$ and $y^2 = 1$, $a = -1 + h + 2v - 2uy$, $v = yu$ and $h = -3uy$, $n = r$ and $c = j$, $q \neq g$ and $r = yq$.

We have entry $f(3gu + qu + gqy + jqy + 2q^y)$, if $q = 0$, then $g = 0$, but $q \neq$, so $q \neq 0$

And we get $-4f^2j + 12ju^2 + qg^2uy + 3j^2uy$ and $-4f^2j + 12ju^2 + qq^2uy + 3j^2uy$, subtracting these gives $q(g - q)(g + q)uy$, so $g = -q$

We get $-4fq^2$ CONTRADICTION.

case2bbaabaab $e = xu$, $i = xs$, $k = v - 1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx - 1$,$f \neq 0$ and $x = 1$ and $m = f - 1$, $t = f$,$s = yf$ and $y^2 = 1$,$a = -1 + h + 2v - 2uy$,$v = yu$ and $h = -3uy$,$n = r$, $c \neq j$ and $q = ry$.

We get $fr(jy + 2r + 4u + gy)$

Split into two cases :

case2bbaabaaba when $r = 0$ and **case2bbaabaabb** when $r \neq 0$ so, $r = -(yg + jy + 4u)/2$

case2bbaabaaba $e = xu$, $i = xs$, $k = v - 1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx - 1$, $f \neq 0$ and $x = 1$ and $m = f - 1$, $t = f$,$s = yf$ and $y^2 = 1$,$a = -1 + h + 2v - 2uy$, $v = yu$ and $h = -3uy$,$n = r$, $c \neq j$ and $r = 0$, $q = ry$.

We get fj^2y, so $j = 0$

We get g^2y, so $g = 0$

We get fc^2, so $c = 0$ but $c \neq j$ CONTRADICTION.

case2bbaabaabb $e = xu$, $i = xs$ $k = v-1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx-1$, $f \neq 0$ and $x = 1$ and $m = f-1$, $t = f$,$s = yf$ and $y^2 = 1$,$a = -1+h+2v-2uy$,$v = yu$ and $h = -3uy$,$n = r$, $c \neq j$ and $q = ry$,$r \neq 0$ and $r = -(yg + jy + 4u)/2$.

We get $fu(2c + qg + j + 12uy)$, so $j = -2c - qg - 12uy$

We get $f(c^2 + 6cg + 9g^2) = (c + 3g)^2$, so $c = -3g$ We get $fu(8uy)$ CONTRADICTION.

case2bbaabab $e = xu$, $i = xs$ $k = v - 1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx - 1$, $f \neq 0$ and $x = 1$ and $m = f-1$, $t = f$,$s = yf$ and $y^2 = 1$,$a = -1+h+2v-2uy$,$v = yu$ and $h = -3uy$,$n \neq r$ and $j = g - 4uy$.

We have entries $2u(4gny + 4qry + 8f^2 - 8nu)$ and $2u(4gny + 4qry + 8f^2 - 8nu)$, subtracting these gives$4y(g - q)(n - r)$, so$g = q$

We also have $n^2 + gq + gry + qry$ and $r^2 + gq + gny + qny$, subtracting these gives$(n - r)(n + r - gy - qy)$, so $r = gy + qy - n$

We get $n^2 + 3q^2 - 2nqy$ and $n^2 + 5q^2 - 2nqy$, subtracting these gives$2q^2 = 0$, so $q = 0$

We get $n = 0$, so $r \neq 0$ because $n \neq r$, but we get r^2 CONTRADICTION.

case2bbaabb $e = xu$, $i = xs$ $k = v - 1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx - 1$,$f \neq 0$ and $x = 1$, $m \neq f - 1$ and $t = \frac{vs}{u}$.

We have entry $u^2 + (1 + a)v - hv - v^2$, so $v \neq 0$

And we have $u^2 + v(1 + a - h) - v^2$ and $u^2(1 + a - h) + vu^2 - v^3$, put $\alpha = 1 + a - h$

We get $u^2 + v\alpha - v^2$ and $u^2\alpha + vu^2 - v^3$ so $u^2 = v^2 - \alpha v$

So $u^2\alpha + vu^2 - v^3 = (v^2 - \alpha v)\alpha + v(v^2 - \alpha v) - v^3 =-\alpha^2 v = 0$, so $\alpha = 0$ and $a = h-1$

We get $u^2(u - v)v(u + v)$, so $v = yu$ when $y^2 = 1$

We get $u^3(h - uy)(h + 3uy)$, so $h = \beta uy$ and $\beta = 1 or - 3$

We get $u(f - sy)(fy + 3s)$, so $f = \alpha sy$ and $\alpha = 1 or - 3$

We also get $fu(1 + m - sy)$, so $m = sy - 1$ but $m \neq f - 1$, so $sy \neq f$, and so $\alpha = -3$.

We get $(s-u)(s+3u)$

Split into two cases :

case2bbaabba when $s=u$ and **case2bbaabbb** when $s=-3u$.

case2bbaabba $e=xu$, $i=xs$, $k=v-1$, $u\neq 0\neq e$ and $x=\pm 1$ and $b=fx-1$,$f\neq 0$ and $x=1$, $m\neq f-1$ and $t=\frac{vs}{u}$, $a=h-1$, $v\neq 0$, $v=yu$, $y^2=1$, $h=\beta uy$ and $\beta=1or-3$, $f=\alpha sy$, $\alpha=-3$, $f=\alpha sy$, $m=sy-1$ and $s=u$.

We get $gn+nq+qu+jqy+r^2y+4nuy-qu\beta$ and $-jn-nq-qu-jqy-r^2y-nuy+qu\beta+nuy\beta$ subtracting these gives $n(g-j+3uy+uy\beta)$

We get $-jn-jr-q^2y-nry-nuy-ruy+nuy\beta+ruy\beta$ and $-jn-gr-q^2y-nry-nuy-4ruy+nuy\beta$ subtracting these gives $r(g-j+3uy+uy\beta)$

We get $-n^2-jq-ru-jry-qry-quy+ru\beta+quy\beta$ and $n^2+gq+ru+jry+qry+4quy-ru\beta$ adding these gives $q(g-j+3uy+uy\beta)$

Split into two cases :

case2bbaabbaa when $j=g+3uy+uy\beta$ and **case2bbaabbab** when $q=r=n=0$.

case2bbaabbaa $e=xu$, $i=xs$, $k=v-1$, $u\neq 0\neq e$ and $x=\pm 1$ and $b=fx-1$,$f\neq 0$ and $x=1$, $m\neq f-1$ and $t=\frac{vs}{u}$, $a=h-1$, $v\neq 0$, $v=yu$, $y^2=1$, $h=\beta uy$ and $\beta=1or-3$, $f=\alpha sy$, $\alpha=-3$, $f=\alpha sy$, $m=sy-1$, $s=u$ and $j=g+3uy+uy\beta$.

Split into two cases:

case2bbaabbaaa when $\beta=-3$ and **case 2bbaabbaab** when $\beta=1$

case2bbaabbaaa $e=xu$, $i=xs$, $k=v-1$, $u\neq 0\neq e$ and $x=\pm 1$ and $b=fx-1$,$f\neq 0$ and $x=1$, $m\neq f-1$ and $t=\frac{vs}{u}$, $a=h-1$, $v\neq 0$, $v=yu$, $y^2=1$, $h=\beta uy$ and $\beta=1or-3$, $f=\alpha sy$, $\alpha=-3$, $f=\alpha sy$, $m=sy-1$, $s=u$ and $j=g+3uy+uy\beta$ and $\beta=-3$.

We get $-cn-gr+4qu-g^2y-gqy+4ruy$ and $-gn-cr+4qu-g^2y-gqy+4nuy$, subtracting these gives $-(n-r)(c-g+4uy)$ Split into two cases:

case2bbaabbaaaa when $n = r$ and **case 2bbaabbaaab** when $n \neq r$, so $g = c+4uy$

case2bbaabbaaaa $e = xu$, $i = xs$, $k = v-1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx-1$,$f \neq 0$ and $x = 1$, $m \neq f-1$ and $t = \frac{vs}{u}$, $a = h-1$, $v \neq 0$, $v = yu$, $y^2 = 1$, $h = \beta uy$ and $\beta = 1 or - 3$, $f = \alpha sy$, $\alpha = -3$, $f = \alpha sy$, $m = sy-1$, $s = u$ and $j = g + 3uy + uy\beta$ and $\beta = -3$ and $n = r$.

Go to **case2bbaabbaaba**

case2bbaabbaaab $e = xu$, $i = xs$, $k = v-1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx-1$,$f \neq 0$ and $x = 1$, $m \neq f-1$ and $t = \frac{vs}{u}$, $a = h-1$, $v \neq 0$, $v = yu$, $y^2 = 1$, $h = \beta uy$ and $\beta = 1 or - 3$, $f = \alpha sy$, $\alpha = -3$, $f = \alpha sy$, $m = sy-1$, $s = u$ and $j = g + 3uy + uy\beta$ and $\beta = -3$, $n \neq r$ and $g = c + 4uy$.

We get $4cuy + c^2 + 4u^2$=$(c + 2uy)^2$, so $c = -2uy$

We get $u^2(n + r + 14u + qy)$and $u^2(-n - r + 2u - qy)$, adding these gives $16u$ CONTRADICTION.

case2bbaabbaab $e = xu$, $i = xs$, $k = v-1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx-1$,$f \neq 0$ and $x = 1$, $m \neq f-1$ and $t = \frac{vs}{u}$, $a = h-1$, $v \neq 0$, $v = yu$, $y^2 = 1$, $h = \beta uy$ and $\beta = 1 or - 3$, $f = \alpha sy$, $\alpha = -3$, $f = \alpha sy$, $m = sy-1$, $s = u$ and $j = g + 3uy + uy\beta$ and $\beta = 1$.

We get $4ru + 4u^2 + 2nqy + 2gry$ and $4nu + 4u^2 + 2ngy + 2qry$ subtracting these gives $2(n-r)(2u + gy - qy)$

Split into two cases:

case2bbaabbaaba when $n = r$ and **case 2bbaabbaabb** when $n \neq r$, so $g = q-2uy$

case2bbaabbaaba $e = xu$, $i = xs$, $k = v-1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx-1$,$f \neq 0$ and $x = 1$, $m \neq f-1$ and $t = \frac{vs}{u}$, $a = h-1$, $v \neq 0$, $v = yu$, $y^2 = 1$, $h = \beta uy$ and $\beta = 1 or - 3$, $f = \alpha sy$, $\alpha = -3$, $f = \alpha sy$, $m = sy-1$, $s = u$ and $j = g + 3uy + uy\beta$ and $\beta = 1$ and $n = r$.

We have $n = r$ when both cases $(\beta = 1 and \beta = -3)$

Split into two cases:

case2bbaabbaabaa when $r = 0$ and **case 2bbaabbaabab** when $r \neq 0$

case2bbaabbaabaa $e = xu$, $i = xs$, $k = v - 1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx - 1$,$f \neq 0$ and $x = 1$, $m \neq f - 1$ and $t = \frac{vs}{u}$, $a = h - 1$, $v \neq 0$, $v = yu$, $y^2 = 1$, $h = \beta uy$ and $\beta = 1 or - 3$, $f = \alpha sy$, $\alpha = -3$, $f = \alpha sy$, $m = sy - 1$, $s = u$ and $j = g + 3uy + uy\beta$ and $\beta = 1$ and $n = r$ and $r = 0$.

