Efficient techniques for calculating scattering amplitudes in non-abelian gauge theories.

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EFFICIENT TECHNIQUES FOR CALCULATING SCATTERING AMPLITUDES IN NON-ABELIAN GAUGE THEORIES

By

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SUBMITTED IN FULFILMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
AT
UNIVERSITY OF WALES SWANSEA
SINGLETON PARK, SWANSEA, SA2 8PP
SEPTEMBER 2006

UNIVERSITY OF WALES SWANSEA

Date: **September 2006**

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Title: **Efficient Techniques for Calculating Scattering Amplitudes in Non-Abelian Gauge Theories**

Department: **Department of Physics**

Degree: **Ph.D.** Year: **2006**

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Declaration

This thesis is based on the following papers, material from which appears throughout this work:

$N = 1$ Supersymmetric One-Loop Amplitudes and the "Holomorphic Anomaly" of Unitarity Cuts - Steven J. Bidder, N.E.J. Bjerrum-Bohr, Lance J. Dixon and David C. Dunbar. Published in Phys. Lett. B**606**:189, (2005).

Twistor Space structure of the Box Coefficients of $N = 1$ One-Loop amplitudes. - Steven J. Bidder, N.E.J. Bjerrum-Bohr, David C. Dunbar and Warren B. Perkins. Published in Phys. Lett. B**608**:151, (2005).

One-Loop Gluon Scattering Amplitudes in Theories with $N < 4$ Supersymmetries - Steven J. Bidder, N.E.J. Bjerrum-Bohr, David C. Dunbar and Warren B. Perkins. Published in Phys. Lett. B**612**:75, (2005).

Supersymmetric Ward Identities and NMHV Amplitudes involving Gluinos - Steven J. Bidder, David C. Dunbar and Warren B. Perkins. Published in JHEP **0508**:055, (2005).

One-Loop NMHV Amplitudes involving Gluinos and Scalars in $N = 4$ Gauge Theory - Kasper Risager, Steven J. Bidder and Warren B. Perkins. Published in JHEP **0510**:003, (2005).

Abstract

Recently, a duality between $N = 4$ Super Yang-Mills Theory and Twistor String Theory has been proposed by Witten [1]. This has led to the development of a number of new techniques for calculating Tree and Loop scattering amplitudes in theories with $N = 4$ Supersymmetries. In this thesis we examine how these techniques can be extended to calculate purely gluonic one-loop scattering amplitudes in theories with $N < 4$ Supersymmetries. We explicitly calculate six-point $N = 1$ next to MHV (NHMV) one-loop amplitudes, and certain n-point NMHV examples, and investigate their twistor structure. We find that the box coefficients of all Supersymmetric amplitudes inherit the same Twistor Space properties, but that the Twistor description does not extend to the coefficients of the Triangle and Bubble functions that also appear in amplitudes in theories with $N < 4$ Supersymmetries.

We also show how to use Supersymmetric Ward Identities to calculate amplitudes involving external fermions and scalars from the known purely gluonic amplitudes with the same helicity structure. We present explicit results for six-point $N = 4$ NMHV one-loop amplitudes and generalise these results to n-point amplitudes, presenting the full generalisation as a series of conversion factors that take amplitudes from the purely gluonic form to the case where there are two external fermions. Finally we discuss how these factors can then be compounded to give amplitudes with more external fermions and scalars.

Acknowledgements

There are a number of people who I must thank for their support and guidance, without whom I would not have been able to complete this work.

PPARC for their financial support.

Warren for his excellent supervision, and Dave for taking such an active mentoring role. You have both always been available to give your advice and have always made the extra effort to explain various aspects of the work to me in a broad context. For taking the time to do so I thank you. I also thank Emil, Kasper and Lance for their collaboration.

Andy, Avtar and Aurora for their friendship. Being able to relieve the difficult moments with a joke, or a chat, has made the last three years much more enjoyable than they would otherwise have been. I wish you all the best in the future, and will miss you enormously.

Rachel, for your love, support and constant encouragement. I hope you feel as loved by me as I do by you. Haydn and Janis for welcoming me into your family and treating me as such. My friends and family for reminding me of the world outside my studies, and for helping me to enjoy it.

My Father. Without doubt I dedicate this thesis to you. You set a remarkable example to me. In spite of the many mistakes I have made, you have never given me anything other than support, encouragement and love. I would not be the person I am today if you were not my father. This has been the single most exhausting and difficult experience of my life. Without your example I would not have completed this work. Thank you.

It is awesome to think how much of the world I will never see. That is not to say that it is any great trick to go round the world these days; you can pay a lot of money and fly around it non-stop in less than forty eight hours. But to know it, to smell it and feel it between your toes, you have to crawl. There is no other way.

- *Jupiter's Travels* - Ted Simon.

Contents

Chapter 1

Introduction

Quantum field theory provides physicists with a mathematical framework for describing the fundamental constituents of matter and their interactions, and has proved to be a very powerful technique. The predictions of quantum field theory match experimental measurements to a high degree of accuracy in many cases. Thus quantum field theory has been remarkably successful at describing physical phenomena at directly accessible energy scales.

In physics, one of the great successes of the 20th Century was the Standard Model of particle physics. This field theory draws together three of the four fundamental forces of nature; the strong, weak and electromagnetic interactions. The theory that describes the strong interactions is called quantum chromodynamics (QCD). The strong interactions bind quarks and gluons together to form hadrons, the constituents of nuclear matter, and also mediate the forces between hadrons, thus controlling the formation of nuclei. In the standard model there exist six flavors of quark. In addition to having flavor, quarks also carry one of three possible charges known as colors. QCD is obtained by taking this color charge to define a local symmetry, and is a non-Abelian gauge theory of the $SU(3)$ gauge group.

The strong interactions are mediated by gluons. These also carry a color charge and thus interact with one another. QCD is therefore a non-linear theory, and is impossible to solve analytically. However, this does not mean that we cannot examine

QCD at all. In particular QCD exhibits properties that allow physicists access to the theory, such as asymptotic freedom. In very high energy reactions, quarks and gluons interact very weakly. For QCD the strength of the interactions, defined by a coupling constant α_s, increases as the distance between interacting elements is increased. Thus, at large energy scales (short distances) the coupling constant α_s is small enough to allow perturbation theory to be used to accurately approximate the theory. To apply perturbation theory we use the technique of Feynman diagrams [2]. The principle is that we represent all of the processes we must consider graphically, by constructing diagrams from vertices and propagators. We do this order by order in the number of loops involved in a process. Each of the vertices we use is given algebraically by an expression that can be derived directly from the Lagrangian of the theory. Thus each diagram we draw can be related in full to an algebraic expression by simply following a set of rules for the form of the vertices and propagators required, where these rules, the Feynman rules, are derived from the Lagrangian of the theory. This approach has provided the most precise tests of QCD to date.

QCD remains a crucial theory. In the near future the Large Hadron Collider is scheduled to begin operation, becoming the world's largest particle accelerator. Since everything at a Hadron Collider involves QCD, LHC is fundamentally a QCD machine. LHC will facilitate collisions between protons with a centre of mass energy of 14 TeV, several times greater than the collisions currently produced at the Tevatron at Fermilab. Likewise, the luminosity available at LHC will be between 10 - 100 times greater than that at Tevatron. Therefore, we can expect LHC to greatly increase the incidence of production of 100 –1000 GeV mass particles due to this increase in energy and luminosity. We expect LHC to produce such particles as top quarks and Higgs bosons. In addition, we also hope that LHC will produce new particles that correspond to physics that extends beyond that described by the Standard Model, such as Supersymmetry, which predicts a variety of new particles in the 100–1000 GeV mass range. To achieve a thorough understanding of LHC processes, including new and undiscovered physics, such as search strategies for the Higgs particle and for manifestations of Supersymmetry, we require a detailed understanding of the production mechanisms and backgrounds calculated by means of QCD.

