Aspects of k-strings and k-domain walls in the AdS/CFT correspondence.

Ridgway, Jefferson Miles

Department of Physics

Swansea University
Prifysgol Abertawe

Aspects of k-strings & k-domain walls in the AdS/CFT correspondence

Jefferson Miles Ridgway

Ph.D.

Submitted to Swansea University in fulfilment of the requirements for the Degree of Doctor of Philosophy.

2009

Abstract

The concepts and behaviours of k-strings and domain-walls are examined in higher representations through the scope of the Gauge-Gravity correspondence.

Tensions of strings in a representation of N-ality k are examined in the strong coupling limit. Confining k-strings are discussed in four dimensional gauge theories using D5 branes in $AdS_5 \times S^5$, and D3 branes in Klebanov-Strassler and Maldacena-Núñez backgrounds. Two main results are presented: The first that confining k-string tensions in $\mathcal{N} = 4$ super Yang-Mills can be calculated using D5 branes in $AdS_5 \times S^5$ with a cut-off in the bulk AdS. It is shown that the D5 brane can replicate a string of rank k in the antisymmetric representation. The second result shows that the S-Dual calculation to string tensions in $\mathcal{N} = 1$ super Yang-Mills gravity duals reproduces the action exactly, while providing a more natural manifestation of the string charge k.

Quantum broadening effects of k-string objects are investigated in both confining and non-confining theories. An old result by Lüscher, Münster and Weisz is generalised to the case of a k-string. When the fundamental string is replaced by a bound state of k strings, the bound state is better described by a wrapped D-brane. The width of the k-string (the wrapped D-brane) is calculated in several confining backgrounds by using a D-brane probe and a universal result is found. The widths of k-strings in $AdS_5 \times S^5$ are examined via connected world-sheet methods, and via the exchange of light supergravity modes, and are shown to disobey the confining string result.

The tension of the deconfining domain walls of $\mathcal{N} = 4$ Super Yang-Mills are studied at weak and strong coupling. The k-wall tension at one loop order is calculated and found to be proportional to $k(N - k)$, the Casimir scaling. The strong coupling calculation is performed by using the Gauge-Gravity correspondence. Arguments are made that the k-wall should be identified with a 5-brane wrapping an S^4 inside S^5 in an AdS-Schwarzschild $\times S^5$ background. The tension at strong coupling is compared with the weak coupling result.

Preliminary results for tension scaling behaviours for thermal gauge theories in two-index representations are presented and briefly discussed.

Contents

Acknowledgements

I would like to thank my long suffering supervisor, A. Armoni, for his continual support over the previous 4 & 1/2 years, and his willingness to answer any question posed. I also wish to thank my collaborators, S. P. Kumar & C. Korthals Altes.

Thanks also goes out to D. Berenstein, N. Drukker, F. Gliozzi, B. Lucini, J.C. Myers, C. Núñez , A. Rago, J. Schmude, M. Ünsal, K. Zarembo for fruitful discussions.

I also wish to thank all the staff and students in the Physics department at Swansea University for making my time there a very pleasant & memorable one.

Finally, I would like to sincerely thank all of my family and friends for being so very supportive, and putting up with an absent minded & preoccupied Jefferson for the last 4 years. Many thanks to you all.

Chapter 1

Introduction

"Nothing is more interesting to the true theorist than a fact which directly contradicts a theory generally accepted up to that time, for this is his particular work."

- Max Planck

"Before I came here I was confused about this subject. Having listened to your lecture I am still confused. But on a higher level."

- Enrico Fermi

"The only things known to go faster than ordinary light is monarchy, according to the philosopher Ly Tin Weedle. He reasoned like this: you can't have more than one king, and tradition demands that there is no gap between kings, so when a king dies the succession must therefore pass to the heir instantaneously. *Presumably, he said, there must be some elementary particles – kingons, or possibly queons – that do this job, but of course succession sometimes fails if, in mid-flight, they strike an anti-particle, or republicon. His ambitious plans to use his discovery to send messages, involving the careful torturing of a small king in order to modulate the signal, were never fully expanded because, at that point, the bar closed."*

- Terry Pratchett, Mort

1.1 Birth of modern string theory

Following Einstein's proposal of General Relativity in 1916 as a description of gravity, and the formulation of quantum mechanics to describe sub-atomic particles in the early 20th century; two highly successful theories of nature, physicists have endeavoured to combine them into a single theory, describing both gravity and particles. However, there is a problem. Of the four fundamental forces of nature; Gravity, electromagnetism, and the strong & weak nuclear forces, all but gravity can quite successfully be described by quantum theories. The inclusion of gravity into a quantised formalism leads to theories that are unrealistic, i.e. the theories produce infinite answers. This is indicative of some unknown underlying physics.

One of the many proposed solutions to this underlying theory was the radical idea of string theory. String theory is a theory of not point-like particles, but extended 1-dimensional objects called, unsurprisingly, strings. These strings live in a higher dimensional space, and different vibrations of these strings in the higher dimensional space, represent different particles to an observer (or more likely, a particle accelerator). The advantages of string theory is that gravity is automatically included in its definition, while the string interactions replicate the particle interactions. However, string theory is very difficult to test. The length of these strings are very small, and because of this, there are no experimental verifications that string theory is indeed a theory of everything.

In 1963, a theory was discovered of combining two of the four fundamental forces of nature; the electromagnetic governing photons, and the weak interactions; the exchange of W and Z bosons. Towards the end of the 1960's, attempts were made to devise a theory of consistently describing the Strong nuclear force, and the myriad of hadrons that had been discovered in particle accelerators the world over.

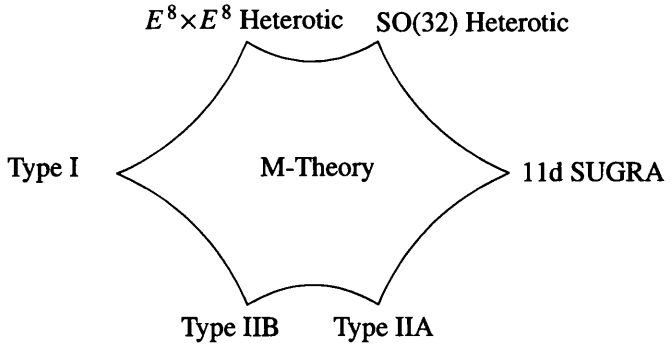
1968 saw the pioneering insight of Veneziano. He discovered that a combination of Euler Beta functions of the Regge trajectories of hadrons, described a scattering amplitude that was later determined to be the scattering amplitude of four tachyonic open strings. Calculations of loop amplitudes using the ideas of Veneziano lead to inconsistencies unless the number of spatial dimensions was 25, which when including time, gives a critical dimension of 26. Bosonic string theory was born.

By the mid 1970's, string theory had been pushed aside following the development of Quantum Chromodynamics, the quantum field theory of the strong interaction. The vast array of hadrons were no longer described by vibrations of a string, but by combinations of more fundamental particles, namely quarks. The revelation of QCD completed the set of interactions of fundamental particles, the $U(1) \times SU(2) \times SU(3)$ model or, more commonly, the Standard Model. Gravity was still unaccounted for.

Following the discovery of supersymmetry in the early 1970's, in which symmetries between bosons and fermions were found to exist, there began a revival in the string theory community, with supersymmetric string, or "superstring", theories. Incorporating fermions reduced the critical dimension of the superstring theory to 10. Originally found in the context of string theory, supersymmetry seemed to be a realistic way of looking at physics beyond the Standard model. Could a superstring theory be not only a theory of the strong interaction, but be the foundation of an underlying theory of all interactions? One issue was the fact that there isn't just one superstring theory, but five.

In the mid-eighties after a number of developments in the superstring arena, superstring theories started to be considered as respectable. One major development was the realisation that an $\mathcal{N}=1$ supersymmetric theory in ten dimensions required one of two gauge groups ($SO(32)$ or $E_8 \times E_8$), and was found to be consistent and anomaly free. String theory started to be accepted by the high energy physics community as a real candidate for the inclusion of gravity into a theory of everything. It was during this first "superstring revolution" that the five superstring theories were discovered, each requiring 10 space-time dimensions to be consistent, and all including supersymmetry. The five theories were recognised as Type I, Types IIA & IIB, and the Heterotic theories, $SO(32)$ & $E_8 \times E_8$. The heterotic models and Type I exhibit $\mathcal{N}=1$ supersymmetry (SUSY) in the full 10 dimensions, while the type II theories have $\mathcal{N}=2$ SUSY.

It was not until the mid-nineties, the era of the second superstring revolution, was it discovered that the five theories along with 11-dimensional supergravity, the supersymmetry extension of Einstein's gravity to which low energy string theory reduces to, were all limits of some single underlying 11-dimensional theory, mysteriously called "M-theory". Dualities where found that would transform one theory



to another, and back again.

1.2 The dawn of AdS/CFT

In 1997, Maldacena [1] made a bold proposal. He conjectured that an exact equivalence exists between Type IIB superstring theory on a 10-dimensional manifold; the product of an Anti de-Sitter 5-space and a 5-sphere, $AdS_5 \times S^5$, and a conformal field theory, namely the supersymmetric gauge theory $\mathcal{N}=4$ super Yang Mills, residing on the 4 dimensional boundary of the AdS_5 space.

The conjecture states that the coupling constants in both theories are identified: The 't Hooft coupling of the field theory is related to the radius of the AdS_5 space, $\sqrt{\lambda} = R^2_{AdS}/\alpha'$ while the string coupling constant is inversely proportional to the number of colors in the field theory, $g_s \sim 1/N$. This implies that in the limit where the number of colors becomes very large, the string interactions essentially vanish, and the string theory becomes essentially classical. This limit is known as the Large-N limit, and plays an important role in the understanding of QCD, as discussed below.

The identification of coupling parameters indicates that a strongly coupled gauge theory is equivalent to a weakly coupled string theory, and vice-versa. As it is easier to compute perturbative observables in weakly coupled theories, to work in the strongly coupled regime of a gauge theory, one simply performs an equivalent, weakly coupled computation in the string theory. The same logic applies to strongly coupled string theories.

It is not simply the identification of the coupling constants between the two theories, but a complete matching of BPS states, symmetries and degrees of freedom.

In the 12 years since its inception, numerous tests of the correspondence have been made, and all have given positive results (See [2] for a review). The AdS/CFT correspondence, or more generally Gauge-Gravity duality, was born.

This was a highly significant development in the string theory community. Since the invention of QCD, precise calculations involving quark and gluon interactions have always proved difficult. Unlike Quantum Electrodynamics (QED), QCD is a non-abelian theory; the gauge bosons (in this case, gluons) interact not only with the fermions (quarks), but with themselves. Together with the higher coupling constant for QCD, higher order corrections for Feynmann diagram computations are larger than the value they are supposed to be correcting!

Inspired by the AdS/CFT correspondence, thought moved to finding a string theory that was dual to QCD. Being able to perform relatively simple weak coupling string theory computations that would be equivalent to strongly coupled QCD results, while modelling other aspects of QCD, including confinement & asymptotic freedom, would be a highly sought prize. It seems that describing the strong force with string theory has come full circle.

In this thesis, a number of gauge-gravity dualities are discussed. In chapters 3 & 4, the full AdS/CFT duality is explored, as well as a restricted, confining version called Hardwall AdS. Another duality between Type IIB string theory and $\mathcal{N} = 1$ super Yang-Mills is also discussed. In chapter 5, non-zero temperature $\mathcal{N} = 4$ SYM is explored. The gravity dual of this is a gauge theory that exhibits a black hole. See the relevant chapters for more detailed information.

1.3 Quantum Chromodynamics

Quantum Chromodynamics, or simply QCD, is a quantum field theory describing the strong force interactions between fermions, called quarks, and the gauge bosons mediating the strong force, the somewhat amusingly named (which seems quite normal for particle physicists in modern times) gluons. Both quarks and gluons carry a charge, separate from any electrical charge, called "color" (Hence *chromo*dynamics). The theory is of the form of a non-abelian Yang-Mills theory. The Lagrangian for QCD can be seen belw:

$$\mathcal{L} = -\frac{1}{4}\left(F_{\mu\nu}\right)^2 + \bar{q}_i^I \left(i\gamma_\mu D^\mu - m\right)_{ij} q_j^I$$

The indices i, j are the color indices ($i, j \in [1, N = 3]$) and are implicitly summed over. The index I represents the number of species or "flavours" of quarks, $N_f = 6$. The gauge symmetry group of QCD is SU(3), 3 denoting the number of color charges; Red, Green & Blue. However, these color charges have never been directly observed. This leads to the first interesting property of QCD: Confinement.

In QCD, mesons & hadrons are comprised from combinations of quarks; hadrons (i.e., protons, neutrons, ...) are assemblies of three quarks, or anti-quarks (qqq or $\bar{q}\bar{q}\bar{q}$), while mesons are comprised of a quark and anti-quark ($q\bar{q}$). As the quarks are in the fundamental representation, each quark has a color charge Red (R), Green (G) or Blue (B) from the **3** triplet, together with anti-Red ($\bar{R}$), anti-Green ($\bar{G}$) and anti-Blue ($\bar{B}$) for anti-quarks from the $\bar{\mathbf{3}}$. As color charges cannot be directly observed, bound states of quarks in hadrons & mesons must be devoid of color charge, a color-singlet state, i.e. colorless. For hadrons, this requires a quark of each color (or anti-color), $q_R\, q_G\, q_B$ (or $q_{\bar{R}}\, q_{\bar{G}}\, q_{\bar{B}}$). Mesons require a matching color and anti-color quark pair, e.g., $q_R\, \bar{q}_{\bar{R}}$. Gluons, however exist in the adjoint representation of SU(3) and are not colour singlets. As the force carrier, they exhibit one of 8 (from the **8** representation of SU(3)) possible color states, commonly referred to as the octet, each a tensor product of color - anti-color charges.

As discussed above, color charges are never seen in isolation, yet quarks have non-zero color charge. This is due to the phenomenon of color confinement. In QED, when two electrically charged sources are separated, the force between them drops off as an inverse power law, so that sources far apart have very little influence over each other. However, in QCD the further apart you take two color sources, the *greater* the potential between them becomes.

$$V_{q\bar{q}} = \sigma r$$

The potential $V_{q\bar{q}}$ increases linearly, proportional to a constant tension, σ. Gluon fields mediating between color sources form a string or tube of color flux. The further the sources are separated, the longer the tube becomes, and the more energy is required to separate them further. For infinite separation, an infinite amount of energy is required.

That is slightly misleading, as this is only true for non-dynamical (infinite mass) quarks. In true QCD, if quarks are separated by imparting energy roughly equivalent to the mass of two quarks, the flux tube with break, forming a pair of quarks from the vacuum. This is known as hadronisation, and is the source of quark jets in particle accelerators, a tell-tale sign of quark interactions.

Another peculiarity of QCD, is the property of Asymptotic freedom discovered in 1973 by, and subsequently earned Nobel prizes in 2004 for, Gross & Wilczek and Politzer. Asymptotic freedom in QCD is the property that the interaction strength of color charged objects get weaker at shorter distance (and accordingly higher energies). It's discovery grew from a calculation on the beta-function of QCD (how the coupling parameter changes with varying energy scales). For non-abelian SU(N) gauge theories with N_f flavours;

$$\beta(g) = -\frac{g^3}{16\pi^2}\left[\frac{11}{3}T(A) - \frac{4}{3}N_f T(R)\right] + O(g^5)$$

For QCD, the gauge group is SU(3), therefore the index of the representation, $T(R) = 1/2$, while the index of the adjoint representation, $T(A) = N = 3$. This leaves the beta function for QCD as

$$\beta(g) = -\frac{g^3}{16\pi^2}\beta_0, \quad \beta_0 \equiv \left[11 - \frac{2}{3}N_f\right]$$

It is clear that the beta-function is only positive, implying increasing coupling strength with increasing energy scale if the number of flavours, N_f is greater than 16. In QCD, this isn't so, with only 6 flavours of quarks (up, down, top, bottom, charmed & strange - another example of particle physicists nomenclature oddities) the beta function is negative. The higher the energy of the system, the more weakly coupled the interactions become. What is the reasoning behind this weak quark interaction at short distances?

In QED, the charge of an electrical source when probed at long distances is screened by virtual electron-positron pairs, thereby producing a lower effective source charge. This effect is also present in QCD, with color sources being screened by virtual quark pairs. However, due to the non-abelian nature of the theory, virtual or "soft" gluons are able to pop into existence from the vacuum, and due to their

color-charge they are able to affect the screening behaviour of sources in a *negative* way - they increase the effective charge of a source at long range. This is know as "anti-screening". At short ranges, sources appear weaker, thus forces between sources at shorter & shorter ranges becomes asymptotically weaker, with quarks becoming effectively free.

Large-N

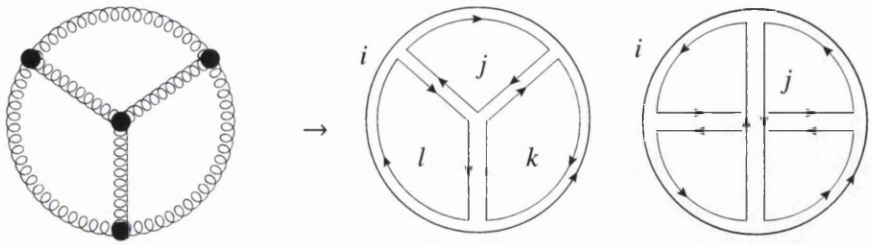
In a QCD-like theory with gauge group SU(N), the scale of the theory, Λ_{QCD} is set by the strength of the coupling constant, α_s;

$$\alpha_s = \frac{2\pi}{\beta_0 \log(Q/\Lambda_{\mathrm{QCD}})}$$

With the momentum of the quarks Q, and $\Lambda_{\mathrm{QCD}} \sim 200\mathrm{MeV}$, there is no obvious parameter about which to expand. In 1973, G. 't Hooft [3] proposed expansions around $1/N$, where aspects of a theory at finite N can studied much more easily in the $N \to \infty$ limit, with additional corrections $\sim O(1/N)$. One often quoted example, is that in a certain large-N limit, it was found that Feynmann diagrams of pure Yang-Mills theory can be classified to overall factors of $(1/N)^n$ for some integer n. Keeping the combination $g^2N \equiv \lambda$, the so-called 't Hooft coupling, fixed while taking the $N \to \infty$ limit, classified loop diagrams into genus. Diagrams which can be drawn on the surface of a sphere (genus 0) will contribute factors of $O(N^2)$, while those that can only be drawn on a torus (genus 1) will contribute factors $O(N^0)$, and a general diagram of genus h, have contributions of $O(N^{2-2h})$.

For example, consider the three loop diagram of pure U(N) Yang-Mills, figure 1.1. Here, two graphs that contribute to the same three loop diagram are shown. The first can be drawn on a sphere or plane, and is thus planar. Such a graph contributes a factor of $(g^2N)^2N^2 = \lambda^2N^2$. The second however, has a gluon propagating out of the plane. Such a diagram can only be drawn on a surface of genus 1 at minimum (i.e. a torus), and only contributes a factor of $g^4N^2 = \lambda^2N^0$. In the large-N limit, the second diagram is suppressed by a factor of $1/N$. The diagrams that can be drawn on a plane dominate at large-N, greatly simplifying computations. This is known as the planar limit.

As was seen earlier, the large-N limit is an important limit not only for gauge theories, but also for string theories in the AdS/CFT correspondence, where large-



$g^4N^4 = (g^2N)^2N^2 = \lambda^2N^2$ $\quad$ $g^4N^2 = \lambda^2N^0$

Figure 1.1: *Planar and non-planar graph examples for 3-loop gluon vacuum amplitude in a U(N) pure Yang-Mills theory. Representing each gluon propagator in the double index notation, each vertex contributes a factor of g, while each loop provides a factor of N . The middle graph has four loops i, j, k, & l, giving a factor of N^4, and four vertices giving g^2. In terms of the 't Hooft coupling, this can be simply written as λ^2N^2. This graph can be drawn on a plane, and hence known as a planar graph. The right-most graph is non-planar, as the single loop j crosses over itself. There are two loops, i & j, and four vertices, giving a total factor of $g^2N^2 = \lambda^2$. This graph is suppressed by $1/N^2$ in the large-N , λ fixed limit.*

N approximations render string theories effectively classical, requiring only consideration of dominant terms, as corrections vanish as $\sim O(1/N)$. The discussions in this thesis will be within this arena of large-N . It shall be seen later that k -strings are very sensitive to the methods of taking N to infinity.

1.4 Thesis outline

This thesis is organised into 3 main result chapters, each one originates from an original research publication.

Chapter 2: A brief introduction to the concepts of k-strings and their properties within the AdS/CFT framework.

Chapter 3: The concepts of string tension computations are introduced, before reviewing k-string computations in $AdS_5 \times S^5$. Following this review, tensions of k-strings in backgrounds that exhibit confinement are discussed, namely Hard-wall AdS and $\mathcal{N} = 1$ super Yang-Mills gravity duals. This chapter is an extended discussion of [4].

Chapter 4: Introduction and review of the quantum broadening effect and it's measurement. Attempts at string width measurements in $AdS_5 \times S^5$ are discussed, and reasoning behind their failings are put forth. Supergravity width descriptions in $AdS_5 \times S^5$ are tested, before discussing widths in confining backgrounds, namely Hard-wall AdS & $\mathcal{N} = 1$ SYM duals. The confining discussions chapter formed the basis of [5].

Chapter 5: The ideas of domain wall tensions at finite temperature are introduced, before investigating the k-domain walls of finite temperature $\mathcal{N} = 4$ SYM in the weak coupling limit. Following the weak coupling result, a strong coupling, string computation is conducted and directly compared with the weak result. These results were summarised in [6]. In addition, some preliminary results are put forward for the tensions of k-walls in weak coupling theories with two index representations. These results are to appear [7].

Chapter 2

Introduction to k-Strings

2.1 What is a k-string?

In the AdS/CFT framework, a quark - anti-quark pair, and the forces between them, can be described by a string. The end points of the string, restricted to $\mathbb{R}^4$, model the quark pair with a force between then proportional to the tension of the string, σ. A single string models a single quark pair in the fundamental representation.

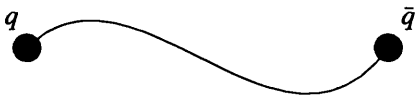


Figure 2.1: *Quark - anti-quark pair represented by the end points of a fundamental string.*

If the number of colors, N of the theory is taken to be greater than 3, then a higher string can arise. If one takes a pair of quarks and a pair of anti-quarks, between the pairs will extend two strings. In a theory with 2 colors, the quarks will form a colorless baryon state, as will the anti-quarks, and the string will break. However, for a theory with more than 3 colors, the strings will interact to form a bound state of two strings, imaginatively called a 2-string[1]. In a more general case, for k interacting strings, where $k \in [1, N]$, $k \in \mathbb{Z}$, they will form a bound state; the k-string.

[1]A 2-string object in a theory of $N = 3$ is equivalent to a fundamental string, due to charge conjugation symmetry, which is introduced later.

k-strings are often referred to as higher representational strings, as k also represents the N-ality of the color representation of the sources. In confining theories, it is thought that the tension of a k-string between sources of N-ality k is independent of the representation of the sources, and only varies with the N-ality, k. This is due to a screening effect by "soft" gluons. A source with representation of N-ality k, can be transformed into a source of any representation with the same N-ality, via the emission or absorption of a relevant number of adjoint gluons. As the adjoint representations of gluons has N-ality 0, the N-ality of the sources is unaffected. As a result, adjoint sources should not exhibit a flux tube between them, as the sources would be completely screened. Take for example the expression below:

$$q^{[ij]} \rightarrow q^{[ik]} + g^j_k \tag{2.1}$$

A quark in the symmetric representation is able to transform into one in the anti-symmetric representation via the emission of a single adjoint gluon.

Via the emission of a maximal number of gluons, the representation of the sources will appear completely anti-symmetric at distances $\sim \Lambda^{-1}$, the dynamical scale of the theory. The k-strings between sources in the anti-symmetric representation are considered to be strings in a ground state (or so-called *bone-fide* k-strings [8]). Strings in a different representation of the same N-ality are said to be excited, and will eventually decay to the anti-symmetric via gluon emission outlined above (see [8–10] and references within). It will be these anti-symmetric, ground state k-string that will be of interest.

2.2 Tensions of k-strings

A highly active area in recent years has been the determination of the properties & tensions of k-strings in various theories, both confining and conformal in nature. A large catalogue of work exists in both string theory [9–21] and lattice communities [22–30]. The basis for these works is the scaling of the k-string tension with the number of colours in the theory, N. Why is this important?

As briefly mentioned above, a k-string is a bound state of k fundamental strings, giving the k-string a binding energy. The binding energy, and therefore the tension of a k-string, σ_k has been shown to depend on the N-ality, k of the source probes,

together with the number of colours in the theory, N. For finite N, σ_k is less than the tension of k non-interacting fundamental strings;

$$\sigma_k < k\sigma_f \tag{2.2}$$

The tension of the k-string, to leading order scales as a function of $1/N$; for a given k, as N increases, the interactions between the strings in the bound k-string become weaker and weaker, such that in the limit where $N \to \infty$, all interactions cease, and $\sigma_k = k\sigma_f$. It is important to stress that this vanishing of the binding energy occurs in the large N limit when k is fixed. When $k \sim O(N)$, binding effects are still apparent. There are two distinct limits of taking N to infinity here:

$$\begin{aligned} N \to \infty, \quad & k = \text{fixed} \\ N, k \to \infty, \quad & \tfrac{k}{N} = \text{fixed} \end{aligned}$$

In both limits, semi-classical methods can be used to analyse observables, however, in each limit the k-string acquires different properties: For $N \to \infty$, k = fixed the k-string becomes a set of k coincident, non-interacting fundamental strings, and can be modelled as such, while k-strings in the $N, k \to \infty$, $\frac{k}{N}$ = fixed limit remain tightly bound states. This can be seen explicitly by studying Wilson loops.

A k-string tracing a contour C, can be expressed as a Wilson loop operator of N-ality k. For a reducible representation of products of k fundamentals, $\mathcal{R}$, the Wilson loop W_k in the large-N limit factorises into k coincident fundamental loops, with leading order corrections of the order $1/N$ raised to some positive integer power.

$$\langle W_k \rangle = \left\langle \text{Tr } e^{i\int_c A^a T^a_{\mathcal{R}}} \right\rangle = \left\langle \text{Tr } e^{ik\int_c A^a T^a_f} \right\rangle \overset{N\to\infty}{\to} \left\langle \text{Tr } e^{i\int_c A^a T^a_f} \right\rangle^k = \langle W_f \rangle^k \tag{2.3}$$

For the $\frac{k}{N}$ = fixed, large-N limit, this factorisation effect is not a viable approximation. Why is this? In such a limit, the tightly bound 1-dimensional strings "blow up" into a higher dimensional object, called a D-brane, via the dielectric or Myers effect [31]. This brane is embedded in a transverse space in such a way as to appear as a string with charge k to an $\mathbb{R}^4$observer. It is the details of dimensionality of this

D-brane, and it's particular embedding in the transverse space which specifies the dependence on the tension to the string charge k. This method of using D-branes to describe k-strings will be discussed in more detail later in this thesis.

There are a number of properties that k-strings exhibit in the theories discussed here, and attempts to model k-strings should reflect these. As above, the tension is highly dependent on the N-ality, with the binding energy increasing with increasing k/N. However, there is a limit: once $k = N$, each end of the k-string forms a colourless baryon vertex, and the force between the quarks & anti-quarks, and therefore the k-string, vanishes. In addition, adding a colourless baryon to each end of the string should not affect the properties of the string. The addition of a colourless baryon, $k \to k + N$ is a symmetry of the system. The system must also be invariant under charge conjugation of the quarks & anti-quarks. Imposing this invariance on the system enforces a second symmetry; $k \to N - k$. A k-string obeying this symmetry will appear to be most tightly bound at $k = N/2$, the self-dual point of the string. Computations of the tension of such strings should reflect these properties.

Throughout the literature of tension computations, expressions for σ_k generally fall into two general forms (either exactly, or closely approximated); Casimir scaling, and the Sine formula [21].

$$\sigma_k \sim \frac{k(N-k)}{N-1} \quad \text{Casimir scaling} \tag{2.4}$$

$$\sigma_k \sim \sin\left(\frac{\pi k}{N}\right) \quad \text{Sine formula} \tag{2.5}$$

With appropriate normalisation, both scaling behaviours only deviate by a maximum of $\sim 3\%$. However, it is not the approximation which is the defining aspect between these two scaling behaviours. Performing a large-N series expansion will reveal that the Casimir has leading order corrections like $O(1/N)$, while the Sine law's corrections have even powers, $O(1/N^2)$. There is a lot of evidence pointing to both scaling behaviours for k-string in many different theories, with lower dimensional theories exhibiting scaling behaviours which are better approximated by Casimir scaling (e.g. 2 & 3d QCD, 3d reduction of $\mathcal{N} = 1$ SYM), while string approaches are more suited to Sine law-like approximations (e.g. MQCD, softly broken $\mathcal{N}=2$ SYM, $\mathcal{N}=1$ SYM).

To correctly model the properties of the k-string a higher dimensional object is required, a D-brane, which is embedded in the spacetime in such a way that any $\mathbb{R}^4$ observer will see a string like object with additional properties. This method of using D-branes to describe strings was hypothesised in [32,33]. Using D-branes wrapping transverse manifolds to model k-string properties was applied in $AdS_5 \times S^5$ for D3 [16] and D5 [15] branes, modelling symmetric and anti-symmetric k-string respectively. This same technique was used for $\mathcal{N} = 1$ super Yang-Mills, a confining theory, via wrapped D3 branes [19,20].

2.3 D-branes

D-branes, or more correctly, Dp-branes, discovered in the mid-nineties [34], are extended $p + 1$ dimensional objects on which strings can end, and are defined by the boundary conditions of these string endpoints. The endpoints of strings that terminate on a D-brane obey Neumann boundary conditions along directions parallel to the branes extended directions, while observing Dirichlet boundary conditions in the transverse directions.

In Type II string theories, D-branes can couple to gauge fields and form stable branes. A n-form gauge field can couple electrically to a Dp-brane of $p = n - 1$, or magnetically to one of $p = 7 - n$. In the Ramond-Ramond (RR) sector of Type II, there are n-form C_n fields, C_1, C_3, & C_5 in IIA, which couple electrically to D0, D2 & D4 branes respectively. In IIB, C_0, C_2, & C_4 fields couple electrically to D-instantons (a point particle localised in time), D1 & D3 branes, and magnetically to D7, D5 & D3 branes respectively. Notice that D3 branes are charged electrically and magnetically; the field strength is said to be self-dual. In the Nevez-Schwarz (NSNS) sector of Type II, there exist 2-form fields parallel to the brane in question, G_{ab}, B_2, and a gauge field living on the brane F_2, together with a scalar, the dilaton Φ [35]. For an introduction to D-branes, see [36,37], while extensive reviews of D-branes and their mechanics can be found in [38,39]

Chapter 3

k-string tensions

3.1 Intro to Wilson loop & tension calculations

One of the proposed applications of the AdS/CFT was that, in the large-N limit, the potential of a infinite mass, static quark - anti-quark pair in $\mathcal{N}=4$ SYM could be determined by computing the expectation value of a Wilson loop operator on the $\mathbb{R}^4$ boundary of AdS_5, traced by a single string moving through the bulk space [40]. The Wilson loop operator for a Yang-Mills theory is

$$W(C) = \frac{1}{N}\mathrm{Tr}\, P\, e^{i\oint_C A} \tag{3.1}$$

with C representing the closed loop on the boundary. The trace is over the representation of the string, which in this case is the fundamental representation. The ends of the string act as sources of chromoelectric flux, and are fixed to the $\mathbb{R}^4$ space, while the string can move in the full AdS_5, commonly referred to as the bulk. As the quarks, separated by a distance L, evolve through a time T, a rectangular Wilson loop is traced out (figure 3.1).