We get q^2uy , so $q = 0$

We get $-g^2u$, so $g = 0$

We get uc^2 , so $c = 0$

We get $4u^3$CONTRADICTION.

case2bbaabbaabab $e = xu$, $i = xs$ $k = v - 1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx - 1$,$f \neq 0$ and $x = 1$, $m \neq f - 1$ and $t = \frac{vs}{u}$, $a = h - 1$, $v \neq 0$, $v = yu$, $y^2 = 1$, $h = \beta uy$ and $\beta = 1 or - 3$, $f = \alpha sy$, $\alpha = -3$, $f = \alpha sy$, $m = sy - 1$, $s = u$ and $j = g + 3uy + uy\beta$ and $\beta = 1$ and $n = r$ and $r \neq 0$.

We get $3r^2 + 4ru + 4u^2$ (**)

And we also get $7r + 8u + 6qy$, so $q = (-7r - 8u)/6$

We get $u(r + 8u)^2(13r + 8u)y$ but it is not solution for (**) CONTRADICTION

case2bbaabbaabb $e = xu$, $i = xs$ $k = v - 1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx - 1$,$f \neq 0$ and $x = 1$, $m \neq f - 1$ and $t = \frac{vs}{u}$, $a = h - 1$, $v \neq 0$, $v = yu$, $y^2 = 1$, $h = \beta uy$ and $\beta = 1 or - 3$, $f = \alpha sy$, $\alpha = -3$, $f = \alpha sy$, $m = sy - 1$, $s = u$ and $j = g + 3uy + uy\beta$ and $\beta = 1$, $n \neq r$ and $g = q - 2uy$.

We get $8cu + 12qu + c^2y + 3q^2y + 60u^2y$ and $-4qu - c^2y - 3q^2y + 20u^2y$ subtracting these gives $q = -c - 10uy$

We get $-63nu - 59ru + 18u^2 - 6cny - 6cry$ and $-59nu - 63ru + 18u^2 - 6cny - 6cry$ subtracting these gives $-4(n - r)$ CONTRADICTION.

case2bbaabbab $e = xu$, $i = xs$ $k = v - 1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx - 1$

$,f \neq 0$ and $x = 1$, $m \neq f-1$ and $t = \frac{vs}{u}$, $a = h-1$, $v \neq 0$, $v = yu$, $y^2 = 1$, $h = \beta uy$ and $\beta = 1 or -3$, $f = \alpha sy$, $\alpha = -3$, $f = \alpha sy$, $m = sy - 1$ and $s = u$ and $q = r = n = 0$.

We get $16u^3$ CONTRADICTION.

case2bbaabbb $e = xu$, $i = xs$ $k = v-1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx - 1$ $,f \neq 0$ and $x = 1$, $m \neq f-1$ and $t = \frac{vs}{u}$, $a = h-1$, $v \neq 0$, $v = yu$, $y^2 = 1$, $h = \beta uy$ and $\beta = 1 or -3$, $f = \alpha sy$, $\alpha = -3$, $f = \alpha sy$, $m = sy - 1$ and $s = -3u$.

We get $cn + jr - qu + j^2y + jqy - nuy - ruy + qu\beta + nuy\beta + ruy\beta$ and $jn + cr - qu + j^2y + jqy - nuy - ruy + qu\beta + nuy\beta + ruy\beta$, subtracting these gives $(c-j)(n-r)$

Split into two cases:

case2bbaabbba when $n = r$ and **case 2bbaabbbb** when $n \neq r$ and $c = j$

case2bbaabbba $e = xu$, $i = xs$ $k = v-1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx - 1$ $,f \neq 0$ and $x = 1$, $m \neq f-1$ and $t = \frac{vs}{u}$, $a = h-1$, $v \neq 0$, $v = yu$, $y^2 = 1$, $h = \beta uy$ and $\beta = 1 or -3$, $f = \alpha sy$, $\alpha = -3$, $f = \alpha sy$, $m = sy - 1$ and $s = -3u$ and $n = r$.

We get $2jr - qu + j^2y + cqy - 2ruy + qu\beta + 2ruy\beta$ and $cr + jr - qu + j^2y + jqy - 2ruy + qu\beta + 2ruy\beta$, subtracting these gives $(c-j)(r-qy)$

Split into two cases:

case2bbaabbbaa when $c = j$ and **case 2bbaabbbab** when $c \neq j$ and $r = qy$

case2bbaabbbaa $e = xu$, $i = xs$ $k = v-1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx - 1$ $,f \neq 0$ and $x = 1$, $m \neq f-1$ and $t = \frac{vs}{u}$, $a = h-1$, $v \neq 0$, $v = yu$, $y^2 = 1$, $h = \beta uy$ and $\beta = 1 or -3$, $f = \alpha sy$, $\alpha = -3$, $f = \alpha sy$, $m = sy - 1$ and $s = -3u$ and $n = r$ and $c = j$.

We get $-3gu + 3gjy + j^2y + 3gu\beta$ and $qu - 3gjy - j^2y + 2ruy - 3gu\beta$, adding these gives $q = 3g - 2ry$

We get $-33ju + 12gr\beta - 9g^2y\beta - j^2y\beta - 6r^2y\beta + ju\beta^2$ and $-33ju - 3g^2y\beta - j^2y\beta + ju\beta^2$, subtracting these gives $g = ry$

We get $3r + u + jy - u\beta$ and $r + 17u + jy - u\beta$, subtracting these gives $ru(r - 8u)$

Split into two cases:

case2bbaabbbaaa when $r = 0$ and **case 2bbaabbbaab** when $r \neq 0$ and $r = 8u$

case2bbaabbbaaa $e = xu$, $i = xs$ $k = v-1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx-1$,$f \neq 0$ and $x = 1$, $m \neq f - 1$ and $t = \frac{vs}{u}$, $a = h - 1$, $v \neq 0$, $v = yu$, $y^2 = 1$, $h = \beta uy$ and $\beta = 1or - 3$, $f = \alpha sy$, $\alpha = -3$, $f = \alpha sy$, $m = sy - 1$ and $s = -3u$ and $n = r$ and $c = j$, $q = 3g - 2ry$, $g = ry$ and $r = 0$.

We get $3uj^2u$, so $j = 0$

We get $3uq^2u$, so $q = 0$

We get $9gu^2y$, so $g = 0$

We get $36u^3y(3 + \beta)$, so $\beta = -3$

We get u^3y CONTRADICTION.

case2bbaabbbaab $e = xu$, $i = xs$, $k = v - 1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx - 1$,$f \neq 0$ and $x = 1$, $m \neq f - 1$ and $t = \frac{vs}{u}$, $a = h - 1$, $v \neq 0$, $v = yu$, $y^2 = 1$, $h = \beta uy$ and $\beta = 1or - 3$, $f = \alpha sy$, $\alpha = -3$, $f = \alpha sy$, $m = sy - 1$ and $s = -3u$ and $n = r$ and $c = j$, $q = 3g - 2ry$, $g = ry$, $r \neq 0$ and $r = 7u$.

We get $16u^2(-j - 25uy + uy\beta)$, so $j = -25uy + uy\beta$

We get $-4u^3y(-33 + \beta)(7 + 2\beta)$ CONTRADICTION.

case2bbaabbbab $e = xu$, $i = xs$, $k = v-1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx-1$,$f \neq 0$ and $x = 1$, $m \neq f - 1$ and $t = \frac{vs}{u}$, $a = h - 1$, $v \neq 0$, $v = yu$, $y^2 = 1$, $h = \beta uy$ and $\beta = 1or - 3$, $f = \alpha sy$, $\alpha = -3$, $f = \alpha sy$, $m = sy - 1$ and $s = -3u$ and $n = r$, $c \neq j$ and $r = qy$.

We get $q(g + j + 2q + uy - uy\beta) = 0$

Split into two cases:

case2bbaabbbaba when $q = 0$ and **case 2bbaabbbabb** when $q \neq 0$, so $g =$

$-(j + uy - uy\beta + 2q)$

case2bbaabbbaba $e = xu$, $i = xs$, $k = v - 1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx - 1$,$f \neq 0$ and $x = 1$, $m \neq f - 1$ and $t = \frac{vs}{u}$, $a = h - 1$, $v \neq 0$, $v = yu$, $y^2 = 1$, $h = \beta uy$ and $\beta = 1or - 3$, $f = \alpha sy$, $\alpha = -3$, $f = \alpha sy$, $m = sy - 1$ and $s = -3u$ and $n = r$, $c \neq j$ and $r = qy$ and $q = 0$.

We get $3ug^2$, so $g = 0$

We get $3(c - j)u^2y$ CONTRADICTION.

case2bbaabbbabb $e = xu$, $i = xs$, $k = v - 1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx - 1$,$f \neq 0$ and $x = 1$, $m \neq f - 1$ and $t = \frac{vs}{u}$, $a = h - 1$, $v \neq 0$, $v = yu$, $y^2 = 1$, $h = \beta uy$ and $\beta = 1or - 3$, $f = \alpha sy$, $\alpha = -3$, $f = \alpha sy$, $m = sy - 1$ and $s = -3u$ and $n = r$, $c \neq j$ and $r = qy$, $q \neq 0$ and $g = -(j + uy - uy\beta + 2q)$.

We get $u^y(c - 4j - 9q - 3uy + 3uy\beta)$, so $c = 4j + 9q + 3uy - 3uy\beta$

We get $3(j + 3q)^2u$, so $j = -3q$

We get $3u^3(-1 + \beta)$, so $\beta = 1$

We get $4qu(q - 8uy)$and $4qu(q + 28uy)$ and $12qu(4u + qy)$ CONTRADICTION.

case2bbaabbbb $e = xu$, $i = xs$, $k = v - 1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx - 1$,$f \neq 0$ and $x = 1$, $m \neq f - 1$ and $t = \frac{vs}{u}$, $a = h - 1$, $v \neq 0$, $v = yu$, $y^2 = 1$, $h = \beta uy$ and $\beta = 1or - 3$, $f = \alpha sy$, $\alpha = -3$, $f = \alpha sy$, $m = sy - 1$ and $s = -3u$, $n \neq r$ and $c = j$.

We have entries $-g^2 - gq - ru - gny - jry + ru\beta$ and $-g^2 - gq - nu - gry - jny + nu\beta$, subtracting these gives $(n - r)(-u + gy - jy + u\beta)$, so $j = g + u\beta y - uy$

We get $3gu(n + r + qy + gy)$

Split into two cases :

case 2bbaabbbba when $g = 0$ and **case 2bbaabbbbb** when $g \neq 0$, so $g = -q - ny - ry$

case2bbaabbbb $e = xu$, $i = xs$, $k = v - 1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx - 1$

,$f \neq 0$ and $x = 1$, $m \neq f - 1$ and $t = \frac{vs}{u}$, $a = h - 1$, $v \neq 0$, $v = yu$, $y^2 = 1$, $h = \beta uy$ and $\beta = 1 or - 3$, $f = \alpha sy$, $\alpha = -3$, $f = \alpha sy$, $m = sy - 1$ and $s = -3u$, $n \neq r$ and $c = j$, $j = g + u\beta y - uy$ and $g = 0$.

We get $3u^3(-1\beta)^2$, so $\beta = 1$

We get $u^2(q + ny + ry)$, so $q = -ny - ry$

We get $6nruy$ and $3(n^2 + nr - r^2)uy$, when $r = 0$, we get $n = 0$ CONTRADICTION.

