Subordination in the sense of Bochner of variable order.

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Abstract

In this thesis we consider subordination (in the sense of Bocnher) of variable order. This work extends previously known results related to operators of variable (fractional) order of differentiation, or variable order fractional powers. The first main result gives a formal backround to the proof that for certain classes of negative definite symbols $q(x, \xi)$ and state space dependent Bernstein functions $f(x, s)$ the pseudo-differential operator $-p(x, D)$ with symbol $-f(x, q(x, \xi))$ extends to the generator of a Feller semigroup. A new concrete example is given. The final result improves upon this result. This is achieved by proving the crucial estimates previously assumed for a large class of symbols and Bernstein functions.

Contents

Notation

$\mathbb{N}$ natural numbers
$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$
$\mathbb{N}_0^n$ set of all multiindicies
$\mathbb{N}_{0,\gamma}^n = \{\beta \in \mathbb{N}_0^n \,||\beta| > 0 \; and \; 0 < |\beta| \leq \gamma\}$
$\mathbb{Z}$ integers
$\mathbb{R}$ real numbers
$\mathbb{R}_+ = \{x \in \mathbb{R} \,| x \geq 0\}$
$\mathbb{R}^n$ euclidean vector space
$\mathbb{C}$ complex numbers
$\mathbb{C}^n$ unitary vector space
$\bar{A}$ closure of a set
$A \setminus B$ set theoretical difference of two sets
$\alpha! = \alpha_1! \ldots \alpha_n!, \;\; \alpha \in \mathbb{N}_0^n$
$x^\alpha = x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}$, $\alpha \in \mathbb{N}_0^n$ and $x \in \mathbb{R}^n$
$\partial^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$
$D^\alpha u = (-i\partial)^\alpha u$
$sign\ a = \begin{cases} \frac{a}{|a|}, a \neq 0 \\ 0, a = 0 \end{cases}$ signum of a
$a \wedge b = min(a, b)$
$a \vee b = max(a, b)$
$diagA = \{(x, x) : x \in A\}$
$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \cdot \ldots \cdot \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix}$ for $\alpha, \beta \in \mathbb{N}_0^n$

χ_A characteristic function of the set A
u^+ positive part of u, i.e. $u^+ = u \vee 0$
u^- negative part of u, i.e. $u^- = -(u \vee 0)$
$Re\ f$ real part of a function
$Im\ f$ imaginary part of a function
$(f_\nu)_{\nu \in \mathbb{N}}$ sequence of functions
$f \circ g$ composition of functions
$f * g$ convolution of functions
$\hat{u},\ Fu$ Fourier transform
$F^{-1}u$ inverse Fourier transform
$supp\ u$ support of a function

$B^{(n)}$ Borel sets in $\mathbb{R}^n$
$\sigma(S)$ σ-field generated by S
$\lambda^{(n)}$ Lebesgue measure in $\mathbb{R}^n$

ϵ_a Dirac measure at $a \in \mathbb{R}^n$
ϵ_0 Dirac measure at $0 \in \mathbb{R}^n$
$\mu_1 \otimes \mu_2$ product of the measures μ_1 and μ_2
$\mu_1 * \mu_2$ convolution of the measures μ_1 and μ_2
$||\mu||$ total mass of a measure μ
$supp\ \mu$ support of a measure μ
$(\mu_t)_{t\geq 0}$ convolution semigroup of subprobabilities
$(\mu_t^f)_{t\geq 0}$ subordinate convolution semigroup
$\int_{a+}^{b} = \int_{(a,b]}$

$B(\Omega)$ Borel measurable function
$C(G)$ continuous functions
$C_0(G)$ continuous functions with compact support
$C_\infty(G)$ continuous functions vanishing at infinity
$C_b(G)$ bounded continuous functions
$C^m(G)$ m-times continuously differentiable functions
$C_0^m(G) = C^m(G) \cap C_0(G)$
C^∞ arbitrarily often differentiable functions
C_0^∞ arbitrarily often differentiable functions with compact support
$L^p(\Omega, \mu)$ space of μ-measurable functions f such that $|f|^p$ is integrable
$H^{\psi,s}(\mathbb{R}^n) = \{u \in S'(\mathbb{R}^n); ||u||_{\psi,s} < \infty\}$
$S(\mathbb{R}^n)$ Schwartz space
$S'(\mathbb{R}^n)$ the dual space of $S(\mathbb{R}^n)$ (tempered distributions)

$|x|$ Euclidean distance in $\mathbb{R}^n$
$|x|_\infty = \max\{|x_1|, \ldots, |x_n|\}$, $x \in \mathbb{R}^n$
$|z|$ Euclidean distance in $\mathbb{C}^n$
$||u||_X$ norm of u in the space X
$||u||_{A,X} = ||u||_X + ||Au||_X$ graph norm with respect to the operator A
$||A|| = ||A||_{X,Y}$ operator norm of the operator A
$||u||_0$, $(u,u)_0$ norm and scalar product in $L^2(\Omega, \mu)$
$||u||_\infty = sup|u(x)|$
$||u||_{\psi,s}$ norm in the space $H^{\psi,s}(\mathbb{R}^n)$
$p_{m_1,m_2}(u) = \sup_{x\in\mathbb{R}^n}((1+|x|^2)^{\frac{m_1}{2}} \sum_{|\alpha|\leq m_2} |\partial^\alpha u(x)|)$
$p_{\alpha,\beta}(u) = \sup_{x\in\mathbb{R}^n} |x^\beta \partial^\alpha u(x)|$

$X \hookrightarrow Y$ continuous embedding of X into Y
$B(X)$ bounded linear operators from X into itself
$(X, ||\cdot||_X)$ Banach space X with norm $||\cdot||_X$
X^* dual space of a topological vector space

$< u,x >$ duality pairing between X^* and X
$(A, D(A))$ linear operator with domain $D(A)$
$D(A)$ domain of an operator
$R(A)$ range of an operator
$\bar{A}$ closure of an operator
$\rho(A)$ resolvent set of an operator
$\rho_+(A) = \rho(A) \cap (0, \infty)$
$(R_\lambda)_{\lambda \geq 0}$ resolvent of an operator
$R_\lambda = (\lambda - A)^{-1}$
$(B, D(B))$ bilinear form with domain $D(B)$
$B(u, v)$ bilinear form
$B_\lambda(u, v) = B(u, v) + \lambda(u, v)_0$
B^{sym} symmetric part of a bilinear form B
B^{asym} antisymmetric part of a bilinear form B
$q(x, D)$ pseudo-differential operator with symbol $q(x, \xi)$
$\psi(D)$ pseudo-differential operator with symbol $\psi(\xi)$
$(T_t)_{t \geq 0}$ one parameter semigroup of operators
$(T_t^*)_{t \geq 0}$ adjoint semigroup of $(T_t)_{t \geq 0}$
$(T_t^f)_{t \geq 0}$ subordinated semigroup
A^f generator of subordinated semigroup
$(T_t^{(\infty)})_{t \geq 0}$ semigroup on C_∞
$A^{(\infty)}$ generator of $(T_t^{(\infty)})_{t \geq 0}$
$(T_t^{(p)})_{t \geq 0}$ semigroup on $L^p(\mathbb{R}^n)$, $1 < p < \infty$
$A^{(p)}$ generator of $(T_t^{(p)})_{t \geq 0}$

Introduction

The main purpose of this thesis is to investigate subordination (in the sense of Bochner) of variable order. S. Bochner introduced a method called subordination which was used to obtain a new process from a given one by a random time change. His original papers are [5] and [6]. We will however, study subordination using an analytic approach by using the books of Chr. Berg and G. Forst [4] and N. Jacob [20]. There is a long history of constructing a functional calculus for generators of subordinate semigroups, with the first general results obtained by R. S. Phillips [34]. Many calculi for this topic have been proposed, however, we only mention the paper of Chr. Berg, Kh. Boyadzhiev and R. de Laubenfels [3], J. Faraut [10], the monograph of R. de Laubenfels [8] and the papers of R. L. Schilling [35] and [36]. We should note that F. Hirsch [15]-[16] had obtained related results prior to this. The representation of fractional powers of generators is the the best known result, this is due to A. V. Balakrishnan [1], see also K. Yosida [39], M. A. Krasnosel'skii et al. [26] and V. Nollau [32]-[33]. At the root of our work is the result that for a continuous negative definite function ψ and a Bernstein function f, $f\circ\psi$ is also a continuous negative definite function. Further, it is already known that for one parameter semigroups, the subordinate semigroup is given by

$$T_t^f u = \int_{\mathbb{R}^n} T_s u \eta_t(ds).$$

where the convolution semigroup $(\eta_t)_{t\geq 0}$ supported by $[0,\infty)$ is linked to f by

$$\mathcal{L}(\eta_t)(x) := \int_0^\infty e^{-xs}\eta_t(ds) = e^{-tf(x)},\ x>0 \ \ and \ \ t>0.$$

It is known that if T_t is a Feller semigroup on $C_\infty(\mathbb{R}^n)$ then T_t^f is also a Feller semigroup on $C_\infty(\mathbb{R}^n)$. Similarly if T_t is a sub-Markovian semigroup on $L^p(\mathbb{R}^n)$ then T_t^f is also a sub-Markovian semigroup on $L^p(\mathbb{R}^n)$. For the translation invariant case we have clear results for the generators of Feller and sub-Markovian semigroups and for the subordinate case.

Subordination has also been studied on the level of pseudo-differential operators. It is illustrated in the thesis that under certain conditions the pseudo-differential operator $-p(x,D)$ with symbol $-f(q(x,\xi))$, where f is a Bernstein function and q belongs to Hoh's symbol class, extends to the generator of a Feller semigroup. A similar result is shown for the sub-Markovian case. This result is not totally unexpected since we know that the function

$$\xi \to f(q(x,\xi))$$

is a continuous negative definite function provided $\xi \longrightarrow q(x,\xi)$ is a continuous negative definite function, hence the pseudo-differential operator $-(f \circ q)(x,D)$ with symbol $-f(q(x,\xi))$ is a candidate for an operator having an extension generating an L^p-sub-Markovian or Feller semigroup.

This now leads us to the next step, i.e. subordination of variable order. By subordination of varibale order we mean the case when we replace a fixed Bernstein function f by a family of Bernstein functions, $f(x,\cdot)$ depending on x. Pseudo-differential operators of variable order of differentiation have already been studied by A. Unterberger and J. Bokobza [40], and in particular by H.-G. Leopold [27], [28]. Feller semigroups obtained from the symbol $(1+|\xi|^2)^{r(x)}$ have been studied by N. Jacob and H.-G. Leopold [23], where further work is due to A. Negoro [31], in particular to K. Kikuchi and A. Negoro [24], [25]. It should also be noted that Weyl-Hörmander calculus can be used to consider operators of variable order of differentiation, see F. Baldus [2]. Moreover Hoh, [19] has shown that when

$$f(x,q(x,\xi)) = (q(x,\xi))^{m(x)}$$

where $0 \leq m(x) \leq 1$, then under certain conditions the pseudo-differential operator

$$-p(x,D)u = -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}(q(x,\xi))^{m(x)}\hat{u}(\xi)d\xi$$

extends to the generator of a Feller semigroup.

The aim of this thesis is to extend these ideas; we want to enlarge the class of examples and obtain a general proof showing that the pseudo-differential $-p(x,D)$ with symbol $-f(x,q(x,\xi))$ extends to the generator of a Feller semigroup. The process of subordination of variable order may also be described as considering pseudo-differential operators of variable order of differentiation as generators of semigroups.

In this work we first meet subordination of variable order in chapter four. Here we give a formal backround to the proof that $-p(x,D)$ generates a Feller semigroup. The aim is to use the Hille-Yosida-Ray theorem to prove this result. The majority of the work in the proof comes from solving the equation $p_{\lambda_0}(x,D)u = g$ which is one of the conditions of the Hille-Yosida-Ray theorem. The essence of the proof is as follows. We first assume

$$\inf_{y\in\mathbb{R}^n} f(y,s) \geq f_0(s) \text{ for all } s \in [0,\infty)$$

and

$$\sup_{y\in\mathbb{R}^n} f(y,s) \leq f_1(s) \text{ for all } s \in [0,\infty)$$

where f_0 and f_1 are Bernstein functions. Further, when $q(x,\xi)$ is comparable with a fixed continuous negative definite function ψ, we find

$$p(x,\xi) \leq f(x,q(x,\xi)) \leq c_1 f_1(\psi(\xi)) = \psi_1(\xi)$$

and similarly

$$p(x,\xi) \geq f(x,q(x,\xi)) \geq c_0 f_0(\psi(\xi)) = \psi_0(\xi).$$

Therefore, we have continuous negative definite functions ψ_0 and ψ_1 as lower and upper bounds for $p(x,\xi)$. Under further assumptions for ψ, ψ_0, ψ_1 and f we get the following embedding results:

$$H^{\psi_1,1}(\mathbb{R}^n) \hookrightarrow H^{\psi_0,1}(\mathbb{R}^n)$$

and

$$H^{\psi_0,m(1+\sigma)}(\mathbb{R}^n) \hookrightarrow H^{\psi_1,m}(\mathbb{R}^n)$$

for $m \geq 0$. These results become crucial further in the proof. We now assume p belongs to the class $S_\rho^{2+\tau_1,\psi_1}(\mathbb{R}^n)$ (for Hoh's symbol class see (2.2)), for some appropriate $\tau \geq 0$. This allows us to derive some important continuity estimates for $p(x,D)$. In order to solve the equation $p_{\lambda_0}(x,D)u = g$ we assume that for the bilinear form $B(u,v) = (p(x,D)u,v)_0$

$$|B(u,v)| \leq \kappa ||u||_{\psi_1,1}||v||_{\psi_1,1}, \ \kappa \geq 0$$

and

$$B(u,u) \geq \gamma ||u||_{\psi_0,1}^2 - \lambda_0 ||u||_0^2, \ \lambda_0 \geq 0, \ \gamma > 0.$$

Since these estimates are in different spaces we introduce an intermediate space $H^{p_{\lambda_0}}(\mathbb{R}^n)$. By the Lax-Milgram theorem we have that for every $g \in H^{\psi_0,-1}(\mathbb{R}^n)$ (since $H^{\psi_0,-1}(\mathbb{R}^n) = (H^{p_{\lambda_0}}(\mathbb{R}^n))^*$) there exists a unique element $u \in H^{p_{\lambda_0}}(\mathbb{R}^n)$ satisfying

$$B_{\lambda_0}(u,v) = < g, v > \tag{0.1}$$

for all $v \in H^{p_{\lambda_0}}(\mathbb{R}^n)$. We next prove that we have a unique variational solution to $p_{\lambda_0}(x,D)u = g$ for $g \in H^{\psi_0,-1}(\mathbb{R}^n)$ and $u \in H^{\psi_0,1}(\mathbb{R}^n)$. More regularity is achieved by assuming that $p_{\lambda_0}^{-1}(x,\xi) = \frac{1}{p(x,\xi)+\lambda_0}$ belongs to the symbol class $S_\rho^{-2+\tau_0,\psi_0}(\mathbb{R}^n)$ for some $\tau_0 > 0$. We now have all the tools we require to prove that $-p(x,D)$ extends to the generator of a Feller semigroup by the Hille-Yosida-Ray theorem. To demonstrate the scope of our result we consider an example where

$$p(x,\xi) = (1+q(x,\xi))^{\frac{\alpha(x)}{2}}(1-e^{-4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}}),$$

i.e. the family of Bernstein funtions is given by

$$f(x,s) = s^{\frac{\alpha(x)}{2}}(1 - e^{-4s^{\frac{\alpha(x)}{2}}}).$$

We prove that $p(x,\xi)$ and $\frac{1}{p_{\lambda_0}(x,\xi)}$ belong to the appropriate symbol classes and then we apply the general framework to find that $-p(x,D)$ extends to the generator of a Feller semigroup.

We improve the results of chapter four in chapter five, namely by proving the crucial estimates that are previously assumed. We prove that p belongs to the symbol class $S_\rho^{2r_1+2\epsilon,\psi}(\mathbb{R}^n)$ and that the symbol $p_{\lambda_0}^{-1}$ belongs to the class $S_\rho^{-2r_0+2\tilde{\eta},\psi}(\mathbb{R}^n)$. This is true assuming that $q(x,\xi)$ is comparable with a fixed continuous negative definite function ψ and f is a Bernstein function such that for appropriate ϵ and δ_0

$$|\partial_x^\alpha \partial_s^k f(x,s)| \leq c_{\alpha,k,\epsilon} \frac{1}{s^k} f(x,s) s^\epsilon$$

holds for all $x \in \mathbb{R}^n$ and $s \geq \delta_0$ with $c_{\alpha,k,\epsilon}$ independent of x and s. This will imply the estimates required in chapter four.

To summarise the content of this thesis, the first chapter begins with some important definitions such as continuous negative definite functions and Bernstein functions. One parameter operator semigroups are also introduced, in particular Feller and sub-Markovian semigroups and their generators. We then conclude this chapter by looking at subordination of operator semigroups.

Chapter two gives a description of Hoh's symbolic calculus; it defines anisotropic Sobolev spaces and introduces estimates for bilinear forms in these spaces. These estimates are needed when trying to solve the equation

$$p(x,D)u = f.$$

We dedicate chapter three to pseudo-differential operators as generators of Feller or sub-Markovian semigroups. Here we first introduce the idea of using the Hille-Yosida-Ray theorem to prove that an operator extends to the generator of a Feller semigroup, a method very important in the chapters to follow.

As mentioned, chapter four gives a general framework of a proof that $-p(x,D)$ generates a Feller semigroup. A new example is then given. In chapter five we prove the estimates required in chapter four for a large class of symbols and Bernstein functions. The final chapter looks at other possibilities of studying variable order subordination, namely Dirichlet forms. Here we only indicate the approach and refer to the literature.

1 Some Considerations on Operator Semigroups

The main purpose of this chapter is to introduce strongly continuous contraction, Feller and sub-Markovian semigroups. We will look closely at their generators and deal with subordination (in the sense of Bochner) of these semigroups. We will begin by considering some introductory material crucial to this work. We follow in our presentation essentially [20], see also [4]

1.1 Introductory Definitions

We will begin by defining a positive definite function.

Definition 1.1.1. *A function* $u : \mathbb{R}^n \longrightarrow \mathbb{C}$ *is called* **positive definite** *if for any* $k \in \mathbb{N}$ *and vectors* $\xi^1, \ldots, \xi^k \in \mathbb{R}^n$ *the matrix* $(u(\xi^j - \xi^l))_{j,l=1,\ldots k}$ *is positive Hermitian, i.e. for all* $\lambda_1 \ldots \lambda_k \in \mathbb{C}$ *we have*

$$\sum_{j,l=1}^{k} u(\xi^j - \xi^l)\lambda_j \bar{\lambda}_l \geq 0.$$

We can now define a negative definite function.

Definition 1.1.2. *A function* $\psi : \mathbb{R}^n \longrightarrow \mathbb{C}$ *is called* **negative definite** *if*

$$\psi(0) \geq 0$$

and

$$\xi \longrightarrow (2\pi)^{-\frac{n}{2}} e^{-t\psi(\xi)} \text{ is positive definite for } t \geq 0.$$

Definition 1.1.3. *A* **convolution semigroup** *on* $\mathbb{R}^n$ *is a family of bounded Borel measures* $(\mu_t)_{t\geq 0}$ *on* $\mathbb{R}^n$ *such that the following conditions hold*

$$\mu_t(\mathbb{R}^n) \leq 1 \text{ for all } t \geq 0;$$

$$\mu_s * \mu_t = \mu_{t+s} \quad s, t \geq 0 \text{ and } \mu_0 = \epsilon_0;$$

$$\mu_t \longrightarrow \epsilon_0 \text{ vaguely as } t \longrightarrow 0.$$

Further we have a relationship between convolution semigroups and continuous negative definite functions.

Theorem 1.1.4. *For a convolution semigroup $(\mu_t)_{t\geq 0}$ on $\mathbb{R}^n$ there exists a unique continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{C}$ such that*

$$\hat{\mu}_t(\xi) = (2\pi)^{-\frac{n}{2}} e^{-t\psi(\xi)} \tag{1.1}$$

holds for all $\xi \in \mathbb{R}^n$ and $t \geq 0$. Conversely, given a continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{C}$ there exists a convolution semigroup $(\mu_t)_{t\geq 0}$ on $\mathbb{R}^n$ such that (1.1) holds for all $\xi \in \mathbb{R}^n$ and $t \geq 0$.

We will now introduce the **Lévy Khinchin formula**. This formula gives a representation for a continuous negative definite function.

Theorem 1.1.5. *Let $\psi : \mathbb{R}^n \to \mathbb{C}$ be a continuous negative definite function. Then the following representation for ψ holds*

$$\psi(\xi) = c + i(d \cdot \xi) + q(\xi) + \int_{\mathbb{R}^n} \left(1 - e^{-ix\cdot\xi} - \frac{ix \cdot \xi}{1 + |x|^2}\right) \nu(dx)$$

where c is a non-negative constant, $d \in \mathbb{R}^n$ a vector, q a symmetric positive semidefinite quadratic form on $\mathbb{R}^n$ and ν a Borel measure integrating $x \to 1 \wedge |x|^2$.

As we will be dealing with subordination in the sense of Bochner it is essential to define a Bernstein function.

Definition 1.1.6. *A real-valued function $f \in C^\infty((0,\infty))$ is called a* ***Bernstein function*** *if*

$$f \geq 0$$

and

$$(-1)^k \frac{d^k f(x)}{dx^k} \leq 0 \qquad k \in \mathbb{N}.$$

Theorem 1.1.7. *Let f be a Bernstein function. Then there exists constants $a, b \geq 0$ and a measure μ on $(0,\infty)$ verifying*

$$\int_{0+}^{\infty} \frac{s}{1+s} \mu(ds) < \infty \tag{1.2}$$

such that

$$f(x) = a + bx + \int_{0+}^{\infty} (1 - e^{-xs}) \mu(ds), \quad x > 0. \tag{1.3}$$

The triple (a, b, μ) is uniquely determined by f. Conversely, given $a, b \geq 0$ and a measure μ on $(0,\infty)$ satisfying (1.2), then (1.3) defines a Bernstein function.

Compare Theorem 3.9.4 in [20].

