Monopoles and complex curves.

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Monopoles and Complex Curves

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This dissertation is submitted in partial
fulfillment of the requirements for the degree of

Doctor of Philosophy

of the University of Wales.

Department of Physics
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Submitted: December 2002

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This work has not previously been accepted for any degree and is not being concurrently submitted in candidature for any degree.

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ACKNOWLEDGEMENTS

I would like to thank my doctoral advisor Professor David Olive for his unusual patience, encouragement, continuous support and for enlightning discussions that we had throughout the course of my PhD studies. Particularly, I would like to express my profound gratitude to him for teaching me many things, sharing his insight and ideas with me and for his permanent advice in the early difficult stage of my research. I am most grateful to him for having faith in me and allowing me complete freedom to wander about in mathematical physics and find my own direction.

I would also like to thank my friends in Swansea for making my stay an enjoyable one, I would like to make particular thanks to Ramzi Al-Saedi, Osama Al-Ayooni, Ahmed Al-Saedi, Saleh Al-Meziel, Maree Al-Bishi, Mahmood Al-Hinai, Sanjeev Shukla, Veronica Dare, Brett Taylor, and Edwin Beggs for their friendship and help throughout the three years.

I would also like to thank my friends outside Swansea, Hania (my sister), Sameer Al-Lamki, Falleh Al-Solami, Andras Asztalos, Mark Emmett , Salman Raza, Domenic Fumo, Peter Johnson and Joseph Sestan. There are many other people in Swansea and Waterloo that I would like to thank but it would take up too much time but then they know who they are anyway.

Finally and by no means least, I would like to thank my parents Raghda Hashem and Mohammad Mahassen, to whom this thesis is dedicated, for their constant encouragement and support which I cannot describe in words.

Abstract

In this thesis I study the geometry of the monopole equations on complex curves. The moduli space is shown to be compact and carries a Kähler structure. Zero dimensional moduli spaces are investigated along with their relation to the uniformization theorem. I then study the non-abelian analogue of the equations which can be regarded as a generalization of Yang-Mills theory on Riemann surfaces. Possible relations with three dimensional topology are discussed. Some background material is provided, including an introduction to differential and spin geometry on complex curves and four-manifolds.

Contents

1 Introduction

In the classical era, associated with names such as Newton and Gauss, little distinction was made between mathematics and physics. The beginning of the modern era however saw a growing separation between these two disciplines. The two, of course, still influenced each other. For example, the theory of electromagnetism developed by Maxwell in the late nineteenth century was one of the origins of Hodge theory. Another known example of this interaction is Einstein's theory of general relativity and Lorentzian differential geometry. From this period onwards, the divergence between physics and mathematics was accentuated. It is only in the last twenty five years or so that the two have come closer together again. This time, understanding physical questions has made a profound impact on contemporary mathematics.

Most people who haven't been trained in physics probably think that the physicist's job is to do incredibly complicated calculations, but this is not really the essence of it. As Witten once said [6], "physics is about concepts, wanting to understand the concepts, the principles by which the world works". Understanding the physical concepts is providing a rich source of inspiration and insight in mathematics. For example the equations which the physicists, even after many approximations, arrive at for their description of the fundamental particles, make such good sense in geometry and topology and are indeed so inevitable that it is a scandal that the mathematicians had not studied them in their own right years ago.

During the 1980's, Simon Donaldson used the Yang-Mills equations, which were discovered in mathematical physics in 1954 [34], to study the differential topology of four-manifolds [9]. These equations are nonabelian and nonlinear and represent a generalization of Maxwell's equations of electromagnetism. Donaldson's theory is elegant and beautiful, but to understand the properties of the space of solutions and give detailed proofs of theorems required a tremendous amount of nonlinear analysis. At our present state of knowledge, understanding smooth structures on four-manifolds seems to require nonlinear as opposed to linear PDE's. It is therefore quite surprising that there is a set of PDE's which are easier to study than the Yang-Mills equations, but can yield most of the important results from Donaldson's theory. These are the Seiberg-Witten monopole equations (not to be confused with the BPS monopoles [2]) introduced by the physicist Edward Witten in the fall of 1994 to tackle problems in four-dimensional topology [33]. Instead of considering the SU(2) Yang-Mills equations, one can study a nonlinear equation (and a linear Dirac equation) with an abelian gauge group. The differential geometry used by Witten is the geometry of spin^c structures, such structures exist on any oriented four-manifold [11].

The origin of the monopole equations came, of course, from quantum field theory, in particular $N = 2$ supersymmetric Yang-Mills theory. Seiberg and Witten [25]-[27] noticed that the quantum field theory involves a parameter u parametrizing a family of elliptic curves which degenerate at $u = \pm 1$ and $u = \infty$. The surprise was that the region near $u = +1$, which is one of the strongly coupled region of the $SU(2)$ theory (the other being at $u = -1$), turned out to be equivalent to a weakly coupled region of abelian gauge fields and spinors. This used the notion of duality introduced in 1977 by

Montonen and Olive [19] in the context of nonabelian gauge theory. This formulation in terms of dual variables makes manifest various properties of the Donaldson invariants. As Witten explained in his paper [33], one gets new proofs of some of the basic results of Donaldson theory; one gets a new description of the basis classes of Kronheimer and Mrowka [14],[16] in terms of zero-dimensional moduli spaces and thus a better understanding of the simple type condition of four-manifolds.

On a four-manifold X one can define classical invariants such as the fundamental group $\pi_1(X)$, the cohomology groups $H^k(X, \mathbf{Z})$ over the integers and so forth. These have been around for a long time and have been studied extensively by mathematicians. Both Donaldson and Seiberg-Witten theories give rise to new invariants of compact smooth four-manifolds with no boundary with $b_2^+ \geq 1$ where b_2^+ is the dimension of the space of self-dual harmonic two-forms on the manifold. These invariants depend upon the differentiable structure, not just the topology and can be thought of as "quantum" invariants since they originate from equations arising in quantum field theory.

So far we have talked about gauge theory in four dimensions. It is natural to see also what the gauge theory point of view produces in two dimensions. The purpose of this thesis is to study the geometry of the monopole equations on complex curves. For us, a complex curve (or a Riemann surface) is a nonsingular compact two-dimensional manifold without a boundary. As we will see, the moduli space of solutions turns out to be a space with an extremely rich geometric structure which will be the focus of our study. The equations we consider relate a pair of objects: a connection A on a complex line bundle L over a Riemann surface Σ_g with Kähler form w, and a section of the positive spin^c bundle ψ. The equations are given by

$$\begin{aligned} D_A\psi &= 0 \\ F_A &= -\frac{1}{2}|\psi|^2 w \end{aligned}$$

These equations are close cousins of the self duality equations introduced by Hitchin [12] and the vortex equations on Riemann surfaces [5]. (See also [21] and [22] where the reduction of the four dimensional Seiberg-Witten equations to two dimensions is analyzed). Other approaches to two dimensional Seiberg-Witten theory can be found in [18] and [24].

The second chapter of the thesis gives a brief review of standard tools in two-dimensional geometry and topology namely complex curves, vector bundles, connections, curvature, gauge transformations chern classes and Hodge theory. In chapter three I give a short introduction to the geometry of spin and spin^c structures on complex curves. Until recently, this geometry appeared unfamiliar and strange to many geometers, although spinors have long been regarded as important in physics.

In the first section of chapter four I briefly describe the equations and their origin. Next, I prove a vanishing theorem related to the solutions of the monopole equations. This uses a Weitzenböck technique to show that certain sections of line bundles must necessarily be zero in the presence of a solution to the equations. In section two I study the geometry of the moduli space of solutions $\mathcal{M}$ of the monopole equations. The Atiyah-Singer index theorem is used to show that the dimension of $\mathcal{M}$ is given by $2(c_1(L)[\Sigma_g]+g-1)$, where $c_1(L)[\Sigma_g]$ is the monopole charge and g is the genus of the Riemann surface. I then show that $\mathcal{M}$ is not only compact but also carries a Kähler structure.

Section three is devoted to the relation between the existence of non-empty zero dimensional monopole moduli spaces and one of the most important theorems in complex analysis, the uniformization theorem which states that every compact Riemann surface of genus $g > 1$ admits a metric of constant negative curvature. In section four I present the nonabelian version of the monopole equations and study the properties of the associated moduli space. I then show that the moduli space of unitary Yang-Mills fields [1] and that of the abelian monopole equations appear as singular regions of the nonabelian moduli space. This permits us to speculate on a "dual" interpretation of unitary Yang-Mills fields over a Riemann surface. This duality is discussed briefly in relation with three dimensional topology in section five.

Finally, in chapter five, as a service to those who are interested but feel intimidated to read the mathematical literature, I present an elementary introduction to Donaldson and Seiberg-Witten theories on four-manifolds.

2 Differential Geometry on Complex Curves

2.1 Complex Curves

Complex curves (Riemann surfaces) are compact, oriented nonsingular two real dimensional manifolds without a boundary. They are classified by their genus g which is essentially the number of holes in the surface. For example when $g = 0$ we get the sphere S^2, $g = 1$ corresponds to the torus T^2 and so on. We will denote a complex curve by Σ_g and give it a local complex coordinate

$$z = x + iy.$$

Define a complex multiplication by

$$J\frac{\partial}{\partial x} = \frac{\partial}{\partial y}\,,\ J\frac{\partial}{\partial y} = -\frac{\partial}{\partial x}.$$

The map J is called a complex structure and satisfies the identity

$$J^2 = -I.$$

Let $\Omega^p(\Sigma_g)$ denote the space of real p-forms on the Riemann surface. The exterior derivative acts on $\Omega^0(\Sigma_g)$ (i.e. the space of smooth real functions) by

$$d : \Omega^0(\Sigma_g) \to \Omega^1(\Sigma_g) \text{ by } \quad df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

df is called the differential of the function f. The operator d can be extended to act on higher forms and satisfies

$$d(\theta_1 \wedge \theta_2) = d\theta_1 \wedge \theta_2 + (-1)^p \theta_1 \wedge d\theta_2 \ \text{ for } \theta_1 \in \Omega^p(\Sigma_g).$$

We mention here that the p-th de Rham cohomology group $H^p(\Sigma_g, \mathbf{R})$ is a topological invariant of the manifold and is given by the set of equivalence classes $[\theta]$ of closed p-forms that differ by exact forms. On a complex curve d splits into two components

$$d = \partial + \overline{\partial} = dz\frac{\partial}{\partial z} + d\overline{z}\frac{\partial}{\partial \overline{z}}$$

In terms of x and y we have the identities

$$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}) \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$$

In general if we set $m + n = p$ we can identify the space of (complex valued) p-forms with the space of (m, n)-forms. These are p-forms spanned by $dz_1 \wedge \ldots \wedge dz_m \wedge d\overline{z_1} \wedge \ldots \wedge d\overline{z_n}$. Note that on a complex curve, m and n are either zero or one. For example a (1, 1)-form is a complex-valued differential form which can be expressed in local coordinate z as

$$f_{z\overline{z}} dz \wedge d\overline{z}.$$

Since d squares to zero, we have the following identities on Σ_g

$$\partial^2 = \partial\overline{\partial} + \overline{\partial}\partial = \overline{\partial}^2 = 0.$$

Before we start discussing metrics, let us recall that a dual vector vector is a linear object that maps a vector to a scalar. This may be generalized to multilinear objects called tensors, which map several vectors and dual vectors to a scalar. Let V be a vector space and V^* be its dual, then a tensor T of type (m, n) is a multilinear map that maps m dual vectors and n vectors to $\mathbf{R}$.

$$T : \otimes^m V^* \otimes^n V^n \to \mathbf{R}$$

where $\otimes$ is called the tensor product.

In terms of the local coordinates x and y, the metric g on the Riemann surface can be written as

$$g = g_{xy} dx \otimes dy.$$

So far we have used x and y as our coordinates and thus dx and dy as our basis of one-forms. There is however an alternative choice of one-forms such that the components of the metric reduce to the components of the identity matrix δ_{ij}. This basis is a non-coordinate basis spanned by the one-forms e^1 and e^2, in which the metric becomes

$$g = \delta_{ij} e^i \otimes e^j.$$

In elementary geometry, the inner product between two vectors u and v is defined by $< u, v >= u^i \delta_{ij} v^j = u^i v^i$ where u^i and v^i are the components of the two vectors. In Riemannian geometry we have instead $< u, v >= u^i g_{ij} v^j$. Recall that a Riemannian metric $<,>$ on a Riemann surface is called Hermitian if

$$< Ju, Jv >=< u, v > \quad \forall\, u, v$$

where u and v live in the tangent space of the manifold. In this case we can define (the components of) the Kähler form on Σ_g by

$$w(u, v) =< Ju, v >$$

To see that the two-form w is antisymmetric notice that

$$w(u,v) =< Ju, v >=< v, Ju >= - < J^2 v, Ju >= - < Jv, u >= -w(v, u).$$

In terms of non-coordinate bases, we get

$$< Ju, v >= u_1 v_2 - u_2 v_1.$$

Thus the components of the Kähler two-form are given by

$$w_{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Since any two-form is written locally as

$$w = \frac{1}{2} w_{ij} e^i \wedge e^j$$

we see that in terms of non-coordinate bases, w is given by

$$w = e^1 \wedge e^2.$$

Since the Kähler form is closed (any two-form in two dimensions is closed), we see that a complex curve is an example of a Kähler manifold.

2.2 Vector Bundles

A vector bundle E over a complex curve (Riemann surface) looks locally like the product of a vector space E_p times the curve Σ_g. When talking about a vector space we have to specify a ground field, for example if the field is $\mathbf{R}$ and the vector space has dimension k, the vector bundle will look locally like $\Sigma_g \times \mathbf{R}^k$. To construct the bundle we start with an open covering $\{U_\alpha\}$ of Σ_g and transition functions

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbf{R})$$

where greek letters α, β etc belong to an index set, an open covering is roughly speaking a collection of two-dimensional open intervals and $GL(k, \mathbf{R})$ is the group of non-singular $k \times k$ matrices. The integer k is called the rank of the bundle.

The transition functions by definition satisfy the cocycle condition

$$g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma} \text{ on } U_\alpha \cap U_\beta \cap U_\gamma.$$

Hence on U_α we have

$$g_{\alpha\alpha}g_{\alpha\alpha} = g_{\alpha\alpha}$$

so $g_{\alpha\alpha}$ is the identity matrix I on U_α. Hence on the overlap $U_\alpha \cap U_\beta$ the transition functions are related by

$$g_{\alpha\beta}g_{\beta\alpha} = g_{\alpha\alpha} = I.$$

Since E is locally the product of Σ_g and a vector space, a typical element of E can be written as (p, v). On $U_\alpha \cap U_\beta$ two such elements are related by

$$p = q, \quad v = g_{\alpha\beta}(p)w.$$

The projection map π acts on E by projecting to the base manifold Σ_g

$$\pi : E \to \Sigma_g \text{ by } \pi(p, v) = p$$

The fiber of the vector bundle E at the point p is the vector space

$$E_p = \pi^{-1}(p) = \mathbf{R}^k.$$

Complex vector bundles are defined similarly, with $\mathbf{C}^k$ replacing $\mathbf{R}^k$ and the transition functions taking their values in $GL(k, \mathbf{C})$ instead of $GL(k, \mathbf{R})$. The fiber in this case is a k dimensional complex vector space. A complex vector bundle of rank one is also called a (complex) line bundle.

Examples of vector bundles are the tangent bundle $T\Sigma_g$ (with fiber the tangent space), the cotangent bundle $T^*\Sigma_g$ (with fiber the cotangent space), tensor bundles $E \otimes E'$, $E^m = E \otimes E \otimes \ldots \otimes E$ (m times), and so forth. Note that if E and E' are vector bundles over Σ_g, the tensor product bundle $E \otimes E'$ is obtained by assigning the tensor product of fibers $F_p \otimes F_p'$ to each point $p \in \Sigma_g$. If $\{e_i\}$ and $\{e_j'\}$ are bases of F and F', $F \otimes F'$ is spanned by $\{e_i \otimes e_j'\}$ and hence

$$\text{rank } E \otimes E' = \text{rank } E \times \text{rank } E'.$$

The vector bundles we are interested in are G-vector bundles where G is a Lie subgroup of the general linear group. Suppose for example that G is the unitary group $U(k)$. A Hermitian metric on a unitary bundle assigns to every point $p \in \Sigma_g$ a map

$$<,>_p : E_p \times E_p \to \mathbf{C}$$

satisfying (in an appropriate basis) the following condition

$$< v, w >_p = \overline{v_1} w_1 + \ldots + \overline{v_k} w_k.$$

From this it follows that

$$\begin{aligned} < v, w >_p &= \overline{< w, v >_p} \\ < v, v >_p &\geq 0, \text{with equality holding when } v = 0. \end{aligned}$$

In the case where G is the orthogonal group $O(k) \subset GL(k, \mathbf{R})$, the bundle E inherits an inner product

$$<,>_p: E_p \times E_p \to \mathbf{R}.$$

A section σ of a vector bundle E over a complex curve Σ_g is a map

$$\sigma : \Sigma_g \longrightarrow E \text{ by } \sigma(p) = (p, v = \sigma_\alpha(p))$$

where σ_α is a map from U_α to the fiber. A section of a vector bundle satisfies

$$\pi \circ \sigma = \text{identity}.$$

From now on we will denote by $\Gamma(E)$ the space of space of sections of a vector bundle E over our base space Σ_g. For example the space of real p-forms on the Riemann surface is given by

$$\Omega^p(\Sigma_g) = \Gamma(\Lambda^p(T^*\Sigma_g))$$

where $\Lambda^p(T^*\Sigma_g)$ (sometimes written as Λ^p) is called the bundle of p-forms on the complex curve. In general if the real forms take their values in the Lie algebra of G we have

$$\Omega^p(E) = \Gamma(\Lambda^p(T^*\Sigma_g) \otimes E).$$

2.3 Connections

A connection is a geometrical structure on G-vector bundles. It is a matrix valued one-form taking its value in the Lie algebra of the structure group G. For example, if E is a $O(k)$-bundle, an orthogonal connection A on E is a one-form whose local representative A_α takes its value in $o(k)$ - the space of $k \times k$ anti-symmetric matrices. Similarly, if E is a $U(k)$ bundle, a unitary connection A on E is a connection such that A_α takes its values in $u(k)$, where $u(k)$ is the space of $k \times k$ anti-Hermitian matrices. A covariant derivative is defined by

$$d_A = d + A$$

where d is the exterior derivative on the base manifold.