And when $n = 0$, we get $r = 0$ CONTRADICTION.

case2bbaabbbb $e = xu$, $i = xs$, $k = v - 1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx - 1$,$f \neq 0$ and $x = 1$, $m \neq f - 1$ and $t = \frac{vs}{u}$, $a = h - 1$, $v \neq 0$, $v = yu$, $y^2 = 1$, $h = \beta uy$ and $\beta = 1 or - 3$, $f = \alpha sy$, $\alpha = -3$, $f = \alpha sy$, $m = sy - 1$ and $s = -3u$, $n \neq r$ and $c = j$, $j = g + u\beta y - uy$, $g \neq 0$ and $g = -q - ny - ry$.

We get $3u^3(-1 + \beta)$, so $\beta = 1$

We get $12uy(q + ny + ry)^2$, so $q = -ry - ny$

We get $2nruy$ and $3(n^2 - nr - r^2)uy$, when $r = 0$, we get $n = 0$ CONTRADICTION.

And when $n = 0$, we get $r = 0$ CONTRADICTION.

case 2bbaac $e = xu$, $i = xs$, $k = v - 1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx - 1$ and $x \neq 1$ so $x = -1$., $f \neq 0$ and $n = r = q = 0$.

We have entries $g(s^2 + u^2)$,$g^2 s$, $j^2 s$, $j(1 + m)$ and $j^2 t$

If $s = 0$ thengu^2 , so $g = 0$ and if $s \neq 0$ then $g = 0$, so must have $g = 0$

Split into two cases :

case 2bbaaca when $j = 0$ and **case 2bbaacb** when $j \neq 0$, so $t = s = 0$ and $m = -1$

case 2bbaaca $e = xu$, $i = xs$ $k = v - 1$, $u \neq 0 \neq e$ and $x = \pm 1$ and $b = fx - 1$ and $x \neq 1$ so $x = -1$., $f \neq 0$ and $n = r = q = j = g = 0$.

We get cfu so $c = 0$

We get$(1 + f + m)(fv + su) = 0$, $(1 + f + m)(fv - su) = 0$,$(f + t)(fv - su) = 0$ and

$(f+t)(fv+su)=0$ .Subtracting these gives $fv(1+f+m)=0$ and $fv(f+t)=0$

Split into two cases :

case 2bbaacaa when $v=0$ and **case 2bbaacab** when $v\neq 0$, so $t=-f$ and $m=-1-f$.

case 2bbaacaa $e=xu$, $i=xs$, $k=v-1$, $u\neq 0\neq e$ and $x=\pm 1$ and $b=fx-1$ and $x\neq 1$ so $x=-1$, $f\neq 0$ and $n=r=q=j=g=v=c=0$.

We get $-hu^2$ so $h=0$

We get u^2 CONTRADICTION.

case 2bbaacab $e=xu$, $i=xs$, $k=v-1$, $u\neq 0\neq e$ and $x=\pm 1$ and $b=fx-1$ and $x\neq 1$ so $x=-1$, $f\neq 0$ and $n=r=q=j=g=c=0$, $t=-f$ and $m=-1-f$ and $v\neq 0$.

We get $f^2(1+a-h-2v)$, so $h=1+a-2v$

We get $((1+a)^2+u^2-2v^2)$, $v(u^2-v^2)$ and $(f^2-s^2)u$, so $u^2=v^2$, $f^2=s^2$ and $(1+a)^2=v^2$

We also have $(1+a-v)(u^2-v-av+2v^2)$ and using $u^2=v^2$ get $v(1+a-v)(3v-1-a)$ so $v=1+a$ or $v=\frac{1+a}{3}$ but we also have $v=1+a$ or $v=-(1+a)$, so $v=1+a$ and $a=v-1$ SOLUTION.

case 2bbaacb $e=xu$, $i=xs$ $k=v-1$, $u\neq 0\neq e$ and $x=\pm 1$ and $b=fx-1$ and $x\neq 1$ so $x=-1$, $f\neq 0$ and $n=r=q=g=t=s=0$ and $m=-1$.

We get f^2u CONTRADICTION.

case2bbab $e=xu$, $i=xs$, $k=v-1$, $u\neq 0\neq e$ and $x=\pm 1$, $b\neq fx-1$ and $m=t-1$.

We have entry $u(h+ah+2hv-2v^2x-u^2x)$ and $u(h+ah+av+hv-v^2-2v^2x)$ subtracting these gives $v(h-a+v)=u^2x$, so $v\neq 0$ and $v\neq a-h$.

And we have $su(x-1)(fx-1-b)$ so $s(x-1)=0$

Split into two cases :

case 2bbaba when $x = 1$ and **2bbabb** when $x \neq 1$, so$x = -1$ and $s = 0$.

case2bbaba $e = xu$, $i = xs$, $k = v - 1$, $u \neq 0 \neq e$ and $x = \pm 1$, $b \neq fx - 1$ and $m = t - 1$ and $x = 1$,$v \neq 0$ and $v \neq a - h$.

We have entries $-fgh + fgj + n^2 + bn^2 + fq^2 + fr^2 + gsu$,$-fgh + fgj + r^2 + br^2 + fq^2 + fn^2 + gsu$ and $-fgh + fgj + q^2 + bq^2 + fn^2 + fr^2 + gsu$ subtracting these gives $(f - 1 - b)(r^2 - n^2)$, $(f - 1 - b)(r^2 - q^2)$ and $(f - 1 - b)(n^2 - q^2)$ and we know $b \neq fx - 1$, so $n^2 = q^2 = r^2$

Put $q = yn$,$r = zn$ and $y^2 = z^2 = 1$ and we let $b = f - 1 - w$

We get $-hu^2 - h^2v + 2u^2v - v^2 - av^2 + v^3$ and $-u^2 - au^2 - h^2v + 2u^2v - hv^2 + v^3$, subtracting these gives $(1 + a - h)(u^-v^2) = 0$.

And we have $v(u^2 + v(1 + a - h) - v^2)$.

Split into two cases: when $h = 1 + a$ and when $u^2 = v^2$, but if $h = 1 + a$, then we get $v(u^-v^2) = 0$, so $u^2 = v^2$ in all cases and so $v = \alpha u$ when $\alpha^2 = 1$

And when $u^2 = v^2$ we get $v^2(1 + a - h) = 0$, so $a = h - 1$

We get $u(-2st\alpha + s^2 + t^2)$ so $u(s - t\alpha)^2$ and $s = t\alpha$, so we get $s^2 = t^2$

We get $tw = 0$, so $t = 0$ and so $s = 0$ because $w \neq 0$

We get f^2u so $f = 0$

We get w^2u CONTRADICTION.

case2bbabb $e = xu$, $i = xs$, $k = v - 1$, $u \neq 0 \neq e$ and $x = \pm 1$, $b \neq fx - 1$ and $m = t - 1$, $x \neq 1$, $x = -1$ and $s = 0$, $v \neq 0$ and $v \neq a - h$.

We get t^2v so $t = 0$

We get uf^2 s0 $f = 0$ and we know $b \neq fx - 1$ so $b \neq -1$

We get $(1 + b)^2u$ CONTRADICTION.

case 2bbb $u \neq 0 \neq e$, $e = xu$ and $i = xs$, $k \neq v - 1$, $a = h - 1$ and $x = \pm 1$.

We have entry $(1 + k - v)(1 + k - vx)$, so $x = -1$, $k = -1 - v$

We get $u(u^2 - v^2)$, so $v^2 = u^2$

We get $u(h^2+u^2-2v^2)$, so $h^2=u^2=v^2$

We get $(1+b+f)su$

Split into two cases :

case 2bbba when $s=0$ and **case 2bbbb** when $s\neq 0$, so $b=-f-1$.

case 2bbba $u\neq 0\neq e$, $e=xu$ and $i=xs$, $k\neq v-1$, $a=h-1$ and $x=-1$, $h^2=u^2=v^2$ and $k=-1-v$ and $s=0$.

We have entry ftu

Split into two cases :

case 2bbbaa when $f=0$ and **2bbbab** when $f\neq 0$, so $t=0$.

case 2bbbaa $u\neq 0\neq e$, $e=xu$ and $i=xs$, $k\neq v-1$, $a=h-1$ and $x=-1$, $h^2=u^2=v^2$ and $k=-1-v$ and $s=f=0$.

We have entries $(1+b)c^2$, $(1+b)n^2$, $(1+b)q^2$ and $(1+b)r^2$.

Split into two cases :

case 2bbbaaa when $b=-1$ and **2bbbaab** when $b\neq -1$, so $r=q=n=c=0$.

case 2bbbaaa $u\neq 0\neq e$, $e=xu$ and $i=xs$, $k\neq v-1$, $a=h-1$ and $x=-1$, $h^2=u^2=v^2$ and $k=-1-v$ and $s=f=0$ and $b=-1$.

We have entries t^2u and $(1+m)^2u$, so $t=0$, $m=-1$

We get $g(c^2+2gj-n^2-q^2-r^2)$

Split into two cases : **caes 2bbbaaaa** when$g=0$ and **caes 2bbbaaab** when$g\neq 0$ and $c^2+2gj-n^2-q^2-r^2=0$.

case 2bbbaaaa $u\neq 0\neq e$, $e=xu$ and $i=xs$, $k\neq v-1$, $a=h-1$ and $x=-1$, $h^2=u^2=v^2$ and $k=-1-v$ and $s=f=t=g=0$ and $b=m=-1$.

We get nqr so at least one of $n,q,r=0$ and we have terms $j(jr+2nq)$, $j(jq+2nr)$ and $j(jn+2rq)$.

Split into two cases : **case 2bbbaaaaa** when $j=0$ and **case 2bbbaaaab** when $j\neq 0$ and at least two of $n,q,r=0$

case 2bbbaaaaa $u \neq 0 \neq e$, $e = xu$ and $i = xs$, $k \neq v - 1$, $a = h - 1$ and $x = -1$, $h^2 = u^2 = v^2$ and $k = -1 - v$ and $s = f = t = g = j = 0$ and $b = m = -1$.

We get chu.

Split into two cases: **case 2bbbaaaaaa** when $c = 0$ and **case 2bbbaaaaab** when $c \neq 0$ and $h = 0$.

case 2bbbaaaaa when $c = 0$ we get $r(q^2 + n^2)$, $q(r^2 + n^2)$ and $n(q^2 + r^2)$, so at least two of $n, q, r = 0$

When $n = q = 0$,we get $u^2 r$ so $r = 0$ SOLUTION.

When $r = q = 0$,we get $u^2 n$ so $n = 0$ SOLUTION.

When $n = r = 0$,we get $u^2 q$ so $q = 0$ SOLUTION.

case 2bbbaaaaab when $c \neq 0$ and $h = 0$, we get cu^2 CONTRADICTION.

case 2bbbaaaab $u \neq 0 \neq e$, $e = xu$ and $i = xs$, $k \neq v - 1$, $a = h - 1$ and $x = -1$, $h^2 = u^2 = v^2$ and $k = -1 - v$ and $s = f = t = g = 0$ and $b = m = -1$and $j \neq 0$ and at least two of $n, q, r = 0$ When $q = r = 0$, we get$u^2 n$, so $n = 0$

When $q = n = 0$, we get$u^2 r$, so $r = 0$

When $n = r = 0$, we get$u^2 q$, so $q = 0$

We get $c(c - j)j$ split into two cases : When $c = 0$ get ju^2, so $j = 0$ CONTRADICTION.

When $c = j$, we get $uj(2h - j)$, so $j = 2h$, we gethu^2, so $h = 0$ but $j \neq 0$ CONTRADICTION.

case 2bbbaaab $u \neq 0 \neq e$, $e = xu$ and $i = xs$, $k \neq v - 1$, $a = h - 1$ and $x = -1$, $h^2 = u^2 = v^2$ and $k = -1 - v$ and $s = f = t = 0$ and $b = m = -1$,$g \neq 0$ and $c^2 + 2gj - n^2 - q^2 - r^2 = 0$.