Remark 1.1.8. *We may extend $f : (0, \infty) \to \mathbb{R}^n$ continuously using the representation (1.3) above into the half plane Re $z \geq 0$, i.e.*

$$f(z) = a + bz + \int_{0+}^{\infty} (1 - e^{-zs})\mu(ds), \quad Re\ z \geq 0.$$

We will now discuss the relation between Bernstein functions and certain convolution semigroups of measures.

Definition 1.1.9. *Let $(\eta_t)_{t\geq 0}$ be a convolution semigroup of measures on $\mathbb{R}$. It is said to be supported by $[0, \infty)$ if supp $\eta_t \subset [0, \infty)$ for all $t \geq 0$.*

Theorem 1.1.10. *Let $f : (0, \infty) \to \mathbb{R}$ be a Bernstein function. Then there exists a unique convolution semigroup $(\eta_t)_{t\geq 0}$ supported by $[0, \infty)$ such that*

$$\mathcal{L}(\eta_t)(x) := \int_0^{\infty} e^{-xs}\eta_t(ds) = e^{-tf(x)}, \ x > 0 \ \ and \ \ t > 0. \tag{1.4}$$

The converse is also true, i.e. for any convolution semigroup $(\eta_t)_{t\geq 0}$ supported by $[0, \infty)$ there exists a unique Bernstein function f such that (1.4) holds.

In the above theorem the convolution semigroup $(\eta_t)_{t\geq 0}$ is associated with the continuous negative definite function given by $y \to f(iy)$ where f is the Bernstein function. If we consider representation (1.3) of a Bernstein function, i.e.

$$f(x) = a + bx + \int_{0+}^{\infty} (1 - e^{-xs})\mu(ds), \quad x > 0.$$

then

$$f(i\xi) = a + ib\xi + \int_{0+}^{\infty} (1 - e^{is\xi})\mu(d\xi),$$

which is the Lévy-Khinchin representation for a continuous negative definite function. Therefore for any Bernstein function f the function $\xi \to \psi(\xi) = f(i\xi)$ is continuous and negative definite. Further using (1.3) we may consider the composition $f \circ \psi$ namely

$$(f \circ \psi)(\xi) = a + b\psi(\xi) + \int_{0+}^{\infty} (1 - e^{-s\psi(\xi)})\mu(ds),$$

for every continuous negative definite function. Since $(1 - e^{-s\psi(\xi)})$ is negative definite, we can derive that $f \circ \psi$ is also negative definite. This leads us on to the following result which is our first encounter with subordination in the sense of Bochner.

Lemma 1.1.11. *For any Bernstein function f and any continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{C}$, the composition function $f \circ \psi$ is also continuous and negative definite.*

To summarise

Proposition 1.1.12. *Let ψ be a continuous negative definite function with associated convolution semigroup $(\mu_t)_{t\geq 0}$ on $\mathbb{R}^n$. Further let f be a Bernstein function with associated convolution semigroup $(\eta_t)_{t\geq 0}$ supported on $[0,\infty)$. Since $f \circ \psi$ is a continuous negative definite function there exists an associated convolution semigroup, which we will denote by $(\mu_t^f)_{t\geq 0}$. We call this convolution semigroup the convolution semigroup subordinate (in the sense of Bochner) to $(\mu_t)_{t\geq 0}$ with respect to $(\eta_t)_{t\geq 0}$ and it is given by*

$$\mu_t^f = \int_0^\infty \mu_s \eta_t(ds) \quad (vaguely).$$

We now introduce complete Bernstein functions.

Definition 1.1.13. *A function $f : (0,\infty) \to \mathbb{R}^n$ is called a* **complete Bernstein function** *if there exists a Bernstein function g such that*

$$f(x) = x^2 \mathcal{L}(g)(x)$$

holds for all $x > 0$.

For $f : (0,\infty) \to \mathbb{R}^n$ the following are equivalent

1. f is a complete Bernstein function.
2. f is a Bernstein function having the representation

$$f(x) = a + bx + \int_{0+}^\infty (1 - e^{-sx})\mu(ds), \quad x > 0,$$

where a and b are non-negative constants and the measure μ is given by $\mu(ds) = m(s)\lambda^{(1)}(ds)$. The density m is given by

$$m(s) = \int_{0+}^\infty e^{-ts}\tau(dt), \quad s > 0,$$

where τ is a measure on $(0,\infty)$ satisfying

$$\int_{0+}^1 \frac{1}{t}\tau(dt) + \int_1^\infty \frac{1}{t^2}\tau(dt) < \infty.$$

1.2 Operator Semigroups

In this section we aim to describe basic facts of the general theory of one parameter semigroups of operators, in particular we will consider Feller semigroups on $C_\infty(\mathbb{R}^n)$ and sub-Markovian semigroups on $L^p(\mathbb{R}^n)$. In the following let $(X, ||\cdot||_X)$ to be a real or complex Banach space.

Definition 1.2.1. *A. A one parameter family $(T_t)_{t\geq 0}$ of bounded linear operators $T_t : X \to X$ which satisfies $T_0 = id$ and $T_{s+t} = T_s \circ T_t$ (semigroup property) for all $s, t \geq 0$ is called a (**one parameter**) **semigroup** of operators.*
*B. We call $(T_t)_{t\geq 0}$ **strongly continuous** if*

$$\lim_{t\to 0} ||T_t u - u||_X = 0$$

for all $u \in X$.
C. The semigroup $(T_t)_{t\geq 0}$ which satisfies $||T_t|| \leq 1$ for all $t \geq 0$ is called a contraction semigroup.

Important for our work is

Definition 1.2.2. *Let $(T_t)_{t\geq 0}$ be a strongly continuous contraction semigroup on $(C_\infty(\mathbb{R}^n), ||\cdot||_\infty)$ which is **positivity preserving**, i.e.*

$$u \geq 0 \ \text{ yields } \ T_t u \geq 0, \quad t \geq 0.$$

*Then we call $(T_t)_{t\geq 0}$ a **Feller semigroup**.*

Further, we require

Definition 1.2.3. *Let $(T_t)_{t\geq 0}$ be a strongly continuous contraction semigroup on $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$. We call $(T_t)_{t\geq 0}$ a **sub-Markovian semigroup** on L^p, $1 \leq p < \infty$ if for $u \in L^p(\mathbb{R}^n)$, such that $0 \leq u \leq 1$ almost everywhere it follows that $0 \leq T_t u \leq 1$ almost everywhere.*

As an illustration we may consider on $S(\mathbb{R}^n)$ the operator

$$T_t u(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-t\psi(\xi)} \hat{u}(\xi) d\xi.$$

Using the convolution theorem and the fact that $\hat{\mu}_t(\xi) = (2\pi)^{-\frac{n}{2}} e^{-t\psi(\xi)}$ we obtain

$$T_t u(x) = \int_{\mathbb{R}^n} u(x-y)\mu_t(dy).$$

When $u \in C_\infty(\mathbb{R}^n)$, $T_t u$ is also in $C_\infty(\mathbb{R}^n)$ and then $(T_t u)_{t\geq 0}$ is an example of a Feller semigroup. Further when $u \in L^2(\mathbb{R}^n)$, $T_t u$ is also in $L^2(\mathbb{R}^n)$ and then $(T_t u)_{t\geq 0}$ is an example of a sub-Markovian semigroup.

Sometimes we write $(T_t^{(p)})_{t\geq 0}, 1 < p < \infty$, to indicate that $(T_t)_{t\geq 0}$ is an L^p-sub-Markovian semigroup. For $p = \infty$, i.e. when writing $(T_t^{(\infty)})_{t\geq 0}$, we always mean that $(T_t)_{t\geq 0}$ is a Feller semigroup.

1.3 Generators of Operator Semigroups

The purpose of this section is to investigate generators of strongly continuous contraction, Feller and sub-Markovian semigroups. Important in this topic are the Hille-Yosida and the Hille-Yosida-Ray theorems.

Definition 1.3.1. *Let $(T_t)_{t\geq 0}$ be a strongly continuous semigroup of operators on a Banach space $(X, ||\cdot||_X)$. The* **generator** *A of $(T_t)_{t\geq 0}$ is definied by*

$$Au := \lim_{t\to 0} \frac{T_t u - u}{t} \quad (\textit{strong limit}) \tag{1.5}$$

with domain

$$D(A) := \left\{ u \in X \,|\, \lim_{t\to 0} \frac{T_t u - u}{t} \textit{ exists as a strong limit} \right\}. \tag{1.6}$$

Compare Definition 4.1.11 in [20]. In order to get a characterisation of all generators of strongly continuous contraction semigroups we first require the following definition.

Definition 1.3.2. *We call a linear operator $A : D(A) \to X$, $D(A) \subset X$* **dissipative** *if*

$$||\lambda u - Au||_X \geq \lambda ||u||_X$$

holds for all $\lambda > 0$ and $u \in D(A)$.

We can now state the Hille-Yosida Theorem.

Theorem 1.3.3. *A linear operator $(A, D(A))$ on a Banach space $(X, ||\cdot||_X)$ is the generator of a strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ if and only if the following three conditions hold:*

1. *$D(A) \subset X$ is dense;*
2. *A is a dissipative operator;*
3. *$R(\lambda - A) = X$ for some $\lambda > 0$.*

Compare Theorem 4.1.33 in [20]. More useful for us is

Theorem 1.3.4. *A linear operator on a Banach space $(X, ||\cdot||_X)$ is closeable and its closure $\bar{A}$ is the generator of a strongly continuous semigroup on X if and only if the following three conditions hold*

1. *$D(A) \subset X$ is dense;*

2. *A is a dissipative operator;*

3. *$R(\lambda - A)$ is dense in X for some $\lambda > 0$.*

Compare Theorem 4.1.37 in [20]. To progress further we introduce the positive maximum principle.

Definition 1.3.5. *Let $A : D(A) \to B(\mathbb{R}^n)$ be a linear operator, $D(A) \subset B(\mathbb{R}^n)$. Then if for any $u \in D(A)$ such that for $x_0 \in \mathbb{R}^n$, $u(x_0) = \sup_{x \in \mathbb{R}^n} u(x) \geq 0$ implies that $Au(x_0) \leq 0$ we say that $(A, D(A))$ satisfies the* **positive maximum principle**.

A linear operator $(A, D(A))$, $D(A) \subset C_\infty(\mathbb{R}^n)$, on $C_\infty(\mathbb{R}^n)$ that satisfies the positive maximum principle on $D(A)$ is a dissipative operator. The fact that the generator $(A, D(A))$ of a Feller semigroup satisfies the positive maximum principle leads us to the Hille-Yosida-Ray theorem.

Theorem 1.3.6. *A linear operator $(A, D(A))$, $D(A) \subset C_\infty(\mathbb{R}^n)$, on $C_\infty(\mathbb{R}^n)$ is closable and its closure is the generator of a Feller semigroup if and only if the following conditions hold:*

1. *$D(A) \subset C_\infty(\mathbb{R}^n)$ is dense;*

2. *$(A, D(A))$ satisfies the positive maximum principle;*

3. *$R(\lambda - A)$ is dense in $C_\infty(\mathbb{R}^n)$ for some $\lambda > 0$.*

Compare Theorem 4.5.3 in [20].

We now continue to consider generators of sub-Markovian semigroups. We cannot use the positive maximum principle here and therefore we cannot use the Hille-Yosida-Ray theorem since pointwise statements do not make sense. We do know however, that generators of sub-Markovian semigroups are Dirichlet operators.

Definition 1.3.7. *A closed, densely defined linear operator $A : D(A) \to L^p(\mathbb{R}^n)$, $1 < p < \infty$, $D(A) \subset L^p(\mathbb{R}^n)$, is called a* **Dirichlet operator** *in $L^p(\mathbb{R}^n)$ if for all $u \in D(A)$*

$$\int_{\mathbb{R}^n} (Au)((u-1)^+)^{p-1} dx \leq 0 \tag{1.7}$$

holds

Definition 1.3.8. *Let* $(A, D(A))$, $D(A) \subset L^p(\mathbb{R}^n)$, *be a linear operator* $A : D(A) \to L^p(\mathbb{R}^n)$. *A is called* **negative definite** *in* $L^p(\mathbb{R}^n)$ *if for all* $u \in D(A)$

$$\int_{\mathbb{R}^n} (Au)(signu)|u|^{p-1}dx \leq 0$$

holds.

We note here that Dirichlet operators are negative definite. Further if $(A, D(A))$ is a negative definite oprerator in $L^p(\mathbb{R}^n)$, $1 < p < \infty$, then it is dissipative. Therefore it follows

Proposition 1.3.9. *Let* $(A, D(A))$ *be a Dirichlet operator in* $L^p(\mathbb{R}^n)$, $1 < p < \infty$. *Then A is dissipative.*

Lemma 1.3.10. *Let* $(T_t)_{t \geq 0}$ *be a sub-Markovian semigroup on* $L^p(\mathbb{R}^n)$, $1 < p < \infty$, *with generator* $(A, D(A))$. *Then for all* $u \in D(A)$ *condition (1.7) holds, i.e. A is a Dirichlet operator.*

Therefore we know that every generator of a sub-Markovian semigroup is a Dirichlet operator, however in general not every Dirichlet operator is the generator of a sub-Markovian semigroup. As a converse we have (Theorem 4.6.17 in [20]).

Theorem 1.3.11. *Let A be a Dirichlet operator on* $L^p(\mathbb{R}^n)$, $1 < p < \infty$, *with the property that* $R(\lambda id - A) = L^p(\mathbb{R}^n)$ *for some* $\lambda > 0$. *Then A generates a sub-Markovian semigroup on* $L^p(\mathbb{R}^n)$.

1.4 Subordination in the Sense of Bochner of Operator Semigroups

In the final section of this chapter we will consider subordination in the sense of Bochner of the semigroups we have dealt with so far.

Let $f : (0, \infty) \to \mathbb{R}$ be a Bernstein function and $(\eta_t)_{t \geq 0}$ be the associated convolution semigroup on $\mathbb{R}$ supported by $[0, \infty)$. Further let $(T_t)_{t \geq 0}$ be a strongly continuous contraction semigroup on the Banach space $(X, || \cdot ||_X)$ with generator $(A, D(A))$.

Definition 1.4.1. *Let* $(T_t)_{t \geq 0}$ *and* $(\eta_t)_{t \geq 0}$ *with corresponding Bernstein function f be as above, then for* $u \in X$ *we define*

$$T_t^f u = \int_0^\infty T_s u \eta_t(ds).$$

We can prove this is a strongly continuous contraction semigroup on X. We call $(T_t^f)_{t\geq 0}$ the semigroup **subordinate** *to $(T_t)_{t\geq 0}$ with repesct to $(\eta_t)_{t\geq 0}$ or with respect to f.*

If $(T_t)_{\geq 0}$ is a Feller semigroup on $C_\infty(\mathbb{R}^n)$ then $(T_t^f)_{t\geq 0}$ is also a Feller semigroup on $C_\infty(\mathbb{R}^n)$. Further if $(T_t)_{\geq 0}$ is a sub-Markovian semigroup on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, then $(T_t^f)_{t\geq 0}$ is also a sub-Markovian semigroup on $L^p(\mathbb{R}^n)$, $1 < p < \infty$.
We will conclude by considering subordination on the level of generators for the translation invariant case.

Example 1.4.2. *Let $(\mu_t)_{t\geq 0}$ be a convolution semigroup associated with the continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{C}$, i.e. $\hat{\mu}_t(\xi) = (2\pi)^{-\frac{n}{2}} e^{-t\psi(\xi)}$. Further, let $f : (0,\infty) \longrightarrow \mathbb{R}$ be a Bernstein function with representation (1.3) and corresponding convolution semigroup $(\eta_t)_{t\geq 0}$, supp $\eta_t \subset [0,\infty)$. A sub-Markovian semigroup $\left(T_t^{(p)}\right)_{t\geq 0}$ can be associated with $(\mu_t)_{t\geq 0}$ on the space $L^p(\mathbb{R}^n)$, also we can associate with $(\mu_t)_{t\geq 0}$ a Feller semigroup $\left(T_t^{(\infty)}\right)_{t\geq 0}$ on the Banach space $(C_\infty(\mathbb{R}^n), ||\cdot||_\infty)$, compare section 1.2. On $S(\mathbb{R}^n)$ we have for these semigroups*

$$T_t u(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\xi} e^{-t\psi(\xi)} \hat{u}(\xi) d\xi$$

or equivalently

$$(T_t u)^\wedge(\xi) = e^{-t\psi(\xi)} \hat{u}(\xi).$$

The generator of the semigroup is given on $S(\mathbb{R}^n)$ by

$$Au(x) = -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\xi} \psi(\xi) \hat{u}(\xi) d\xi$$

After subordination we have

$$T_t^f u(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\xi} e^{-tf(\psi(\xi))} \hat{u}(\xi) d\xi$$

or

$$(T_t^f)^\wedge(\xi) = e^{-t(f\circ\psi)(\xi)} \hat{u}(\xi)$$

and for its generator A^f we get

$$A^f u(x) = -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\xi} f(\psi(\xi)) \hat{u}(\xi) d\xi.$$

It is possible to deduce that the generator of the subordinate semigroup $(T_t^f)_{t\geq 0}$ is given by $-f(-A)$. This however requires a more formal definition of $-f(-A)$ in the sense of a closed operator and some more involved discussions of domains.

2 Hoh's Symbolic Calculus

The aim of this chapter is to introduce Hoh's symbolic calculus and then use this theory to obtain estimates for pseudo-differentail operators. For Hoh's symbolic calculus, see W. Hoh [17] or [18], compare also [21]. For the related estimates we will follow [21].

2.1 Essential Material

This section is dedicated to introducing the key points of Hoh's symbolic calculus and defining Sobolev spaces, spaces in which we will now frequently work.

Definition 2.1.1. *An arbitrarily often differentiable continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{R}$ belongs to the class Λ if for all $\alpha \in \mathbb{N}_0^n$ it satisfies*

$$|\partial_\xi^\alpha (1+\psi(\xi))| \leq c_{|\alpha|}(1+\psi(\xi))^{\frac{2-\rho(|\alpha|)}{2}}, \tag{2.1}$$

where $\rho(k) = k \wedge 2$ for $k \in \mathbb{N}_0^n$.

Definition 2.1.2. *A. Let $m \in \mathbb{R}$ and $\psi \in \Lambda$. We then call a C^∞-function $q : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{C}$ a symbol in the class $S_\rho^{m,\psi}(\mathbb{R}^n)$ if for all $\alpha, \beta \in \mathbb{N}_0^n$ there are constants $c_{\alpha,\beta} \geq 0$ such that*

$$|\partial_x^\beta \partial_\xi^\alpha q(x,\xi)| \leq c_{\alpha,\beta}(1+\psi(\xi))^{\frac{m-\rho(|\alpha|)}{2}} \tag{2.2}$$

holds for all $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$. We call $m \in \mathbb{R}$ the order of the symbol $q(x,\xi)$.
B. Let $\psi \in \Lambda$ and suppose that for an arbitrarily often differentiable function $q : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{C}$ the estimate

$$|\partial_\xi^\alpha \partial_x^\beta q(x,\xi)| \leq \tilde{c}_{\alpha,\beta}(1+\psi(\xi))^{\frac{m}{2}} \tag{2.3}$$

holds for all $\alpha, \beta \in \mathbb{N}_0^n$ and $x, \xi \in \mathbb{R}^n$. In this case we call q a symbol of the class $S_0^{m,\psi}(\mathbb{R}^n)$.

Note that $S_\rho^{m,\psi}(\mathbb{R}^n) \subset S_0^{m,\psi}(\mathbb{R}^n)$. For $q \in S_0^{m,\psi}(\mathbb{R}^n)$, hence also for $q \in S_\rho^{m,\psi}(\mathbb{R}^n)$, we can define on $S(\mathbb{R}^n)$ the pseudo-differential operator $q(x,D)$ by

$$q(x,D)u(x) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} q(x,\xi)\hat{u}(\xi)d\xi \tag{2.4}$$

and we denote the classes of these operators by $\Psi_\rho^{m,\psi}(\mathbb{R}^n)$ and $\Psi_0^{m,\psi}(\mathbb{R}^n)$, respectively.

Theorem 2.1.3. *Let $q \in S_0^{m,\psi}(\mathbb{R}^n)$ then $q(x,D)$ maps $S(\mathbb{R}^n)$ continuously into itself.*

We now introduce anisotropic Sobolev spaces associated with continuous negative definite functions.

Definition 2.1.4. *Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a fixed continuous negative definite function. For $s \in \mathbb{R}$ and $u \in S(\mathbb{R}^n)$ (or $u \in S'(\mathbb{R}^n)$) we define the norm*

$$||u||_{\psi,s}^2 = ||(1+\psi(D))^{\frac{s}{2}}u||_0^2 = \int_{\mathbb{R}^n} (1+\psi(s))^s |\hat{u}(\xi)|^2 d\xi. \tag{2.5}$$

The space $H^{\psi,s}(\mathbb{R}^n)$ is defined as

$$H^{\psi,s}(\mathbb{R}^n) := \{u \in S'(\mathbb{R}^n); ||u||_{\psi,s} < \infty\}. \tag{2.6}$$

We call $H^{\psi,s}(\mathbb{R}^n)$ an anisotropic Sobolev space.

The scale $H^{\psi,s}(\mathbb{R}^n)$, $s \in \mathbb{R}$, and more general spaces have been systematically investigated in [11] and [12], see also [21]. In particular we know that if for some $\rho_1 > 0$ and $\tilde{c}_1 > 0$ the estimate $\psi(\xi) \geq \tilde{c}_1 |\xi|^{\rho_1}$ holds for all $\xi \in \mathbb{R}^n$, $|\xi| \geq R$, $R \geq 0$, then the space $H^{\psi,s}(\mathbb{R}^n)$ is continuously embedded into $C_\infty(\mathbb{R}^n)$ provided $s > \frac{n}{2\rho_1}$.
The following result is of most importance to us

Theorem 2.1.5. *Let $\psi \in \Lambda$. For $q_1 \in S_\rho^{m_1,\psi}(\mathbb{R}^n)$ and $q_2 \in S_\rho^{m_2,\psi}(\mathbb{R}^n)$ the symbol q of the operator $q(x,D) := q_1(x,D) \circ q_2(x,D)$ is given by*

$$q(x,\xi) = q_1(x,\xi) \cdot q_2(x,\xi) + \sum_{j=1}^n \partial_{\xi_j} q_1(x,\xi) D_{x_j} q_2(x,\xi) + q_{r_1}(x,\xi) \tag{2.7}$$

with $q_{r_1} \in S_0^{m_1+m_2-2,\psi}(\mathbb{R}^n)$.