Let E be a G-vector bundle over Σ_g defined by the open covering $\{U_\alpha\}$ and the transition functions $g_{\alpha\beta}$. On U_α, a section $\sigma \in \Gamma(E)$ has local representative σ_α given by

$$\sigma_\alpha = \begin{pmatrix} \sigma_{1\alpha} \\ \vdots \\ \sigma_{k\alpha} \end{pmatrix} : U_\alpha \to (\mathbf{R}^k \text{ or } \mathbf{C}^k)$$

where σ_α is a k-tuple of functions on U_α. The covariant derivative is given by

$$d_A \begin{pmatrix} \sigma_{1\alpha} \\ \vdots \\ \sigma_{k\alpha} \end{pmatrix} = \begin{pmatrix} d\sigma_{1\alpha} \\ \vdots \\ d\sigma_{k\alpha} \end{pmatrix} + \begin{pmatrix} A_{11\alpha} & \dots & A_{1k\alpha} \\ \vdots & \dots & \vdots \\ A_{k1\alpha} & \dots & A_{kk\alpha} \end{pmatrix} \begin{pmatrix} \sigma_{1\alpha} \\ \vdots \\ \sigma_{k\alpha} \end{pmatrix}$$

This equation can be written in a more abbreviated way

$$d_A \sigma_\alpha = d\sigma_\alpha + A_\alpha \sigma_\alpha .$$

We say that a connection is trivial when $A = 0$. In this case the covariant derivative reduces to the exterior derivative on the manifold,

$$d_A \begin{pmatrix} \sigma_{1\alpha} \\ \vdots \\ \sigma_{k\alpha} \end{pmatrix} = \begin{pmatrix} d\sigma_{1\alpha} \\ \vdots \\ d\sigma_{k\alpha} \end{pmatrix}.$$

The covariant derivative can be regarded as a map from bundle valued zero-forms to one-forms,

$$d_A : \Omega^0(E) \rightarrow \Omega^1(E)$$

and can be extended to all p-forms.

As an example let $E = T\Sigma_g$ be the tangent bundle of a complex curve Σ_g. This bundle can be thought of as a complex line bundle with structure group $U(1) \cong SO(2)$. Let M be the generator of the $SO(2)$ group

$$M = \begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{pmatrix}$$

where θ is a real function on the base space. Linearizing we get

$$M \approx I + \theta m$$

where I is the identity matrix and m is the generator of the $so(2)$ Lie algebra given by

$$m = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The Levi-Civita connection W is constructed as follows. Let W^i_{jk} be the components of the connection, here i and j are $so(2)$ indices. The connection one-form $W \in \Omega^1 \otimes so(2)$ is then given by

$$W = \begin{pmatrix} 0 & W^1_{21} \\ -W^1_{21} & 0 \end{pmatrix} e^1 + \begin{pmatrix} 0 & W^1_{22} \\ -W^1_{22} & 0 \end{pmatrix} e^2$$

If we let $W^i_j \equiv \delta^{ik} W_{kj}$ be the components of the matrix W then in two dimensions we have the relations

$$W^1_2 = -W^2_1, \quad W^1_1 = W^2_2 = 0.$$

These components can be calculated using the first structural equation

$$de^i = -W^i_j \wedge e^j.$$

2.4 Curvature

The curvature of a vector bundle E is a matrix valued two-form given by

$$\Omega = d^2_A$$

Unlike the exterior derivative, the square of the covariant derivative is not necessarily zero. If we cover the complex curve by the open covering $\{U_\alpha\}$ and let σ_α be local representatives of sections, we have

$$\begin{aligned} d^2_A \sigma_\alpha &= (d + A_\alpha)(d\sigma_\alpha + A_\alpha \sigma_\alpha) \\ &= d^2\sigma_\alpha + (dA_\alpha)\sigma_\alpha - A_\alpha d\sigma_\alpha + A_\alpha d\sigma_\alpha + A_\alpha \wedge A_\alpha \sigma_\alpha \\ &= (dA_\alpha + A_\alpha \wedge A_\alpha)\sigma_\alpha, \end{aligned}$$

and hence

$$\Omega_\alpha = dA_\alpha + A_\alpha \wedge A_\alpha.$$

In the case of an orthonormal connection on a $O(k)$-bundle, it follows directly from the equation above that the matrix Ω_α is antisymmetric. In the case of a unitary connection on a $U(k)$-bundle, the matrices Ω_α are anti-Hermitian.

The case of complex line bundles is particularly important. In this case the matrices Ω_α are all 1×1 and, as we will see in the next section, the gauge transformation formula implies that $\Omega_\alpha = \Omega_\beta$ on overlaps. Thus the Ω_α's fit together to make a globally defined purely imaginary two-form on Σ_g. We can write $\Omega = -iF_A$ where F_A is a real-valued two-form. Differentiation of Ω yields the Bianchi identity:

$$dF_A = 0.$$

As an example, let us consider the case of the tangent bundle over a complex curve. The connection one-form is defined by the first structural equation

$$de^i = -W^i_j \wedge e^j$$

The components Ω^i_j of the curvature (matrix) two-form are defined by the second structural equation

$$\Omega^i_j = dW^i_j + W^i_k \wedge W^k_j.$$

The coefficients of the curvature two-form in the $e^k \wedge e^l$ basis are the components of the Riemann tensor,

$$\Omega^i_j = \frac{1}{2} R^i_{jkl} e^k \wedge e^l.$$

Since the only nonvanishing components of the connections are W^1_2 andW^2_1, the term $W^i_k \wedge W^k_j$ vanishes. The components Ω^i_j are then given by

$$\Omega^i_j = dW^i_j \equiv K e^1 \wedge e^2$$

where K is called the Gaussian curvature and $e^1 \wedge e^2$ can be thought of as the area two-form of the manifold. The Ricci tensor R_{ij} of the Riemann surface is defined by

$$R_{ij} \equiv R^k_{ikj}.$$

From this we can define an important invariant of complex curves Σ_g, the scalar curvature s, given by the formula

$$s \equiv R_{ii}.$$

2.5 Gauge Transformations

The gauge principle states that physics should not depend on how we describe it. This means that the functional describing the gauge theory should be invariant under gauge transformations. Suppose that E is a (real or complex) vector bundle of rank k over Σ_g and transition functions $g_{\alpha\beta}$. In gauge theory, $g_{\alpha\beta}$ represents an element of the gauge group. Any element $\sigma \in \Gamma(E)$ has local representatives σ_α related on $U_\alpha \cap U_\beta$ by

$$\sigma_\alpha = g_{\alpha\beta}\sigma_\beta.$$

Writing the above condition as

$$\sigma_\beta = g^{-1}_{\alpha\beta}\sigma_\alpha$$

we see that covariant derivatives on the overlap $U_\alpha \cap U_\beta$ satisfy

$$\begin{aligned} d\sigma_\alpha + A_\alpha\sigma_\alpha &= g_{\alpha\beta}(d\sigma_\beta + A_\beta\sigma_\beta) \\ &= g_{\alpha\beta}[d(g^{-1}_{\alpha\beta}\sigma_\alpha) + A_\beta g^{-1}_{\alpha\beta}\sigma_\alpha] \\ &= d\sigma_\alpha + (g_{\alpha\beta}dg^{-1}_{\alpha\beta} + g_{\alpha\beta}A_\beta g^{-1}_{\alpha\beta})\sigma_\alpha. \end{aligned}$$

Thus we conclude that the connection is given by

$$A_\alpha = g_{\alpha\beta} d g_{\alpha\beta}^{-1} + g_{\alpha\beta} A_\beta g_{\alpha\beta}^{-1} \text{ on } U_\alpha \cap U_\beta.$$

The curvature (matrix) two-form satisfies

$$\Omega_\alpha \sigma_\alpha = g_{\alpha\beta} \Omega_\beta \sigma_\beta = g_{\alpha\beta} \Omega_\beta g_{\alpha\beta}^{-1} \sigma_\alpha \text{ on } U_\alpha \cap U_\beta.$$

Thus the transformation rule for Ω is given by

$$\Omega_\alpha = g_{\alpha\beta} \Omega_\beta g_{\alpha\beta}^{-1}.$$

The case of complex line bundles L is particularly important. In this case a gauge transformation of L is just a smooth map g

$$g : \Sigma_g \quad \rightarrow \quad S^1$$

where S^1 is regarded as the complex numbers of length one. On the overlap $U_\alpha \cap U_\beta$ we have

$$\begin{aligned} A_\alpha &= A_\beta + g_{\alpha\beta} d g_{\alpha\beta}^{-1} \\ \Omega_\alpha &= \Omega_\beta. \end{aligned}$$

Thus the curvature Ω_α's fit together to make a globally defined real two-form on Σ_g. Let A_0 be an abelian connection on a complex line bundle L chosen as a base point, then any other connection A on L can be written in the form

$$A = A_0 - ia \text{ where } a \in \Omega^1(\Sigma_g).$$

On the overlap $U_\alpha \cap U_\beta$ we have

$$\begin{aligned} A_{0\alpha} &= A_{0\beta} + g_{\alpha\beta} d g_{\alpha\beta}^{-1} \\ A_{0\alpha} - i a_\alpha &= A_{0\beta} - i a_\beta + g_{\alpha\beta} d g_{\alpha\beta}^{-1} \end{aligned}$$

Thus on the overlap $U_\alpha \cap U_\beta$ we get

$$a_\alpha = a_\beta$$

so the a_α's fit together into a real-valued one-form a on Σ_g. In other words, the space of abelian connections is isomorphic to the space of one-forms. Note that constant gauge transformations act trivially on the space of abelian connections.

2.6 Chern Classes

Let E be a $U(k)$ vector bundle, a complex vector bundle of rank k with a Hermitian metric over a complex curve Σ_g. Let A be a unitary connection on E with local representative A_α and curvature matrix Ω_α. As we saw in the preceding section, Ω_α is anti-Hermitian, so $i\Omega_\alpha$ is Hermitian. It is possible to construct polynomials in Ω which are invariant under gauge transformations and these give rise to topological and geometric invariants. An example of these invariants is the Chern class of E defined by

$$c(E) \equiv \det\,(I + \frac{i\Omega}{2\pi})$$

where I is the $(k \times k)$ identity matrix. In the case of a complex line bundle, the Chern class is given by

$$c(L) = 1 + c_1(L)$$

where $c_1(L)$ is called the first Chern class of L. It is a closed two-form given by

$$c_1(L) = \frac{F_A}{2\pi}.$$

This characteristic class is clearly invariant under gauge transformations and satisfies the identity

$$dc_1(L) = 0.$$

Note that higher characteristic classes (i.e. $2k$-forms) can also be defined, but they automatically vanish on manifolds of dimension ≤ 2.

Let L_1 and L_2 be two complex line bundles over Σ_g with structure groups $e^{i\theta_1}$ and $e^{i\theta_2}$ respectively. The structure group of the tensor product bundle $L_1 \otimes L_2$ is simply the group $e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$. If A_1 and A_2 are connections on L_1 and L_2 respectively, the connection on the bundle $L_1 \otimes L_2$ is just $A_1 + A_2$ and hence the curvature on the tensor product bundle $L_1 \otimes L_2$ is $-i(F_{A_1} + F_{A_2})$. Thus we have the important identity

$$c_1(L_1 \otimes L_2) = \frac{1}{2\pi}F_{A_1} + \frac{1}{2\pi}F_{A_2} = c_1(L_1) + c_1(L_2).$$

The first Chern class of a line bundle L over a complex curve is "quantized", i.e. $c_1(L)$ integrates to an integer over Σ_g,

$$c_1(L) \in H^2(\Sigma_g, \mathbf{Z}).$$

This is the famous Dirac quantization condition [8] which states that the magnetic charge (i.e. the integral of the Faraday tensor F_A over the two-dimensional manifold) is an integer multiple of 2π,

$$\int_{\Sigma_g} F_A = 2\pi n \;\; n \in \mathbf{Z}.$$

As an example let us consider the special case where L is the tangent bundle of a two-dimensional sphere of radius r and metric g given by

$$g = r^2 \sin^2(\theta)(d\phi) \otimes (d\phi) + r^2(d\theta) \otimes (d\theta).$$

The non-coordinate bases of one forms e^1 and e^2 are given by

$$e^1 = r\,\sin(\theta)(d\phi) \quad e^2 = r(d\theta).$$

The first structural equation gives

$$\begin{aligned} de^1 &= -(W^1_{21}e^1 + W^1_{22}e^2) \wedge e^2 \\ &= -W^1_{21}e^1 \wedge e^2. \end{aligned}$$

Hence we have

$$r\cos(\theta)(d\theta) \wedge (d\phi) = W^1_{21} r^2 \sin(\theta)(d\theta) \wedge (d\phi)$$

Similarly,

$$\begin{aligned} de^2 &= (W^1_{21}e^1 + W^1_{22}e^2) \wedge e^1 \\ &= W^1_{22}e^2 \wedge e^1 = 0 \end{aligned}$$

Hence the connection one-form W is given by

$$W = \begin{pmatrix} 0 & r^{-1}\cot(\theta) \\ -r^{-1}\cot(\theta) & 0 \end{pmatrix} e^1 = \begin{pmatrix} 0 & \cos(\theta)(d\phi) \\ -\cos(\theta)(d\phi) & 0 \end{pmatrix}$$

The coefficients Ω^i_j of the curvature two-form Ω are given by

$$\Omega^1_2 = dW^1_2 = -\sin(\theta) d\theta \wedge d\phi = \frac{1}{r^2} e^1 \wedge e^2 = -\Omega^2_1.$$

Hence the curvature becomes

$$\Omega = \begin{pmatrix} 0 & r^{-2} \\ -r^{-2} & 0 \end{pmatrix} e^1 \wedge e^2 = \begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix} e^1 \wedge e^2$$

where K is the Gaussian curvature of the manifold. In terms of the Ricci tensor R^i_{jkl} we have

$$\begin{aligned}\Omega^1_2 &= \frac{1}{2}R^1_{212}e^1 \wedge e^2 + \frac{1}{2}R^1_{221}e^2 \wedge e^1 = R^1_{212}e^1 \wedge e^2 = \frac{1}{r^2}e^1 \wedge e^2 \\ \Omega^2_1 &= \frac{1}{2}R^2_{112}e^1 \wedge e^2 + \frac{1}{2}R^2_{121}e^2 \wedge e^1 = R^2_{121}e^2 \wedge e^1 = \frac{1}{r^2}e^2 \wedge e^1\end{aligned}$$

Hence the scalar curvature of the complex curve is given by

$$s = R^1_{212} + R^2_{121} = 2R^1_{212} = \frac{2}{r^2}.$$

Note that when $r = 1$, the first Chern class of the tangent bundle of the manifold becomes

$$c_1(TS^2) = \int_{S^2} \frac{1}{2\pi} K e^1 \wedge e^2 = \frac{1}{2\pi}(4\pi) = 2.$$

In general, when $L = T\Sigma_g$ the tangent bundle of a compact oriented Riemann surface, the Gauss-Bonnet theorem gives the value of the first Chern class evaluated over Σ_g,

$$c_1(T\Sigma_g)[\Sigma_g] = \frac{1}{2\pi}\int_{\Sigma_g} F_A = \frac{1}{2\pi}\int_{\Sigma_g} Ke^1 \wedge e^2 = \chi(\Sigma_g) \equiv 2 - 2g$$

where $\chi(\Sigma_g)$ is a topological invariant called the Euler characteristic of the Riemann surface.