From this semi-classical, large-N string perspective, the Wilson loop is described by the minimised surface area of the string world-sheet tracing the loop at the boundary. The expectation value of the Wilson loop can be expressed in terms of the minimal world-sheet action, S:

$$\langle W(C)\rangle = e^{-S} \tag{3.2}$$

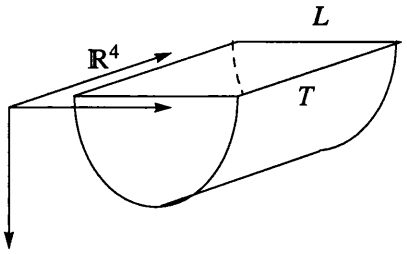


Figure 3.1: *A quark - anti-quark pair is restricted to the* $\mathbb{R}^4$ *boundary of* AdS_5*, and separated by a distance L. They evolve through a time T, tracing a rectangular Wilson loop, TL. The string, which is not restricted in the AdS bulk sweeps out a valley-like world-sheet, leading to an inter-quark potential linear in L:* $V(L) = \sqrt{\lambda}/L$.

For theories that exhibit confinement, the action is expressed as the minimised area of the string world-sheet $\mathcal{A}$, and a pre-factor σ defined as the tension of the string:

$$S = \sigma\mathcal{A} \tag{3.3}$$

This known as the "Area Law" for Wilson loop expectation values. The Wilson loop acts as an effective order parameter, signalling confinement in a system with an area law behaviour. What does this imply? As the separation, and thus the potential of the quark - anti-quark pair increases, so does the size of the Wilson loop, along with the minimal area of the string world-sheet ending on the loop. Ergo, the energy of a quark pair is directly related to the area of the world-sheet, factored with the string tension σ.

Using the Wilson loop as an order parameter for confinement in this way is perfectly acceptable for a system with non-dynamical, infinite mass quarks. However, for finite mass quarks like those found in real QCD, the Wilson loop area law will only hold up to a point. Upon separating a quark pair, an area law effect will be manifest until enough energy is imparted to the pair to generate a $q\bar{q}$ pair from the vacuum. At which the string breaks, a $q\bar{q}$ pair is formed via hadronisation, and the area law is lost. At quark separation above this point, the Wilson loop is no longer a reliable order parameter. Even though no area law would be present, the theory would still be confining.

In non-confining theories, the area law does not apply. For the AdS_5 case in figure 3.1, the minimised action for the string world-sheet and the subsequent inter-quark potential are inversely proportional to their separation L.

$$S = \frac{\sqrt{\lambda}T}{L}, \quad V(L) = \frac{\sqrt{\lambda}}{L} \tag{3.4}$$

The area law is not present, and the pre-factor is no longer a tension, and is simply the 't Hooft coupling, $\sqrt{\lambda}$.

For the purposes of this thesis, the pre-factors in both confining and non-confining cases will be referred to as tensions, and will be usually represented by σ. This is to simplify direct comparisons between the Wilson loop & string width computations in $AdS_5 \times S^5$ and Hardwall AdS_5 . However the reader must bear in mind that the pre-factor is only strictly a tension in confining theories.

The problem with the method by Maldacena of computing quark potentials & string tensions via minimal world-sheet computations, is that it produces infinite results. These divergencies are due to the infinite mass of the static quarks. For $AdS_5 \times S^5$, the infinities can be simply isolated, and only finite parts examined, but this is rather ad-hoc. This method was refined [41] by proposing that using the Hamiltonian, as opposed the Lagrangian for the string action, automatically regularised the configuration, and eliminated the infinities due to the non-dynamical quark masses.

Although developed in conformal $AdS_5 \times S^5$, these computations have also been extended to confining theories [42], and produce the area law effect as described above.

Section 3.2 introduces the explicit calculation of the string action for a single fundamental string in the AdS_5 background.

Section 3.3 deals with the introduction of k -strings into the calculation scheme, and how they can be described in $AdS_5 \times S^5$ at the large N, k/N fixed limit by a 6 dimensional, D5 brane. The brane is wrapped on a 4-cycle in the space transverse to the $\mathbb{R}^4$space-time, effectively acting as a string-like object to any $\mathbb{R}^4$observer. This section reviews the computation of [15]

Section 3.4 introduces some original work, in which the method of [15], outlined in section 3.3, is applied to $AdS_5 \times S^5$ with an Infrared cut-off, so-called Hardwall AdS, which is non-conformal. The work produces the expected area law behaviour for confining theories, and a k-string tension like that of normal $AdS_5 \times S^5$, with an overall factor related to the IR cut-off. It is shown that the ratios of fundamental & k-string tensions is unchanged from section 3.3. These results were published in [4].

Section 3.5 introduces k-string tension computations in the $\mathcal{N}=1$ SYM gravity duals, where the k-string is represented as D3 branes wrapped on a 3-cycle in the space transverse to the gauge theory. The calculations of [19, 20] are discussed in the NS sector.

Following this review, original work is presented, in which the method of [14–16] is applied in the the S-dual (I.e. the Ramond-Ramond sector) of the $\mathcal{N}=1$ SYM gravity duals. The D3 brane action, and subsequently the k-string tension, of [19] is reproduced exactly, as expected via S-duality. This was published along with the results from section 3.4 in [4].

3.2 Fundamental strings

Consider a circular Wilson loop that sits on the boundary of AdS_5 space, with a single string describing the contour of the loop. The ends of the string obey Neumann boundary conditions in space-time at the boundary, tracing the loop at the boundary, while the string itself is allowed to move throughout the entire AdS_5 space. This system can be continuously deformed via conformal transformations to that of the rectangular Wilson loop discussed in section 3.1 [40]. Whereas the rectangular loop represents a static quark – anti-quark pair evolving through time, the circular loop describes the creation and annihilation of a non-dynamical quark – anti-quark pair.

Taking the metric of AdS_5 in the Poincaré patch in units of the AdS radius ($R^2_{AdS} = 1$);

$$ds^2 = \frac{1}{y^2}(dy^2 + dr^2 + r^2 d\eta^2 + dx_2^2 + dx_3^2) \tag{3.5}$$

with y representing the AdS radial "direction". As the fundamental string is restricted to only the AdS_5 manifold, the transverse 5-sphere is a spectator to the

system and can be ignored. The Wilson loop is parameterised in x_1, x_2 by the radial co-ordinate r, and the angular measure η, where $r \in [0, R]$, and $\eta \in [0, 2\pi]$, R being the radius of the Wilson loop on the boundary. Let y and r become scalar fields of a variable ρ, and remain unchanged about rotations in η. Taking the Nambu-Goto action for the string;

$$S_{\text{N.G.}} = -\frac{1}{2\pi\alpha'}\int d^2\xi\ \sqrt{-\det(g_{ab})} \tag{3.6}$$

$$= -\frac{1}{2\pi\alpha'}\int d\rho\, d\eta\, \frac{r}{y^2}\sqrt{(\partial_\rho r)^2 + (\partial_\rho y)^2} \tag{3.7}$$

The equations of motion obtained by minimising the action with respect to ρ can be solved using the ansatz

$$\rho = r, \quad R^2 = y^2 + r^2 \tag{3.8}$$

Using this ansatz, it is clear to see that the string sweeps out a hemisphere world-sheet within the AdS_5 space, ending on the Wilson loop at the boundary. Applying this solution to the action gives

$$S_{\text{N.G.}} = -\frac{1}{2\pi\alpha'}R\int_0^{2\pi} d\eta \int_0^R dr\, \frac{r}{(R^2 - r^2)^{3/2}} \tag{3.9}$$

Integrating over r and η, one will find that the action is unbounded, due to the AdS boundary at infinity. To regularise this, introduce a UV cut-off, by shifting the upper integration limit by an infinitesimal value, ϵ; $R \to R - \epsilon$:

$$S_{\text{N.G.}} = -\frac{R}{\alpha'}\int_0^{R-\epsilon} dr\, \frac{r}{(R^2 - r^2)^{3/2}} = \frac{1}{\alpha'}\left(1 + \sqrt{\frac{R}{2\epsilon}} + O(\epsilon)\right) \tag{3.10}$$

It is clear to see where the divergence arises. Ignoring the infinite ϵ term, one finds the result to be simply $1/\alpha'$, which in units of the AdS radius, is equal to the square root of the 't Hooft coupling, $1/\alpha' = \sqrt{\lambda}$.

This truncation of the result seems quite ad-hoc. A more rigorous method of regularisation is to perform the computation using the Hamiltonian rather than the

Lagrangian as the string action [41]. This is equivalent to adding conjugate momentum boundary terms for each field in the system, in this case, y.

$$S = \int d^2\xi\, \mathcal{H} = \int d^2\xi \left(\mathcal{L} - y' \frac{\partial \mathcal{L}}{\partial y'}\right) \tag{3.11}$$

This approach self regulates, and provides the same finite result as eq. 3.10:

$$S = -\frac{1}{\alpha'} \int_0^R dr \frac{r^3}{R(R^2 - r^2)^{3/2}} - \frac{Rr}{(R^2 - r^2)^{3/2}} = -\frac{1}{\alpha'} \int_0^R dr \frac{r}{R\sqrt{R^2 - r^2}} = \frac{1}{\alpha'} \tag{3.12}$$

Thus, the string 'tension' for a fundamental string in the strong coupling limit is simply $\sigma_f = (\alpha')^{-1} = \sqrt{\lambda}$. It should be noted that in the rectangular Wilson loop computation, the result is $\sqrt{\lambda}T/L$. This is consistent, as with a circular Wilson loop, both T and L are replaced by the radius of the loop, R, and would cancel. Thus the action of a string tracing a circular Wilson loop in AdS_5 is constant with respect to the size of the loop.

In the following sections, it will be shown that a k-string exhibits the appearance of a single fundamental string, with a modified, "effective" string tension, dependent on the charge or N-ality of the k-string, k, and the number of colors in the theory, N.

3.3 k-strings in $AdS_5 \times S^5$

In this section, the tension of anti-symmetric k-strings will be discussed in the $AdS_5 \times S^5$ background, namely the strong coupling limit of $\mathcal{N}=4$ super Yang-Mills theory. Tension computations of both anti-symmetric and symmetric k-strings have been discussed extensively in $AdS_5 \times S^5$ [14–17, 43]. The work of Hartnoll & Kumar [14], and Yamaguchi [15] will be reviewed.

In these works, it was shown that in the anti-symmetric representation, the on-shell action of a D5-brane embedded in $AdS_5 \times S^5$ is equivalent to the expectation value of a Polyakov/Wilson loop in $\mathcal{N} = 4$ super Yang-Mills theory. This is analogous to the Wilson Loop traced by the fundamental string with string tension $\sim \frac{2N}{3\pi} \sin^3 \theta_k$, with θ_k as the embedding angle of the D5 in the transverse S^5.

In the prescription, a D5-brane probe has two of it's world-volume co-ordinates set in $\mathbb{R}^2$ of AdS_5 , such that it will appear to a $\mathbb{R}^4$ observer as a fundamental string, tracing the loop. The remaining four world-volume directions wrap an $S^4 \subset S^5$, sitting at an azimuthal angle θ_k in the S^5 , which is directly related to the charge or the N-ality of the string, k.

Taking the definition of the AdS_5 space in eq. 3.5, include the transverse S^5

$$ds^2 = \frac{1}{y^2}(dy^2 + dr^2 + r^2 d\eta^2 + dx_2^2 + dx_3^2) + d\Omega_5^2 \tag{3.13}$$

As previously, y is the radial AdS direction, and r, η parameterise the Wilson loop (of radius R). The measure $d\Omega_5^2$ symbolises the S^5, and for convenience, can be re-written as a 4-sphere & an angular dependence, θ; $d\theta^2 + \sin^2\theta\, d\Omega_4^2$. Insert into this space, a D5-brane, with its world-volume identified as ρ, ϕ, and the S^4, Ω_4. θ is identified as the angle at which the brane, wrapping the S^4 , sits in the S^5 , $\theta \equiv \theta_k$. Once again, y and r are allowed to become functions of ρ, unchanged about rotations in η.

In the $AdS_5 \times S^5$ background, there exists a Ramond-Ramond 4-form potential, C_4, which satisfies $G_5 = dC_4$. As only components parallel to the world-volume are important, the relevant part of the 4-form is given as:

$$C^4_{\text{rel}} = 4\left[\left(\frac{3}{8}\theta - \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta\right) d\Omega_4^2\right] \tag{3.14}$$

This satisfies $G^5_{\text{rel}} = dC^4_{\text{rel}} = 4\sin^4\theta\, d\theta \wedge d\Omega_4^2$. In addition to the 4-form, an electric field exists on the string, $\mathcal{F} \equiv \mathcal{F}_{\rho\eta}$. The action for the brane in the $AdS_5 \times S^5$ bulk is given by the Dirac-Born-Infeld, (or DBI) and Wess-Zumino terms:

$$\begin{aligned} S_{\text{Bulk}} &= S_{\text{DBI}} + S_{\text{WZ}} \\ &= T_{D5}\int d^6\xi \sqrt{\det(\mathcal{G} + \mathcal{F}_{\rho\eta})} - iT_{D5}\int C^4_{rel} \wedge \mathcal{F}_{\rho\eta} \end{aligned} \tag{3.15}$$

$\mathcal{G}$ is the pullback of the background metric to the brane world-volume, & T_{D5} is the tension of the D5-brane; expressed as $T_{D5} = \frac{1}{(2\pi)^5\alpha'^3} = \frac{N}{8\pi^4\alpha'}$. Applying the metric and four-form potential, the bulk action becomes:

$$S_{\text{Bulk}} = T_{\text{D5}} \int d^6\xi \left[\sin^4\theta_k \sqrt{\frac{r^2}{y^4}(y'^2 + r'^2) + \mathcal{F}_{\rho\eta}^2} - i\mathcal{F}_{\rho\eta} G(\theta_k) \right] \tag{3.16}$$

Where $G(\theta_k) = \left(\frac{3}{2}\theta_k - \sin 2\theta_k + \frac{1}{8}\sin 4\theta_k\right)$, and $\theta_k = \theta$ = constant. Primes denote derivatives with respect to ρ.

As was first seen in section 3.2, to ensure the action is regularised correctly, additional boundary terms must be included in the action (The use of these terms to regularise tension computations with branes can be seen in the papers of Drukker *et al.* [16,41]). These terms take the form:

$$\begin{aligned} S_{\text{Bdy,y}} &= -\frac{\delta S_{\text{Bulk}}}{\delta y'} y' \\ &= -T_{\text{D5}} \int d^6\xi \frac{r^2}{y^4}(y')^2 \frac{\sin^4\theta_k}{\sqrt{\frac{r^2}{y^4}(y'^2 + r'^2) + \mathcal{F}_{\rho\eta}^2}} \end{aligned} \tag{3.17}$$

$$\begin{aligned} S_{\text{Bdy,A}} &= -\frac{\delta S_{\text{Bulk}}}{\delta \mathcal{F}_{\rho\eta}} \mathcal{F}_{\rho\eta} \\ &= -T_{\text{D5}} \int d^6\xi \left[\frac{\sin^4\theta_k \mathcal{F}_{\rho\eta}^2}{\sqrt{\frac{r^2}{y^4}(y'^2 + r'^2) + \mathcal{F}_{\rho\eta}^2}} - i\mathcal{F}_{\rho\eta} G(\theta_k) \right] \end{aligned} \tag{3.18}$$

Summing these boundary terms with the bulk action, eq.3.16

$$S_{\text{Total}} = S_{\text{Bulk}} + S_{\text{Bdy,y}} + S_{\text{Bdy,A}} \tag{3.19}$$

$$= T_{\text{D5}} \int d^6\xi \sin^4\theta_k \frac{(r\,r')^2}{y^4} \left[\frac{r^2}{y^4}(y'^2 + r'^2) + \mathcal{F}_{\rho\eta}^2 \right]^{-1/2} \tag{3.20}$$

The equations of motion for the bulk action (which are unchanged for the total action) are solved by the following assignments for $r(\rho)$ and $y(\rho)$;

$$y(\rho) = \rho; \quad r(\rho) = \sqrt{R^2 - \rho^2}. \tag{3.21}$$

As defined earlier, R denotes the radius of the Wilson loop at the boundary of AdS_5. Notice how these solutions mirror the solutions for the fundamental string,

eq.3.8. However, in addition, there are solutions for the electric field strength $\mathcal{F}_{\rho\eta}$.

$$\mathcal{F}_{\rho\eta} = -i\cos\theta_k \frac{R}{\rho^2}, \tag{3.22}$$

$$\frac{\pi k}{N} = \left(\theta_k - \frac{1}{2}\sin 2\theta_k\right). \tag{3.23}$$

The field strength solution is rooted to the azimuthal angle, θ_k, in the S^5; the angle at which the S^4, wrapped by the brane, sits in the S^5. The imaginary nature of the field strength is due to the Euclidean signature [14]. The solutions provide a relation between θ_k and the electric charge on the string, k, which is defined by the variation of the action with $\mathcal{F}_{\rho\eta}$.

$$k = 2\pi\alpha' \frac{\partial S_{\text{Bulk}}}{\partial \mathcal{F}_{\rho\eta}} \tag{3.24}$$

In [17], Hartnoll provides a general result for antisymmetric Wilson loops in a general type IIB background of the form $M \times S^5$. Substituting the manifold M with AdS_5 provides the exact solutions produced here (it shall be shown later that the string tension is in strict accordance with this general result) Applying solutions eqs. 3.21, 3.22, & 3.23 to the total action eq.3.20, and integrating over the brane world-volume (Integration over the S^4 provides a factor of $\frac{8\pi^2}{3}$);

$$S_{\text{Total}} = T_{\text{D5}} \frac{1}{R} \int d\Omega_4 \int_0^{2\pi} d\eta \int_0^R d\rho \sin^3\theta_k \tag{3.25}$$

$$= \frac{2N}{3\pi\alpha'} \sin^3\theta_k \tag{3.26}$$

Studying this result, and comparing to the fundamental string case in section 3.2, it is clear to see that the D5-brane, acting as a k-string, replicates a fundamental string with an effective string tension of $\sigma_k = \frac{2N}{3\pi\alpha'} \sin^3\theta_k$, with θ_k related to k by eq.3.23.

In the introduction, is was seen that the tension of k interacting fundamental strings was greater than the tension of a single k-string, $k\sigma_f > \sigma_k$. A useful method of comparison is to look at the ratio of the fundamental tension and the k-string tension;

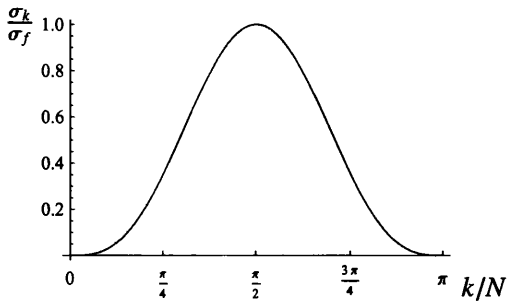


Figure 3.2: *Plot of approximation (3.29) for σ_k/σ_f against k. The k-string tension reaches a maximum at $k = N/2$, and via charge conjugation, $k \to N - k$, is symmetric about $k = N/2$.*

$$\frac{\sigma_k}{\sigma_f} = \frac{2N}{3\pi} \sin^3 \theta_k \tag{3.27}$$

The expression, together with eq.3.23, satisfies the criteria of invariance under charge conjugation, $k \to N - k$ and the addition of a colourless baryon to the state, $k \to k + N$. These symmetries of k translate into symmetries about θ_k:

$$\theta_k \to \pi - \theta_k \quad \theta_k \to \pi + \theta_k \tag{3.28}$$

Noting how θ_k is the S^5 azimuthal angle, it is clear to see how these symmetries arise.

Taking the large N limit, the ratio eq.3.27, will tend to k, as it should. This can be seen more clearly using an approximation. Within 3% error for $k = 0 \ldots N/2$, the ratio can be expressed as a function of powers of $\sin \pi k/N$ [44]:

$$\frac{\sigma_k}{\sigma_f} \sim \frac{N}{\pi} \sin \frac{\pi k}{N} \left[1 - \frac{1}{3} \left(\sin \frac{\pi k}{N} \right)^{1/2} \right] \tag{3.29}$$

This is manifestly invariant under $k \to N - k$ and $k \to N + k$. This is illustrated in figure 3.2

This expression for the k-string tension is in contrast to the conjectured sine law of Douglas & Shenker for softly broken $\mathcal{N} = 2$ [18, 21] where the tension can be expressed exactly as $\sim \sin \frac{\pi k}{N}$. As seen in the approximation above for strongly coupled $\mathcal{N} = 4$, additional corrections of $\sin \frac{\pi k}{N}$ are required to model

the dynamics correctly (In the work of Armoni & Shifman [8], they show that in $\mathcal{N} = 1$ SYM at the saturation limit, the sine law does not exactly replicate string tensions, and higher order corrections are required. However, these corrections are highly suppressed, more so than the approximation given here).

As commented upon earlier, the calculation performed here is in direct correlation with the general result of Hartnoll [17], thus showing the application of this result to k string tension calculations.

One should be reminded, that the D5-brane description of the k-string is only suitable in the large N limit, where k/N is kept fixed. At large N, when k is kept fixed, $1/N$ effects vanish, the interaction between the strings drops to zero, and the case is reached where $\sigma_k = k\sigma_f$.

In section 3.4, this calculation will be adapted to a confining theory, namely $AdS_5 \times S^5$ with a cutoff in the radial AdS_5 direction. This background is also referred to as Hard-wall AdS_5.

3.4 k-string tensions in Hard-wall $AdS_5 \times S^5$

To examine the effect of the k-string tension in a confining background, focus shifts to the case of a k-string in Hard-wall AdS_5. Hard-wall AdS_5 is defined as an $\mathbb{R}^4 \times S^5$ subset of $AdS_5 \times S^5$, that is to say, that a 4d flat space slice is selected in the interior of AdS_5 at a constant value of the AdS radius, $y = y_\Lambda$ (An alternative description, would be that a system in Hard-wall AdS is considered at the IR cut-off in $AdS_5 \times S^5$). Restricting the computation to this $\mathbb{R}^4$ slice and the transverse S^5, provides an effective confining background, that can be directly compared to the non-confining $AdS_5 \times S^5$ case.

As in the full $AdS_5 \times S^5$ case, the Wilson loop in this system sits on the boundary of Hard-wall AdS_5, namely $y = 0 < y_\Lambda$. Ergo, the string world-sheet tracing the loop at the boundary is split into two regimes. In the range $y \in [0, y_\Lambda)$, the string is unaffected by the IR cut-off y_Λ, and follows the complete AdS_5 solution, $R^2 = r^2 + y^2$. However, if the radius of the Wilson loop is greater than the depth of the cutoff, $R > y_\Lambda$, there will exist a second regime, where variation in y is lost, namely the cut-off at y_Λ. The world-sheet sits at y_Λ, and forms a disk of radius

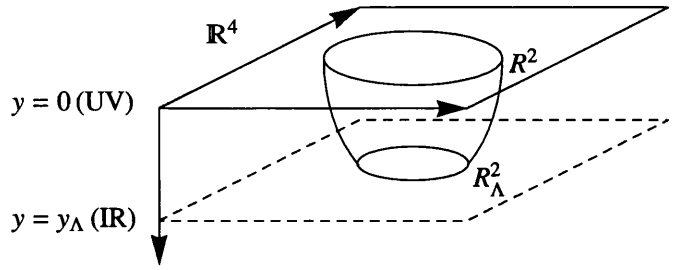


Figure 3.3: *A circular Wilson loop of radius R sitting at the* $\mathbb{R}^4$ *UV boundary of* AdS_5 *produces an effective loop at the AdS cutoff* y_Λ *of radius* R_Λ*, where* $R > y_\Lambda$*. The cut-off at* y_Λ *acts as an IR cut-off of the theory, eliminating effects deep in the AdS bulk.*

R_Λ, where $R^2 = R_\Lambda^2 + y_\Lambda^2$. This is illustrated in figure 3.3. Ultimately, the large radius limit of the boundary Wilson loop is to be investigated (i.e. large quark separations). In such a case, for a fixed y_Λ, as $R \to \infty$, $R_\Lambda \to R$. It is therefore sufficient to consider simply the disk worldsheet at y_Λ, neglecting the minimal effects at $y < y_\Lambda$.

The 9d metric of the Hard-wall slice, in the Euclidean signature with AdS radius at unity, is expressed in the Poincarè patch as:

$$ds_\Lambda^2 = \frac{1}{y_\Lambda^2}(dr^2 + r^2 d\eta^2 + dx_2^2 + dx_3^2) + d\theta^2 + \sin^2\theta\, d\Omega_4^2 \qquad (3.30)$$

Here, y from the $AdS_5 \times S^5$ metric is replaced by the constant y_Λ, and is the position of the $\mathbb{R}^4$slice in the AdS_5 interior. This overall factor of y_Λ^{-2} effectively acts as a scaling on the background, independent of any dynamics. As previously, r, η re-parameterises x_1, x_2 in polar co-ordinates, with the circular Wilson loop centred at $r = 0$. Again the S^5 is written as $S^4 \times S^1$.

A D5 brane is inserted into this background, and as in section 3.3, the world-volume of the brane is identified as ρ, η, and the S^4. As the system is restricted to the $y_\Lambda$slice, the brane's degree of freedom in y is lost, effectively identifying ρ & r. However, for now consider r as simply a general function of ρ. The Ramond-Ramond 4-form potential in the S^5 is unaffected by the selection of the flat-space slice, and thus remains unchanged from section 3.3, eq. 3.14.

The bulk action is again given by the DBI and Wess-Zumino terms, including the electric field strength $\mathcal{F} \equiv \mathcal{F}_{\rho\eta}$, and simplifies to:

$$S_{\text{Bulk},\Lambda} = T_{\text{D5}} \int d^6\xi \left[\sin^4\theta_k \sqrt{\frac{r^2}{y_\Lambda^4} r'^2 + \mathcal{F}_{\rho\eta}^2} - i\mathcal{F}_{\rho\eta} G(\theta_k) \right] \tag{3.31}$$

Where once again $G(\theta_k) = \left(\frac{3}{2}\theta_k - \sin 2\theta_k + \frac{1}{8}\sin 4\theta_k\right)$, and $\theta_k = \theta$ = constant. It is clear to see, that the only modifications to the calculations of the full $AdS_5 \times S^5$ case are due to the loss of the variation in y. Examining the boundary terms, one finds that boundary effects for y vanish ($\partial_\rho y_\Lambda = 0$), leaving simply the term relating to the electric field strength;

$$S_{\text{Bdy,A},\Lambda} = -\frac{\delta S_{\text{Bulk}}}{\delta \mathcal{F}_{\rho\eta}} \mathcal{F}_{\rho\eta} \tag{3.32}$$

$$= -T_{\text{D5}} \int d^6\xi \left[\frac{\sin^4\theta_k \mathcal{F}_{\rho\eta}^2}{\sqrt{\frac{r^2}{y_\Lambda^4} r'^2 + \mathcal{F}_{\rho\eta}^2}} - i\mathcal{F}_{\rho\eta} G(\theta_k) \right] \tag{3.33}$$

Summing both the bulk action and boundary terms, the total action for the brane at the cut-off becomes

$$\begin{aligned} S_{\text{Tot},\Lambda} &= S_{\text{Bulk},\Lambda} + S_{\text{Bdy,A},\Lambda} \\ &= T_{\text{D5}} \int d^6\xi \left[\frac{r^2}{y_\Lambda^4} \frac{r'^2 \sin^4\theta_k}{\sqrt{\frac{r^2}{y_\Lambda^4} r'^2 + \mathcal{F}_{\rho\eta}^2}} \right] \end{aligned} \tag{3.34}$$

Still keeping the ρ dependence of r undefined, the only equation of motion that remains is that for the electric field strength;

$$\mathcal{F}_{\rho\eta} = -i \cos\theta_k \frac{r}{y_\Lambda^2} r' \tag{3.35}$$

The dependence of k with θ_k is unmodified from eq. 3.23. This is consistent with the argument that the dependence of k on θ_k should be independent of the cut-off in the AdS region. Application of eq. 3.23 & 3.35 into eq. 3.34 gives

$$S_{\text{Tot},\Lambda} = T_{\text{D5}} \frac{1}{y_\Lambda^2} \int d^6\xi \, r \, r' \sin^3\theta_k \tag{3.36}$$

The integration interval is over the complete S^4, as previously, together with the flat space area described by r and η at y_Λ. Integrating over the S^4, and explicitly identifying r to ρ:

$$\begin{aligned} S_{\text{Tot},\Lambda} &= \frac{2N}{3\pi\alpha'} \frac{1}{y_\Lambda^2} \sin^3\theta_k \int_{r=0}^{r=R} d\rho \left(\frac{dr}{d\rho}\right) r \\ &= \frac{2N}{3\pi\alpha'} \sin^3\theta_k \frac{R_\Lambda^2}{2y_\Lambda^2} \end{aligned} \tag{3.37}$$

$R_\Lambda$is the radius of the Wilson loop on the slice y_Λ. For comparison, consider the computation for the tension of a fundamental string in the same Hard-wall background. Using the Nambu-Goto string action,

$$S_{\text{N.G.},\Lambda} = \frac{1}{\alpha'} \int_0^{R_\Lambda} dr \frac{r}{y_\Lambda^2} = \frac{1}{\alpha'} \frac{R_\Lambda^2}{2y_\Lambda^2} \equiv \sigma_f R_\Lambda^2 \tag{3.38}$$

Thus, the D5-brane system reduces to that of a fundamental string with a modified string tension with a k dependence of the form $\sigma_k = \frac{2N}{3\pi\alpha'} \frac{1}{2y_\Lambda^2} \sin^3\theta_k$.

To complete the computation, the large radius limit is taken to negate the additional constant term from the AdS_5 region of the space. Taking the $R \to \infty$ limit, $R_\Lambda \to R$. This is interpreted as almost the entirety of the brane world-sheet sitting on the flat space slice, while only a very small proportion of the brane stretches between $y = 0$ and $y = y_\Lambda$ (i.e. in AdS_5).

$$S_{\text{Tot}}|_{R\to\infty} = \sigma_k R^2 \tag{3.39}$$

with the k string tension σ_k

$$\sigma_k = \frac{2N}{3\pi\alpha'} \sin^3\theta_k \frac{1}{2y_\Lambda^2} \tag{3.40}$$

In the large R limit, the world-sheet area becomes equivalent to the area of the Wilson loop, hence making the area law, and thus confinement manifest. This is illustrated in figure 3.4.

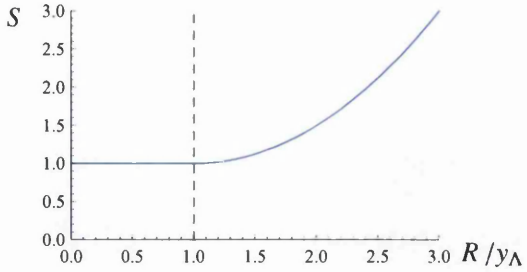


Figure 3.4: *At small R, the total action of the brane, S is constant, as the entire world-sheet sits in the AdS region. As R increases, the world-sheet gets closer to, and eventually touches, the cut-off, y_Λ. As R increases further, the action begins to increase as the square the radius of the world-sheet disk that sits at y_Λ. For $R \to \infty$, the radius of the disk at y_Λ, $R_\Lambda \to R$.*

As was discussed in chapter 2.1, $k\,\sigma_f > \sigma_k$ for k interacting strings. Result 3.40 gives the ratio of the k string tension and the fundamental string tension as;

$$\frac{\sigma_k}{\sigma_f} = \frac{2N}{3\pi}\sin^3\theta_k \tag{3.41}$$

This is the exact same ratio as was found for the anti-symmetric k-string in full $AdS_5 \times S^5$. This illustrates that the ratios of k-string tension in $AdS_5 \times S^5$ are unaffected by the addition of a cut-off to the AdS_5 space. This is of course not true of the tension itself (eq.3.40). Moving to the non-conformal theory, the IR cut-off scale that was introduced into the AdS bulk becomes an effective energy scale to which the tension is now proportional to.

On a technical point, it is interesting to note that even when considering flat space in the large R limit, the effect of the electric field at the boundary must be included for the expressions to be finite, and thus consistent with those of the fundamental string. This may be explained by the $R \to \infty$, y_Λ = finite limit being equivalent to R = finite, $y_\Lambda \to 0$; (i.e. the boundary). It would seem obvious why the boundary effects would be required in this case.