A number of the QCD backgrounds we are most interested in, particularly in the context of Higgs search strategies, involve the production of jets of particles. Therefore, to successfully detect and interpret any processes that correspond to new physics at LHC we must be able to quantitatively account for the Standard Model backgrounds for processes that produce jets of particles.

Asymptotic freedom gives us the opportunity to calculate high energy QCD scattering amplitudes by using a perturbative expansion in the coupling constant α_s. However, for Hadron Collider cross sections the leading order terms in the α_s expansion - corresponding to Tree amplitudes - are not sufficient to reduce the uncertainty to around 10%. In fact the corrections due to the next-to-leading-order terms are typically between 30 – 100% [3].

Thus, the desired quantitative understanding of LHC events will require cross sections to be evaluated at next-to-leading-order in the perturbative expansion. In addition to Tree amplitudes, this will require the calculation of one-loop amplitudes to reduce uncertainties to 10%. What's more, if we require a more precise evaluation we must reduce the uncertainties to less than 10%. This would require the additional calculation of two loop amplitudes.

In principle, Feynman rules are all we need to evaluate the tree and loop amplitudes. Indeed, for fifty years theoretical physicists have had use of this standard calculation technique. In the last twenty five years this has included the full development of how these rules are applied to non-Abelian gauge theories such as QCD. It would seem a fair expectation that all significant Standard Model scattering processes should by now have been calculated to the experimentally required accuracy. This is not the case, however. In particular most QCD scattering processes have been calculated only to leading-order in the strong coupling constant.

In field theory, amplitudes serve as the input to leading-order and next-to-leading-order cross section calculations. As opposed to the full cross sections, it is these amplitudes that field theorists focus on calculating. The computation of tree and loop amplitudes should in principle be a straightforward exercise given knowledge of the Feynman rules. The practise is to draw all relevant Feynman diagrams for a given process and use standard loop integral reduction techniques to evaluate the subsequent expressions. However, in practise this quickly becomes both laborious and cumbersome. The process is increasingly inefficient as the number of external legs grows for a number of reasons.

The calculation of these processes is made particularly difficult by the large number of Feynman diagrams which appear in the perturbative expansion. As an example Table 1.1 shows the number of diagrams contributing to the process *gluon-gluon* $\rightarrow$ *n-gluons* [4].

n	2	3	4	5	6	7	8
# of diagrams	4	25	220	2485	34300	559405	10525900

Table 1.1: The number of Feynman diagrams contributing to the scattering process $gg \rightarrow \mathrm{n}\ g$.

Additionally, non-Abelian gauge boson self interactions are so complicated that the structure of these vertices leads to an almost uncontrollable inflation in the number of terms which are generated. Likewise the large number of kinematic variables leads to arbitrarily complicated expressions. Indeed the intermediate expressions are significantly more complicated than the final result. Given the number of diagrams contributing and the complexity of the calculations involved, it is clear that the prospect of calculating multiple jet events using the Feynman diagram approach is unrealistic as the standard techniques of numerical evaluation and algebraic manipulation will quickly become redundant. It is the focus of many research groups to develop new calculational techniques to overcome these obstacles.

Efficient techniques for calculating tree amplitudes have existed for a number of years. These include the organisation of amplitudes in terms of spinor helicity, color ordering and Supersymmetry, as discussed in Chapter 3. At one-loop, although these techniques still form an important part of the calculation, the picture is even more complex and calculations become more intricate. Exploiting the techniques so successful at tree level is often not enough to complete a calculation. Therefore, the development of new techniques for calculating loop amplitudes, to compliment the existing techniques for tree level calculations, forms an essential area of perturbative field theory research.

In this work we investigate new techniques for calculating one-loop scattering amplitudes in Yang-Mills theories. As we discuss in chapter 5, the recent interest in perturbative field theory was stimulated by the work of Ed Witten. In particular, Witten chose to examine why many QCD amplitudes are more simple than we would expect. For instance, Parke and Taylor [5] showed that tree level gluon scattering amplitudes have a particularly simple form. If the helicities of the gluons are all the same, or all the same bar one, the amplitude vanishes. We write this relation as,

$$A_n^{\text{tree}}(1^{\pm}, 2^+, 3^+, ..., n^+) = 0. \tag{1.1}$$

The first example of a non-vanishing tree amplitude occurs when exactly two of the gluons, labelled s and r here, have helicity that is opposite to the helicity of all of the other gluons. For the case where exactly two gluons have negative helicity, and the rest are positive, we call the amplitude a Maximally Helicity Violating (MHV) amplitude. Parke and Taylor proposed [5], and Berends and Giele later proved [6], that these MHV amplitudes have a particularly simple form,

$$A_n^{\text{MHV}} \equiv A_n^{\text{tree}}(1^+, 2^+, \ldots, s^-, ..., r^-, ..., n^+) = i\frac{\langle s\, r\rangle^4}{\langle 1\, 2\rangle \langle 2\, 3\rangle \ldots \langle n\, 1\rangle}, \tag{1.2}$$

where the notation is as described in chapter 3, and momentum conservation is implicit. We also have the case where exactly two of the gluons have positive helicity and the rest are negative. We call these amplitudes Googly MHV or $\overline{\text{MHV}}$ amplitudes. For the case above where s and r label the gluons of helicity different to the rest, the $\overline{\text{MHV}}$ amplitude can be written as

$$A_n^{\overline{\text{MHV}}} \equiv A_n^{\text{tree}}(1^-,2^-,\ldots,s^+,...,r^+,...,n^-) = i\frac{[s\,r]^4}{[1\,2]\,[2\,3]\,...\,[n\,1]}, \tag{1.3}$$

The sheer simplicity of equations (1.2) and (1.3) convinced Witten that there must be some underlying structure that was not yet apparent. From his investigations, Witten proposed a duality between $N = 4$ Super Yang-Mills theory and a topological string theory [1]. This becomes manifest by transforming amplitudes into twistor space where they are supported on simple curves, the degree of which is related to the number of negative helicities. Consequently, when expressed in terms of spinor variables $p_{a\dot{a}} = \lambda_a\tilde{\lambda}_{\dot{a}}$, tree amplitudes are annihilated by various differential operators corresponding to the localisation of points on lines and planes in twistor space. This has led to significant progress in the computation of amplitudes in gauge theories.

At tree level, the structure of amplitudes would appear to be inherited from this twistor string description. This has resulted in many reformulations of tree level amplitudes. Most notably, Cachazo, Svercek and Witten [7] have proposed the use of MHV vertices instead of three and four-point Feynman vertices. Using this approach one obtains simpler, more compact expressions for tree amplitudes. This has been extended to include other particles, including fermions and scalars.

Over a number of years, various techniques have been developed to calculate loop amplitudes more efficiently than the conventional Feynman diagram approach. Many of these are simply continuations of the techniques used at tree level, although naturally the application of these techniques becomes more complex at one-loop level.

These techniques include the idea that an amplitude can be cut constructible - the principle that an entire amplitude can be reconstructed from a knowledge of its four dimensional two particle cuts. This was developed by Bern, Dixon, Dunbar and Kosower [8, 9].

A further technique is the use of generalised unitarity. The idea of generalised unitarity cuts has been investigated within various contexts since the 1960's. However a recent breakthrough was achieved by Britto, Cachazo and Feng [10] with the observation that by analytically continuing tree amplitudes to a signature $(--++)$ and using these to calculate quadruple cuts, coefficients of integral functions that appear in the amplitude can be determined algebraically from products of on-shell tree amplitudes.