Remark 2.1.6. *An easy calculation yields $q_1 \cdot q_2 \in S_\rho^{m_1+m_2,\psi}(\mathbb{R}^n)$, $\partial_{\xi_j} q_1 \in S_\rho^{m_1-1,\psi}(\mathbb{R}^n)$ and $D_{x_j} q_2 \in S_\rho^{m_2,\psi}(\mathbb{R}^n)$. Hence the second term on the right hand side in (2.7) belongs to $S_\rho^{m_1+m_2-1,\psi}(\mathbb{R}^n)$.*

2.2 Estimates for Pseudo-Differential Operators using Hoh's Symbolic Calculus

We will begin this section with the theorem of Calderón and Vaillancourt.

Theorem 2.2.1. *Let $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ be a function such that for all $\alpha, \beta \in \mathbb{N}_0^n$, $|\alpha|, |\beta| \leq 3$, the partial derivatives $\partial_x^\beta \partial_\xi^\alpha q(x,\xi)$ exist, are continuous and satisfy the estimates*

$$|\partial_x^\beta \partial_\xi^\alpha q(x,\xi)| \leq c_{\alpha,\beta}.$$

Then the pseudo-differential operator $q(x,D)$ which is defined on $S(\mathbb{R}^n)$ extends to a bounded operator from $L^2(\mathbb{R}^n)$ into itself.

Theorem 2.2.2. *Let $q \in S_0^{m,\psi}(\mathbb{R}^n)$ and let $q(x,D)$ be the corresponding pseudo-differential operator. For all $s \in \mathbb{R}$ the operator $q(x,D)$ maps the space $H^{\psi,m+s}(\mathbb{R}^n)$ continuously into the space $H^{\psi,s}(\mathbb{R}^n)$, and for all $u \in H^{\psi,m+s}(\mathbb{R}^n)$ we have the estimate*

$$||q(x,D)u||_{\psi,s} \leq c||u||_{\psi,m+s}. \tag{2.8}$$

On $S(\mathbb{R}^n)$ we may define the bilinear form

$$B(u,v) := (q(x,D)u,v)_0, \quad q \in S_\rho^{m,\psi}(\mathbb{R}^n). \tag{2.9}$$

Theorem 2.2.3. *Let $q \in S_\rho^{m,\psi}(\mathbb{R}^n)$ be real valued and $m > 0$. It follows that*

$$|B(u,v)| \leq c||u||_{\psi,\frac{m}{2}}||v||_{\psi,\frac{m}{2}} \tag{2.10}$$

holds for all $u,v \in S(\mathbb{R}^n)$. Hence the bilinear form B has a continuous extension onto $H^{\psi,\frac{m}{2}}(\mathbb{R}^n)$. If in addition for all $x \in \mathbb{R}^n$

$$q(x,\xi) \geq \delta_0(1+\psi(\xi))^{\frac{m}{2}} \text{ for } |\xi| \geq R \tag{2.11}$$

with some $\delta_0 > 0$ and $R \geq 0$, and

$$\lim_{|\xi|\to\infty} \psi(\xi) = \infty \tag{2.12}$$

holds, then we have for all $u \in H^{\psi,\frac{m}{2}}(\mathbb{R}^n)$ the Gårding inequality

$$ReB(u,u) \geq \frac{\delta_0}{2}||u||^2_{\psi,\frac{m}{2}} - \lambda_0||u||_0^2. \tag{2.13}$$

Furthermore we have

Theorem 2.2.4. *If we assume (2.11) and (2.12) then for $s > -m$ we have*

$$\frac{\delta_0}{2}||u||_{\psi,m+s} \leq ||q(x,D)u||^2_{\psi,s} + ||u||^2_{\psi,m+s-\frac{1}{2}} \tag{2.14}$$

for $q \in S_\rho^{m,\psi}(\mathbb{R}^n)$ real-valued and all $u \in H^{\psi,s+m}(\mathbb{R}^n)$.

For solving the equation $q_\lambda(x,D)u = q(x,D)u + \lambda u = f$ we now set

$$B_\lambda(u,v) = (q(x,D)u,v)_0 + \lambda(u,v)_0.$$

By Theorem 2.2.3 the bilinear form B_λ extends to a continuous bilinear form on $H^{\psi,\frac{m}{2}}(\mathbb{R}^n)$ denoted again by B_λ i.e. we have

$$|B_\lambda(u,v)| \leq c||u||_{\psi,\frac{m}{2}}||v||_{\psi,\frac{m}{2}}.$$

Important for us is the Lax-Milgram theorem which we have taken from [14], Theorem I.14.1

Theorem 2.2.5. *Let B be a sesquilinear form on a complex Hilbert space $(H,(.,.)_H)$. Suppose that*

$$|B(u,v)| \leq c||u||_H||v||_H$$

and

$$|B(u,u)| \geq \gamma||u||_H^2$$

hold for all $u,v \in H$ with some $\gamma > 0$. In addition, let $l : H \to \mathbb{C}$ be a continuous linear functional, i.e. $l \in H^*$. Then there exist unique elements $v,w \in H$ such that

$$l(u) = B(u,v) = \overline{B(w,u)}$$

holds for all $u \in H$.

Definition 2.2.6. *We call $u \in H^{\psi,\frac{m}{2}}$ a variational solution to the equation*

$$q_\lambda(x,D)u = f \tag{2.15}$$

for all $\lambda \in \mathbb{R}$ and $f \in L^2(\mathbb{R}^n)$ if

$$B_\lambda(u,\phi) = (\phi,f)_0$$

holds for all $\phi \in C_0^\infty(\mathbb{R}^n)$, or $\phi \in H^{\psi,\frac{m}{2}}(\mathbb{R}^n)$.

Therefore using Theorem 2.2.3 and the Lax-Milgram theorem there exists for all $f \in L^2(\mathbb{R}^n)$ a unique variational solution $u \in H^{\psi,\frac{m}{2}}$ to (2.15). For more regularity we have

Theorem 2.2.7. *Let $q \in S_\rho^{m,\psi}(\mathbb{R}^n)$ be as in Theorem 2.2.4, $m \geq 1$. Further suppose that for $f \in H^{\psi,s}(\mathbb{R}^n)$, $s \geq 0$, there exists $u \in H^{\psi,\frac{m}{2}}(\mathbb{R}^n)$ such that*

$$B(u,\phi) = (f,\phi)_{L^2} \tag{2.16}$$

holds for all $\phi \in H^{\psi,\frac{m}{2}}(\mathbb{R}^n)$ (or $\phi \in S(\mathbb{R}^n)$). Then u already belongs to the space $H^{\psi,m+s}(\mathbb{R}^n)$.

So far we have used properties of symbols to establish mapping properties and estimates for operators. The real power of a symbolic calculus is that it reduces calculations for operators to calculations for symbols.

3 Pseudo-Differential Operators with Negative Definite Symbols $q \in S_\rho^{2,\psi}(\mathbb{R}^n)$

In this chapter we want to deal with pseudo-differential operators with symbols from Hoh's class which extend to generators of Feller or L^p sub-Markovian semigroups. Further we will investigate subordination of semigroups constructed using this method.

3.1 Pseudo-Differential Operators as Generators of Feller or L^p Sub-Markovian Semigroups.

In this section we will use the estimates considered in the previous chapter in order to extend certain operators as generators of Feller and certain L^p sub-Markovian semigroups. In particular when considering generators of Feller semigroups we aim to use the Hille-Yosida-Ray theorem, Theroem 1.3.6.

Recall that the main characteristics of the Hille-Yosida-Ray theorem are $D(A) \subset C_\infty(\mathbb{R}^n)$, $(A, D(A))$ satisfies the positive maximum principle and the range condition; $R(\lambda - A)$ is dense in $C_\infty(\mathbb{R}^n)$. In our case we consider $q(x, D)$ on $C_0^\infty(\mathbb{R}^n)$ with **negative definite symbol** i.e. $\xi \to q(x, \xi)$ is for $x \in \mathbb{R}^n$ fixed a continuous negative definite function. Since $C_0^\infty(\mathbb{R}^n)$ is dense in $C_\infty(\mathbb{R}^n)$ the first condition of the theorem is satisfied. Theorem 4.5.6 in [20] shows us that $q(x, D)$ satisfies the positive maximum principle on $C_0^\infty(\mathbb{R}^n)$, see also [7] . Therefore our problem is reduced to tackling the range condition, or equivalently to solve for some $\lambda > 0$ the equation

$$q_\lambda(x, D)u = q(x, D)u + \lambda u = f \tag{3.1}$$

in $C_\infty(\mathbb{R}^n)$ for $f \in C_0^\infty(\mathbb{R}^n)$. This is too difficult to solve on the domain $C_0^\infty(\mathbb{R}^n)$. To overcome this problem we consider $q(x, D)$ on a larger domain $H^{\psi,s}(\mathbb{R}^n)$ where (3.1) is easier to deal with. For the positive maximum principle to hold on this larger domain we use

Theorem 3.1.1. *Let $D(A) \subset C_\infty(\mathbb{R}^n)$, and suppose that $A : D(A) \to C_\infty(\mathbb{R}^n)$ is a linear operator. In addition assume that $C_0^\infty(\mathbb{R}^n) \subset D(A)$ is an operator core of A in the sense that to every $u \in D(A)$ there exists a sequence $(\phi_k)_{k\in\mathbb{N}}$, $\phi_k \in C_0^\infty(\mathbb{R}^n)$, such that*

$$\lim_{k\to\infty} ||\phi_k - u||_\infty = \lim_{k\to\infty} ||A\phi_k - Au||_\infty = 0$$

If $A|_{C_0^\infty}$ satisfies the positive maximum principle on $C_0^\infty(\mathbb{R}^n)$, then it satisfies the positive maximum principle also on $D(A)$.

Comapare Theorem 2.6.1 in [21]. The following results are due to W. Hoh [17] and [18]. Using the estimates introduced in the last chapter together with the above theorem we get

Theorem 3.1.2. *Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a continuous negative definite function in the class Λ such that $\psi(\xi) \geq c_0|\xi|^{r_0}$, $r_0 > 0$, holds. If $q(x,\xi) \in S_\rho^{2,\psi}(\mathbb{R}^n)$ is a negative definite symbol satisfying*

$$q(x,\xi) \geq \delta(1+\psi(\xi)) \tag{3.2}$$

for some $\delta > 0$ and all $\xi \in \mathbb{R}^n$, $|\xi|$ sufficiently large, then $-q(x,D)$ defined on $C_0^\infty(\mathbb{R}^n)$ is closeable in $C_\infty(\mathbb{R}^n)$ and its closure is a generator of a Feller semigroup.

Further for L^2 sub-Markovian semigroups we have

Theorem 3.1.3. *Using the same assumptions as the previous theorem the operator $(-q_\lambda(x,D), H^{\psi,2}(\mathbb{R}^n))$, $\lambda > \lambda_0$ as in (2.13), is the generator of an L^2-sub-Markovian semigroup. Hence $(-q_\lambda(x,D), H^{\psi,2}(\mathbb{R}^n))$ is a Dirichlet operator.*

3.2 Subordination of Semigroups

In this section we will continue to develop the ideas of subordination that we have already met in section 1.4. However now we will apply subordination to the semigroups constructed using Hoh's symbolic calculus.

If we recall the translation invariant case of section 1.4. It was shown that for a continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{C}$

$$(T_t u)^\wedge(\xi) = e^{-t\psi(\xi)}\hat{u}(\xi)$$

for $u \in S(\mathbb{R}^n)$. Now if f is a Bernstein function with corresponding convolution semigroup $(\eta_t)_{t\geq 0}$ supported on $(0,\infty)$ the Fourier transform of the subordinate semigroup is given by

$$(T_t^f)^\wedge(\xi) = e^{-t(f\circ\psi)(\xi)}\hat{u}(\xi).$$

On the level of generators we saw that on $S(\mathbb{R}^n)$ the generator of $(T_t)_{t\geq 0}$ is given by

$$-\psi(D)u(x) = -(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\psi(\xi)\hat{u}(\xi)d\xi.$$

Further the generator of the subordinated semigroup is given by

$$-\psi^f(D)u(x) = -(f\circ\psi)(D)u(x) = -(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{ix\cdot\xi}f(\psi(\xi))\hat{u}(\xi)d\xi.$$

We also illustrated that if $A^{(p)}$ is the generator of $(T_t^{(p)})_{t\geq 0}$, then $A^{(p),f} = -f(-A^{(p)})$ is the generator of $(T_t^{(p),f})_{t\geq 0}$, where $(T_t^{(p)})_{t\geq 0}$ is an L^p-sub-Markovian semigroup for $1 < p < \infty$ and a Feller semigroup for $p = \infty$. Further $(T_t^{(p),f})_{t\geq 0}$ is again an L^p-sub-Markovian or Feller semigroup respectively

Now considering the pseudo-differential operator $q(x, D)$ with symbol $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ such that $\xi \to q(x, \xi)$ is a continuous negative definite function. Suppose the operator $-q(x, D)$ extends from $S(\mathbb{R}^n)$ to a generator of an L^p-sub-Markovian semigroup, $1 < p < \infty$, and a Feller semigroup for $p = \infty$. We denote the generated semigroup by $(T_t)_{t\geq 0}$. When subordinating with respect to a Bernstein function f we no longer get the representation we had in the translation invariant case, i.e.

$$(T_t^f u)^\wedge(\xi) \neq e^{-tf(q(x,\xi))}\hat{u}(\xi)$$

and

$$A^f u \neq -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(q(x,\xi))\hat{u}(\xi)d\xi. \tag{3.3}$$

In the following we write $-f(q(x, D))$ for the right hand side of (3.3). This should be understood as a shorthand for $(f \circ q)(x, D)$. We are now interested when $-f(q(x, D))$ extends to the generator of a Feller or sub-Markovian semigroup. We will see that often if $q \in S_\rho^{m,\psi}(\mathbb{R}^n)$ and f is a Bernstein function, then $f \circ q$ is also a symbol belonging to Hoh's class. Further, if $\xi \to q(x, \xi)$ is a continuous negative definite function then so is the function $\xi \to f(q(x, \xi))$, therefore it is sensible to investigate whether $-(f \circ q)(x, D)$ extends to the generator of a Feller or sub-Markovian semigroup. Of course, this procedure is closely linked to subordination in the sense of Bochner. By modifying the proof to Theorem 2.6.4 in [21] we find

Theorem 3.2.1. *Let $q \in S_\rho^{2,\psi}(\mathbb{R}^n)$ be a continuous negative definite symbol satisfying (3.2) such that $-q(x, D)$ generates an L^p-sub-Markovian semigroup $(T_t)_{t\geq 0}$. Further let f be a Bernstein function with corresponding semigroup $(\eta_t)_{t\geq 0}$ supported on $[0, \infty)$, then $-(f \circ q)(x, D)$ extends to the generator of a Feller or sub-Markovian semigroup. We will denote the generated semigroup by $(S_t)_{t\geq 0}$.*

Remark 3.2.2. *It is important to note that the semigroup generated by $-(f \circ q)(x, D)$, $(S_t)_{t\geq 0}$ is not, in general equal to the original subordinated semigroup $(T_t^f)_{t\geq 0}$. However knowing $(S_t)_{t\geq 0}$ helps us to approximate $(T_t^f)_{t\geq 0}$.*

4 Subordination of Variable Order - Part I

This chapter will follow [14] closely in order to consider two important topics related to subordination of variable order. Formally, subordination of variable order means that we replace a fixed Bernstein function f by a family of Bernstein functions depending on x. Firstly, we suggest a method to study "variable order subordination" for more general Bernstein functions than the example studied by Hoh $f_\alpha(s) = s^\alpha$, $0 < \alpha < 1$. More precisely, we consider symbols of the form

$$p(x,\xi) = f(x, q(x,\xi)) \tag{4.1}$$

where q is a suitable symbol from Hoh's class and $f : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ is a smooth function such that for fixed $x \in \mathbb{R}^n$ the function $s \to f(x,s)$ is a Bernstein function. Our method uses some ideas from the theory of t-coercive (differential) operators as investigated by I. S. Louhivaara and C. Simader [29]-[30] in order to establish the result that $-p(x,D)$ generates a Feller semigroup. A different view to our approach is to interpret $f(x,r)$ as a state space dependent (family of) Bernstein functions which is obtained by making parameters state space dependent i.e. consider a function $f(r)_{a,b,c,\ldots}$ depending on parameters $a, b, c, \ldots$. By making these parameters state space dependent we obtain for a negative definite symbol $q(x,\xi)$ a new negative definite symbol by $f_{a(x),b(x),c(x),\ldots}(q(x,\xi))$. More precisely, let $\tilde{f}$ be a Bernstein function with representation

$$\tilde{f}(r) = \int_{0+}^{\infty} (1 - e^{-sr})\mu(ds) = \int_{0+}^{\infty} (1 - e^{-sr})\tilde{m}(s)ds. \tag{4.2}$$

Suppose that $\tilde{m}$ depends on parameters $a, b, c\ldots$ i.e. $\tilde{m}(s) = \tilde{m}_{a,b,c\ldots}(s)$. Now we may let the parameters depend on x, i.e. we switch to $(x,s) \to \tilde{m}_{a(x),b(x),c(x)\ldots}(s)$ and consider the family of Bernstein functions

$$f^*(x,r) = \int_{0+}^{\infty} (1 - e^{-sr})\tilde{m}_{a(x),b(x),c(x)\ldots}(s)ds.$$

Thus we may consider the symbol $(x,\xi) \to \tilde{p}(x, q(x,\xi))$ defined by

$$\tilde{p}(x, q(x,\xi)) = \int_{0+}^{\infty} (1 - e^{-sq(x,\xi)})\tilde{m}_{a(x),b(x),c(x)\ldots}(s)ds.$$

More generally, let us consider with a suitable function $\tau : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$

$$f(x,r) = \int_{0+}^{\infty} (1 - e^{-sr})\tau(x,s)ds \tag{4.3}$$

and the associated symbol

$$p(x,\xi) = f(x,q(x,\xi)) = \int_{0+}^{\infty} (1 - e^{-sq(x,\xi)})\tau(x,s)ds. \tag{4.4}$$

The second purpose of this chapter is to enrich the class of examples by studying the Bernstein function

$$s \mapsto s^{\frac{\alpha}{2}}(1 - e^{-4s^{\frac{\alpha}{2}}}).$$

4.1 Jusification of the Phrase "Variable Order Subordination"

This section will follow section 2.10 of [21]. Discussing "variable order subordination" is related to the study of pseudo-differential operators with variable order of differentiation. To illustrate this we consider the example where the Bernstein function $s \mapsto f(s)$ is substituted by $(x,s) \mapsto s^{r(x)}$ with $r : \mathbb{R}^n \to \mathbb{R}$ being a continuous function such that $0 \leq r(x) \leq 1$. We now let s be the continuous negative definite function $|\xi|^2$. We know $(|\xi|^2)^{r(x)}$ is also a continuous negative definite function. This implies that the pseudo-differential operator

$$Au(x) = (-\Delta)^{r(x)} = -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}(|\xi|^2)^{r(x)}\hat{u}(\xi)d\xi$$

is a candidate for a genarator of a Feller semigroup. Note that when $n = 1$ and instead of using the symbol $|\xi|^2$ we use $\pm i\xi$ and we get the operator $(\mp\frac{d}{dx})$, hence the phrase "operator of variable order of differentiation".

4.2 The Formal Background of our Proof that $-p(x,D)$ Generates a Feller Semigroup

The proof that $-p(x,D)$ extends to a generator of a Feller semigroup depends on various estimates which might be different for different operators. However, once these estimates are established we only need to apply a piece of "soft" analysis. In this section we discuss this part of the proof, i.e. we will assume all crucial estimates hold.

Let $f : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ be an arbitrarily often differentiable function such that for $y \in \mathbb{R}^n$ fixed the function $s \mapsto f(y,s)$ is a Bernstein function. Moreover we assume

$$\inf_{y\in\mathbb{R}^n} f(y,s) \geq f_0(s) \quad \text{for all } s \in [0,\infty) \tag{4.5}$$

as well as

$$\sup_{y \in \mathbb{R}^n} f(y,s) \leq f_1(s) \quad \text{for all } s \in [0,\infty) \tag{4.6}$$

where f_0 and f_1 are Bernstein functions. For a given real-valued negative definite symbol $q(x,\xi)$ it follows that

$$p(y;x,\xi) := f(y,q(x,\xi)) \tag{4.7}$$

give rise to a further negative definite symbol by defining

$$p(x,\xi) := p(x;x,\xi). \tag{4.8}$$

In case where $q(x,\xi)$ is comparable with a fixed continuous negative definite function ψ, i.e.

$$0 < c_0 \leq \frac{q(x,\xi)}{\psi(\xi)} \leq c_1, \qquad c_1 \geq 1, \tag{4.9}$$

for all $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$, we find using Lemma 3.9.34.B in [20]

$$p(x,\xi) \leq f(x,q(x,\xi)) \leq c_1 f_1(\psi(\xi)) \tag{4.10}$$

and we define

$$\psi_1(\xi) := c_1 f_1(\psi(\xi)). \tag{4.11}$$

Moreover it holds

$$p(x,\xi) \geq f(x,q(x,\xi)) \geq c_0' f_0(\psi(\xi))$$

and we set

$$\psi_0(\xi) := c_0' f_0(\psi(\xi)). \tag{4.12}$$

Clearly, ψ_0 and ψ_1 are continuous negative definite functions. Later on we assume that for $|\xi|$ large

$$\psi(\xi) \geq \tilde{c}_1 |\xi|^{\rho_1}, \quad \tilde{c}_1 > 0 \text{ and } \rho_1 > 0 \tag{4.13}$$

holds as well as

$$f(y_0,s) \geq \tilde{c}_0 s^{\rho_0}, \quad \tilde{c}_0 > 0 \text{ and } \rho_0 > 0. \tag{4.14}$$

This implies for $|\xi|$ large that

$$\psi_0(\xi) \geq \tilde{c}_2 |\xi|^{\rho_0 \rho_1}, \quad \tilde{c}_2 > 0, \tag{4.15}$$

holds. Since $\psi_0(\xi) \leq \psi_1(\xi)$ we have

$$H^{\psi_1,1}(\mathbb{R}^n) \hookrightarrow H^{\psi_0,1}(\mathbb{R}^n). \tag{4.16}$$

We add the assumption that there exists $0 < \sigma < \frac{1}{2}$ such that

$$(1 + \psi_1)^{\frac{1}{2}} \in S_\rho^{1+\sigma,\psi_0}(\mathbb{R}^n). \tag{4.17}$$

This will imply that

$$H^{\psi_0,m(1+\sigma)}(\mathbb{R}^n) \hookrightarrow H^{\psi_1,m}(\mathbb{R}^n) \tag{4.18}$$

holds for $m \geq 0$. Further, (4.17) implies that if $p_1(x,\xi)$ is any symbol belonging to $S_\rho^{m,\psi_1}(\mathbb{R}^n)$ then it also belongs to $S_\rho^{m(1+\sigma),\psi_0}(\mathbb{R}^n)$ which follows from

$$\begin{aligned} |\partial_\xi^\alpha \partial_x^\beta p_1(x,\xi)| &\leq c_{\alpha,\beta}(1+\psi_1(\xi))^{\frac{m-\rho(|\alpha|)}{2}} \\ &\leq \tilde{c}_{\alpha,\beta}(1+\psi_0(\xi))^{\frac{(m-\rho(|\alpha|))(1+\sigma)}{2}} \\ &\leq \tilde{c}_{\alpha,\beta}(1+\psi_0(\xi))^{\frac{(1+\sigma)m-\rho(|\alpha|)}{2}}. \end{aligned}$$

The pseudo-differential operator $q(x,D)$ has the symbol $q \in S_\rho^{2,\psi}(\mathbb{R}^n)$. We assume that the pseudo-differential operator $p(x,D)$, defined on $S(\mathbb{R}^n)$ by

$$\begin{aligned} p(x,D)u(x) &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x,\xi)\hat{u}(\xi)d\xi \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(x,q(x,\xi))\hat{u}(\xi)d\xi \end{aligned} \tag{4.19}$$

has a symbol $p \in S_\rho^{2+\tau_1,\psi_1}(\mathbb{R}^n)$ for some appropriate $\tau_1 \geq 0$. This implies together with (4.17) that the operator $p(x,D)$ is continuous from $H^{\psi_0,2+\tau_1+2\sigma+\tau_1\sigma+s}(\mathbb{R}^n)$ to $H^{\psi_0,s}(\mathbb{R}^n)$, in particular it is continuous from $H^{\psi_0,1}(\mathbb{R}^n)$ to $H^{\psi_0,-1-\tau_1-2\sigma-\tau_1\sigma}(\mathbb{R}^n)$.