3.5 k-string tensions in $\mathcal{N}=1$ SYM gravity duals

Following the string tension calculations in the Hard-wall AdS confining background, focus shifts to a theory more closely related to QCD, while still exhibiting a gauge-gravity correspondence in which semi-classical string methods can be em-

ployed. The theory to be utilised is $\mathcal{N}=1$ super Yang-Mills, and it's gravity-duals.

Anti-symmetric k-strings in $\mathcal{N}=1$ SYM Gravity Duals (hereafter referred to simply as $\mathcal{N}=1$) at the IR limit, are described by wrapped D3-branes. The tensions of k-strings in $\mathcal{N}=1$ was given serious thought in the 2001 paper of Klebanov & Herzog [19]. In their paper, the authors use a D3-brane to describe an anti-symmetric k-string in both the Klebanov-Strassler (KS) [45] and Maldacena-Núñez D5 (MN) [46] $\mathcal{N}=1$ SYM Gravity Dual backgrounds [1], following closely the method of Bachas *et al.* [20]. The computations performed are in the Neveu-Schwarz - Neveu-Schwarz (or Magnetic) sector of the theory, which contains a Neveu-Schwarz (NS) 2-form field, and a chromoelectric field strength, related to the magnetic monopole number, parallel to it.

Although differing in nature, both the Maldacena-Núñez & Klebanov-Strassler gravity duals of $\mathcal{N}=1$ are achieved via the wrapping of N D5 branes on an S^2 inside the resolved conifold, a Calabi-Yau 3-form manifold, CY_3, and allowing the branes to back-react on the geometry. As shall be seen, taking the IR limits and making a set of specific identifications, the two different backgrounds reduce to $\mathbb{R}^4 \times S^3$ (For the MN background this is an exact reduction, while for KS, this is exact up to a $\sim 6\%$ error. This is shown explicitly below).

In section 3.6, the computation of Klebanov & Herzog will be explicitly performed in the MN background, while being carefully reviewed in the KS background. In section 3.7, an analogous computation to that of Klebanov & Herzog is performed: The MN & KS backgrounds are S-dualised, and it is shown that the exact same action can be produced following the method used in section 3.4, without the requirement of a *a priori* selected field strength.

3.6 Tensions in NS Sector

3.6.1 Maldacena-Nunez NS Background

The first background to be considered is the Maldacena-Núñez background. The system again calls for a circular Wilson loop to be placed in the $\mathbb{R}^4$spacetime. A D3 brane is embedded into the background and is wrapped over two compact

[1] The Maldacena-Núñez gravity solution was first discovered by Chamseddine *et al.* [47], but was correctly interpreted as the gravity dual of $\mathcal{N}=1$ SYM by Maldacena and Núñez [46].

directions transverse to $\mathbb{R}^4$. This allows the two remaining directions to sweep a string world-sheet in the $\mathbb{R}^4$, tracing the Wilson loop.

In the IR limit, the Maldacena-Núñez background has the topology of $\mathbb{R}^4 \times \mathbb{R} \times M^5$, where M^5 is an $S^2 - S^3$ fibration. For the D3 brane to correctly wrap the transverse space, a 3-cycle is chosen within the $S^2 - S^3$ fibration, reducing the system to $\mathbb{R}^5 \times S^3$. It is the angle at which the D3 sits within the S^3 that becomes related to the world-volume field strength charge, and thus the k-dependency of the system. This is analogous to the θ_k angle in the S^5 in the AdS_5 cases (Sections 3.3 & 3.4). The metric for the 10 dimensional (Euclidean) spacetime is given as:

$$ds_{10}^2 = \left[dr^2 + r^2 d\eta^2 + dx_3^2 + dx_4^2 + N\alpha' \left(d\rho^2 + e^{2h(\rho)} \left(d\theta_1^2 + \sin\theta_1 d\phi_1^2\right) + \frac{1}{4}(\omega_i - A_i)^2\right)\right] \tag{3.42}$$

Where:

$$A_1 = -a(\rho)d\theta_1, \quad A_2 = a(\rho)\sin\theta_1 d\phi_1, \quad A_3 = -\cos\theta_1 d\phi_1. \tag{3.43}$$

$$\begin{aligned} \omega_1 &= \cos\psi d\theta_2 + \sin\psi \sin\theta_2 d\phi_2, \\ \omega_2 &= -\sin\psi d\theta_2 + \cos\psi \sin\theta_2 d\phi_2, \\ \omega_3 &= d\psi + \cos\theta_2 d\phi_2 \end{aligned} \tag{3.44}$$

The parameters r, η, x_3 & x_4 represent the $\mathbb{R}^4$ spacetime, and ρ, the effective energy level of the system, analogous to y_Λ in AdS_5. The angles θ_1, ϕ_1 & θ_2, ϕ_2, ψ parametrise the S^2 and fibered S^3 respectively. The ranges of the angles being:

$$\theta_1 \in [0,\pi], \quad \phi_1 \in [0,2\pi), \quad \theta_2 \in [0,\pi], \quad \phi_2 \in [0,2\pi], \quad \psi \in [0,4\pi).$$

Within the background, there exists a B_2 Neveu-Schwarz (NS) potential, which obeys $H_3 = dB_2$, and is given by:

$$\begin{aligned} B_2 = \frac{N\alpha'}{4}[&\psi(\sin\theta_1 d\theta_1 \wedge d\phi_1 - \sin\theta_2 d\theta_2 \wedge d\phi_2) \\ &- \cos\theta_1 \cos\theta_2 d\phi_1 \wedge d\phi_2 - (d\theta_1 \wedge \omega_1 - \sin\theta_1 d\phi_1 \wedge \omega_2)] \end{aligned} \tag{3.45}$$

To select a 3-cycle in the transverse space for the D3 to wrap, a set of identifications are made. These reduce the S^2 and fibered S^3 to another S^3, which shall

be denoted as $\tilde{S}^3$. There are two possible sets of identifications, both of which, for this computation, provide identical 3-cycles.[2] The identifications are grouped thus:

$$\theta \equiv \theta_1 = \theta_2, \quad \phi \equiv \phi_1 = 2\pi - \phi_2, \quad \psi \to 2\Psi + \pi; \tag{3.46}$$

$$\theta \equiv \theta_1 = \pi - \theta_2, \quad \phi \equiv \phi_1 = \phi_2, \quad \psi \to 2\Psi. \tag{3.47}$$

For this calculation, eq. 3.46 will be the identification applied, as in [19].

The background needs to be taken to the infra-red limit. This is applied by taking the zero-limit of the energy scale, $\rho \to 0$. In this limit the functions of ρ become trivial, $a(\rho) \to 1$ & $e^{2h(\rho)} \to 0$. Following this running of the system to the IR limit, along with the selection of the 3-cycle, the background and B_2 simplify;

$$ds^2 = dr^2 + r^2 d\eta^2 + dx_3^2 + dx_4^2 + N\alpha' \left[d\rho^2 + d\Psi^2 + \sin^2\Psi \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right] \tag{3.48}$$

$$B_2 = N\alpha' \left(\Psi - \frac{1}{2}\sin 2\Psi \right) \sin\theta \, d\theta \wedge d\phi \tag{3.49}$$

The D3 is embedded along r, η (effectively tracing out a flat disk world-sheet in $\mathbb{R}^4$), and θ, ϕ in the selected $\tilde{S}^3$, with Ψ being the angle at which the brane sits in the $\tilde{S}^3$.

A world-volume gauge field is turned on along θ & ϕ, parallel to the NS 2-form potential. Let this gauge field be designated by $F_2 \equiv F_{\theta\phi}$. The total energy of the D3 brane will have contributions from this gauge field, together with the NS 2-form field. These two fields generate the invariant, quantised entity, $\mathcal{F} = B_2 + 2\pi F_2$. Ensuring consistency, it is possible to select the gauge field in the form

$$F_{\theta\phi} = -\frac{k}{2} \sin\theta \, d\theta \wedge d\phi \tag{3.50}$$

Here, k is referred to as the "Magnetic monopole number", and as shall be seen below, equates to the string charge. The total action for the brane is given by the DBI action

[2]The symmetry between these identifications is realised as a symmetry in the geometry, and moving from one idenification to the other is equivalent to a "flop" in the geometry. See [48] and references there-in for discussions on this symmetry.

$$S_{\text{DBI}} = T_{\text{D3}} \int d^4\xi \sqrt{\det(\mathcal{G} + \mathcal{F})} \tag{3.51}$$

Where $\mathcal{F} = B_2 + 2\pi F_2$, and $\mathcal{G}$ is the usual pullback to the world-volume. T_{D3} is the D3 brane tension and is given by $T_{\text{D3}} = 1/(2\pi)^3 \alpha'^2$. Applying the metric, B_2 and F_2, the action becomes.

$$S = T_{\text{D3}} \int d^4\xi \, N\alpha' r \sin\theta \left[\sin^4\Psi + \left(\frac{\pi k}{N} - \left[\Psi - \frac{1}{2} \sin 2\Psi \right] \right)^2 \right]^{\frac{1}{2}} \tag{3.52}$$

Minimizing the action with respect to Ψ, the string charge k is related to the azimuthal angle of the D3 in the $\tilde{S}^3$ by

$$\Psi = \frac{\pi k}{N} \tag{3.53}$$

Applying this solution for Ψ into eq. 3.52, integrating over η, θ & ϕ, together with the integration over $r \in [0, R]$ (R the Wilson loop radius, as usual);

$$S = \frac{N}{\pi\alpha'} \sin\left[\frac{\pi k}{N}\right] \int_0^R dr\, r = \frac{N}{2\pi\alpha'} \sin\left[\frac{\pi k}{N}\right] R^2 \tag{3.54}$$

Studying the action, is it clear to see that it exhibits the area law discussed previously;

$$S = \sigma_k R^2; \quad \sigma_k = \frac{N}{2\pi\alpha'} \sin\left[\frac{\pi k}{N}\right]. \tag{3.55}$$

The result provides a tension similar to the sine law proposed by Douglas & Shenker [21]. Taking the ratios of two tensions for differing string charge, k & k', the exact sine law behaviour is revealed.

$$\frac{\sigma_k}{\sigma_{k'}} = \frac{\sin\left[\frac{\pi k}{N}\right]}{\sin\left[\frac{\pi k'}{N}\right]} \tag{3.56}$$

In section 3.7, it will be shown that this result can be obtained via an equivalent method, namely that of section 3.4, in the S-dual picture. Before moving to the

S-dual case, first consider the k-string tension calculation in the KS background in the Nevez-Schwarz sector.

3.6.2 Klebanov-Strassler NS Background

In the Klebanov-Strassler background [45,49], the same procedure is employed as the Maldacena-Núñez background case. The metric is expressed as the product of Euclidean spacetime $\mathbb{R}^4$ and the 6-dimensional Calabi-Yau metric of the deformed conifold; with an associated warp factor $\tilde{h}(\rho)$:

$$\begin{aligned} ds^2 &= \tilde{h}(\rho)^{-1/2}(dx^2)_4 + \tilde{h}(\rho)^{1/2}ds_6^2 \qquad (3.57)\\ ds_6^2 &= \frac{1}{2}\epsilon^{4/3}K(\rho)\left[\frac{1}{3K(\rho)^3}(d\rho^2+g_5^2)+\cosh^2\left(\frac{\rho}{2}\right)(g_3^2+g_4^2)+\sinh^2\left(\frac{\rho}{2}\right)(g_1^2+g_2^2)\right] \end{aligned}$$

where

$$K(\rho) = \frac{[\sinh(2\rho)-2\rho]^{1/3}}{2^{1/3}\sinh\rho} \qquad (3.58)$$

and g_i are angular 1-forms. These are given as:

$$\begin{aligned} g_1 &= \frac{1}{\sqrt{2}}(-\sin\theta_1 d\phi_1 + \sin\psi d\theta_2 - \cos\psi\sin\theta_2 d\phi_2)\\ g_2 &= \frac{1}{\sqrt{2}}(d\theta_1 - \sin\psi\sin\theta_2 d\phi_2 - \cos\psi d\theta_2)\\ g_3 &= \frac{1}{\sqrt{2}}(-\sin\theta_1 d\phi_1 - \sin\psi d\theta_2 + \cos\psi\sin\theta_2 d\phi_2)\\ g_4 &= \frac{1}{\sqrt{2}}(d\theta_1 + \sin\psi\sin\theta_2 d\phi_2 + \cos\psi d\theta_2)\\ g_5 &= d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2 \end{aligned}$$

Working in units of g_s, the warp factor [49] is $\tilde{h}(\rho) = 2^{2/3}\epsilon^{-8/3}(N\alpha')^2 I(\rho)$, where

$$I(\rho) = \int_\rho^\infty dx \frac{x\coth x - 1}{\sinh^2 x}[\sinh(2x)-2x]^{1/3} \qquad (3.59)$$

Applying the identifications used previously, eqs.3.46; namely $\theta \equiv \theta_1 = \theta_2$, $\phi \equiv \phi_1 = 2\pi - \phi_2$, & $\psi \to 2\Psi + \pi$, while taking the IR limit ($\rho \to 0$), the functions $\tilde{h}(\rho)$, and $K(\rho)$ are found to become constants:

$$\begin{aligned} K(\rho \to 0) &= (2/3)^{1/3} \\ \tilde{h}(\rho \to 0) &= a_0 (N\alpha')^2 2^{2/3} \epsilon^{-8/3} \end{aligned}$$

Thus causing the metric to simplify to

$$ds^2 = \frac{\epsilon^{4/3}}{a_0^{1/2} 2^{1/3} N\alpha'} (dx^2)_4 + N\alpha' b \left[d\psi^2 + \sin^2\psi (d\theta^2 + \sin^2\theta d\phi^2) \right] \tag{3.60}$$

The metric collapses to a $\mathbb{R}^4 \times S^3$ slice at $\rho = 0$. For notational convenience, $b = \frac{2^{2/3}}{3^{1/3}} a_0^{1/2}$, where

$$a_0 = \int_0^\infty dx \frac{x \coth x - 1}{\sinh^2 x} [\sinh(2x) - 2x]^{1/3} \approx 0.71805$$

Thus $b \approx 0.93266$ [19,49]. There also exists a Nevez-Schwarz 2-form potential within the background, B_2, which satisfies $H_3 = dB_2$, and is expressed as:

$$B_2 = N\alpha' \left[\Psi - \frac{1}{2} \sin(2\Psi) \right] \sin\theta \, d\theta \wedge d\phi \tag{3.61}$$

As previously, a magnetic field strength in θ & ϕ is also turned on, $F_{\theta\phi}$, and is parallel to the 2-form potential B_2

$$F_{\theta\phi} = -\frac{k}{2} \sin\theta \, d\theta \wedge d\phi. \tag{3.62}$$

The D3-brane that will trace the Wilson loop is embedded in the background with world-volume co-ordinates in the $\mathbb{R}^4$along r and η, and wrapping θ & ϕ in the transverse S^3, again with Ψ as the angle the wrapped brane sits in the S^3. Integrating the DBI action over the world-volume;

$$\begin{aligned} S_{\text{DBI}} &= T_{\text{D3}} \int d^4\xi \sqrt{\det(\mathcal{G} + \mathcal{F})} \\ &= \frac{\epsilon^{4/3} R^2}{2^{2/3} 3^{1/3} b\pi \alpha'^2} \sqrt{b^2 \sin^4\Psi + \left(\Psi - \frac{1}{2} \sin 2\Psi - \frac{\pi k}{N} \right)^2} \end{aligned} \tag{3.63}$$

The equation of motion for Ψ provides an expression for k:

$$\frac{\pi k}{N} = \Psi + \frac{1}{2}(b^2 - 1)\sin 2\Psi \tag{3.64}$$

Notice how the action, eq.3.63 is equivalent, up to an overall factor, to the Maldacena-Núñez computation, eq.3.52, when $b = 1$. Although not completely solvable, applying the solution, eq.3.64 to the action, the tension can be expressed in terms of Ψ;

$$S = \frac{\epsilon^{4/3} R^2 \sin\Psi}{2^{2/3} 3^{1/3} \pi \alpha'^2} \sqrt{b^2 \cos^2\Psi + \sin^2\Psi} \tag{3.65}$$

$$\sigma_k \sim \sin\Psi \sqrt{1 + (b^2 - 1)\cos^2\Psi} \tag{3.66}$$

As for the MN background, the action once again exhibits an area law, as expected. The action and tension are also invariant under charge conjugation and additions of colourless baryons. Although not manifest, these symmetries are visible from the symmetry of Ψ around $\pi/2$, $\Psi \to \pi - \Psi$ is equivalent to $k \to N - k$. This is more clearly visible if the approximation is taken that b is exactly 1 (i.e., MN background case).

In the next section, the S-dual case of the calculations considered here will be discussed, namely tension calculations in the Ramond-Ramond sector. The computation will show that while being non-trivially equivalent to that in the NS sector, identical dynamics, tensions and numerical factors can be reproduced in a more endogenous regime.

3.7 Tensions in RR Sector

In this section, the resultant actions for the wrapped D3 branes in section 3.6 will be replicated in the after performing an S-duality on the $\mathcal{N} = 1$ dual backgrounds.

S-duality

S-duality, or more informatively, strong-weak duality, is an equivalence between two theories (in this case, string theories), one at weak coupling, the other strongly coupled. S-duality maps the properties of a theory with coupling g, to a theory of coupling $1/g$. The duality operation also exchanges topological and

local charges, mapping fields between the electric (Ramond-Ramond) and magnetic (Nevez-Schwarz) sectors. In the $\mathcal{N} = 1$ duals, the NS 2-form field is transformed into a Ramond-Ramond 2-form potential, along the same directions at the NS 2-form. The chromoelectric field strength of the NS-NS-sector, associated with the magnetic monopole number, is replaced by a field strength in the $\mathbb{R}^4$, and becomes related directly to the electric charge of the k-string. S-duality also affects the branes that exist in the background. NS5-branes in the magnetic sector are mapped to D5-branes in the electric sector, fundamental strings map to D0 branes (1-dimensional branes - string-like), while D3 branes map to themselves.

The MN background computation will be discussed in detail, to allow direct comparison with section 3.6.1, while the KS background will be discussed briefly.

3.7.1 Maldacena-Nunez RR Background

The action of S-duality on the background metric introduces an overall factor related to the dilaton field, Φ, while the remainder of the metric remains unchanged.

$$\begin{aligned} ds_{10}^2 &= e^{\frac{\Phi}{2}}\Big[dr^2 + r^2 d\eta^2 + dx_3^2 + dx_4^2 + \\ &\quad N\alpha'\left(d\rho^2 + e^{2h(\rho)}\left(d\theta_1^2 + \sin\theta_1 d\phi_1^2\right) + \frac{1}{4}(\omega_i - A_i)^2\right)\Big] \end{aligned} \tag{3.67}$$

The factors A_i and ω_i remain unchanged and are those given in eqs. 3.43 & 3.45. As discussed earlier, the B_2 NS field transforms into a C_2 Ramond-Ramond potential, which obeys $F_3 = dC_2$, and is given by:

$$\begin{aligned} C_2 &= \frac{N\alpha'}{4}[\psi(\sin\theta_1 d\theta_1 \wedge d\phi_1 - \sin\theta_2 d\theta_2 \wedge d\phi_2) \\ &\quad - \cos\theta_1 \cos\theta_2 d\phi_1 \wedge d\phi_2 - (d\theta_1 \wedge \omega_1 - \sin\theta_1 d\phi_1 \wedge \omega_2)] \end{aligned} \tag{3.68}$$

Notice that this is identical to the NS B_2 field previously. Applying the identifications (3.46), and taking the IR limit, the metric and C_2 simplify to:

$$ds^2 = dr^2 + r^2 d\eta^2 + dx_3^2 + dx_4^2 + N\alpha'\left[d\rho^2 + d\Psi^2 + \sin^2\Psi\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] \tag{3.69}$$

$$C_2 = N\alpha'\left(\Psi - \frac{1}{2}\sin 2\Psi\right)\sin\theta\, d\theta \wedge d\phi \tag{3.70}$$

In the IR limit, $\rho \to 0$, the dilaton, Φ tends to a constant. For convenience, the system is expressed in units of g_s; ($g_s = e^{\frac{\Phi}{2}} = 1$). In such units, two S-dual theories operate at the same coupling scale ($g_s = \frac{1}{g_s} = 1$).

As before, the D3 world-volume is set along r, η, θ, & ϕ. An electric field is turned on, that runs along the length of the D3 in the $\mathbb{R}^4$spacetime. The field strength $F_{r\eta}$ acts as a measure of the chromoelectric flux travelling along the "string" in $\mathbb{R}^4$. This field strength is transverse to the field strength in the NS case.

The action governing the D3 brane includes the same DBI term, plus an additional Wess Zumino term:

$$\begin{aligned} S_{\text{Bulk}} &= S_{\text{DBI}} + S_{\text{WZ}} \\ &= T_{\text{D3}}\int d^4\xi\,\sqrt{\det(\mathcal{G}+\mathcal{F})} - iT_{\text{D3}}\int d^4\xi\, C_2 \wedge \mathcal{F} \end{aligned} \tag{3.71}$$

As there is no B_2 field, $\mathcal{F}$ consists only of field strength components, therefore $\mathcal{F} = 2\pi\alpha' F_{r\eta}$. Due to the Euclidean signature [14], the field strength is imaginary, so let $F_{r\eta} \to iF$. Applying the metric, C_2 and F;

$$S_{\text{Bulk}} = T_{\text{D3}}N\alpha'\int d^4\xi\,\sin\theta\left[\sin^2\Psi\sqrt{r^2 - 4\pi^2\alpha'^2F^2} + 2\pi\alpha' F\left(\Psi - \frac{1}{2}\sin 2\Psi\right)\right] \tag{3.72}$$

As was performed in the $AdS_5 \times S^5$ and Hardwall AdS_5 computations of sections 3.3 & 3.4 respectively, to find the string charge, k, take the variation of the action with respect to the field strength.

$$k = \frac{\delta S_{\text{Bulk}}}{\delta F} = T_{\text{D3}}N\alpha'\int d^4\xi\,\sin\theta\left[-\frac{\sin^2\Psi\, 4\pi^2\alpha'^2F}{\sqrt{r^2 - 4\pi^2\alpha'^2F^2}} + 2\pi\alpha'\left(\Psi - \frac{1}{2}\sin 2\Psi\right)\right] \tag{3.73}$$

Finding a solution for F in terms of k:

$$F = \frac{r}{2\pi\alpha'} \frac{\left(\Psi - \frac{1}{2}\sin 2\Psi\right) - \frac{\pi k}{N}}{\sqrt{\sin^4\Psi + \left[\left(\Psi - \frac{1}{2}\sin 2\Psi\right) - \frac{\pi k}{N}\right]^2}} \tag{3.74}$$

This a more natural expression for F than that of the magnetic, NS-NS sector [19], as there is no freedom in this choice for F. Using this expression, the bulk action, Eq.(3.72), becomes;

$$S_{\text{Bulk}} = \frac{N}{2\pi^2\alpha'} \int dr\, d\eta\, r \frac{\sin^4\Psi + \left[(\Psi - \frac{1}{2}\sin 2\Psi) - \frac{\pi k}{N}\right](\Psi - \frac{1}{2}\sin 2\Psi)}{\sqrt{\sin^4\Psi + \left[(\Psi - \frac{1}{2}\sin 2\Psi) - \frac{\pi k}{N}\right]^2}} \tag{3.75}$$

As above, to correctly regularise the action, a boundary term for the field strength is required.

$$\begin{aligned} S_{\text{Bdy}} &= -\int d^4\xi \frac{\delta L_{\text{Bulk}}}{\delta F} F \qquad (3.76) \\ &= -\frac{N}{2\pi^2\alpha'} \int dr\, d\eta\, r \left(\frac{\pi k}{N}\right) \frac{\left[(\Psi - \frac{1}{2}\sin 2\Psi) - \frac{\pi k}{N}\right]}{\sqrt{\sin^4\Psi + \left[(\Psi - \frac{1}{2}\sin 2\Psi) - \frac{\pi k}{N}\right]^2}} \end{aligned}$$

Adding the bulk and boundary terms together, and integrating over the Wilson loop, r, & η, the total action becomes:

$$\begin{aligned} S_{\text{Tot}} &= S_{\text{Bulk}} + S_{\text{Bdy}} \qquad (3.77) \\ &= \frac{N}{2\pi^2\alpha'} \int_0^R dr \int_0^{2\pi} d\eta\, r \sqrt{\sin^4\Psi + \left[(\Psi - \frac{1}{2}\sin 2\Psi) - \frac{\pi k}{N}\right]^2} \\ &= \frac{N}{2\pi\alpha'} R^2 \sqrt{\sin^4\Psi + \left[(\Psi - \frac{1}{2}\sin 2\Psi) - \frac{\pi k}{N}\right]^2} \qquad (3.78) \end{aligned}$$

Applying the solution for k, from the NS-NS sector, eq. 3.53;

$$S_{\text{Tot}} = \frac{N}{2\pi\alpha'} R^2 \sin\frac{\pi k}{N} \tag{3.79}$$

the total action in the R-R sector reduces identically to the total action in the NS-NS sector, eq.3.54. It can be seen that the action of S-duality on the system

leaves not only the dynamics, but the entire action and numerical factors unaffected. Before discussing further, briefly consider the k-string tension in the Klebanov-Strassler background, Ramond-Ramond sector.

3.7.2 Klebanov-Strassler RR Background

In this section, the k-string tension computation in the KS background in the RR sector will be briefly outlined, and shown to be identical to the NSNS sector result. Under S-duality, the background metric, eq.3.58 is unaffected. The field content is however: The NS 2-form field is replaced by a Ramond-Ramond 2-form potential, C_2, that obeys $F_3 = dC_2$. The F_3 is expressed as:

$$F_3 = N\alpha' \left[g_5 \wedge g_3 \wedge g_4 + d\{F(\rho)(g_1 \wedge g_3 + g_2 \wedge g_4)\} \right] \tag{3.80}$$

where g_i are the angular 1 forms found in the metric, eq.3.58. The function $F(\rho)$ interpolates between 0 and $1/2$ in the IR and UV limits respectively.

Again, working in units of $g_s = 1$, and taking the usual angular identifications, the metric and C_2 simplify in the IR limit to

$$\begin{aligned} ds^2 &= \frac{\epsilon^{4/3}}{a_0^{1/2} 2^{1/3} N\alpha'} (dx_4^2 + d\rho^2) \\ &\quad + N\alpha' b \left[d\Psi^2 + \sin^2\Psi (d\theta^2 + \sin^2\theta d\phi^2) \right] && (3.81) \\ F_3 &= dC_2 = N\alpha' \left[1 - \cos(2\Psi) \right] \sin\theta \, d\theta \wedge d\phi \wedge d\Psi && (3.82) \\ C_2 &= N\alpha' \left[\Psi - \frac{1}{2} \sin(2\Psi) \right] \sin\theta \, d\theta \wedge d\phi && (3.83) \end{aligned}$$

Embed the D3-brane with world-volume co-ordinates along r, η, θ & ϕ, again with Ψ as the angle the brane sits in the S^3. Turn on an electric field strength in r and η, $F_{r\eta}$, which is set as imaginary, as before, due to the Euclidean signature; $F_{r\eta} \to iF$.

Before any computation is attempted, one can see that the construction is almost entirely equivalent to that in the Maldacena-Núñez case. Computation of the DBI and Wess-Zumino actions, plus an additional boundary term for the description of the electric field at the boundary, reproduce the exact result obtained via the S-Dual arguments, eq. 3.63

$$S_{\text{Tot}} = \frac{1}{2\pi^2\alpha'^2}\frac{\epsilon^{4/3}}{a_0^{1/2}2^{1/3}}\int dr d\eta\, r\left[b^2\sin^4\psi + \left[(\psi - \frac{1}{2}\sin 2\psi) - \frac{\pi k}{N}\right]^2\right]^{1/2} \quad (3.84)$$

$$= \frac{\epsilon^{4/3}R^2}{2^{2/3}3^{1/3}b\pi\alpha'^2}\sqrt{b^2\sin^4\Psi + \left(\Psi - \frac{1}{2}\sin 2\Psi - \frac{\pi k}{N}\right)^2} \quad (3.85)$$

The reader should note two points of interest here; firstly the k-string action and subsequently the dynamics are invariant under S-duality, as expected. Secondly, there are some interesting technical differences between the computations.

It is easily seen that the S-Dual calculation reproduces not just the dynamics of the action, but the exact numerical factors. As expected, the dynamics of the k-string, moving between sectors is unaffected, and as the system is considered in units of g_s, it is also comforting to note that the exact overall numerical factors arising from the world-sheet dynamics are also left invariant.

What is also of interest here, is how in the RR sector computations, the world-volume electric field strength on $\mathbb{R}^2 \subset \mathbb{R}^4$ is determined, not *a priori* as with the NS sector method, but directly from the variation of the world-sheet action itself.

3.8 Discussions

As demonstrated in this chapter, k-string tension computations in various confining backgrounds are not universal. They appear to be of very similar forms, namely exact factors, or approximations of $\sin(\pi k/N)$. This is true not only for the results in this thesis, but backgrounds (MQCD, softly broken $\mathcal{N} = 2, \ldots$). There is no reason why the tensions should be universal across various theories. What about universality between conformal & non-conformal theories?

For k-strings in Hardwall AdS, the ratio of tensions, eq.3.41 is identical to that obtained from the full $AdS_5 \times S^5$ background. Is this a universal result? Should the ratios be the same from a conformal model and a non-conformal modification? Moving from the conformal case of $AdS_5 \times S^5$, where there is no area law effect, and no string tension in the strict sense, to a non-conformal model where the Wilson loop as an order parameter signals confinement, naively there is no reason to

generally expect the ratios of σ's to remain universal. However, looking closely at the technical method of determining the k-string tension, one will find that to impose a scale on the conformal system required an IR cut-off in the AdS region of space, while the k & N scaling behaviours arise from dynamics in the transverse S^5 space. It would seem that the factorisation of the k & N scaling dynamics and the string world-sheet dynamics was unaffected by the application of the IR cut-off, and the k dynamics in the transverse space were unaffected.

It would seem that provided the k & N scaling dynamics were unaffected by the application of a non-conformal limit or cut-off, and the world-sheet dynamics remained decoupled from the k scaling, it would seems plausible that the invariance of the ratios might hold moving between conformal & non-conformal phases, but this is not guaranteed.

As was seen in section 3.5, the minimised action and subsequent dynamics of the k-string in $\mathcal{N} = 1$ SYM Gravity dual theories was invariant under the action of S-duality on the systems.

Of interest in this section was not only the invariance of the models results under S-duality, which is not only expected but required, but also the dynamics of the world-volume electric field strength on $\mathbb{R}^2 \subset \mathbb{R}^4$ in the RR sector. The world-volume field is determined directly from the variation of the world-sheet action, and not prior to the minimisation of the action, as with the NSNS sector. This seems a more natural way to introduce the k scaling behaviour to the system, than via the requirement of gauge invariance within the DBI action [20]. The method of determining the string charge here seems more endogenous, and less manufactured as the gauge invariance argument suggests.

What is also intriguing is how in the RR sector computations, boundary terms are required to correctly capture the dynamics, and eliminate divergencies due to the non-dynamical quarks. This was seen in the computations detailed in section 3.3 & 3.4. In the NS sector discussions, no such boundary terms are required. Regularisation must be performed in another intrinsic way.

In addition to the $\mathcal{N} = 1$ computations outlined here, attempts were made to embed NS5 and D5 branes in the MN background in an attempt to model k-strings

of representation different to the anti-symmetric. The computation called for additional wrapping in the space transverse to $\mathbb{R}^5$. The NS5 brane method achieved a constant result for the action, independent of the string charge k, or the angle of embedding, Ψ. For the D5 brane complications arose as it was found the determinant of the S^2 & S^3 in MN vanish in the infrared limit. Electric fields were inserted in the space in an attempt to prevent this collapse, however no sensible results could be obtained. It may be that 6 dimensional brane structures in this background are unstable, or simply do not correspond to a representation of the Wilson Loop. This area is left open for future work.