2.7 Hodge Theory

Let p be a point on a Riemann surface Σ_g and let $V = T_p\Sigma_g$ be the tangent space at p. As a vector space, V can be identified with the cotangent space

$T_p^*\Sigma_g$. If we let (e^1, e^2) be an orthonormal basis for V, then we can define the Hodge star operator $\star$ as a map

$$\star \quad \Lambda^p V \to \Lambda^{2-p} V \text{ by}$$

$$\star \quad 1 = e^1 \wedge e^2, \qquad \star \quad e^1 = e^2, \qquad \star \quad e^2 = -e^1, \qquad \star \quad e^1 \wedge e^2 = 1.$$

Thus on one-forms, the Hodge star satisfies $\star^2 = -1$ and in general equals $(-1)^p$ on p-forms. On a compact Riemann surface Σ_g we can use the Hodge star to define a bilinear form on Ω^p

$$(\ ,\ ) : \Omega^p \times \Omega^p \to \mathbf{R} \text{ by } (\theta_1, \theta_2) = \int_{\Sigma_g} \theta_1 \wedge \star\theta_2.$$

If we write θ_1 and θ_2 in terms of non-coordinate bases, we easily verify that this bilinear form is symmetric and positive-definite, and hence it is an inner product on Ω^p. If θ is a p-form, we write

$$|\theta|^2 dV \equiv \theta \wedge \star\theta$$

where dV is the volume element on Σ_g defined by

$$dV \equiv \sqrt{|\det g_{xy}|}dx \wedge dy = e^1 \wedge e^2.$$

We can also use the Hodge star to define the adjoint of the exterior derivative

$$\delta = - \star d \star : \Omega^p(\Sigma_g) \to \Omega^{p-1}(\Sigma_g)$$

We then have

$$\begin{aligned} (d\theta_1, \theta_2) &= \int_{\Sigma_g} d\theta_1 \wedge \star\theta_2 = \int_{\Sigma_g} d(\theta_1 \wedge \star\theta_2) - (-1)^p \int_{\Sigma_g} \theta_1 \wedge d \star \theta_2 \\ &= -\int_{\Sigma_g} \theta_1 \wedge \star \star d \star \theta_2 = \int_{\Sigma_g} \theta_1 \wedge \star\delta\theta_2 = (\theta_1, \delta\theta_2), \end{aligned}$$

where $p = \deg \theta_1$ and we have used Stokes' theorem. Finally, we can define the Hodge Laplacian

$$\Delta = d\delta + \delta d \, : \, \Omega^p(\Sigma_g) \to \Omega^p(\Sigma_g).$$

In the case of $\Omega^0(\Sigma_g)$, the Hodge Laplacian reduces to the Laplace operator on functions. To see this, let f be a function on the Riemann surface and let (e^1, e^2) be positively oriented non-coordinate bases. Since δf vanishes, the Hodge Laplacian reduces to the standard Laplace operator on functions,

$$\begin{aligned} \Delta f &= -\star d \star [(\partial_1 f)e^1 + (\partial_2 f)e^2] = -\star d[(\partial_1 f)e^2 - (\partial_2 f)e^1] \\ &= -\star(\partial_1^2 f + \partial_2^2 f)e^1 \wedge e^2 = -(\partial_1^2 + \partial_2^2)f. \end{aligned}$$

Using the fact that d and δ are adjoints of each other, we find that on a compact Riemann surface,

$$(\Delta\theta, \theta) = ((d\delta + \delta d)\theta, \theta) = (\delta\theta, \delta\theta) + (d\theta, d\theta) \geq 0.$$

Hence,

$$\Delta\theta = 0 \;\Leftrightarrow\; (\Delta\theta, \theta) = 0 \;\Leftrightarrow\; d\theta = 0 \text{ and } \delta\theta = 0.$$

These last two equations can be thought of as analogs of the (4-dimensional) Maxwell's equations of electromagnetism in the vacuum,

$$dF_A = 0, \quad d \star F_A = 0$$

Hence we see that an electromagnetic connection is just a connection whose curvature form is harmonic. Maxwell's equations can be generalized to

$SU(2)$-bundles leading to nonabelian gauge theory, which is the foundation of Donaldson's approach to study the topology and geometry of four-manifolds.

A differential p-form on a complex curve Σ_g is harmonic if $\Delta\theta = 0$. Let $\mathcal{H}^p(\Sigma_g)$ denote the space of harmonic p-forms on Σ_g. Hodge's theorem states that every de Rham cohomology class on a complex curve Σ_g possesses a unique harmonic representative. Thus

$$H^p(\Sigma_g, \mathbf{R}) \cong \mathcal{H}^p(\Sigma_g).$$

Let b_p denotes the p-th Betti number of Σ_g, so $b_p = \dim H^p(\Sigma_g, \mathbf{R})$. The zero-th cohomology group $H^0(\Sigma_g, \mathbf{R})$ is the set of closed functions f. Since the manifold is connected, the condition $df = 0$ is satisfied if and only if f is constant over Σ_g. Hence $H^0(\Sigma_g, \mathbf{R})$ is isomorphic to the vector space $\mathbf{R}$, thus we have $b_0 = 1$. By Poincaré duality we have

$$H^p(\Sigma_g, \mathbf{R}) \cong H^{2-p}(\Sigma_g, \mathbf{R})$$

hence $b_0 = b_2$. In terms of harmonic forms, we get

$$\mathcal{H}^p(\Sigma_g) \cong \mathcal{H}^{2-p}(\Sigma_g).$$

To calculate the first Betti number it is easier to think about homology groups instead of cohomology groups (by de-Rham's theorem the two are isomorphic). The torus T^2 can be thought of as the product of two circles a and b. These two circles are called one-cycles and generate the first homology group of T^2. In general on a genus g Riemann surfaces we will have g of the a circles and g of the b circles for a total of $2g$. Hence the dimension of $H^1(\Sigma_g)$ is $b_1 = 2g$. In the case of the sphere S^2, all one-cycles are homologous to the (trivial) zero-cycle. Hence the dimension of $H^1(S^2)$ is zero.

3 Spin Geometry on Complex Curves

3.1 Clifford Algebra

Let V be the two-dimensional Euclidean vector space $\mathbf{R}^2$ over $\mathbf{R}$ with orthonormal basis $\{e_1, e_2\}$. The Clifford algebra $Cl(V)$ is the algebra over $\mathbf{R}$ generated by $\{e_1, e_2\}$ subject to the relations

$$\begin{aligned} \gamma(e_i)\gamma(e_j) + \gamma(e_j)\gamma(e_i) = -2\delta_{ij} = -2 \text{ if } i &= j \\ = 0 \text{ if } i &\neq j \end{aligned}$$

Here $\gamma_i \equiv \gamma(e_i)$ are complex 2×2 matrices taken for example to be

$$\gamma_1 = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & +i \\ +i & 0 \end{pmatrix}.$$

In particular, every element of $Cl(V)$ can be written uniquely as a sum of $1, \gamma_1, \gamma_2$ and $\gamma_1\gamma_2$. Hence, $Cl(V)$ is isomorphic to the quaternion algebra $\mathbf{H}$. Thus the dimension of $Cl(V)$ as a real vector space is $2^d = 4$, where d is the dimension of V over $\mathbf{R}$. Thus we can identify γ_1 with j, γ_2 with k and $\gamma_1\gamma_2$ with i. Since any quaternion $Q \in \mathbf{H}$ can be written as

$$Q = \begin{pmatrix} a - id & -b + ic \\ b + ic & a + id \end{pmatrix}$$

where a, b, c and d are real numbers, the determinant of the quaternion Q is given by the formula

$$\det Q = a^2 + b^2 + c^2 + d^2.$$

So the unit 3-sphere S^3 in Euclidean four-space can be identified with the special unitary group

$$SU(2) = \{Q \in \mathbf{H}, \det Q = 1\}.$$

The sphere of unit quaternions $SU(2)$ forms a Lie group whose Lie algebra - the tangent space to the unit sphere at the identity - is the space of purely imaginary quaternions spanned by i, j and k, or equivalently the space of trace-free 2×2 anti-Hermitian matrices. The exterior algebra of V is defined by

$$\begin{aligned} \Lambda^*(V) &= \Lambda^0(V) \oplus \Lambda^1(V) \oplus \Lambda^2(V) \\ &\cong \text{span}\,\{1, e^1, e^2, e^1 \wedge e^2\}. \end{aligned}$$

Hence as vector spaces, $Cl(V)$ and $\Lambda^*(V)$ are isomorphic. Finally, we define a Clifford bundle over Σ_g to be a vector bundle whose fiber is the Clifford algebra $Cl(V) \cong \Lambda^*(V)$.

3.2 Spin Structures

A very important characteristic of subatomic particles is their spin. If an ordinary spinning body is rotated in space through 360^0 it returns to its original configuration. A spinor (a particle with spin $1/2$) however will not do this. To return to its initial state, it is necessary to rotate (its wavefunction) through 720^0. In other words, a spinor requires a double rotation relative to everyday objects before it comes back to its starting state. Mathematically, in two dimensions, we say that Spin(2) is the double cover of the $SO(2)$

group. In terms of γ matrices, every element of Spin(2) can be written as $cos(\theta/2) + \gamma_1\gamma_2 sin(\theta/2)$ where θ is a real function on the Riemann surface. Hence the group Spin(2) is isomorphic to S^1. Thus we have

$$SO(2) = \begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{pmatrix}, \quad \text{Spin}(2) = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}.$$

Similarly, we define the Spin$(2)^c$ group as

$$\text{Spin}(2)^c = \begin{pmatrix} \lambda e^{-i\theta/2} & 0 \\ 0 & \lambda e^{i\theta/2} \end{pmatrix}, \text{ where } \lambda \in U(1).$$

Let Σ_g be a Riemann surface, thought of as an oriented Riemannian manifold of dimension two. The structure group of its tangent bundle $T\Sigma_g$ is simply the group $SO(2)$ of rotations in two dimensions. Thus we can choose a cover $\{U_\alpha : \alpha \in I\}$ for Σ_g so that the corresponding transition functions take their values in $SO(2)$:

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to SO(2) \subset GL(2, \mathbf{R}).$$

A spin structure on Σ_g is given by an open covering $\{U_\alpha : \alpha \in I\}$ and a collection of transition functions

$$\tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to \text{Spin}(2)$$

such that the cocycle condition

$$\tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma} = \tilde{g}_{\alpha\gamma} \text{ on } U_\alpha \cap U_\beta \cap U_\gamma.$$

is satisfied. Manifolds which admit spin structures are called spin manifolds. We will see later that all Riemann surfaces are spin. This result can be derived by showing that the index of the Dirac operator is always an integer.

Recall that once we have transition functions satisfying the cocycle condition, we get a corresponding vector bundle. Thus given a spin structure on Σ_g defined by

$$\tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{Spin}(2)$$

the transition functions

$$\tilde{g}^{+}_{\alpha\beta} : U_\alpha \cap U_\beta \to e^{-i\theta/2}, \quad \tilde{g}^{-}_{\alpha\beta} : U_\alpha \cap U_\beta \to e^{i\theta/2}$$

determine complex vector bundles of rank one over Σ_g, which we denote by S^+ and S^-. Similarly, a spin^c structure on Σ_g is given by an open covering $\{U_\alpha : \alpha \in I\}$ and a collection of transition functions

$$\tilde{g}^{c}_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{Spin}(2)^c$$

such that the cocycle condition is satisfied.

Starting with a spin structure on Σ_g defined by the transition functions

$$\tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{Spin}(2)$$

and a complex line bundle L over $\Sigma_g$with transition functions

$$h_{\alpha\beta} : U_\alpha \cap U_\beta \to U(1),$$

we can define a spin^c structure on Σ_g by taking the transition functions to be

$$h_{\alpha\beta}\tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Spin}(2)^c.$$

The transition functions

$$\tilde{g}^{c+}_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \lambda e^{-i\theta/2}, \quad \tilde{g}^{c-}_{\alpha\beta} : U_\alpha \cap U_\beta \to \lambda e^{i\theta/2}$$

determine complex line bundles $S^+ \otimes L$ and $S^- \otimes L$ whose sections are called spinor fields of positive or negative chirality respectively.

One of the most important line bundles on complex manifolds is the canonical line bundle K. We construct this bundle as follows. Let $T^*\Sigma_g$ be the cotangent bundle over Σ_g with local sections dx and dy. The complexified cotangent bundle splits as

$$T^*\Sigma_g \otimes \mathbf{C} = T^{*1,0}\Sigma_g \oplus T^{*0,1}\Sigma_g$$

with local sections $dz = dx + idy$ and $d\bar{z} = dx - idy$ respectively. The canonical line bundle is defined in general as the top exterior power of $T^{*1,0}\Sigma_g$ but in two dimensions K is the same as $T^{*1,0}\Sigma_g \cong T^*\Sigma_g$ (here we are thinking of the cotangent bundle as a complex line bundle over the Riemann surface and identifying dx and dy with $dx + idy$). The cotangent bundle is defined to be the dual of the tangent bundle, this means that if the structure group of $T\Sigma_g$ (considered as a complex line bundle) is $e^{i\theta}$, that of $T^*\Sigma_g$ is $e^{-i\theta}$. As we saw earlier, the structure group of S^+ is $e^{-i\theta/2}$ hence we have the important fact that $S^{+2} \equiv S^+ \otimes S^+ = K$ which is the same as saying $S^+ = K^{1/2}$.

3.3 The Spin Connection

A connection ϕ on $S \equiv S^+ \oplus S^-$ is called a spin(2) connection if it can be expressed in terms of each local trivialization as a one-form with values in the Lie algebra of Spin(2). The covariant derivative on spinors is expressed locally as

$$(d_\phi \sigma)_\alpha = d\sigma_\alpha + \phi_\alpha \sigma_\alpha.$$

Since the Lie algebra of Spin(2) is generated by $\gamma_1\gamma_2$, the condition that ϕ is a spin(2) connection is simply

$$\phi_\alpha = \sum_{i,j=1}^{2} \phi_{\alpha ij}\gamma_i\gamma_j, \quad \phi_{\alpha ij} = -\phi_{\alpha ji}$$

where $\phi_{\alpha ij}$ are ordinary real-valued one-forms.

Given a spin(2) connection ϕ on S, there is a unique connection $\tilde{\phi}$ on the Clifford bundle over Σ_g which satisfies the Leibniz rule

$$d_\phi(\gamma\sigma) = (d_{\tilde{\phi}}\gamma)\sigma + \gamma d_\phi \sigma$$

Here γ is any linear combination of the γ_i-matrices and $\sigma \in \Gamma(S)$. We will now show that, when the above equation is satisfied, the spin connection on S induces the Levi-Civita connection on the Clifford bundle. To start, we trivialize the vector bundle S over an open set $U \subset \Sigma_g$ and let (σ_1, σ_2) be the orthonormal sections of $S|_U$ defined by

$$\sigma_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Suppose now that ϕ is a spin(2) connection on S, given over U by

$$\phi = \sum_{i,j=1}^{2} \phi_{ij}\gamma_i\gamma_j, \quad \phi_{ij} = -\phi_{ji}.$$

Since the components of σ_λ and $\gamma_k\sigma_\lambda$ are constant we get,

$$d_\phi\sigma_\lambda = \sum_{i,j=1}^{2} \phi_{ij}\gamma_i\gamma_j\sigma_\lambda, \quad d_\phi(\gamma_k\sigma_\lambda) = \sum_{i,j=1}^{2} \phi_{ij}\gamma_i\gamma_j\gamma_k\sigma_\lambda.$$

Hence it follows from the Leibniz condition that

$$\sum_{i,j=1}^{2} \phi_{ij}\gamma_i\gamma_j\gamma_k\sigma_\lambda = (\tilde{\phi}\gamma_k)\sigma_\lambda + \gamma_k \sum_{i,j=1}^{2} \phi_{ij}\gamma_i\gamma_j\sigma_\lambda,$$

or equivalently,

$$(\tilde{\phi}\gamma_k)\sigma_\lambda = \sum_{i,j=1}^{2} \phi_{ij}(\gamma_i\gamma_j\gamma_k - \gamma_k\gamma_i\gamma_j)\sigma_\lambda.$$

Hence,

$$\begin{aligned}
\tilde{\phi}\gamma_1 &= \phi_{12}(\gamma_1\gamma_2\gamma_1 - \gamma_1\gamma_1\gamma_2) + \phi_{21}(\gamma_2\gamma_1\gamma_1 - \gamma_1\gamma_2\gamma_1) \\
&= 2\phi_{12}(-\gamma_2\gamma_1\gamma_1 - \gamma_1\gamma_1\gamma_2) = 2\phi_{12}(\gamma_2 + \gamma_2) = -4\phi_{21}\gamma_2 \\
\tilde{\phi}\gamma_2 &= \phi_{12}(\gamma_1\gamma_2\gamma_2 - \gamma_2\gamma_1\gamma_2) + \phi_{21}(\gamma_2\gamma_1\gamma_2 - \gamma_2\gamma_2\gamma_1) \\
&= 2\phi_{12}(-\gamma_1\gamma_2\gamma_2 + \gamma_2\gamma_2\gamma_1) = 2\phi_{12}(\gamma_1 + \gamma_1) = -4\phi_{12}\gamma_1
\end{aligned}$$

Thus, given the connection $\tilde{\phi}$ satisfying

$$\tilde{\phi}\gamma_k = -4\sum_{i=1}^{2} \phi_{ik}\gamma_i$$

on the Clifford bundle, we can define a corresponding spin(2)-connection on S by setting

$$\phi_{ij} = -\frac{1}{4}W_{ij}.$$

This is the unique spin(2)-connetion on S which induces the $SO(2)$-connection. The spin(2) covariant derivative is simply

$$d - \frac{1}{4}\sum_{i,j=1}^{2} W_{ij}\gamma_i\gamma_j.$$

The curvature of this connection is given by

$$\begin{aligned}
\Omega &= (d - \frac{1}{4}\sum_{i,j=1}^{2} W_{ij}\gamma_i\gamma_j)^2 \\
&= d(-\frac{1}{4}\sum_{i,j=1}^{2} W_{ij}\gamma_i\gamma_j) \equiv -\frac{1}{4}\sum_{i,j=1}^{2} \Omega_{ij}\gamma_i\gamma_j
\end{aligned}$$

where

$$\Omega_{ij} = dW_{ij}.$$

Given a unitary connection A on a complex line bundle L over a Riemann surface Σ_g, we can define a spin$(2)^c$ connection on the bundle $S \otimes L$, expressed in terms of each local trivialization as

$$\phi_\alpha^c = AI - \frac{1}{4} \sum_{i,j=1}^{2} W_{ij} \gamma_i \gamma_j.$$

Its covariant derivative is given locally by

$$(d_{\phi^c} \sigma)_\alpha = d\sigma_\alpha + \phi_\alpha^c \sigma_\alpha.$$

Moreover, since $F_A = idA$ where $-iF_A$ is the curvature of the connection on L, we see that the curvature of $S \otimes L$ is

$$\Omega_A = -iF_A I + \Omega = -iF_A I - \frac{1}{4} \sum_{i,j=1}^{2} \Omega_{ij} \gamma_i \gamma_j.$$

In this formula, the Ω_{ij}'s are the curvature forms for the Levi-Civita connection on Σ_g.