With $p(x,D)$ we can associate the bilinear form

$$B(u,v) := (p(x,D)u,v)_0, \quad u,v \in S(\mathbb{R}^n). \tag{4.20}$$

Assuming the estimate

$$|B(u,v)| \leq \kappa ||u||_{\psi_1,1}||v||_{\psi_1,1}, \quad \kappa \geq 0, \tag{4.21}$$

to hold for all $u,v \in S(\mathbb{R}^n)$, we may extend B to a continuous bilinear form on $H^{\psi_1,1}(\mathbb{R}^n)$. This extension is again denoted by B. For $u \in H^{\psi_1,1}(\mathbb{R}^n)$ we assume in addition

$$B(u,u) \geq \gamma ||u||_{\psi_0,1}^2 - \lambda_0 ||u||_0^2, \ \lambda_0 \geq 0, \gamma > 0. \tag{4.22}$$

Following ideas from I.S. Louhivaara and Chr. Simader, [29] and [30], we consider an intermediate space. To do this we consider

$$\tilde{B}_{\lambda_0}(u,v) := \tilde{B}(u,v) + \lambda_0(u,v)_0, \tag{4.23}$$

where $\tilde{B}$ is the symmetric part of B, i.e.

$$\tilde{B}_{\lambda_0}(u,v) = \frac{1}{2}(B_{\lambda_0}(u,v) + B_{\lambda_0}(v,u)$$

on $H^{\psi_1,1}(\mathbb{R}^n)$. We have

$$|\tilde{B}_{\lambda_0}(u,v)| \leq \kappa ||u||_{\psi_1,1} ||v||_{\psi_1,1}$$

and

$$\tilde{B}_{\lambda_0}(u,u) \geq \gamma ||u||^2_{\psi_0,1}.$$

Since $\tilde{B}_{\lambda_0}(u,v)$ is a scalar product on $H^{\psi_1,1}(\mathbb{R}^n)$ we may consider the completion of $H^{\psi_1,1}(\mathbb{R}^n)$ with respect to $\tilde{B}_{\lambda_0}(u,v)$. We denote this new intermediate space by $H^{p_{\lambda_0}}(\mathbb{R}^n)$. We have

$$H^{\psi_1,1}(\mathbb{R}^n) \hookrightarrow H^{p_{\lambda_0}}(\mathbb{R}^n) \hookrightarrow H^{\psi_0,1}(\mathbb{R}^n) \tag{4.24}$$

in the sense of continuous embeddings.

Lemma 4.2.1. *The bilinear form B_{λ_0} is continuous on $H^{p_{\lambda_0}}(\mathbb{R}^n)$.*

Proof. We find by using Corollary 2.4.23 in [21] that

$$\frac{1}{2}(p_{\lambda_0}(x,D) + p^*_{\lambda_0}(x,D)) = \frac{1}{2}(p_{\lambda_0}(x,D) + \bar{p}_{\lambda_0}(x,D)) + r_1(x,D)$$
$$= p_{\lambda_0}(x,D) + r_1(x,D)$$

where $r_1 \in S^{1+\tau_1,\psi_1}_\rho(\mathbb{R}^n)$ and we used that $p(x,\xi)$ is real-valued. Consider

$$|B_{\lambda_0}(u,v)| = |(p_{\lambda_0}(x,D))u,v)_0|$$
$$\leq \frac{1}{2}|((p_{\lambda_0}(x,D) + p^*_{\lambda_0}(x,D))u,v)_0| + |(r_1(x,D)u,v)_0|$$
$$= |\tilde{B}_{\lambda_0}(u,v)| + |(r_1(x,D)u,v)_0|.$$

We know that $\tilde{B}_{\lambda_0}(u,v)$ is continuous on $H^{p_{\lambda_0}}(\mathbb{R}^n)$ therefore our calculations are reduced to estimating $|(r_1(x,D)u,v)_0|$.
We know that $r_1 \in S^{\tau_1+1,\psi_1}_\rho(\mathbb{R}^n)$ therefore $r_1 \in S^{1+\tau_1+\sigma+\tau_1\sigma,\psi_0}_\rho(\mathbb{R}^n)$, this implies by Theorem 2.2.3 that

$$|(r_1(x,D)u,v)_0| \leq c||u||_{\psi_0,\frac{1+\tau_1+\sigma+\tau_1\sigma}{2}} ||v||_{\psi_0,\frac{1+\tau_1+\sigma+\tau_1\sigma}{2}}.$$

If $\tau_1 + \sigma + \tau_1\sigma < 1$ we get

$$||u||_{\psi_0, \frac{1+\tau_1+\sigma+\tau_1\sigma}{2}} \leq ||u||_{\psi_0,1} \leq c||u||_{p_{\lambda_0}}$$

implying the result by (4.24). □

Now, by the Lax-Milgram theorem, for every $g \in (H^{p_{\lambda_0}}(\mathbb{R}^n))^*$ exists a unique element $u \in H^{p_{\lambda_0}}(\mathbb{R}^n)$ satisfying

$$B_{\lambda_0}(u,v) = < g, v > \tag{4.25}$$

for all $v \in H^{p_{\lambda_0}}(\mathbb{R}^n)$. This element we call the variational solution to the equation $p(x,D)u + \lambda_0 u = g$.
From (4.24) we derive

$$H^{\psi_0,-1}(\mathbb{R}^n) = \left(H^{\psi_0,1}(\mathbb{R}^n)\right)^* \hookrightarrow (H^{p_{\lambda_0}}(\mathbb{R}^n))^*, \tag{4.26}$$

hence for $g \in H^{\psi_0,-1}(\mathbb{R}^n)$ there exists a unique $u \in H^{p_{\lambda_0}}(\mathbb{R}^n)$ satisfying (4.25). We claim now that for every $g \in H^{\psi_0,-1}(\mathbb{R}^n)$ there exists a unique $u \in H^{\psi_0,1}(\mathbb{R}^n)$ such that

$$p_{\lambda_0}(x,D)u = p(x,D)u + \lambda_0 u = g \tag{4.27}$$

holds. Denote by $u \in H^{p_{\lambda_0}}(\mathbb{R}^n)$ the unique solution to (4.25) for $g \in H^{\psi_0,-1}(\mathbb{R}^n)$ given and take a sequence $(u_k)_{k\in\mathbb{N}}$, $u_k \in S(\mathbb{R}^n)$, converging in $H^{p_{\lambda_0}}(\mathbb{R}^n)$ to u. It follows from

$$(p_{\lambda_0}(x,D)u_k, v)_0 = B_{\lambda_0}(u_k, v), \quad v \in S(\mathbb{R}^n),$$

and the continuity of $p_{\lambda_0}(x,D)$ from $H^{\psi_0,1}(\mathbb{R}^n)$ into $H^{\psi_0,(-1-2\sigma-\tau_1-\tau_1\sigma)}(\mathbb{R}^n)$ that for $k \to \infty$

$$< p_{\lambda_0}(x,D)u, v > = B_{\lambda_0}(u,v) = < g, v >$$

for all $v \in S(\mathbb{R}^n)$. Thus $p_{\lambda_0}(x,D)u = g$ in $S'(\mathbb{R}^n)$. The uniqueness follows of course once again from (4.22).
In order to get more regularity for variational solutions or equivalently for solutions to (4.27) we assume that for $\lambda \geq \lambda_0$ the function $p_\lambda^{-1}(x,\xi) := \frac{1}{p(x,\xi)+\lambda}$ belongs to $S_\rho^{-2+\tau_0,\psi_0}(\mathbb{R}^n)$ for some $\tau_0 > 0$. In this case we can prove

Theorem 4.2.2. *Let $p(x,\xi)$ be given by (4.8) where we assume for q condition (4.9) and for f we require (4.5), (4.6) to hold. In addition we suppose that $p \in S_\rho^{2+\tau_1,\psi_1}(\mathbb{R}^n) \subset S_\rho^{2+\tau_1+2\sigma+\tau_1\sigma,\psi_0}(\mathbb{R}^n)$ and $p_\lambda^{-1} \in S_\rho^{-2+\tau_0,\psi_0}(\mathbb{R}^n)$, $\tau_1 + \tau_0 + 2\sigma + \tau_1\sigma < 1$. Let $u \in H^{p_{\lambda_0}}(\mathbb{R}^n) \subset H^{\psi_0,1}(\mathbb{R}^n)$ be the solution to (4.27) for $g \in H^{\psi_0,k}(\mathbb{R}^n)$, $k \geq 0$. Then it follows that $u \in H^{\psi_0,2+k-\tau_0}(\mathbb{R}^n)$.*

Proof. From Theorem 2.1.5 it follows that

$$p_{\lambda_0}^{-1}(x,D)\circ p_{\lambda_0}(x,D) = id + r(x,D) \tag{4.28}$$

with $r \in S_0^{-1+\tau_1+\tau_0+2\sigma+\tau_1\sigma,\psi_0}(\mathbb{R}^n)$. Since $p_{\lambda_0}(x,D)u = g$ we deduce from (4.28) that

$$\begin{aligned} u &= p_{\lambda_0}^{-1}(x,D)\circ p_{\lambda_0}(x,D)u - r(x,D)u \\ &= p_{\lambda_0}^{-1}(x,D)g - r(x,D)u. \end{aligned}$$

Now, $p_{\lambda_0}^{-1}(x,D)g \in H^{\psi_0,k+2-\tau_0}(\mathbb{R}^n)$ and $r(x,D)u \in H^{\psi_0,2-\tau_1-\tau_0-2\sigma-\tau_1\sigma}(\mathbb{R}^n)$ implying that $u \in H^{\psi_0,t}(\mathbb{R}^n)$ for $t = (k+2-\tau_0)\wedge(2-\tau_1-\tau_0-2\sigma-\tau_1\sigma) > 1$. With a finite number of iterations we arrive at $u \in H^{\psi_0,2+k-\tau_0}(\mathbb{R}^n)$. □

Remark 4.2.3. *From $\tau_1+\tau_0+2\sigma+\tau_1\sigma < 1$ the necessary condition $\sigma < \frac{1}{2}$ follows.*

Corollary 4.2.4. *In the situation of Theorem 4.2.2, if $2+k-\tau_0 > \frac{n}{2\rho_0\rho_1}$, compare (4.15), then $u \in C_\infty(\mathbb{R}^n)$.*

Finally we can collect all preparatory material to prove

Theorem 4.2.5. *Let $f : \mathbb{R}^n\times[0,\infty) \to \mathbb{R}$ be an arbitrarily often differentiable function such that for $y \in \mathbb{R}^n$ fixed, the function $s \to f(y,s)$ is a Bernstein function. Moreover assume (4.5), (4.6) and (4.14). In addition let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a continuous negative definite function in the class Λ which satisfies in addition (4.13). For an elliptic symbol $q \in S_\rho^{2,\psi}(\mathbb{R}^n)$ satisfying (4.9) we define $p(x,\xi)$ by (4.8). For ψ_1 and ψ_2 defined by (4.11) and (4.12), respectively we assume (4.18). Suppose that $p \in S_\rho^{2+\tau_1,\psi_1}(\mathbb{R}^n)$ and $\frac{1}{p+\lambda} \in S_\rho^{-2+\tau_0,\psi_0}(\mathbb{R}^n)$. If $\tau_1+\tau_0+\sigma(2+\tau_1) < 1$, σ as in (4.18), then $-p(x,D)$ extends to a generator of a Feller semigroup on $C_\infty(\mathbb{R}^n)$.*

Proof. We want to apply the Hille-Yosida-Ray theorem, Theorem 1.3.6 We know that $p(x,D)$ maps $H^{\psi_0,2+k+2\sigma+\tau_1+\tau_1\sigma}(\mathbb{R}^n)$ into $H^{\psi_0,k}(\mathbb{R}^n)$. Hence if $k > \frac{n}{2\rho_0\rho_1}$ the operator $(-p(x,D), H^{\psi_0,2+k+2\sigma+\tau_1+\tau_1\sigma}(\mathbb{R}^n))$ is densely defined on $C_\infty(\mathbb{R}^n)$ with range in $C_\infty(\mathbb{R}^n)$. That $-p(x,D)$ satisfies the positive maximum principle on $H^{\psi_0,2+k+2\sigma+\tau_1+\tau_1\sigma}(\mathbb{R}^n)$ follows from Theorem 3.1.1 Now, for $\lambda \geq \lambda_0$ we know that for $g \in H^{\psi_0,k+1}(\mathbb{R}^n)$ we have a unique solution to $p_\lambda(x,D)u = g$ belonging to $H^{\psi_0,2+k+1-\tau_0}(\mathbb{R}^n)$. But $\tau_1+\tau_0+2\sigma+\tau_1\sigma < 1$ implies that $H^{\psi_0,2+k+1-\tau_0}(\mathbb{R}^n) \subset H^{\psi_0,2+k+2\sigma+\tau_1+\tau_1\sigma}(\mathbb{R}^n)$, hence for $g \in H^{\psi_0,k+1}(\mathbb{R}^n)$ we always have a (unique) solution
$u \in H^{\psi_0,2+k+2\sigma+\tau_1+\tau_1\sigma}(\mathbb{R}^n)$ implying the theorem. □

4.3 Some Concrete Examples

The first part of this section will consider the work W. Hoh has done on pseudo-differential operators with variable order of differentiation. We will consider the case where the Bernstein function $s \to f(s)$ is substituted by $(x, s) \to s^{m(x)}$ with $m : \mathbb{R}^n \to \mathbb{R}$ being a smooth function such that $0 < m(x) \leq 1$ holds. Let $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ be a continuous function such that $\xi \to q(x, \xi)$ is a continuous negative definite function. It then follows that

$$\xi \to q(x, \xi)^{m(x)} \tag{4.29}$$

is once again a continuous negative definite function implying that the pseudo-differential operator

$$Au(x) := -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} q(x, \xi)^{m(x)} \hat{u}(\xi) d\xi \tag{4.30}$$

is a candidate for a generator of a Feller semigroup. We now meet Hoh's result:

Theorem 4.3.1. *Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a fixed continuous negative definite function such that*

$$\psi(\xi) \geq c_0 |\xi|^r, \quad |\xi| \textit{ large and } r > 0, \tag{4.31}$$

holds. Let $q \in S_\rho^{2,\psi}(\mathbb{R}^n)$ be a real-valued negative definite symbol which is elliptic, in the sense that we have

$$q(x, \xi) \geq \delta_0 (1 + \psi(\xi)). \tag{4.32}$$

Further let $m : \mathbb{R}^n \to (0, 1]$ be an element in $C_b^\infty(\mathbb{R}^n)$ satisfying

$$M - \mu < \frac{1}{2} \tag{4.33}$$

where $M := \sup m(x)$ and $0 < \mu := \inf m(x)$. Consider the symbol

$$(x, \xi) \to p(x, \xi) := q(x, \xi)^{m(x)} \tag{4.34}$$

which has the property that $\xi \to p(x, \xi)$ is a continuous negative definite function. The operator

$$-p(x, D)u(x) := -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} p(x, \xi) \hat{u}(\xi) d\xi \tag{4.35}$$

maps $C_0^\infty(\mathbb{R}^n)$ into $C_\infty(\mathbb{R}^n)$, is closeable in $C_\infty(\mathbb{R}^n)$ and its closure is a generator of a Feller semigroup.

For a proof see W. Hoh [19], compare also [18].

We are now going to consider a further example. First note that the function $s \to \sqrt{s}(1-e^{-4\sqrt{s}})$ is a Bernstein function. Hence, using Corollary 3.9.36 in [20], it follows that for $0 \le \alpha \le 1$ the function $s \to s^{\frac{\alpha}{2}}(1-e^{-4s^{\frac{\alpha}{2}}})$ is also a Bernstein function. Thus, given a negative definite symbol $q \in S_\rho^{2,\psi}(\mathbb{R}^n)$ we may consider the new symbol

$$p(x,\xi) = (1+q(x,\xi))^{\frac{\alpha(x)}{2}}(1-e^{-4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}})$$

for $\alpha(\cdot)$ being an appropriate function.

Lemma 4.3.2. *Let $q \in S_\rho^{2,\psi}(\mathbb{R}^n)$ be a real-valued negative definite symbol which is elliptic in the sense that*

$$q(x,\xi) \ge \delta_0(1+\psi(\xi)),\ \delta_0 > 0. \tag{4.36}$$

Also let $\alpha(\cdot) : \mathbb{R}^n \to (0,1]$ be an element in $C_b^\infty(\mathbb{R}^n)$ satisfying

$$m - \mu < \frac{1}{2}$$

where $m = \sup \frac{\alpha(x)}{2}$ and $\mu = \inf \frac{\alpha(x)}{2} > 0$.
Now if we let $p(x,\xi) = (1+q(x,\xi))^{\frac{\alpha(x)}{2}}(1-e^{-4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}})$, then we have for all $\epsilon > 0$ the estimates

$$|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \le c_{\alpha,\beta,\epsilon} p(x,\xi)(1+\psi(\xi))^{\frac{-\rho(|\alpha|)+\epsilon}{2}} \tag{4.37}$$

i.e. $p \in S_\rho^{2m+\epsilon,\psi}(\mathbb{R}^n)$.