Likewise, we can use Supersymmetric Ward Identities (SWI) [11], which relate amplitudes with the same helicity structure but with different external particles types. As we discuss in chapter 2, Supersymmetric Ward Identities place powerful constraints on amplitudes, particularly MHV amplitudes where knowledge of the all gluon amplitude appearing within a particular SWI set ultimately determines the other two-gluino amplitudes that appear in the same set. For NMHV amplitudes (i.e. exactly three negative helicity gluons and the rest positive helicity) the SWI do not lead to such simple solutions. Applying the Supersymmetry operator leads to a system that has rank 2, so it can only directly give two of the amplitudes appearing in a particular SWI set in terms of the other amplitudes appearing in the same set. By itself, this relationship does not solve for the fermionic amplitudes unambiguously from the purely gluonic. However, with the application of further constraints we can obtain the fermionic amplitudes, as we discus in the results presented in chapter 7. Additional SWI can be used to determine amplitudes involving scalars or two flavours of gluino.

In addition to these, other techniques we shall discuss in later chapters include the application of MHV vertices, and exploiting the "holomorphic anomaly" of unitarity cuts. This derives from the observation by Cachazo, Svercek and Witten that the one-loop amplitudes are not annihilated by the collinear operator F. They interpreted this as the MHV vertex approach not working at one-loop. However Brandhuber, Spence and Travaglini [12] were able to use the MHV vertex approach to caluclate a one-loop coefficient. This paradox was resolved by Witten et al [13] by observing that for theories with $N = 4$ Supersymmetries differential operators acting within the loop-momentum integral yield δ functions, producing rational functions as a result. Consequently it was observed that acting with the collinear operator upon both the cut and the imaginary part of the amplitude, and demanding consistency via the optical theorem, leads to algebraic equations for the coefficients of integral functions which appear in the amplitude, and which are extremely helpful in computing the entire amplitude.

The focus of this research is to continue the development of new techniques, in conjunction with those that already exist, that solve or simplify the calculation of one-loop amplitudes.

This thesis is organised as follows.

In chapter 2 we discuss Yang-Mills theory and introduce the idea of localising or gauging a symmetry transformation. We discuss how Yang and Mills used this principle to derive a non-Abelian gauge theory, defined by invariance under transformations characterised by a continuous symmetry group. We review the formalism of Yang-Mills theory and state the Yang-Mills Lagrangian.

We also discuss the origins of Supersymmetry, and define the Supersymmetric algebra for normal and extended Supersymmetry. Finally we discuss the use of Supersymmetric Ward Identities to relate amplitudes with the same helicity structure but with different external particle types. In particular, we describe how to generate Supersymmetric Ward Identities using the $N = 1$ and $N = 2$ Supersymmetry algebra, including a simple example to demonstrate their application.

In chapter 3 we introduce a series of traditional techniques used to simplify calculations, that depend on keeping track of all information about external particles. We review the formalism of spinor helicity and set the notation used throughout this work. We explicitly define the spinor algebra in appendix A. We describe the color decomposition of amplitudes at tree level and state how this is extended to one-loop level. We define the properties and relationships satisfied by color ordered amplitudes and state how at one-loop level we can generate sub-leading color structures from leading order terms, thus reducing the amount of explicit calculation required. We describe how one-loop amplitudes can be organised by considering the contributions from different Supersymmetric multiplets, focusing on manipulating the sum over internal spins of particles circulating in the loop and rearranging the terms that appear in the sum to simplify a calculation. We apply this to the $N = 1$ chiral and vector multiplets and the $N = 4$ multiplet and explicitly show that the contributions from these three multiplets are not independent.

In chapter 4 we discuss the analytic properties of amplitudes, in particular the factorisation properties of tree and loop amplitudes. We examine the unitarity properties of loop amplitudes and discuss the optical theorem and Cutkosky's rules, explaining how these are used to solve for amplitudes using unitarity cuts. We state how, when performing a unitarity cut, we can relate loop amplitudes with different particles circulating in the loop via Supersymmetric ρ factors.
We also describe the traditional loop integral reduction techniques that allow amplitudes to be expanded as a sum of known scalar integral functions multiplied by unknown rational coefficients and discuss the effect Supersymmetry has on restricting the type of scalar integral functions that may appear in the expansion.
Finally we describe the different bases of functions that we use in our calculations and how these are defined and labelled.

In chapter 5 we introduce the recent progress that acted as the stimulus for the work presented in this thesis. We begin by discussing Witten's original conjecture that there is a duality between $N = 4$ Super Yang-Mills theory and string theory. We discuss the motivation for this conjecture and describe the derivation of this duality at tree level, focusing on Witten's principle of Fourier transforming amplitudes into twistor space. We discuss the geometric description of tree level amplitudes in twistor space, deriving the MHV case and stating the NMHV case. We also define the differential operators that act on amplitudes, defining their twistor properties.

We discuss the continuation of this work by Cachazo, Svercek and Witten to give numerical values for amplitudes. We describe the CSW construction in terms of using MHV tree amplitudes as fundamental building blocks from which more complicated amplitudes can be constructed. We include a simple example of this technique to demonstrate its application.

We describe the Generalised Unitarity technique introduced by Britto, Cachazo and Feng, including a discussion of the key steps taken. In particular we discuss the motivation for continuing from Minkowski signature to signature $(- \; - \; + \; +)$. Again we include a simple example to illustrate the application of this technique.

Finally we discuss the continuation of this picture to loop level amplitudes. We discuss the origin of the "holomorphic anomaly" of unitarity cuts and how this can be exploited to solve for amplitudes in theories with $N = 4$ Supersymmetries. Finally we describe the twistor structure of the box coefficients for MHV and NMHV $N = 4$ one-loop amplitudes.

Chapter 5 completes the theoretical introduction to this thesis. We then move on to explicitly describe the new results we have obtained during the last three years.

In chapter 6 we present the first set of results for gluonic one-loop amplitudes in theories with $N < 4$ Supersymmetries, based on the following publications:

- **$N = 1$ Supersymmetric One-Loop Amplitudes and the "Holomorphic Anomaly" of Unitarity Cuts [14]**

Witten *et al.* resolved the paradox discovered by Brandhuber, Spence and Travaglini by suggesting the existence of a "holomorphic anomaly" in $N = 4$ Supersymmetric amplitudes. Indeed he noted that the existence of such a feature could be used to derive algebraic equations for the coefficients of integral functions which appear in the amplitude, a very useful spin off. We extend this analysis to examine how the "holomorphic anomaly" acts upon the cuts of $N = 1$ Supersymmetric one-loop amplitudes, focusing on a six-gluon NMHV amplitude which had been previously calculated independently from cut constructibility and collinear limit methods (this had not been published previously). We show that the anomaly must be taken into account when acting with differential operators on the cuts in order to satisfy the optical theorem. We suggest that as a calculational tool to evaluate amplitudes, application of the "holomorphic anomaly" yields differential equations for the coefficients of the integral functions in the $N = 1$ case, as opposed to the algebraic equations of the $N = 4$ case, and that the general solution to these differential equations contain homogeneous parts which can be fixed by boundary conditions or physical conditions such as collinear limits.