Chapter 4

k-string widths

4.1 Intro to width & correlator calculations

The idea of a "width", or by the more revealing name "quantum broadening", of a string was first seriously discussed in the 1981 paper *"How thick are chromo-electric flux tubes?"* of Lüscher, Münster & Weisz [50]. They propose a method of measuring this broadening effect by evaluating the chromo-electric field density, $\mathcal{P}(x)$, above that of the vacuum, of an infinitely heavy, non-dynamical quark - anti-quark pair. The measure of the width is defined as follows:

$$\Sigma^2 = \frac{\int dx_\perp x_\perp^2 \mathcal{P}(x)}{\int dx_\perp \mathcal{P}(x)} \tag{4.1}$$

The electric field density of the quarks is evaluated over all transverse directions, $x_\perp$ to the flux tube.

For a confining theory, the flux tube is expected to be a localised object (quark separation being larger than the scale of the theory, i.e. Λ_{QCD}), with the width growing slowly with increasing quark separation. For non-confining theories, the "flux tube" acts like a dipole, with a width increasing as the square of the separation. Examples of both confining and non-confining string widths shall be shown later in this chapter.

Lüscher *et al* performed width calculations in 4d flat space in the strong coupling limit on an SU(2) lattice model (modern lattice investigations include [51]), and via string arguments. They proposed that the energy density can be expressed as the

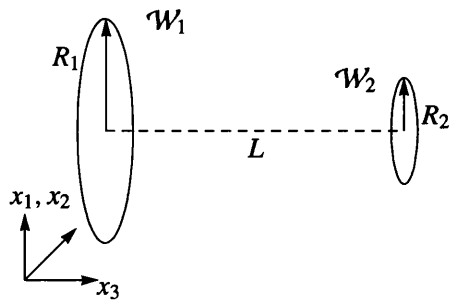


Figure 4.1: *Circular Wilson loops, $\mathcal{W}_1$ & $\mathcal{W}_2$ of radius R_1 & R_2, sit at the origin in the x_1, x_2 plane, and at $x_3 = 0$ & $x_3 = L$ respectively.*

correlation function between a pair of Wilson loops. As the point of interest is the string model, and shall be used as a basis for later calculations, consider the system of two circular Wilson loops.

Take two, circular Wilson loops, $\mathcal{W}_1$ & $\mathcal{W}_2$, of radius R_1 & R_2 respectively, in the 2d plane x_1, x_2, but separated by a transverse distance, L, in the x_3 direction. This is illustrated in figure 4.1. Let $\mathcal{W}_1$ represent the creation and annihilation of the quark pair, and when taking the $R_2 \to 0$ limit, $\mathcal{W}_2$ represents an electric field strength operator, F^2. For the string model, the correlator is described by the minimal surface area of the world-sheet of a closed string propagating between the two loops. In a 3 dimensional system, the minimal area forms a catenoid, and has the topology of an annulus.

For such a system with minimal area $A(L)$ and string tension σ, the energy density is given by the following correlation function

$$\mathcal{P}(L) = \frac{\langle \mathcal{W}_1 \mathcal{W}_2 \rangle - \langle \mathcal{W}_1 \rangle \langle \mathcal{W}_2 \rangle}{\langle \mathcal{W}_1 \rangle} \propto e^{-\sigma A(L)} \tag{4.2}$$

Such a computation, in the large quark separation limit, namely R_1 & $R_1/R_2 \to \infty$ provides a width which grows logarithmically with R_1/R_2, to leading order in R_1.

$$\Sigma^2 = \frac{1}{2\pi\sigma} \log[R_1/R_2] \tag{4.3}$$

Notice that the width is inversely proportional to the tension of the string.

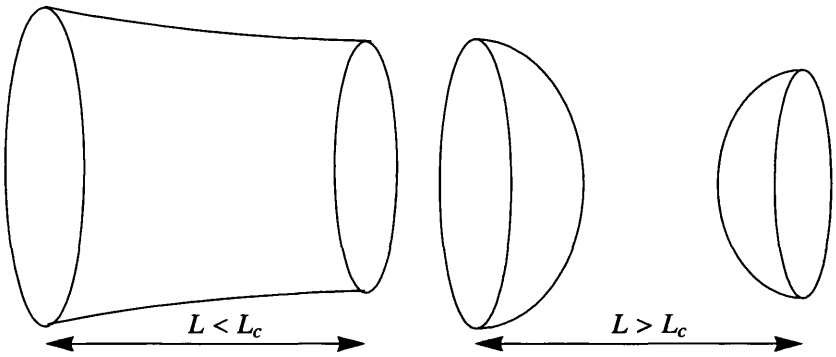


Figure 4.2: *Connected ($L < L_c$) and disconnected ($L > L_c$) worldsheet phases of the Wilson loop correlator in AdS_5.*

A natural area to experiment with this computation is within the AdS/CFT correspondence. In the works of Zarembo [52], and Olesen & Zarembo [53], the Wilson loop correlator in $AdS_5 \times S^5$ is discussed from an analytical perspective. However a problem is encountered.

Using the configurations seen above, and applying them to an AdS_5 space, it is found that as one takes $\mathcal{W}_1$ to infinite extent, the minimal surface solution is found to become inconsistent at some point. In fact, as any one, or combination, of the parameters (L, R_1, R_2) of the system is taken towards infinite extent, the solution encounters inconsistencies at parameter ratios of around 1–2 orders of magnitude. This would sound the death knell for the AdS_5 Wilson loop correlator, was it not for the Gross-Ooguri, or string breaking phase transition [54].

In AdS_5, when $\mathcal{W}_1$ & $\mathcal{W}_2$ are separated by a small distance $L \lessapprox R_1, R_2$, the global minimum of the world-sheet is the connected phase, stretching between the loops. However, as L increases, a critical point is reached $L \equiv L_c$, above which the connected world-sheet breaks into two world-sheets, each tracing one of the loops. The topology changes from the annulus to that of two disks. This is illustrated in figure 4.2. In the disconnected phase, the correlator is described by interactions via the exchange of light supergravity modes between the world-sheet surfaces.

Moving through the parameter space, the correlator in AdS_5 will encounter the Gross-Ooguri phase transition, before reaching the inconsistency. In the work of Greensite & Olesen [55], the width of strings in the AdS_5/CFT are discussed from

a numerical perspective. The inconsistency in AdS_5 is again found, however taking the limit of an $\mathbb{R}^4$ slice in the AdS_5 bulk, and evaluating the correlator, and hence the width, the general result of Lüscher *et al.* [50] is found.

It was not until Gliozzi *et al.* [56] that the quantum broadening effect was applied to higher representational, k-strings. They proposed that the width of a k-string required a Wilson loop correlator between a set of k co-incident Wilson loops, and a single probe loop. Their conclusion showed that the k-string width is independent of k, as the fundamental loop couples to one of the k coincident loops, while the remaining $k-1$ loops are spectators to the system. There are two significant issues here; one of limits, and one of suitable probes.

Consider first the case of limits. As was discussed in chapter 2, there are two large N regimes that can be explored for a string of N-ality k; $k, N \to \infty$, k/N fixed & $N \to \infty$, k fixed. In the k/N fixed limit of large N, the k-string is represented as a wrapped D-brane, whereas in the k fixed limit, the k-string is expressed simply as a collection of k co-incident, weakly interacting strings. The work of Gliozzi *et al.* falls into this second regime. A complete investigation of a k-string width requires consideration of both limits.

A second, more concerning point, is the selection of probe. What type of probe should be used in calculating the k-string width? There are two main choices; the use of a fundamental probe, or a probe in the kth antisymmetric representation, just like the k-string itself. The width will be dependent upon the type of the probe, so the aim is to find which is the most "physical" probe.

Consider the two point function, representing the connected world-sheet;

$$\langle \mathcal{W}_1\ (k)\, \mathcal{W}_2\ (f) \rangle_{\text{conn.}} \tag{4.4}$$

namely $\mathcal{W}_1$ being N-ality k and the probe loop $\mathcal{W}_2$ in the fundamental representation. Considering the large N, fixed k limit, $\mathcal{W}_1$ can be expressed as the k-th tensor product of the fundamental representation (k coincident loops), namely that equation 4.4 takes the form

$$\langle [\mathcal{W}_1\ (f)]^k\, \mathcal{W}_2\ (f) \rangle_{\text{conn.}} \tag{4.5}$$

At large-N, the correlator can be factorised, 4.5 becomes

$$k\langle \mathcal{W}_1(f)\rangle^{k-1}\langle \mathcal{W}_1(f)\mathcal{W}_2(f)\rangle_{\text{conn.}} \tag{4.6}$$

Notice that the probe connects to only one of the co-incident loops in $\mathcal{W}_1$, while the remaining $k-1$ loops become spectators to the correlator. Thus, if the probe Wilson loop is in the fundamental representation, it will necessarily "see" only one constituent of $\mathcal{W}_1$ at a time. This leads to the conclusion that the width of $\mathcal{W}_1$ is the same as the width of a Wilson loop in the fundamental representation. This is exactly the case in [56].

Alternatively, consider a probe of N-ality k, again in the large N, fixed k limit. In such a case the two point function 4.4 is replaced by

$$\langle \mathcal{W}_1(k)\mathcal{W}_2(k)\rangle_{\text{conn.}} \tag{4.7}$$

At large-N this takes the form of

$$\langle [\mathcal{W}_1(f)]^k\,[\mathcal{W}_2(f)]^k\rangle_{\text{conn.}} = k!\langle \mathcal{W}_1(f)\mathcal{W}_2(f)\rangle^k_{\text{conn.}} \tag{4.8}$$

Here, the probe interacts simultaneously with all the constituents of the k-string and the resultant measurement is that the k-string is a factor of k narrower than the fundamental string. Namely, σ in equation 4.3 is replaced by $k\sigma$.

Motivated by the above analysis, a system of a k-string with a probe loop in the same representation, in the limit of $k, N \to \infty$, k/N fixed is considered. This enables the width of the k-string to be measured, and not the width of one of its constituents, ergo determining the true nature of the string width dependence on the N-ality k.

Section 4.2 reviews Wilson loop correlators via connected string world-sheets in $AdS_5 \times S^5$ for fundamental strings, following the paper of Olesen & Zarembo [53]. By taking one of the Wilson loops to the probe limit, the connected world-sheet is shown to undergo a Gross-Ooguri phase transition before reaching a point at which the string equations of motion become inconsistent, thus saving the model. The inconsistency argument is briefly discussed with extension to the case of the k-string.

Motivations & discussions are made as to the suitability of probe loops of different representations, before disconnected Wilson loop correlators from light Supergravity mode exchanges are examined and used to determine string widths in $AdS_5 \times S^5$ after the Gross-Ooguri phase transition.

In section 4.3, original work is presented. The method of Lüscher *et al.* [50] is employed to determine the fundamental string width in Hard-wall AdS_5 briefly, before expanding to the k-string width computation in Hardwall AdS using wrapped D5-branes. Finally, in section 4.4, the work of section 4.3 is extended to determine k-string widths within $\mathcal{N} = 1$ gravity dual backgrounds. These two sections of original work were published with A. Armoni in [5].

4.2 String widths in $AdS_5 \times S^5$

4.2.1 Fundamental String Correlator in $AdS_5 \times S^5$

In this section, the paper of Olesen & Zarembo is reviewed [53]. The minimal surface Wilson loop correlator in $AdS_5 \times S^5$ will be examined, in an attempt to determine the flux tube width in the fundamental representation. However, it will be shown that it is not possible to form a Wilson loop correlator in AdS_5 via a string worldsheet for general loop size and transverse separation.

Consider again the setup of section 4.1, namely two concentric, circular, spatial Wilson loops, $\mathcal{W}_1$ & $\mathcal{W}_2$, of general radii R_1 & R_2 respectively, separated by a transverse distance L (visualised in figure 4.1). To calculate the minimal surface area of the worldsheet that stretches between the two loops, employ the Nambu-Goto string action, minimise, and find solutions for the connected sheets.

Taking the AdS_5 metric in the Poincaré patch, with the spectator S^5 ignored:

$$ds^2 = \frac{1}{y^2}(dy^2 + dr^2 + r^2 d\eta^2 + dz^2 + dx_4^2) \tag{4.9}$$

The radius of AdS_5 is set to 1, with the boundary at $y = 0$. The x_1, x_2 plane is reparametrised in circular co-ordinates by r & η, with x_3 relabelled as z. The Wilson loops sit on the boundary, and are separated in this z direction, where $\mathcal{W}_1$ & $\mathcal{W}_2$ sit at $z = 0$ & $z = L$ respectively. Allowing the string to move through the interior of AdS_5, y, along with the spatial directions r, η, z, use the string

embedding; $r \to r(\tau)$, $y \to y(\tau)$, $z \to z(\tau)$, & $\eta \to \eta(\sigma) = \sigma$. This embedding provides the string action[1]

$$S_{NG} = \frac{1}{2\pi}\int_0^{2\pi} d\sigma \int_{\tau_1}^{\tau_2} d\tau \frac{r}{y^2}\sqrt{r'^2 + y'^2 + z'^2} \tag{4.10}$$

The Wilson loops are circular, thus the system is invariant under rotations in η, simplifing the action to:

$$S_{NG} = \int_{\tau_1}^{\tau_2} d\tau \frac{r}{y^2}\sqrt{r'^2 + y'^2 + z'^2} \tag{4.11}$$

Minimising the action, provides the following equations of motion:

$$\frac{r}{y^2}\frac{z'}{\sqrt{r'^2 + y'^2 + z'^2}} = l \tag{4.12}$$

$$\left(\frac{r}{y^2}\frac{r'}{\sqrt{r'^2 + y'^2 + z'^2}}\right)' - \frac{1}{y^2}\sqrt{r'^2 + y'^2 + z'^2} = 0 \tag{4.13}$$

$$\left(\frac{r}{y^2}\frac{y'}{\sqrt{r'^2 + y'^2 + z'^2}}\right)' + \frac{2r}{y^2}\sqrt{r'^2 + y'^2 + z'^2} = 0 \tag{4.14}$$

Where l is a constant of integration. As z increases monotonically from 0 to L, a gauge choice can be made, identifying $z = \tau$, thus expressing the equations of motions as:

$$rr'' - \frac{r^2}{y^4 l^2} = 0; \quad yy'' + \frac{2r^2}{y^4 l^2} = 0; \quad r'^2 + y'^2 + 1 = \frac{r^2}{y^4 l^2} \tag{4.15}$$

$$rr'' + yy'' + r'^2 + y'^2 + 1 = 0 \tag{4.16}$$

The final expression is a sum of the previous three. Simplifying, and integrating twice;

$$(r^2 + y^2)'' + 2 = 0 \tag{4.17}$$

$$r^2 + y^2 + (z + b)^2 = a^2 \tag{4.18}$$

With a & b as integration constants. These are determined using the boundary conditions on each Wilson Loop. Considering first the loop, $\mathcal{W}_1$ at $z = 0$, $y = 0$

[1] In this chapter, unless stated, the string tension $1/\alpha'$ is set to 1 for notational convenience.

and $r = R_1$, and the second loop, $\mathcal{W}_2$ at $z = L$, $y = 0$ and $r = R_2$, find

$$a^2 = R_1^2 + b^2 \tag{4.19}$$

$$b = \frac{R_1^2 - R_2^2 - L^2}{2L} \tag{4.20}$$

Eq.4.18 can be reparameterised into trigonometric functions of an angular parameter ϕ.

$$\begin{aligned} r &= \sqrt{a^2 - (z+b)^2}\cos\phi \\ y &= \sqrt{a^2 - (z+b)^2}\sin\phi \end{aligned} \tag{4.21}$$

At $z = 0$, $r = R_1$, and moving along z toward $z = L$, r will become R_2. As both loops sit on the boundary, y will be zero. However, in the interval $0 < z < L$, the minimal surface is not restricted to the boundary, as the string can leave the boundary of AdS_5 and can travel through the bulk where $y \neq 0$. This implies that, while at both $z = 0$ and $z = L$, ϕ will be zero, between the loops ϕ is non-zero and can exist anywhere in the interval $\phi \in [0, \pi/2]$. This further causes r to be generally less than R_1 & R_2 when not at the boundary. It must be noted, that ϕ is not single valued, and will reach some maximal value within the allowed region. This will be considered fully below.

Applying the parameterisations back into Eq.4.15 the following expression is obtained for the behaviour of ϕ

$$\frac{d\phi}{dz} = \pm\frac{1}{l\,[a^2 - (b+z)^2]}\sqrt{\cot^2\phi\csc^2\phi - l^2a^2} \tag{4.22}$$

The sign choice represents the two branches of the function; ϕ increasing from 0 at $z = 0$ to some maximum value, ϕ_0, and ϕ decreasing to 0 at $z = L$. The maximal value, ϕ_0 is given when $\cot^2\phi\csc^2\phi - l^2a^2$ vanishes:

$$\cot^2\phi_0\csc^2\phi_0 - l^2a^2 = 0 \tag{4.23}$$

There are four possible solutions for ϕ_0 from the above relation, but as ϕ_0 is positive, real, and restricted to the interval $\phi \in [0, \pi/2]$, there is only one consistent

solution:

$$\phi_0 = \arccos\left(1 - \frac{2}{1+\sqrt{1+4l^2a^2}}\right)^{1/2} \tag{4.24}$$

From Eq.4.22, separate variables and obtain the following integral relation

$$\int \frac{dz}{l\,[a^2-(b+z)^2]} = \pm \int \frac{d\phi}{\sqrt{\cot^2\phi\csc^2\phi - l^2a^2}} \tag{4.25}$$

Again there is a sign choice. To remove this, consider the integral limits. In the region $z \in [0, z_0]$, where $\phi(z = z_0) = \phi_0$, ϕ is increasing, so take the positive root. In the region $z \in [z_0, L]$, ϕ decreases, take the negative root.

$$\begin{aligned} \int_0^{z_0} \frac{dz}{l\,[a^2-(b+z)^2]} &= \int_0^{\phi_0} \frac{d\phi}{\sqrt{\cot^2\phi\csc^2\phi - l^2a^2}} \\ \int_{z_0}^{L} \frac{dz}{l\,[a^2-(b+z)^2]} &= -\int_{\phi_0}^{0} \frac{d\phi}{\sqrt{\cot^2\phi\csc^2\phi - l^2a^2}} \end{aligned} \tag{4.26}$$

Summing both regions, and performing the integration over z provides:

$$\begin{aligned} \int_0^{L} \frac{dz}{l\,[a^2-(b+z)^2]} &= \left(\int_0^{\phi_0} - \int_{\phi_0}^{0}\right) \frac{d\phi}{\sqrt{\cot^2\phi\csc^2\phi - l^2}} \\ &= 2\int_0^{\phi_0} \frac{d\phi}{\sqrt{\cot^2\phi\csc^2\phi - l^2a^2}} \end{aligned} \tag{4.27}$$

$$\frac{1}{4}\log\left[\frac{a+b+L}{a-b-L}\right] - \frac{1}{4}\log\left[\frac{a+b}{a-b}\right] = la\int_0^{\phi_0} \frac{d\phi}{\sqrt{\cot^2\phi\csc^2\phi - l^2a^2}} \tag{4.28}$$

Define a pair of functions, $\mathcal{F}_R$ & $\mathcal{F}_\phi$, for the left & right hand sides of 4.28 respectively. The function $\mathcal{F}_\phi$ is defined with the combination la as a variable, as l & a only appear in the $d\phi$ integral in this combination, while $\mathcal{F}_R$ is expressed in R_1, R_2 & L.

$$\mathcal{F}_R\,(R_1, R_2, L) = \frac{1}{2}\log\left[\frac{R_1^2 + R_2^2 + L^2 + \sqrt{L^4 + (R_1^2 - R_2^2)^2 + 2L^2(R_1^2 + R_2^2)}}{2R_2R_2}\right] \tag{4.29}$$

$$\mathcal{F}_\phi(la) = la\int_0^{\phi_0}\frac{d\phi}{\sqrt{\cot^2\phi\csc^2\phi - l^2a^2}} \tag{4.30}$$

A consistent solution is one that equates $\mathcal{F}_\mathcal{R}$ & $\mathcal{F}_\phi$.

$$\mathcal{F} = \mathcal{F}_\mathcal{R} \equiv \mathcal{F}_\phi \tag{4.31}$$

It can be seen that $\mathcal{F}_\mathcal{R}$ is unbounded from above, so now consider the behaviour of $\mathcal{F}_\phi$. Performing the ϕ integration provides an expression for $\mathcal{F}_\phi(la)$ in terms of elliptical integrals:

$$\mathcal{F}_\phi(la) = \sqrt{-\frac{2l^2a^2}{\beta_-}}\left\{F\left[\arcsin\left(\sqrt{\frac{2l^2a^2}{\beta_+}}\tan\phi_0\right),\frac{\beta_+}{\beta_-}\right]\right. \tag{4.32}$$

$$\left. - \Pi\left[-\frac{\beta_+}{2l^2a^2}, -\arcsin\left(\sqrt{\frac{2l^2a^2}{\beta_+}}\tan\phi_0\right),\frac{\beta_+}{\beta_-}\right]\right\} \tag{4.33}$$

Where $\beta_\pm = 1 \pm \sqrt{1+4l^2a^2}$, and F and Π are incomplete elliptical integrals of the 1st and 3rd kind respectively. Applying the value of ϕ_0 found above, the function simplifies to become.

$$\mathcal{F}_\phi(la) = \sqrt{\frac{\beta_+}{2}}\left\{K\left[\frac{\beta_+}{\beta_-}\right] - \Pi\left[-\frac{\beta_+}{2l^2a^2},\frac{\beta_+}{\beta_-}\right]\right\} \tag{4.34}$$

Where the elliptical integrals are now complete. This expression for $\mathcal{F}_\phi(la)$ increases from $la = 0$, but reaches a maximum at $la \approx 0.581$, where $\mathcal{F}_\phi(la) \approx 0.501$. The behaviour of $\mathcal{F}_\phi(la)$ can be seen in fig.4.3. As $\mathcal{F}_\mathcal{R}$ is unbounded above, there exists a interval space for R_1, R_2, & L, outside of which, eq. 4.31 is no longer consistent.

What are the consistency limits of $\mathcal{F}$? $\mathcal{F}_\mathcal{R}$ can be re-expressed in terms of ratios of R_1, R_2, & L. Letting $\frac{R_2}{L} = \zeta\frac{R_1}{L} = \zeta\rho$, and solving for ρ^2 we find:

$$\rho^2 = \frac{1}{2\zeta\cosh[2\mathcal{F}] - \zeta^2 - 1} \tag{4.35}$$

Assuming ρ can be taken small, but non-zero (L can be large but not infinite), the limits to the magnitude of ζ are;

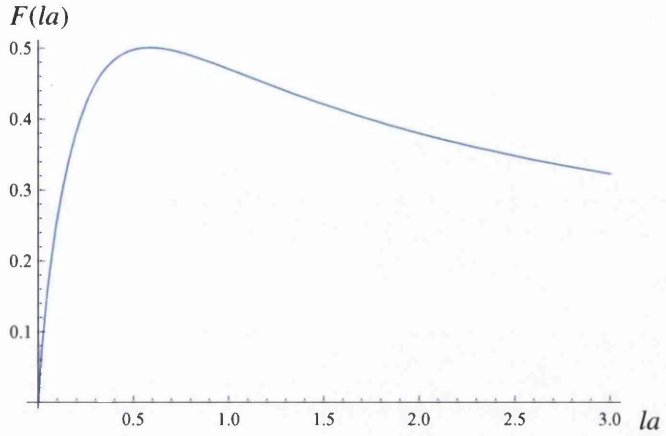


Figure 4.3: *Plot of $\mathcal{F}_\phi(la)$ against the combination la. It is clearly seen that $\mathcal{F}_\phi(la)$ reaches a maximum of ≈ 0.501 at $la \approx 0.581$.*

$$1 + 2\sinh^2\mathcal{F} - 2\cosh\mathcal{F}\sinh\mathcal{F} < \zeta < 1 + 2\sinh^2\mathcal{F} + 2\cosh\mathcal{F}\sinh\mathcal{F} \quad (4.36)$$

Using the upper limit for $\mathcal{F} = \mathcal{F}_\phi(la)$, the range of $\zeta = R_2/R_1$ that obeys eq. 4.31:

$$0.367 \lesssim \zeta \lesssim 2.72$$

Implying there exists a limit to how small the probe loop can be taken. Arranging eq.4.35 to give a function of L

$$L = R_1\sqrt{2\zeta\cosh[2\mathcal{F}] - \zeta^2 - 1} \quad (4.37)$$

For both limits of ζ, L vanishes. Note that taking $\zeta \approx 0.367$ and $\mathcal{F} \approx 0.501$, sends L to zero. What is the maximal value of L, for a given ζ? Maximising with respect to ζ:

$$\partial_\zeta L = R_1\frac{\cosh[2\mathcal{F}] - \zeta}{\sqrt{2\zeta\cosh[2\mathcal{F}] - \zeta^2 - 1}} = 0 \quad (4.38)$$

$$\zeta = 1 + 2\sinh^2\mathcal{F} \quad (4.39)$$

Giving the maximal value of L as:

$$L_{\max} = R_1 \sinh 2\mathcal{F} \tag{4.40}$$

Thus, for our upper bound of $\mathcal{F}_{\mathcal{R}}$, the value of $L_{\max} \approx 1.178 R_1$. Any value of L greater than this, and the solution becomes inconsistent.

This inconsistency must not be reached,other wise the solution is not acceptable. Before reaching the inconsistency, the connected world-sheet should collapse to two disconnected surfaces by the Gross-Oguri Phase Transition. To check find the minimal surface area of the connected world-sheet, and find the point in R_1, R_2, L space at which it is equal to the disconnected surface. Providing this point is reached before the inconsistent region, the solution is acceptable.

Applying the re-parameterisations, Eq.4.21, to the action, Eq.4.11;

$$S = \pm \int \frac{\cot^2 \phi \, d\phi}{\sqrt{\cos^2 \phi - l^2 a^2 \sin^4 \phi}} \tag{4.41}$$

Again, there exists a sign choice relating to integration branches. In a similar vein as the eq. 4.27, integrate over the two regions separately. However, the integral is divergent at $\phi = 0$. To regularise, a boundary term is included in the action, of the form of a Legandre transform, to remove this divergence.

$$S_{\text{bdy}} = -\frac{\partial S}{\partial y'} y = \frac{r}{y} \frac{y'}{\sqrt{r'^2 + y'^2 + 1}} \tag{4.42}$$

Introduce a cut-off near the boundary, $y = \epsilon$, where $0 < \epsilon \ll 1$, at which ϕ becomes very small. Near the boundary, $r \approx R$, where R can be R_1 or R_2, depending on the limit of z under consideration. This implies that both r' and y' tend toward zero near the boundary. In this limit, with $y = \epsilon$, and $r = R$, the boundary term reduces to

$$S_{\text{bdy}}|_{y=\epsilon} = \frac{R}{\epsilon} \tag{4.43}$$

From Eq.4.21, $\epsilon \approx R\phi$. The total action, adding boundary terms for both loops, becomes:

$$S_{\text{Total}} = S + S_{\text{bdy},\mathcal{W}_1} + S_{\text{bdy},\mathcal{W}_2} \tag{4.44}$$

$$= \left(\int_{\epsilon/R_1}^{\phi_0} + \int_{\epsilon/R_2}^{\phi_0}\right) \frac{\cot^2\phi\, d\phi}{\sqrt{\cos^2\phi - l^2a^2\sin^4\phi}} + \frac{R_1 + R_2}{\epsilon} \tag{4.45}$$

This integrates in terms of elliptical integrals to:

$$S_{\text{Conn.}} \equiv S_{\text{Total}} = 2\frac{\sqrt{2}\,la}{\beta_+}\left(E\left[\beta_+/\beta_-\right] - K\left[\beta_+/\beta_-\right]\right) \tag{4.46}$$

Here, E is the complete elliptical of the second kind (The expression for ϕ_0, shown explicitly above, has been applied). This is the complete action for the connected minimal surface area, as a function of la, of two Wilson loops $\mathcal{W}_1$ & $\mathcal{W}_2$ of radii R_1 & R_2 respectively. For the disconnected surface area, the action is simply the sum of the areas of the two loops individually.

The action for a circular loop of radius R, S_R:

$$S_W = \frac{1}{2\pi}\int dy\, d\phi \frac{r}{y^2}\sqrt{1 + \left(\frac{\partial r}{\partial y}\right)^2} = \int_0^R dr \frac{r}{y^2}\sqrt{1 + \left(\frac{\partial y}{\partial r}\right)^2} \tag{4.47}$$

Using the solution $R^2 = y^2 + r^2$, and regularising as above:

$$S_W = \int_0^{\sqrt{R^2-\epsilon^2}} \frac{R\,r\,dr}{(R^2 - r^2)^{3/2}} + \frac{R}{\epsilon} \tag{4.48}$$

$$= 1 \tag{4.49}$$

Thus, the disconnected area is given as

$$S_{\text{Disconn.}} = S_{W_1} + S_{W_2} \tag{4.50}$$

$$= 2 \tag{4.51}$$

The phase transition will occur when the connected and disconnected surface areas are equal. If this occurs before the connected solution becomes inconsistent, then the system is safe, as the inconsistency will never be probed.

Equating both actions, the two actions are found to be equal at $la \approx 1.316\,(\equiv la_{\text{crit}})$. This is illustrated in fig.4.4.

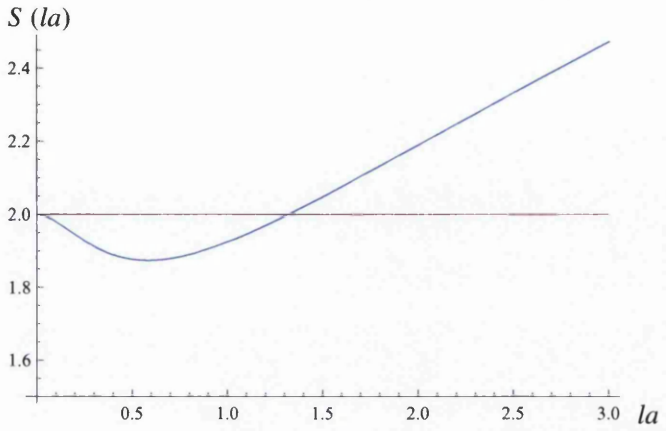


Figure 4.4: *Plot of the connected action against the combination la. Also shown is the disconnected area. The Gross Ooguri Phase Transition occurs where the two lines intersect, $S = 2$, at $la \approx 1.316$.*

The value of la_{crit} translates into a value of $\mathcal{F}_\phi\,(la_{\text{crit}}) \approx 0.438$, which is within the limits computed above. This provides a range limit to ζ (from eq.4.36) of:

$$0.416 \lesssim \zeta \lesssim 2.40,$$

and a subsequent critical value for $L_{\text{max,crit}} \approx 0.99\,R_1$. Therefore, taking limits in an L, R_1, R_2 parameter space, one will encounter the Gross-Ooguri Phase Transition before any inconsistent sectors of the system are reached, thus protecting the model and method.

This provides a restriction to using the minimal surface area approach to calculating the Wilson Loop correlator, when using one loop as a probe. Ultimately, to determine the correlator for k-strings, a generalisation of the arguments outlined here are required to extend to that of the k-string case.

4.2.2 k-String extension

Section 4.2.1 illustrated that in AdS_5, the connected world-sheet between two Wilson loops does not exist across the entire parameter space of the loop radii, R_1, R_2, and their transverse separation L. To refresh, to allow the string width to be calculated accurately, one of the loops must be taken to infinitesimal size, and evaluated over all possible transverse distances, $L \in [0, \infty]$.

For the extension to the k-string, the fundamental string is replaced by a wrapped higher dimensional object. As was learnt in section 3.3, in $AdS_5 \times S^5$ the k-string is modelled by a D5 brane, wrapped on the transverse $S^4 \subset S^5$, at an angle related to the string charge k.