Proof. We have to estimate

$$\begin{aligned} \partial_\xi^\alpha \partial_x^\beta p(x,\xi) &= \partial_\xi^\alpha \partial_x^\beta((1+q(x,\xi))^{\frac{\alpha(x)}{2}}(1-e^{-4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}})) \\ &= \partial_\xi^\alpha \partial_x^\beta(e^{\frac{\alpha(x)}{2}\log(1+q(x,\xi))}(1-e^{-4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}})). \end{aligned}$$

Using (2.19) in [20] we get

$$\partial_\xi^\alpha \partial_x^\beta(e^{\frac{\alpha(x)}{2}\log(1+q(x,\xi))}(1-e^{-4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}})) =$$

$$\sum_{\alpha' \le \alpha}\sum_{\beta' \le \beta}\binom{\alpha}{\alpha'}\binom{\beta}{\beta'}(\partial_\xi^{\alpha'}\partial_x^{\beta'}e^{\frac{\alpha(x)}{2}\log(1+q(x,\xi))})$$

$$\times(\partial_\xi^{\alpha-\alpha'}\partial_x^{\beta-\beta'}(1-e^{-4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}})). \tag{4.38}$$

First consider

$$|(\partial_\xi^{\alpha'}\partial_x^{\beta'}e^{\frac{\alpha(x)}{2}\log(1+q(x,\xi))})|.$$

By (2.28) in [20] with $l=|\alpha'|+|\beta'|$ we get

$$|(\partial_\xi^{\alpha'}\partial_x^{\beta'}e^{\frac{\alpha(x)}{2}\log(1+q(x,\xi))})| \leq$$

$$e^{\frac{\alpha(x)}{2}\log(1+q(x,\xi))} \sum_{\substack{\alpha'^1+\ldots+\alpha'^{l'}=\alpha' \\ \beta'^1+\ldots+\beta'^{l'}=\beta' \\ l'=0,1,\ldots,l}} |c_{\{\alpha'^j,\beta'^j\}}\prod_{j=1}^{l'} q_{\alpha'^j\beta'^j}(x,\xi)|, \tag{4.39}$$

where

$$\begin{aligned} q_{\alpha'^j\beta'^j}(x,\xi) &= \partial_\xi^{\alpha'^j}\partial_x^{\beta'^j}\left(\frac{\alpha(x)}{2}\log(1+q(x,\xi))\right) \\ &= \sum_{\bar\beta'^j\leq\beta'^j}\begin{pmatrix}\beta'^j\\ \bar\beta'^j\end{pmatrix}\left(\partial_x^{\beta'^j-\bar\beta'^j}\frac{\alpha(x)}{2}\right)\partial_\xi^{\alpha'^j}\partial_x^{\bar\beta'^j}\log(1+q(x,\xi)). \end{aligned}$$

Now, using (2.26) in [20] with $k=|\alpha'^j|+|\bar\beta'^j|>0$ we get

$$\partial_\xi^{\alpha'^j}\partial_x^{\bar\beta'^j}\log(1+q(x,\xi)) =$$

$$\sum_{\substack{\tilde\alpha'^1+\ldots+\tilde\alpha'^{l'} \\ \tilde\beta'^1+\ldots+\tilde\beta'^{l'}=\bar\beta'^j}} c_{\{\tilde\alpha'^j,\tilde\beta'^j\}}\prod_{i=1}^{k}\frac{\partial_\xi^{\tilde\alpha'^i}\partial_x^{\tilde\beta'^i}(1+q(x,\xi))}{(1+q(x,\xi))}.$$

Since we assume that $q(x,\xi)$ is an elliptic symbol (in the sense of (4.36)) in $S_\rho^{2,\psi}(\mathbb{R}^n)$, we get

$$\left|\partial_\xi^{\alpha'^j}\partial_x^{\bar\beta'^j}\log(1+q(x,\xi))\right|$$

$$\begin{aligned} &\leq c_{\alpha'^j,\bar\beta'^j}\sum_{\substack{\tilde\alpha'^1+\ldots+\tilde\alpha'^{l'}=\alpha'^j \\ \tilde\beta'^1+\ldots+\tilde\beta'^{l'}=\bar\beta'^j}}\prod_{i=1}^{k}(1+\psi(\xi))^{\frac{-\rho(|\tilde\alpha'^i|)}{2}} \\ &\leq c_{\alpha^j,\bar\beta^j}(1+\psi(\xi))^{\frac{-\rho(|\alpha'^j|)}{2}}, \end{aligned}$$

where we used the subadditivity of ρ.We always have

$$|\log(1+q(x,\xi))| \leq c_\epsilon(1+\psi(\xi))^{\frac{\epsilon}{2l}}.$$

It follows for $\alpha \in C_b^\infty(\mathbb{R}^n)$ that

$$|q_{\alpha'^j,\beta'^j}(x,\xi)| \leq c_{\alpha'^j,\beta'^j,\epsilon} \begin{cases} (1+\psi(\xi))^{\frac{-\rho(|\alpha'^j|)}{2}}, & \alpha'^j \neq 0 \\ (1+\psi(\xi))^{\frac{\epsilon}{2l}}, & \alpha'^j = 0\ . \end{cases} \tag{4.40}$$

Putting (4.39) and (4.40) together we get

$$|(\partial_\xi^{\alpha'}\partial_x^{\beta'} e^{\frac{\alpha(x)}{2}\log(1+q(x,\xi))})| \leq c_{\alpha',\beta',\epsilon} e^{\frac{\alpha(x)}{2}\log(1+q(x,\xi))}(1+\psi(\xi))^{\frac{-\rho(|\alpha'|)+\epsilon}{2}}. \tag{4.41}$$

For the desired result we need

$$|\partial_\xi^{\alpha-\alpha'}\partial_x^{\beta-\beta'}(1-e^{-4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}})|$$

$$\leq c_{\alpha',\beta',\alpha,\beta}(1-e^{-4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}})(1+\psi(\xi))^{-\frac{\rho(|\alpha-\alpha'|)}{2}}.$$

When $\alpha-\alpha'=0$ and $\beta-\beta'=0$ there is nothing to prove.
Otherwise, by (2.28) in [20] with $l_2 = |\alpha-\alpha'|+|\beta-\beta'|$, we get

$$|\partial_\xi^{\alpha-\alpha'}\partial_x^{\beta-\beta'}(1-e^{-4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}})| \leq$$

$$e^{-4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}}|\sum c_{\{(\alpha-\alpha')^j,(\beta-\beta')^j\}}\prod_{j=1}^{l_2'} q_{(\alpha-\alpha')^j(\beta-\beta')^j}(x,\xi)|, \tag{4.42}$$

where the sum is such that

$$\begin{gathered}(\alpha-\alpha')^1+\ldots+(\alpha-\alpha')^{l_2'} = (\alpha-\alpha') \\ (\beta-\beta')^1+\ldots+(\beta-\beta')^{l_2'} = (\beta-\beta') \\ l_2' = 1,\ldots,l_2\end{gathered}$$

and where

$$q_{(\alpha-\alpha')^j(\beta-\beta')^j}(x,\xi) = \partial_\xi^{(\alpha-\alpha')^j}\partial_x^{(\beta-\beta')^j}(4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}).$$

Since $q(x,\xi)$ is a symbol in the class $S_\rho^{2,\psi}(\mathbb{R}^n)$ and satisfies (4.36) we have the estimate

$$|q_{(\alpha-\alpha')^j(\beta-\beta')^j}(x,\xi)| \leq \tilde{L}(1+q(x,\xi)) \ \text{ for all } (\alpha-\alpha')^j,(\beta-\beta')^j \in \mathbb{N}_0^n,$$

where $\tilde{L}(\lambda)$ is a suitable polynomial ≥ 0 which might depend on $(\alpha-\alpha')^j$ and $(\beta-\beta')^j$. Now returning to (4.42) we get

$$|\partial_\xi^{(\alpha-\alpha')}\partial_x^{(\beta-\beta')}(1-e^{-4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}})| \leq \tilde{L}(1+q(x,\xi))e^{-4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}}$$

$$= \frac{4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}}{1+4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}} \cdot \frac{1+4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}}{4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}}\tilde{L}(1+q(x,\xi))e^{-4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}}$$

$$\times(1+\psi(\xi))^{-\frac{\rho(|\alpha-\alpha'|)}{2}}(1+\psi(\xi))^{\frac{\rho(|\alpha-\alpha'|)}{2}}$$

$$\leq \frac{4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}}{1+4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}}(1+\psi(\xi))^{-\frac{\rho(|\alpha-\alpha'|)}{2}} \cdot c_0$$

since

$$\left|\frac{1+4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}}{4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}}(1+\psi(\xi))^{\frac{\rho(|\alpha-\alpha'|)}{2}}\tilde{L}(1+q(x,\xi))e^{-4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}}\right| \leq c_0.$$

Now using (2.7) in [20] i.e for all $a \geq 0$ and $t \geq 0$ the estimate

$$\frac{at}{1+at} \leq 1-e^{-at},$$

we get

$$|\partial_\xi^{(\alpha-\alpha')}\partial_x^{(\beta-\beta')}(1-e^{-4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}})| \leq c_0(1-e^{-4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}})(1+\psi(\xi))^{-\frac{\rho(|\alpha-\alpha'|)}{2}} \tag{4.43}$$

Substituting (4.41) and (4.43) into (4.38)

$$|\partial_\xi^\alpha\partial_x^\beta(e^{\frac{\alpha(x)}{2}\log(1+q(x,\xi))}(1-e^{-4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}}))|$$

$$\begin{aligned}
&\leq \sum_{\alpha'\leq\alpha}\sum_{\beta'\leq\beta}\binom{\alpha}{\alpha'}\binom{\beta}{\beta'}c_{\alpha',\beta',\epsilon}e^{\frac{\alpha(x)}{2}\log(1+q(x,\xi))}\\
&\quad\times(1+\psi(\xi))^{\frac{-\rho(|\alpha'|)+\epsilon}{2}}(1-e^{-4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}})(1+\psi(\xi))^{-\frac{\rho(|\alpha-\alpha'|)}{2}}\\
&\leq c_{\alpha,\beta,\epsilon}e^{\frac{\alpha(x)}{2}\log(1+q(x,\xi))}(1-e^{-4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}})\\
&\quad\times(1+\psi(\xi))^{\frac{-\rho(|\alpha|)+\epsilon}{2}}\\
&\leq c_{\alpha,\beta,\epsilon}p(x,\xi)(1+\psi(\xi))^{\frac{-\rho(|\alpha|)+\epsilon}{2}}.
\end{aligned}$$

The proof now follows from the estimate $p(x,\xi) \leq (1+\psi(\xi))^m$.

□

Lemma 4.3.3. *The function* $p_\lambda^{-1}(x,\xi) = \frac{1}{p(x,\xi)+\lambda}$ *belongs to the class* $S_\rho^{-2\mu+\epsilon,\psi}(\mathbb{R}^n)$.

Proof. Using (2.27) in [20] we find with $l = |\alpha| + |\beta|$ that

$$|\partial_\xi^\alpha \partial_x^\beta p_\lambda^{-1}(x,\xi)| \leq \frac{1}{p_\lambda(x,\xi)} \sum_{\substack{\alpha^1+\cdots+\alpha^l=\alpha \\ \beta^1+\cdots+\beta^l=\beta}} c_{\{\alpha^j,\beta^j\}} \prod_{j=1}^{l} \left| \frac{\partial_\xi^{\alpha^j}\partial_x^{\beta^j} p_\lambda(x,\xi)}{p_\lambda(x,\xi)} \right|.$$

For any $\epsilon > 0$ we find using (4.37)

$$\left| \frac{\partial_\xi^{\alpha^j}\partial_x^{\beta^j} p_\lambda(x,\xi)}{p_\lambda(x,\xi)} \right| \leq \tilde{c}_{\alpha^j,\beta^j}(1+\psi(\xi))^{\frac{-\rho(|\alpha^j|)+\epsilon}{2}}$$

and the ellipticity assumption of $q(x,\xi)$ together with the subadditivity of ρ yields

$$\left|\partial_\xi^\alpha \partial_x^\beta p_\lambda^{-1}(x,\xi)\right| \leq \tilde{c}_{\alpha,\beta,\epsilon}(1+\psi(\xi))^{-\mu}(1+\psi(\xi))^{\frac{-\rho(|\alpha|)+\epsilon}{2}}$$

which proves the lemma. □

Now applying the general framework in section 4.2 to Lemma 4.3.2 and Lemma 4.3.3 we get

Theorem 4.3.4. *Let* $q \in S_\rho^{2,\psi}(\mathbb{R}^n)$ *be a real-valued negative definite symbol which is elliptic in the sense that*

$$q(x,\xi) \geq \delta_0(1+\psi(\xi)),\ \delta_0 > 0,$$

where $\psi \in \Lambda$ *satisfies*

$$\psi(\xi) \geq c_0|\xi|^r.$$

Also let $\alpha(\cdot) : \mathbb{R}^n \to (0,1]$ *be an element in* $C_b^\infty(\mathbb{R}^n)$ *satisfying*

$$m - \mu < \frac{1}{2}$$

where $m = \sup \frac{\alpha(x)}{2}$ *and* $\mu = \inf \frac{\alpha(x)}{2} > 0$.
Now set $p(x,\xi) = (1+q(x,\xi))^{\frac{\alpha(x)}{2}}(1-e^{-4(1+q(x,\xi))^{\frac{\alpha(x)}{2}}})$ *which implies that* $\xi \to p(x,\xi)$ *is a continuous negative definite function since* $\xi \to q(x,\xi)$ *is a continuous negative definite function.. For all* $\epsilon > 0$ *we have* $p \in S_\rho^{2m+\epsilon,\psi}(\mathbb{R}^n)$ *and* $p_\lambda^{-1}(x,\xi) \in S_\rho^{-2\mu+\epsilon,\psi}(\mathbb{R}^n)$ *then the operator* $-p(x,D)$ *extends to the generator of a Feller semigroup.*

5 Subordination of Variable Order - Part II

In this chapter we aim to improve the ideas already met in the previous chapter. This is achieved by proving the "crucial" estimates that are assumed in the formal background of the proof described in section 4.2. Let $f : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}$ be an arbitrarily often differentiable function such that for $y \in \mathbb{R}^n$ fixed the function $s \to f(y, s)$ is a Bernstein function. We assume that with some $0 < r_1 \leq 1$ we have

$$\sup_{y \in \mathbb{R}^n} f(y, s) \leq c_1 s^{r_1} \text{ for } s \geq \gamma_0 \tag{5.1}$$

as well as for some $0 < r_0$ and $\tilde{\eta} > 0$ such that $0 < r_0 - \tilde{\eta} < r_1$ it holds

$$\inf_{y \in \mathbb{R}^n} f(y, s) \geq c_2 s^{r_0 - \tilde{\eta}} \text{ for } s \geq \gamma_0. \tag{5.2}$$

In our applications we will consider symbols $f(x, q(x, \xi))$ where $q(x, \xi) \geq \lambda_0(1 + \psi(\xi))$ for some real-valued continuous negative definite function ψ. Thus we can always confine ourselves to the case where $\gamma_0 > 1$. Consider again the negative definite symbol

$$p(x, \xi) = f(x, q(x, \xi)) \tag{5.3}$$

where the symbol $q(x, \xi)$ is comparable with a fixed continuous negative definite function ψ satisfying $\lim_{\xi \to \infty} \psi(\xi) = \infty$, i.e.

$$0 < c_3 \leq \frac{q(x, \xi)}{\psi(\xi)} \leq c_4, \tag{5.4}$$

for all $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$. Note that the lower bounds imply $\psi(\xi) > 0$. Since $\xi \to \psi(\xi) - \psi(0)$ is also a continuous negative definite function the lower bound in (5.4) corresponds to an estimate $\tilde{c}(1 + \tilde{\psi}(\xi)) \leq q(x, \xi)$ for $\tilde{\psi}$ being a continuous negative function which might have a zero. We find using Lemma 3.9.34.B in [20]

$$p(x, \xi) = f(x, q(x, \xi)) \leq \tilde{c}_1 (1 + \psi(\xi))^{r_1} \tag{5.5}$$

and

$$p(x, \xi) \geq \tilde{c}_2 (1 + \psi(\xi))^{r_0 - \tilde{\eta}} \tag{5.6}$$

i.e $p(x, \xi)$ is bounded above and below by continuous negative definite functions.
The pseudo-differential operator

$$p(x, D)u(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} p(x, \xi) \hat{u}(\xi) d\xi$$

$$= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(x, q(x,\xi))\hat{u}(\xi)d\xi \qquad (5.7)$$

has a symbol $p \in S_\rho^{2r_1+2\epsilon,\psi}(\mathbb{R}^n)$. The following section gives a detailed proof of this result.

5.1 Estimates for $p(x,\xi)$

Let $f : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}$ be an arbitrarily often differentiable function such that for each $x \in \mathbb{R}^n$ the function $f(x,\cdot) : (0,\infty) \to \mathbb{R}$ is a Bernstein function. For every Bernstein function $h : (0,\infty) \to \mathbb{R}$ the estimates

$$|h^{(k)}(s)| \leq \frac{k!}{s^k} h(s), \quad s > 0 \ \textit{and} \ k \in \mathbb{N}_0 \qquad (5.8)$$

hold, compare [20], Lemma 3.9.34.D. Hence for f as above we find

$$|f^{(k)}(x,s)| \leq \frac{k!}{s^k} f(x,s), \quad s > 0, \quad x \in \mathbb{R}^n \ \textit{and} \ k \in \mathbb{N}_0, \qquad (5.9)$$

where

$$f^{(k)}(x,s) = \frac{\partial^k f(x,s)}{\partial s^k}.$$

We assume now in addition:
There exists $\eta > 0$ and $\delta_0 > 0$ such that for $\epsilon \in (0,\eta)$ and for all $s \geq \delta_0$ it follows that

$$|\partial_x^\alpha \partial_s^k f(x,s)| \leq c_{\alpha,k,\epsilon} \frac{1}{s^k} f(x,s) s^\epsilon \qquad (5.10)$$

holds for all $x \in \mathbb{R}^n$ and $s \geq \delta_0$ with $c_{\alpha,k,\epsilon}$ independent of x and s.

Example 5.1.1. *Consider $f(x,s) = s^{m(x)}$ for $0 < m \leq m(x) \leq M < 1$. It follows that*

$$\partial_s^k s^{m(x)} = P_k(m(x)) \frac{1}{s^k} s^{m(x)} \qquad (5.11)$$

where $P_k(t)$ is a polynomial of degree less or equal to k. If we assume in addition that $m(\cdot) \in C^\infty(\mathbb{R}^n)$ and $|\partial^\alpha m(x)| \leq m_\alpha$ for all $\alpha \in \mathbb{N}_0^n$ we find using (5.11) that

$$\partial_x^\alpha \partial_s^k s^{m(x)} = \partial_x^\alpha (P_k(m(x)) \frac{1}{s^k} s^{m(x)})$$

$$= \frac{1}{s^k} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta} P_k(m(x)) \sum_{\substack{\beta_1 + \cdots + \beta_{l'} = \beta \\ l' = 1, \cdots, |\beta|}} c_{\{\beta_j\}} \prod_{j=1}^{l} (\partial^{\beta_j}(m(x)\ln s)) s^{m(x)}.$$

Thus we arrive at

$$|\partial_x^\alpha \partial_s^k s^{m(x)}| \leq c_{\alpha,k,m} \frac{Q(\ln s)}{s^k} s^{m(x)}$$

provided $s \geq 1$ (otherwise, if $s \geq \delta_0 > 0$, we need to treat the terms involving $\ln s$ a bit differently, for example we may switch to $|\ln s|$) where Q is a suitable polynomial. Since for $\epsilon > 0$ we find a constant c_ϵ such that $Q(\ln s) \leq c_\epsilon s^\epsilon$ holds we arrive at (5.10).

Example 5.1.2. *Consider $f(x,s) = s^{\frac{m(x)}{2}}(1 - e^{-4s^{\frac{m(x)}{2}}})$ where $m : \mathbb{R}^n \to \mathbb{R}$ is arbitrarily often differentiable such that $0 < m \leq m(x) \leq M < 2$ and $|\partial_x^\alpha m(x)| \leq m_\alpha$. Now we consider $\partial_x^\alpha \partial_s^k f(x,s)$ assuming $0 < \eta < 2 - M$ and $\gamma_0 \geq 1$. It follows with the notation used in Example 5.1.1 that*

$$\begin{aligned}
&\partial_x^\alpha \partial_s^k (s^{\frac{m(x)}{2}}(1 - e^{-4s^{\frac{m(x)}{2}}})) \\
&= \partial_x^\alpha \left(\sum_{l \leq k} \binom{k}{l} \partial_s^{k-l} s^{\frac{m(x)}{2}} \partial_s^l (1 - e^{-4s^{\frac{m(x)}{2}}}) \right) \\
&= \partial_x^\alpha \left(\sum_{l \leq k} \binom{k}{l} P_{k-l}\left(\frac{m(x)}{2}\right) \frac{1}{s^{k-l}} s^{\frac{m(x)}{2}} \partial_s^l (1 - e^{-4s^{\frac{m(x)}{2}}}) \right).
\end{aligned}$$

For $l \neq 0$ we find further

$$\begin{aligned}
&\partial_s^l (1 - e^{-4s^{\frac{m(x)}{2}}}) = -\partial_s^l (e^{-4s^{\frac{m(x)}{2}}}) \\
&= \left(- \sum_{\substack{r_1 + \cdots + r_{l'} = l \\ l' = 0, \cdots l}} c_{\{r_j\}} \prod_{j=1}^{l'} \partial_s^{r_j} (-4s^{\frac{m(x)}{2}}) \right) e^{-4s^{\frac{m(x)}{2}}} \\
&= \left(- \sum_{\substack{r_1 + \cdots + r_{l'} = l \\ l' = 0, \cdots , l}} c_{\{r_j\}} \prod_{j=1}^{l'} (-4) P_{r_j}\left(\frac{m(x)}{2}\right) \frac{1}{s^{r_j}} s^{\frac{m(x)}{2}} \right) e^{-4s^{\frac{m(x)}{2}}}.
\end{aligned}$$

This leads to

$$\begin{aligned}
&\partial_x^\alpha \partial_s^k (s^{\frac{m(x)}{2}}(1 - e^{-4s^{\frac{m(x)}{2}}}) \\
&= \partial_x^\alpha \Bigg\{ \sum_{0 < l \leq k} \Bigg(\tilde{P}_{k-l}(m(x)) \frac{1}{s^{k-l}} s^{\frac{m(x)}{2}} \sum_{\substack{r_1 + \cdots + r_{l'} = l \\ l' = 0, \cdots , l}} c_{\{r_j\}} \prod_{j=1}^{l'} P_{r_j}^*(m(x)) \frac{1}{s^{r_j}} s^{\frac{m(x)}{2}} \Bigg) e^{-4s^{\frac{m(x)}{2}}}
\end{aligned}$$

$$+\tilde{P}_k(m(x))\frac{1}{s^k}s^{\frac{m(x)}{2}}(1-e^{-4s^{\frac{m(x)}{2}}})\Big\}$$

$$= \partial_x^\alpha(S_1(x)+S_2(x)).$$

Consider the terms in $S_1(x)$. *First we note that the powers of* s *always add up to* $\frac{1}{s^k}$. *Next, any derivative of* $\tilde{P}_{k-l}(m(x))$ *or* $P^*_{r_j}(m(x))$ *will result in a function which is bounded in* x *and independent of* s. *Each derivative of* $s^{\frac{m(x)}{2}}$ *will give a term which we can write as*

$$s^{\frac{m(x)}{2}}R(\ln s, m(x),\cdots,\partial_x^\gamma m(x)) \tag{5.12}$$

with a suitable multi-index $\gamma\in\mathbb{N}_0^n$ *and* R *is a polynomial. The derivatives* $\partial_x^{\tilde{\gamma}}(e^{-4s^{\frac{m(x)}{2}}})$ *are of the type*

$$e^{-4s^{\frac{m(x)}{2}}}\tilde{R}(\ln s, m(x),\cdots,\partial_x^\gamma m(x)) \tag{5.13}$$

where $\tilde{R}$ *is again a polynomial. Taking into account that* $s\geq 1$, *we find that for every* $\delta>0$ *there exists a constant such that*

$$|\partial_x^\alpha S_1(x)|\leq c_\delta\frac{1}{s^k}s^{\frac{m(x)+\delta}{2}}e^{-4s^{\frac{m(x)}{2}}}. \tag{5.14}$$

We split the second term into two terms, the first is the one where we take no derivatives of $(1-e^{-4s^{\frac{m(x)}{2}}})$. *The second term then becomes similar to the terms already treated and for this term we get an estimate of type (5.14). The first term leads to terms of type*

$$\frac{1}{s^k}(\partial_x^\beta(\tilde{P}_k(m(x)s^{\frac{m(x)}{2}}))(1-e^{-4s^{\frac{m(x)}{2}}})$$

and using previous calculations, for every $\epsilon>0$ *this term is bounded by*

$$\tilde{c}_\delta\frac{1}{s^k}s^{\frac{m(x)+\delta}{2}}(1-e^{-4s^{\frac{m(x)}{2}}}). \tag{5.15}$$

Assuming $s\geq 1$ *implies*

$$e^{-4s^{\frac{m(x)}{2}}}\leq\left(\frac{1}{e^4-1}\right)(1-e^{-4s^{\frac{m(x)}{2}}}).$$

Hence we have proved that for all $\alpha\in\mathbb{N}_0^n$ *and* $k\in\mathbb{N}_0$ *there exists* $\epsilon>0$, $0<\epsilon<\eta$, *such that*

$$|\partial_x^\alpha\partial_s^k s^{\frac{m(x)}{2}}(1-e^{-4s^{\frac{m(x)}{2}}})|\leq c_{\alpha,k,\epsilon}\frac{1}{s^k}s^{\frac{m(x)+\epsilon}{2}}(1-e^{-4s^{\frac{m(x)}{2}}}). \tag{5.16}$$

We now return to the general case. It should be noted here that due to their length calculations may have to be split over many lines. To avoid any confusion if a calculation is written as

$$\sum$$

$$\times \sum$$

it means

$$\sum \sum$$

and not

$$\left(\sum\right)\left(\sum\right).$$

Eventually we need various controls on the symbol

$$(x,\xi) \to f(x, q(x,\xi))$$

where $q(x,\xi)$ comes from a certain symbol class which we will introduce later. For this we use a formula to calculate higher order derivatives of composed functions which is due to L. E. Fraenkel [13], compare also [20], p15.