- **Twistor Space structure of the Box Coefficients of $N = 1$ One-Loop amplitudes [15]**

Many fascinating geometric features appear in the twistor space realisation of gauge theory amplitudes, which are of particular interest when determining scattering amplitudes. The coefficients of integral functions contained in an amplitude exhibit interesting structure in twistor space, particularly the coefficients of the I_4 integral functions. In theories with $N = 4$ Supersymmetries it has been shown that these I_4 coefficients for next to MHV amplitudes have planar support in twistor space, behaviour that is analogous to that of the tree amplitudes. We investigate whether similar behaviour exists for theories with $N < 4$ Supersymmetries by computing the I_4 coefficients for all six-point $N = 1$ amplitudes and examining their twistor space structure. We find that for next to MHV amplitudes these coefficients have planar support in twistor space, confirming explicitly that the structure at $N = 4$ persists to $N = 1$. We are able to extend this analysis to include certain classes of n-point $N = 1$ amplitudes, where we find further support for the twistor space structure described.

- **One-Loop Gluon Scattering Amplitudes in Theories with $N < 4$ Supersymmetries [16]**

Although Witten's proposed relationship between twistor string theory and perturbative field theory has been observed at $N = 4$, it is as yet unresolved as to what degree this relationship extends to theories with less or indeed no Supersymmetry. It is therefore reasonable to continue gathering information by studying the properties of amplitudes in such theories until a direct connection is uncovered. By focusing on the I_4 integral functions that appear in specific example amplitudes, and exploiting the generalised unitarity technique of Britto, Cachazo and Feng by using quadruple cuts, we compute the coefficients of these functions, and show that they satisfy the same collinearity and coplanarity conditions independent of the Supersymmetry. We demonstrate by means of a relatively simple proof that the $N = 4$, $N = 1$ and $N = 0$ cases for amplitudes that are "MHV-deconstructible" are inherently related, and as such one must only demonstrate that the expected twistor space properties are exhibited in two of the above cases to conclude that the third case must also satisfy these properties. Furthermore, we exploit the approach of Britto, Cachazo and Feng by using triple cuts to determine the coefficients of the I_3 and I_2 integral functions and present the full expression for an example one-loop amplitude.

In chapter 7 we present the final set of results for fermionic one-loop amplitudes in theories with $N = 4$ Supersymmetries, based on the publications:

- **Supersymmetric Ward Identities and NMHV Amplitudes involving Gluinos [17]**

On-shell Supersymmetric Ward Identities (SWI) impose powerful constraints on amplitudes in gauge theories, giving algebraic relationships between amplitudes with the same helicity configuration but different external particle types. These constraints apply at any order in perturbation theory. From a Feynman diagram perspective, these relationships are most naturally employed to obtain purely gluonic amplitudes from amplitudes involving fermions. Motivated by the recent advances in calculating purely gluonic amplitudes, we reverse this process and generate amplitudes involving fermions from the purely gluonic ones. For some helicity configurations the SWI contain sufficient information to simply solve for the fermionic amplitudes. For example in $N = 4$ gauge theory the Supersymmetric Ward Identities for MHV amplitudes could be easily solved and amplitudes with any external particles obtained from the purely gluonic MHV amplitudes by a simple multiplicative factor.

For other configurations, such as NMHV amplitudes, the SWI do not allow such simple solutions. However, we show how the SWI can be solved in a natural way to obtain amplitudes with two gluinos in terms of the purely gluonic case. We first apply this to the six-point tree amplitudes where we can connect to known computations. Secondly we determine the one-loop six-point NMHV amplitudes in $N = 4$ Supersymmetric Yang-Mills theory which involve two gluinos. More generally there also exist SWI which involve amplitudes with two gluinos, four gluinos, two scalars and two gluinos plus a scalar. We explicitly determine the two scalar amplitudes. The SWI then give the remaining amplitudes directly in terms of known amplitudes.

- **One-Loop NMHV Amplitudes involving Gluinos and Scalars in $N = 4$ Gauge Theory [18]**

One-loop NMHV amplitudes in $N = 4$ gauge theory can be expressed in terms of MHV-deconstructible diagrams and so can be evaluated using quadruple cuts and known MHV tree amplitudes. These amplitudes also satisfy SWI which can be employed to minimise the number of independent diagrams that must be computed explicitly. We use these techniques to determine a set of conversion factors that relate two-gluino box coefficients to purely gluonic ones. Analysis of quadruple cuts is then used to show how these factors can be compounded to give two-scalar and scalar-gluino-gluino box coefficients. Amplitudes involving more external fermions and scalars then follow from SWI.

Finally, we finish with some conclusions in chapter 8. The appendix contains the explicit spinor helicity notation and spinor algebra introduced in chapter 3.

Chapter 2

Yang-Mills Theory and Supersymmetry

2.1 Yang-Mills Theory

The modern development of non-Abelian gauge theories began when Yang and Mills attempted to make hadronic isospin into a local symmetry. Although they were unable to achieve precisely this, the eventual formalism they developed did turn out to describe a fundamental theory. Instead of describing the interactions between hadrons, they developed a theory that describes the interactions between the constituents of hadrons, namely quarks. To introduce the basic principles of their work we must consider the idea of gauging, or localising a symmetry transformation.

We can write a global symmetry transformation as,

$$\psi \to \exp\left[i\alpha^i \frac{\sigma^i}{2}\right]\psi, \tag{2.1}$$

where the parameters α^i are independent of space-time [19]. With such a transformation, fields at different points in space-time transform by the same amount. We can promote this global symmetry to a local symmetry by insisting that the invariance of the theory hold for the same transformation except with α^i now a function of x, i.e. the symmetry transformations are now space-time dependent. We write the local symmetry transformation as [2],

$$\psi \to V(x)\psi(x), \qquad \text{where } V(x) = \exp\left[i\alpha^i(x)\frac{\sigma^i}{2}\right]. \tag{2.2}$$

In 1954 Yang and Mills used this principle of gauging, or localising, a symmetry transformation [20]. They suggested that the central idea of the only existing field theory of the time, QED, i.e. the invariance under local phase rotation, could be generalised to invariance under transformations characterised by any continuous symmetry group. This exhibits a fundamental difference to that of QED. The symmetry generators for transformations under a continuous symmetry group do not commute with each other. We refer to a field theory characterised by a non-commuting local symmetry as a non-Abelian gauge theory, the simplest example of which is called Yang-Mills Theory.

QED has an Abelian $U(1)$ gauge symmetry defined by the transformations,

$$\begin{aligned} \psi &\to U\psi \\ A_\mu &\to A_\mu - \frac{i}{e}U\partial_\mu U^{-1} \\ F_{\mu\nu} &\to F_{\mu\nu}, \end{aligned} \tag{2.3}$$

where,

$$U = \exp[i\alpha(x)]. \tag{2.4}$$

The covariant derivative is defined as,

$$D_\mu\psi = (\partial_\mu + ieA_\mu)\psi, \tag{2.5}$$

with symmetry transformation,

$$D_\mu\psi \to UD_\mu\psi. \tag{2.6}$$

In 1954 Yang and Mills extended the $U(1)$ symmetry of the QED model and generalised this to non-Abelian groups such as $SU(N)$. In such a case we write,

$$U = \exp[i\alpha^a(x)T^a], \tag{2.7}$$

where T^a are the generators satisfying,

$$[T^a, T^b] = if^{abc}T^c, \tag{2.8}$$

and f^{abc} are the structure constants for the non-Abelian group. The T^a are normalised such that Tr $T^aT^b = \frac{1}{2}\delta^{ab}$.

Therefore wave-functions transform according to

$$\psi \to \exp[i\alpha^a(x)T^a]\psi. \tag{2.9}$$

To define a suitable covariant derivative, we must introduce definite force fields. For each generator of the Lie algebra there is one independent gauge field A^a_μ, often called a Yang-Mills field, which lies in the adjoint representation of the group. For solutions of the free particle field equations we write A^a_μ in the form,

$$A_\mu(x) = \epsilon_\mu(p)e^{-ip.x} \tag{2.10}$$

where $\epsilon_\mu(p)$, are the polarisation vectors.