Following the work of Gilozzi *et al.* [56], and their idea of a k-string being probed by a fundamental loop, and the motivated idea of section 4.1 to probe a k-string with a loop in the same representation, there are two possible methods of calculating the width:

D5-F correlator

Take a k-string in the anti-symmetric representation, and probe with a string in the fundamental representation

Use a D5-brane to model a k-string tracing a single Wilson loop, $\mathcal{W}_1$, then find the connected world-sheet of a fundamental string propagating from this D5-brane/k-string Wilson loop world-sheet within AdS_5 to the probe Wilson loop on the boundary, $\mathcal{W}_2$.

In such a case, the connected world-sheet only reaches the boundary at $\mathcal{W}_2$, and terminates on the brane in the AdS_5 interior. The world-sheet is described by the Nambu-Goto string action, with boundary conditions at the brane end governed by the brane's minimal area. This is a similar calculation to that performed in section 4.2.1, with the brane boundary conditions imposed on the equations of motion of the fundamental string. There is an additional complexity imposed by the angle that the brane sits at in the S^5. The angle at which the string sits in the S^5 is independent of the brane. This introduces an additional degree of freedom, imposing an additional equation of motion. This calculation was performed by Yamaguchi [57], and this construction was also shown to undergo a Gross-Ooguri phase transition to a disconnected world-sheet state, before the solution becomes inconsistent.

D5-D5 correlator: Probe a k-string with k-string

Take a k-string in the anti-symmetric representation, and probe with a second k-string in the same representation, with the same N-ality k.

Use a single D5-brane to model the entire connected world-sheet between two Wilson loops, $\mathcal{W}_1$ & $\mathcal{W}_2$, effectively modelling the propagation of a k-string from $\mathcal{W}_1$ to $\mathcal{W}_2$.

In this case, the brane moves through the interior of AdS_5 , and reaches the boundary at both Wilson loops. As the same brane models the entire connected world-sheet, both loops sit at the same angle in S^5, and thus have the same k, therefore not introducing the additional degree of freedom seen in the D5-F correlator. Such a calculation, once integrated over the transverse S^4, will reduce to the calculation in section 4.2.1, with the action exhibiting a suitably modified string tension, proportional to k . Ergo, the Gross-Ooguri phase transition will be reached before a desirable probe system is reached.

It is clear that the connected string world-sheet cannot be used to generate accurate string widths within AdS_5 . However, after transition to the disconnected state, the two disconnected Wilson loop world-sheets are able to interact via light supergravity mode exchange. In the next section, this exchange will be examined for application to a width calculation.

4.2.3 Fundamental String widths via dilaton exchange

As shown previously, the world-sheet two point correlator in $AdS_5 \times S^5$ cannot be used to calculate the width of a string. Taking one of the Wilson loops towards zero size, while the second has finite extent, a Gross-Ooguri transition is passed, beyond which a world-sheet between the two loops ceases to exist. In such a case, the system will collapse to two separate Wilson loop world-sheets, interacting via the exchange of light supergravity modes. In the limit of the small loop becoming point-like, the system is a Wilson loop described by a string, with an operator insertion at a transverse distance L. The operator insertion under consideration will be the gauge invariant operator, F^2, measuring the chromoelectric field strength at a distance L from the Wilson loop.

To calculate such a correlator, the following object must be evaluated:

$$\frac{\langle W(C)F^2\rangle_{\text{con.}}}{\langle W(C)\rangle} \tag{4.52}$$

The numerator selects only connected graphs. For an F^2 operator, the lightest exchanged mode is the dilaton. To compute such a correlator, it is necessary to perform an Operator Product Expansion of the Wilson Loop. This was done for single dilaton exchange in [58]. It was shown that for a chiral primary operator of conformal dimension Δ, the correlator

$$\frac{\langle W(C)O\rangle_{con.}}{\langle W(C)\rangle} = \text{Const.}Y(\theta)\frac{1}{2\pi}\int dA\, G(y,\vec{x},y',\vec{x}';\Delta) \tag{4.53}$$

$G(y,\vec{x},y',\vec{x}';\Delta)$ represents the Green's function of the propagator from the operator insertion point $(y,\vec{x})$ to a point on the string world-sheet that ends on the finite Wilson loop $(y',\vec{x}')$. This function is integrated over the surface of the world-sheet $\int dA$. There is an additional spherical harmonic factor, and an overall constant related to the conformal dimension Δ.

Considering a system akin to that reached after passing the Gross-Ooguri Phase transition in section 4.2.1, namely that of the probe loop, $\mathcal{W}_2$, taken to infinitesimal size, and effectively becoming an F^2 operator insertion at the boundary, a distance L transverse to the plane of the loop $\mathcal{W}_1$, and aligned centrally, ($r = y = 0, x_3 = L$). The Green's function for such a system produces:

$$G(\vec{x},y',\vec{x}') = \text{Const.}\frac{y^\Delta}{[(\vec{x}-\vec{x}')^2 - y^2]^\Delta} \tag{4.54}$$

For the chiral primary operator F^2, $\Delta = 4$, using the area element of a circular Wilson loop, $\int dA = \int \frac{R_1}{y^2}\, dy\, d\psi$, and solution of the world-sheet surface of $\mathcal{W}_1$ as $R_1^2 = r^2 + y^2$, the correlator becomes

$$\frac{\langle W(C)O\rangle_{\text{con.}}}{\langle W(C)\rangle} = \text{Const} \times Y(\theta)\int_0^{R_1} dy \frac{R_1}{y^2}\frac{y^4}{\left[L^2+R_1^2\right]^4} \approx \frac{R_1^4}{(L^2+R_1^2)^4} \tag{4.55}$$

Letting $\mathcal{P} \propto \frac{\langle W(C)O\rangle_{\text{con.}}}{\langle W(C)\rangle}$, the width is computed as:

$$\Sigma^2 = \frac{\int d\mathbb{L}\, L^2\mathcal{P}}{\int d\mathbb{L}\,\mathcal{P}} = \frac{1}{5}R_1^2 \tag{4.56}$$

Due to the normalisation factor, all overall constants in the correlator $\mathcal{P}$ cancel. The width of the string in this system exhibits a dipole-like behaviour, growing like R^2, faster than the quark separation. This illustrates that the "flux tube" in $AdS_5 \times S^5$, and thus $\mathcal{N} = 4$ SYM is non-localised. This is reasonable, as there is no confinement exhibited in $\mathcal{N} = 4$ SYM, and ergo no confining flux tube. Also note how the width is completely independent of the string tension. This seems to indicate that such a SUGRA computation is not suitable for capturing the required string dynamics. Nevertheless, a further interesting question is how this width, in a k-string extension, would vary with respect to k, if at all.

4.2.4 k-String widths via dilaton exchange

To extend to a k-string case, consider the two methods of section 4.2.2:

D5-F correlator

The world-sheet ending on $\mathcal{W}_1$ is no longer traced by a fundamental string, but by a D-brane. The propagator is still that of the dilaton, and the operator insertion is unchanged to that of section 4.2.3. Subsequently, the only modification to the calculation is the modified area element of $\mathcal{W}_1$, namely the DBI action of the D5-brane tracing the loop:

$$\int dA = T_{D5} \int d^6\xi \sqrt{\det(\mathcal{G} + \mathcal{F})} \tag{4.57}$$

$$\approx \int dy\, d\psi \sin^5\theta_k \frac{R_1}{y^2} \tag{4.58}$$

Computing the k-string width with this modified area element provides the same result as that for the fundamental string, i.e.,

$$\mathcal{P} \approx \sin^5\theta_k \frac{R_1^4}{[R_1^2 + L^2]^4} \tag{4.59}$$

$$\Sigma^2 = \frac{\int dL\, L^2 \mathcal{P}}{\int dL\, \mathcal{P}} = \frac{1}{5} R_1^2 \tag{4.60}$$

It is obvious to see that for this method, the width computation does not capture any k scaling behaviour exhibited by the addition of the D-brane, and once again,

is seeming independent of the k-string tension. In the next computation, using D5-branes for both loops may yield some form of k scaling, if not dependence on the string tension.

D5-D5 correlator

This method requires both $\mathcal{W}_1$ & $\mathcal{W}_2$ to be traced by D-branes, and thus requires a little more thought. The simplist approach is the following: Consider the dilaton mode exchange between the world-sheets of $\mathcal{W}_1$ & $\mathcal{W}_2$, traced by two D5-branes, $D5_1$ & $D5_2$, *before* the probe limit is taken. The expression for such a correlator is given in [58] as:

$$\frac{\langle \mathcal{W}_1 \, \mathcal{W}_2 \rangle}{\langle \mathcal{W}_1 \rangle \langle \mathcal{W}_2 \rangle} = \exp\left[Y(\theta) \frac{1}{4} \int \frac{dA_1}{2\pi\alpha'} \frac{dA_2}{2\pi\alpha'} G[w(\sigma_1, \sigma_2)] \right] \tag{4.61}$$

The bulk-to-bulk propagator between the $D5_1$ & $D5_2$ world-sheets, σ_1 & σ_2, tracing $\mathcal{W}_1$ & $\mathcal{W}_2$ respectively is governed by the Green's function:

$$G(w) = \frac{\alpha_0}{\beta} w^{\Delta} \, {}_2F_1(\Delta, \Delta + \frac{1-d}{2}; 2\Delta - d + 1; -4w) \tag{4.62}$$

$$w = \frac{y_1 y_2}{(y_1 - y_2) + \sum (x_1 - x_2)^2} \tag{4.63}$$

where d is the dimensionality of the space-time (here $d = 4$), ${}_2F_1$ is the generalised hypergeometric function, and α_0 & β constants. Now consider the probe limit, taking R_2 (and subsequently y_2 & r_2) towards zero, letting $R_2 \cong y_2 \cong r_2 = \epsilon$ with $\epsilon \to 0$.

$$G(w) \approx \frac{y_1^4 \epsilon^4}{(L^2 + R_1^2)^4} \tag{4.64}$$

The area element of $\mathcal{W}_2$, dA_2, will reduce to an overall constant, $\sim \sin^5 \theta_k$, thus reducing the correlator to that of the D5-F case, but with a different multiplicative constant dependent on k. And as overall constants inflict no changes on the width, the result is the same, $\Sigma^2 = \frac{1}{5} R_1^2$, once more independent of the string tension, and any form of k scaling. This would seem to sound the death knell for using such SUGRA mode exchange methods to determine string widths in $AdS_5 \times S^5$.

May this issue lie with the non-localised flux tube in a conformal theory, or with the method of using SUGRA mode exchanges? It may even lie with the definition of the string widths in conformal theories being ill-defined. Extending the SUGRA computation to include higher orders may give some indications to this issue. This area is left open for future work.

In the following section, connected string world-sheet methods will be employed to determine string widths in non-conformal Hardwall AdS.

4.3 String widths in Hardwall $AdS_5 \times S^5$

In section 4.2 attempts were made to determine the width of fundamental and k-strings in the conformal $AdS_5 \times S^5$ background. As was noted, the width of a string in such a background should provide a non-localised flux tube width, as the theory the background is dual to $\mathcal{N}=4$ SYM. As this is a non-confining theory, no true flux tube should exist.

A more revealing calculation would be one in a confining background. As a demonstration to the change in width behaviour between a conformal and confining background, consider an $\mathbb{R}^4$ slice of AdS_5, more commonly referred to as Hard-wall AdS_5. The width of a fundamental string will be determined in Hard-wall AdS_5 before generalising to the k-string case.

4.3.1 Fundamental String Width in Hardwall $AdS_5 \times S^5$

Hard-wall AdS_5 consists of an $\mathbb{R}^4$ slice in the AdS_5 interior, with a transverse S^5, as with true $AdS_5 \times S^5$. The $\mathbb{R}^4$ slice sits at a constant value, or "cut-off" of the AdS radius, $y = \text{Const.} \equiv y_\Lambda$, where y_Λ acts as the scale of the theory (C.F. $\sim \Lambda_{QCD}$). The metric in the Euclidean Poincaré patch becomes:

$$ds^2 = \frac{1}{y_\Lambda}^2 (dr^2 + r^2 d\eta^2 + dz^2 + dx_4^2) + d\Omega_5^2 \tag{4.65}$$

It is apparent from section 4.1 that the fundamental string width in such a metric will be equivalent to that of Lüscher *et al.*, bar a modified string tension dependent on the theory scale y_Λ. Following the set-up & procedure of section 4.1, the fundamental string width is simply

$$\Sigma^2 = y_\Lambda^2 \log[R_1/R_2] = \frac{1}{2\pi\sigma_f} \log[R_1/R_2] \tag{4.66}$$

To generalise this method to consider k-strings, replace the string world sheet by the world sheet of a D-brane wrapping a suitable manifold, as was encountered with regards to k-string tensions in section 3.4.

4.3.2 k-String Width in Hardwall $AdS_5 \times S^5$

As in $AdS_5 \times S^5$, the anti-symmetric k-string is described by a D5 brane wrapping a 4-cycle inside the transverse S^5, in Hard-wall AdS, the same configuration can be used, with the brane having one less degree of freedom in the AdS ($\mathbb{R}^4$) region.

In section 4.2.2, there were two possible cases for the measurement of a k-string width, namely a fundamental string probing a k-string (D5-F correlator), and a k-string probing a k-string (D5-D5 correlator). For the D5-F correlator in Hard-wall AdS_5, one finds that the string width calculation will not extract any k dependence. Why is this? From a technical aspect, as the probe is in the fundamental representation, (i.e., a fundamental string), the correlator is the connected worldsheet from the probe loop to the surface of the D5 brane. As the fundamental string has degrees of freedom only in the $\mathbb{R}^4$, the boundary conditions that give rise to a k dependence, namely the angle of the $S^4 \subset S^5$, θ_k, are ignored. Thus, a D5-F correlator computation will produce a width equivalent to eq.4.66. For a D5-D5 correlator, the probe loops is intimately dependent on θ_k. It is this computation that will be of interest.

In analogy with the $AdS_5 \times S^5$ case, consider a D5 brane wrapping an $S^4 \subset S^5$, with the remaining two directions along τ & σ, the string co-ordinates. The action of the brane is described by DBI & Wess-Zumino parts;

$$S_{\text{Bulk}} = T_{D5} \int d^6\xi \sqrt{\det(\mathcal{G} + \mathcal{F})} - iT_{D5} \int d^6\xi\, C_4 \wedge \mathcal{F} \tag{4.67}$$

The integration is performed over the $S^4 \subset S^5$, τ & σ, with T_{D5} as the brane tension, $\mathcal{G}$ as the induced metric on the D5, C_4 the Ramond-Ramond 4-form potential that exists in the S^4, which satisfies $G_5 = dC_4$. $\mathcal{F} = 2\pi\alpha' F_{\tau\sigma}$, where $F_{\tau\sigma}$ is the quantised chromoelectric field strength that sits on the brane in τ, σ space.

As previously, in addition to the bulk action, there is an additional conjugate momentum term due to the effects of the field strength at the boundary. The addition of this term provides a total action for the brane;

$$S_{\text{Total}} = S_{\text{Bulk}} - \mathcal{F}\frac{\partial}{\partial \mathcal{F}} S_{\text{Bulk}} \tag{4.68}$$

It is this total action that will describe the minimal area of the catenoid between $\mathcal{W}_1$ and $\mathcal{W}_2$. This shall be employed to provide the k-string width. Using the metric, eq. 4.65, and re-expressing the S^5 as $S^1 \times S^4$;

$$ds^2 = \frac{1}{y_\Lambda^2}\left(dr^2 + r^2 d\eta^2 + dz^2 + z^2 dx_4^2\right) + d\theta^2 + \sin^2\theta d\Omega_4^2 \tag{4.69}$$

As previously, the loops $\mathcal{W}_1$ & $\mathcal{W}_2$ lie in r, η space, separated in the z direction with $\mathcal{W}_1$ at $z = 0$, and $\mathcal{W}_2$ at $z = L$. Note that due to the angular symmetry of the problem, the distance between the centres of the Wilson loops does not depend on x_4. As the loops are concentric, the centre of each loops lies at $r = 0$. At $\mathcal{W}_1$ $r = R_1$, and as z increases, r will interpolate towards $r = R_2$ at $\mathcal{W}_2$, likely reaching some minimum in-between. $d\Omega_4^2$ represents the $S^4 \subset S^5$ which is wrapped by the brane, while the angle θ is the constant angle which the S^4 sits in the S^5, and is related to k.

There exists a Ramond-Ramond 4-form potential, C_4, which in this co-ordinate system is of the form

$$C_4 = \left(\frac{3}{2}\theta - \sin 2\theta + \frac{1}{8}\sin 4\theta\right) d\Omega_4^2 \tag{4.70}$$

Due to the symmetry of the system, let η to be identified with σ, and allowed to vary across $[0, 2\pi]$. As the radius, r, of the catenoid varies with z, allow both r and z to be general functions of τ.

$$r \to r(\tau), \quad z \to z(\tau) \tag{4.71}$$

Using this string embedding, the bulk action is expressed as

$$S_{\text{Bulk}} = T_{D5}\int d^6\xi \left[\sin^4\theta\sqrt{\frac{r^2}{y_\Lambda^4}(z'^2+r'^2) - 4\pi\alpha'^2F^2} + 2\pi\alpha' FG(\theta)\right] \tag{4.72}$$

With T_{D5} as the D-brane tension, and $d^6\xi = d\tau d\sigma d\Omega_4$. For simplicity of notation, let $G(\theta) = \left(\frac{3}{2}\theta - \sin 2\theta + \frac{1}{8}\sin 4\theta\right)$. The primes denote derivatives with respect to τ. Due to the euclidean signature, $\mathcal{F}$ is re-expressed in terms of F, where $\mathcal{F} = 2\pi\alpha' F_{\tau\sigma} = i\,2\pi\alpha' F$.

As was seen in section 3.3, in $AdS_5 \times S^5$ it is known that $\theta = \theta_k$ = constant is related to $k = \frac{\partial S_{\text{Bulk}}}{\partial F}$, by the following relation

$$k = \frac{N}{\pi}\left(\theta_k - \frac{1}{2}\sin 2\theta_k\right) \tag{4.73}$$

This relation can be shown to hold in this string width system also. $k = \frac{\partial S_{\text{Bulk}}}{\partial F}$ leads to the expression of F in terms of θ_k

$$2\pi\alpha' F = \cos\theta_k \frac{r}{y_\Lambda^2}\sqrt{z'^2+r'^2} \tag{4.74}$$

As mentioned earlier, there exists a conjugate momentum term due to the electric field strength F which must be added to the bulk action to provide the total action of the system.

$$S_{\text{Total}} = S_{\text{Bulk}} - kF \tag{4.75}$$

Applying the expression for F & k into the total action, using $T_{D5} = \frac{N}{8\pi^4\alpha'}$ (let α'become explicit), and integrating over the S^4, the action simplifies to

$$S_{\text{Total}} = \frac{2N}{3\pi\alpha'}\frac{1}{2\pi\alpha'}\sin^3\theta_k\int d\tau d\eta\,\frac{r}{y_\Lambda^2}\sqrt{z'^2+r'^2} \tag{4.76}$$

The classical equations of motion for r & z are

$$\sqrt{z'^2+r'^2} - \left(\frac{r\,r'}{\sqrt{z'^2+r'^2}}\right)' = 0 \tag{4.77}$$

$$\frac{rz'}{\sqrt{z'^2 + r'^2}} = m \tag{4.78}$$

m is defined as a constant in τ. It is now appropriate to make the same gauge choice as was made in section 4.2: As both τ and z increase monotonically, let $z(\tau) = \tau = z$. Using this choice, the equations of motion combine and simplify, and using the fact m is now constant in z;

$$1 + r'^2 - rr'' = 0 \tag{4.79}$$

This is solved by

$$r(z) = B\cosh\left[\frac{z - z_0}{B}\right] \tag{4.80}$$

where z_0 is defined as the value of z at the minimum radius of the catenoid, and B is a constant. Applying this solution to the total action, and integrating over $z \in [0, L]$, & $\eta \in [0, 2\pi]$.

$$S_{\text{Total}} = \frac{2N}{3\pi}\frac{1}{2\pi\alpha'}\frac{B}{y_\Lambda^2}\sin^3\theta_k \int dz\, d\eta \cosh^2\left[\frac{z - z_0}{B}\right] \tag{4.81}$$

$$= \frac{N}{3\pi\alpha'}\frac{B}{y_\Lambda^2}\sin^3\theta_k \left(L - \frac{B}{2}\left[\sinh\left(\frac{2(z_0 - L)}{B}\right) - \sinh\left(\frac{2z_0}{B}\right)\right]\right) \tag{4.82}$$

At this stage, eq. 4.81 is entirely equivalent to that of the Lüscher *et al.* case, with a string tension modified by the integration over the wrapped portion of the D5 brane. Continuing the computation, from the solution for $r(z)$, the expression for z_0 can be found by using the boundary conditions for z & r.

$$R_1 = B\cosh\left[\frac{z_0}{B}\right], \quad R_2 = B\cosh\left[\frac{L - z_0}{B}\right] \tag{4.83}$$

$$z_0 = \frac{1}{2}\left[L - B\left(\operatorname{arccosh}\left[\frac{R_1}{B}\right] + \operatorname{arccosh}\left[\frac{R_2}{B}\right]\right)\right] \tag{4.84}$$

An expression for B cannot be determined in an analytic fashion, therefore an approximation is required.

Stepping back for a moment to consider the model, for the string width $\mathcal{W}_2$ must be considered as a probe loop, and thus must be very small, as outlined earlier.

Therefore consider the limit of R_1 & the ratio R_1/R_2 becoming large. In such a limit, the value of B is approximated to leading order as

$$B = \frac{L}{\log[R_1/R_2]} + O\left(\log[R_1/R_2]\right)^{-2} \tag{4.85}$$

Substituting into the total action the expressions for B and z_0

$$S_{\text{Total}} = \frac{N}{3\pi\alpha'}\frac{1}{y_\Lambda^2}\sin^3\theta_k\left[(R_2^2 - R_1^2) + \frac{L^2}{\log[R_1/R_2]}\right] \tag{4.86}$$

As only the second term has a dependence on the loop separation, L, when the string width is computed, in the numerator the exponent will cause the first term to cancel with an identical term from the denominator, thus only the second term is relevant. The width is calculated as

$$\Sigma_k^2 = \frac{\int_0^\infty e^{-S_{\text{Total}}}L^2 dL}{\int_0^\infty e^{-S_{\text{Total}}}dL} \tag{4.87}$$

$$= \frac{3\pi y_\Lambda^2\alpha'}{2N\sin^3\theta_k}\log[R_1/R_2] \tag{4.88}$$

$$= \frac{1}{2\pi\sigma_k}\log[R_1/R_2] \tag{4.89}$$

Comparing this result to that of the fundamental string, 4.66, it is obvious that the direct replacement of the fundamental string tension with the k-string tension seems to hold in this case, $\sigma_f \to \sigma_k$. For a general k & k', the ratio of widths becomes

$$\frac{\Sigma_k^2}{\Sigma_{k'}^2} = \frac{\sin^3\theta_{k'}}{\sin^3\theta_k} \tag{4.90}$$

It would appear that this method captures the correct, and most expected, dynamics of the k-string width. The width is slowly increasing for increasing quark separation, signalling a localised object in $\mathbb{R}^4$, as expected in a confining theory.

One can see how the ratio of the widths of the k-strings is the reciprocal of the tension ratios for the Hardwall calculation. For increasing k/N, the tube width shrinks, to a minimum at $k = N/2$, reflecting the increasing inter-string interaction

within the k-string when described as an assembly of k interacting strings. The string become maximally bound at the smalled string width, at $k = N/2$.

Following through the computation steps, the k dynamics factorise from the minimised area of the world-sheet, thus reducing the system to the computation of a fundamental string, with a modified string tension. It will be interesting to see if this is a universal effect of the computation in various confining backgrounds.

As interesting as it is to look at an AdS hard-wall background, it would be more meaningful to study the flux tube width in a theory with a greater resemblance to QCD (albeit with only one flavour), namely $\mathcal{N} = 1$ super Yang-Mills. In the next section, discussions with turn to the computation of k-string widths within the gravity-dual of $\mathcal{N} = 1$ SYM.

4.4 k-string width in $\mathcal{N} = 1$ SYM Gravity Dual

In this section it will be shown that the k-string width can be calculated in the Maldacena-Núñez background in the R-R sector at the IR limit using the same method presented in the previous section. It must be noted that although the R-R sector has a greater cross over to the string width calculation of hard wall AdS, the width can also easily be calculated in the NS-NS sector via S-duality.

As was seen in the string tension calculations of sections 3.6 & 3.7, the anti-symmetric k-string is described by a D3 brane wrapping an S^2 in a 3-cycle of a space transverse to the $\mathbb{R}^4$ in which the k-string resides.

Again, using the same Wilson loop construction, with the probe in the anti-symmetric representation, the 10 dimensional cylindrical Euclidean MN metric is most easily be expressed as

$$ds_{10}^2 = (dr^2+r^2d\eta^2+dz^2+dx_4^2)+N\alpha'\left(dy^2 + e^{2h(y)}\left(d\theta_1^2 + \sin\theta_1 d\phi_1^2\right) + \frac{1}{4}(\omega_i - A^i)^2\right) \tag{4.91}$$

Where:

$$A^1 = -a(y)d\theta_1, \quad A^2 = a(y)\sin\theta_1 d\phi_1, \quad A^3 = -\cos\theta_1 d\phi_1. \tag{4.92}$$

$$\begin{aligned}\omega_1 &= \cos\psi d\theta_2 + \sin\psi\sin\theta_2 d\phi_2, \\ \omega_2 &= -\sin\psi d\theta_2 + \cos\psi\sin\theta_2 d\phi_2, \\ \omega_3 &= d\psi + \cos\theta_2 d\phi_2\end{aligned} \tag{4.93}$$

with $a(y)$ and $h(y)$ functions dependent on the radial co-ordinate y. The topology of the transverse space is of two 2-spheres, S_1^2 & S_2^2, with an S^1 fibration between them. The angles θ_1, ϕ_1 & θ_2, ϕ_2 parametrise the S_1^2 & S_2^2 respectively, while the fibered S^1 by ψ. Along with the metric there exists a C_2 Ramond-Ramond potential, which obeys $F_3 = dC_2$, and is given by:

$$\begin{aligned}C_2 = \frac{N\alpha'}{4}[&\psi(\sin\theta_1 d\theta_1 \wedge d\phi_1 - \sin\theta_2 d\theta_2 \wedge d\phi_2) \\ &- \cos\theta_1\cos\theta_2 d\phi_1 \wedge d\phi_2 - (d\theta_1 \wedge \omega_1 - \sin\theta_1 d\phi_1 \wedge \omega_2)]\end{aligned} \tag{4.94}$$

The IR limit is defined as $y \to 0$, causing the functions $a(y) \to 1$ and $e^{2h(y)} \to 0$. Making the choice $\theta \equiv \theta_1 = \theta_2$, $\phi \equiv \phi_1 = 2\pi - \phi_2$, & $\psi \to 2\Psi + \pi$, a 3-cycle, S^3 is selected from the transverse space, causing the metric and C_2 to reduce to;

$$ds^2 = (dr^2 + r^2 d\eta^2 + dz^2 + dx_4^2) + N\alpha'\left[dy^2 + d\Psi^2 + \sin^2\Psi(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{4.95}$$

$$C_2 = N\alpha'\left(\Psi - \frac{1}{2}\sin 2\Psi\right)\sin\theta d\theta \wedge d\phi \tag{4.96}$$

The action is that of the previous section, namely the DBI, Wess-Zumino and chromoelectric field strength momentum terms. With $\mathcal{W}_1$ and $\mathcal{W}_2$ set in r, η space, while being separated in z, the D3 is wrapped along σ & τ, with $\eta \to \sigma$, $r \to r(\tau)$ and $z \to z(\tau)$, while the remaining two directions are wrapped upon an S^2 in the transverse 3-cycle, S^3. Turning on an electric field strength $\mathcal{F} = 2\pi\alpha' F_{\tau\sigma} = 2\pi\alpha' iF$, and allowing $k = \frac{\partial S_{\text{Bulk}}}{\partial F} = \frac{\Psi N}{\pi}$, the total action becomes

$$S_{\text{Total}} = S_{\text{DBI}} + S_{\text{WZ}} - \mathcal{F}\frac{\partial}{\partial\mathcal{F}}(S_{\text{DBI}} + S_{\text{WZ}}) \tag{4.97}$$

$$\begin{aligned}&= T_{D3}\int d^4\xi\left[\sqrt{\det(\mathcal{G}+\mathcal{F})} - iC_2 \wedge \mathcal{F}\right] - \mathcal{F}\frac{\partial}{\partial\mathcal{F}}(S_{\text{DBI}} + S_{\text{WZ}}) \\ &= \frac{N}{2\pi^2\alpha'}\sin\frac{\pi k}{N}\int d\eta\, d\tau\, r\sqrt{z'^2 + r'^2}\end{aligned} \tag{4.98}$$

Again, primes denote derivatives with respect to τ, and $T_{D3} = 1/(2\pi)^3\alpha'^2$. The equations of motion for r and z are identical to those from the hard-wall case, implying that the solution and gauge choice for r and z respectively will directly apply to this computation. Once more, the systematics of the k dependence in the transverse space, and those minimal area in 4d spacetime, factorise completely and do not influence each other.

Using the solution for r, the total action simplifies to become

$$S_{\text{Total}} = \frac{BN}{2\pi^2\alpha'} \sin\frac{\pi k}{N}\int d\eta\, dz\, \cosh^2\left[\frac{z-z_0}{B}\right] \tag{4.99}$$

Integrating over z & η ($z \in [0, L]$ & $\eta \in [0, 2\pi]$), using the boundary conditions for r & z to eliminate z_0, and taking the limit where R_1, & the ratio R_1/R_2, both become very large, the total action to leading order is expressed as

$$S_{\text{Total}} = \frac{N}{2\pi\alpha'} \sin\frac{\pi k}{N}\left[(R_2^2 - R_1^2) + \frac{L^2}{\log[R_1/R_2]}\right] \tag{4.100}$$

Calculation of the string width provides a result of the same form as for the hard-wall AdS, and ultimately the fundamental string case.

$$\omega_k^2 = \frac{\pi\alpha'}{N\sin\frac{\pi k}{N}}\log[R_1/R_2] \tag{4.101}$$

$$= \frac{1}{2\pi\sigma_k}\log[R_1/R_2] \tag{4.102}$$

Performing the equivalent calculation in the KS background provides a string width of the exact same form; namely $\propto 1/\sigma_k$. Both R-R & NS-NS sectors of both $\mathcal{N}=1$ backgrounds provide the same result. For a general k & k', the ratio becomes

$$\frac{\omega_k^2}{\omega_{k'}^2} = \frac{\sin\frac{\pi k'}{N}}{\sin\frac{\pi k}{N}}\,. \tag{4.103}$$

Once more, the ratio of the widths of the k-strings is the reciprocal of the tension ratios, as for the Hardwall calculation, signalling smaller string widths as k increased towards k/N. The localisation of the string, and thus the flux tube, is also apparent as expected for non-confining $\mathcal{N}=1$ SYM.

4.5 Discussions

It appears that a universal result has been found. It would seem, following from these calculations in Hardwall AdS, and the Maldacena-Núñez , and Kelbanov-Strassler gravity duals of $\mathcal{N}=1$ SYM, that string width computations in a confining background will be yield a width of the general form of the reciprocal of the string tension. Indeed this seems correct when performing a generalised string width calculation on a $\mathbb{R}^4 \times M$ spacetime, for a given manifold, M.

For a k-string system, this universal result would also apply, providing the k dynamics factorise. This would occur automatically, provided the mechanism for introducing the k scaling operates solely within the transverse space (I.e. manifold M above).

Following the results of string widths in both Hardwall AdS, and $\mathcal{N}=1$ gravity duals, it has been learned that the method of determining k-string tensions using a fundamental probe [56] was fatally flawed, and only in the large-N, k fixed limit could it be used to provide accurate results, where the k-string is described by a fundamental string. To correctly gain the k scaling dynamics in which $\sigma_f \to \sigma_k$ for the transition from a fundamental string to a k-string, the probe must exist in the same representation as the string it is probing.