Let $u : \mathbb{R}^m \to \mathbb{C}$ and $v_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, \ldots, m$, be smooth functions. Then for $\alpha \in \mathbb{N}_0^n$ it holds with $v = (v_1, \ldots v_m)$

$$\partial^\alpha u(v(x)) = \partial^\alpha u(v_1(x), \ldots v_m(x)) \tag{5.17}$$

$$= \sum_{\substack{1 \le |\sigma| \le |\alpha| \\ \sigma \in \mathbb{N}_0^m}} \frac{(\partial^\sigma u)(v(x))}{\sigma!} \sum_{\substack{\gamma^1 + \cdots + \gamma^m = \alpha \\ \gamma^j \in \mathbb{N}_0^n}} P_{\gamma^1}(\sigma_1, v_1; x) \cdot \ldots \cdot P_{\gamma^m}(\sigma_m, v_m; x)$$

where for $\gamma \in \mathbb{N}_0^n$

$$P_\gamma(N, v; x) := \sum_{\rho \in R(\gamma, N)} \frac{N!}{\rho!} \left(\frac{\partial^{\beta(1)} v(x)}{\beta(1)!}\right)^{\rho_1} \cdot \ldots \cdot \left(\frac{\partial^{\beta(r)} v(x)}{\beta(r)!}\right)^{\rho_r} \tag{5.18}$$

with

$$R(\gamma, N) := \left\{ \rho \in \mathbb{N}_0^r \,\middle|\, \sum_{j=1}^r \rho_j \beta(j) = \gamma \text{ and } |\rho| = N \right\}, \tag{5.19}$$

$$\mathbb{N}_{0,\gamma}^n := \{\beta \in \mathbb{N}_0^n |\ 0 < \beta \le \gamma) \tag{5.20}$$

and with $|\mathbb{N}_{0,\gamma}^n| = r$ an enumeration of $\mathbb{N}_{0,\gamma}^n$ is given by $\beta(1), \ldots, \beta(r)$. In our concrete problem many reductions happen. We consider first $f : \mathbb{R}^n \times$

$(0,\infty) \to \mathbb{R}$ artificial as $f : \mathbb{R}^n \times \mathbb{R}^n \times (0,\infty) \to \mathbb{R}$ by setting $f(x,s) = f(x_1,\ldots,x_n,1,\ldots,1,s)$. Next we introduce the $2n+1$ functions

$$v_j(x,\xi) = \begin{cases} x_j, & 1 \le j \le n \\ 1, & n+1 \le j \le 2n \\ q(x,\xi), & j = 2n+1 \end{cases} \tag{5.21}$$

In the following multi-indices in $\mathbb{N}_0^{2n}$ will be split as $\alpha = (\alpha^{(1)}, \alpha^{(2)})$ where $\alpha^{(1)}$ acts on the x-variables and $\alpha^{(2)}$ acts on the ξ variables. Our problem is to estimate

$$\partial_x^\beta \partial_\xi^\alpha f(v_1(x,\xi),\ldots,v_{2n+1}(x,\xi)) = \partial_x^\beta \partial_\xi^\alpha f(x_1,\ldots,x_n,1,\ldots,1,q(x_1,\ldots,x_n,\xi_1,\ldots\xi_n))$$

$$= \sum_{\substack{1 \le |\sigma| \le |\alpha| + |\beta| \\ \sigma \in \mathbb{N}_0^{2n+1}}} \frac{(\partial^\sigma f)(v(x,\xi))}{\sigma!}$$

$$\times \sum_{\substack{\gamma^1 + \cdots + \gamma^{2n+1} = \omega \\ \gamma^j \in \mathbb{N}_0^{2n}}} P_{\gamma^1}(\sigma_1, v_1; x, \xi) \cdot \ldots \cdot P_{\gamma^{2n+1}}(\sigma_{2n+1}, v_{2n+1}; x, \xi) \tag{5.22}$$

where $\omega = (\beta, \alpha)$. If $\gamma^j = (\delta_1^j, \delta_2^j)$ then

$$\mathbb{N}_{0,\gamma^j}^{2n} = \{(\zeta,\tau) \in \mathbb{N}_0^{2n} \mid |\zeta| + |\tau| > 0 \text{ } and \text{ } 0 < \zeta \le \delta_1^j, 0 < \tau \le \delta_2^j\}.$$

Let an enumeration of $\mathbb{N}_{0,\gamma^j}^{2n} : \eta(1) = (\zeta(1),\tau(1)),\ldots,\eta(r_j) = (\zeta(r_j),\tau(r_j))$ where $r_j = r_j(\gamma^j)$ be given. Then we have with $\sigma = (\sigma_1,\ldots,\sigma_{2n+1})$

$$R(\gamma^j,\sigma_j) = \{\rho \in \mathbb{N}_0^{r_j} \,\Big|\, \sum_{l=1}^{r_j} \rho_l \eta(l) = \gamma^j \text{ } and \text{ } |\rho| = \sigma_j\}$$

$$= \{\rho \in \mathbb{N}_0^{r_j} \,\Big|\, \sum_{l=1}^{r_j} \rho_l (\zeta(l),\tau(l)) = \gamma^j \text{ } and \text{ } |\rho| = \sigma_j\}$$

$$= \{\rho \in \mathbb{N}_0^{r_j} \,\Big|\, \sum_{l=1}^{r_j} \rho_l \zeta(l) = \delta_1^j, \sum_{l=1}^{r_j} \rho_l \tau(l) = \delta_2^j \text{ } and \text{ } |\rho| = \sigma_j\}$$

and

$$P_{\gamma^j}(\sigma_j, v_j, x, \xi) = \sum_{\rho \in R(\gamma^j,\sigma_j)} \frac{\sigma_j!}{\rho!} \left(\frac{\partial_x^{\zeta(1)} \partial_\xi^{\tau(1)} v_j(x,\xi)}{\zeta(1)!\tau(1)!} \right)^{\rho_1} \cdot \ldots \cdot \left(\frac{\partial_x^{\zeta(r_j)} \partial_\xi^{\tau(r_j)} v_j(x,\xi)}{\zeta(r_j)!\tau(r_j)!} \right)^{\rho_{r_j}}$$

where $\sigma \in \mathbb{N}_0^{2n+1}$ such that $\sigma = (\sigma_x, \sigma_\xi, \sigma_{2n+1})$, $\sigma_x, \sigma_\xi \in \mathbb{N}_0^n$ and $\sigma_{2n+1} \in \mathbb{N}_0$. When $\sigma_\xi \neq 0$ then $\partial^\sigma f = 0$ therfore

$$\sum_{\substack{1 \leq |\sigma| \leq |\alpha| + |\beta| \\ \sigma \in \mathbb{N}_0^{2n+1}}}$$

reduces to

$$\sum_{\substack{1 \leq |\sigma_x| + \sigma_{2n+1} \leq |\alpha| + |\beta| \\ \sigma \in \mathbb{N}_0^{2n+1},\ \sigma_\xi = 0}}$$

Consider $P_{\gamma^j}(\sigma_j, v_j, x, \xi)$

i) $1 \leq j \leq n$ implies

$$\partial_x^{\zeta(k)} \partial_\xi^{\tau(k)} v_j(x,\xi) = \partial_x^{\zeta(k)} \partial_\xi^{\tau(k)} x_j = \begin{cases} 1, & \zeta(k) = \epsilon_j, \tau(k) = 0 \\ 0, & \mathit{otherwise} \end{cases}$$

therfore for $1 \leq j \leq n$

$$P_{\gamma_j}(\sigma_j, v_j, x, \xi) = \sum_{\rho \in R(\gamma^j, \sigma_j)} \frac{\sigma_j!}{\rho!} \left(\frac{\partial_x^{\zeta(1)} \partial_\xi^{\tau(1)} v_j(x,\xi)}{\zeta(1)!\tau(1)!} \right)^{\rho_1} \cdot \ldots \cdot \left(\frac{\partial_x^{\zeta(r_j)} \partial_\xi^{\tau(r_j)} v_j(x,\xi)}{\zeta(r_j)!\tau(r_j)!} \right)^{\rho_{r_j}}$$

$$= \sum_{\rho \in \{\rho' \in \mathbb{N}_0^{r_j} | \sum_{l=1}^{r_j} \rho'_l \zeta(l) = \sigma_j \ \mathit{and}\ \zeta(l) = \epsilon^j\}} \frac{\sigma_j!}{\rho!} \left(\frac{\partial_x^{\zeta(1)} \partial_\xi^{\tau(1)} v_j(x,\xi)}{\zeta(1)!\tau(1)!} \right)^{\rho_1} \cdot \ldots \cdot \left(\frac{\partial_x^{\zeta(r_j)} \partial_\xi^{\tau(r_j)} v_j(x,\xi)}{\zeta(r_j)!\tau(r_j)!} \right)^{\rho_{r_j}}$$

i.e. in this case $P_{\gamma_j}(\sigma_j, v_j, x, \xi) = c_{j,\sigma}$

ii) $n+1 \leq j \leq 2n$ implies

$$\partial_x^{\zeta(k)} \partial_\xi^{\tau(k)} v_j(x,\xi) = 0 \ \ \mathit{whenever}\ \ \zeta(k) \neq 0 \ \ \mathit{or}\ \ \tau(k) \neq 0,\ \mathit{i.e.}\ (\zeta(k), \tau(k)) \neq 0,$$

i.e. for $n+1 \leq j \leq 2n$

$$P_{\gamma_j}(\sigma_j, v_j, x, \xi) = \sum_{\rho \in R(\gamma^j, \sigma_j)} \frac{\sigma_j!}{\rho!} \left(\frac{\partial_x^{\zeta(1)} \partial_\xi^{\tau(1)} v_j(x,\xi)}{\zeta(1)!\tau(1)!} \right)^{\rho_1} \cdot \ldots \cdot \left(\frac{\partial_x^{\zeta(r_j)} \partial_\xi^{\tau(r_j)} v_j(x,\xi)}{\zeta(r_j)!\tau(r_j)!} \right)^{\rho_{r_j}} = 0$$

iii) Finally let $j = 2n+1$ and set $r = r_j$, i.e.

$$R(\gamma^{2n+1}, \sigma_{2n+1}) = \{\rho \in \mathbb{N}_0^r \left| \sum_{l=1}^r \rho_l(\zeta(l), \tau(l)) = \gamma^{2n+1}, |\rho| = \sigma_{2n+1} \right\}$$

then

$$P_{\gamma^{2n+1}}(\sigma_{2n+1}, v_{2n+1}, x, \xi) = P_{\gamma^{2n+1}}(\sigma_{2n+1}, q(x,\xi))$$

$$= \sum_{\rho \in R(\gamma^{2n+1}, \sigma_{2n+1})} \frac{\sigma_{2n+1}!}{\rho!} \left(\frac{\partial_x^{\zeta(1)} \partial_\xi^{\tau(1)} q(x,\xi)}{\zeta(1)!\tau(1)!} \right)^{\rho_1} \cdot \ldots \cdot \left(\frac{\partial_x^{\zeta(r_j)} \partial_\xi^{\tau(r_j)} q(x,\xi)}{\zeta(r_j)!\tau(r_j)!} \right)^{\rho_{r_j}}.$$

We observe that

$$(\partial^\sigma f)(x_1, \ldots x_n, s) = 0 \quad if \quad \sigma_\xi \neq 0 \in \mathbb{N}_0^n.$$

Thus we find using the previous calculations and (5.22)

$$\partial_x^\beta \partial_\xi^\alpha f(x, q(x,\xi)) = \sum_{\substack{1 \le |\sigma_x| + \sigma_{2n+1} \le |\alpha| + |\beta| \\ \sigma = (\sigma_x, 0, \sigma_{2n+1}) \in \mathbb{N}_0^{2n+1}}} \left(\frac{\partial^\sigma f}{\sigma!} \right)(x, q(x,\xi))$$

$$\times \sum_{\substack{\gamma^1 + \cdots + \gamma^{2n+1} = (\beta, \alpha) \\ \gamma^j \in \mathbb{N}_0^{2n}}} P_{\gamma^1}(\sigma_1, x_1) \cdot \ldots \cdot P_{\gamma^{2n+1}}(\sigma_{2n+1}, q(x,\xi))$$

$$= \sum_{\substack{1 \le |\sigma_x| + \sigma_{2n+1} \le |\alpha| + |\beta| \\ \sigma = (\sigma_x, 0, \sigma_{2n+1}) \in \mathbb{N}_0^{2n+1}}} \left(\frac{\partial^\sigma f}{\sigma!} \right)(x, q(x,\xi))$$

$$\times \sum_{\substack{\delta_1^1 + \cdots + \delta_1^{2n+1} = \beta \\ \delta_2^1 + \cdots + \delta_2^{2n+1} = \alpha \\ \delta_k^j \in \mathbb{N}_0^n}} P_{(\delta_1^1, \delta_2^1)}(\sigma_1, x_1) \cdot \ldots \cdot P_{(\delta_1^{2n+1}, \delta_2^{2n+1})}(\sigma_{2n+1}, q(x,\xi))$$

$$= \sum_{\substack{1 \le |\sigma_x| + \sigma_{2n+1} \le |\alpha| + |\beta| \\ \sigma = (\sigma_x, 0, \sigma_{2n+1}) \in \mathbb{N}_0^{2n+1}}} \left(\frac{\partial^\sigma f}{\sigma!} \right)(x, q(x,\xi))$$

$$\times \sum_{\substack{\delta_1^1 + \cdots + \delta_1^n + \delta_1^{2n+1} = \beta \\ \delta_2^1 + \cdots + \delta_2^n + \delta_2^{2n+1} = \alpha \\ \delta_k^j \in \mathbb{N}_0^n, \delta_k^j = 0, \ n+1 \le j \le 2n}} P_{(\delta_1^1, \delta_2^1)}(\sigma_1, x_1) \cdot \ldots \cdot P_{(\delta_1^{2n+1}, \delta_2^{2n+1})}(\sigma_{2n+1}, q(x,\xi))$$

$$= \sum_{\substack{1 \le |\sigma_x| + \sigma_{2n+1} \le |\alpha| + |\beta| \\ \sigma = (\sigma_x, 0, \sigma_{2n+1}) \in \mathbb{N}_0^{2n+1}}} \left(\frac{\partial^\sigma f}{\sigma!} \right)(x, q(x,\xi))$$

$$\times \sum_{\substack{\delta_1^1 + \cdots + \delta_1^n + \delta_1^{2n+1} = \beta \\ \delta_2^{2n+1} = \alpha \\ \delta_k^j \in \mathbb{N}_0^n, \delta_k^j = 0, \ n+1 \le j \le 2n \\ \delta_1^j \in \{0, \epsilon^j\}, \ 1 \le j \le n, \ \delta_2^j = 0, \ 1 \le j \le n}} P_{(\delta_1^1, \delta_2^1)}(\sigma_1, x_1) \cdot \ldots \cdot P_{(\delta_1^{2n+1}, \delta_2^{2n+1})}(\sigma_{2n+1}, q(x,\xi))$$

$$
= \sum_{\substack{1 \le |\sigma_x| + \sigma_{2n+1} \le |\alpha| + |\beta| \\ \sigma = (\sigma_x, 0, \sigma_{2n+1}) \in \mathbb{N}_0^{2n+1}}} \left(\frac{\partial^\sigma f}{\sigma!} \right)(x, q(x,\xi))
$$

$$
\times \sum_{\substack{\delta_1^1 + \cdots + \delta_1^n + \delta_1^{2n+1} = \beta \\ \delta_2^{2n+1} = \alpha \\ \delta_k^j \in \mathbb{N}_0^n, \delta_k^j = 0,\ n+1 \le j \le 2n \\ \delta_1^j \in \{0, \epsilon^j\},\ 1 \le j \le n,\ \delta_2^j = 0,\ 1 \le j \le n}} C(\delta_1^1, \ldots, \delta_1^n, \delta_2^1, \ldots, \delta_2^n) P_{(\delta_1^{2n+1}, \alpha)}(\sigma_{2n+1}, q(x,\xi))
$$

$$
= \sum_{\substack{1 \le |\sigma_x| + \sigma_{2n+1} \le |\alpha| + |\beta| \\ \sigma = (\sigma_x, 0, \sigma_{2n+1}) \in \mathbb{N}_0^{2n+1}}} \left(\frac{\partial^\sigma f}{\sigma!} \right)(x, q(x,\xi))
$$

$$
\times \sum_{\substack{\delta_1^1 + \cdots + \delta_1^n + \delta_1^{2n+1} = \beta \\ \delta_k^j \in \mathbb{N}_0^n, \delta_k^j = 0,\ n+1 \le j \le 2n \\ \delta_1^j \in \{0, \epsilon^j\},\ 1 \le j \le n,\ \delta_2^j = 0,\ 1 \le j \le n}} C(\delta_1^1, \ldots, \delta_1^n)
$$

$$
\times \sum_{\rho \in R((\delta_1^{2n+1}, \alpha), \sigma_{2n+1})} \frac{\sigma_{2n+1}!}{\rho!} \left(\frac{\partial_x^{\zeta(1)} \partial_\xi^{\tau(1)} q(x,\xi)}{\zeta(1)!\tau(1)!} \right)^{\rho_1} \cdot \ldots \cdot \left(\frac{\partial_x^{\zeta(r_j)} \partial_\xi^{\tau(r_j)} q(x,\xi)}{\zeta(r_j)!\tau(r_j)!} \right)^{\rho_{r_{2n+1}}}
$$

In order to estimate $\partial_x^\beta \partial_\xi^\alpha f(x, q(x,\xi))$ we assume further

$$
|\partial_x^\zeta \partial_\xi^\tau q(x,\xi)| \le c_{\zeta\tau} (1 + \psi(\xi))^{\frac{2-\rho(|\tau|)}{2}} \tag{5.23}
$$

and the ellipticity condition

$$
q(x,\xi) \ge \gamma_0 (1 + \psi(\xi)). \tag{5.24}
$$

Taking (5.10) into account we find for every $\epsilon > 0$ but sufficiently small that

$$
|\partial_x^\beta \partial_\xi^\alpha f(x, q(x,\xi))| \le C'_{\alpha,\beta,\epsilon} \sum_{\substack{1 \le |\sigma_x| + \sigma_{2n+1} \le |\alpha| + |\beta| \\ \sigma = (\sigma_x, 0, \sigma_{2n+1}) \in \mathbb{N}_0^{2n+1}}} \frac{1}{q(x,\xi)^{\sigma_{2n+1}}} f(x, q(x,\xi)) q(x,\xi)^\epsilon
$$

$$
\times \sum_{\substack{\delta_1^1 + \cdots + \delta_1^n + \delta_1^{2n+1} = \beta \\ \delta_k^j \in \mathbb{N}_0^n, \delta_k^j = 0,\ n+1 \le j \le 2n \\ \delta_1^j \in \{0, \epsilon^j\},\ 1 \le j \le n,\ \delta_2^j = 0,\ 1 \le j \le n}}
$$

$$
\times \sum_{\rho \in R((\delta_1^{2n+1}, \alpha), \sigma_{2n+1})} (1 + \psi(\xi))^{\frac{(2-(2\wedge\tau(1)))}{2}\rho_1} \cdot \ldots \cdot (1 + \psi(\xi))^{\frac{(2-(2\wedge\tau(r)))}{2}\rho_{r_{2n+1}}}
$$

$$\le C'_{\alpha,\beta,\epsilon} \sum_{\substack{1\le |\sigma_x|+\sigma_{2n+1}\le |\alpha|+|\beta| \\ \sigma=(\sigma_x,0,\sigma_{2n+1})\in\mathbb{N}_0^{2n+1}}} \frac{1}{(1+\psi(\xi))^{\sigma_{2n+1}}} f(x,q(x,\xi))q(x,\xi)^{\epsilon}$$

$$\times \sum_{\substack{\delta_1^1+\cdots+\delta_1^n+\delta_1^{2n+1}=\beta \\ \delta_k^j\in\mathbb{N}_0^n,\, \delta_k^j=0,\ n+1\le j\le 2n \\ \delta_1^j\in\{0,\epsilon^j\},\ 1\le j\le n,\ \delta_2^j=0,\ 1\le j\le n}}$$

$$\times \sum_{\rho\in R((\delta_1^{2n+1},\alpha),\sigma_{2n+1})} (1+\psi(\xi))^{\rho_1+\cdots+\rho_{r_{2n+1}}}(1+\psi(\xi))^{-(\frac{(2\wedge\tau(1))}{2}\rho_1+\cdots+\frac{(2\wedge\tau(r))}{2}\rho_{r_{2n+1}})}.$$