With the gauging of the symmetry transformation the covariant derivative generalises to,

$$D_\mu\psi = (\partial_\mu - igT^a A^a_\mu)\psi. \tag{2.11}$$

where g is a coupling constant.

Under a gauge transformation, with $U = \exp[i\alpha^a(x)T^a]$, we have that,

$$\psi \rightarrow U\psi, \tag{2.12}$$

and we require

$$D_\mu\psi \rightarrow UD_\mu\psi. \tag{2.13}$$

This fixes the gauge transformation of A^a_μ. Writing $A_\mu = A^a_\mu T^a$ we have simply that,

$$A_\mu \rightarrow UA_\mu U^{-1} + \frac{i}{g}U\partial_\mu U^{-1} \tag{2.14}$$

or in component form,

$$A_\mu^a \rightarrow A_\mu^a + \frac{i}{g}\partial_\mu \alpha^a + f^{abc} A_\mu^b \alpha^c. \tag{2.15}$$

The field strength tensor $F_{\mu\nu} = F_{\mu\nu}^a T^a$ is defined as the commutator of two covariant derivatives,

$$[D_\mu, D_\nu] = -igF_{\mu\nu} \tag{2.16}$$

Therefore we can write this as,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu], \tag{2.17}$$

or in components,

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c. \tag{2.18}$$

Under a gauge transformation,

$$F_{\mu\nu} \rightarrow U F_{\mu\nu} U^{-1}, \tag{2.19}$$

so that the term Tr $[F_{\mu\nu}F^{\mu\nu}]$ is a gauge invariant and Lorentz invariant quantity, suitable for using in a Lagrangian.

We can now write down a Lagrangian which generalises to a non-Abelian gauge theory. We write the Yang-Mills Lagrangian as,

$$\mathcal{L} = \bar{\psi}(i \not{D})\psi - \frac{1}{4}(F^a_{\mu\nu})^2 - m\bar{\psi}\psi. \tag{2.20}$$

This describes the interaction of Yang-Mills vector fields with fermions and depends on two parameters, the coupling constant g and the fermion mass m.

2.2 Supersymmetry

Despite a lack of experimental evidence to support the idea, theories that include Supersymmetry remain the most accepted framework for describing physics beyond the Standard Model. That this is the case is due to the number of attractive properties of Supersymmetry, perhaps the most significant of which is that it appears to play a vital role in the only quantum theories that appear able to describe strong, electroweak and gravitational interactions; so called Super-string theories [21].

The origins of Supersymmetry are found in considering the allowed space-time symmetries of particle physics. In 1967 Coleman and Mandula [22] proved that, given certain assumptions, the only possible space-time symmetries of particle physics are "internal" global symmetries that depend on certain quantum numbers being conserved; the discrete symmetries C, P and T, and invariance under Poincare transformations.

Coleman and Mandula's conclusion can be avoided if we consider weakening their assumptions, in particular that the symmetry algebra can involve only commutators. By allowing generators that were anti-commuting lead to the postulation of Supersymmetry, which is defined as the presence of such anti-commuting generators that transform in the spinor representation of the Lorentz group. It was subsequently proved by Haag, Lopuszanski and Sohnius [23] in 1975 that Supersymmetry represents the only new symmetry the revised set of assumptions allow. As such, Supersymmetry is considered to be the only possible extension of the known space-time symmetries of particle physics [24].

2.2.1 $N = 1$ Supersymmetry

The essential property of Supersymmetry is that it relates bosons to fermions. The Supersymmetry generators Q_α, with spinor index α, are tightly constrained by Lorentz symmetry, which implies,

$$\begin{aligned} [P^\mu, Q_\alpha] &= 0 \\ [M^{\mu\nu}, Q_\alpha] &= -i(\sigma^{\mu\nu})_\alpha^\beta Q_\beta \\ [M^{\mu\nu}, \bar{Q}^{\dot{\alpha}}] &= -i(\bar{\sigma}^{\mu\nu})_{\dot{\beta}}^{\dot{\alpha}} \bar{Q}^{\dot{\beta}}, \end{aligned} \tag{2.21}$$

where we define $\bar{Q}_{\dot{\alpha}} \equiv (Q_\alpha)^\dagger$.

Haag, Lopuszanski and Sohnius [23] proved that the Supersymmetric generators Q_α must anti-commute. If Q_α are to be non-trivial hermitian operators then the most general possibility is,

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu. \tag{2.22}$$

where the σ^μ matrices are defined as in reference [21],

$$\sigma^\mu \equiv (I_2, \overline{\sigma}) \tag{2.23}$$

and $\overline{\sigma}$ are the Pauli matrices.

2.2.2 Extended Supersymmetry

We can consider the introduction of more than one Supersymmetry generator Q_α^i, where $i = 1, ..., \mathrm{N}$ labels the number of generators. In this case the Supersymmetry is said to be extended. The additional index i does not change the commutation relations with the Poincare algebra,

$$[P^\mu, Q_\alpha^i] = 0$$

$$\begin{aligned} [M^{\mu\nu}, Q^i_\alpha] &= -i(\sigma^{\mu\nu})_\alpha^\beta Q^i_\beta \\ [M^{\mu\nu}, \bar{Q}^{\dot{\alpha}}_i] &= -i(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}_{\dot{\beta}} \bar{Q}^{\dot{\beta}}_i . \end{aligned} \tag{2.24}$$

The anti-commutator eq. (2.22) now takes the form,

$$\{Q^i_\alpha, \bar{Q}_{\dot{\beta}j}\} = 2\delta^i_j \sigma^\mu_{\alpha\dot{\beta}} P_\mu . \tag{2.25}$$

The most obvious difference from $N = 1$ Supersymmetry is found in the anti-commutators $\{Q^i_\alpha, Q^j_\beta\}$, $\{\bar{Q}^i_{\dot{\alpha}}, \bar{Q}^j_{\dot{\beta}}\}$. These are now given by,

$$\begin{aligned} \{Q^i_\alpha, Q^j_\beta\} &= \epsilon_{\alpha\beta} Z^{ij} \\ \{\bar{Q}^i_{\dot{\alpha}}, \bar{Q}^j_{\dot{\beta}}\} &= \epsilon_{\dot{\alpha}\dot{\beta}} (Z^{ij})^* \end{aligned} \tag{2.26}$$

where the ϵ matrices are defined as in reference [21],

$$\epsilon_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{2.27}$$

The Z^{ij} are antisymmetric, i.e. $Z^{ij} = -Z^{ji}$, and are called central charges. The central charges do not occur in the case of $N = 1$ Supersymmetry, where the anti-commutators $\{Q^i_\alpha, Q^j_\beta\}$ and $\{\bar{Q}^i_{\dot{\alpha}}, \bar{Q}^j_{\dot{\beta}}\}$ are therefore zero.