We have entries $(g-j)(2qr+(g+j)n)$, $(g-j)(2nr+(g+j)q)$, $(g-j)(2qn+(g+j)r)$

We have three possibolties :

case 2bbbaaab1 when $g = j \neq 0$ and

case 2bbbaaab4 when $g \neq j$ and $g = -j \neq 0$ and at least 2 of n, q, r are zero.

case 2bbbaaab2 when $g \neq j$ and $g \neq -j$ and $n = q = r = 0$

case 2bbbaaab3 when $g \neq j$ and $g \neq -j$, so$n^2 = q^2 = r^2 \neq 0$ and $(g+j)^2 = 4n^2$.

case 2bbbaaab1 $u \neq 0 \neq e$,$e = xu$ and $i = xs$,$k \neq v - 1$, $a = h - 1$ and $x = -1$, $h^2 = u^2 = v^2$ and $k = -1 - v$ and $s = f = t = 0$ and $b = m = -1$,$g \neq 0$ and $g = j \neq 0$.

We have $c^2 + 2g^2 = n^2 + r^2 + q^2$ and $g^2h + h(n^2 + q^2 + r^2) + cu^2 + gu^2$,so we get $hc^2 + 3hg^2 + (c+g)u^2$

And we have $hc^2 + 3hg^2 + (-c+3g)u^2$, subtract them to get $(2c - 2g)u^2 = 0$, so $c = g \neq 0)$

We get $2g(2gh + u^2)$, so $h = \frac{-u^2}{2g}$ and we get $(g(3g^2 - n^2 - q^2 - r^2)$ and $(g^3 - nqr)$ deduce all of n ,q, r are non zero and $3g^2 = n^2 + r^2 + q^2$.

And we have entries $g^2n - n(q^2+r^2) + gqr$, $g^2r - r(q^2+n^2) + gqn$, $g^2q - q(n^2+r^2) + gnr$.

So $g^2 + \frac{gqr}{n} = q^2 + r^2$, $g^2 + \frac{gqn}{r} = q^2 + n^2$, $g^2 + \frac{gnr}{q} = n^2 + r^2$ and (from $3g^2 = n^2 + r^2 + q^2$) get

$2g^2 = n^2 + g\frac{qr}{n} = q^2 + g\frac{nr}{q} = r^2 + g\frac{qn}{r}$

And from $g^3 = nqr$ get $\frac{qr}{n} = \frac{g^3}{n^2}$, so $g^4 - 2g^2n^2 + n^4 = 0$ and so $g^2 = n^2 = q^2 = r^2$

we get $q = yg$, $n = zg$, $r = yzg$ and $y = \pm 1$, $z = \pm 1$

Now when we put $h = \frac{-u^2}{2g}$, we get $\frac{u^3}{4g^2}(u + 2g)(u - 2g)$, so $u = 2gw$ when $w^2 = 1$

We get $6g + 2gy + vwz(y - 1)$, so $vwz(y - 1) \neq 0$, and so $y = -1$ and $v \neq 0$

We get $v - 6gwz$

But $v(2g - v)(2g + v)$ CONTRADICTION.

case 2bbbaaab4 $u \neq 0 \neq e$, $e = xu$ and $i = xs$, $k \neq v - 1$, $a = h - 1$ and $x = -1$, $h^2 = u^2 = v^2$ and $k = -1 - v$ and $s = f = t = 0$ and $b = m = -1$ and $g \neq 0$, $g \neq j$

and $g = -j \neq 0$ and at least 2 of n, q, r are zero.

We get $g^2n = g^2q = g^2r = 0$, so all of $n, q, r = 0$

We get gu^2 CONTRADICTION.

case 2bbbaaab2 $u \neq 0 \neq e$, $e = xu$ and $i = xs$, $k \neq v - 1$, $a = h - 1$ and $x = -1$, $h^2 = u^2 = v^2$ and $k = -1 - v$ and $s = f = t = 0$ and $b = m = -1$ and $g \neq 0$, $g \neq j$ and $g \neq -j$ and all three of n, q, r are zero.

We get gu^2 CONTRADICTION.

case 2bbbaaab3 $u \neq 0 \neq e$, $e = xu$ and $i = xs$, $k \neq v - 1$, $a = h - 1$ and $x = -1$, $h^2 = u^2 = v^2$ and $k = -1 - v$ and $s = f = t = 0$ and $b = m = -1$ and $g \neq 0$, $g \neq j$ and $g \neq -j$, $n^2 = q^2 = r^2 \neq 0$ and $(g + j)^2 = 4n^2$.

We put $r = zn$,$q = yn$ when $y^2 = 1$ and $z^2 = 1$

We get $(g - j)(zg + zj + 2ny)n$and $(g - j)(g + j + 2nyz)n$. Subtracting these gives$(g + j - 2ny)(z - 1) = 0$

Split into two cases:

case 2bbbaaab3a when $z = 1$ and **case 2bbbaaab3b** when $z \neq 1$ so ,$z = -1$ and $j = 2ny - g$.

case 2bbbaaab3a $u \neq 0 \neq e$, $e = xu$ and $i = xs$, $k \neq v - 1$, $a = h - 1$ and $x = -1$, $h^2 = u^2 = v^2$ and $k = -1 - v$ and $s = f = t = 0$ and $b = m = -1$ and $g \neq 0$, $g \neq j$ and $g \neq -j$, $n^2 = q^2 = r^2 \neq 0$ and $(g + j)^2 = 4n^2$ and $r = zn$, $q = yn$ when $y^2 = 1$ and $z^2 = 1$ and $z = 1$.

We get $u^2(g + ny)$, so $g = -ny$

We get $hn(n + jy)$ and $hn(n - cy)$

Split into two cases:

case 2bbbaaab3aa when $h = 0$ and **case 2bbbaaab3ab** when $h \neq 0$ so, $c \neq 0$ and $j \neq 0$ and $c = -j$.

case 2bbbaaab3aa $u \neq 0 \neq e$, $e = xu$ and $i = xs$, $k \neq v - 1$, $a = h - 1$ and $x = -1$,

$h^2 = u^2 = v^2$ and $k = -1 - v$ and $s = f = t = h = 0$ and $b = m = -1$ and $g \neq 0$, $g \neq j$ and $g \neq -j$, $n^2 = q^2 = r^2 \neq 0$ and $(g + j)^2 = 4n^2$ and $r = zn$,$q = yn$ when $y^2 = 1$ and $z^2 = 1$ and $z = 1$.

We get u^3 CONTRADICTION.

case 2bbbaaab3ab $u \neq 0 \neq e$, $e = xu$ and $i = xs$, $k \neq v - 1$, $a = h - 1$ and $x = -1$, $h^2 = u^2 = v^2$ and $k = -1 - v$ and $s = f = t = 0$ and $b = m = -1$ and $g \neq 0$, $g \neq j$ and $g \neq -j$, $n^2 = q^2 = r^2 \neq 0$ and $(g + j)^2 = 4n^2$ and $r = zn$, $q = yn$ when $y^2 = 1$ and $z^2 = 1$ and $z = 1$,$h \neq 0$, $c \neq 0$, $j \neq 0$ and $c = -j$.

We get $hj^2(3 + y^4)$ CONTRADICTION.

case 2bbbaaab3b $u \neq 0 \neq e$, $e = xu$ and $i = xs$, $k \neq v - 1$, $a = h - 1$ and $x = -1$, $h^2 = u^2 = v^2$ and $k = -1 - v$ and $s = f = t = 0$ and $b = m = -1$ and $g \neq 0$, $g \neq j$ and $g \neq -j$, $n^2 = q^2 = r^2 \neq 0$ and $(g + j)^2 = 4n^2$ and $r = zn$,$q = yn$ when $y^2 = 1$ and $z^2 = 1$ and $z \neq 1$, $z = -1$ and $j = 2ny - g$.

We get $(hn + uv)(-g + ny)$

Split into two cases:

case 2bbbaaab3ba when $g = ny$ and **case 2bbbaaab3bb** when $g \neq ny$ so ,$h = \frac{-uv}{n}$.

case 2bbbaaab3ba $u \neq 0 \neq e$, $e = xu$ and $i = xs$, $k \neq v - 1$, $a = h - 1$ and $x = -1$, $h^2 = u^2 = v^2$ and $k = -1 - v$ and $s = f = t = 0$ and $b = m = -1$ and $g \neq 0$, $g \neq j$ and $g \neq -j$, $n^2 = q^2 = r^2 \neq 0$ and $(g + j)^2 = 4n^2$ and $r = zn$, $q = yn$ when $y^2 = 1$ and $z^2 = 1$ and $z \neq 1$,$z = -1$ and $j = 2ny - g$ and $g = ny$.

We get $2nu(-v + uy)$, $n^2(c + ny)$, so $v = uy$ and $c = -ny$

We get $4n^2u$ CONTRADICTION.

case 2bbbaaab3bb $u \neq 0 \neq e$, $e = xu$ and $i = xs$, $k \neq v - 1$, $a = h - 1$ and $x = -1$, $h^2 = u^2 = v^2$ and $k = -1 - v$ and $s = f = t = 0$ and $b = m = -1$ and $g \neq 0$, $g \neq j$ and $g \neq -j$, $n^2 = q^2 = r^2 \neq 0$ and $(g + j)^2 = 4n^2$ and $r = zn$,$q = yn$ when $y^2 = 1$

and $z^2 = 1$ and $z \neq 1$,$z = -1$ and $j = 2ny - g$, $g \neq ny$ and $h = \frac{-uv}{n}$.

We get $\frac{(n-u)(n+u)u^3}{n^2}$ so $n = wu$ when $w^2 = 1$

We get $\frac{uv(gw-uy)}{w}$ so $g = \frac{uy}{w}$

We get $\frac{u^3(w-1)(1+3w+4w^2)y}{w}$ so $w = 1$

We get $-4u^3y$ CONTRADICTION.

case 2bbbaab $u \neq 0 \neq e$, $k \neq v - 1$, $e = xu$ and $i = xs$, $a = h - 1$ and $x = -1$, $h^2 = u^2 = v^2$ and $k = -1 - v$ and $s = f = r = q = n = c = 0$ and $b \neq -1$.

We get gu^2, so $g = 0$

We get $(1 + b)ju$, so $j = 0$

We get t^2u, so $t = 0$

We get $(1 + m)^2u$, so $m = -1$

We get $u(1 + b)$ COTADICTION.

case 2bbbab $u \neq 0 \neq e$, $e = xu$ and $i = xs$, $k \neq v - 1$, $a = h - 1$ and $x = -1$, $h^2 = u^2 = v^2$ and $k = -1 - v$ and $s = t = 0$ and $f \neq 0$.

We get f^2u CONTRADICTION.

case 2bbbb $u \neq 0 \neq e$, $e = xu$ and $i = xs$, $k \neq v - 1$, $a = h - 1$ and $x = -1$, $h^2 = u^2 = v^2$ and $k = -1 - v$ and $s \neq 0$ and $b = -f - 1$.

We have entry $(1 + f + m)(tu + sv)$

Split into two cases:

case 2bbbba when $m = -f - 1$ and **case 2bbbbb** when $m \neq -f - 1$ so, $t = \frac{-sv}{u}$

case 2bbbba $u \neq 0 \neq e$, $e = xu$ and $i = xs$,$k \neq v - 1$, $a = h - 1$ and $x = -1$, $h^2 = u^2 = v^2$ and $k = -1 - v$ and $s \neq 0$ and $b = -f - 1$ and $m = -f - 1$.