Since for $\rho\in R((\delta_1^{2n+1},\delta_2^{2n+1}),\sigma_{2n+1})$ it follows that $\rho_1+\cdots+\rho_{r_{2n+1}}=\sigma_{2n+1}$ we arrive at

$$|\partial_x^\beta\partial_\xi^\alpha f(x,q(x,\xi))| \le \tilde{C}_{\alpha,\beta,\epsilon} f(x,q(x,\xi))q(x,\xi)^{\epsilon} \sum_{\substack{1\le |\sigma_x|+\sigma_{2n+1}\le |\alpha|+|\beta| \\ \sigma=(\sigma_x,0,\sigma_{2n+1})\in\mathbb{N}_0^{2n+1}}}$$

$$\times \sum_{\substack{\delta_1^1+\cdots+\delta_1^n+\delta_1^{2n+1}=\beta \\ \delta_k^j\in\mathbb{N}_0^n,\, \delta_k^j=0,\ n+1\le j\le 2n \\ \delta_1^j\in\{0,\epsilon^j\},\ 1\le j\le n,\ \delta_2^j=0,\ 1\le j\le n}}$$

$$\times \sum_{\rho\in R((\delta_1^{2n+1},\alpha),\sigma_{2n+1})} (1+\psi(\xi))^{-\frac{1}{2}((2\wedge\tau(1))\rho_1+\cdots+(2\wedge\tau(r))\rho_r)}.$$

Denote the last sum by $U_{R((\delta_1^{2n+1},\delta_2^{2n+1}),\sigma_{2n+1})}$, i.e.

$$U^{(\psi)}_{R((\delta_1^{2n+1},\delta_2^{2n+1}),\sigma_{2n+1})} = \sum_{\rho\in R((\gamma_1^{2n+1},\gamma_2^{2n+1})\sigma_{2n+1})} (1+\psi(\xi))^{-\frac{1}{2}((2\wedge\tau(1))\rho_1+\cdots+(2\wedge\tau(r))\rho_r)}.$$

If $|\alpha|=1$ we get a contribution from $R((\delta_1^{2n+1},\delta_2^{2n+1}),1)$ of the type $|\tau(l)|=1$, $\rho_l=1$, and we have the estimate

$$|U^{(\psi)}_{R((\delta_1^{2n+1},\delta_2^{2n+1}),1)}| \le (1+\psi(\xi))^{-\frac{1}{2}} = (1+\psi(\xi))^{-\frac{1}{2}(2\wedge|\alpha|)}.$$

If $|\alpha|=2$ we get at least one contribution from $R((\delta_1^{2n+1},\delta_2^{2n+1}),2)$ of the type $|\tau(l)|=2$, $\rho_l=1$ or $|\tau(l)|=|\tau(k)|=1$ and $\rho_l=\rho_k=1$, $l\ne k$, hence we get that the estimate

$$|U^{(\psi)}_{R((\delta_1^{2n+1},\delta_2^{2n+1}),2)}| \le (1+\psi(\xi))^{-1} = (1+\psi(\xi))^{-\frac{1}{2}(2\wedge|\alpha|)}$$

holds. Finally, for $|\alpha| \geq 3$ we find the estimate by analysing the possible terms in $R((\gamma_1^{2n+1}, \gamma_2^{2n+1})\sigma_{2n+1})$ for $\sigma_{2n+1} \geq 3$

$$|U^{(\psi)}_{R((\delta_1^{2n+1}, \delta_2^{2n+1}), k)}| \leq (1 + \psi(\xi))^{-\frac{1}{2}(2 \wedge |\alpha|)}$$

with $k \geq 3$. Thus we have proved

Theorem 5.1.3. *Suppose that $f : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}$ is arbitrarily often differentiable and that $f(x, \cdot) : (0, \infty) \to \mathbb{R}$ is a Bernstein function. Suppose further that (5.10) holds. Let $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be an arbitrarily often differentiable function satisfying with a fixed continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{R}$ condition (5.23) and (5.24). Then for every $\epsilon > 0$ sufficiently small and all $\alpha, \beta \in \mathbb{N}_0^n$ it holds*

$$|\partial_x^\beta \partial_\xi^\alpha f(x, q(x, \xi))| \leq \tilde{C}_{\alpha,\beta,\epsilon} f(x, q(x, \xi)) q(x, \xi)^\epsilon (1 + \psi(\xi))^{-\frac{1}{2}\rho(|\alpha|)}. \qquad (5.25)$$

This implies together with Theorem 2.5.4 in [21] and (5.1) that $p \in S_\rho^{2r_1+2\epsilon,\psi}(\mathbb{R}^n)$ and that $p(x, D)$ maps the space $H^{\psi,2r_1+2\epsilon+s}(\mathbb{R}^n)$ to the space $H^{\psi,s}(\mathbb{R}^n)$ i.e.

$$||p(x, D)u||_{\psi,s} \leq c||u||_{\psi,2r_1+2\epsilon+s}.$$

In particular $p(x, D)$ is continuous from $H^{\psi,s}(\mathbb{R}^n)$ to $H^{\psi,s-2r_1-2\epsilon}(\mathbb{R}^n)$ i.e.

$$||p(x, D)u||_{\psi,s-2r_1-2\epsilon} \leq c'||u||_{\psi,s}.$$

We now consider the bilinear form

$$B(u, v) := (p(x, D)u, v)_0, \quad u, v \in S(\mathbb{R}^n).$$

Since $p \in S_\rho^{2r_1+2\epsilon,\psi}(\mathbb{R}^n)$ we may apply Theorem 2.2.3 to get

$$|B(u, v)| \leq \kappa ||u||_{\psi,r_1+\epsilon} ||v||_{\psi,r_1+\epsilon}, \qquad (5.26)$$

for some $\kappa > 0$ and all $u, v \in S(\mathbb{R}^n)$ i.e. B has a continuous extension onto $H^{\psi,r_1+\epsilon}(\mathbb{R}^n)$ again denoted by B. Furthermore we have

Proposition 5.1.4. *For $u \in H^{\psi,2r_1+2\epsilon}(\mathbb{R}^n)$ we have the Gårding inequality*

$$B(u, u) \geq \delta_1 ||u||^2_{\psi,r_0-\tilde{\eta}} - \lambda_0 ||u||_0^2, \qquad (5.27)$$

for some $\lambda_0 \geq 0$.

Proof. We have the lower bound

$$p(x,\xi) \geq \tilde{c}(1+\psi(\xi))^{r_0-\tilde{\eta}}.$$

This implies

$$r(x,\xi) = p(x,\xi) - \tilde{c}(1+\psi(\xi))^{r_0-\tilde{\eta}} \geq 0.$$

Now Theorem 2.5.5 in [21], which is due to W. Hoh, see [17], gives

$$(r(x,D)u,u)_0 \geq -k||u||^2_{\frac{2r_1+2\epsilon-1}{2}} \tag{5.28}$$

since $(r(x,D)u,u)_0$ is real-valued, where

$$(r(x,D)u,u)_0 = ReB(u,u) - \tilde{c}||u||^2_{\psi,r_0-\tilde{\eta}}. \tag{5.29}$$

Putting (5.28) and (5.29) together we get

$$ReB(u,u) \geq \tilde{c}||u||^2_{\psi,r_0-\tilde{\eta}} - k||u||^2_{\frac{2r_1+2\epsilon-1}{2}}.$$

Under the assumption that

$$\psi(\xi) \geq c_0|\xi|^{\rho_0}$$

and $r_1+\epsilon-\frac{1}{2} < r_0-\tilde{\eta}$ we get for every $\epsilon_0 > 0$

$$(1+\psi(\xi))^{\frac{2r_1+2\epsilon-1}{2}} \leq \epsilon_0^2(1+\psi(\xi))^{r_0-\tilde{\eta}} + c^2(\epsilon_0)$$

which leads to

$$||u||^2_{\frac{2r_1+2\epsilon-1}{2}} \leq \epsilon_0||u||^2_{\psi,r_0-\tilde{\eta}} + c(\epsilon_0)||u||^2_0$$

implying the result. □

We are now dealing with the space $H^{\psi,r_0-\tilde{\eta}}(\mathbb{R}^n)$ and the space $H^{\psi,r_1+\epsilon}(\mathbb{R}^n)$ which is the smaller of the two. Since our estimates for B are in different space we seek to introduce an intermediate space.

Firstly, we consider the symmetric part $\tilde{B}$ of B i.e.

$$\tilde{B}_{\lambda_0}(u,v) = \frac{1}{2}(B_{\lambda_0}(u,v) + B_{\lambda_0}(v,u)$$

on $H^{\psi,r_1+\epsilon}(\mathbb{R}^n)$. Then

$$\tilde{B}_{\lambda_0}(u,v) := \tilde{B}(u,v) + \lambda_0(u,v)_0, \tag{5.30}$$

We have

$$|\tilde{B}_{\lambda_0}(u,v)| \leq \kappa||u||_{\psi_1,1}||v||_{\psi_1,1}$$

and

$$\tilde{B}_{\lambda_0}(u,u) \geq \gamma ||u||^2_{\psi_0,1}.$$

Since $\tilde{B}_{\lambda_0}(u,v)$ is a scalar product on $H^{\psi,r_1+\epsilon}(\mathbb{R}^n)$ we may consider the completion of $H^{\psi,r_1+\epsilon}(\mathbb{R}^n)$ with respect to $\tilde{B}_{\lambda_0}(u,v)$. We denote this new intermediate space by $H^{p_{\lambda_0}}(\mathbb{R}^n)$. Clearly we have

$$H^{\psi,r_1+\epsilon}(\mathbb{R}^n) \hookrightarrow H^{p_{\lambda_0}}(\mathbb{R}^n) \hookrightarrow H^{\psi,r_0-\tilde{\eta}}(\mathbb{R}^n) \tag{5.31}$$

in the sense of continuous embeddings.

Lemma 5.1.5. *The bilinear form B_{λ_0} is continuous on $H^{p_{\lambda_0}}(\mathbb{R}^n)$.*

Proof. We find by using Corollary 2.4.23 in [21] that

$$\begin{aligned}\frac{1}{2}(p_{\lambda_0}(x,D) + p^*_{\lambda_0}(x,D)) &= \frac{1}{2}(p_{\lambda_0}(x,D) + \bar{p}_{\lambda_0}(x,D)) + r_1(x,D)\\ &= p_{\lambda_0}(x,D) + r_1(x,D)\end{aligned}$$

where $r_1 \in S^{2r_1+2\epsilon-1,\psi}_{\rho}(\mathbb{R}^n)$ and we used that $p(x,\xi)$ is real-valued. Consider

$$\begin{aligned}|B_{\lambda_0}(u,v)| &= |(p_{\lambda_0}(x,D))u,v)_0|\\ &\leq \frac{1}{2}|((p_{\lambda_0}(x,D) + p^*_{\lambda_0}(x,D))u,v)_0| + |(r_1(x,D)u,v)_0|\\ &= |\tilde{B}_{\lambda_0}(u,v)| + |(r_1(x,D)u,v)_0|.\end{aligned}$$

We know that $\tilde{B}_{\lambda_0}(u,v)$ is continuous on $H^{p_{\lambda_0}}(\mathbb{R}^n)$ therefore our calculations are reduced to estimating $|(r_1(x,D)u,v)_0|$.
We know that $r_1 \in S^{2r_1+2\epsilon-1,\psi}_{\rho}(\mathbb{R}^n)$ which implies by Theorem 2.2.3 that

$$|(r_1(x,D)u,v)_0| \leq c||u||_{\psi,r_1+\epsilon-\frac{1}{2}}||v||_{\psi,r_1+\epsilon-\frac{1}{2}}.$$

If $r_1 + \epsilon - \frac{1}{2} < r_0 - \tilde{\eta}$ we get

$$||u||_{\psi,r_1+\epsilon-\frac{1}{2}} \leq ||u||_{\psi,r_0-\tilde{\eta}} \leq c||u||_{p_{\lambda_0}}$$

implying the result by (5.31). □

By the Lax-Milgram theorem, for every $g \in (H^{p_{\lambda_0}})^* \subset S'(\mathbb{R}^n)$ there exists a unique element $u \in H^{p_{\lambda_0}}$ satisfying

$$B_{\lambda_0}(u,v) =< g,v > \tag{5.32}$$

for all $v \in H^{p_{\lambda_0}}$. We call u the variational solution to $p(x,D)u + \lambda_0 u = g$. From (5.31) we get

$$H^{\psi,-(r_0-\tilde{\eta})}(\mathbb{R}^n) = (H^{\psi,r_0-\tilde{\eta}}(\mathbb{R}^n))^* \hookrightarrow (H^{p_{\lambda_0}}(\mathbb{R}^n))^*,$$

hence for $g \in H^{\psi,-(r_0-\tilde{\eta})}(\mathbb{R}^n)$ there exists a unique $u \in H^{p_{\lambda_0}}(\mathbb{R}^n)$ satisfying (5.32).

Proposition 5.1.6. *For every $g \in H^{\psi,-(r_0-\tilde{\eta})}(\mathbb{R}^n)$ there exists a unique $u \in H^{\psi,r_0-\tilde{\eta}}(\mathbb{R}^n)$ such that*

$$p_{\lambda_0}(x,D)u = p(x,D)u + \lambda_0 u = g \tag{5.33}$$

holds as an equality in $S'(\mathbb{R}^n)$.

Proof. Denote by $u \in H^{p_{\lambda_0}}(\mathbb{R}^n)$ the unique solution to (5.33) for $g \in H^{\psi,r_0-\tilde{\eta}}(\mathbb{R}^n)$ given and take a sequence $(u_k)_{k\in\mathbb{N}}$, $u_k \in S(\mathbb{R}^n)$ converging in $H^{p_{\lambda_0}}(\mathbb{R}^n)$ to u. It follows from

$$(p_{\lambda_0}(x,D)u_k, v)_0 = B_{\lambda_0}(u_k, v), \quad v \in S(\mathbb{R}^n)$$

and the continuity of $p_{\lambda_0}(x,D)$ from $H^{\psi,s}(\mathbb{R}^n)$ to $H^{\psi,s-2r_1-2\epsilon}(\mathbb{R}^n)$ that for $k \to \infty$

$$< p_{\lambda_0}(x,D)u, v >= B_{\lambda_0}(u,v) =< g, v >$$

for all $v \in S(\mathbb{R}^n)$. Thus $p_{\lambda_0}(x,D)u = g$ in $S'(\mathbb{R}^n)$ and the uniqueness follows from (5.27). □

In order to get more regularity for variational solutions we have

Theorem 5.1.7. *Assume (5.10), (5.23), (5.24) and in addition let $\tilde{\eta} = 0$ in (5.2) to give for $\tilde{\gamma} > 0$ that*

$$\tilde{\gamma}(1+\psi(\xi))^{r_0} \leq f_0(\gamma_0(1+\psi(\xi))) \tag{5.34}$$

holds where

$$f_0(s) := \inf_{x\in\mathbb{R}^n} f(x,s). \tag{5.35}$$

Then for every $\tilde{\eta} > 0$ sufficiently small the function $p_\lambda^{-1}(x,\xi)$ belongs to the class $S_\rho^{-2r_0+2\tilde{\eta},\psi}(\mathbb{R}^n)$.

Proof. Let us assume for simplicity that $\delta_0 = \gamma_0$, compare (5.10). For $\lambda > 0$ let $p_\lambda(x,\xi) = \lambda + f(x, q(x,\xi))$. From (2.27) in [20] we find with $l = |\alpha| + |\beta|$

$$\left|\partial_\xi^\alpha \partial_x^\beta \frac{1}{p_\lambda(x,\xi)}\right| \leq \frac{1}{p_\lambda(x,\xi)} \sum_{\substack{\alpha^1+\cdots+\alpha^l=\alpha \\ \beta^1+\cdots+\beta^l=\beta}} c_{\{\alpha^j,\beta^j\}} \prod_{j=1}^{l} \left| \frac{\partial_\xi^{\alpha^j} \partial_x^{\beta^j} p_\lambda(x,\xi)}{p_\lambda(x,\xi)} \right|$$

$$= \frac{1}{\lambda + f(x,q(x,\xi))} \sum_{\substack{\alpha^1+\cdots+\alpha^l=\alpha \\ \beta^1+\cdots+\beta^l=\beta}} c_{\{\alpha^j,\beta^j\}} \prod_{j=1}^{l} \left| \frac{\partial_\xi^{\alpha^j} \partial_x^{\beta^j} p_\lambda(x,\xi)}{p_\lambda(x,\xi)} \right|.$$

From the definition of $f_1(s)$, (5.35), together with (5.24) we get

$$\left|\partial_\xi^\alpha \partial_x^\beta \frac{1}{p_\lambda(x,\xi)}\right| \leq \frac{1}{f_1(\gamma_0(1+\psi(\xi)))} \sum_{\substack{\alpha^1+\cdots+\alpha^l=\alpha \\ \beta^1+\cdots+\beta^l=\beta}} c_{\{\alpha^j,\beta^j\}} \prod_{j=1}^{l} \left| \frac{\partial_\xi^{\alpha^j} \partial_x^{\beta^j} p_\lambda(x,\xi)}{p_\lambda(x,\xi)} \right|$$

$$\leq \hat{c}_{\alpha,\beta} \frac{1}{f_1(\gamma_0(1+\psi(\xi)))} \sum_{\substack{\alpha^1+\cdots+\alpha^l=\alpha \\ \beta^1+\cdots+\beta^l=\beta}} \prod_{j=1}^{l} \left| \frac{\partial_\xi^{\alpha^j} \partial_x^{\beta^j} (\lambda + f(x,q(x,\xi)))}{\lambda + f(x,q(x,\xi))} \right|$$

$$= \hat{c}_{\alpha,\beta} \frac{1}{f_1(\gamma_0(1+\psi(\xi)))} \sum_{\substack{\alpha^1+\cdots+\alpha^l=\alpha \\ \beta^1+\cdots+\beta^l=\beta}} \prod_{j=1}^{l} \left| \frac{\partial_\xi^{\alpha^j} \partial_x^{\beta^j} \lambda + \partial_\xi^{\alpha^j} \partial_x^{\beta^j} f(x,q(x,\xi))}{\lambda + f(x,q(x,\xi))} \right|.$$

When $\alpha = \beta = 0$ we find

$$\left|\partial_\xi^\alpha \partial_x^\beta \frac{1}{p_\lambda(x,\xi)}\right| \leq \tilde{c}_{0,0} \frac{1}{f_1(\gamma_0(1+\psi(\xi)))}.$$

Moreover whenever $\alpha^j = \beta^j = 0$, then

$$\left| \frac{\partial_\xi^{\alpha^j} \partial_x^{\beta^j} \lambda + \partial_\xi^{\alpha^j} \partial_x^{\beta^j} f(x,q(x,\xi))}{\lambda + f(x,q(x,\xi))} \right| = 1.$$

Therefore we now only have to consider

$$\left|\partial_\xi^\alpha \partial_x^\beta \frac{1}{p_\lambda(x,\xi)}\right| \leq \hat{c}_{\alpha,\beta} \frac{1}{f_1(\gamma_0(1+\psi(\xi)))} \sum_{\substack{\alpha^1+\cdots+\alpha^l=\alpha \\ \beta^1+\cdots+\beta^l=\beta \\ \alpha^j+\beta^j>0}} \prod_{j=1}^{l} \left| \frac{\partial_\xi^{\alpha^j} \partial_x^{\beta^j} f(x,q(x,\xi))}{\lambda + f(x,q(x,\xi))} \right|$$

Further, using (5.25) we get

$$\left|\partial_\xi^\alpha \partial_x^\beta \frac{1}{p_\lambda(x,\xi)}\right| \leq \frac{1}{f_1(\gamma_0(1+\psi(\xi)))} \sum_{\substack{\alpha^1+\cdots+\alpha^l=\alpha \\ \beta^1+\cdots+\beta^l=\beta \\ \alpha^j+\beta^j>0}}$$

$$\times \prod_{j=1}^{l} \frac{c_{\alpha^j\beta^j\epsilon} f(x,q(x,\xi)) q(x,\xi)^\epsilon (1+\psi(\xi))^{-\frac{\rho(|\alpha^j|)}{2}}}{\lambda + f(x,q(x,\xi))}$$

$$= \frac{1}{f_1(\gamma_0(1+\psi(\xi)))} \sum_{\substack{\alpha^1+\cdots+\alpha^l=\alpha \\ \beta^1+\cdots+\beta^l=\beta \\ \alpha^j+\beta^j>0}} \left[\frac{f(x,q(x,\xi)) q(x,\xi)^\epsilon}{\lambda + f(x,q(x,\xi)))} \right]^l \left(\prod_{j=1}^{l} c_{\alpha^j\beta^j\epsilon} \right)$$

$$\times(1+\psi(\xi))^{-\frac{1}{2}(\rho(|\alpha^1|)+\cdots+\rho(|\alpha^l|))}$$

$$\leq \gamma_{\alpha,\beta,\epsilon}\frac{1}{f_1(\gamma_0(1+\psi(\xi)))}\sum_{\substack{\alpha^1+\cdots+\alpha^l=\alpha\\ \beta^1+\cdots+\beta^l=\beta\\ \alpha^j+\beta^j>0}}\left[\frac{f(x,q(x,\xi))q(x,\xi)^\epsilon}{\lambda+f(x,q(x,\xi))}\right]^l(1+\psi(\xi))^{-\frac{1}{2}\rho(|\alpha|)}$$

using the sub-additivity of ρ. Since $\left|\frac{f(x,q(x,\xi))}{\lambda+f(x,q(x,\xi))}\right|\leq 1$ we find

$$\left|\partial_\xi^\alpha\partial_x^\beta\frac{1}{p_\lambda(x,\xi)}\right|\leq\frac{c_{\alpha,\beta,\epsilon}}{f_1(\gamma_0(1+\psi(\xi)))}\sum_{\substack{\alpha^1+\cdots+\alpha^l=\alpha\\ \beta^1+\cdots+\beta^l=\beta\\ \alpha^j+\beta^j>0}}q(x,\xi)^{l\epsilon}(1+\psi(\xi))^{-\frac{1}{2}\rho(|\alpha|)}. \quad (5.36)$$

Since $q(x,\xi)\leq c_{0,0}(1+\psi(\xi))$ it follows further that given $\tilde{\eta}>0$ we can find $\tilde{c}_{\alpha,\beta,\tilde{\eta}}>0$ such that

$$\left|\partial_\xi^\alpha\partial_x^\beta\frac{1}{p_\lambda(x,\xi)}\right|\leq\frac{\tilde{c}_{\alpha,\beta,\tilde{\eta}}}{f_0(\gamma_0(1+\psi(\xi)))}(1+\psi(\xi))^{\tilde{\eta}-\frac{1}{2}\rho(|\alpha|)}. \quad (5.37)$$