2.2.3 Supersymmetric Ward Identities

Supersymmetric Ward Identities (SWI) relate amplitudes with the same helicity structure but with different external particle types [11]. The Ward Identities can be obtained by acting with the Supersymmetry generator Q_α on a string of operators, z_i, which has vanishing vacuum expectation value. Typical choices are strings with an odd number of fermionic operators. Since Q_α annihilates the vacuum we obtain,

$$0 = \left\langle \left[Q_\alpha, \prod_i z_i\right] \right\rangle = \sum_i \left\langle z_1 \cdots [Q_\alpha, z_i] \cdots z_n \right\rangle . \tag{2.28}$$

For $N = 1$ Supersymmetry the commutation relations of the fields $g^{\pm}(p)$, $\Lambda^{\pm}(p)$ with the supercharge Q_α, where $g^{\pm}(p)$ creates a gluon of momentum p and helicity $\pm$ and $\Lambda^{\pm}(p)$ creates a gluino of momentum p and helicity $\pm$, are given by the Supersymmety relations [11],

$$\begin{aligned} [Q(\eta), g^+(p)] &= -\Gamma^+(p,\eta)\bar{\Lambda}^+, & [Q(\eta), g^-(p)] &= \Gamma^-(p,\eta)\Lambda^-, \\ [Q(\eta), \bar{\Lambda}^+(p)] &= -\Gamma^-(p,\eta)g^+, & [Q(\eta), \Lambda^-(p)] &= \Gamma^+(p,\eta)g^-, \end{aligned} \tag{2.29}$$

where the Supersymmetry generator Q multiplied by a arbitrary spinor parameter $\bar{\eta}$ defines $Q(\eta) \equiv \bar{\eta}^\alpha Q_\alpha$, and the $\Gamma^{\pm}(p,\eta)$ are linear in η. The Γ are given by [3],

$$\Gamma^+(p,\eta) \equiv [\eta\, p] \quad , \quad \Gamma^-(p,\eta) \equiv \langle p\, \eta\rangle . \tag{2.30}$$

where the notation is explained in the next chapter.

As an example of how effective the SWI can be, consider applying the Supersymmetry operator Q_α to the MHV amplitude $A_n(g_1^-, g_2^-, \Lambda_3^+, g_4^+, \ldots, g_n^+)$ (i.e. a string of gluonic creation operators with a single gluino creation operator). We obtain,

$$\begin{aligned} 0 &= \langle 1\,\eta\rangle\, A_n(\Lambda_1^-, g_2^-, \bar{\Lambda}_3^+, g_4^+, \ldots, g_n^+) \\ &+ \langle 2\,\eta\rangle\, A_n(g_1^-, \Lambda_2^-, \bar{\Lambda}_3^+, g_4^+, \ldots, g_n^+) \\ &- \langle 3\,\eta\rangle\, A_n(g_1^-, g_2^-, g_3^+, g_4^+, \ldots, g_n^+), \end{aligned} \tag{2.31}$$

where we have used the fact that amplitudes with two fermions (one flavour) of the same helicity vanish. Choosing $\eta = 1$, for example, gives,

$$A_n(g_1^-, \Lambda_2^-, \bar{\Lambda}_3^+, g_4^+, \ldots, g_n^+) = \frac{\langle 3\,1\rangle}{\langle 2\,1\rangle} A_n(g_1^-, g_2^-, g_3^+, g_4^+, \ldots, g_n^+), \tag{2.32}$$

and we can thus obtain the MHV two-gluino amplitudes from the gluonic amplitude.

For NMHV amplitudes the SWI do not lead to such simple solutions. To see this consider applying the Supersymmetry operator to $A_n(g_1^-, g_2^-, g_3^-, \bar{\Lambda}_4^+, g_5^+, \ldots, g_n^+)$. We obtain,

$$\begin{aligned} 0 = \langle 1\,\eta\rangle\, A_n(\Lambda_1^-, g_2^-, g_3^-, \bar{\Lambda}_4^+, g_5^+, \ldots, g_n^+) + \langle 2\,\eta\rangle\, A_n(g_1^-, \Lambda_2^-, g_3^-, \bar{\Lambda}_4^+, g_5^+, \ldots, g_n^+) \\ + \langle 3\,\eta\rangle\, A_n(g_1^-, g_2^-, \Lambda_3^-, \bar{\Lambda}_4^+, g_5^+, \ldots, g_n^+) - \langle 4\,\eta\rangle\, A_n(g_1^-, g_2^-, g_3^-, g_4^+, g_5^+, \ldots, g_n^+)\,. \end{aligned} \tag{2.33}$$

In this example the SWI can only directly give two of the amplitudes in terms of the other two. Thus, by itself this does not solve for the fermionic amplitudes unambiguously from the purely gluonic. However, when further constraints are applied, the fermionic amplitudes can be obtained, as we shall see later.

We can also consider $N = 2$ Supersymmetric Ward Identities [11, 25]. Using Supersymmetry generators Q_i, $i = 1, 2$, we have,

$$\begin{aligned} [Q_i(\eta), g^+(p)] &= -\Gamma^+(p,\eta)\bar{\Lambda}_i^+, \\ [Q_i(\eta), g^-(p)] &= \Gamma^-(p,\eta)\Lambda_i^-, \\ [Q_i(\eta), \bar{\Lambda}_j^+(p)] &= -\Gamma^-(p,\eta)\delta_{ij}g^+ - i\Gamma^+(p,\eta)\epsilon_{ij}\phi^+, \\ [Q_i(\eta), \Lambda_j^-(p)] &= \Gamma^+(p,\eta)\delta_{ij}g^- + i\Gamma^-(p,\eta)\epsilon_{ij}\phi^-, \\ [Q_i(\eta), \phi^+(p)] &= -i\Gamma^-(p,\eta)\epsilon_{ij}\bar{\Lambda}_j^+, \\ [Q_i(\eta), \phi^-(p)] &= +i\Gamma^+(p,\eta)\epsilon_{ij}\Lambda_j^-. \end{aligned} \tag{2.34}$$

where the Γ are given by eq. (2.30). We use these identities to determine amplitudes involving scalars or two flavours of gluino.

No perturbative approximations are necessary to derive the SWI. They therefore apply order by order in the perturbative expansion. Since QCD is effectively Supersymmetric at tree level, these identities can be applied directly to QCD tree amplitudes, and are also a valuable tool that can be used to reduce the number of independent calculations required at loop level.

Chapter 3

Organising Amplitudes

One of the major developments in calculating perturbative QCD scattering amplitudes has been the use of techniques that decompose amplitudes into simpler pieces by using the quantum numbers of external states, such as helicity and color. Calculations can be simplified significantly by exploiting this idea, which is especially useful for computing amplitudes at loop level.

3.1 Spinor Helicity

The existence of compact representations for tree and loop amplitudes in QCD is largely due to the development of the spinor helicity formalism [26, 27, 28]. In this section we review the notation we use throughout this thesis [29]. For the full formalism, notation and definitions of the spinor algebra see Appendix A.

The principle behind this choice of notation is straightforward. Traditionally, when we describe an amplitude, we use the four-momentum vectors p_i^μ as the arguments of the amplitude, i.e. $A = A(p_i)$. From these four-momentum vectors we construct the Lorentz inner products $s_{ij} = 2p_i \cdot p_j$ as the relativistic invariants. In the massless case these are given by $s_{ij} = 2p_i \cdot p_j = (p_i + p_j)^2$, since $p_i^2 = 0$.

However, instead of using p_i^μ, we can choose to use massless Dirac spinors. These are written as $u_\pm(p_i)$, where $u_\pm(p_i)$ defines positive energy solutions of the massless

Dirac equation, see Appendix A. The $\pm$ sign labels the helicity. The two-component (Weyl) version of these spinors is written as [29],

$$(\lambda_i)_\alpha \equiv \left[u_+(p_i)\right]_\alpha, \qquad (\tilde{\lambda}_i)_{\dot{\alpha}} \equiv \left[u_-(p_i)\right]_{\dot{\alpha}}. \tag{3.1}$$

Using the positive energy projector for massless spinors, $u(p)\bar{u}(p) = \not{p}$, the associated momenta can always be reconstructed from the spinors. In the two-component notation we write this as [29],

$$p_i^\mu(\sigma_\mu)_{\alpha\dot{\alpha}} = (\not{p}_i)_{\alpha\dot{\alpha}} = (\lambda_i)_\alpha(\tilde{\lambda}_i)_{\dot{\alpha}}\,. \tag{3.2}$$

where σ_μ are the Pauli matrices. Eq. (3.2) reflects the property that a massless momentum vector, when written as a bi-spinor, is simply the product of a left-handed and a right-handed spinor.