3.4 The Dirac Operator

Let Σ_g be a Riemann surface with a spin^c structure and a spin$(2)^c$-connection ϕ^c on the spinor bundle S. The Dirac operator $D_A : \Gamma(S \otimes L) \longrightarrow \Gamma(S \otimes L)$ is defined by

$$D_A \psi = \gamma_i (d_{\phi^c})_i \psi$$

where the subscript i in the covariant derivative denotes the i-th component. Considered as an operator on the space of sections of the full spinor bundle $S \otimes L = (S^+ \otimes L) \oplus (S^- \otimes L)$, the Dirac operator D_A of the form

$$\begin{pmatrix} 0 & D_A^- \\ D_A^+ & 0 \end{pmatrix}$$

is formally self adjoint, thus

$$\int_{\Sigma_g} < D_A\psi, D_A\psi > dV = \int_{\Sigma_g} < \psi, D_A^2\psi > dV.$$

The two operators D_A^+ and D_A^- satisfy

$$D_A^+ : \Gamma(S^+ \otimes L) \to \Gamma(S^- \otimes L), \quad D_A^- : \Gamma(S^- \otimes L) \to \Gamma(S^+ \otimes L).$$

Hence in the case where $\psi \in \Gamma(S^+ \otimes L)$ the operator D_A coincides with D_A^+.

On flat $\mathbf{R}^2$ with coordinates x and y, the Dirac operator is just

$$D\psi = \gamma_1 \frac{\partial}{\partial x}\psi + \gamma_2 \frac{\partial}{\partial y}\psi$$

the γ's being the the Clifford algebra matrices satisfying

$$\begin{aligned} \gamma_i\gamma_j + \gamma_j\gamma_i = -2\delta_{ij} = -2 \text{ if } i &= j \\ = 0 \text{ if } i &\neq j. \end{aligned}$$

Thus we find that

$$D^2\psi = -(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})\psi.$$

In this case, the Dirac operator is a square root of the usual Euclidean Laplacian.

The main result of this section, Weitzenböck's formula (which works in any dimension), gives a similar relationship between the Dirac operator and the Laplacian acting on spinors $\Delta_A : \Gamma(S \otimes L) \rightarrow \Gamma(S \otimes L)$ defined by the formula

$$\Delta_A = -\sum_{i=1}^{2} (d_{\phi^c})_i (d_{\phi^c})_i.$$

The square of the Dirac operator D_A is related to the Laplace operator Δ_A by the formula

$$D_A^2 \psi = \Delta_A \psi + \frac{s}{4}\psi - \sum_{k<l} i F_{Akl} \gamma_k \gamma_l \psi.$$

This can be derived by noticing that

$$\begin{aligned} D_A^2 \psi &= \gamma_k \gamma_l (d_{\phi^c})_k (d_{\phi^c})_l \psi = (\frac{1}{2}\{\gamma_k, \gamma_l\} + \frac{1}{2}[\gamma_k, \gamma_l])(d_{\phi^c})_k (d_{\phi^c})_l \psi \\ &= -(d_{\phi^c})_k (d_{\phi^c})_k \psi + \frac{1}{2}\gamma_k \gamma_l [(d_{\phi^c})_k, (d_{\phi^c})_l] \psi \\ &= \Delta_A \psi + \frac{1}{2}\gamma_k \gamma_l (-i F_{Akl} - \frac{1}{4}\sum_{i,j=1}^{2} R_{ijkl} \gamma_i \gamma_j) \psi. \end{aligned}$$

Here we used the fact that

$$\Omega_{ij} = \frac{1}{2} R_{ijkl} e^k \wedge e^l$$

and that the k-th and l-th components of Ω are nothing but $-R_{ijkl}\gamma_i\gamma_j/4$. The square of the Dirac operator now becomes

$$D_A^2 \psi = \Delta_A \psi - i \sum_{k<l} F_{Akl} \gamma_k \gamma_l \psi - \frac{1}{8} R_{ijkl} \gamma_k \gamma_l \gamma_i \gamma_j \psi.$$

Using the fact that $\Omega_{12} = -\Omega_{21}$ and that $R_{ijkl} = -R_{ijlk}$ we can write the last term in the above expression as

$$\begin{aligned} -\frac{1}{8} R_{ijkl} \gamma_k \gamma_l \gamma_i \gamma_j \psi &= -\frac{1}{8}(-R_{1212} + R_{1221} + R_{2112} - R_{2121}) I \psi \\ &= \frac{1}{4}(R_{1212} + R_{2121}) \psi = \frac{s}{4}\psi \end{aligned}$$

where s is the scalar curvature of Σ_g. This proves the Weitzenböck formula.

The Dirac operator we have defined is a Dirac operator acting on S and $S \otimes L$. This can be extended further to general vector bundles on Σ_g of the form $S \otimes E$ where E is a complex vector bundle. The Dirac operator with coefficients in E is defined locally by the formula

$$D_A\psi^i = \gamma_m(\partial_m\psi^i - \frac{1}{4}\sum_{k,l=1}^{2} W_{klm}\gamma_k\gamma_l\psi^i + A_m^a(T^a)^{ij}\psi^j)$$

where T^a are the generators of the Lie algebra of the structure group G of the bundle E and the indices i and j run from 1 to N where N is the dimension of the representation of G.

4 The Geometry of Monopoles

4.1 The Monopole Equations

Let Σ_g be a complex curve with a spin^c structure and a corresponding positive spin^c bundle $K^{1/2} \otimes L$. We seek pairs (A, ψ), where A is a connection on the line bundle L and ψ is a section of $K^{1/2} \otimes L$ such that

$$D_A^+ \psi = 0, \qquad F_A = -\frac{1}{2}|\psi|^2 w$$

where $F_A = idA$, D_A^+ is the Dirac operator on $K^{1/2} \otimes L$ and w is the Kähler form on Σ_g. Here ψ is a commuting bosonic variable that transforms as a spinor, the reason for this comes from topological field theory [30] but I will not discuss this here as this will take us too far afield. The $-1/2$ factor on the right hand side of the second equation is important because with this choice of normalization and sign, strong restrictions can be placed on the set of solutions. I will call these equations the monopole equations since they are the two-dimensional analogue of the Seiberg-Witten monopole equations [33]. They can be thought of as describing a massless charged spinor moving on a Riemann surface in the background of an electromagnetic field. I will sometimes write the equations in the form

$$D_A^+ \psi = 0, \qquad F_A = \sigma(\psi).$$

These equations fail to be linear in ψ only because of the presence of the term $\sigma(\psi)$. Note that since any antisymmetric two-form in two dimensions is of type $(1,1)$, the $(2,0)$ and $(0,2)$ components of F_A vanish.

We define a real-valued functional on the space

$$\mathcal{A} = \{(A,\psi) : A \text{ is a connection on } L, \psi \in \Gamma(K^{1/2} \otimes L)\}$$

by the formula

$$S(A,\psi) = \int_{\Sigma_g} (|D_A^+\psi|^2 + |F_A - \sigma(\psi)|^2) dV$$

where dV denotes the volume element on Σ_g. Clearly the absolute minimum of this functional is attained when the solutions of the monopole equations exist.

Let λ_1 and λ_2 be two complex numbers then we have the identity

$$2Re(\overline{\lambda_1}\lambda_2) = \overline{\lambda_1}\lambda_2 + \overline{\lambda_2}\lambda_1.$$

Hence

$$< \Delta_A\psi, \psi > + < \psi, \Delta_A\psi >= 2Re < \psi, \Delta_A\psi > .$$

If we denote by Δ the usual Laplacian on functions, then a simple calculation in local coordinates shows that

$$- < (d_{\phi^c})_i\psi, (d_{\phi^c})_i\psi > + Re < \psi, \Delta_A\psi >= \frac{1}{2}\Delta|\psi|^2.$$

Integrating over Σ_g and noticing that $\Delta|\psi|^2$ is a total derivative, we get

$$\int_{\Sigma_g} < (d_{\phi^c})_i\psi, (d_{\phi^c})_i\psi > dV \equiv \int_{\Sigma_g} |(d_{\phi^c})_i\psi|^2 dV = \int_{\Sigma_g} Re < \psi, \Delta_A\psi > dV.$$

Note that the integral over Σ_g of the real function $< D_A^+\psi, D_A^+\psi >$ gives

$$\int_{\Sigma_g} |D_A^+\psi|^2 dV = \int_{\Sigma_g} (< \psi, \Delta_A\psi > + \frac{s}{4}|\psi|^2 + < \psi, -i\sum_{k<l} F_{Akl}\gamma_k\gamma_l\psi >) dV.$$

It then follows from the Weitzenböck formula that

$$\begin{aligned}
S(A,\psi) &= \int_{\Sigma_g} [|D_A^+\psi|^2 dV + (F_A - \sigma(\psi)) \wedge \star(F_A - \sigma(\psi))] \\
&= \int_{\Sigma_g} [(|D_A^+\psi|^2 + |F_A|^2 + |\sigma(\psi)|^2) dV - F_A \wedge \star\sigma(\psi) - \sigma(\psi) \wedge \star F_A] \\
&= \int_{\Sigma_g} (<\psi, \Delta_A\psi> + \frac{s}{4}|\psi|^2 + |F_A|^2 + |\sigma(\psi)|^2) dV \\
&- iF_{A12} <\psi, \gamma_1\gamma_2\psi> e^1 \wedge e^2 - 2(-1)F_A(\frac{1}{2}|\psi|^2) \\
&= \int_{\Sigma_g} (<\psi, \Delta_A\psi> + \frac{s}{4}|\psi|^2 + |F_A|^2 + |\sigma(\psi)|^2) dV \\
&+ (-iF_{A12}(-i)|\psi|^2 + F_{A12}|\psi|^2) e^1 \wedge e^2 \\
&= \int_{\Sigma_g} (<\psi, \Delta_A\psi> + \frac{s}{4}|\psi|^2 + |F_A|^2 + |\sigma(\psi)|^2) dV.
\end{aligned}$$

Since the functional $S(A,\psi)$ is real, we see that $<\psi, \Delta_A\psi>$ is also real. Thus

$$S(A,\psi) = \int_{\Sigma_g} (|(d_{\phi^c})_i\psi|^2 + \frac{s}{4}|\psi|^2 + |F_A|^2 + |\sigma(\psi)|^2) dV.$$

In particular, if the genus of Σ_g is less than two i.e. the scalar curvature $s \geq 0$, all the terms in the last expression are non-negative and there are no nonzero solutions to the monopole equations. Moreover, since

$$|\sigma(\psi)|^2 = \frac{1}{4}|\psi|^4, \quad \int_{\Sigma_g} (|\psi|^2 + \frac{s}{2})^2 dV \geq 0,$$

we conclude that if (A,ψ) is a solution to the monopole equations, one has the inequality for the norm of F_A

$$\int_{\Sigma_g} F_A \wedge \star F_A \leq -\int_{\Sigma_g} (\frac{s}{4}|\psi|^2 + \frac{1}{4}|\psi|^4) dV \leq \int_{\Sigma_g} \frac{s^2}{16} dV.$$

In the next section we will study the properties of the space of solutions to the monopole equations. We will see that the moduli space is compact and carries a Kähler structure.

4.2 The Moduli Space

Recall that the space of connections A on L and sections of the positive Spin bundle ψ is given by

$$\mathcal{A} = \{(A, \psi) : A \text{ on } L, \psi \in \Gamma(K^{1/2} \otimes L)\}.$$

The group of gauge transformations

$$\mathcal{G} = \mathrm{Map}(\Sigma_g, S^1) = \{\text{maps } g : \Sigma_g \to S^1\}$$

acts on the space $\mathcal{A}$ by

$$g : (A, \psi) \to (A + gd(g^{-1}), g\psi).$$

The quotient space $\mathcal{A}/\mathcal{G}$ has singularities at the reducible elements. Recall that this happens whenever the gauge group does not act freely on the quotient space. In our case the $U(1)$ group of constant gauge transformations acts freely on the quotient space except at the points $(A, 0)$. Note that a solution with $\psi = 0$ necessarily has $F_A = 0$. We can avoid the reducible elements by perturbing the monopole equations around $(A, 0)$. We can then define the moduli space $\mathcal{M}$ of monopoles to be the space

$$\mathcal{M} = \{[A, \psi] \in \mathcal{A}/\mathcal{G} : (A, \psi) \text{ satisfying the monopole equations}\}.$$

To determine the dimension of $\mathcal{M}$ we fix a point in $\mathcal{M}$ and see in how many directions we can go while remaining in $\mathcal{M}$. Let δ' be a small variation then if (A, ψ) satisfy the monopole equations we want $(A + \delta' A, \psi + \delta' \psi)$ to also satisfy the monopole equations. The number of linearly independent

$(\delta' A, \delta' \psi)$ tells us the dimension of the moduli space. After linearizing the equations, we end up with

$$D_A^+ \delta' \psi = -\gamma_i \delta' A_i \psi \equiv -\gamma.(\delta' A)\psi, \quad d\delta' A = \frac{i}{2}(\overline{\psi}\delta' \psi + \delta' \overline{\psi}\psi)w.$$

We can identify $\delta' \psi$ and $\delta' A$ with an infinitesimal action of the gauge group where $\delta' \psi = i\lambda\psi$ and $\delta' A = -id\lambda$ where $\lambda \in \Omega^0(\Sigma_g, \mathbf{R})$.

If we define two operators d_1 and d_2 by

$$\begin{aligned} d_1\lambda &= (-id\lambda, i\lambda\psi) \\ d_2(\alpha, \beta) &= (d\alpha + \frac{i}{2}(\overline{\psi}\beta + \overline{\beta}\psi)w,\ D_A^+\beta + (\gamma.\alpha)\psi) \end{aligned}$$

where $\alpha \equiv \delta' A$ and $\beta \equiv \delta' \psi$, it follows that

$$\begin{aligned} d_2 d_1 \lambda &= (0 + \frac{i}{2}(i\overline{\psi}\lambda\psi - i\lambda\overline{\psi}\psi)w,\ D_A^+(i\lambda\psi) - i\gamma_i((d)_i\lambda)\psi) \\ &= (0 + 0,\ i\lambda D_A^+\psi + i\gamma_i((d)_i\lambda)\psi - i\gamma_i((d)_i\lambda)\psi) \\ &= (0, 0) \end{aligned}$$

Thus we can define a complex C by

$$\Lambda^0 \rightarrow \Lambda^1 \oplus (K^{1/2} \otimes L) \rightarrow \Lambda^2 \oplus (K^{-1/2} \otimes L)$$

where the first map is given by d_1 and the second by d_2. If we define the operator T by

$$T = d_1^* \oplus d_2$$

where d_1^* is the adjoint of d_1, then C is equivalent to

$$\Lambda^1 \oplus (K^{1/2} \otimes L) \rightarrow \Lambda^0 \oplus \Lambda^2 \oplus (K^{-1/2} \otimes L).$$

In terms of components, the map T is given by

$$(\delta' A, \delta' \psi) \to (- \star d \star (\delta' A), d\delta' A, D_A^+ \delta' \psi).$$

where $- \star d \star (\delta' A) = 0$ fixes the gauge. To calculate the dimension of the moduli space, we will have to make use of the Atiyah-Singer index theorem. In reference [3] it was shown rigorously how to compute the dimension of a moduli space in terms of data of the given vector bundles and base manifold.

Going back to our equations, since the equations are now linear, the index of the operator T splits into the sum of the index of the operator $d + \delta$ (here δ is the adjoint of d) acting on one forms, and of the Dirac operator from $K^{1/2} \otimes L$ to $K^{-1/2} \otimes L$.