We have entries $s(f + t)u$ and $(f^2 - s^2)u$, so $t = -f$ and $f^2 = s^2$

We get$g(c^2 + 2gj - n^2 - q^2 - r^2)$

Split into two cases:

case 2bbbbaa when $g = 0$ and **case 2bbbbab** when $g \neq 0$ so ,$c^2+2gj-n^2-q^2-r^2 =$

0.

case 2bbbbaa $u \neq 0 \neq e$, $e = xu$ and $i = xs$,$k \neq v - 1$, $a = h - 1$ and $x = -1$, $h^2 = u^2 = v^2$ and $k = -1 - v$ and $s \neq 0$ and $b = -f - 1$ and $m = -f - 1$ $t = -f$ and $f^2 = s^2$ and $g = 0$.

We get $c^2 f$

Split into two cases:

case 2bbbbaaa when $f = 0$ and **case 2bbbbaab** when $f \neq 0$ so , $c = 0$

case 2bbbbaaa $u \neq 0 \neq e$, $e = xu$ and $i = xs$, $k \neq v - 1$, $a = h - 1$ and $x = -1$, $h^2 = u^2 = v^2$ and $k = -1 - v$ and $s \neq 0$ and $b = -f - 1$ and $m = -f - 1$ $t = -f$ and $f^2 = s^2$ and $g = f = 0$.

We get $s^2 u$ CONTRADICTION.

case 2bbbbaab $u \neq 0 \neq e$, $e = xu$ and $i = xs$, $k \neq v - 1$, $a = h - 1$ and $x = -1$, $h^2 = u^2 = v^2$ and $k = -1 - v$ and $s \neq 0$ and $b = -f - 1$ and $m = -f - 1$ $t = -f$ and $f^2 = s^2$ and $f \neq 0$ and $g = c = 0$.

We have $f(n^2 - q^2 + r^2)$ and $f(n^2 - q^2 - r^2)$. Subtracting these gives $-2q^2 f$, so $q = 0$

We have also $f(n^2 + q^2 + r^2)$ and $f(n^2 - q^2 - r^2)$. Subtracting these gives $2n^2 f$, so $n = 0$ and so $r = 0$

We get $f^2 j$, so $j = 0$ SOLUTION.

case 2bbbbab $u \neq 0 \neq e$, $e = xu$ and $i = xs$, $k \neq v - 1$, $a = h - 1$ and $x = -1$, $h^2 = u^2 = v^2$ and $k = -1 - v$ and $s \neq 0$ and $b = -f - 1$ and $m = -f - 1$ and $m = -f - 1$, $t = -f$ and $f^2 = s^2$ and $g \neq 0$ and, $c^2 + 2gj - n^2 - q^2 - r^2 = 0$.

Split into 4 cases :

case 2bbbaaab1 when $n = q = r = 0$ and **case 2bbbaaab2** + **case 2bbbaaab3** + **case 2bbbaaab4** at least one of n, r, q are non zero (two or three of n, r, q are non zero)

case 2bbbbab1 $u \neq 0 \neq e$, $e = xu$ and $i = xs$, $k \neq v - 1$, $a = h - 1$ and $x = -1$,

$h^2 = u^2 = v^2$ and $k = -1 - v$ and $s \neq 0$ and $b = -f - 1$ and $m = -f - 1$ and $m = -f - 1$ $t = -f$ and $f^2 = s^2$ and $g \neq 0$ and $c^2 + 2gj - n^2 - q^2 - r^2 = 0$ and $n = q = r = 0$.

We get $g^2 s$ CONTRADICTION.

case 2bbbbab2 $u \neq 0 \neq e$, $e = xu$ and $i = xs$, $k \neq v - 1$, $a = h - 1$ and $x = -1$, $h^2 = u^2 = v^2$ and $k = -1 - v$ and $s \neq 0$ and $b = -f - 1$ and $m = -f - 1$ and $m = -f - 1$ $t = -f$ and $f^2 = s^2$ and $g \neq 0$ and, $c^2 + 2gj - n^2 - q^2 - r^2 = 0$.

If exacitly one of n , q ,r are non zero (eg r)

We get $g^2 r$, so$r = 0$ CONTRADICTION.

case 2bbbbab3 $u \neq 0 \neq e$, $e = xu$ and $i = xs$, $k \neq v - 1$, $a = h - 1$ and $x = -1$, $h^2 = u^2 = v^2$ and $k = -1 - v$ and $s \neq 0$ and $b = -f - 1$ and $m = -f - 1$ and $m = -f - 1$ $t = -f$ and $f^2 = s^2$ and $g \neq 0$ and, $c^2 + 2gj - n^2 - q^2 - r^2 = 0$.

If exacitly two of n , q ,r are non zero (eg r , n)

We get$2(g - j)nr$, so $g = j$ and also get $nr(c + g - j)$, so $c = 0$

Then we get $n(g^2 - r^2)$ and $r(g^2 - r^2)$, so $g^2 = r^2 = n^2$

We get $g^2 s + fgn$ and $g^2 s - fgr$. Subtracting these gives $fgn = -fgr$, so $n = -r$

We get$g^2 s$ CONTRADICTION.

case 2bbbbab4 $u \neq 0 \neq e$, $e = xu$ and $i = xs$, $k \neq v - 1$, $a = h - 1$ and $x = -1$, $h^2 = u^2 = v^2$ and $k = -1 - v$ and $s \neq 0$ and $b = -f - 1$ and $m = -f - 1$ and $m = -f - 1$, $t = -f$ and $f^2 = s^2$ and $g \neq 0$ and $c^2 + 2gj - n^2 - q^2 - r^2 = 0$.

When all n , q ,r are non zero.

We have entries $(g-j)(q(g+j)+2nr)$, $(g-j)(n(g+j)+2qr)$ and $(g-j)(r(g+j)+2nq)$

Split into two cases :

case2bbbbab4a when $g = j$ and **case2bbbbab4b** when $g \neq j$

case 2bbbbab4a $u \neq 0 \neq e$, $e = xu$ and $i = xs$, $k \neq v - 1$, $a = h - 1$ and $x = -1$, $h^2 = u^2 = v^2$ and $k = -1 - v$ and $s \neq 0$ and $b = -f - 1$ and $m = -f - 1$ and $t = -f$

and $f^2 = s^2$ and $g \neq 0$ and ,$c^2 + 2gj - n^2 - q^2 - r^2 = 0$ and $g = j$ and all n , q ,r are non zero.

We get $s(gu+g^2w-ghw+n^2w+q^2w-r^2w)$ and $s(-gu+g^2w-ghw+n^2w+q^2w-r^2w)$. Subtracting these gives $2sgu$ CONTADICTION.

case 2bbbbab4b $u \neq 0 \neq e$, $e = xu$ and $i = xs$,$k \neq v - 1$, $a = h - 1$ and $x = -1$, $h^2 = u^2 = v^2$ and $k = -1 - v$ and $s \neq 0$ and $b = -f - 1$ and $m = -f - 1$ and $t = -f$ and $f^2 = s^2$ and $g \neq 0$ and ,$c^2 + 2gj - n^2 - q^2 - r^2 = 0$ and $g \neq j$ and all n , q ,r are non zero.

We have $-\frac{(g-j)}{2} = \frac{nr}{q} = \frac{nq}{r}\frac{qr}{n}$, so $r^2 = q^2 = n^2$ and $q = yn$,$r = zn$, $y^2 = 1$ and$z^2 = 1$

We get $g - j = \frac{-2nz}{y}$, so $j = g + 2nzy$

We get $4n^2yz(g + 2nzy)$, so $g = -2nyz$

We get $2nyz(-c^2 + 3n^2)$, so $c^2 = 3n^2 \neq 0$ and we also get $4cn^2$ CONTADICTION.

case 2bbbbb $u \neq 0 \neq e$, $e = xu$ and $i = xs$,$k \neq v - 1$, $a = h - 1$ and $x = -1$, $h^2 = u^2 = v^2$ and $k = -1 - v$ and $s \neq 0$ and $b = -f - 1$ and $m \neq -f - 1$ and ,$t = \frac{-sv}{u}$.

We have entries $su + v(1 + m)$ and $su - fv$ so we get $v(f + 1 + m)$, and so $v = 0$

We get s^2u CONTRADICTION.

case 2baab $u = e = 0$, $s = yi \neq 0$ and $y^2 = 1$.

We have entries $iv(f - ty)$ and $iv(1 + b - t$

Split into two cases :

case2baaba when $v = 0$ and **case2baabb** when $v \neq 0$ so $f = ty$ and$b = t - 1$

case 2baaba $u = e = v = 0$, $s = yi \neq 0$ and $y^2 = 1$.

We get $h(1+k)^2$, $(f^2-i^2)(1+k)$,$i(1+k)(b-m)$, $(1+k)(1+m-fy)$,$i(1+k)(f-ty)$ and $i(1 + k)(1 + b - t)$.

Split into two cases :

case2baabaa when $k = -1$ and **case2baabab** when $k \neq -1$ so $f = ty$,$b = t - 1$,

$h = 0$, $f = zi$ and $z^2 = 1$, $m = fy - 1$ and $b = m$.

case 2baabaa $u = e = v = 0$, $s = yi \neq 0$ and $y^2 = 1$ and $k = -1$.

We get $(1 + a - h)i^2$, so $a = h - 1$

We get $fg + fgm + ir + bir + fgt + finy$, $fg + fgm + in + bin + fgt + firy$. Subtracting these gives $i(1 + b - fy)(r - n)$

Split into two cases :

case2baabaaa when $n = r$ and **case2baabaab** when $n \neq r$ so$b = fy - 1$

case 2baabaaa $u = e = v = 0$, $s = yi \neq 0$ and $y^2 = 1$ and $k = -1$ and $a = h - 1$, $n = r$.

We have entries $gi^2 + qt^2 + 2iry + 2imry$ and $qi^2 + gt^2 + 2iry + 2imry$, subtracting these gives $(i^2 - t^2)(g - q)$

And we have $iq + biq + fr + fmr + frt + fgiy$ and $ig + big + fr + fmr + frt + fqiy$, subtracting these gives $i(1 + b - fy)(g - q)$

Split into two cases :

case2baabaaaa when $q = g$ and **case2baabaaab** when $q \neq g$ so$b = fy - 1$,$t = zi$ and $z^2 = 1$.

case 2baabaaaa $u = e = v = 0$, $s = yi \neq 0$ and $y^2 = 1$ and $k = -1$ and $a = h - 1$, $n = r$ and $q = g$.

We have entries $g^2(1 + m) + ir(-h + j + gy) + r^2t = 0$ and $g^2t + ir(-h + j + gy) + r^2(1 + m) = 0$, subtracting these gives$(g^2 - t^2)(1 + m - t) = 0$

Split into three cases :

case 2baabaaaa1 when $g = 0$ and **case 2baabaaaa2** when $g \neq 0$ so$g^2 = r^2$ and **case 2baabaaaa3** when $g \neq 0$, $g^2 \neq r^2$ and $m = t - 1$

case 2baabaaaa1 $u = e = v = g = 0$, $s = yi \neq 0$ and $y^2 = 1$ and $k = -1$ and $a = h - 1$, $n = r$ and $q = g$.

We get r^3, ij^2y and ci^2y, s0 $r = c = j = 0$

And we get $2f(1+b+f)h$

Split into three cases :

case 2baabaaaa1a when $h=0$ and **case 2baabaaaa1b** when $h \neq 0$ and$f=0$ and **case 2baabaaaa1c** when $h \neq 0$, $f \neq 0$ and $b=-f-1$

case 2baabaaaa1a $u=e=v=g=c=j=r=h=0$, $s=yi \neq 0$ and $y^2=1$ and $k=-1$ and $a=h-1$, $n=r$ and $q=g$.