With this notation, we replace the Lorentz inner products, $s_{ij} = 2p_i \cdot p_j$, with the spinor products [29],

$$\begin{aligned} \langle i\, j\rangle &\equiv \langle i^-|j^+\rangle = \bar{u}_-(p_i)u_+(p_j) = \epsilon^{\alpha\beta}(\lambda_i)_\alpha(\lambda_j)_\beta, \\ [i\, j] &\equiv \langle i^+|j^-\rangle = \bar{u}_+(p_i)u_-(p_j) = \epsilon^{\dot{\alpha}\dot{\beta}}(\tilde{\lambda}_i)_{\dot{\alpha}}(\tilde{\lambda}_j)_{\dot{\beta}}, \end{aligned} \tag{3.3}$$

where $\epsilon^{\alpha\beta}$ and $\epsilon^{\dot{\alpha}\dot{\beta}}$ are the $SU(2)$ antisymmetric tensors defined in eq. (2.27). The spinor products are antisymmetric, i.e.

$$\langle i\, j\rangle = -\langle j\, i\rangle\,, \qquad [i\, j] = -[j\, i]\,, \qquad \langle i\, i\rangle = [i\, i] = 0. \tag{3.4}$$

and satisfy the identity

$$\langle i\, j\rangle\,[j\, i] = \frac{1}{2}\mathrm{Tr}[\not{p}_i\,\not{p}_j] = 2p_i \cdot p_j = s_{ij}. \tag{3.5}$$

Thus the spinor products are, up to a phase, simply the square roots of the Lorentz inner products, i.e.

$$\langle i\,j\rangle = \sqrt{s_{ij}}e^{i\phi_{ij}}\,, \qquad [i\,j] = \pm\sqrt{s_{ij}}e^{-i\phi_{ij}}\,. \tag{3.6}$$

Therefore, by replacing the Lorentz invarients s_{ij} we can rewrite amplitudes entirely in terms of these spinor products. Using spinor helicity, amplitudes do have an overall phase. However, physically this is meaningless as the cross section ultimately depends on the modulus of the amplitude squared i.e. $|A|^2$. Writing amplitudes in this notation can drastically simplify calculations, as we discuss in Appendix A.

3.2 Color ordering

3.2.1 Tree Level

We begin by describing the color decomposition of amplitudes at tree level [3, 30, 6, 4]. For the purposes of this work, we need only consider the color ordering of amplitudes where all particles transform in the adjoint representation. For a more general discussion, see [4]. Although the gauge group for QCD is $SU(3)$, we often generalise this to $SU(N_c)$, where N_c represents the number of colors. The generators of $SU(N_c)$ in the fundamental representation are T^a, the index $a = 1, ..., N_c^2 - 1$ refers to the adjoint color index carried by gluons. The T^a are traceless hermitian $N_c \times N_c$ matrices, normalised as

$$\mathrm{Tr}(T^a T^b) = \delta^{ab}. \tag{3.7}$$

For a generic QCD Feynman diagram there will be a number of standard vertices which we must rewrite using the above prescription. For each purely gluonic three-vertex there will be a group theory structure constant f^{abc} from the Feynman term [2],

$$V_3 = g f^{abc} \left[g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(k-p)^\mu + g^{\rho\mu}(k-p)^\nu \right] \tag{3.8}$$

where we define this structure constant by,

$$[T^a, T^b] = i f^{abc} T^c. \tag{3.9}$$

For each purely gluonic four vertex there will be pairs of structure constants $f^{abc} f^{cde}$ from terms [2],

$$V_4 = -ig^2 \; [f^{abc}f^{cde}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) + f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}) + f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})]. \tag{3.10}$$

Many indices are contracted together by factors δ_{ab} appearing in the gluon propagators, written as [2],

$$-\frac{i}{p^2}g^{\mu\nu}\delta_{ab} \tag{3.11}$$

Eliminating the structure constants in favour of the group generators T^a exposes the general color structure of an amplitude. At any arbitrary vertex the color structure function can be replaced using eq. (3.9), rewritten as,

$$f_{abc} = -i\left(\text{Tr}(T^aT^bT^c) - \text{Tr}(T^cT^bT^a)\right). \tag{3.12}$$

Each leg attached to this vertex is either an external leg, or is an internal leg and is thus connected to another vertex. The color structure function associated with this next vertex is replaced using the T^c associated with the internal leg from the previous vertex by writing it as $f_{cde}T^c = -i[T^d, T^e]$. This process is continued until all vertices in the Feynman diagram have been replaced in this way. We are left with a large number of traces of the form $\text{Tr}(...T^a...)\text{Tr}(...T^a...)...\text{Tr}(...T^a...)$. We can use the $SU(N_c)$ Fierz identity [31],

$$\text{Tr}(T^aX)\text{Tr}(T^aY) = \text{Tr}(XY) - \frac{1}{N_c}\text{Tr}(X)\text{Tr}(Y), \tag{3.13}$$

to rearrange the contracted T^a's and reduce the number of traces. Often, it may be more convenient to consider the gauge theory as $U(N_c) = SU(N_c) \times U(1)$. The new $U(1)$ generator is proportional to the identity matrix. The $U(N_c)$ generators still obey eq. (3.13) only now without the $-1/N_c$ term. As the supplementary $U(1)$ gauge field commutes with $SU(N_c)$ i.e. $f^{a_{U(1)}bc} = 0$ for all b, c; it carries no color charge and is called the photon.

By making the substitutions eq. (3.12) and eq. (3.13), remembering to neglect the $-1/N_c$ term where appropriate, we are left with all possible permutations of a single trace. In this way any tree diagram for n-gluon scattering can be reduced to the sum of a single trace. We write the color decomposition of an n-gluon tree amplitude as [6],

$$A_n^{\text{tree}}(k_i, \lambda_i, a_i) = g^{n-2} \sum_{\sigma \in \frac{S_n}{Z_n}} \text{Tr}(T^{a_{\sigma(1)}}....T^{a_{\sigma(n)}}) A_n^{\text{tree}}(\sigma(1^{\lambda_1}), ..., \sigma(n^{\lambda_n})), \tag{3.14}$$

where k_i and λ_i are the gluon momenta and helicity respectively, and g is the gauge coupling given by $\alpha_s = \frac{g^2}{4\pi}$. S_n is the set of all permutations of n objects and Z_n is the subset of cyclic permutations which preserves the trace. Summing over the set S_n/Z_n includes all possible distinct cyclic orderings that appear in the trace.

By construction, the amplitude has been expanded into color dependent and color independent pieces. We may now focus on manipulating the partial amplitudes $A_n^{\text{tree}}(1^{\lambda_1}, ..., n^{\lambda_n})$, which contain only kinematic information. Calculating these partial amplitudes still requires a lot of work. However, because these are now color ordered amplitudes they are much simpler, as they only receive contributions from amplitudes in which the gluons have a particular cyclic ordering. The possibilities for singularities of the partial amplitudes, such as poles (and for loop amplitudes cuts), are restricted as they must occur when cyclically adjacent momenta form the momentum channels.