The studies of string width attempts via light Supergravity mode exchanges did not produce the expected results of inverse dependence on the string tension. There are a number of possible causes for this. Firstly, the computation considered only primary operators, and only light exchanged modes. It may be that higher order exchanges must be considered to unveil hidden dynamics. Secondly, as was briefly mentioned earlier, the definition of the string width detailed in this thesis may not be applicable to SUGRA methods, and would require redefinition. Finally, there might be no width to compute. In a conformal theory, there is no strict localised flux tube with a tension. This might be all that the mode exchange computation is revealing. This area is still of interest and is left open to further work.

After the publication of the k-string width results, a number of lattice based experiments have been discussed to model the results obtained here. It will be very interesting to see the results of such experiments.

For future work, it would be interesting to see how the k-string width would be affected, if at all, when a k-string is probed by a Wilson loop of the same N-ality, but different representation. One such system might be an antisymmetric string, a D3 brane, probed by a symmetric probe, namely a D5 brane [16]. Does the width depend only on the N-ality, just like the tension computations, or does the representation of the probe come into play? This question is left open.

Chapter 5

Domain walls at Finite Temperature

5.1 Intro to domain walls at finite temp

In SU(N) gauge theories with matter in the adjoint representation, there exists a deconfinement transition as a temperature is applied to the system (See [59] and references within). At zero temperature, the system sits in a confined phase, with an unbroken $\mathbb{Z}_N$ gauge group centre of SU(N). At non-zero temperature, the centre spontaneously breaks as the system becomes deconfined, generating N distinct vacua. Each vacua is separated from its neighbour by a physical interface, which is more commonly known as a domain wall. These "fundamental" domain walls interpolate between two adjacent vacua.

The phase transition has an associated order parameter, the expectation value of the Polyakov loop [60,61]. In the confining phase, the order parameter is zero, $P = 0$, while in the deconfined phase it takes on a phase of one of the Nth root of unity, $P = \exp(i2\pi l/N)$, where l is the label of the vacua, $l \in [0, N-1]$. Fundamental walls interpolate between vacua of value $P = \exp i2\pi l/N$, and a neighbouring vacua, $P = \exp i2\pi(l+1)/N$, giving a phase difference of $P = \exp i2\pi/N$.

To impose a temperature on the system, which causes the spontaneous breaking of $\mathbb{Z}_N$, the system is compactified on a temporal circle. For the systems considered in this thesis, take a four dimensional Euclidean space and impose periodicity along the Euclidean time direction of length $2\pi\beta$, giving a topology of $S^1 \times \mathbb{R}^3$. This

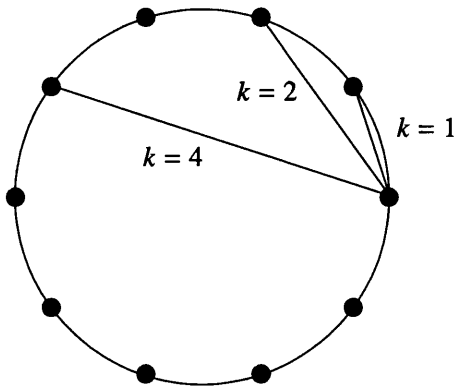


Figure 5.1: *Examples of $k = 1, 2, 4$ domain walls for the vacuum structure of SU(10), with each dot representing a distinct vacua & straight lines representing domain walls.*

imposes a temperature on the system, $T = 1/\beta$, where β is the radius of the temporal circle. The Polyakov order parameter can be explicitly defined for a gauge field $A(x,t)$, along the periodic Euclidean time direction t as:

$$P(x) = \frac{1}{N}\mathrm{Tr}\mathcal{P}\left(ig_{YM}\int_0^\beta A_0(x,t)dt\right), \quad \beta = 1/T. \tag{5.1}$$

In a more general case, there also exist "k-walls". A k-wall is defined as a domain wall interpolating between vacua of phase difference $P = \exp i2\pi k/N$, namely between vacua of values $P = \exp i2\pi l/N$ and $P = \exp i2\pi(l+k)/N$. A fundamental wall is equivalent to a $k = 1$ wall. A vacuum diagram illustrating fundamental and k-wall examples can be seen for SU(10) in fig. 5.1.

In this chapter, the tensions of k domain walls will be computed and discussed in the context of $\mathcal{N}=4$ super Yang-Mills theory. It will be shown that at the one-loop effective action level at weak coupling, the domain wall tension obeys the Casimir scaling law, namely $\mathcal{T}_k \sim k(N-k)$. An argument will also be put forward in an attempt to justify the expected existence of the Casimir scaling at the two-loop level for $\mathcal{N}=4$ SYM.

There has been a lot of interest and progress in the area of high temperature domain walls at weak coupling [62–67]. It has been shown that the tension of domain walls in pure Yang-Mills theory exhibit Casimir scaling at one-loop [62,63]

and two-loop [64], but this breaks down at three-loops, where Casimir scaling is no longer present. The tensions of these domain walls were also investigated form lattice perspectives [68–70], and the Casimir scaling was found to hold, within errors, at low temperatures close to the critical temperature.

In this chapter, the procedure employed in [65] to determine k domain wall tensions (and their Casimir scaling behaviour) in pure Yang-Mills, is extended in original work to include the adjoint scalars and fermions of $\mathcal{N}=4$ SYM at the 1-loop level. Following this result for 1-loop, a plausibility argument is presented in which it is argued that Casimir scaling will remain at 2-loops. In section 5.4, k domain walls are examined from the strong coupling limit. Applying the procedure of [14] to the AdS-Schwarzschild black hole in $AdS_5 \times S^5$, an original result of the k domain wall tension in the strong coupling limit is obtained. Following this, some preliminary results are presented for original work, where the procedure of [65] is once again used to determine domain wall tensions in a theory with matter in two index representations.

5.2 k-wall scaling in $\mathcal{N}=4$ SYM at 1-loop

Before looking at the $\mathcal{N}=4$ SYM case, the tensions of $\mathbb{Z}_N$ interfaces will be reviewed from a semiclassical description, in the context of pure Yang-Mills, before introducing the matter fields in adjoint representation of $\mathcal{N}=4$ SYM.

The first step in the domain wall tension computation involves the parameterisation of the expectation value of the Polyakov loop as it varies across the domain wall. This is done by considering a classical, background value for the temporal gauge field A_0, which can subsequently be factored in classical and quantum parts.

$$A_0 = A_0^{\text{cl}} + A_0^{\text{qu}} \tag{5.2}$$

Application of this factorisation will be briefly discussed in the context of the fundamental interface, before expanding the discussion to the general k-wall.

5.2.1 Fundamental wall

From the three dimensional effective theory viewpoint, the domain wall interface is 2-dimensional, therefore it can be though of as a string-like object moving

through time. Taking this interface to span the x_1 & x_2 directions, the different vacua will sit at different points along the $x_3 \equiv z$-direction. Regarding the background field, an ansatz can be chosen where $A_0^{\rm cl}$ can be expressed in terms of a diagonal traceless generator t_N,

$$A_0^{\rm cl} = \frac{2\pi T}{g_{YM}N}\frac{t_N}{B(N)}q(z); \quad t_N = \mathrm{Diag}[\underbrace{1,1,1,\ldots,1}_{N-1\text{ entries}},1-N] \tag{5.3}$$

$$B(N) = \frac{1}{\sqrt{2N(N-1)}} \tag{5.4}$$

The spatial components of the gauge field are set to zero, $A_i = 0$ for $i = 1,2,3$. The function $q(z)$ acts as a profile function, parameterising the N different vacua. Evaluating the Polyakov loop with the above expression for the classical gauge reveals the profile function role of q explicitly (the z dependence will be dealt with later);

$$P = \frac{1}{N}\left[(N-1)e^{\frac{2\pi i}{N}q} + e^{\frac{2\pi i(1-N)}{N}q}\right] \tag{5.5}$$

Allowing q to take on integer values, $q = 0,1,2,\ldots,N-1$, one can select each of the N vacua, labelled by $P = e^{\frac{2\pi i}{N}q}$. As the vacua sit at integer values of q, non-integer values of q correspond to positions within the domain interface that interpolates between two vacua.

The ansatz for the background field, eq.5.3, can be used to describe a single wall, interpolating between two $\mathbb{Z}_N$ phases labelled by consecutive integers (i.e. two neighbouring vacua). As all vacua are physically equivalent with regards to wall tensions, a single interface can be focussed upon without loss of generality. It is simplest to select the interface between $q = 0$ and $q = 1$, with $q(z)$ interpolating between $q(z = 0) = 0$ and $q(z = L) = 1$, where L is the extent of the interface between neighbouring vacua.

Up to this point, the walls under scrutiny are fundamental or $k = 1$ walls, i.e. they exist between vacua of singular phase difference. To consider walls between vacua of multiple $\mathbb{Z}_N$ "charge" difference, k, the ansatz for $A_0^{\rm cl}$ must be modified.

5.2.2 k-wall ansatz

Before proceeding, consider the conventions for the N^2-1 generators of SU(N). Using the Cartan basis, they take the form of $N-1$ diagonal generators and $N(N-1)$ off-diagonal or ladder generators. The $N-1$ diagonal generators are of the form,

$$t_{\text{diag}} \equiv t_i = B(i)\ \text{Diag}[\underbrace{1,1,1,\ldots,1}_{i-1\text{ entries}},1-i,\underbrace{0,0,\ldots,0}_{N-i\text{ entries}}],\quad i\in[2,N]. \tag{5.6}$$

The normalisation $B(i)$ is as defined in eq.5.4, and ensures that

$$\text{Tr}(t_i t_j) = \frac{1}{2}\delta_{ij} \tag{5.7}$$

For every diagonal generator t_i, there exist $2(i-1)$ ladder generators, t_{ij}, with one non-zero element;

$$t_{ij}^{mn} = \frac{1}{\sqrt{2}}\delta_i^n\delta_j^m;\quad j\in[1,i-1]. \tag{5.8}$$

t_{ij} provides the off-diagonal generators with non-zero matrix elements in the upper right half, while the lower left off-diagonal generators are given by the transpose, t_{ji}. Together, the ladder generators provide $N(N-1)$ generators, and combined with the $N-1$ diagonal ones, provide the total of N^2-1 generators required.

The off-diagonal generators are normalised in the following way

$$\text{Tr}(t_{ij}\,t_{j'i'}) = \frac{1}{2}\,\delta_{ii'}\,\delta_{jj'} \tag{5.9}$$

The algebra of the generators simplifies significantly in this basis, with the only non-vanishing generators being of the form

$$[t_i,t_{ij}] = NB(i)\,t_{ij}\,;\quad [t_i,t_{ji}] = -NB(i)\,t_{ji} \tag{5.10}$$

Returning to the idea of an interface between multiple charged vacua, a so called k-wall, a modified ansatz is required. As a k-wall is an interface between two vacua with a charge difference k, as opposed to a charge difference of 1 for a fundamental wall, the generator t_N in A_0^{cl} is replaced with a hypercharge matrix Y_k

$$A_0^{\rm cl} = \frac{2\pi T}{gN} qY_k. \tag{5.11}$$

The matrix Y_k is defined as

$$Y_k \equiv \text{Diag}[\underbrace{k,k,k,\ldots,k}_{N-k \text{ entries}},\underbrace{k-N,k-N,\ldots,k-N}_{k \text{ entries}}], \quad k \in [1,N] \tag{5.12}$$

As with t_N, Y_k is traceless, and by cyclic invariance, the system obeys charge conjugation $k \leftrightarrow N-k$, as required by the $\mathbb{Z}_N$ invariance of the theory.

Applying these modifications to the order parameter, the role of q is now clear. Previously, the parameter q had a dual role, defining each vacua individually when integer valued, with non-integer q characterising a point within an interface. For k-walls with $k > 1$, it is no longer q, but the product kq that specifies a given vacua for integer q; q now becomes a parameter varying across the k-wall, as before from $q(0) = 0$ to $q(L) = 1$.

The Polyakov loop order parameter for the modified ansatz becomes;

$$P = \frac{1}{N}\left[(N-k)e^{\frac{2\pi i}{N}kq} + k\,e^{\frac{2\pi i}{N}(k-N)q}\right] \tag{5.13}$$

with $P = 1$ at $q = 0$ as before, but $P = e^{\frac{2\pi i}{N}k}$ at $q = 1$.

5.2.3 k-wall Tension in $\mathcal{N}=4$ SYM

Having specified the ansatz for the k-wall solutions, the aim is to determine the interface tension of the k-wall in $\mathcal{N}=4$ SYM. The idea is to insert the classical profile for the $\mathbb{Z}_N$ instanton, and use weak coupling to expand around the background configuration. The quantum fluctuations induce a one-loop effective potential for the profile functions $q(z)$. This is used to determine the solution of the wall and its tension. It is also necessary to ensure the resulting system is self-consistent at weak coupling.

At the classical level, there is no interface solution. Therefore one-loop effects must be included. The only terms in the action of $\mathcal{N} = 4$ SYM that are relevant are those that involve interactions of the background gauge field $A_0^{\rm cl}$ with quantum fluctuations:

$$S = \int d^3x \int_0^\beta d\tau \left\{ \mathrm{Tr}\left[\frac{1}{2}(F_{\mu\nu})^2\right] + \mathrm{Tr}\left[\sum_{A=1}^{4} \bar{\psi}_A \not{D} \psi_A\right] + \mathrm{Tr}\left[\sum_{i=1}^{6} \frac{1}{2} D_\mu \phi_i D^\mu \phi_i\right] + \ldots \right\} \tag{5.14}$$

The relevant portion of the action includes only kinetic terms for four Majorana fermions and six real scalars, and their interactions with the background field through the covariant derivative. Yukawa couplings, and the $\mathcal{N}=4$ quartic scalar potential term are omitted. Ultimately, the action will be integrated over the fluctuating quantum fields ϕ, ψ, the fluctuating gauge field, A_μ^{qu}, and associated gauge fixing ghosts; thus providing an effective potential for the classical gauge fields.

Working in $4d$ Euclidean space, on $\mathbb{R}^3 \times S^1$, with the S^1 of radius $\beta \equiv 1/T$ as the time direction, and the remaining space spanned by x_1, x_2 and z. Each of the quantum fluctuations will be considered separately below.

Gauge field fluctuations

The one-loop calculations outlined in this section follow essentially standard steps, however they are included here as a review and for completeness. As previously seen, the gauge field A^μ consists of classical and quantum parts, thus the gauge part of the action can be separated accordingly.

$$S_A = S_A^{\mathrm{cl}} + S_A^{\mathrm{qu}} \tag{5.15}$$

Letting q be a general function of z, and using the fact that the only non-zero classical gauge field is A_0^{cl}, the classical action can be calculated simply on this background.

$$\mathcal{L}_A^{\mathrm{cl}} = \mathrm{Tr}\left[\frac{1}{2}(F_{\mu\nu})^2\right] = \mathrm{Tr}\left[(\partial_z A_0^{\mathrm{cl}})^2\right] = \mathrm{Tr}\left[\frac{4\pi^2 T^2}{g_{YM}^2 N^2}(\partial_z q)^2 Y_k^2\right] \tag{5.16}$$

The trace over Y_k^2 gives $\mathrm{Tr}[Y_k^2] = (N-k)k^2 + k(N-k)^2 = Nk(N-k)$. Thus the classical action reduces to

$$S_A^{\mathrm{cl}} = \frac{4\pi^2 T^2}{g_{YM}^2 N} k(N-k) \int d^3x \int_0^\beta d\tau (\partial_z q)^2 \tag{5.17}$$

As mentioned briefly at the start of section 5.2.3, the classical action alone is not enough to show the existence of k-walls, since it is only sensitive to the gradient energy, as taking the Hamiltonian only yields a constant q solution. To find a k-wall solution, the action must be calculated beyond tree level; in this case, 1-loop, while remaining self-consistent at weak coupling.

To compute the contribution from the gauge fluctuations at one-loop order, shift attention to the quantum gauge part of the action. Here gauge fixing must be imposed. Employing the usual background field R_ξ gauges to obtain the action for these gauge fluctuations;

$$\mathcal{L}_A^{\rm qu} = \mathrm{Tr}\left[\frac{1}{2}(F_{\mu\nu}^{\rm qu})^2\right] + \mathrm{Tr}\left[\frac{1}{\xi}\left(D_{\rm cl}^\mu A_\mu^{\rm qu}\right)^2\right] + \mathrm{Tr}\left[\overline{\eta}\left(-D_{\rm cl}^\mu D_\mu\right)\eta\right] \tag{5.18}$$

with $\overline{\eta}$ & η being the Fadeev-Popov ghosts, and the adjoint covariant derivates D_μ & $D_\mu^{\rm cl}$ defined thus;

$$D_\mu = \partial_\mu - ig_{YM}[A_\mu,], \quad D_\mu^{\rm cl} = \partial_\mu - ig_{YM}[A_\mu^{\rm cl},]. \tag{5.19}$$

At the level of one-loop, interactions between different fluctuations are negligible, and can be ignored. This allows the full covariant derivative D_μ, with its background field equivalent $D_\mu^{\rm cl}$ which is gauge covariant with respect to the background. Integration by parts and assuming the background field is constant;

$$S_A^{\rm qu} = \int d^3x \int_0^\beta d\tau\, \mathrm{Tr}\left[A_\mu^{\rm qu}\left(-D_{\rm cl}^2 g^{\mu\nu} + (1-\frac{1}{\xi})D_{\rm cl}^\mu D_{\rm cl}^\nu\right)A_\nu^{\rm qu}\right] + \mathrm{Tr}\left[\overline{\eta}\left(-D_{\rm cl}^\mu D_\mu\right)\eta\right] \tag{5.20}$$

It is important to note a technical point. The background field is assumed to be constant, thus providing an effective potential for constant background field configurations only. However, the ultimate aim is to apply this to non-constant domain wall profiles. Therefore this requires that the profile function is slowly varying.

Performing the functional integral over the gauge fluctuations $A_\mu^{\rm cl}$, and the ghost fields, the one-loop contribution becomes;

$$S_A^{\rm qu} = \int d^3x \int_0^\beta d\tau \frac{1}{2} \mathrm{Tr}\left[\ln\left(-D_{\rm cl}^2 g^{\mu\nu} + (1-\frac{1}{\xi})D_{\rm cl}^\mu D_{\rm cl}^\nu\right)\right] - \mathrm{Tr}\left[\ln\left(-D_{\rm cl}^2\right)\right] \quad (5.21)$$

The effective action can be shown to be independent of the gauge fixing parameter ξ due to the commutativity of the covariant derivatives for constant background fields. Therefore for slowly varying wall profiles, gauge invariant results are effectively guaranteed. Taking the Feynman gauge, $\xi = 1$;

$$S_A^{\rm qu} = -\int d^3x \int_0^\beta d\tau \frac{1}{2} \mathrm{Tr}\ln(-D_{\rm cl}^2) \quad (5.22)$$

As the background field present $A_0^{\rm cl}(z)$ in the adjoint covariant derivative is only non-zero along the temporal direction, τ, it reduces to an ordinary derivative in the transverse directions, x_1, x_2 and z. In the compact temporal direction the background field is proportional to the matrix Y_k, and being diagonal with N elements, there exist non-trivial contributions to the covariant derivative when acting upon ladder generators t_{ij} and t_{ji} (E.g. eq.(5.10)). Following the notation of [66];

$$D_0^{\rm cl}\, t_{ij} = (\partial_0 - 2\pi i T q)\, t_{ij} \equiv D_0^+ t_{ij} \quad (5.23)$$

$$D_0^{\rm cl}\, t_{ji} = (\partial_0 + 2\pi i T q)\, t_{ji} \equiv D_0^- t_{ji} \quad (5.24)$$

The commutator of Y_k with the ladder generators of SU(N) has very similar properties to the commutator in eq.(5.10). However, there are significant differences. In the diagonal generators, t_N, there is only one non-unity element (namely $1-N$). However, in Y_k, there are k elements which take the form $k-N$; thus:

$$[Y_k, t_{ij}] = N t_{ij}, \quad [Y_k, t_{ji}] = -N t_{ji}, \quad i \in [N-k, N], \quad j \in [1, N-k]. \quad (5.25)$$

All other commutators vanish. Full non-trivial q dependence arises from the action of the covariant derivates on the ladder generators, or equivalently, integrating out all off-diagonal fluctuations that do not commute with Y_k. Therefore one can write:

$$D_{\rm cl} \to (D_0^\pm, \vec{\partial}) \quad (5.26)$$

Fourier transforming to Euclidean momentum space, the classical temporal derivative ∂_0 may be replaced by the Matsubara frequencies p_0.

$$i\partial_0 \to p_0 = 2\pi nT, \quad n \in \mathbb{Z}. \tag{5.27}$$

On ladder operator-like fluctuations, the covariant derivative from eq.(5.24) acts as;

$$iD_0^{\pm} \to p_0^{\pm} = 2\pi T(n \pm q) \tag{5.28}$$

As the sum over the Matsubara frequencies, n includes both positive and negative values, the sum over p_0^+ is equivalent, up to an overall sign, to the sum over p_0^-. Hence it is reasonable to consider only the sum over n for p_0^+, and subsequently introduce an overall factor of 2 to the action. Also, since there are exactly $k(N-k)$ non-zero fluctuations, eq.(5.25), the effective action for the gauge fluctuations becomes:

$$S_A^{\mathrm{qu}} = 2k(N-k)V_{\mathrm{tr}}LT \sum_{n=-\infty}^{+\infty} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \ln\left((p_0^+)^2 + \mathbf{p}^2\right) \tag{5.29}$$

Here V_{tr} is the volume transverse to the z-axis,

$$V_{\mathrm{tr}} = L_1 L_2 \beta \tag{5.30}$$

with L_1 & L_2 being the length of the system in x_1& x_2 directions respectively. As the Euclidean time circle is compactified, a factor of T has been introduced cancelling the factor of β, thus effectively reducing the system to a 3 dimensional problem. In such a case, the k-wall becomes smeared along this direction.

The next step is to determine the q-dependence of the one-loop effective action. Up to additive constants, that can be discarded, the q-dependence is determined by the variation of S_A^{qu} with respect to q,

$$\frac{1}{2\pi T}\frac{\partial S_A^{\mathrm{qu}}}{\partial q} = 4k(N-k)V_{\mathrm{tr}}LT \sum_{n=-\infty}^{+\infty} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \left(\frac{p_0^+}{(p_0^+)^2 + \mathbf{p}^2}\right) \tag{5.31}$$

Using the standard expressions for integration using dimensional regularisation, the spatial momentum integration transforms into a product of n and q addi-

tions;

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3}\left(\frac{p_0^+}{(p_0^+)^2+\mathbf{p}^2}\right) = \frac{p_0^+}{(4\pi)^{3/2}}\frac{\Gamma(-1/2)}{\Gamma(1)}\left[\frac{1}{(p_0^+)^2}\right]^{-\frac{1}{2}} \tag{5.32}$$

$$= -\frac{1}{4\pi}p_0^+|p_0^+| \tag{5.33}$$

$$= -\pi T^2(n+q)|n+q| \tag{5.34}$$

Thus the variation of the action simplifies, with the sum over the Matsubara frequencies, n, now explicitly shown.

$$\frac{\partial S_A^{\mathrm{qu}}}{\partial q} = -4k(N-k)(2\pi T)V_{tr}L\pi T^3\sum_{n=-\infty}^{+\infty}(n+q)|n+q| \tag{5.35}$$

Using zeta function regularisation, the explicitly divergent sum over n can be controlled quite elegantly. Using the zeta function definition,

$$\zeta(l,m) = \sum_{n=0}^{+\infty}(n+m)^{-l} \tag{5.36}$$

and re-expressing the sum over n from $n \in [-\infty,\infty]$ to a sum in the region $n \in [0,\infty]$, the variation becomes:

$$\sum_{n=-\infty}^{+\infty}(n+q)|n+q| = \sum_{n=0}^{+\infty}\left[(n+q)^2-(n+1-q)^2\right] = [\zeta(-2,q)-\zeta(-2,1-q)] \tag{5.37}$$

$$\frac{\partial S_A^{\mathrm{qu}}}{\partial q} = -4k(N-k)(2\pi T)V_{\mathrm{tr}}L\pi T^3\left[\zeta(-2,q)-\zeta(-2,1-q)\right]. \tag{5.38}$$

The particular form of the zeta function is a simple polynomial in q,

$$\zeta(-2,q) = -\frac{1}{12}\frac{d}{dq}\left[q^2(1-q^2)\right] \tag{5.39}$$

Negating any addition-wise constant, thus concentrating on the q dependence only, and letting L revert to an integral over z, the one-loop effective action for slowly varying $q(z)$ becomes:

$$S_A^{\text{qu}} = \frac{4}{3}k(N-k)V_{\text{tr}}\pi^2 T^4 \int_0^L dz\, q^2(1-q)^2 \tag{5.40}$$

It can be seen that this one-loop gauge effect acts like a wall in $q \in [0,1]$, with minima at both $q = 0$ and $q = 1$; the factor $q^2(1-q)^2$ is invariant under the shift $q \to 1-q$, illustrating a form of symmetric 'kink'-like configuration interpolating between the two minima.

Combining the quantum action at one-loop with the classical kinetic term calculated earlier, the total effective action can be compactly expressed, using a coordinate rescaling of $z \to z' = \sqrt{g_{YM}^2 N/3}Tz$ as;

$$S_A = \frac{4\pi^2 T^3}{\sqrt{3N}g_{YM}}k(N-k)V_{\text{tr}} \int_0^{L'} dz' \left[\left(\frac{\partial q}{\partial z'}\right)^2 + q^2(1-q)^2\right] \tag{5.41}$$

The double well potential, represents the so-called "q-valley". Due to the rescaling of $z \to z'$, the upper limit of integration L is also effectively rescaled; $L' = (\sqrt{N/3}gTL \to \infty)$. The large volume limit corresponds to $\sqrt{g_{YM}^2 NTL} \to \infty$, which can also be viewed at the three dimensional limit when the thermal circle shrinks to zero size.

It is now possible to self-consistently justify the use of the constant-q effective potential to infer the existence of the spatially varying domain wall. Considering the z' coordinate, it is clear that the width of the domain wall is simply a number $\sim O(1)$. In physical units, the width of the domain wall is subsequently set by $\left(\sqrt{g_{\text{YM}}^2 NT}\right)^{-1}$, which is the Debye or electric screening length. At weak coupling this is much larger than the typical thermal wavelength T^{-1}. Thus the domain wall is a thick and slowly varying configuration.

Following this review of the perturbative gauge field contributions, the next section will introduce the matter fields in the adjoint representation of $\mathcal{N} = 4$ SYM theory.

Scalar field contributions

Due to the similarity in the calculation to the gauge field contribution, consider now the 6 Hermitian scalar fields transforming in the adjoint representation of the

gauge group, before turning to the fermion fields in the next section. The scalar part of the action coupled to the classical background field is simply the kinetic term for the scalars with the background field covariant derivative:

$$S_S = \int d^3x \int_0^\beta d\tau \,\mathrm{Tr}\left[\sum_{i=1}^{n_s} \frac{1}{2}\left(D^{\mathrm{cl}}_\mu \phi_i D^\mu_{\mathrm{cl}} \phi_i\right)\right] \tag{5.42}$$

where n_s represents the total number of real adjoint scalars, here $n_s = 6$, however it will be kept explicit for illustrative purposes. Integrating out the scalar fluctuations

$$S_S = \frac{n_s}{2}\int d^3x \int_0^\beta d\tau \,\mathrm{Tr}\ln(-D^2_{\mathrm{cl}}) \tag{5.43}$$

This expression is equivalent to eq.(5.22) up to the overall number of scalars. Following the procedure used with the gauge fluctuations, the scalar action reduces to effectively $n_s/2$ times the expression for S_A^{qu};

$$S_S = \frac{4\pi^2 T^3}{\sqrt{3Ng}} k(N-k)\frac{n_s}{2} V_{tr} \int_0^{L'} dz' q^2(1-q)^2 \tag{5.44}$$

Fermionic field contributions

The fermion contributions arise from $n_f = 4$ Majorana fermions, which transform as a **4** of the SO(6) R-symmetry. These play a crucial role at finite temperature as they exhibit anti-periodic boundary conditions around the thermal circle, which breaks the supersymmetry of the system. At one-loop order;

$$S_F = \int d^3x \int_0^\beta d\tau \mathrm{Tr}\left[\sum_{a=1}^{n_f} \overline{\psi}_a \not{D} \psi_a\right] \tag{5.45}$$

where n_f is left explicit. Working in Euclidean space, the Gamma matrices convention taken is taken as;

$$\gamma^{1,2,3} = \begin{pmatrix} 0 & -i\sigma_{1,2,3} \\ i\sigma_{1,2,3} & 0 \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \tag{5.46}$$

with the Pauli matrices convention is taken as

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.47}$$

The functional integral over the fermion fields yields the Pfaffian of the Dirac operator, since ψ_a and $\overline{\psi_a}$ are not independent, via the Majorana condition.

$$S_F = -\int d^3x \int_0^\beta d\tau n_f \mathrm{Tr}\left[-((D_0^{\mathrm{cl}})^2 + \nabla^2)\right] \tag{5.48}$$

So far, the formal treatment has been similar to the bosonic contributions; however at this point, the analysis departs from that of the gauge and scalar contributions.

Consider firstly the case at zero temperature, where the compact direction has *periodic* (SUSY preserving) boundary conditions for the fermions. The fluctuation determinant eq. 5.48 would be identical to that of the bosons, but of the opposite sign. This produces a one-loop action of the form $S_F = -n_f S_A^{\mathrm{qu}}$. With supersymmetric, periodic boundary conditions the three fluctuation terms at the one-loop level would cancel, leaving only the classical action,

$$S_{\mathrm{Total}} = S_A^{\mathrm{cl}} + S_A^{\mathrm{qu}} + S_S + S_F = S_A^{\mathrm{cl}} + (1 + n_s/2 - n_f)S_A^{\mathrm{qu}} = S_A^{\mathrm{cl}} \tag{5.49}$$

This cancellation effect between the bosons and fermions will persist to all loops for SUSY-preserving periodic boundary conditions.

However, in the Euclidean thermal theory, this cancelation effect is not an issue; as the fermions exhibit anti-periodic boundary conditions around the thermal circle. This causes the Matsubara frequencies to be shifted to half-integer values, $n \to n + \frac{1}{2}, n \in \mathbb{Z}$. Therefore, for fermions, the definition of the covariant derivative, eq. (5.28) is modified thus

$$iD_0^{\pm} \to p_0^{\pm} = 2\pi T\left(n + \frac{1}{2} \pm q\right) \tag{5.50}$$

It is this shift that removes the fermion-boson cancellation in the one-loop effective potential. Under the half-integer shift of the Matsubara frequencies, the sums over n for p_0^+ and p_0^- are no longer equivalent, therefore, each sum must be evaluated separately.

$$S_F = -k(N-k)V_{\text{tr}}LTn_f \sum_{n=-\infty}^{+\infty} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \left(\ln\left[(p_0^+)^2 + \mathbf{p}^2\right] + \ln\left[(p_0^-)^2 + \mathbf{p}^2\right] \right) \quad (5.51)$$

As previously with the gauge term, taking the variation of the action with q in order to determine the q-dependence;

$$\frac{\partial S_F}{\partial q} = -2k(N-k)(2\pi T)V_{\text{tr}}LTn_f \sum_{n=-\infty}^{+\infty} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \left(\frac{p_0^+}{(p_0^+)^2 + \mathbf{p}^2} - \frac{p_0^-}{(p_0^-)^2 + \mathbf{p}^2} \right) \quad (5.52)$$

Again, integrating over the spatial momenta $\mathbf{p}$, employing dimensional regularisation;

$$\frac{\partial S_F}{\partial q} = 2k(N-k)(2\pi T)V_{\text{tr}}LTn_f(\pi T^2) \quad (5.53)$$

$$\times \sum_{n=-\infty}^{+\infty} \left[(n+1/2+q)\,|n+1/2+q| - (n+1/2-q)\,|n+1/2-q| \right] \quad (5.54)$$

To perform regularisation of the sum over n with zeta functions, consider the sum in two separate regions of q, $q \in [0, 1/2]$ & $q \in [1/2, 1]$

$$q \in [0, 1/2] \rightarrow 2 \sum_{n=-\infty}^{+\infty} \left[(n+1/2+q)^2 - (n+1/2-q)^2 \right] \quad (5.55)$$

$$q \in [1/2, 1] \rightarrow 2 \sum_{n=-\infty}^{+\infty} \left[(n-1/2+q)^2 - (n+3/2-q)^2 \right] \quad (5.56)$$

Applying the shift $q \rightarrow 1-q$ swaps the two terms and their respective regions of validity in q. As there is no loss in generality, it is acceptable to only consider the region $0 \leq q \leq 1/2$ and introduce an overall doubling factor. The definition of the Hurwitz zeta function as a derivative, eq.5.39, allows the action to be explicitly determined, up to integration constants,

$$S_F = -\frac{4\pi^2 T^3}{\sqrt{3N}g_{YM}} k(N-k)V_{\text{tr}}n_f \times 2 \int dz \left(\frac{1}{2}+q\right)^2 \left(\frac{1}{2}-q\right)^2. \quad (5.57)$$

The integration region is now only defined over $0 \leq q \leq 1/2$. It is obvious that this fermionic action will not cancel with the quantum gauge and scalar parts.