SOLUTION.

case 2baabaaaa1b $u=e=v=g=r=c=j=f=0$, $s=yi \neq 0$ and $y^2=1$ and $k=-1$ and $a=h-1$, $n=r$ and $q=g$ and $h \neq 0$.

We get $hi(1+m+t)(1+y)$ Split into two cases :

case 2baabaaaa1ba when $y=-1$ and **case2baabaaaa1bb** when $y \neq -1$,so $m=-1-t$ and $y=1$.

case 2baabaaaa1ba $u=e=v=g=r=c=j=f=0$, $s=yi \neq 0$ and $y^2=1$ and $k=-1$ and $a=h-1$, $n=r$ and $q=g$ and $h \neq 0$ and $y=-1$.

We get $(1+m)^2+t^2-2i^2=0$, so $i^2=((1+m)^2+t^2)/2$

We get $h(b-m-t)(2+b+m+t)$

When $m=t-b$ SOLUTION.

When $m=-2-b-t$ SOLUTION.

case 2baabaaaa1bb $u=e=v=g=r=c=j=f=0$, $s=yi \neq 0$ and $y^2=1$ and $k=-1$ and $a=h-1$, $n=r$ and $q=g$ and $h \neq 0$, $y \neq -1$, $m=-1-t$ and $y \neq 1$.

We get $2h(t^2+i^2)$, so $t^2=-i^2$

We get $h(1+b-2i)(1+b+2i)$

When $b=2i-1$ SOLUTION.

When $b=2i+1$ SOLUTION.

case 2baabaaaa1c $u=e=v=g=r=c=j=0$, $s=yi \neq 0$ and $y^2=1$ and

$k=-1$ and $a=h-1$, $n=r$ and $q=g$, $h\neq 0$, $f\neq 0$ and $b=-f-1$

We get $2h(2f^2-i^2-t(1+m))$, so $i^2=2f^2-t(1+m)$

We get $hi(1+m+t)(1+y)$

Split into two cases :

case 2baabaaaa1ca when $y=-1$ and **case2baabaaaa1cb** when $y\neq -1$, so $m=-1-t$ and $y=1$.

case 2baabaaaa1ca $u=e=v=g=r=c=j=0$, $s=yi\neq 0$ and $y^2=1$ and $k=-1$ and $a=h-1$, $n=r$ and $q=g$, $h\neq 0$, $f\neq 0$ and $b=-f-1$ and $y=-1$

We get $h(1-2f+m+t)(1+2f+m+t)$

When $m=2f-t-1$ SOLUTION.

When $m=-2f-t-1$ SOLUTION.

case 2baabaaaa1cb $u=e=v=g=r=c=j=0$, $s=yi\neq 0$ and $y^2=1$ and $k=-1$ and $a=h-1$, $n=r$ and $q=g$, $h\neq 0$, $f\neq 0$ and $b=-f-1$, $y\neq -1$, $m=-1-t$ and $y=1$

We get $2h(i^2+t^2)$ and $2h(2f^2-i^2+t^2)$, so $f^2=i^2/2$ and $t^2=-i^2$

We get $2hi^2$ CONTRADICTION .

case 2baabaaaa2 $u=e=v=0$, $s=yi\neq 0$ and $y^2=1$ and $k=-1$ and $a=h-1$, $n=r$ and $q=g$, $g\neq 0$ and $g^2=r^2$.

We have $g^2=r^2$ so $g=xr$, when $x^2=1$

We get $iry(c-j)$, so $c=j$

We get $r(t^2+2ix(1+m)+i^2)$ and $r((i^2+t^2)x+2iy(1+m))$. Subtracting these gives $2i(1+m)(1-y)$

We get $r((1+m)^2+2itx+i^2)$ and $r(x(1+m)^2+2it+xi^2)$. Subtracting these gives $2it(1-y)+i^2(x-1)$

Split into two cases :

case 2baabaaaa2a when $m = -1$ and **case2baabaaaa2b**when $m \neq -1$, so $y = 1$ and so $x = 1$.

case 2baabaaaa2a $u = e = v = 0$, $s = yi \neq 0$ and $y^2 = 1$ and $k = -1$ and $a = h - 1$, $n = r$ and $q = g$, $g \neq 0$ and $g^2 = r^2$ and $m = -1$.

We get $i^2 + t^2 = 0$ so $t \neq 0$ and $i^2 = -t^2$

And we get $i(2tx + i)$, os $i = -2tx$ and $i^2 = 4t^2$ but $i^2 = -t^2$ CONTRADICTION.

case 2baabaaaa2b $u = e = v = 0$, $s = yi \neq 0$ and $y^2 = 1$ and $k = -1$ and $a = h - 1$, $n = r$ and $q = g$, $g \neq 0$ and $g^2 = r^2$, $m \neq -1$, $y = 1$ and $x = 1$.

We have $t^2 + 2i(1 + m) + i^2 = 0$ and $(1 + m)^2 + 2it + i^2 = 0$. Subtracting these gives$(t - 1 - m)(t + 1 + m - 2i) = 0$

Split into two cases :

case2baabaaaa2ba when $m = t - 1$ and **case2baabaaaa2bb**when $m \neq t - 1$, so $m = 2i - t - 1$.

case 2baabaaaa2ba $u = e = v = 0$, $s = yi \neq 0$ and $y^2 = 1$ and $k = -1$ and $a = h - 1$, $n = r$ and $q = g$, $g \neq 0$ and $g^2 = r^2$,$m \neq -1$, $y = 1$ and $x = 1$ and $m = t - 1$.

We get $(t + i)^2 = 0$, so $t = -i$

We get $ir(1 + b - f)$ and $ir(h - j + r)$, so $b = f - 1$ and $h = j - r$

We get fr^2 so $f = 0$.

We get $4i^2r$ CONTRADICTION.

case 2baabaaaa2bb $u = e = v = 0$, $s = yi \neq 0$ and $y^2 = 1$ and $k = -1$ and $a = h - 1$, $n = r$ and $q = g$, $g \neq 0$ and $g^2 = r^2$, $m \neq -1$, $y = 1$ and $x = 1$, $m \neq t - 1$ and $m = 2i - t - 1$.

We get $t^2 - 2it + 5i^2$, we let $t = zi$, so we will have $z^2 - 2z + 5 = 0$.

But we get $i^2r(z - 3)(1 + z)$ CONTRADICTION.

case 2baabaaaa3 $u = e = v = 0$, $s = yi \neq 0$ and $y^2 = 1$ and $k = -1$ and $a = h - 1$,

$n = r$ and $q = g$,$g \neq 0$, $g^2 \neq r^2$ and $m = t - 1$

We get $gi^2 + gt^2 + 2irty$ and $gi^2 + gt^2 + irt + irty$. Subtracting these gives $irt(y-1)$.

Split into three cases :

case 2baabaaaa3a when $r = 0$ and **case 2baabaaaa3b**when $r \neq 0$, so $y = -1$ and **case 2baabaaaa3c**when $r \neq 0$, $y \neq -1$, so $y = 1$ and $t = 0$.

case 2baabaaaa3a $u = e = v = r = 0$, $s = yi \neq 0$ and $y^2 = 1$ and $k = -1$ and $a = h - 1$, $n = r$ and $q = g$, $g \neq 0$, $g^2 \neq r^2$ and $m = t - 1$

We get $g^2 i$ CONTRADICTION.

case 2baabaaaa3b $u = e = v = 0$, $s = yi \neq 0$ and $y^2 = 1$ and $k = -1$ and $a = h - 1$, $n = r$ and $q = g$, $g \neq 0$, $g^2 \neq r^2$ and $m = t - 1$, $r \neq 0$ and $y = -1$

We get $gi^2 + 2irt + gt^2$ and $ri^2 + 2igt + rt^2$, subtracting these gives $(g-r)(i-t)^2$, so $t = i$

We get $2i^2(g + r)$, so $g = -r$ but $g^2 \neq r^2$ CONTRADICTION.

case 2baabaaaa3c $u = e = v = t = 0$, $s = yi \neq 0$ and $y^2 = 1$ and $k = -1$ and $a = h - 1$, $n = r$ and $q = g$,$g \neq 0$, $g^2 \neq r^2$ and $m = t - 1$,$r \neq 0$, $y \neq -1$ and $y = 1$.

We get $i^2 g$ CONTRADICTION.

case 2baabaaab $u = e = v = 0$, $s = yi \neq 0$ and $y^2 = 1$ and $k = -1$ and $a = h - 1$, $n = r$ and $q \neq g$, $b = fy - 1$, $t = zi$ and $z^2 = 1$.

We have entries $f(g(1 + m) + 2iry + iqz)$ and $f(q(1 + m) + 2iry + igz)$, subtracting these gives $f(1 + m - iz)(g - q)$

Split into two cases :

case 2baabaaaba when $f = 0$ and **case 2baabaaabb** when $f \neq 0$ so$m = iz - 1$

case 2baabaaaba $u = e = v = f = 0$, $s = yi \neq 0$ and $y^2 = 1$ and $k = -1$ and $a = h - 1$, $n = r$ and $q \neq g$, $b = fy - 1$,$t = zi$ and $z^2 = 1$.

We get $i(giy + ry + mry + qz + mqz + irz)$ and $i(qiy + ry + mry + qz + mqz + irz)$, subtracting these gives $(iy - z - mz)(g - q)$, so $m = iyz - 1$

We get $i^2(g + 2rz + q)$, so $g = -2rz - q$

And we get $i(gh - gj - r^2y - qrz - qryz)$ and $i(qh - qj - r^2y - grz - gryz)$. Subtracting these gives $(h - j + ryz)(g - q)$, so $h = j - ryz$

We also get $i(hr - jr - gry - q^2z - r^2yz)$ and $i(hr - jr - qry - qgz - r^2yz)$. Subtracting these gives $(qz - ry)(g - q)$, so $r = qzy$

We get $iq^2z(-1 + 2 - 2y + 1)$=$iq^2z(1 - y) = 0$

Split into two cases :

case2baabaaabaa when $y = 1$ and **case2baabaaabab** when $y \neq 1$ so$q = 0$

case 2baabaaabaa $u = e = v = f = 0$, $s = yi \neq 0$ and $y^2 = 1$ and $k = -1$ and $a = h - 1$, $n = r$ and $q \neq g$, $b = fy - 1$, $t = zi$ and $z^2 = 1$,$m = iyz - 1$,$h = j - ryz$, $g = -2rz - q$,$r = qzy$ and $y = 1$.

We get $iq(c - j + 4q)$

When $q = 0$, so $r = g = 0$ but $q \neq g$ CONTRADICTION.

When $j = c + 4q$, we get $3q(c - 9q)(c + 3q)$, split into two cases:

When $c = 9q$, we get iq^2 CONTRADICTION.

When $c = -3q$ SOLUTION.

case 2baabaaabab $u = e = v = f = 0$, $s = yi \neq 0$ and $y^2 = 1$ and $k = -1$ and $a = h - 1$, $n = r$ and $q \neq g$, $b = fy - 1$, $t = zi$ and $z^2 = 1$,$m = iyz - 1$,$h = j - ryz$, $g = -2rz - q$,$r = qzy$,$y \neq 1$,$q = 0$ and $y = -1$.