Recall from Hodge theory that the square of the operator $d + \delta$ is the Hodge Laplacian $\Delta = d\delta + \delta d$. Let θ be a p-form then since

$$(d + \delta)\theta = 0 \Leftrightarrow d\theta = 0 \text{ and } \delta\theta = 0 \Leftrightarrow \Delta\theta = 0,$$

we see that the kernel of $d + \delta$ is the finite-dimensional space of harmonic forms. Setting

$$\Lambda^+ = \Lambda^0 \oplus \Lambda^2, \;\; \Lambda^- = \Lambda^1$$

and dividing the operator $d + \delta$ into two pieces,

$$(d + \delta)^+ : \Lambda^+ \to \Lambda^-, \;\; (d + \delta)^- : \Lambda^- \to \Lambda^+,$$

we define the index of $(d + \delta)^+$ to be

$$\text{index of } (d + \delta)^+ = \text{dim Ker } (d + \delta)^+ - \text{dim Ker } (d + \delta)^-.$$

But we know that

$$\begin{aligned}\text{Ker }(d+\delta)^+ &= \mathcal{H}^0(\Sigma_g)\oplus\mathcal{H}^2(\Sigma_g)\\ \text{Ker }(d+\delta)^- &= \mathcal{H}^1(\Sigma_g)\end{aligned}$$

thus we get

$$\text{index }(d+\delta)^+ = b_0+b_2-b_1 = \chi(\Sigma_g),$$

the Euler characteristic of Σ_g.

In a similar spirit, the Atiyah-Singer index theorem gives a formula for the index of the Dirac operator in terms of topological data. Just as in Hodge theory, we define the index of D_A^+ to be

$$\text{index } D_A^+ = \text{dim Ker }(D_A^+) - \text{dim Ker }(D_A^-).$$

To state the Atiyah-Singer index theorem for the Dirac operator, we need a special combination of the Chern classes, called the Chern character, and the Dirac genus $\hat{A}$.

If L is a complex line bundle on Σ_g with first Chern class $c_1(L)$, the Chern character is

$$ch(L) = e^{c_1(L)} = 1 + c_1(L) + \ldots \quad \in H^*(\Sigma_g, \mathbf{R}),$$

where $H^*(\Sigma_g, \mathbf{R})$ denotes the direct sum of all real cohomology groups of Σ_g and the dots indicate terms which will automatically vanish in two-dimensions.

The Dirac genus Â is given by

$$\hat{A} = 1 + 4k \text{ forms}$$

where k is an integer. We are now ready to state one of the monuments of twentieth century mathematics, the Atiyah-Singer index theorem for Dirac operators [4]. If D_A is a Dirac operator coupled with an abelian connection on a complex line bundle L over a manifold Σ_g, then

$$\begin{aligned} \text{index } D_A^+ &= \int_{\Sigma_g} \hat{A}\, ch(L) \\ &= \int_{\Sigma_g} c_1(L). \end{aligned}$$

In the integral above, only two-forms are picked up, so that the integration makes sense. Note that in the case where the Dirac operator is not coupled with a connection, the index formula gives

$$\text{index } D^+ = 0.$$

Since index D^+ is not a fraction, we say that Σ_g is a spin manifold.

Recall that we want to calculate the real dimension of the moduli space $\mathcal{M}$. To do this we will have to think about the vector bundles $S^\pm \otimes L$ as real vector bundles of rank two, and thus multiply the index of D_A^+ by two, thus we have

$$\begin{aligned} \dim \mathcal{M} &= -\text{index } (d+\delta)^+ + 2 \text{ index } D_A^+ \\ &= 2(c_1(L)[\Sigma_g] + g - 1). \end{aligned}$$

where $c_1(L)[\Sigma_g]$ in the above expression denotes the integral of the first Chern class over the complex curve.

Note that for the moduli space to have non-negative dimension, we need the magnetic charge to satisfy

$$c_1(L)[\Sigma_g] \geq 1 - g.$$

The first remarkable property of the moduli space $\mathcal{M}$ is that it is compact as I will show now. The claim is that if (A, ψ) is a solution to the monopole equations and the maximum value of $|\psi|^2$ is assumed at a point $p \in \Sigma_g$, then

$$|\psi|^2(p) \leq -\frac{1}{2}s(p)$$

where s is the scalar curvature. To see this, let Δ be the usual Laplace operator on functions, then $\Delta(f) \geq 0$ at a local maximum of a function f. Since p is a maximum for $|\psi|^2$, we have

$$\frac{1}{2}\Delta(|\psi|^2)(p) \geq 0$$

Thus

$$-< (d_{\phi^c})_i\psi, (d_{\phi^c})_i\psi > + < \psi, \Delta_A\psi > \geq 0.$$

Since $< (d_{\phi^c})_i\psi, (d_{\phi^c})_i\psi >$ is positive definite, we see that

$$< \psi, \Delta_A\psi > \geq 0.$$

It follows from the Dirac equation along with the Weitzenböck formula that

$$0 = D_A^{+^2}\psi = \Delta_A\psi + \frac{s}{4}\psi - \sum_{k<l} iF_{Akl}\gamma_k\gamma_l\psi.$$

Taking the inner product with ψ and applying the second of the monopole equations we obtain

$$-\frac{s}{4}|\psi|^2 - \frac{1}{2}|\psi|^4 = < \psi, \Delta_A\psi > \geq 0$$

Dividing by $|\psi|^2$ we obtain the desired result.

Note that since F_A is a non-positive two-form, its integral over the Riemann surface should also be non-positive hence we get the following bound on the value of the first Chern class

$$1 - g \leq c_1(L)[\Sigma_g] \leq 0.$$

To see that the moduli space is compact, recall that in a Yang-Mills-Higgs theory with a Higgs field ϕ taking its value in the Lie algebra of $SO(3)$ the potential term $V(\phi)$ is proportional to

$$V(\phi) \sim (|\phi|^2 - 1)^2.$$

When $V(\phi)$ vanishes, the Higgs field ϕ lies in the compact space S^2. Thus the modulus of ϕ is always bounded (and equals one in this case). Returning to our monopole moduli space, $|\psi|^2$ is always bounded by $-s/2$ hence by analogy with the Higgs field, $\mathcal{M}$ is compact. Note that the bound on $|\psi|^2$ gives a bound on F_A, the derivative of A, when substituted in the second monopole equation. Results from analysis can be used to get a bound on the gauge field itself. The compactness of the moduli space boils down to showing that any sequence of solutions of the monopole equations possesses a convergent subsequence. An outline for a rigorous proof (in the four dimensional case but the arguments apply here as well) can be found in [15], see also [22] where some of the arguments above are independently derived.

The second remarkable property of the moduli space $\mathcal{M}$ is that it is Kähler. To start recall that the infinite dimensional space $\mathcal{A}$ is the space of abelian connections and charged spinor fields (A, ψ). Let (a_i, ϕ_i) and (a_j, ϕ_j) be tangent vectors (to points (A_i, ψ_i) and (A_j, ψ_j)) in the tangent space of

$\mathcal{A}$. We are going to show that the objects $J_{\mathcal{A}}$ and Θ_{ij} (defined below) give $\mathcal{A}$ a Kähler structure. We then use a thechnique due to Marsden and Weinstein [17] that allows us to carry the structure on $\mathcal{A}$ to our moduli space $\mathcal{M}$.

We define the map $J_{\mathcal{A}}$ by

$$J_{\mathcal{A}} : (a_i, \phi_i) \longrightarrow (\star a_i, i\phi_i)$$

It is then clear that $J_{\mathcal{A}}^2 = -1$. We now define Θ_{ij} by

$$\Theta_{ij} = \int_{\Sigma_g} (-a_i \wedge a_j + Im < \phi_i, \phi_j > w)$$

Here $<,>$ is the inner product on the fiber of the bundle $K^{1/2} \otimes L$, thus

$$\overline{< \phi_i, \phi_j >} = < \phi_j, \phi_i > .$$

Thus Θ_{ij} is antisymmetric and can be considered as the components of a two-form Θ. This follows from the identities

$$\begin{aligned} a_i \wedge a_j &= -a_j \wedge a_i \\ Im < \phi_i, \phi_j > &= -Im < \phi_j, \phi_i > . \end{aligned}$$

Notice that since the a_i's are one forms with values in the Lie algebra of $U(1)$ we see that the two form Θ is real. Moreover since

$$-\int_{\Sigma_g} a_i \wedge \star a_i > 0 \ \text{ for } \delta_i \mathrm{A} \neq 0$$

we see that

$$\Theta((a_i, \phi_i), J_{\mathcal{A}}(a_i, \phi_i)) > 0.$$

If we write the a_i's in terms of non-coordinate bases, we get the identity

$$\begin{aligned} \star(a_{i1}e^1 + a_{i2}e^2) \wedge \star(a_{j1}e^1 + a_{j2}e^2) &= -(a_{i1}a_{j2}e^2 \wedge e^1 + a_{i2}a_{j1}e^1 \wedge e^2) \\ &= a_{i1}e^1 \wedge a_{j2}e^2 + a_{i2}e^2 \wedge a_{j1}e^1 \end{aligned}$$

Thus

$$\star a_i \wedge \star a_j = a_i \wedge a_j$$

Hence the two form Θ satisfies

$$\Theta(J_{\mathcal{A}}(a_i, \phi_i), J_{\mathcal{A}}(a_j, \phi_j)) = \Theta((a_i, \phi_i), (a_j, \phi_j))$$

Note that the components Θ_{ij} are constant thus the two form Θ is closed. To see this we consider a generic point (A, ψ) in the space $\mathcal{A}$ of connections and sections (so we are not assuming yet that the components satisfy and PDEs). At this point we can define the tangent space $T\mathcal{A}$. This space can be written as a direct sum of two vector spaces, the first isomorphic to the infinite dimensional space of one forms on the Riemann surface with values in $i\mathbf{R}$ and the second isomorphic to the infinite dimensional space of zero forms on the Riemann surface with values in $\mathbf{C}$,

$$T\mathcal{A} = \Omega^1(\Sigma_g, i\mathbf{R}) \oplus \Omega^0(\Sigma_g, \mathbf{C}).$$

Thus the exterior derivative on $T\mathcal{A}$ is the same as the exterior derivative on the complex curve, namely d. A typical point in $T\mathcal{A}$ can be written as (a_i, ϕ_i). If we let x_1 and x_2 be local coordinates on the Riemann surface then a_i and ϕ_i can be written locally as

$$\begin{aligned} a_i &= f_1(x_1, x_2)dx_1 + f_2(x_1, x_2)dx_2 \\ \phi_i &= f(x_1, x_2), \end{aligned}$$

where f_1, f_2 and f are functions on the complex curve. Since the components Θ_{ij}, defined above, are given by an integral of a combination of a's and ϕ's over the Riemann surface, they are indeed constants. Since the exterior derivative on $T\mathcal{A}$ is just d we conclude that the two form Θ is closed.

A Riemannian metric G can be defined on the infinite dimensional space $\mathcal{A}$ by setting

$$G((a_i,\phi_i),(a_j,\phi_j)) = -\Theta(J_{\mathcal{A}}(a_i,\phi_i),(a_j,\phi_j))$$

It is then easy to see that the metric G satisfies

$$\Theta((a_i,\phi_i),(a_j,\phi_j)) = G(J_{\mathcal{A}}(a_i,\phi_i),(a_j,\phi_j))$$

and

$$\begin{aligned} G((a_i,\phi_i),(a_i,\phi_i)) &= -\Theta(J_{\mathcal{A}}(a_i,\phi_i),(a_i,\phi_i)) \\ &= \Theta((a_i,\phi_i),J_{\mathcal{A}}(a_i,\phi_i)) \end{aligned}$$

hence G is positive definite. Writing the a_i's in terms of non-coordinate basis we get

$$\begin{aligned} \star a_i \wedge a_j &= \star(a_{i1}e^1 + a_{i2}e^2) \wedge (a_{j1}e^1 + a_{j2}e^2) \\ &= a_{i1}a_{j1}e^2 \wedge e^1 - a_{i2}a_{j2}e^1 \wedge e^2 \\ &= a_{j1}e^2 \wedge a_{i1}e^1 - a_{j2}e^1 \wedge a_{i2}e^2 \\ &= \star a_j \wedge a_i. \end{aligned}$$

This, along with the identity

$$Im < i\phi_i, \phi_j >= Im < i\phi_j, \phi_i >$$

imply that

$$G((a_i,\phi_i),(a_j,\phi_j)) = G((a_j,\phi_j),(a_i,\phi_i)).$$

Hence the metric G on the space $\mathcal{A}$ is Riemannian. It is easy to see that G is also Hermitian, this is because

$$\begin{aligned} G(J_{\mathcal{A}}(a_i,\phi_i),J_{\mathcal{A}}(a_j,\phi_j)) &= -\Theta(J_{\mathcal{A}}^2(a_i,\phi_i),J_{\mathcal{A}}(a_j,\phi_j)) \\ &= \Theta((a_i,\phi_i),J_{\mathcal{A}}(a_j,\phi_j)) \\ &= G(a_i,\phi_i),(a_j,\phi_j)). \end{aligned}$$

Putting everything together, we conclude that the space $\mathcal{A}$ Kähler.

So far we have discussed the properties of the space $\mathcal{A}$. The space we are interested in however is a subspace of the quotient of the space $\mathcal{A}$ by the Lie group $\mathcal{G}$, namely $\mathcal{M}$. One might think that even if we start with a Kähler space there would be no reason for its quotient with a Lie group to be Kähler as well. Marsden and Weinstein showed that in the case that the space is symplectic, one can define its symplectic quotient (also called symplectic reduction) by a Lie group in terms of moment maps. In [13] it was shown that the MW reduction can also be applied to the Kähler (and hyperKähler) case as well, and this is what we will use to transfer the Kähler structure from the space $\mathcal{A}$ to its quotient by $\mathcal{G}$.

To start I shall write the monopole equations in a different way. The gauge bundle L can be written as the direct product $K^{-1/2}\otimes(K^{1/2}\otimes L)$. If we fix a base connection A' on $K^{-1/2}$ then any connection A can be written as $A'+B$ where B is a one form connection on the twisted positive spin bundle $K^{1/2}\otimes L$. An elementary calculation (included in the next section of the thesis) shows that if $A^{'} = -iW_{12}/2$ then the Dirac operator coupled to $A^{'}$ is just $\overline{\partial}$. In terms of the connection B, the first monopole equation can be written as

$$\overline{\partial}_B \psi = 0$$

where $\overline{\partial}_B$ is the Dolbeault aperator coupled to B. The second monopole equation takes the form

$$F_B = F_A - idA^{'} = -\frac{1}{2}|\psi|^2 w - \frac{1}{2}Kw = -\frac{1}{2}[|\psi|^2 + \frac{s}{2}]w$$

where K and s are respectively the Gaussian and scalar curvature of the complex curve. Mathematically, if the two form F_B is of type (1, 1), we say that the vector bundle $K^{1/2} \otimes L$ is holomorphic and ψ is a holomorphic section of it. Thus the monopole equations can be interpreted in terms of holomorphic geometry, the data being the pair ψ and B. Let us define the space $\mathcal{H}^{1,1}$ by

$$\mathcal{H}^{1,1} = \{(B, \psi) : \overline{\partial}_B \psi = 0, F_B \text{ is of type } (1,1)\}.$$

We will also define a map μ by

$$\mu(A, \psi) \equiv F_A + \frac{1}{2}|\psi|^2 w = F_B + \frac{1}{2}[|\psi|^2 + \frac{s}{2}]w.$$

We can easily see that the moduli space of monopoles modulo gauge transformations can be written as

$$\mathcal{M} = (\mathcal{H}^{1,1} \cap \mu^{-1}(0))/\mathcal{G}.$$

In other words, we can think of the moduli space $\mathcal{M}$ as simply the space of solutions of holomorphic pairs satisfying the second monopole equation.

This equation has a deep geometric interpretation that I shall now describe. Given the symplectic space $\mathcal{A}$ (with its Kähler form Θ) and the Lie group $\mathcal{G}$ acting on it, the moment map μ of this action satisfies

$$d\mu(\zeta) = \Theta(X_\zeta, \cdot).$$

Few remarks about the above equation follow: The symbol $-i\zeta$, for a real function ζ on the complex curve, represents an element of the Lie algebra of the gauge group. It generates the (Hamiltonian) vector field X_ζ on the space $\mathcal{A}$. The (slightly confusing) symbol $d\mu$ is thought of as a small variation of the moment map and the left hand side of the equation above is an integral of $\zeta d\mu$ over the Riemann surface. To proceed we define the vector field X_ζ and then show that the equation above is satisfied. These are the requirements to transfer the Kähler structure from $\mathcal{A}$ to $\mathcal{M}$. I shall write the quantity $d\mu$ above as

$$\begin{aligned} d\mu &= \frac{d}{dt}[(F_{A+ta} - F_A) + \frac{1}{2}(|\psi + t\phi|^2 - |\psi|^2)w]_{t=0} \\ &= ida + \frac{1}{2}(<\psi,\phi> + <\phi,\psi>)w \\ &= ida + Re <\psi,\phi> w \end{aligned}$$

where a and ϕ are small variations in the gauge field A and spinor ψ respectively. The Lie algebra valued zero form $-i\zeta$ generates a vector field on $\mathcal{A}$ as follows: $X_\zeta(A,\psi) \equiv (-id\zeta, -i\zeta\psi)$. Since $-i\zeta \in \Omega^0(\Sigma_g, i\mathbf{R})$, and using the fact that $d(i\zeta \wedge a)$ is a closed form that integrates over the Riemann surface to zero, we can write

$$d\mu(\zeta) \equiv \int_{\Sigma_g} \zeta(ida + Re <\psi,\phi> w)$$

$$
\begin{aligned}
&= \int_{\Sigma_g} [(-id\zeta \wedge a) + (Re < \zeta\psi, \phi > w) \\
&= \int_{\Sigma_g} [(-id\zeta \wedge a) + (Im < -i\zeta\psi, \phi > w)] \\
&= \Theta(X_\zeta, \cdot)
\end{aligned}
$$

which is what we wanted to prove. This finishes our discussion about the Kähler property of $\mathcal{M}$.