Taking into account (5.34) we eventually arrive at

$$\left|\partial_\xi^\alpha\partial_x^\beta\frac{1}{p_\lambda(x,\xi)}\right|\leq\hat{c}_{\alpha,\beta,\tilde{\eta}}(1+\psi(\xi))^{\frac{-2r_0+2\tilde{\eta}-\rho(|\alpha|)}{2}}, \quad (5.38)$$

i.e. $p_\lambda^{-1}\in S_\rho^{-2r_0+2\tilde{\eta},\psi}(\mathbb{R}^n)$ proving the theorem. □

We can now prove

Theorem 5.1.8. *Let $p(x,\xi)$ be given by (5.3) where we assume for q condition (5.4). For f it is supposed that (5.1) and (5.2) hold. Then we have that $p\in S_\rho^{2r_1+2\epsilon,\psi}(\mathbb{R}^n)$ and $p_\lambda^{-1}\in S_\rho^{-2r_0+2\tilde{\eta},\psi}(\mathbb{R}^n)$ where we assume that $r_1-r_0<\frac{1}{2}$. Let $u\in H^{p_{\lambda_0}}(\mathbb{R}^n)\subset H^{\psi,r_0-\tilde{\eta}}(\mathbb{R}^n)$ be the solution to (5.33) for $g\in H^{\psi,k}(\mathbb{R}^n)$, $k\geq 0$. Then it follows that $u\in H^{\psi,k+2r_0-2\tilde{\eta}}(\mathbb{R}^n)$.*

Proof. The statements for p and p_λ^{-1} have already been proved, i.e. Theorem 5.1.3 and Theorem 5.1.7, respectively. From Theorem 2.1.5 it follows that

$$p_{\lambda_0}^{-1}(x,D)\circ p_{\lambda_0}(x,D)=id+r(x,D) \quad (5.39)$$

with $r\in S_0^{2r_1+2\epsilon-2r_0+2\tilde{\eta}-1,\psi}(\mathbb{R}^n)$. Since $p_{\lambda_0}(x,D)u=g$ we deduce from (5.39) that

$$\begin{aligned} u &= p_{\lambda_0}^{-1}(x,D)\circ p_{\lambda_0}(x,D)u-r(x,D)u\\ &= p_{\lambda_0}^{-1}(x,D)g-r(x,D)u. \end{aligned}$$

Now, $p_{\lambda_0}^{-1}(x, D)g \in H^{\psi,2r_0-2\tilde{\eta}+k}(\mathbb{R}^n)$ and $r(x, D)u \in H^{\psi,3r_0-3\tilde{\eta}-2r_1-2\epsilon+1}(\mathbb{R}^n)$ implying that $u \in H^{\psi,t}(\mathbb{R}^n)$ for $t = (k + 2r_0 - 2\tilde{\eta}) \wedge (3r_0 - 3\tilde{\eta} - 2r_1 - 2\epsilon + 1) > r_0 - \tilde{\eta}$. With a finite number of iterations we arrive at $u \in H^{\psi,k+2r_0-2\tilde{\eta}}(\mathbb{R}^n)$. □

Corollary 5.1.9. *If $k + 2r_0 - 2\tilde{\eta} > \frac{n}{2\rho_1}$, compare (4.13) in the situation of Theorem 5.1.8, then $u \in C_\infty(\mathbb{R}^n)$.*

We finally arrive at

Theorem 5.1.10. *Let $f : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}$ be an arbitrarily often differentiable function such that for $y \in \mathbb{R}^n$ fixed, the function $s \to f(y, s)$ is a Bernstein function. Moreover assume (5.1) and (5.2). In addition let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a continuous negative definite function in the class Λ which satisfies in addition (4.13). For an elliptic symbol $q \in S_\rho^{2,\psi}(\mathbb{R}^n)$ satisfying (5.4) we define $p(x, \xi)$ by (5.3). We know that $p \in S_\rho^{2r_1+2\epsilon,\psi}(\mathbb{R}^n)$ and $\frac{1}{p+\lambda} \in S_\rho^{-2r_0+2\tilde{\eta},\psi}(\mathbb{R}^n)$. If $r_1 - r_0 < \frac{1}{2}$ then $-p(x, D)$ extends to a generator of a Feller semigroup on $C_\infty(\mathbb{R}^n)$.*

Proof. The statements for p and p_λ^{-1} have already been proved, i.e. Theorem 5.1.3 and Theorem 5.1.7, respectively. We want to apply the Hille-Yosida-Ray theorem, Theorem 1.3.6 We know that $p(x, D)$ maps $H^{\psi,2r_1+2\epsilon+k}(\mathbb{R}^n)$ into $H^{\psi,k}(\mathbb{R}^n)$. Hence if $k > \frac{n}{2\rho_1}$ the operator $(-p(x, D), H^{\psi,2r_1+2\epsilon+k}(\mathbb{R}^n))$ is densely defined on $C_\infty(\mathbb{R}^n)$ with range in $C_\infty(\mathbb{R}^n)$. That $-p(x, D)$ satisfies the positive maximum principle on $H^{\psi,2r_1+2\epsilon+k}(\mathbb{R}^n)$ follows from Theorem 3.1.1. Now, for $\lambda \geq \lambda_0$ we know that for $g \in H^{\psi,k+2r_1-2r_0+2\tilde{\eta}+2\epsilon}(\mathbb{R}^n)$ we have a unique solution to $p_\lambda(x, D)u = g$ belonging to $H^{\psi,k+2r_1+2\epsilon}(\mathbb{R}^n)$ implying the theorem. □

6 Dirichlet Forms

The purpose of this chapter is to study Dirichlet forms, we will indicate how Dirichlet forms may lead to a different approach of variable order subordination.

6.1 Dirichlet Forms: A few Remarks in Relation to Subordination of Variable Order

Let $(T_t)_{t\geq 0}$ be a sub-Markovian semigroup in $L^2(\mathbb{R}^n)$, i.e. $(T_t)_{t\geq 0}$ is a strongly continuous contraction semigroup on $L^2(\mathbb{R}^n)$ and if $0 \leq u \leq 1$ holds almost everywhere then $0 \leq T_t u \leq 1$ holds almost everywhere for $u \in L^2(\mathbb{R}^n)$. Denote by A the generator of T_t, we know by Lemma 4.6.6 that A satisfies

$$\int_{\mathbb{R}^n} (Au)(u-1)^+ dx \leq 0$$

for all $u \in D(A)$, i.e. A is a **Dirichlet operator**.
Further, by Definition 1.3.8 a negative definite operator A satisfies on its domain $D(A) \subset L^2(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} (Au)(signu)|u| dx \leq 0 \tag{6.1}$$

i.e.

$$\int_{\mathbb{R}^n} (Au) u dx \leq 0$$

or

$$\int_{\mathbb{R}^n} (-Au) u dx \geq 0$$

for all $u \in D(A) \subset L^2(\mathbb{R}^n)$. This implies that $(-A)$ is a non-negative definite operator. If we now let $(T_t)_{t\geq 0}$ be a symmetric sub-Markovian semigroup in $L^2(\mathbb{R}^n)$ then $T_t = T_t^*$ and by Corollary 4.1.46 in [20] it follows that $A = A^*$ as closed operators. Therefore $(A, D(A))$ is a self-adjoint operator, i.e.

$$D(A^*) = D(A) \quad and \quad (Au, v)_0 = (u, Av)_0.$$

To summarise, we have a non-negative self-adjoint operator $(-A)$ such that

$$\mathcal{E}(u,v) := ((-A)^{\frac{1}{2}} u, (-A)^{\frac{1}{2}} v)_0$$

is a continuous bilinear form on $D(\mathcal{E}) = D((-A)^{\frac{1}{2}})$ with respect to the norm $||u||_{\mathcal{E}}^2 = ||u||_0^2 + \mathcal{E}(u,u)$. Consider the bilinear form on $u \in D(A), v \in D(\mathcal{E})$

$$\mathcal{E}(u,v) = (-Au, v)_0 = \int_{\mathbb{R}^n} (-Au) v dx = \int_{\mathbb{R}^n} (-A)^{\frac{1}{2}} (-A)^{\frac{1}{2}} u \cdot v dx$$

$$= \int_{\mathbb{R}^n} (-A)^{\frac{1}{2}} u (-A)^{\frac{1}{2}} v dx$$
$$= ((-A)^{\frac{1}{2}} u, (-A)^{\frac{1}{2}} v)_0.$$

Using the Cauchy-Schwarz inequality we find

$$|(-Au, v)_0| \leq ||(-A)^{\frac{1}{2}} u||_0 ||(-A)^{\frac{1}{2}} v||_0$$

or

$$|(-Au, v)_0| \leq ((-Au, u)_0)^{\frac{1}{2}} ((-Av, v)_0)^{\frac{1}{2}}. \tag{6.2}$$

In the sense of continuous embeddings we have

$$D(A) \hookrightarrow D(\mathcal{E}) \hookrightarrow L^2(\mathbb{R}^n).$$

We also find that

$$|\mathcal{E}(u, v)| \leq (\mathcal{E}_1(u, v))^{\frac{1}{2}} (\mathcal{E}_1(u, v))^{\frac{1}{2}} \tag{6.3}$$

where

$$\mathcal{E}_1(u, v) = \mathcal{E}(u, v) + (u, v)_0.$$

Theorem 6.1.1. *Let $(A, D(A))$ be a densely defined operator on $L^2(\mathbb{R}^n)$ satisfying (6.1) and (6.2). Then there exists a closed bilinear form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(\mathbb{R}^n)$ such that $D(A) \subset D(\mathcal{E}) \subset L^2(\mathbb{R}^n)$. Thus $\mathcal{E}$ is densely defined and for $u \in D(A)$, $v \in D(\mathcal{E})$ we have $\mathcal{E}(u, v) = (-Au, v)_0$. Moreover, $\mathcal{E}$ satisfies (6.3).*

Compare Theorem 4.7.5 in [20].

It is not assumed that $(A, D(A))$ is a closed operator, however in the situation of Theorem 6.1.1 it is closeable, where we denote the closure by $\tilde{A}$ and the domain of its closure $D(\tilde{A})$ is a subspace of $D(\mathcal{E})$. Theorem 6.1.1 also holds for any Dirichlet operator satisfying (6.2). We aim to find a certain converse to Theorem 6.1.1. Before we can state the main results between Dirichlet operators and certain types of bilinear forms, i.e. Dirichlet forms, we have to introduce the following definitions

Definition 6.1.2. *A bilinear form $(\mathcal{E}, D(\mathcal{E}))$ is a* **closed form** *on $L^2(\mathbb{R}^n)$ if $(D(\mathcal{E}), \mathcal{E}_1^{sym})$, where $\mathcal{E}_1^{sym}(u, v) := \frac{1}{2}(\mathcal{E}_1(u, v) + \mathcal{E}_1(v, u))$, is a Hilbert space and $\mathcal{E}$ is continuous with respect to $\mathcal{E}_1^{sym}$, i.e.*

$$|\mathcal{E}(u, v)| \leq (\mathcal{E}_1^{sym}(u, u))^{\frac{1}{2}} (\mathcal{E}_1^{sym}(v, v))^{\frac{1}{2}}$$

holds for all $u, v \in D(\mathcal{E})$.

Definition 6.1.3. *Let $(\mathcal{E}, D(\mathcal{E}))$ be a closed form on $L^2(\mathbb{R}^n)$.*
A. We call $(\mathcal{E}, D(\mathcal{E}))$ a **semi-Dirichlet form** *if for all $u \in D(\mathcal{E})$ it follows that $u^+ \wedge 1 \in D(\mathcal{E})$ and*

$$\mathcal{E}(u + (u^+ \wedge 1), u - (u^+ \wedge 1)) \geq 0$$

holds.
B. The form $(\mathcal{E}, D(\mathcal{E}))$ is called a **non-symmetric Dirichlet form** *if for all $u \in D(\mathcal{E})$ it follows that $u^+ \wedge 1 \in D(\mathcal{E})$ and*

$$\begin{aligned} &\mathcal{E}(u + (u^+ \wedge 1), u - (u^+ \wedge 1)) \geq 0, \\ &\mathcal{E}(u - (u^+ \wedge 1), u + (u^+ \wedge 1)) \geq 0. \end{aligned} \tag{6.4}$$

C. If $(\mathcal{E}, D(\mathcal{E}))$ satisfies (6.4) and also is a symmetric form, then we call $(\mathcal{E}, D(\mathcal{E}))$ a **symmetric Dirichlet form***.*

We are now in a position to give some main results for Dirichlet operators and Dirichlet forms.

Firstly, if $(A, D(A))$ is a Dirchlet operator on $L^2(\mathbb{R}^n)$ satisfying (6.2) and generating a sub-Markovian semigroup $(T_t)_{t\geq 0}$ then the bilinear form $(\mathcal{E}, D(\mathcal{E}))$ is a semi-Dirichlet form. Conversely, suppose that $(\mathcal{E}, D(\mathcal{E}))$ is a semi-Dirichlet form on $L^2(\mathbb{R}^n)$ then the associated operator $(A, D(A))$ is a Dirichlet operator and the associated semigroup $(T_t)_{t\geq 0}$ is sub-Markovian on $L^2(\mathbb{R}^n)$.

Next we consider non-symmetric Dirichlet forms. If $(A^*, D(A^*))$ is also a Dirichlet operator , then $(\mathcal{E}, D(\mathcal{E}))$ is a non-symmetric Dirichlet form. Conversely, If $(\mathcal{E}, D(\mathcal{E}))$ is a non-symmetric Dirichlet form, then $(A^*, D(A^*))$ is a Dirichlet operator and the associated semigroup $(T_t)_{t\geq 0}$ is sub-Markovian.

Finally, for symmetric Dirichlet form the following holds. If $(A, D(A))$ is selfadjoint then $(\mathcal{E}, D(\mathcal{E}))$ is a symmetric Dirichlet form. The converse is also true.

We now consider some examples with the aim of demonstrating how the Fourier transform and continuous negative definite functions come into play.

Example 6.1.4. *Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a continuous negative definite function with associated convolution semigroup $(\mu_t)_{t\geq 0}$. The operator $-\psi(D)$ defined on $C_0^\infty(\mathbb{R}^n)$ by*

$$-\psi(D)u(x) = -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \psi(\xi)\hat{u}(\xi)d\xi \tag{6.5}$$

extends to a selfadjoint Dirichlet operator $(A, H^{\psi,2}(\mathbb{R}^n))$ (recall that the graph norm $||u||_{\psi(D),0} = ||u||_0 + ||\phi(D)u||_0$ is equivalent to $||u||_{\psi,2}$). Further for

$u \in H^{\psi,2}(\mathbb{R}^n)$ *we find*

$$\mathcal{E}(u,u) := \int_{\mathbb{R}^n} (-Au)(x)u(x)dx$$

$$= \int_{\mathbb{R}^n} \psi(\xi)\hat{u}(\xi)\overline{\hat{u}(\xi)}d\xi = \int_{\mathbb{R}^n} \psi(\xi)|\hat{u}(\xi)|^2 d\xi,$$

which implies that the symmetric Dirichlet form corresponding to $(A, H^{\psi,2}(\mathbb{R}^n))$ *has the domain* $D(\mathcal{E}) = H^{\psi,1}(\mathbb{R}^n)$ *and it is given by*

$$\mathcal{E}(u,v) = \int_{\mathbb{R}^n} \psi(\xi)\hat{u}(\xi)\overline{\hat{v}(\xi)}d\xi$$

$$= \int_{\mathbb{R}^n} [\psi(D)]^{\frac{1}{2}}u \cdot [\psi(D)]^{\frac{1}{2}}v d\xi, \tag{6.6}$$

where $[\psi(D)]^{\frac{1}{2}}u$ *is given on* $C_0^\infty(\mathbb{R}^n)$ *by (6.5) but with* $\psi(\xi)^{\frac{1}{2}}$ *instead of* $\psi(\xi)$. *In particular, since* $(1+\psi(\xi))^{\frac{1}{2}}$ *is also a continuous negative definite function with values only in* $\mathbb{R}$, *we see that for any continuous negative definite function* $\psi : \mathbb{R}^n \to \mathbb{R}$ *the space* $(H^{\psi,1}(\mathbb{R}^n), (.,.)_1)$ *is a symmetric Dirichlet space and therfore* $u \in H^{\psi,1}(\mathbb{R}^n)$ *implies always that* $u^+ \wedge \lambda$ *and* $u \wedge \lambda$, $\lambda \geq 0$ *belongs to* $H^{\psi,1}(\mathbb{R}^n)$ *too.*

We will now take a different approach, we will consider $\mathcal{E}(u,v)$ using a Lévy- Khinchin representation for the continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{R}$.

Example 6.1.5. *The continuous negative definite function* $\psi : \mathbb{R}^n \to \mathbb{R}$ *has the following Lévy-Khinchin representation*

$$\psi(\xi) = c + q(\xi) + \int_{\mathbb{R}^n\setminus\{0\}} (1 - \cos(x\cdot\xi))\nu(dx) \tag{6.7}$$

where $c \geq 0$ *is a constant,* q *a symmetric positive semidefinite quadratic form on* $\mathbb{R}^n$ *and* ν *is a measure on* $\mathbb{R}^n$ *integrating* $x \to |x|^2 \wedge 1$. *Compare Corollary 3.7.9 in [20]. We now substitute (6.7) into (6.6) to get*

$$\mathcal{E}^\psi(u,v) = \int_{\mathbb{R}^n} \left(c + \sum_{j,k=1}^n q_{lk}\xi_l\xi_k + \int_{\mathbb{R}^n\setminus\{0\}} (1 - \cos(x\cdot\xi))\nu(dx) \right) \tag{6.8}$$

$$\times \hat{u}(\xi)\overline{\hat{v}(\xi)}d\xi.$$

Using Plancherel's theorem, Corollary 3.1.3 in [20] and the proof of Theorem 3.10.17 in [20] to find

$$\mathcal{E}^{\psi}(u,v) = c\int_{\mathbb{R}^n} u(x)v(x)dx + \int_{\mathbb{R}^n}\sum_{k,l=1}^{n} q_{kl}\frac{\partial u(x)}{\partial x_k}\frac{\partial v(x)}{\partial x_l}dx \qquad (6.9)$$

$$+\frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(u(x+y)-u(x))(v(x+y)-v(x))\nu(dy)dx.$$

In the case that $q_{kl} = 0$ *for* $k, l = 1, \ldots n$ *and* $c = 0$ *we have*

$$D(\mathcal{E}) = \left\{ u \in L^2(\mathbb{R}^n) \middle| \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(u(x+y)-u(x))^2\nu(dy)dx < \infty \right\}$$

and (6.9) becomes

$$\mathcal{E}^{\psi}(u,v) = \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(u(x+y)-u(x))(v(x+y)-v(x))\nu(dy)dx. \qquad (6.10)$$

Remark 6.1.6. *If in the situation of Example 6.1.5 we have* $c = 0$, *the measure* $\nu(dy) = 0$ *and*

$$q_{lk} = \delta_{lk} = \begin{cases} 1, & l = k \\ 0, & l \neq k \end{cases}$$

then we have

$$\mathcal{E}(u,v) = \sum_{l=1}^{n}\int_{\mathbb{R}^n}\frac{\partial u}{\partial x_l}\frac{\partial v}{\partial x_l}dx. \qquad (6.11)$$

If in the situation of Example 6.1.5 we can set $\nu(dy) = N(y)dy$, where $N(y)$ is a density, then we may rewrite (6.10) as

$$\mathcal{E}^{\psi}(u,v) = \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(u(x+y)-u(x))(v(x+y)-v(y))N(y)dydx. \qquad (6.12)$$

Or equvalently

$$\mathcal{E}^{\psi}(u,v) = \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(u(x)-u(y))(v(x)-v(y))N(x-y)dydx. \qquad (6.13)$$

We now want to compare our representation for a Dirichet form (6.10) with the Beurling-Deny representation for symmetric Dirichlet forms.

Theorem 6.1.7. *(Beurling-Deny) If $\mathcal{E}$ is a symmetric, regular Dirichlet form on $L^2(\mathbb{R}^n)$ then we have the following representation for $\mathcal{E}$*

$$\mathcal{E}(u,v) = \int_{\mathbb{R}^n} cu(x)v(x)dx + \int_{\mathbb{R}^n} \sum_{k,l=1}^{n} a_{kl}(x)\frac{\partial u}{\partial x_l}\frac{\partial v}{\partial x_k}dx$$

$$+ \int_{\mathbb{R}^n - D}\int_{\mathbb{R}^n - D} (u(x)-u(y))(v(x)-v(y))J(dx,dy) \qquad (6.14)$$

where $J(dx,dy)$ is a measure on $\mathbb{R}^n \times \mathbb{R}^n - D$ and D is the diagonal $D = \{(x,x), x \in \mathbb{R}^n\}$. Further, if $\mathcal{E}$ has no local part then we have

$$\mathcal{E}(u,v) = \int_{\mathbb{R}^n - D}\int_{\mathbb{R}^n - D} (u(x)-u(y))(v(x)-v(y))J(dx,dy). \qquad (6.15)$$

Recall that the generator A of $\mathcal{E}^\psi$ is given by

$$Au(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\psi(\xi)\hat{u}(\xi)d\xi.$$

Our next step is to use Dirichlet form techniques to study stable-like processes. We study the following symmetric quadratic form on $L^2(\mathbb{R}^n)$

$$D(\mathcal{E}^\alpha) = \left\{u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))^2}{|x-y|^{n+\alpha(x)}}dxdy < \infty\right\}$$

$$\mathcal{E}^\alpha(u,v) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+\alpha(x)}}dxdy,$$

where $0 < a_1 \le \alpha(x) \le a_2$.

A further extension is to consider the form

$$D(\mathcal{E}^\alpha) = \left\{u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))^2}{|x-y|^{n+\alpha(x,y)}}dxdy < \infty\right\}$$

$$\mathcal{E}^\alpha(u,v) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+\alpha(x,y)}}dxdy,$$

where $0 < a_1 \le \alpha(x,y) \le a_2$.

In the case that $(\mathcal{E}^\alpha, D(\mathcal{E}^\alpha))$ is closeable its closure is a Dirichlet form with some generator $(A, D(A))$. Considered as a pseudo-differential operator A will be of variable order, i.e. the "order" of its symbol $q(x,\xi)$ will depend on x. Hence considering such type of Dirichlet forms will open a further way to consider variable order subordination. For considerations along these lines we refer to [37] and [38].

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