We get ij^2 and i^2c so $c = j = 0$ SOLUTION .

case 2baabaaabb $u = e = v = 0$, $s = yi \neq 0$ and $y^2 = 1$ and $k = -1$ and $a = h - 1$, $n = r$ and ,$q \neq g$, $b = fy - 1$, $t = zi$ and $z^2 = 1$, $f \neq 0$ and$m = iz - 1$.

We get $2r + qz + gyz$ and $2r + gz + qz$. Subtracting these gives $g(1 - y)$

And we get $g + qy + 2rz$ and $g + q + 2rz$. Subtracting these gives $q(1 - y)$, so $y = 1$ because $q \neq g$ and $q \neq -q - 2rzy$, so $q \neq -ryz$

We get $q(c-h+q+2rz)$, so $c=h-q-2rz$

We get $hq-jq+q^2+2r^2+2hrz-2jrz$ and $hq-jq+r^2+2rqz+2hrz-2jrz$.

Subtracting these gives $(q-zr)^2$, so $q=rz$ and $q\neq 0$ and $r\neq 0$

We get $fr(h-j+rz)$, so $j=h+rz$

We get $hi(h+8rz)$ and $hf(h-24rz)$, when $h\neq 0$. Subtracting these gives $rz=0$

CONTRADICTION.

And when $h=0$ SOLUTION.

case 2baabaab $u=e=v=0$, $s=yi\neq 0$ and $y^2=1$ and $k=-1$ and $a=h-1$, $n\neq r$ and $b=fy-1$.

We have entries $f(r+mr+nt+giy+iqy)$, $f(n+mr+rt+giy+iqy)$. Subtracting these gives $f(1+m-t)(n-r)$

Split into two cases :

case2baabaaba when $f=0$ and **case2baabaabb** when $f\neq 0$ so$m=t-1$

case 2baabaaba $u=e=v=f=0$, $s=yi\neq 0$ and $y^2=1$ and $k=-1$ and $a=h-1$,$n\neq r$ and $b=fy-1$.

We get $r(1+m)^2+ity(g+q)+i^2n=0$ and $n(1+m)^2+ity(g+q)+i^2r=0$. Subtracting these gives $(1+m)^2=i^2$, so $m=xi-1$ when $x^2=1$

We get $i^2r+nt^2+i^2xy(g+q)$ and $i^2n+rt^2+i^2xy(g+q)$. Subtracting these gives $(n-r)(i^2-t^2)$ so $t=iz$ when $z^2=1$

We get $i^2((g+q)+z(n+r))=0$, so $g=z(n+r)-q$

We get $2i^2(n+r)=0$, so$n=-r$

We get $iq(c-h-rx+qy+rz)$ // Split into two cases :

case 2baabaabaa when $q=0$ and **case 2baabaabab** when $q\neq 0$ so$c=h+rx-qy-rz$

case 2baabaabaa $u=e=v=f=q=0$, $s=yi\neq 0$ and $y^2=1$ and $k=-1$ and $a=h-1$,$n\neq r$ and $b=fy-1$.

We get ir^2z, so $r = 0$ but $n \neq r$ and $n = -r$, so $n = -r = 0$ CONTRADICTION.

case 2baabaabab $u = e = v = f = 0$, $s = yi \neq 0$ and $y^2 = 1$ and $k = -1$ and $a = h - 1$,$n \neq r$, $b = fy - 1$, $q \neq 0$ and $c = h + rx - qy - rz$.

We have $iqr(x - z)$ and $(q^2 + r^2)(x - z)$, so when $r = 0$ we get $x = z$ and when $r \neq 0$, we get $x = z$, so $x = z$ in all cases.

We get $(rz + q)(1 - y) = 0$

Split into two cases :

case 2baabaababa when $y = 1$ and **case 2baabaababb** when $y \neq 1$ so$y = -1$ and $r = -qz$

case 2baabaababa $u = e = v = f = 0$, $s = yi \neq 0$ and $y^2 = 1$ and $k = -1$ and $a = h - 1$, $n \neq r$, $b = fy - 1$, $q \neq 0$ and $c = h + rx - qy - rz$, $x = z$ and $y = 1$.

We get $h(h - 4q)q$

Split into two cases :

case 2baabaababaa when $h = 0$ and **case 2baabaababab** when $h \neq 0$ so$h = 4q$

case 2baabaababaa $u = e = v = f = h = 0$, $s = yi \neq 0$ and $y^2 = 1$ and $k = -1$ and $a = h - 1$,$n \neq r$, $b = fy - 1$, $q \neq 0$ and $c = h + rx - qy - rz$,$x = z$ and $y = 1$.

We get $i^2(3j - q)z$, so $q = 3j$

We get $ij^2 = 0$, so $j = 0$

We get $ir^2 = 0$ CONTRADICTION.

case 2baabaababab $u = e = v = f = 0$, $s = yi \neq 0$ and $y^2 = 1$ and $k = -1$ and $a = h - 1$, $n \neq r$, $b = fy - 1$, $q \neq 0$ and $c = h + rx - qy - rz$,$x = z$ and $y = 1$, $h \neq 0$ and $h = 4q$.

We get $8i(j - q)q$, so $j = q$

We get $-8iq^2$ CONTRADICTION.

case 2baabaababb $u = e = v = f = 0$, $s = yi \neq 0$ and $y^2 = 1$ and $k = -1$ and $a = h - 1$, $n \neq r$, $b = fy - 1$, $q \neq 0$ and $c = h + rx - qy - rz$, $y \neq 1$,$x = z$,$y = -1$

and $r = -qz$.

We get $2iq(1+z^2) = 4iq = 0$ CONTRADICTION.

case 2baabaabb $u = e = v = 0$, $s = yi \neq 0$ and $y^2 = 1$ and $k = -1$ and $a = h - 1$,$n \neq r$ and $b = fy - 1$,$f \neq 0$ and $m = t - 1$.

We get $i^2r + nt^2 + gity + iqty$ and $i^2n + rt^2 + gity + iqty$. Subtracting these gives $(n-r)(t^2 - i^2)$, so$t^2 = i^2$, $t = zi$ when $z^2 = 1$

We get $q+g+zn+zr$ and $q+g+zny+zry$. Subtracting these gives $(y-1)(n+r)z = 0$

Split into two cases :

case2baabaabba when $y = 1$ and **case2baabaabbb** when $y \neq 1$ so $n = -r$ and $y = -1$.

case 2baabaabba $u = e = v = 0$, $s = yi \neq 0$ and $y^2 = 1$ and $k = -1$ and $a = h-1$, $n \neq r$ and $b = fy - 1$, $f \neq 0$ and $m = t-1$, $t = zi$, $z^2 = 1$ and $y = 1$.

We have entry $fi(g + q + nz + rz)$, so $g = -q - nz - rz$

We get$ir(c - h + q + nz + rz)$, so$h = c + q + nz + nr$

We get $i(c - 2j - q - nz - rz)(c + q + nz + rz)$

Split into two cases :

case2baabaabbaa when $c = 2j + q + nz + rz$ and **case2baabaabbab** when $c = -q - nz - rz$.

case 2baabaabbaa $u = e = v = 0$, $s = yi \neq 0$ and $y^2 = 1$ and $k = -1$ and $a = h - 1$,$n \neq r$ and $b = fy - 1$,$f \neq 0$ and $m = t - 1$, $t = zi$, $z^2 = 1$ and $y = 1$, $g = -q - nz - rz$, $h = c + q + nz + nr$ and $c = 2j + q + nz + rz$.

We get $i(j+q+nz+rz)(j+2q+2nz+3rz)$ and $i(j+q+nz+rz)(j+2q+3nz+2rz)$

Split into two cases :

case2baabaabbaaa when $j = -q - nz - rz$ and **case2baabaabbaab** when $j + 2q + 2nz + 3rz = 0$ and $j + 2q + 3nz + 2rz = 0$.

case 2baabaabbaaa $u = e = v = 0$, $s = yi \neq 0$ and $y^2 = 1$ and $k = -1$ and

$a = h - 1$,$n \neq r$ and $b = fy - 1$,$f \neq 0$ and $m = t - 1$, $t = zi$, $z^2 = 1$ and $y = 1$, $g = -q - nz - rz$, $h = c + q + nz + nr$ and $c = 2j + q + nz + rz$ and $j = -q - nz - rz$.
We get $i(n^2 - q^2 - nr - r^2 - nqz - qrz)$ and $i(n^2 + q^2 + nr - r^2 + nqz + qrz)$, subtracting these gives $n^2 - r^2 = 0$, so $n = -r$

We get $4f^2q$ and $i(q^2 - r^2)$ and we have ,$n \neq r$, so $r \neq 0$ CONTRADICTION.

case 2baabaabbaab $u = e = v = 0$, $s = yi \neq 0$ and $y^2 = 1$ and $k = -1$ and $a = h - 1$, $n \neq r$ and $b = fy - 1$, $f \neq 0$ and $m = t - 1$, $t = zi$, $z^2 = 1$ and $y = 1$, $g = -q - nz - rz$, $h = c + q + nz + nr$ and $c = 2j + q + nz + rz$.

We have entries $j + 2q + 2nz + 3rz = 0$ and $j + 2q + 3nz + 2rz = 0$. Subtracting these gives $z(n - r) = 0$, so$n = r$ CONTRADICTION.

case 2baabaabbab $u = e = v = 0$, $s = yi \neq 0$ and $y^2 = 1$ and $k = -1$ and $a = h - 1$, $n \neq r$ and $b = fy - 1$, $f \neq 0$ and $m = t - 1$, $t = zi$, $z^2 = 1$ and $y = 1$, $g = -q - nz - rz$, $h = c + q + nz + nr$ and $c = -q - nz - rz$.

We get $i(n^2 - jq + nr - r^2)$ and $i(-n^2 - jq + nr + r^2)$. Subtracting these gives $n^2 - r^2 = 0$, so $n = -r$

We get $f^2(3j - q)$, so $q = 3j$

We get $8fjr = 0$ and $2f(3j^2 + r^2) = 0$, so $j = 0$ but$f \neq 0$ and $r \neq= 0$ CONTRADICTION.

case 2baabaabbb $u = e = v = 0$, $s = yi \neq 0$ and $y^2 = 1$ and $k = -1$ and $a = h - 1$, $n \neq r$ and $b = fy - 1$, $f \neq 0$ and $m = t - 1$, $t = zi$, $z^2 = 1$, $n = -r$ and $y = -1$.

We get $fi(g + q)$, so $g = -q$

We get $f(h - j + q)r$, so $j = h + q$

We get$2f(q^2 + r^2)$ and $2f(q^2 - r^2)$. Subtracting these gives $-2r^2$ CONTRADICTION.

case 2baabab $u = e = v = 0$, $s = yi \neq 0$ and $y^2 = 1$, $k \neq -1$, $f = ty$, $b = t - 1$, $h = 0$, $f = zi$ and $z^2 = 1$, $m = fy - 1$ and $b = m$.

We get $i^2(a-k)yz$ and $i(1+k)(-t+iyz)$, so $a=k$ and $t=iyz$

We get $i(cn+jry+j^2yz+jqyz)$ and $i(cr+jny+j^2yz+jqyz)$. Subtracting these gives $(r-n)(jy-c)$

Split into two cases :

case2baababa when $r=n$ and **case2baababb** when $r \neq n$ so $c=jy$.

case 2baababa $u=e=v=0$, $s=yi \neq 0$ and $y^2=1$, $k \neq -1$, $f=ty$,$b=t-1$, $h=0$, $f=zi$ and $z^2=1$, $m=fy-1$ and $b=m$, $r=n$,$a=k$ and $t=iyz$.