Recall that $\mathcal{M}$ is an even dimensional space of dimension, say, $2n$. If we let α^i and β^i (where $i = 1, \ldots, n$) form a basis of non-coordinate one-forms on the cotangent space of $\mathcal{M}$ then the Kähler form can be written as

$$\Theta = \sum_{i=1}^{n} \alpha^i \wedge \beta^i$$

where $\wedge$ is the wedge product on $\mathcal{M}$. Thus if we take the wedge product of Θ with itself n-times we get

$$\Theta^n = n!\, \alpha^1 \wedge \beta^1 \wedge \ldots \wedge \alpha^n \wedge \beta^n.$$

Since $\mathcal{M}$ is complex, it is oriented. The $2n$ form $\alpha^1 \wedge \beta^1 \wedge \ldots \wedge \alpha^n \wedge \beta^n$ will serve as a volume form on $\mathcal{M}$ and lives in $H^{2n}(\mathcal{M}, \mathbf{R})$.

4.3 Zero Dimensional Moduli Spaces

Recall that the dimension d of the moduli space is given by $2(c_1(L)[\Sigma_g]+g-1)$. If $d > 0$, the moduli space is as described in the previous section. When $d < 0$ there are generically no solutions to the monopole equations. In the case where $d = 0$ the moduli space consists of a finite number of points (because $\mathcal{M}$ is compact). Let us consider this case in this section. From the dimension formula we see that for d to vanish we need the condition $c_1(L)[\Sigma_g] = 1 - g$. This means that the gauge bundle L is isomorphic to the line bundle $K^{-1/2}$. On the bundle $K^{-1/2}$ there is a unique choice of connection A' such that the Dirac operator reduces to the Dolbeault operator $\overline{\partial}$. If we write the Levi-Civita connnection on the tangent bundle as a purely imaginary one form iW_{12} then the gauge field A' is given by

$$A' = -\frac{1}{2} i W_{12}.$$

Recall that the spin$(2)^c$ covariant derivative on the Riemann surface is given by

$$d_{\phi^c} = d + IA' - \frac{1}{4} \sum_{i,j=1}^{2} W_{ij} \gamma_i \gamma_j.$$

The covariant derivative can be rewritten in matrix form as

$$d_{\phi^c} = d + \begin{pmatrix} -iW_{12}/2 & 0 \\ 0 & -iW_{12}/2 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -iW_{12} & 0 \\ 0 & +iW_{12} \end{pmatrix}.$$

In this case, the Dirac operator corresponding to $A' = -iW_{12}/2$ becomes

$$D^+_{A'} = \gamma_i (d)_i.$$

If we work locally on an open set U_α with local coordinates $z = x_1 + ix_2$ we see that the Dirac operator is written in matrix form as

$$D^+_{A'} = \begin{pmatrix} 0 & -\frac{\partial}{\partial_{x_1}} + i\frac{\partial}{\partial_{x_2}} \\ \frac{\partial}{\partial_{x_1}} + i\frac{\partial}{\partial_{x_2}} & 0 \end{pmatrix}.$$

Acting on sections ψ of $K^{1/2} \otimes L$, the Dirac operator becomes

$$D^+_{A'} \begin{pmatrix} \psi \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{\partial}{\partial_{x_1}} \psi + i \frac{\partial}{\partial_{x_2}} \psi \end{pmatrix}$$

Thus the Dirac equation becomes nothing but

$$\overline{\partial}\psi = 0.$$

So we see that ψ, considered as a $(0,0)$ form, is a holomorphic section of the line bundle $K^{1/2} \otimes L$. In the case of a zero dimensional moduli space, the line bundle L is isomorphic to $K^{-1/2}$ hence ψ is a section of the trivial bundle $T \equiv K^{1/2} \otimes K^{-1/2}$. Since any holomorphic function on a Riemann surface is constant, we can think of ψ as a globally defined section of T. Let $|\psi|^2 = c$ where c is a positive constant, then using the second monopole equation we get

$$F_{A'} = -\frac{1}{2}|\psi|^2 w = -\frac{1}{2}cw = idA' = \frac{1}{2}dW_{12} = \frac{1}{2}Kw = \frac{1}{4}sw$$

where s, K and w are the scalar curvature, Gaussian curvature and the Kähler form of Σ_g respectively. Hence $c = -s/2$ or in other words the metric on the Riemann surface has negative constant curvature. Thus in the case where the dimension of the moduli space vanish, solutions of the monopole equations exists only when s is a constant. Combining this fact with the uniformization theorem (which states that every Riemann surface with $g > 1$ admits a metric with constant negative curvature) we get the following

Existence Theorem: Every Riemann surface of positive genus admits a nonempty zero dimensional monopole moduli space.

4.4 Nonabelian Monopoles

In this section I present the nonabelian version of the monopole equations on Riemann surfaces of genus $g > 1$ and study the associated moduli space.

Unlike Yang-Mills theory which involves the study of the moduli space of flat connections, nonabelian monopoles involve the study of spinors and Yang-Mills connections and thus appear to be a natural generalization of Yang-Mills theory.

To start let us recall some basic facts about Yang-Mills theory on Riemann surfaces [1]. Let E be an $SU(2)$ vector bundle on a closed oriented surface Σ_g and let $\mathcal{A}_{YM}$ be the space of unitary connections in E. The Yang-Mills functional is the map $YM : \mathcal{A}_{YM} \longrightarrow \mathbf{R}^+$ defined by

$$YM(A) = -\frac{1}{2}\int_{\Sigma_g} tr(F_A \wedge \star F_A)$$

where $F_A = dA + A \wedge A$ is the curvature of the connection. Thus if A is an $su(2)$ connection, the Yang-Mills action becomes

$$YM(A) = \int_{\Sigma_g} (|F_A^1|^2 + |F_A^2|^2 + |F_A^3|^2) dV$$

where F_A^1, F_A^2 and F_A^3 are three real two-forms on the Riemann surface. Clearly $YM(A) \geq 0$, with equality if and only if the connection A is flat, i.e. $F_A = 0$. The group $\mathcal{G}$ of gauge transformations acts on the space $\mathcal{A}_{YM}$ by

$$(g, A) \rightarrow gAg^{-1} + gdg^{-1}.$$

Finally, the moduli space $\mathcal{M}_{YM}$ is given by the quotient space of flat $su(2)$ connections modulo gauge transformations and has dimension $6(g-1)$. In general, if the structure group is taken to $SU(k)$, the dimension of $\mathcal{M}_{YM}$ becomes $\dim G(2g-2) = 2(k^2-1)(g-1)$.

We now move on to study the geometry of nonabelian monopoles. To start, fix a complex line bundle $\mathcal{L} \equiv K^{1/2}$ on the Riemann surface. We also need to consider a nonabelian structure group, for example $SU(k)$, on a complex vector bundle E . We will see that with this choice of $\mathcal{L}$ we get interesting results about the moduli space of nonabelian monopoles. These equations are simply

$$\begin{aligned} D_A^+\psi^i &= 0 \\ F_A^a &= -\frac{i}{2}\overline{\psi}^i(T^a)^{ij}\psi^j w. \end{aligned}$$

Here A is a connection on $\mathcal{L}\otimes E$, ψ^i are the components of the section ψ of the bundle $K^{1/2}\otimes(\mathcal{L}\otimes E)$, T^a are the generators of the Lie algebra of the structure group G of the vector bundle $\mathcal{L}\otimes E$, w is the Kähler form on the Riemann surface and the indices i and j run from 1 to N where N is the dimension of the representation of G. From now on, I will restrict the study to the case where the structure group of E is $SU(k)$. We can think of G as the Lie group $U(k)$ and hence $\mathcal{L}\otimes E$ is a $U(k)$ vector bundle. Note that the first monopole equation is simply the Dirac equation with the Dirac operator coupled to the gauge connection A

$$D^+_{A_0,A}\psi^i = \gamma_m(\partial_m\psi^i - \frac{1}{4}\sum_{k,l=1}^{2} W_{klm}\gamma_k\gamma_l\psi^i + A^a_m(T^a)^{ij}\psi^j)$$

I will now define the space of connections and sections by

$$\mathcal{A} = \{(A,\psi) : A \text{ on } \mathcal{L}\otimes E, \psi\in\Gamma(K^{1/2}\otimes(\mathcal{L}\otimes E))\}.$$

The group $\mathcal{G}$ of gauge transformations acts on the space $\mathcal{A}$ by

$$g:(A,\psi)\to(gAg^{-1}+gd(g^{-1}),g\psi).$$

The moduli space of non abelian monopoles is the defined by

$$\mathcal{M}_{NA} = \{[A,\psi]\in\mathcal{A}/\mathcal{G} : (A,\psi) \text{ satisfying the nonabelian equations}\}.$$

A first step to understand the structure of the moduli space of solutions modulo gauge transformations is to compute its dimension. To determine the dimension of $\mathcal{M}_{NA}$ we fix a point in it and see in how many directions we can go while remaining in the moduli space. Hence if (A, ψ) satisfy the nonabelian monopole equations, we want $(A + \delta' A, \psi + \delta' \psi)$ to also satisfy the nonabelian monopole equations. After linearizing the equations we get

$$\begin{aligned} \dim \mathcal{M}_{NA} &= \dim \mathcal{M}_{YM} + 2 \,\text{index}\, D_A^+ \\ &= k^2(2g-2) + 2Rc_1(K^{1/2}) \end{aligned}$$

where R is the dimension of the representation of G. The dimension of the moduli space of nonabelian monopoles is thus given by

$$\dim \mathcal{M}_{NA} = 2[k^2 + R](g-1).$$

It is easy to see that the spave $\mathcal{M}_{NA}$ is larger than that of the Yang-Mills fields.

As in the abelian case, the moduli space $\mathcal{M}_{NA}$ is Kähler. Let (a_i, ϕ_i) and (a_j, ϕ_j) be tangent vectors (to points (A_i, ψ_i) and (A_j, ψ_j)) in the tangent space of $\mathcal{A}$. We define the map $J_{\mathcal{A}}$ by

$$J_{\mathcal{A}} : (a_i, \phi_i) \to (\star a_i, i\phi_i)$$

It is then clear that $J_{\mathcal{A}} = -1$. We define the Kähler two-form Θ_{ij} by

$$\Theta_{ij} = \int_{\Sigma_g} [tr(-a_i \wedge a_j) + Im < \phi_i, \phi_j > w]$$

Here $<,>$ is the inner product on the fiber of the bundle $K^{1/2} \otimes (\mathcal{L} \otimes E)$ and it is understood that the inner product is on the components of the spinor.

This Kähler form Θ descends to a Kähler form on the space $\mathcal{M}_{NA}$. The trick is to use moment maps however I shall not include the details here since they are very similar to the ones we considered in the abelian case. Recall the $\mathcal{M}_{NA}$ is an even dimensional space. Hence if we let m be such that $2m = \dim\mathcal{M}_{NA}$ and take α^i and β^i (where $i = 1, \ldots, m$) to form a basis of non-coordinate one-forms on the cotangent space of $\mathcal{M}_{NA}$, then the volume form on the moduli space can be expressed as

$$dV = \frac{1}{m!}\Theta^m = \alpha^1 \wedge \beta^1 \wedge \ldots \wedge \alpha^m \wedge \beta^m.$$

Since $\mathcal{M}_{NA}$ is complex, it is oriented, the orientation being given by the volume form dV.

It is important to note that the nonabelian monopole moduli space is, as the Yang-Mills moduli space, not compact. One will have to use sophisticated methods of algebraic geometry to find a suitable compactification. This space has two types of singularities. The first type occurs whenever a constant gauge transformation acts at the points $(A, 0)$. In this case, the nonabelian monopole equations reduce to the equation which defines flat connections $F_A = 0$. The second type of singularity occurs whenever the vector bundle E splits as a sum of smaller rank vector bundles. For example if E is an $SU(2)$ bundle, it will split as $V \oplus V^{-1}$ where V^{-1} is the inverse (or dual) line bundle of V. In this case we obtain the abelian monopole equations for a connection A on $\mathcal{L} \otimes V$ and a section $\psi \in \Gamma(K^{1/2} \otimes \mathcal{L} \otimes V)$ where we identify $\mathcal{L} \otimes V$ with the gauge bundle in the abelian case L. Thus the Yang-Mills and abelian monopole moduli spaces appear as the singular points of the moduli space of nonabelian monopoles.

Finally, it seems that Yang-Mills theory and the abelian monopole theory are related within a mixed theory of nonabelian monopoles. This suggests that instead of working with flat unitary connections on Riemann surfaces, one can get an alternative description using the abelian monopole equations. This is a two dimensional analogue of the electromagnetic duality conjecture [19] and duality in $N = 2$ supersymmetric Yang-Mills theory [25]-[27].

4.5 Three Dimensional Topology

I finish this chapter with some speculations about some possible relation with three dimensional topology. The modern way to study three dimensional manifolds (and knots) is via Witten's approach to quantum SU(2) Chern Simons theory [31], [32]. If we consider a 4-manifold X of the form $Y \times S^1$, where Y is a 3-manifold, and study instantons over X (which we shall cover in the next chapter), we find that the anti-self-duality equation reduces to $F_A = 0$ over Y. This equation minimizes the Chern Simons functional

$$CS = \int_Y A \wedge dA + \frac{2}{3} A \wedge A \wedge A$$

In order to study the behavior of Chern Simons theory on an arbitrary three manifold, we chop Y into pieces, we solve the theory on the pieces, and then we "glue" the pieces back together. If we cut Y along a Riemann surface Σ_g, then Y looks near the cut like the product manifold $\Sigma_g \times \mathbf{R}$. Canonically quantizing the theory gives a constraint on the gauge fields in the form of Poisson brakets. If we fix the gauge in such a way that the gauge field is zero in the $\mathbf{R}$ direction, the critical points of the Chern Simons functional reduce to flat connections on Σ_g. As we can see, we are led to the study of the moduli space of flat connections on Riemann surfaces. As we saw before this space, along with the moduli space of abelian monopoles, appear as singular points on the moduli space of non abelian monopoles. We were also led to speculate that the two singular theories are "dual" to each other. This means that instead of studying the moduli space of SU(2) connections, one might get an alternate point of view by using the moduli space of abelian monopoles.

One should note that the SU(2) Chern Simons action is metric independent since it only involves the gauge field A. If one considers the path integral $Z(Y)$ of (a constant multiple of) the CS action, a metric has to be introduced for gauge fixing. Witten introduces a counterterm that makes $Z(Y)$ independent of the metric, and hence a topological invariant. On the other hand, in the monopole case going from four to three dimensions, the dimensional reduction leads to a pair of coupled partial differential equations whose solutions are the critical points of the Chern Simons Dirac functional CSD [15]. This functional is metric dependent since it contains the Dirac operator. Perhaps the way to proceed is to take the path integral of CSD and then search for a counterterm that will get rid of the metric dependency.

Quantization can be achieved, from a mathematical point of view, via geometric quantization. This consists of constructing a connection on a holomorphic line bundle over the space $\mathcal{A}$ whose curvature is the Kähler form on $\mathcal{A}$. Here $\mathcal{A}$ is the space of $U(1)$ connections and positive chirality spinors on Σ_g. One then has to show that this holomorphic line bundle when restricted to the moduli space $\mathcal{M}$ has a curvature given by the Kähler form on $\mathcal{M}$. Finally, if we write our (space-time) manifold Y near the cut as $\Sigma_g \times \mathbf{R}$ then a quantum field theory must assign a Hilbert space to each Riemann surface Σ_g. Holomorphic sections in this Hilbert space will represent possible states of the universe given that the space manifold is Σ_g.

This is only half of the story however. What we are also interested in is knots in 3-manifolds. The gauge theory approach to studying knot theory is by considering Riemann surfaces with marked points. Let us consider a manifold Y with a knot in it. If we chop Y along a Riemann surface, then at any point in $\mathbf{R}$, the knot leaves marks along Σ_g. From a physicist's point of view, the inclusion of knots amounts to multiplying the Wilson loops to the exponential of the CSD functional. These Wilson loops are metric independent quantities of the form $exp(\int_{C_i} A)$ where C_i is topologically a circle in Y. Links can also be studied by considering products of Wilson loops with non-intersecting knots. Note that at points where the spinor vanishes, we are reduced to the abelian Chern Simons functional with Wilson loops. This scenario was already discussed in [23], [31] and [32]. This is another hint that the monopole equations might shed some light about three dimensional topology.

5 Donaldson and Seiberg-Witten Theories

In this section I will give a brief introduction to four dimensional manifolds. Good references for the material discussed here can be found in [7], [10], [20] and [28].