Full $\mathcal{N}=4$ SYM one-loop effective action

Summing all the above contributions, the one-loop effective action for the interface is obtained, with the integration range adjusted accordingly;

$$\begin{aligned} S_{\text{Total}} &= S_A + S_S + S_F && (5.58)\\ &= \frac{4\pi^2 T^3}{\sqrt{3N}g_{YM}} k(N-k)V_{\text{tr}} 2\int dz' && (5.59)\\ &\left[\left(\frac{\partial q}{\partial z'}\right)^2 + \left(1+\frac{n_s}{2}\right)q^2(1-q)^2 - n_f\left(\frac{1}{2}+q\right)^2\left(\frac{1}{2}-q\right)^2\right] && (5.60)\end{aligned}$$

Letting n_s and n_f take their explicit values, the total quantum effective action simplifies to

$$S_{\text{Total}} = \frac{4\pi^2 T^3}{\sqrt{3N}g_{YM}} k(N-k)V_{\text{tr}} \times 2\int dz' \left[\left(\frac{\partial q}{\partial z'}\right)^2 + 2q^2(3-4q) - 1/4\right] \quad (5.61)$$

It is a simple exercise using the Hamiltonian to obtain the minimum configuration that interpolates between the two vacua $q = 0$ & $q = 1$, satisfying $(\partial q/\partial z')^2 = 2q^2(3-4q)$, so that the action for the interface is

$$\begin{aligned} S_{\text{Total}} &= \frac{4\pi^2 T^3}{\sqrt{3N}g_{YM}} k(N-k)V_{\text{tr}} \times 4\int_0^{1/2} dq\sqrt{2q^2(3-4q)} && (5.62)\\ &= \frac{4\pi^2 T^3}{\sqrt{3N}g_{YM}} k(N-k)V_{\text{tr}}\frac{\sqrt{2}}{5}(3\sqrt{3}-2) && (5.63)\end{aligned}$$

One concludes, therefore that the tension of the k-wall in the Euclidean high temperature limit, $\mathcal{N}=4$ SYM theory at weak coupling is;

$$\begin{aligned} \mathcal{T}_k &= \frac{4\pi^2 T^2}{\sqrt{3N}g_{YM}} k(N-k)\frac{\sqrt{2}}{5}(3\sqrt{3}-2) && (5.64)\\ &\approx 0.904\frac{4\pi^2 T^2}{\sqrt{3N}g_{YM}} k(N-k) && (5.65)\end{aligned}$$

where one factor of T has been cancelled against the size of the thermal circle in $V_{\rm tr}$, leaving the tension of a 1 + 1 dimensional interface in three dimensions.

5.2.4 Discussions

Focussing on eq.5.65, note that the dependence on the gauge coupling is the same as in ordinary Yang-Mills theory. One difference however, is that unlike in pure Yang-Mills, the gauge coupling does not run, and therefore does not depend on the temperature. Nevertheless, one is free to chose an arbitrarily weak coupling in $\mathcal{N}=4$ theory, $g_{YM} \ll 1$. All other qualitative aspects of the solution are similar to that of pure YM. Specifically, the wall is "fat", with a width set by the Debye screening length $\left(\sqrt{g_{YM}^2 NT}\right)^{-1}$.

Finally, the one-loop calculation demonstrates a Casimir scaling law for the k-wall tension in $\mathcal{N}=4$ SYM, as found in the pure Yang-Mills theory. it is not *a priori* clear that Casimir scaling will persist at higher loop orders, since at one-loop its origin is essentially kinematic.

In the next section, a plausibility argument will be put forward for the Casimir scaling behaviour to hold at 2-loops.

5.3 k-wall scaling in $\mathcal{N}=4$ SYM at 2-loop

It has been shown [64,65] in pure Yang-Mills that Casimir scaling of $\mathbb{Z}_N$ domain walls remains at two-loops, but is lost at three-loops. Below, the calculation for the two-loop contribution to pure Yang-Mills is reviewed, before being adapted to argue that the scaling behaviour for domain walls in $\mathcal{N}=4$ SYM will also be Casimir-like at two-loop order.

5.3.1 Pure Yang-Mills at 2-loop

Consider the two-loop calculation for the domain wall tension in the deconfined phase of pure Yang-Mills theory. Setting out definitions and notations, let the structure constants of SU(N) & their normalisation be defined thus:

$$if^{a,b,c} = 2\mathrm{Tr}\left(\left[t^a, t^b\right] t^c\right), \quad \left(f^{a,b,c}\right)^2 = \frac{1}{2} \tag{5.66}$$

The indices a, b, c can represent either the diagonal generators, t_{diag}, or the ladder generators, t_{ij}, t_{ji}. It follows from this definition, that the only non-zero values of $f^{a,b,c}$ occur when no more than one of the generators is diagonal, due to the commutativity of a diagonal generator with itself. These non-zero cases are explored in more detail below.

In [64–66], it was demonstrated that all possible graphs at the two-loop level, including three & four-gluon interaction vertices, and gluon ghost interactions, all generate contributions to the k-wall action of the form

$$\sim \sum_{a,b,c} f^{a,b,c} f^{a,b,c} B_2(C_a) B_2(C_b) \tag{5.67}$$

The function B_2 is related to the second Bernoulli polynomial, which is even in C_a

$$B_2(C_a) \sim \left(C_a^2 - |C_a| + 1/6\right) \tag{5.68}$$

The variables C_a represent a function C_{ij}, that introduces all the q dependencies.

$$C_{ij} = A_{0i} - A_{0j} \sim q\left[(Y_k)_i - (Y_k)_j\right] \tag{5.69}$$

Obviously $C_{ii} = 0$, while C_{ij} is only non-zero when i & j sit in different sectors of Y_k. Thus $C_{ij} = 0$ or $\pm q$ up to an overall factor.

Explicit computation of eq. (5.67) provides a Casimir-like scaling, like that at the one-loop level, by summing all non-zero terms that express non-trivial q dependence (i.e. ignoring all $B_2(0)^2$ terms). All non-trivial cases can be classified and accounted for, as seen in Appendix A. The final result after summing all contributing terms:

$$\sim Nk(N-k)\left[B_2(q)^2 + 2B_2(q)B_2(0)\right] \tag{5.70}$$

As outlined in the Appendix, there are two key technical reasons for the Casimir scaling to arise in the final result. Firstly, all q-dependence in eq.5.67 arises form terms where at least one of the two indices a or b are off-diagonal generators. Secondly, and perhaps more importantly, the combination $B_2(C_a)B_2(C_b)$ is a even function of C_a.

5.3.2 Argument for Casimir scaling for $\mathcal{N}=4$ SYM at 2-loops

The two factors outlined in the previous section, coupled with the structure constants, effectively guarantee Casimir scaling in pure Yang-Mills. For this scaling to be present in $\mathcal{N}=4$ theory, the same factors must apply. It can be argued that the propagators for adjoint scalars in $\mathcal{N}=4$ SYM, are equivalent (up to an overall factor) to the ghost propagators in pure YM. Therefore, inclusion of the adjoint scalars are not expected to change the Casimir scaling at two-loops.

As for the adjoint fermions in $\mathcal{N}=4$, a SUSY-based argument can be employed. In the 1-loop case, it was seen that the action for the adjoint fermions with periodic boundary conditions cancelled the quantum gauge field and adjoint scalar contributions when considering a constant or slowly varying background field A_0. For SUSY to hold, this cancellation effect must hold for all perturbative levels. Since the bosonic fluctuations at two-loops yield the Casimir scaling of the effective potential, the fermionic contributions will also exhibit this scaling.

Switching focus to the thermal interpretation; imposing anti-periodic boundary conditions on the fermions, there will simply be a shift in the q dependence, $q \to q' = q \pm 1/2$ due to the Matsubara frequency shift. It is expected that this will have no effect on the overall N scaling, leading to terms of the form

$$\sim (f^{ij,ji,\text{diag}})^2 B_2(C^F_{ij}) B_2(C_{ji}) + \text{permutations} \tag{5.71}$$

where C^F_{ij} is the shifted difference:

$$C^F_{ij} \sim (A_{0i} - A_{0j}) \pm \frac{1}{2} \sim q \pm \frac{1}{2} \tag{5.72}$$

Such a term would arise from a two-loop graph as seen in figure 5.2. This would then imply

$$\sim \tfrac{1}{2} k(N-k) B_2(q) \left[B_2\left(q + \tfrac{1}{2}\right) + B_2\left(q - \tfrac{1}{2}\right) \right] \tag{5.73}$$

$$= \tfrac{1}{2} k(N-k) q^2 (1-q)^2 \left[\left(\tfrac{1}{2} + q\right)^2 \left(\tfrac{1}{2} - q\right)^2 + \left(-\tfrac{1}{2} + q\right)^2 \left(\tfrac{3}{2} - q\right)^2 \right] \tag{5.74}$$

representing a pair of gluon - fermion vertices, as in figure 5.2. These arguments make it plausible that Casimir scaling of the k-wall tensions persist at the

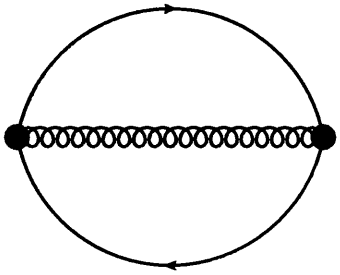


Figure 5.2: *A two-loop fermion contribution to the effective action.*

two-loop level in $\mathcal{N} = 4$ SYM, as with pure Yang-Mills theory. However, there is no reason for this Casimir scaling behaviour to exist at higher loops.

5.4 k-wall scaling in $\mathcal{N}=4$ SYM at strong coupling

At this point attention turns to the domain walls in the strongly coupled $\mathcal{N} = 4$ SYM theory in the large N limit and at finite temperature. The deconfined phase of the four dimensional field theory at strong coupling is described by Type IIB string theory in the Schwarzschild black hole in $AdS_5 \times S^5$ [71]. In the euclidean picture, the conformal boundary of the geometry is $\mathbb{R}^3 \times S^1$, with the S^1 being identified as the Euclidean thermal circle of the strongly coupled field theory. The spontaneous breaking of the $\mathbb{Z}_N$ centre symmetry in the deconfined phase arises as the thermal circle can shrink smoothly to zero size, and does this at a radial co-ordinate in the geometry corresponding to the horizon of the black hole. This resulting "cigar" shape, or D_2 ($\equiv \mathbb{R} \times S^1$) black hole, can be wrapped by a string world-sheet of finite action, generating a non-zero Polyakov loop (the order parameter), which spontaneously breaks the $\mathbb{Z}_N$ symmetry.

5.4.1 D-string as domain wall

In the deconfined phase, with the $\mathbb{Z}_N$ symmetry spontaneously broken, it should be possible to identify the N distinct vacua of the strongly coupled theory in the IIB string dual. Shifting the phase of the Polyakov loop through the N-th roots of unity, moves through the N possible vacua states. In the string dual picture this is realised as a $2\pi k/N$ shift of the two-form NS field B_2 integrated over the black hole cigar [72]

$$\int_{D_2} B_2 \to \int_{D_2} B_2 + \frac{2\pi k}{N}, \quad k = 1, \ldots N. \tag{5.75}$$

World-sheets wrapping the cigar will pick up a phase $\exp(i2\pi k/N)$, determining the expectation value of the Polyakov loop for a given vacua k.

Across a domain wall, the phase of the Polyakov loop jumps. It was discussed in [72] that a D-string world-sheet $\Sigma \subset \mathbb{R}^3$ and point-like in the transverse D_2 and S^5, provides a suitable jump. Across Σ, the RR three form flux, F_3 changes by one unit for a single D-string.

There is another argument establishing the connection between $\mathbb{Z}_N$ domain walls at the D-string. This exploits the direct relation between $\mathbb{Z}_N$ interfaces and spatial 't Hooft loops [67, 73]. The spatial 't Hooft loop operator $V(C)$ along a contour C, creates an infinitely thin tube of chromoelectric flux along C. The spatial loop bounds a surface, a "Dirac sheet", across which the gauge potential A_0 is discontinuous. The explicit perturbative computation of the expectation value of the 't Hooft loop in the deconfined phase can be shown to reduce to the computations for the domain wall in section 5.2.3. Specifically, the leading contribution to the large 't Hooft loop in $\mathbb{R}^3$ is proportional to the minimal surface area, A bounded by the contour C.

$$V(C) \sim \exp(-\mathcal{T}A)$$

Therefore the tension $\mathcal{T}$ of a domain wall of infinite volume is computed by a spatial 't Hooft loop of infinite size.

From the AdS/CFT correspondence, the 't Hooft loop is a Euclidean D-string disc world-sheet, whose boundary traces the spatial loop in the field theory on the conformal boundary of the spacetime. In the AdS-Schwarzschild black hole geometry, as with the Hardwall case previously, the D-string world-sheet dips into the bulk, but as the loop is scaled upwards, a greater proportion of the world-sheet sits on the horizon where the geometry smoothly ends. Thus the spatial 't Hooft loop exhibits an area law. Taking the loop to infinite extent, a Euclidean D-string world-sheet Σ is obtained, located at the horizon of the Euclidean black hole. This is the $\mathbb{Z}_N$ interface.

5.4.2 k-wall tensions

The metric for the finite temperature, Euclidean AdS-Schwarzschild black hole in $AdS_5 \times S^5$ (explicitly including the AdS radius, R) is:

$$ds^2 = R^2\left[f(y)dt^2 + f^{-1}(y)dy^2 + y^2 dx_i^2 + d\Omega_5^2\right] \tag{5.76}$$

$$f(y) = y^2 - \frac{\pi^4 T^4}{y^2} \tag{5.77}$$

where $R^4 = 4\pi(g_s N)\alpha'^2 = (g_{YM}^2 N)\alpha'^2$. The D1-brane world-sheet is described by the DBI action with no coupling to background RR potentials.

$$S_{D1} = \frac{1}{2\pi\alpha'}\int d^2\xi\, e^{-\Phi}\sqrt{\det \mathcal{G}} \tag{5.78}$$

The dilaton is constant, with $e^{-\Phi} = 1/g_s = 1/g_{YM}^2$. Embedding the world-sheet Σ along the x_1, x_2 plane, let $\xi_1 = x_1$ & $\xi_2 = x_2$. Minimising the action, the D-string will sit at the smallest possible value of y, which in this geometry is $y = \pi T$. The tension of the $k = 1$ wall at strong coupling is

$$\mathcal{T}_1 = \frac{1}{2\pi\alpha' g_s} R^2\pi^2 T^2 = 2\pi^2 \frac{N}{\sqrt{g_{YM}^2 N}} T^2. \tag{5.79}$$

Remarkably, the parametric dependence of this formula on g_{YM} & N, closely resembles eq.5.65. The dependence on the temperature is guaranteed to be quadratic by the underlying conformal invariance of $\mathcal{N}=4$ SYM. The N dependence is consistent with the domain wall being a D-brane in the large-N limit and the fact that the tension of the D-string in AdS is proportional to $1/\sqrt{g_{YM}^2 N}$ is also obvious from supergravity. It is also interesting to note that the formula at weak coupling has the same dependence on the 't Hooft coupling, $\sqrt{\lambda} \equiv \sqrt{g_{YM}^2 N}$.

In the large-N limit, for a collection of k D-strings, with $k \sim O(1)$, the tension is simply k times that of a single D1-brane. For the limit of large-N, fixed k/N, the collection blows up into a higher dimensional brane via the analogue of the dielectric effect [31]. In analogy with the tension computations of chapter 3, there are two possible brane configurations to consider in the $AdS_5 \times S^5$ black hole geometry carrying k units of D-string charge. At zero temperature, in $AdS_5 \times S^5$, electric

Wilson loops in the k^{th} rank antisymmetric and symmetric tensor representations of $SU(N)$, are computed by a D5-brane wrapped on an $S^4 \subset S^5$ and a D3-brane wrapping an $S^2 \subset AdS_5$, respectively [14–16, 43]. Hence, in a S-dual picture, a collection of k D-strings representing 't Hooft loops can expand into wrapped NS5 and D3-branes.

The k-wall as a 5-brane

The correct configuration describing a k-wall is expected to be a wrapped 5-brane. This yields the k^{th} rank antisymmetric tensor representation of Wilson/'t Hooft loop. As seen previously, this system is invariant under $k \to N-k$, which is required for $\mathbb{Z}_N$ interfaces.

It is most convenient to consider the D5-brane configuration, carrying k units of fundamental string charge, before S-dualising the system to obtain the D-string domain wall. The action for the probe D5-brane has both DBI & Wess-Zumino terms, and closely following the analysis of [14]:

$$S_{\text{Bulk}} = \frac{1}{(2\pi)^5 \alpha'^3 g_s} \int d^6\xi \left[\sqrt{\det(\mathcal{G} + 2\pi\alpha' F)} - i g_s 2\pi\alpha' F \wedge C_4 \right] \quad (5.80)$$

$$C_4 = \frac{R^4}{g_s} \left[\frac{3}{2}(\theta - \pi) - \sin^3\theta \cos\theta - \frac{3}{2} \cos\theta \sin\theta \right] d\Omega_4 \quad (5.81)$$

The C_4 is the relevant part of the RR four-form potential parallel to the S^4. The D5 wraps an S^4 at the azimuthal angle $\theta \in [0, \pi]$ in the S^5. The D5 have a world-volume $\Sigma \times S^4$ where Σ is embedded in the x_1, x_2 plane. A world-volume electric field is turned on along Σ, $F_{x_1 x_2}$, giving the D5 an f-string charge, and $F_{x_1 x_2} = iF$ due to the Euclidean signature.

Letting the AdS radius be re-expressed, $R^4 = 4\pi(g_s N)\alpha'^2$, the D5-brane action becomes

$$S_{\text{Bulk}} = \frac{N\sqrt{\lambda}}{3\pi^2} \int dx_1 dx_2 \left(\sin^4\theta \sqrt{y^4 - \frac{4\pi^2 F^2}{\lambda}} - \frac{2\pi F}{\sqrt{\lambda}} G(\theta) \right) \quad (5.82)$$

The 't Hooft coupling has been defined as $\lambda = g_{YM}^2 N$ and

$$G(\theta) = -\frac{3}{2}(\theta - \pi) + \sin^3\theta\cos\theta + \frac{3}{2}\cos\theta\sin\theta \tag{5.83}$$

The equation of motion for the gauge field associated with F provides the quantised total f-string charge k. Thus

$$k = -\frac{\delta S}{\delta F} = \frac{2N}{3\pi}\left(\frac{2\pi F}{\sqrt{\lambda}}\frac{\sin^4\theta}{\sqrt{y^4 - \frac{4\pi^2F^2}{\lambda}}} + G(\theta)\right) \tag{5.84}$$

Together with the equation of motion for the azimuthal angle in the S^4, the angle θ is completely determined in terms of the string charge k.

$$\frac{2\pi F}{\sqrt{\lambda}\sqrt{y^4 - \frac{4\pi^2F^2}{\lambda}}} = -\cot\theta, \quad \cos\theta\sin\theta - (\theta - \pi) = \frac{\pi k}{N} \tag{5.85}$$

This location of the S^4 inside the S^5 is explicitly linked to the charge k. Importantly, the system is invariant under $k \to N - k$, as this maps to a shift in the angle, $\theta \to \pi - \theta$. This is of course necessary for the $\mathbb{Z}_N$ interface.

Since the system is entirely determined at this point from a world-volume field and S^4 perspective, the effective action can be computed by simply applying eqs. 5.85;

$$S_{\text{Bulk}} = \frac{N\sqrt{\lambda}}{3\pi^2}\int dx_1 dx_2 y^2\left[\sin^3\theta + \frac{3}{2}\left(\frac{\pi k}{N}\right)\cos\theta\right] \tag{5.86}$$

As seen in previous chapters, boundary terms must be included for the world-volume field;

$$S_{\text{Bdy}} = k\int dx_1 dx_2 F \tag{5.87}$$

Thus providing a total action;

$$S_{\text{Bulk}} + S_{\text{Bdy}} = \frac{N\sqrt{\lambda}}{3\pi^2}\int dx_1 dx_2\left\{y^2\left[\sin^3\theta + \frac{3}{2}\left(\frac{\pi k}{N}\right)\cos\theta\right] + 3\pi^2\frac{k}{N}\frac{F}{\sqrt{\lambda}}\right\} \tag{5.88}$$

It is necessary to include such a boundary term due to the infinite extent to which the spatial Wilson/'t Hooft loop is taken for correct interpretation of domain walls. Inclusion of the term also ensures invariance under $k \to N-k$. As the equation of motion 5.85 implies $F = -y^2\sqrt{\lambda}\cos\theta/2\pi$, the complete Lagrangian only depends on y^2, and as the action is minimal at $y = \pi T$, the resulting formula for the tension is

$$\mathcal{T}_{F1} = N\sqrt{\lambda}\frac{T^2}{3}\sin^3\theta. \tag{5.89}$$

Tension of the 5-brane k-wall

So far, the tension of a wrapped D5-brane carrying k-units of fundamental string charge has been computed. This configuration can be interpreted as a domain wall associated to the breaking of the magnetic $\mathbb{Z}_N$ symmetry of $\mathcal{N}=4$ SYM at finite temperature. Performing S-duality on this yields the domain wall in the electric picture as a wrapped NS5-brane carrying k units of D-string charge.

S-dualising eq.5.89 by sending $g_s \to 1/g_s$, obtains the tension of the k-wall at strong coupling;

$$\mathcal{T}_k = \frac{4}{3}\pi N^2\frac{T^2}{\sqrt{g^2_{YM}N}}\sin^3\theta, \quad \cos\theta\sin\theta-(\theta-\pi) = \frac{\pi k}{N} \tag{5.90}$$

It is obvious that this result bears little resemblance to the weakly coupled result, eq.5.65. Nevertheless there are a number of significant remarks that can be made. The dependence of the 't Hooft coupling is surprisingly similar to that of the weak coupled Yang-Mills regime, and in the large N, k/N fixed limit, the tension scales as N^2 which is the expected from a classical soliton in a large-N theory, such as a NS5-brane in the IIB dual. This feature is also visible at weak coupling from the Casimir scaling. In the large-N, fixed k limit, the tension can be expanded in terms of fractional powers of (k/N);

$$\mathcal{T}_k \sim 2\pi^2 Nk\frac{T^2}{\sqrt{\lambda}}\left[1-O\left(\frac{k}{N}\right)^{2/3}+\ldots\right] \tag{5.91}$$

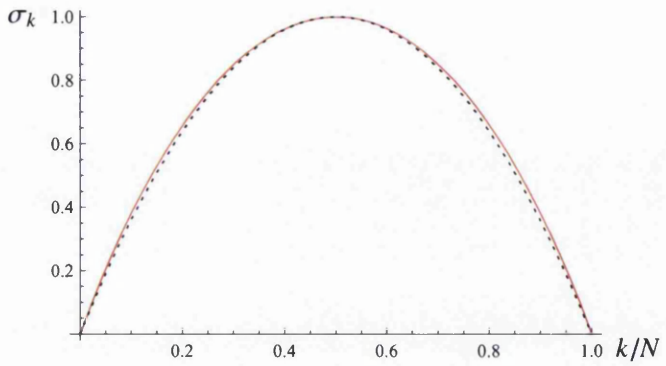


Figure 5.3: *Casimir and* $\sin^3$ *behaviours. The solid red line is* $\sin^3$ *(strong coupling), whereas the dashed black line is the Casimir scaling (weak coupling).*

5.4.3 Discussions

In this chapter the tension of domain walls in the deconfined phase of $\mathcal{N}=4$ super Yang-Mills theory on $\mathbb{R}^3 \times S^1$ was discussed . In the weak coupling limit, the tension of the domain wall exhibits Casimir scaling, $k(N-k)$ at the one loop level, and proposed to hold at two loops. This was confirmed after publication of the results in the paper or Korthals-Altes [74]. however, it was shown that the domain wall tension exhibits a different behaviour in the strong coupling regime. This is expected since it has already been shown that at three loops the $\mathbb{Z}_N$ domain wall tensions are not expected to exhibit a Casimir scaling [75].

However, a quantitative comparison of the weak and strong coupling behaviours of the tensions reveals interesting features.

In figure 5.3 the Casimir scaling (weak coupling behaviour) and the supergravity result (strong coupling) are plotted together as a function of k/N for $N \to \infty$. The two graphs are normalised such that $\mathcal{T}_{k/N=1/2} = 1$. The maximum difference between the two graphs is about 4%.

These results can be compared to lattice simulations, which were performed for the pure YM theory. Within the measurement error, the lattice results are compatible with a Casimir scaling [69, 70], even at low temperature (but above the deconfinement transition) where perturbation theory is not applicable and there is no reason to expect an exact Casimir scaling behaviour. It will be interesting to perform a more accurate simulation, in order to see a deviation from Casimir scaling

at low temperatures.

Although the results presented here were obtained for $\mathcal{N} = 4$ super Yang-Mills, they might be able to shed light on the expected tension of the pure Yang-Mills theory at strong coupling (low-temperatures). Qualitatively, it is expected that as the temperature of pure Yang-Mills theory decreases the ratio of the k-wall tension to the fundamental wall tension will increase, but only by a very small amount. this is illustrated in figure 5.3.

5.5 Domain walls with two-index matter

Until this point, only theories at weak coupling with adjoint matter have been discussed. In a finite temperature SU(N) theory with matter in the symmetric or anti-symmetric (or tensor sum of both) representations, the center symmetry is $\mathbb{Z}_2$, generating two degenerate vacua separated by a phase of π (i.e. vacua at 1 & -1). As with the adjoint case, there exists a domain wall interpolating between these vacua. In this section, the tension of a domain wall in two index representation theories is to be examined, following the same procedure as section 5.2.

As in the adjoint case, the order parameter for the deconfinement transition is the Polyakov Loop around the compactified time direction;

$$P(x) = \frac{1}{N}\mathrm{Tr}\ \mathcal{P}\ \exp\left(ig\int_0^\beta A_0(x,t)dt\right) \tag{5.92}$$

In the high temperature phase, the Polyakov loop acquires an expectation value across the deconfinement transition, signalling the spontaneous breaking of the centre symmetry, and the generation of distinct vacua separated by phases. For anti-symmetric or symmetric (from here on referred to as Asym or Sym) matter, the centre is broken to $\mathbb{Z}_2$, as opposed to $\mathbb{Z}_N$ for matter in the adjoint (Adj).

In contrast to the adjoint case where multiple domain walls can exist, for Asym & Sym, there exists only a single domain wall. The aim is to compare the tension of this wall in the weak coupling regime, with an equivalent domain wall in the adjoint case

As in the adjoint case, the temporal gauge field A_0 can be factored into classical and quantum parts, $A_0 = A_0^{\rm cl} + A_0^{\rm qu}$. As only a single domain wall exists, consider the following ansatz for the background field $A_0^{\rm cl}$, expressed in terms of the diagonal traceless matrix Y_N,

$$A_0^{\rm cl} = \frac{2\pi T}{gN} q\, Y_N. \tag{5.93}$$

With spatial A_μ vanishing, $A_i = 0$ for $i = 1, 2, 3$. Y_N is the hypercharge matrix;

$$Y_N = \text{Diag}[\underbrace{N/2, N/2, \ldots, N/2}_{N/2 \text{ entries}}, \underbrace{-N/2, -N/2, \ldots, -N/2}_{N/2 \text{ entries}}] \tag{5.94}$$

Notice how Y_N is split into two sections, each of equal size; one containing positive values of $N/2$, and the other negative. These sections will be referred to as the positive and negative sectors of Y_N respectively. In the context above, q parameterises the 2 different vacua. This can be seen more explicitly using the order parameter with the background field $A_0^{\rm cl}$ (The explicit x dependence is dealt with later).

$$P = \frac{1}{N}\left[\left(N - \frac{N}{2}\right) e^{\frac{2\pi i q}{N}\frac{N}{2}} + \frac{N}{2} e^{\frac{2\pi i q}{N}\left(\frac{N}{2} - N\right)}\right] = \cos \pi q \tag{5.95}$$

Here q is allowed to take one of two integer values, 0 or 1, specifying which of the two vacua are chosen, with $P(q = 0) = 1$ & $P(q = 1) = -1$

After compactification, the theory is effectively 3 dimensional with a temperature. In such a space, a domain interface can be represented as a string-like object moving through time. Consider this interface to sit in an x_1, x_2 plane, with the different vacua sitting at different values of $x_3 \equiv z$. As the vacua sit at integer values of q, non-integer values must therefore sit within the domain interfaces. Allow q to be a function of z, where $q(z = 0) = 0$ and $q(z = L) = 1$, where L is the extent of the interface between neighbouring vacua.

5.5.1 Bi-Fundamental Fermions

The aim is to determine the interface tension of the domain wall in a theory with adjoint gluons and anti-symmetric or symmetric fermions. To simplify the discussion, firstly consider the tensor product of two fundamental representations,

effectively the sum of the symmetric and antisymmetric cases, before focussing on the cases individually. As previously discussed in section 5.2, to perform this calculation the quantum action at one loop is required. At the classical level, as shall be shown, there is no interface solution, so 1-loop effects must be included. The only terms under scrutiny are those involving interactions with the background field, $A_0^{\rm cl}$:

$$S = \int d^3x \int_0^\beta d\tau \left\{ Tr\left[\frac{1}{2}(F_{\mu\nu})^2\right] + Tr\left[\sum_{i=1}^{n_f} \overline{\psi}_i \not{D} \psi_i\right] + \ldots \right\} \tag{5.96}$$

The action includes only a kinetic term for n_f Dirac fermions, with the only interactions through the covariant derivative. Ultimately, the action will be integrated over the fluctuating quantum fields ψ, $A_\mu^{\rm qu}$, and associated ghosts; thus providing an effective potential for the classical gauge fields. Working in Euclidean $\mathbb{R}^3 \times S^1$ space with anti-periodic fermion boundary conditions, consider each part of the action separately.

Gauge Fields

As the gauge fields exist in the adjoint representation, one can simply use the gauge part of the $\mathcal{N} = 4$ SYM action, eq. 5.41, with k set equal to $N/2$. This is equivalent to executing the gauge contribution computation using Y_N as opposed to Y_k. The gauge action with such a replacement becomes:

$$S_A = \frac{4\pi^2 T^3}{\sqrt{3N} g_{YM}} \left(\frac{N}{2}\right)^2 V_{\rm tr} \int_0^{L'} dz' \left[\left(\frac{\partial q}{\partial z'}\right)^2 + q^2(1-q)^2\right] \tag{5.97}$$

Fermionic Term

For the Sym/Asym fermionic term, the calculations will differ significantly from those previously encountered for adjoint fermions. To simplify the discussions, the tensor sum of the symmetric and anti-symmetric representations will be considered first, before considering each irreducible representation separately.