We get $g^2+ci^2+3i^2j+g^k$ and $g^2+ci^2+i^2j+g^k+2i^2jy$ Subtracting these gives $2i^2j(1-y)$.

Split into two cases :

case2baabababaa when $y=1$ and **case2baabababab** when $y \neq 1$ so $y=-1$ and $j=0$.

case 2baabababaa $u=e=v=0$, $s=yi \neq 0$ and $y^2=1$, $k \neq -1$, $f=ty$,$b=t-1$, $h=0$, $f=zi$ and $z^2=1$, $m=fy-1$ and $b=m$, $r=n$,$a=k$, $t=iyz$ and$y=1$

.

We get$g^2+ci^2+3i^2j+g^2k$, $n^2+ci^2+3i^2j+n^2k$ and $q^2+ci^2+3i^2j+q^2k$. Subtracting these gives$(1+k)(n^2-q^2)$ and $(1+k)(n^2-g^2)$, so $q^2=n^2=g^2$

And we also have $n(-g^+n^2-cq-gq+jq+q^2)$ and$n(-j^+n^2-cq-gq+jq+q^2)$. Subtracting these gives $n(j^2-g^2)$. Split into two cases :

case2baabababaaa when $n=0$, so $q=n=g=0$ and **case2baabababaab** when $n \neq 0$ so $q^2=n^2=g^2=j^2 \neq 0$.

case 2baabababaaa $u=e=v=q=n=g=0$, $s=yi \neq 0$ and $y^2=1$, $k \neq -1$, $f=ty$,$b=t-1$, $h=0$, $f=zi$ and $z^2=1$, $m=fy-1$ and $b=m$, $r=n$,$a=k$, $t=iyz$ and$y=1$.

We get j^2i and c^2i so $j=c=0$ SOLUTION.

case 2baabababaab $u=e=v=0$, $s=yi \neq 0$ and $y^2=1$, $k \neq -1$, $f=ty$

$,b = t-1$, $h=0$, $f = zi$ and $z^2 = 1$, $m = fy-1$ and $b = m$, $r = n$,$a = k$, $t = iyz$ and$y = 1$,$n \neq 0$ and $q^2 = n^2 = g^2 = j^2 \neq 0$.

We have entry $i(gj+2n^2+q^2)z$ and because $q^2 = n^2 = g^2 = j^2 \neq 0$ we get $ig(j+3g)z$, so $j = -3g$

We get $gn(g-q)$ and $g^2(5c+12g+3q)$ so, $q = g$ and $c = -3g$

We get $gi(g-nz)$ so $g = nz$

We get $g^2 + gk - 12i^2nz$ and $g^2 + gk + 4i^2nz$ Subtracting these gives $16i^2nz$ CONTRADICTION.

case 2baababab $u = e = v = 0$, $s = yi \neq 0$ and $y^2 = 1$, $k \neq -1$, $f = ty$,$b = t-1$, $h = 0$, $f = zi$ and $z^2 = 1$, $m = fy-1$ and $b = m$, $r = n$,$a = k$, $t = iyz$,$y \neq 1$, $y = -1$ and $j = 0$.

We get ci^2 , so $c = 0$

We get $g^2(1+k)$, $q^2(1+k)$ and $n^2(1+k)$, so $g = q = n = 0$ SOLUTION.

case 2baababb $u = e = v = 0$, $s = yi \neq 0$ and $y^2 = 1$, $k \neq -1$, $f = ty$,$b = t-1$, $h = 0$, $f = zi$ and $z^2 = 1$, $m = fy-1$ and $b = m$,$a = k$ and $t = iyz$, $r \neq n$ and $c = jy$.

We get $n^2 + jyi^2 + 3i^2j + n^2k$ and $r^2 + jyi^2 + 3i^2j + r^2k$ Subtracting these gives $(1+k)(n-r)(n+r)$, so $r = -n$

We get $(1+k)(n^2-j^2) = 0$, $gi^2 + j^2 + j^2k + i^2q$ and $gi^2 + q^2 + q^2k + i^2q$ Subtracting these gives $(1+k)(j^2-q^2) = 0$,so $j^2 = q^2 = n^2$.

We also get $ij(j+q)$ and $ij(3g+jy)z$

Split into two cases :

case2baababba when $j = 0$, so $q = n = 0$ and **case2baababbb** when $j \neq 0$ so $q^2 = n^2 = j^2 \neq 0$,$j = -3gy$ and $q = -j$.

case 2baababba $u = e = v = j = q = n = 0$, $s = yi \neq 0$ and $y^2 = 1$, $k \neq -1$, $f = ty$,$b = t-1$, $h = 0$, $f = zi$ and $z^2 = 1$, $m = fy-1$ and $b = m$,$a = k$ and

$t = iyz$, $r \neq n$ and $c = jy$, $r = -n$.

We get $g^2 i$, so $g = 0$ SOLUTION.

case 2baabababbb $u = e = v = 0$, $s = yi \neq 0$ and $y^2 = 1$, $k \neq -1$, $f = ty$,$b = t-1$, $h = 0$, $f = zi$ and $z^2 = 1$, $m = fy - 1$ and $b = m$,$a = k$ and $t = iyz$, $r \neq n$ and $c = jy$,$r = -n$, $j \neq 0$, $q^2 = n^2 = j^2 \neq 0$,$j = -3gy$ and $q = -j$.

We get $2g^2 n$, so $g = 0$ and $j = 0$ but we have $q^2 = n^2 = j^2 \neq 0$ CONTRADICTION.

case 2baabb $u = e = 0$, $s = yi \neq 0$ and $y^2 = 1$ and $v \neq 0$, $f = ty$ and$b = t - 1$.

We have entries $i(1 + m - t)v$ and $vy(i^2 - t^2)$, so $m = t - 1$,$t = zy$ and $y^2 = 1$

And we also have entries $h^2(1 + k) + v^2 h = 0$ and $h^2(1 + k) - (1 + k)^2 v + v^2 h = 0$.

Subtracting these gives $(1 + k)^2 v$, so $k = -1$

We get hv^2 , so $h = 0$

We get $i^2(1 + a - v)yz$ so $a = v - 1$

We get $gnv + i^2ny + i^2ry + gi^2z + i^2qz$ and $grv + i^2ny + i^2ry + gi^2z + i^2qz$, subtracting these gives $gv(n - r) = 0$

Split into two cases :

case2baabba when $g = 0$ and **case 2baabbb** when $g \neq 0$ so $n = r$.

case 2baabba $u = e = 0$, $s = yi \neq 0$ and $y^2 = 1$ and $v \neq 0$, $f = ty$ and$b = t - 1$, $m = t - 1$,$t = zy$ and $y^2 = 1$, $k = -1$ and $h = g = 0$,$a = v - 1$.

We have entries $invyz$, $irvyz$, $iqvyz$ so $n = r = q = 0$

We get $icyz$, so $c = 0$

We get $(c - j)ivyz$, so $j = 0$ SOLUTION.

case 2baabbb $u = e = 0$, $s = yi \neq 0$ and $y^2 = 1$ and $v \neq 0$, $f = ty$ and$b = t - 1$, $m = t - 1$,$t = zy$ and $y^2 = 1$, $k = -1$ and $h = 0$,$a = v - 1$,$g \neq 0$ and $n = r$.

We get $ci^2 + 3i^2j + g^2v$, $ci^2 + 3i^2j + q^2v$ and $ci^2 + 3i^2j + r^2v$. Subtracting these gives $r^2 = q^2 = g^2 \neq 0$

We get $i^2r + jrv + i^2ry + gi^2z + iqyz$ and $i^2r + qrv + i^2ry + gi^2z + iqyz$. Subtracting these gives $vr(j-q)$, so $j=q$ and $j\neq 0$

We get $ij(3j+cy)$ and $ij(3j+gy)$. Subtracting these gives $c=g$

We get $ij(g+3jy)$, so $g=-3jy$

We get $2j^2(1-y)$, so $y=1$

We get j^2v CONTRADICTION.

Appendix B

We assume $1+a=x$, $1+b=y$, $1+m=z$ and $1+k=w$, so $a=x-1$, $b=y-1$

,$m=z-1$ and $k=w-1$

We get $sx+u(f+t+z)$

Split into two cases:

case A when $u=s=0$ and **case B** when $u=s^{*}\neq 0$

case A when $u=s=0$

We get $xy=1$ and we have $x=\overline{y}$ and $y=\frac{1}{x}$, so $\overline{x}=\frac{1}{x}$, and so $lxl^2=1$ and$x\neq 0$.

Split into two cases:

case A1 when $v=t=0$ and **case A2** when $v=t^{*}\neq 0$.

case A1 $u=s=t=v=0$ and $y=\frac{1}{x}$

We get fw and hz

Split into two cases:

case A1a when $h=f=0$, because $f=h^{*}$ and **case B2b** when $h\neq 0 and f\neq 0$,

$w=z=0$, because $w=z^{*}$

case A1a $u=s=t=v=h=f=0$

We get $c^2+3gj=0$, and we have $cisrealandg=j^{*}$, so $c=g=j=0$

We get $n^2+r^2+q^2=0$, so $n=r=q=0$, because all them are real

We get $wz=1$,so $z=\frac{1}{z}$

We get ie , so $i=e=0$ because $i=e^{*}$ SOLUTION.

case A1b $u=s=t=v=w=z=0$ and $h\neq 0 and f\neq 0$

We get $fh=1$,so $f=\frac{1}{h}$

We get $3+c^2+3gj=0$ but c is real and $g=j^{*}$ CONTRADICTION.

case A2 $u=s=0$ and $v=t^{*}\neq 0$ and $y=\frac{1}{x}$

We get $c^2+3fh+3gj=0$ and we have c is real and $g=j^{*}$ and $f=h^{*}$, so

$c = g = j = f = h = 0$

We get $tw = 0$and$vz = 0$, so $z = w = 0$

We get $tv = 1$, so $t = \frac{1}{v}$

We get $n^2 + r^2 + q^2 = 0$, so $n = r = q = 0$ because all them are real.

We get $iv = 0$and $\frac{e}{v} = 0$, so $i = e = 0$ SOLUTION.

case B $u = s^* \neq 0$

We have entries

$A = -1 + fh + es + tv + wz = 0.$

$B = -1 + gj + n^2 + q^2 + r^2 + es + iu + tv + wz = 0.$

$D = -1 + c^2 + 3fh + 3gj + xy = 0.$

$E = -1 + 3iu + xy = 0.$

When we use $[D - 3(A - B) - E] = 0$, we get $c^2 + 6gj + 3n^2 + 3q^2 + 3r^2 = 0$, so $c = n = r = q = n = g = j = 0$ because c , r, q , n are real and $g = j^*$

We get $-1 + es + iu + tv + wz$ and $-1 + fh + es + tv + wz$ .Subtract these gives $-fh + iu = 0$

Split into two cases:

case B1 when $f = h = 0$ and **case B2** when$fh \neq 0$.

case B1 $u = s^* \neq 0$ and $c = n = r = q = n = g = j = f = h = 0$.

We get $i = e = 0$

We get $tw = 0$, $vz = 0$ and $su + tw + vz = 0$, so $su = 0$ CONTRADICTION.

case B2 $u = s^* \neq 0$ and $c = n = r = q = n = g = j = 0$ and $fh \neq 0$.

We get $f = \frac{iu}{h} \neq 0$.

We get $\frac{iu(2h+x)}{h} + hy = 0$ and $\frac{iux}{h} + \frac{ehs\left(\frac{2iu}{h}+y\right)}{iu} = 0$, when we solve them get$x = 0, y = -\frac{2iu}{h}$, but $x = y^*$ CONTRADICTION .