5.1 Four-Manifolds

Let p be a point in a compact oriented four-manifold X and let $V \equiv T_p^* X$ be the cotangent space at p. The Betti numbers of X are determined by b_0 (which is 1 if X is connected), b_1 (which is 0 if X is simply connected) and b_2. If we let (e^1, e^2, e^3, e^4) be an orthonormal basis for V then

$$e^1 \wedge e^2, \;\; e^1 \wedge e^3, \;\; e^1 \wedge e^4, \;\; e^2 \wedge e^3, \;\; e^2 \wedge e^4, \;\; e^3 \wedge e^4$$

span $\Lambda^2 V$. We define the four dimensional Hodge star as a map

$$\star \, \Lambda^p V \longrightarrow \Lambda^{4-p} V$$

by

$$\star 1 = e^1 \wedge e^2 \wedge e^3 \wedge e^4; \;\; \star(e^i \wedge e^j) = \epsilon \, e^k \wedge e^l; \;\; \star e^i = \epsilon \, e^j \wedge e^k \wedge e^l$$

where $\epsilon = +1$ if (i, j, k, l) is an even permutation of $(1, 2, 3, 4)$ and -1 otherwise. The second Betti number b_2 possesses a further decomposition which I will now describe. Since $\star^2 = 1$ on two-forms, we can divide $\Lambda^2 V$ into

$$\Lambda^2 V = \Lambda^{2,+} V \oplus \Lambda^{2,-} V,$$

where

$$\Lambda^{2,+} V = \{w \in \Lambda^2 V : \star w = w\}; \;\; \Lambda^{2,-} V = \{w \in \Lambda^2 V : \star w = -w\}.$$

Thus $\Lambda^{2,+}V$ is generated by

$$e^1 \wedge e^2 + e^3 \wedge e^4, \;\; e^1 \wedge e^3 + e^4 \wedge e^2, \;\; e^1 \wedge e^4 + e^2 \wedge e^3,$$

while $\Lambda^{2,-}V$ is generated by

$$e^1 \wedge e^2 - e^3 \wedge e^4, \;\; e^1 \wedge e^3 - e^4 \wedge e^2, \;\; e^1 \wedge e^4 - e^2 \wedge e^3.$$

Sections of the bundles whose fibers are $\Lambda^{2,+}V$ and $\Lambda^{2,-}V$ are called self-dual and anti-self-dual forms respectively. If w is a two-form on X, it can be divided into

$$\begin{aligned} w^+ &= \frac{1}{2}(w + \star w) \in \Omega^{2,+} = \{\text{ self} - \text{dual two} - \text{forms on } X\} \\ w^- &= \frac{1}{2}(w - \star w) \in \Omega^{2,-} = \{\text{ anti} - \text{self} - \text{dual two} - \text{forms on } X\} \end{aligned}$$

Note that the self-dual and anti-self-dual components of a harmonic two-form are again harmonic. Thus the space of harmonic two-forms on X can be written as

$$\mathcal{H}^2(X) \cong \mathcal{H}^{2,+}(X) \oplus \mathcal{H}^{2,-}(X),$$

the two summands consisting of self-dual and anti-self-dual harmonic two-forms, respectively. Let

$$b_2^+ = \dim \mathcal{H}^{2,+}(X), \;\; b_2^- = \dim \mathcal{H}^{2,-}(X).$$

Then $b_2^+ + b_2^- = b_2$, the second Betti number of X, while

$$\tau(X) = b_2^+ - b_2^-$$

is called the signature of X.

In four-dimensions, we can regard the four dimensional spin group Spin(4) as the space of (4×4) matrices

$$\begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix}, \text{where } A_\pm \in SU_\pm(2).$$

Topologically, Spin(4) is a product $S^3 \times S^3$, hence simply connected. This Lie group of Dimension six is contained in an important Lie group of dimension seven,

$$\text{Spin}(4)^c = \begin{pmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{pmatrix} : A_+ \in SU_+(2),\ A_- \in SU_-(2),\ \lambda \in U(1).$$

A spin structure on X is given by an open covering $\{U_\alpha \subset X\}$ and a collection of transition functions

$$\tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to \text{Spin}(4)$$

such that the cocycle condition

$$\tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma} = \tilde{g}_{\alpha\gamma} \text{ on } U_\alpha \cap U_\beta \cap U_\gamma$$

is satisfied. Manifolds which admit spin structures are called spin manifolds. Thus if we have a spin structure on X, the transition functions

$$\tilde{g}^+_{\alpha\beta} : U_\alpha \cap U_\beta \to SU_+(2),\ \ \tilde{g}^-_{\alpha\beta} : U_\alpha \cap U_\beta \to SU_-(2)$$

determine complex vector bundles of rank two over X, which we denote by S^+ and S^-.

A spin^c structure on X is given by an open covering $\{U_\alpha\}$ and a collection of transition functions

$$\tilde{g}^c_{\alpha\beta} : U_\alpha \cap U_\beta \to \text{Spin}(4)^c$$

such that the cocycle condition is satisfied.

If X has a spin structure defined by the transition functions

$$\tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \mathrm{Spin}(4)$$

and L is a complex line bundle over X with transition functions

$$h_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow U(1),$$

we can define a spin^c structure on X by taking the transition functions to be

$$h_{\alpha\beta}\tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \mathrm{Spin}(4)^c.$$

Note that the bundles $S^+ \otimes L$ and $S^- \otimes L$ exist as genuine vector bundles even though the factors S^+, S^- and L do not exist unless X is spin. In a sense S^+, S^- and L are virtual vector bundles.

Let V be the four-dimensional real vector space $\mathbf{R}^4$ with orthonormal basis $\{e_1, e_2, e_3, e_4\}$. The Clifford algebra $Cl(V)$ is the algebra over $\mathbf{R}$ generated by $\{e_1, e_2, e_3, e_4\}$ subject to the relations

$$\begin{aligned} \gamma(e_i)\gamma(e_j) + \gamma(e_j)\gamma(e_i) = -2\delta_{ij} = -2 \text{ if } i &= j \\ = 0 \text{ if } i &\neq j \end{aligned}$$

Here $\gamma_i \equiv \gamma(e_i)$ are complex 4×4 matrices taken for example to be

$$\gamma_1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & 0 & +i & 0 \\ 0 & 0 & 0 & -i \\ +i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix},$$

$$\gamma_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & +1 & 0 \\ 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 0 & 0 & 0 & +i \\ 0 & 0 & +i & 0 \\ 0 & +i & 0 & 0 \\ +i & 0 & 0 & 0 \end{pmatrix}.$$

As we saw before, $\Lambda^2 V$ can be divided into

$$\Lambda^2 V = \Lambda^{2,+} V \oplus \Lambda^{2,-} V,$$

the self-dual part being generated by

$$e^1 \wedge e^2 + e^3 \wedge e^4, e^1 \wedge e^3 + e^4 \wedge e^2, e^1 \wedge e^4 + e^2 \wedge e^3.$$

The corresponding elements in the Clifford Algebra are

$$\begin{pmatrix} -2i & 0 & 0 & 0 \\ 0 & +2i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & +2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & -2i & 0 & 0 \\ -2i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus we see that $\Lambda^{2,+}$ is just the Lie algebra of $SU_+(2)$.

A connection ϕ on $S = S^+ \oplus S^-$ is called a spin(4) connection if it can be expressed in terms of each local trivialization as a one-form with values in the Lie algebra of Spin(4). The covariant derivative on spinors is expressed locally as

$$(d_\phi \sigma)_\alpha = d\sigma_\alpha + \phi_\alpha \sigma_\alpha.$$

Given a unitary connection A on a complex line bundle L over X, we can define a spin$(4)^c$ connection on the bundle $S \otimes L$, expressed in terms of each local trivialization as

$$\phi_\alpha^c = A_\alpha I + \phi_\alpha$$

where I denotes the 4×4 identity matrix. The spin$(4)^c$ covariant derivative acting on spinors is given locally by

$$(d_{\phi^c}\sigma)_\alpha = d\sigma_\alpha + \phi_\alpha^c \sigma_\alpha.$$

We can now define the Dirac operator $D_A^+ : \Gamma(S^+ \otimes L) \rightarrow \Gamma(S^- \otimes L)$ acting on sections of the positive spin bundle $S^+ \otimes L$ by

$$D_A^+\psi = \gamma_i (d_\phi^c)_i \psi.$$

On a compact oriented four-manifold X, the Atiyah-Singer index theorem for the Dirac operator is

$$\text{index of } D_A^+ = -\frac{1}{8}\tau(X) + \frac{1}{2}\int_X c_1(L) \wedge c_1(L),$$

where $\tau(X) = b_2^+ - b_2^-$ is the signature of X.

5.2 Donaldson Theory

The starting point in Donaldson theory is non-abelian gauge theory with gauge group $SU(2)$. To start, let X be a closed oriented Riemannian 4-manifold and let E be an $SU(2)$ bundle over X. If A is a connection on E (i.e. a matrix-valued one-form taking its value in $su(2)$, the Lie algebra of $SU(2)$) with curvature F_A given by

$$F_A = dA + A \wedge A$$

then we define the Yang-Mills functional as

$$S_{YM} = \frac{1}{8\pi^2}\int_X Tr(F_A \wedge \star F_A).$$

A critical point of this functional is a Yang-Mills connection or an instanton. Let $t \in \mathbf{R}$ and a be an $su(2)$ valued one-form then expanding in the vicinity of $t = 0$ gives

$$\begin{aligned}
S(A+ta) &= \frac{1}{8\pi^2}\int_X Tr\{[(F_A + t(da + A\wedge a + a\wedge A) + t^2 a\wedge a]\wedge \\
\star\ & [(F_A + t(daA\wedge a + a\wedge A) + t^2 a\wedge a]\} \\
&= \frac{1}{8\pi^2}\int_X Tr(F_A\wedge\star F_A) \\
&+ \frac{t}{8\pi^2}\int_X Tr[F_A\wedge\star(da + A\wedge a + a\wedge A)] \\
&+ \frac{t}{8\pi^2}\int_X Tr[(da + A\wedge a + a\wedge A)\wedge\star F_A] + \dots \\
&= S(A) + \frac{2t}{8\pi^2}\int_X Tr[(da + A\wedge a + a\wedge A)\wedge\star F_A] + \dots \\
&= S(A) + \frac{2t}{8\pi^2}\int_X Tr(a\wedge\star\star d\star(-F_A) - a\wedge A\wedge\star F_A \\
&+ a\wedge\star F_A\wedge A) + \dots \\
&= S(A) - \frac{2t}{8\pi^2}\int_X Tr(a\wedge(d\star F_A + [A,\star F_A])) + \dots
\end{aligned}$$

Since a is arbitrary, the critical points of the Yang-Mills functional are solutions to the Yang-Mills equation

$$d\star F_A + [A,\star F_A] = 0.$$

Differentiation of $F_A = dA + A\wedge A$ yields

$$\begin{aligned}
dF_A &= dA\wedge A - A\wedge dA \\
&= (dA + A\wedge A)\wedge A - A\wedge(dA + A\wedge A)
\end{aligned}$$

$$\begin{aligned} &= F_A \wedge A - A \wedge F_A \\ &= [F_A, A]. \end{aligned}$$

We have just proved the Bianchi identity,

$$dF_A + [A, F_A] = 0.$$

If $\star F_A$ is a scalar multiple of F_A then the Bianchi identity implies the Yang-Mills equations. Since $\star^2 = 1$, the only possibilities are $\star F_A = F_A$, in which case we say that A is a self-dual connection, or $\star F_A = -F_A$, in which case we say that A is an anti-self-dual (ASD) connection. These connections, if they exist, are absolute minima for the Yang-Mills functional. To see this, decompose F_A into its self-dual and anti-self-dual parts F_A^+ and F_A^-,

$$\begin{aligned} F_A &= \frac{1}{2}(F_A + \star F_A) + \frac{1}{2}(F_A - \star F_A) \\ &= F_A^+ + F_A^- \\ \Rightarrow S(A) &= \frac{1}{8\pi^2}\int_A Tr(F_A^+ \wedge \star F_A^+) + \frac{1}{8\pi^2}\int_A Tr(F_A^- \wedge \star F_A^-). \end{aligned}$$

Since the first Chern class of the $SU(2)$ bundle E vanishes, the only topological number that classifies instantons on four-manifolds is the instanton number

$$\begin{aligned} n &= -\frac{1}{8\pi^2}\int_X Tr(F_A \wedge F_A) \\ &= -\frac{1}{8\pi^2}\int_X Tr(F_A^+ \wedge \star F_A^+) + \frac{1}{8\pi^2}\int_X Tr(F_A^- \wedge \star F_A^-). \end{aligned}$$

Thus for ASD gauge fields we have the condition

$$n = S(A) \geq 0.$$

This gives a topological constraint on the existence of anti-self-dual connections.

We denote by $\mathcal{A}$ the space of all connections on E. The gauge group acts on the connection in the usual way. In a local trivialization a gauge transformation is a map $g : U_\alpha \to SU(2)$ such that

$$g : A \to g^{-1}Ag + g^{-1}\mathrm{d}g.$$

After picking a metric on X, we can consider anti-self-dual connections, satisfying the following equation

$$F_A^+ \equiv \frac{1}{2}(F_A + \star F_A) = 0.$$

Let $\mathcal{M}_n$ be the moduli space of ASD connections defined as the space of solutions to this equation modulo gauge equivalence (after gauge fixing). The dimension of $\mathcal{M}_n$ is calculated using the index theorem,

$$\begin{aligned}\dim \mathcal{M}_n &= 8n - \frac{3}{2}(\chi(X) + \tau(X)) \\ &= 8n - 3(1 - b_1 + b_2^+).\end{aligned}$$

As we will see later, instanton invariants are best defined for simply connected 4-manifolds. In this case, for dim $\mathcal{M}_n \geq 0$, we require n to be positive.

In Donaldson theory, the moduli spaces of anti-self-dual connections in an $SU(2)$ bundle over a compact oriented four-manifold are oriented but not compact, and much effort is needed towards finding a suitable compactification $\overline{\mathcal{M}_n}$. After finding a compactification of $\mathcal{M}_n$ one can define a fundamental class $[\overline{\mathcal{M}_n}]$ which pairs with suitable cohomology classes. These cohomology classes are given by maps

$$\mu_i : H_i(X, \mathbf{Z}) \to H^{4-i}(\overline{\mathcal{M}_n}, \mathbf{Z}).$$

The μ_i maps are defined as follows. We start by constructing an $SU(2)$ vector bundle $\tilde{E}$ on the product space $X \times \overline{\mathcal{M}_n}$ with connection $\tilde{A}$ given (at every point (p, A) in $X \times \overline{\mathcal{M}_n}$) by

$$\tilde{A}|_{(p,A)} = A(p).$$

Starting with a homology class $\beta \in H_i(X, \mathbf{Z})$ we can switch to cohomology and get a differential form $b \in H^i(X, \mathbf{Z})$. By Poincaré duality, we can construct a dual form $\theta \in H^{4-i}(X, \mathbf{Z})$. Let $\tilde{c_2}$ be the second Chern class of $\tilde{E}$. We can now consider the differential form $\theta \wedge \tilde{c_2}$ and integrate it over X to get

$$\mu_i(\beta) = \tilde{\theta} = \int_X \theta \wedge \tilde{c_2} \in H^{4-i}(\overline{\mathcal{M}_n}, \mathbf{Z}).$$

Note that since Donaldson theory is mostly studied on simply connected four-manifolds, i-dimensional cycles exist only for $i = 0, 2$ and 4. Recall that since X is connected, dim $H^4(X, \mathbf{Z}) = 1$. Using a non-coordinate basis $\{e^1, e^2, e^3, e^4\}$ for the cotangent space of X, one can write the volume form on X as $e^1 \wedge e^2 \wedge e^3 \wedge e^4$ whose Hodge dual is the zero-form $\theta = 1$. Thus

$$\mu_4(X) = \int_X \tilde{c_2} = n \in \mathbf{Z}.$$

We conclude that the only non-trivial classes are the ones corresponding to $i = 0$ and 2. The case $i = 4$ yields an elementary topological invariant of the bundle $\tilde{E}$ over $X \times \overline{\mathcal{M}_n}$, but gives no information about the differentiable structure on X.

Let us assume that $b_2^+(X)$ is odd. Then the moduli spaces are even dimensional and yield invariants

$$q_{p,r}(\alpha) = < \mu_2(\alpha)^p u^r, [\overline{\mathcal{M}_n}] >,$$

where $\alpha \in H_2(X, \mathbf{Z})$, $u = \mu_0(\text{point}) \in H^4(\overline{\mathcal{M}_n}, \mathbf{Z})$ and $2p + 4r$ equals the dimension of the moduli space. These q's are defined using a choice of Riemannian metric on X, but the essential point is that in the end one gets differential topological invariants, independent of metrics. This idea is similar to integrating the first Chern class of S^2 over the two-sphere. The final answer is a topological invariant (the Euler characteristic of S^2).

We should note that the moduli spaces $\overline{\mathcal{M}_n}$ can have singularities. These singularities occur whenever the gauge group does not act freely. This is the case when $SU(2)$ is broken to $U(1)$. In this case, we are dealing with an abelian connection, so the only singularities are the abelian instantons, i.e. connections with curvature F satisfying $F^+ = 0$. Since $F/2\pi$ represents the first Chern class of the $U(1)$ bundle, it is integral; in particular if $F^+ = 0$ then $F/2\pi$ lies in the intersection of the integral lattice in $H^2(X, \mathbf{R})$ with the anti-self-dual subspace $H^{2,-}(X, \mathbf{R})$.

When do these singularities form a serious problem? Recall that we are free to pick a metric. If we choose a generic metric then when $b_2^+ = 0$, i.e. when $H^{2,+}(X, \mathbf{R}) = 0$, the intersection of $H^{2,-}(X, \mathbf{R})$ with the integer lattice will be non-zero.