The fermionic term at finite temperature is crucial, as they have anti-periodic boundary conditions around the Euclidean time circle, hence breaking supersymmetry. Taking n_f dirac fermions, the action is simply the kinetic term:

$$S_F = \int d^3x \int_0^\beta d\tau \mathrm{Tr}\left[\sum_{A=1}^{n_f} \overline{\psi}_A \not{D} \psi_A\right] \tag{5.98}$$

As the covariant derivative is no longer in the adjoint representation, but Sym $\oplus$ Asym, the commutator is replaced by a simple additive operator. Looking at individual elements, the commutator is replaced thus:

$$(A_0^{\mathrm{cl}})_{ii}\psi_{ij} - (A_0^{\mathrm{cl}})_{jj}\psi_{ij} \quad \rightarrow \quad (A_0^{\mathrm{cl}})_{ii}\psi_{ij} + (A_0^{\mathrm{cl}})_{jj}\psi_{ij} \tag{5.99}$$

where i & j run from 1 to N, ψ_{ij} is an element of an unconstrained, $N \times N$ matrix for ψ_A It is now obvious that diagonal elements of ψ will contribute, unlike the adjoint gluons and adjoint fermions in section 5.2. Focus for a moment on the q dependency, and the degree of freedom counting.

For the adjoint fermions to contribute a factor of q , i & j must correspond to different sectors of Y_N, therefore contributions only occur when $i \in [1, N/2]$ & $j \in [N/2+1, N]$ and vice-versa, giving a total of $N^2/2$ degrees of freedom, all of which come from off-diagonal elements of ψ.

However, from eq.5.99, two-index fermions contribute a factor only when i & j are in the same sector of Y_N, namely $i, j \in [1, N/2]$ or $i, j \in [N/2+1, N]$. This will give $\frac{1}{2}(N^2 - 2N)$ off diagonal contributions, but unlike the adjoint case, there is an additional N contributions from the diagonal, totalling $N^2/2$ degrees of freedom, as expected. The fermionic action can now be written as

$$S_F = -\int d^3x \int_0^\beta d\tau \frac{N^2}{2} n_f \mathrm{Tr}\left[-((D_0^{\mathrm{cl}})^2 + \mathbf{D}^2)\right] \tag{5.100}$$

As the fermions are specified by anti-periodic boundary conditions around the Euclidean time circle, the Matsubara frequencies suffer a half-integer shift, $n \rightarrow n + \frac{1}{2}, n \in \mathbb{Z}$. The eigenvalues of the covariant derivative, eq. (5.28) are modified thus;

$$iD_0^{\pm} \rightarrow p_0^{\pm} = 2\pi T\left(n + \frac{1}{2} \pm q\right) \tag{5.101}$$

This shift has a non-trivial effect on the effective potential. Under such a shift, the sums over n for p_0^+ and p_0^- are no longer equivalent,and each sum must be evaluated separately.

$$S_F = -\frac{N^2}{2} V_{tr} L T n_f \sum_{n=-\infty}^{+\infty} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \left(\ln\left[(p_0^+)^2 + \mathbf{p}^2\right] + \ln\left[(p_0^-)^2 + \mathbf{p}^2\right] \right) \quad (5.102)$$

As with the gauge term, taking the variation of the action with q;

$$\frac{\partial S_F}{\partial q} = -N^2(2\pi T) V_{tr} L T n_f \sum_{n=-\infty}^{+\infty} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \left(\frac{p_0^+}{(p_0^+)^2 + \mathbf{p}^2} - \frac{p_0^-}{(p_0^-)^2 + \mathbf{p}^2} \right) \quad (5.103)$$

Again, integrating over the spatial momenta $\mathbf{p}$, employing the usual dimensional regularisation;

$$\begin{aligned} \frac{\partial S_F}{\partial q} &= N^2(2\pi T) V_{\mathrm{tr}} L T n_f (\pi T^2) \sum_{n=-\infty}^{+\infty} \left[(n + 1/2 + q) \left| n + 1/2 + q \right| \right. \\ &\quad \left. - (n + 1/2 - q) \left| n + 1/2 - q \right| \right] \end{aligned} \quad (5.104)$$

To regulate with zeta functions, consider the sum in each region of q individually.

$$2 \sum_{n=0}^{+\infty} \left[(n + 1/2 + q)^2 - (n + 1/2 - q)^2 \right], \qquad q \in [0, 1/2] \quad (5.105)$$

$$2 \sum_{n=0}^{+\infty} \left[(n - 1/2 + q)^2 - (n + 3/2 - q)^2 \right], \qquad q \in [1/2, 1] \quad (5.106)$$

Applying a shift $q \leftrightarrow 1 - q$ swaps the two terms, and the associated regions of q. Thus, there is no loss of generality to consider only the region $q \in [0, 1/2]$, including an overall doubling factor. Using the Hurwitz zeta function once more, the fermionic action can be determined explicitly:

$$S_F = -\frac{4\pi^2 T^3}{\sqrt{3N}g} \frac{N^2}{4} V_{\mathrm{tr}} n_f \times 2 \int dz \left(\frac{1}{2} + q \right)^2 \left(\frac{1}{2} - q \right)^2 . \quad (5.107)$$

Notice that the invariance under unit shifts in q is retained, with the two terms interchanging. Upon inspection, one will find that this action is exactly equivalent to that for the adjoint fermion case, when $k = N/2$. The next step is to consider the symmetric and antisymmetric, irreducible representations.

Irreducible representations

For the symmetric and anti-symmetric cases, there is only a small yet significant change to the above computation, and that difference resides with ψ. Previously, the $N\times N$ matrix for ψ was considered entirely general, with no restrictions. For the anti-symmetric case, $\psi_{ij} = -\psi_{ji}$, & $\psi_{ii} = 0$, therefore, i can take on $N/2$ values in each sector, while j can only take on $N/2-1$ values in the same sector, as diagonal terms are vanishing. There is an overall factor of $1/2$ due to the reduced degrees of freedom, which is cancelled by the doubling due to there being two sectors of Y_N, leading to a total of;

$$2\times\frac{1}{2}\times\frac{N}{2}\times\left(\frac{N}{2}-1\right)=\frac{1}{4}N(N-2) \tag{5.108}$$

With the fermionic action modified thus;

$$S_F^{\text{ASym}}=-\frac{4\pi^2T^3}{\sqrt{3Ng}}\left[\frac{N}{8}(N-2)\right]V_{tr}n_f\times 2\int dz\left(\frac{1}{2}+q\right)^2\left(\frac{1}{2}-q\right)^2. \tag{5.109}$$

For the symmetric case, $\psi_{ij} = -\psi_{ji}$ & $\psi_{ii} \neq 0$. Counting off diagonal terms first, i can take on $N/2$ values in each sector, while j, $N/2-1$ values in the same sector. As with the anti-symmetric case, there is a factor of $1/2$ cancelled by the sector doubling. There is the final addition of the diagonal terms, $i = j$, a factor of N;

$$2\times\frac{1}{2}\times\frac{N}{2}\times\left(\frac{N}{2}-1\right)+N=\frac{1}{4}N(N+2) \tag{5.110}$$

$$S_F^{\text{Sym}}=-\frac{4\pi^2T^3}{\sqrt{3Ng}}\left[\frac{N}{8}(N+2)\right]V_{tr}n_f\times 2\int dz\left(\frac{1}{2}+q\right)^2\left(\frac{1}{2}-q\right)^2. \tag{5.111}$$

It is easy to see that the sum of the fermionic terms from the irreducible representations totals to that for the tensor sum action, as required.

5.5.2 Total action

Summing all parts of the action, one can obtain the one-loop effective action for the domain wall;

$$
\begin{aligned}
S_{\text{Total}} &= S_A + S_F && (5.112)\\
&= \frac{4\pi^2 T^3}{\sqrt{3N}g} V_{\text{tr}} 2 \int dz' \left\{ \left(\frac{N}{2}\right)^2 \left[\left(\frac{\partial q}{\partial z'}\right)^2 + q^2(1-q)^2 \right] \right. && (5.113)\\
&\quad \left. -n_f \frac{N}{8}(N+\beta) \left[\left(\frac{1}{2}+q\right)^2 \left(\frac{1}{2}-q\right)^2 \right] \right\} && (5.114)
\end{aligned}
$$

Here β is introduced as a representation parameter, $\beta = -2, +2, +N$ for anti-symmetric, symmetric, and reducible tensor sum representations respectively.

5.5.3 Discussions

In the limit of large N, both the Symm & ASymm actions converge, as expected. It is interesting to note that for the tensor sum ($\beta = N$), the total action is identical to that obtained from a $\mathcal{N} = 1$ SYM computation (namely, the $\mathcal{N} = 4$ action with no scalars, $n_s = 0$, and the number of fermions, n_f left explicit), where k is set to $k = N/2$.

It is hoped that an exact equivalence between $\mathcal{N} = 1$ SYM and a bi-fundamental fermionic theory can be shown and proved. This may prove useful for developing further dualities between $\mathcal{N} = 1$ SYM and theories of the Planar Equivalence (See [76]and references there-in). It must be noted that the results shown here are only preliminary results, and together with further discussions on two index representation domain walls, further work will be released [7].

Chapter 6

Final Discussions

As has been shown in this thesis, the ideas and concepts of higher representational objects provide very interesting and expansive environments for study. These objects have allowed a rich structure of interactions of quarks to become apparent, modelling interactions which are not manifest in real QCD.

The work on k-string tensions in chapter 3 has shown that the methods applied to determine Wilson loop expectation values in conformal $AdS_5 \times S^5$ can also be applied with only minor alterations to non-conformal Hardwall AdS, showing the existence of a true string tension, area law, and thus confinement in this background. This side-by-side comparison also illustrated the transition of the value of the Wilson Loop as it acts an order parameter, shifting from an electrostatic-like $\sim 1/R$ inter-quark potential in $AdS_5 \times S^5$ to a confining phase, linear potential, $\sim R$ in Hardwall AdS.

Universality between the ratios of tensions or Wilson loop pre-factors between the conformal $AdS_5 \times S^5$ & non-conformal Hardwall AdS has been shown, but this is by no means indicative of further universalities between conformal theories, and their possible non-conformal limits or truncations. It has also been shown that k-string tensions are not invariant of background, and in-fact vary significantly, even for different gravity duals of the same field theory. This can be seen by the differing result for the tension in the two duals of $\mathcal{N}=1$ SYM, Klebanov-Strassler and Maldacena-Núñez .

The S-duality invariance on k-string tension computations in $\mathcal{N}=1$ SYM gravity duals was not unexpected, however the differences in computation technique varied

significantly from a technical view point. Shifting from the NSNS sector to the RR, repealed the requirement of selecting the explicit form of the world volume gauge field via gauge quantisation arguments in the DBI action. This provided a more endogenous selection of the world-volume field, determined by the background itself. However, this price was paid for by the inclusion of additional boundary terms due to the variance of this generated gauge field. It is fascinating how two different, separate methods of computation, each an S-dual to the other, are able to produce identical actions, numerical factors and dynamics.

Chapter 4 introduced the concepts of quantum broadening of strings from the 1980's to the framework of the AdS/CFT. Motivated by the first, incomplete attempts by Giudice *et al.*, the correct method of determining the width of k-string flux tubes was found, where the probe of the flux tube was required to be of the same representation as the flux tube itself to correctly capture the k dynamics. Fundamental probe loops did not capture any of the k dynamics apparent in the large N, fixed k/N limit. It was universally found that changing widths from a fundamental to a k-string requires a simple replacement of the fundamental tension with the k tension, $\sigma_f \to \sigma_k$.

From the conformal theory perspective, the method of determining the width of strings breaks down. Connected world-sheet Wilson loop correlators do not exist over the full parameter space, so taking the probe limit is not possible in a connected state. Passing the Gross-Ooguri phase transition to the disconnected state allowed "widths" to be computed, which were invariant of the string tension, and any form of scaling with respect to k or N. This is most likely to be indicative of the lack of a confining flux tube in $AdS_5 \times S^5$, or that this method of string width computation breaks down in the conformal phase.

It would be a fascinating prospect to determine flux tube widths from the weak coupling limit in $\mathcal{N} = 1$ SYM. Would widths in the weak coupling limit exhibit an almost inverse Casimir scaling, or would scaling like $\sim N^2$ still be apparent at the weak limit. This is an open question.

Finally, in chapter 5, the scaling behaviour of the domain walls of finite temperature $\mathcal{N} = 4$ SYM were discussed. It was shown that following the domain wall tension computation of [65], in which k domain walls in pure Yang-Mills were

investigated, the system could be expanded to determine the tension of k domain walls in $\mathcal{N}=4$ SYM, by the inclusion of adjoint scalars and fermions. The added difficulty with this computation was the half-integer shifts of the Matsubara frequencies, this requiring the system to be evaluated in two different regions and eventually summed. The k domain walls where found to exhibit the same Casimir scaling behaviour of k & N at 1-loop, as found in the pure Yang-Mills case, along with the same dependence on the temperature and coupling. This Casimir scaling is expected to remain at the 2-loop level, but there is no reason for it to remain at higher orders. The profile of the domain wall was slightly modified, with the $\mathcal{N}=4$ SYM k-wall being a narrowing interface than that for pure Yang-Mills.

In the strong coupling limit, the k domain wall was found to be described by a wrapped D5-brane in the AdS-Schwarzchild black hole. This is, of course, similar in concept to the k-string described by a wrapped D5-brane in $AdS_5 \times S^5$ from chapter 3. The tension of the k-wall at strong coupling is of course different to that from the weak coupling limit, however the results from both limits share the same dependence on the 't Hooft coupling and the temperature.

Preliminary results show that a possible duality may exist between a finite temperature $\mathcal{N}=1$ SYM theory with 1 adjoint fermion, and a theory with matter in a two-index representation (be it Symmetric, Anti-symmetric, or a reducible tensor product of both) in the large-N limit. This is motivated by the equivalence of the domain wall tensions between the two theories in the large-N limit. Work is still on-going with these models, and results should be released in the foreseeable future.

Bibliography

[1] J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," Adv. Theor. Math. Phys. **2** (1998) 231 [Int. J. Theor. Phys. **38** (1999) 1113] [arXiv:hep-th/9711200].

[2] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, "Large N field theories, string theory and gravity," Phys. Rept. **323** (2000) 183 [arXiv:hep-th/9905111].

[3] G. 't Hooft, "A Planar Diagram Theory for Strong Interactions," Nucl. Phys. B **72** (1974) 461.

[4] J. M. Ridgway, "Confining k-string tensions with D-branes in super Yang-Mills theories," Phys. Lett. B **648** (2007) 76 [arXiv:hep-th/0701079].

[5] A. Armoni and J. M. Ridgway, "Quantum Broadening of k-Strings from the AdS/CFT Correspondence," Nucl. Phys. B **801** (2008) 118 [arXiv:0803.2409 [hep-th]].

[6] A. Armoni, S. P. Kumar and J. M. Ridgway, "Z(N) Domain walls in hot N=4 SYM at weak and strong coupling," JHEP **0901** (2009) 076 [arXiv:0812.0773 [hep-th]].

[7] A. Armoni, C. Korthals-Altes and J. M. Ridgway, *To appear.*

[8] A. Armoni and M. Shifman, "Remarks on stable and quasi-stable k-strings at large N," Nucl. Phys. B **671** (2003) 67 [arXiv:hep-th/0307020].

[9] A. Armoni and B. Lucini, "Universality of k-string tensions from holography and the lattice," JHEP **0606** (2006) 036 [arXiv:hep-th/0604055].

[10] M. Shifman, "k strings from various perspectives: QCD, lattices, string theory and toy models," Acta Phys. Polon. B **36** (2005) 3805 [arXiv:hep-ph/0510098].

[11] C. G. . Callan, A. Guijosa, K. G. Savvidy and O. Tafjord, "Baryons and flux tubes in confining gauge theories from brane actions," Nucl. Phys. B **555** (1999) 183 [arXiv:hep-th/9902197].

[12] A. Armoni and M. Shifman, "On k-string tensions and domain walls in N = 1 gluodynamics," Nucl. Phys. B **664** (2003) 233 [arXiv:hep-th/0304127].

[13] C. A. Doran, L. A. P. Zayas, V. G. J. Rodgers and K. Stiffler, "Tensions and Luscher Terms for (2+1)-dimensional k-strings from Holographic Models," arXiv:0907.1331 [hep-th].

[14] S. A. Hartnoll and S. P. Kumar, "Multiply wound Polyakov loops at strong coupling," [arXiv:hep-th/0603190]

[15] S. Yamaguchi, "Wilson loops of anti-symmetric representation and D5-branes," JHEP **0605** (2006) 037 [arXiv:hep-th/0603208].

[16] N. Drukker and B. Fiol, "All-genus calculation of Wilson loops using D-branes," JHEP **0502** (2005) 010 [arXiv:hep-th/0501109].

[17] S. A. Hartnoll, "Two universal results for Wilson loops at strong coupling," Phys. Rev. D **74** (2006) 066006 [arXiv:hep-th/0606178].

[18] A. Hanany, M. J. Strassler and A. Zaffaroni, "Confinement and strings in MQCD," Nucl. Phys. B **513** (1998) 87 [arXiv:hep-th/9707244].

[19] C. P. Herzog and I. R. Klebanov, "On string tensions in supersymmetric SU(M) gauge theory," Phys. Lett. B **526** (2002) 388 [arXiv:hep-th/0111078].

[20] C. Bachas, M. R. Douglas and C. Schweigert, "Flux stabilization of D-branes," JHEP **0005** (2000) 048 [arXiv:hep-th/0003037].

[21] M. R. Douglas and S. H. Shenker, "Dynamics of SU(N) supersymmetric gauge theory," Nucl. Phys. B **447**, 271 (1995) [arXiv:hep-th/9503163].

[22] B. Lucini and M. Teper, "The k = 2 string tension in four dimensional SU(N) gauge theories," Phys. Lett. B **501** (2001) 128 [arXiv:hep-lat/0012025].

[23] B. Lucini and M. Teper, "Confining strings in SU(N) gauge theories," Phys. Rev. D **64** (2001) 105019 [arXiv:hep-lat/0107007].

[24] L. Del Debbio, H. Panagopoulos, P. Rossi and E. Vicari, "k-string tensions in SU(N) gauge theories," Phys. Rev. D **65** (2002) 021501 [arXiv:hep-th/0106185].

[25] L. Del Debbio, H. Panagopoulos, P. Rossi and E. Vicari, "Spectrum of confining strings in SU(N) gauge theories," JHEP **0201** (2002) 009 [arXiv:hep-th/0111090].

[26] L. Del Debbio, H. Panagopoulos, P. Rossi and E. Vicari, "Spectrum of k-string tensions in SU(N) gauge theories," Nucl. Phys. Proc. Suppl. **106** (2002) 688 [arXiv:hep-lat/0110101].

[27] L. Del Debbio, H. Panagopoulos and E. Vicari, "Confining strings in representations with common n-ality," JHEP **0309**, 034 (2003) [arXiv:hep-lat/0308012].

[28] B. Lucini, M. Teper and U. Wenger, "Glueballs and k-strings in SU(N) gauge theories: Calculations with improved operators," JHEP **0406**, 012 (2004) [arXiv:hep-lat/0404008].

[29] P. Giudice, F. Gliozzi and S. Lottini, "Quantum fluctuations of k-strings: A case study," PoS **LAT2006** (2006) 065 [arXiv:hep-lat/0609055].

[30] B. Bringoltz and M. Teper, "Closed k-strings in SU(N) gauge theories : 2+1 dimensions," Phys. Lett. B **663** (2008) 429 [arXiv:0802.1490 [hep-lat]].

[31] R. C. Myers, "Dielectric-branes," JHEP **9912** (1999) 022 [arXiv:hep-th/9910053].

[32] C. G. Callan and J. M. Maldacena, "Brane dynamics from the Born-Infeld action," Nucl. Phys. B **513** (1998) 198 [arXiv:hep-th/9708147].

[33] G. W. Gibbons, "Born-Infeld particles and Dirichlet p-branes," Nucl. Phys. B **514** (1998) 603 [arXiv:hep-th/9709027].

[34] J. Polchinski, "Dirichlet-Branes and Ramond-Ramond Charges," Phys. Rev. Lett. **75** (1995) 4724 [arXiv:hep-th/9510017].

[35] J. Polchinski, "String Theory,Vol I & II" Cambridge University Press (1998), ISBN 0-521-63303-6, 0-521-67228-7.

[36] B. Zwiebach, "A First Course in String Theory," Cambridge University Press (2004), ISBN 0-521-83143-1.

[37] K. Becker, M. Becker and J. H. Schwarz, "String Theory and M-Theory," Cambridge University Press (2007), ISBN 0-521-86069-5.

[38] C. P. Bachas, "Lectures on D-branes," arXiv:hep-th/9806199.

[39] C. V. Johnson, "D-brane primer," arXiv:hep-th/0007170.

[40] J. M. Maldacena, "Wilson loops in large N field theories," Phys. Rev. Lett. **80** (1998) 4859 [arXiv:hep-th/9803002].

[41] N. Drukker, D. J. Gross and H. Ooguri, "Wilson loops and minimal surfaces," Phys. Rev. D **60** (1999) 125006 [arXiv:hep-th/9904191].

[42] A. Brandhuber, N. Itzhaki, J. Sonnenschein and S. Yankielowicz, "Wilson loops, confinement, and phase transitions in large N gauge theories from supergravity," JHEP **9806** (1998) 001 [arXiv:hep-th/9803263].

[43] S. A. Hartnoll and S. P. Kumar, "Higher rank Wilson loops from a matrix model," JHEP **0608** (2006) 026 [arXiv:hep-th/0605027].

[44] A. Armoni, "Anomalous dimensions from a spinning D5-brane," JHEP **0611** (2006) 009 [arXiv:hep-th/0608026].

[45] I. R. Klebanov and M. J. Strassler, "Supergravity and a confining gauge theory: Duality cascades and chiSB-resolution of naked singularities," JHEP **0008** (2000) 052 [arXiv:hep-th/0007191].

[46] J. M. Maldacena and C. Nunez, "Towards the large N limit of pure N = 1 super Yang Mills," Phys. Rev. Lett. **86** (2001) 588 [arXiv:hep-th/0008001].

[47] A. H. Chamseddine and M. S. Volkov, "Non-Abelian solitons in N = 4 gauged supergravity and leading order string theory," Phys. Rev. D **57** (1998) 6242 [arXiv:hep-th/9711181].

[48] M. Bertolini and P. Merlatti, "A note on the dual of N = 1 super Yang-Mills theory," Phys. Lett. B **556** (2003) 80 [arXiv:hep-th/0211142].

[49] C. P. Herzog, I. R. Klebanov and P. Ouyang, "Remarks on the warped deformed conifold," [arXiv:hep-th/0108101].

[50] M. Luscher, G. Munster and P. Weisz, "How Thick Are Chromoelectric Flux Tubes?," Nucl. Phys. B **180**, 1 (1981).

[51] M. N. Chernodub and F. V. Gubarev, "Confining string and its widening in HP(1) embedding approach," Phys. Rev. D **76** (2007) 016003 [arXiv:hep-lat/0703007].

[52] K. Zarembo, "Wilson loop correlator in the AdS/CFT correspondence," Phys. Lett. B **459** (1999) 527 [arXiv:hep-th/9904149].

[53] P. Olesen and K. Zarembo, "Phase transition in Wilson loop correlator from AdS/CFT correspondence," arXiv:hep-th/0009210.

[54] D. J. Gross and H. Ooguri, "Aspects of large N gauge theory dynamics as seen by string theory," Phys. Rev. D **58** (1998) 106002 [arXiv:hep-th/9805129].

[55] J. Greensite and P. Olesen, "Broadening of the QCD(3) flux tube from the AdS/CFT correspondence," JHEP **0011** (2000) 030 [arXiv:hep-th/0008080].

[56] P. Giudice, F. Gliozzi and S. Lottini, "Quantum broadening of k-strings in gauge theories," JHEP **0701**, 084 (2007) [arXiv:hep-th/0612131].

[57] T. S. Tai and S. Yamaguchi, "Correlator of fundamental and anti-symmetric Wilson loops in AdS/CFT JHEP **0702** (2007) 035 [arXiv:hep-th/0610275].

[58] D. E. Berenstein, R. Corrado, W. Fischler and J. M. Maldacena, "The operator product expansion for Wilson loops and surfaces in the large N limit," Phys. Rev. D **59** (1999) 105023 [arXiv:hep-th/9809188].

[59] D. J. Gross, R. D. Pisarski and L. G. Yaffe, "QCD And Instantons At Finite Temperature," Rev. Mod. Phys. **53** (1981) 43.

[60] L. G. Yaffe and B. Svetitsky, "First Order Phase Transition In The SU(3) Gauge Theory At Finite Temperature," Phys. Rev. D **26**, 963 (1982).

[61] B. Svetitsky and L. G. Yaffe, "Critical Behavior At Finite Temperature Confinement Transitions," Nucl. Phys. B **210**, 423 (1982).

[62] T. Bhattacharya, A. Gocksch, C. Korthals Altes and R. D. Pisarski, "Interface tension in an SU(N) gauge theory at high temperature," Phys. Rev. Lett. **66**, 998 (1991).

[63] T. Bhattacharya, A. Gocksch, C. Korthals Altes and R. D. Pisarski, "Z(N) interface tension in a hot SU(N) gauge theory," Nucl. Phys. B **383**, 497 (1992) [arXiv:hep-ph/9205231].

[64] P. Giovannangeli and C. P. Korthals Altes, "'t Hooft and Wilson loop ratios in the QCD plasma," Nucl. Phys. B **608**, 203 (2001) [arXiv:hep-ph/0102022].

[65] P. Giovannangeli and C. P. Korthals Altes, "Spatial 't Hooft loop to cubic order in hot QCD," Nucl. Phys. B **721**, 1 (2005) [arXiv:hep-ph/0212298].

[66] C. P. Korthals Altes, "Constrained effective potential in hot QCD," Nucl. Phys. B **420**, 637 (1994) [arXiv:hep-th/9310195].

[67] C. Korthals-Altes, A. Kovner and M. A. Stephanov, "Spatial 't Hooft loop, hot QCD and Z(N) domain walls," Phys. Lett. B **469**, 205 (1999) [arXiv:hep-ph/9909516].

[68] P. de Forcrand, B. Lucini and M. Vettorazzo, "Measuring interface tensions in 4d SU(N) lattice gauge theories," Nucl. Phys. Proc. Suppl. **140**, 647 (2005) [arXiv:hep-lat/0409148].

[69] F. Bursa and M. Teper, "Casimir scaling of domain wall tensions in the deconfined phase of D = 3+1 SU(N) gauge theories," JHEP **0508**, 060 (2005) [arXiv:hep-lat/0505025].

[70] P. de Forcrand, B. Lucini and D. Noth, "'t Hooft loops and perturbation theory," PoS **LAT2005**, 323 (2006) [arXiv:hep-lat/0510081].

[71] E. Witten, "Anti-de Sitter space and holography," Adv. Theor. Math. Phys. **2** (1998) 253 [arXiv:hep-th/9802150].

[72] O. Aharony and E. Witten, "Anti-de Sitter space and the center of the gauge group," JHEP **9811** (1998) 018 [arXiv:hep-th/9807205].

[73] C. Korthals-Altes and A. Kovner, "Magnetic Z(N) symmetry in hot QCD and the spatial Wilson loop," Phys. Rev. D **62** (2000) 096008 [arXiv:hep-ph/0004052].

[74] C. P. Korthals Altes, "Spatial 't Hooft loop in hot SUSY theories at weak coupling," arXiv:0904.3117 [hep-ph].

[75] P. Giovannangeli and C. P. Korthals Altes, "Spatial 't Hooft loop to cubic order in hot QCD. II," Nucl. Phys. B **721** (2005) 25 [arXiv:hep-ph/0412322].

[76] A. Armoni and M. Shifman, "Planar equivalence 2006," arXiv:hep-th/0702045.

Appendix A

Combinatorics for 2-loop YM domain wall

Case I: $C_a, C_b \neq 0$

C_a and C_b are only non-zero when a & b represent ladder generators. The remaining index on $f^{a,b,c}$, c is left unchosen. There are two possible subcases

c is diagonal

$$f^{ij,ji,d} f^{ij,ji,d} B_2(C_{ij}) B_2(C_{ji}) \tag{A.1}$$

i, j must be in separate sectors of Y_k for C_{ij} and C_{ji} not to vanish. If C_i is let to take on k, there are $N-k$ possible choices, while C_j is forced to be $k-N$, for which there are k choices. And as i and j can swap sectors, there is a doubling effect. Totalling all possible choices for ij;

$$\rightarrow 2k(N-k) B_2(q)^2 \tag{A.2}$$

Due to the even quality of B_2, any sign change from $C_{ij} = -C_{ji}$ is irrelevant.

a, b & c are off-diagonal

$$f^{il,lj,ji} f^{il,lj,ji} B_2(C_{il}) B_2(C_{lj}) \tag{A.3}$$

i, j sit in the same sector of Y_k, while l sits in the opposing sector. Letting C_i be k, there are $N-k$ choices as in the diagonal case. However, as C_j sits in the same sector, there are only $N-k-1$ choices. C_l is $k-N$ and has k choices. In a similar vein for the swapped sectors, this gives a total of

$$\rightarrow [k(N-k)(N-k-1) + k(k-1)(N-k)]\, B_2(q)^2 \tag{A.4}$$

Case II: $C_a = 0, C_b \neq 0$ and $C_a \neq 0, C_b = 0$

For this case, either a or b is diagonal, or a and b are off-diagonal, but either C_a or C_b has both indices defined in the same sector.

a or b is diagonal

$$f^{ij,d,ji} f^{ij,d,ji} B_2(C_{ij}) B_2(C_d) \quad \text{or} \quad f^{d,ij,ji} f^{d,ij,ji} B_2(C_d) B_2(C_{ij}) \tag{A.5}$$

Following equivalent combinatorics to the diagonal case in I , plus an additional factor of 2 from $a \leftrightarrow b$;

$$\rightarrow 4k(N-k) B_2(q) B_2(0) \tag{A.6}$$

a, b & c are off-diagonal

$$f^{il,lj,ji} f^{il,lj,ji} B_2(C_{il}) B_2(C_{lj}) \tag{A.7}$$

Here i and j are in different sectors, therefore forcing l to be in a matching sector to one of them. Thus, either C_{il} or C_{lj} will vanish. Swapping sectors for i and j gives a factor of 2.

$$\rightarrow 2\,[k(N-k)(N-k-1) + k(k-1)(N-k)]\, B_2(q) B_2(0) \tag{A.8}$$

Total of I & II

Summing all non-vanishing, q dependent terms from both cases provides:

$$\begin{aligned}
\rightarrow \quad & (2k(N-k) + [k(N-k)(N-k-1) + k(k-1)(N-k)])\, B_2(q)^2 \\
& + (4k(N-k) + 2\,[k(N-k)(N-k-1) + k(k-1)(N-k)])\, B_2(q)B_2(0) \\
= \quad & Nk(N-k)\left[B_2(q)^2 + 2B_2(q)B_2(0)\right] \qquad \text{(A.9)}
\end{aligned}$$

Casimir scaling remains at 2-loops in pure Yang-Mills. There are two main factors which lead to the Casimir-like scaling: Firstly only non-trivial q-dependence arises from either one, or both of C_a & C_b being non-zero. Secondly, and more importantly, the combined function $B_2(C_a)B_2(C_b)$ is even. Thus for the cases where the indices of f^2, a, b & c are off-diagonal (as in both of the cases above) the two contributions from l being in different sectors sum to give the scaling. Explicitly, consider a general function of C_a and C_b, $H(C_a, C_b)$, where all q-dependence vanishes only for $H(0,0)$. Focussing on the analogous arguments to Case I:

$$f^{ij,ji,d} f^{ij,ji,d} H(C_{ij}, C_{ji}) = k(N-k)\,[H(q,-q) + H(-q,q)] \qquad \text{(A.10)}$$

The diagonal contribution produces the Casimir scaling. However, for the off-diagonal contributions:

$$f^{il,lj,ji} f^{il,lj,ji} H(C_{il}, C_{lj}) = k(N-k)\,[(N-k-1)H(q,-q) + (k-1)H(-q,q)] \qquad \text{(A.11)}$$

Due to the non-even properties of $H(C_a, C_b)$, there remain additional N factors together with the overall $k(N-k)$.

The result, eq.A.9, is the pure Yang-Mills result, however there is a plausibility argument that the Casimir scaling remains at 2-loops for $\mathcal{N}=4$ SYM.