The case $b_2^+ = 1$ is also interesting but before discussing it, I would like to give a small introduction about intersection forms on 4-manifolds. Let X be a compact oriented closed 4-manifold with second integral homology group $H_2(X, \mathbf{Z})$ spanned by two-cycles Σ_i. The intersection form Q (also called the intersection matrix) of X is a $b_2 \times b_2$ matrix given by

$$Q(p, q) \equiv \int_X p \wedge q$$

where p and q are elements of $H^2(X, \mathbf{Z})$. In other words, the intersection form can be regarded as a map

$$Q : H^2(X, \mathbf{Z}) \times H^2(X, \mathbf{Z}) \to \mathbf{Z}.$$

As an example, let us consider the manifold $X = S^2 \times S^2$. Using the Künneth formula

$$H^r(S^2 \times S^2) = \oplus_{m+n=r} H^m(S^2) \otimes H^n(S^2)$$

we can calculate to calculate the second Betti number of $S^2 \times S^2$,

$$\begin{aligned} b_2(S^2 \times S^2) &= \sum_{m+n=2} b_m(S^2) b_n(S^2) \\ &= 1 \times 1 + 0 \times 0 + 1 \times 1 = 2. \end{aligned}$$

Let η_1 and η_2 be the basic two-forms on the first and second S^2 respectively normalized by

$$\int_{S_i^2} \eta_j = \delta_{ij}.$$

The Künneth formula tells us that these two-forms span the second cohomology group of $S^2 \times S^2$. Since $Q_{11} \equiv Q(\eta_1, \eta_1) = Q(\eta_2, \eta_2) \equiv Q_{22} = 0$, the only non-zero components of Q are

$$Q_{12} = \int_{S_1^2} \eta_1 \int_{S_2^2} \eta_2 = \int_{S_2^2} \eta_2 \int_{S_1^2} \eta_1 = Q_{21} = 1.$$

Thus the intersection form is given by

$$Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Note that since $Q_{ij} = Q_{ji}$, the intersection form is symmetric. It has two eigenvalues, one positive and one negative. In general, Q has c_+ positive eigenvalues and c_- negative eigenvalues.

On a 4-manifold, the Hodge star satisfies $\star^2 = 1$ when acting on two-forms and hence $\star$ has eigenvalues ± 1. As we saw before, the space of harmonic two-forms splits as

$$\mathcal{H}^2(X) \cong \mathcal{H}^{2,+}(X) \oplus \mathcal{H}^{2,-}(X)$$

according to the eigenvalue of $\star$. For $w^\pm \in \mathcal{H}^{2,\pm}(X)$ we have,

$$\begin{aligned} Q(w^+, w^+) &= \int_X w^+ \wedge w^+ = \int_X w^+ \wedge \star w^+ > 0 \\ Q(w^-, w^-) &= \int_X w^- \wedge w^- = -\int_X w^- \wedge \star w^- < 0 \\ Q(w^+, w^-) &= 0 = Q(w^-, w^+). \end{aligned}$$

Hence Q is block diagonal with respect to $\mathcal{H}^{2,+}(X) \oplus \mathcal{H}^{2,-}(X)$ and, moreover, for $X = S^2 \times S^2$ we have $b_2^+ = 1 = c_+$ and $b_2^- = 1 = c_-$.

We now turn to the metric on $S^2 \times S^2$. We can take it to satisfy

$$\star\, \eta_1 = \eta_2 \qquad \star\, \eta_2 = \eta_1.$$

In this case we obviously have $\star^2 = 1$. Let $F/2\pi \in H^2(S^2 \times S^2, \mathbf{Z})$ be a closed two-form on $S^2 \times S^2$. It can be written as an integer linear combination of η_1 and η_2,

$$\frac{F}{2\pi} = n_1 \eta_1 + n_2 \eta_2.$$

Its self-dual part is given by

$$\begin{aligned} \frac{F^+}{2\pi} &= \frac{1}{2}(n_1\eta_1 + n_2\eta_2) + \frac{1}{2} \star\ (n_1\eta_1 + n_2\eta_2) \\ &= \frac{1}{2}(n_1 + n_2)(\eta_1 + \eta_2) \end{aligned}$$

where $\eta_1 + \eta_2$ is the generator of $H^{2,+}(S^2 \times S^2, \mathbf{Z})$. From the above equation one sees that we have an abelian instanton when $n_1 = -n_2$. Thus we have $F^+ = 0$ whenever we hit the line $n_1 = -n_2$ in the two-dimensional space $H^2(S^2 \times S^2, \mathbf{Z})$.

Let us try to perturb the metric a little by setting

$$\star\, \eta_1 = \lambda \eta_2 \quad \star\ \eta_2 = \frac{1}{\lambda}\eta_1$$

where λ is a non-zero real number close to unity. With this metric the self-dual part of $F/2\pi$ becomes

$$\frac{F^+}{2\pi} = \frac{\lambda n_1 + n_2}{2\lambda}(\eta_1 + \lambda \eta_2)$$

where $\eta_1 + \lambda\eta_2$ is the generator of $H^{2,+}(S^2 \times S^2, \mathbf{Z})$. Since n_1 and n_2 are fixed, as long as $\lambda \neq 1$, F^+ is always non-zero.

In Donaldson theory, one wishes to write down topological invariants independent of the choice of metric. In the example of $X = S^2 \times S^2$, if one follows a path in the space of metrics with $\lambda < 1$ to $\lambda > 1$ then one cannot avoid passing through $\lambda = 1$. On a general four-manifold X, we will have problems finding a family of metrics g_λ interpolating between two metrics g_{λ_1} and g_{λ_2} with non-singular moduli spaces. Somewhere along the path we can have a moduli space with singularities. Therefore we restrict ourselves to manifolds with $b_2^+ > 1$.

After a period involving considerable amount of difficult calculations of Donaldson invariants, Kronheimer and Mrowka proved a deep structure theorem for a large class of 4-manifolds, those of simple type. By definition, this condition holds if the invariants are related by

$$q_{p,r+2} = 4q_{p,r}.$$

To keep the notation simple, I will write $\mu_2(\alpha)$ as μ. The equality above can now be written as

$$\int_{\overline{\mathcal{M}_{n+1}}} \mu^p u^{r+2} = 4 \int_{\overline{\mathcal{M}_n}} \mu^p u^r.$$

They showed that this condition holds for many 4-manifolds and there are no examples of simply-connected manifolds which are known not to be of simple type. If X has simple type, then under this assumption, the only interesting invariants are $q_{p,0}$ and $q_{p,1}$.

It is useful to index the spaces $\mathcal{M}_n$ by their dimension $2d$. In terms of the instanton number n, d is given by

$$d = 4n - \frac{3}{2}(1 + b_2^+).$$

Note that if $2p + 4r$ is not equal to $2d$, or if d is negative, we set $q_{p,r} = 0$. In the following I will write the instanton invariants as

$$q_{p,r} = \int_{2d} \mu^p u^r$$

where it is understood that the integration is carried over a $2d$ dimensional moduli space.

Let us first look at $q_{p,0}$. Since $r = 0$, the invariants take the simple form

$$q_d \equiv \int_{2d} \mu^d.$$

As an example, let us consider a 4-manifold with $b_2^+ = 3$. Recall that the dimension of the spaces we are integrating over is an integer modulo eight. Therefore in this case, the only non-vanishing $q_{p,0}$ are

$$\int_4 \mu^2 , \quad \int_{12} \mu^6 , \quad \int_{20} \mu^{10} \ldots$$

corresponding to $d = 2, 6, 10, \ldots$. It is convenient to combine all the invariants $q_{p,0}$ into a formal power series by summing over d,

$$D_0 \equiv \sum_d q_d/d!$$

We now turn to the $q_{p,1}$ invariants. In this case the invariants take the form

$$q_d \equiv \int_{2d} \mu^{d-2} u.$$

If we set $d' = d - 2$ then we have

$$q_{d'} = \int_{2d'+4} \mu^{d'} u.$$

In the example of a 4-manifold with $b_2^+ = 3$, the only non-vanishing $q_{d'}$ are

$$\int_4 u , \quad \int_{12} \mu^4 u , \quad \int_{20} \mu^8 u \ldots$$

corresponding to $d' = 0, 4, 8, \ldots$. Combining the invariants $q_{d'}$ into a formal power series by summing over d' gives

$$D_1 \equiv \frac{1}{2} \sum_{d'} q_{d'}/d'!$$

where the factor $1/2$ was chosen by Kronheimer and Mrowka to make some later formulas come out in a standard form. We can now define a new generating function $\mathbf{D}(\alpha)$ by

$$\mathbf{D}(\alpha) \equiv D_0 + D_1.$$

Few remarks about the new instanton invariants are in order. $\mathbf{D}(\alpha)$ is an even or odd formal power series according to the parity of

$$\mathbf{P} = -\frac{3}{2}(1 + b_2^+) \mod 2.$$

When the dimension of the moduli space is zero, i.e. when b_2^+ satisfies

$$b_2^+ = \frac{8n}{3} - 1$$

the first Donaldson invariant is defined by counting the points in $\overline{\mathcal{M}}_n$ with appropriate signs, ± 1, according to the orientation of the moduli space at the points. Note that since $\overline{\mathcal{M}}_n$ is compact, a zero-dimensional moduli space will consist of a finite number of points. I will explain the counting procedure in the next section when discussing the monopole invariants.

From the general form of $\mathbf{D}(\alpha)$ one might guess that it is some sort of an exponential "function" and this is indeed the case. Kronheimer and Mrowka's structure theorem, for manifolds of simple type, asserts that there is a finite number of classes indexed by i - called basic classes - $k_i \in H^2(X, \mathbf{Z})$ and coefficients $a_i \in \mathbf{Q}$ such that

$$\mathbf{D}(\alpha) = e^{\alpha \cdot \alpha / 2} \sum_i a_i e^{k_i \alpha}$$

where $k_i\alpha$ is defined by

$$k_i\alpha \equiv \int_\alpha k_i.$$

In the expression of $\mathbf{D}(\alpha)$ above, $\alpha \cdot \alpha$ is defined as follows. Working with integer homology, we can write α as $s_i\Sigma_i$ where the Σ_i's are the generators of $H_2(X, \mathbf{Z})$. We now have

$$\alpha \cdot \alpha = s_i\Sigma_i \cdot \Sigma_j s_j \equiv s_i Q_{ij}^{-1} s_j.$$

Thus, from the above discussion, we see that the information from the instanton invariants boils down to the basic classes in $H^2(X, \mathbf{Z})$ and the coefficients a_i. The basic classes, studied on a large number of 4-manifolds, were conjectured by Kronheimer and Mrowka to satisfy

$$k_i^2 \equiv \int_X k_i \wedge k_i = 2\chi + 3\tau.$$

for each i in the finite set of basic classes mentioned above.

5.3 Seiberg-Witten Theory

In the fall of 1994, Witten noticed that S-duality in four dimensions gives a new point of view about Donaldson theory of four-manifolds. Instead of defining four-manifold invariants by integrating over the (compactification of the) moduli space of $SU(2)$ instantons, one can define equivalent four-manifold invariants by counting solutions of the (Seiberg-Witten) monopole equations.

Let X be a closed oriented four-dimensional Riemannian manifold with a spin^c structure and corresponding positive spin bundle $S^+ \otimes L$. In Seiberg-Witten theory the ingredients are an abelian gauge field (a connection on L) and a charged spinor (a section of $S^+ \otimes L$) such that

$$D_A^+ \psi = 0, \quad F_A^+ = -\frac{i}{2} \sum_{i<j} < \psi, \gamma_i \gamma_j \psi > e^i \wedge e^j \equiv \sigma(\psi).$$

Here F_A^+ is the self-dual part of the curvature of the connection A and $<,>$ is the inner product on the fibers of $S^+ \otimes L$.

We define a real-valued functional on the space

$$\mathcal{A}' = \{(A, \psi) : A \text{ is a connection on } L, \psi \in \Gamma(S^+ \otimes L)\}$$

by the formula

$$S(A,\psi) = \int_X (|D_A^+\psi|^2 + |F_A^+ - \sigma(\psi)|^2) dV$$

where dV denotes the volume element on X. It then follows from the Weitzenböck formula that

$$S(A,\psi) \;=\; \int_X [|(d_\phi^c)_i\psi|^2 + |F_A^+|^2 + \frac{R}{4}|\psi|^2 + \frac{1}{2}|\psi|^4] dV$$

where R is the scalar curvature of X. In particular, if R is positive, there are no non-zero solutions to the Seiberg-Witten equations.

Let us work out the dimension of the moduli space $\mathcal{M}$ of solutions of the equations up to gauge transformations. After gauge fixing and linearizing the equations we can define an operator T by,

$$\begin{aligned} T : \Lambda^1(X) \oplus (S^+ \otimes L) &\to \Lambda^0 \oplus \Lambda^{2,+}(X) \oplus (S^- \otimes L) \\ T : (\delta' A, \delta' \psi) &\to (-\star d \star (\delta' A), d^+(\delta' A), D_A^+ \delta' \psi) \end{aligned}$$

where δ' denotes a small variation and d^+ is the projection of the d operator to self-dual forms.

Since we wish to determine the real dimension of $\mathcal{M}$, we temporarily think of $S^\pm \otimes L$ as real vector bundles of rank four. Thus we have

$$\begin{aligned} \dim \mathcal{M} &= -\text{index}\,(-\star d \star + d^+) + 2\,\text{index}\, D_A^+ \\ &= -(b_0 - b_1 + b_2^+ - b_3 + b_4) + 2(-\frac{\tau}{8} + \frac{1}{2}\int_X c_1(L) \wedge c_1(L)) \\ &= -\frac{(\chi + \tau)}{2} - \frac{\tau}{4} + \int_X c_1(L) \wedge c_1(L) \\ &= -\frac{2\chi + 3\tau}{4} + \int_X c_1(L) \wedge c_1(L) \end{aligned}$$

where χ and τ are the Euler charateristic and signature of X respectively.

The moduli space $\mathcal{M}$ can be shown to be compact. This follows immediately from the Weitzenböck formula. It can also be oriented by trivializing the top exterior power of the tangent bundle of $\mathcal{M}$ (i.e. the line bundle $\Lambda^d T^* \mathcal{M}$ where d is the dimension of $\mathcal{M}$.

Just as in Donaldson theory, an obstruction arises when the gauge group acts freely on the space of solutions of the monopole equations. A solution (A, ψ) of the Seiberg-Witten equations is singular (also called reducible) if $\psi = 0$. A solution with vanishing spinor necessarily has $F_A^+ = 0$. Thus to define topological invariants, assume that $b_2^+(X) > 1$.

Note that when the dimension d of $\mathcal{M}$ is negative, there are generically no solutions of the monopole equations. Let $x \equiv c_1(L)$, then in the interesting case of $d = 0$ that is when

$$x^2 \equiv 4 \int_X c_1(L) \wedge c_1(L) = 2\chi + 3\tau$$

the moduli space generically consists of a finite set of points i. With each such point, one can associate a sign $\epsilon_{i,x} = \pm 1$ depending on the orientation of $\mathcal{M}$ at that point. Once this is done, define for each x obeying the equation above an integer n_x by

$$n_x = \sum_i \epsilon_{i,x}.$$

These n_x are the Seiberg-Witten invariants. The case where $x = 0$ is an exception to the above discussion since it cannot be avoided by picking a generic metric on manifolds with $b_2^+ > 1$. To define n_0 one will have to perturb the Seiberg-Witten equations.

To see why these n_x's are indeed topological invariants, let us study an elementary example taken from [29]. Consider a single variable u and an equation

$$u^2 - a = 0$$

where a is a non-vanishing real number. Obviously, the number of solutions to this equation is not a topological invariant; there are two for positive a and none for negative a. Let $f(u) = u^2 - a$. What we want to calculate first is df/du at the zeroes of f. We then want to calculate the quantity

$$n = \sum_{zeroes} \epsilon_{zeroes}.$$

For $a < 0$, $n = 0$ since there are no zeroes. For $a > 0$, there are two zeroes, at $u = \pm\sqrt{a}$. n still vanishes since $\epsilon = \pm 1$ for $u = \pm\sqrt{a}$. Thus the point is that, while the total number of solutions is not a topological invariant, the number of solutions weighted with signs is such an invariant.

Thus the Seiberg-Witten equations give rise to new invariants of four-dimensional smooth manifolds. As Witten noticed, a compact four-manifold with positive scalar curvature must have vanishing Seiberg-Witten invariants.

The main difference between Donaldson and Seiberg-Witten theory is that the gauge group associated to the monopole equations is abelian and the main calculations involve the curvature of complex line bundles, as opposed to rank two complex vector bundles. Note that in Yang-Mills theory, curvature calculations do not give much useful information, whereas in Seiberg-Witten theory they give the crucial compactness results as well as vanishing theorems in the case of positive scalar curvature.

Witten has conjectured that, in case X is of simple type, the x's are exactly the Kronheimer-Mrowka basic classes k_i and has also conjectured the precise form of the rational numbers a_i in the Kronheimer-Mrowka formula,

$$\mathbf{D}(\alpha) = 2^{m(X)} e^{\alpha\cdot\alpha/2} \sum_x n_x e^{x\alpha}$$

where

$$m(X) = 2 + \frac{1}{4}(7\chi(X) + 11\tau(X))$$

is chosen in such a way to agree with calculations of special cases of Donaldson invariants. Thus we see that the coefficients a_i are just the Seiberg-Witten invariants, up to an overall factor $2^{m(X)}$. The challenge now is to come to grips with how these mathematical arguments fit in with quantum field theory.

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