These partial amplitudes satisfy a number of important properties and relationships. They are gauge invariant, so we have the freedom to choose different gauges for different partial amplitudes, simplifying calculations. The number of independent partial amplitudes that must be computed is reduced as they are invariant under cyclic permutations of legs and have symmetries such as parity; which allows us to reverse all helicities in an amplitude simultaneously; and reflection, i.e. $A_n^{\text{tree}}(n^{\lambda_1}, ..., 1^{\lambda_n}) = (-1)^n A_n^{\text{tree}}(1^{\lambda_1}, ..., n^{\lambda_n})$. They also obey group theory relations, such as dual Ward identities, that allow us to write partial amplitudes where two negative helicities are not adjacent in terms of a partial amplitude in which they are adjacent [3].

Exploiting these various symmetries and group theory properties greatly reduces the number of independent partial amplitudes that must be computed.

3.2.2 Loop Level

The color decomposition of loop amplitudes [31] is similar. Since all structure constants involving an extra $U(1)$ field must vanish, the $U(1)$ gauge boson decouples, and once again we can work with $U(N)$ instead of $SU(N)$. This simplifies the process as the $U(1)$ Fierz identities [31],

$$\text{Tr}(T^a X)\text{Tr}(T^a Y) = \text{Tr}(XY), \tag{3.15}$$

$$\text{Tr}(T^a X T^a Y) = \text{Tr}(X)\text{Tr}(Y). \tag{3.16}$$

are simpler than their $SU(N)$ counterpart eq. (3.13). At one-loop we now generate both single and double trace structures. For a closed loop we are contracting the indices of two generators in the same trace. There are two possibilities for this process. The first case is when the two generators sit next to each other, i.e.

$$\text{Tr}(T^{a_1}...T^{a_m}T^{a_l}T^{a_l}T^{a_{m+1}}...T^{a_n}), \tag{3.17}$$

in which case the contraction on $T^{a_1}T^{a_1}$ will produce a Casimir operator, in the $U(N)$ representation written as $T^aT^a = N_C$. In this case we derive a single trace multiplied by the number of colors,

$$N_c \mathrm{Tr}(T^{a_1}...T^{a_n}). \tag{3.18}$$

The alternative is that the two generators we contract are separated by other generators, i.e.

$$\mathrm{Tr}(T^{a_1}...T^{a_m}T^{a_1}T^{a_2}...T^{a_3}T^{a_1}T^{a_{m+1}}...T^{a_n}). \tag{3.19}$$

In this case we can use the second $U(N)$ Fierz identity eq. (3.16) to derive two traces,

$$\mathrm{Tr}(T^{a_1}...T^{a_m}T^{a_{m+1}}...T^{a_n})\mathrm{Tr}(T^{a_2}...T^{a_3}). \tag{3.20}$$

Following this prescription leads to the one-loop color decomposition. In this case there are two traces over color matrices and we must also sum over the different spins, J, of the internal particles circulating in the loop. When all particles transform as color adjoints, the result takes the form [31],

$$A_n(\{k_i, \lambda_i, a_i\}) = g^n \sum_J \sum_{c=1}^{n/2+1} \sum_{\sigma \in \frac{S_n}{S_{n;c}}} Gr_{n;c}(\sigma) A_{n;c}^{[J]}(\sigma), \tag{3.21}$$

where Z_n are subsets of S_n that leave the single and double traces invariant, and the sum over c runs up to the largest integer that is less than or equal to $n/2 + 1$.

The leading color structure factor,

$$Gr_{n;1}(1) = N_c \mathrm{Tr}(T^{a_1}...T^{a_n}), \tag{3.22}$$

is just N_c times the tree color structure factor, and the sub-leading color structures ($c > 1$) are given by

$$Gr_{n;c}(1) = \mathrm{Tr}(T^{a_1}...T^{a_{c-1}})\mathrm{Tr}(T^{a_c}...T^{a_n}). \tag{3.23}$$

The partial amplitudes $A_{n;c}$ are not the most basic objects in eq. (3.21). This role is taken by the $A_{n;1}$, called primitive amplitudes, which are color ordered just like the tree partial amplitudes A_n^{Tree}. We can generate the one-loop partial amplitudes $A_{n;c>1}$ as sums of the $A_{n;1}$ using an appropriate permutation sum [8],

$$A_{n;j}(1,2,...,j-1,j,j+1,...,n) = (-1)^{j-1} \sum_{\sigma \in COP\{\alpha\}\{\beta\}} A_{n;1}(\sigma) \tag{3.24}$$

where $\alpha_i \in \{\alpha\} \equiv \{j-1,...,1\}$, and $\beta_i \in \{\beta\} \equiv \{j,...,n\}$. $COP\{\alpha\}\{\beta\}$ represents the set of all permutations of $\{1...n\}$ where n is held fixed that maintain the cyclic ordering of the α_i and the β_i within $\{\alpha\}$ and $\{\beta\}$ respectively.

Therefore, to reconstruct the full one-loop amplitude, it is sufficient to consider only the $A_{n;1}$ as they contain all the information required.

3.3 Supersymmetric Decomposition of Loop Amplitudes

For the one-loop calculations required, even with the use of the spinor helicity notation and color ordering we have discussed, the algebra is still considerably complicated. We require an additional tool for organising loop amplitudes to help calculations go through.

QCD is a non-Supersymmetric theory, where, for purely gluonic amplitudes, this non-Supersymmetry becomes manifest at loop level. We can, however, still make use of Supersymmetry to manipulate the sum over internal spins of particles circulating around the loop, rearranging the terms that appear in the sum to simplify a calculation. It has been shown [32, 33, 34] that the most simple way to determine gluon amplitudes is by evaluating contributions from different Supersymmetric multiplets.

For $N = 1$ Super Yang-Mills with external gluons there are two possible Supersymmetric multiplets contributing to the loop amplitude, the vector and the chiral matter multiplets. These can be decomposed into the contribution from single particle spins (for simplicity we consider the leading-in-color components of color ordered one-loop amplitudes),

$$\begin{aligned} A_n^{N=1\ vector} &\equiv A_n^{[1]} + A_n^{[1/2]} \\ A_n^{N=1\ chiral} &\equiv A_n^{[1/2]} + A_n^{[0]} \end{aligned} \tag{3.25}$$

where $A_n^{[J]}$ is the one-loop amplitude with n external gluons and particles of spin J circulating in the loop. (We represent a complex scalar as spin 0 in this notation). In particular the $N = 1$ chiral or matter multiplet contains one fermion and one complex (two real) scalars.

For $N = 4$ Super Yang-Mills theory there is a single multiplet given by

$$A_n^{N=4} \equiv A_n^{[1]} + 4A_n^{[1/2]} + 3A_n^{[0]}. \tag{3.26}$$

where the $N = 4$ Super Yang-Mills multiplet contains a gluon, four gluinos and three complex (six real) scalars.

This decomposition has certain advantages that make calculations simpler. The Supersymmetric terms, $A_n^{N=1\ vector}$, $A_n^{N=1\ chiral}$ and $A_n^{N=4}$, obey Supersymetric Ward identities, as discussed in chapter 2. In addition to this they also have generic cancellations between terms on a diagram by diagram basis. Therefore they are much simpler to compute than the non-Supersymmetric terms.

This technique allows us to replace a gluon propagating in the loop, i.e. a term $A_n^{[1]}$, with a scalar, i.e. a term $A_n^{[0]}$, plus Supersymmetric terms. For example [3], using eq. (3.25) and eq. (3.26) we can rewrite the internal gluon loop g, of a QCD loop amplitude in terms of a scalar loop s, and Supersymmetric contributions, i.e.

$$g = (g + 4f + 3s) - 4(f + s) + s = A_n^{N=4} - 4A_n^{N=1\ chiral} + A_n^{[0]} \tag{3.27}$$

Thus we can solve for the QCD amplitude by calculating the Supersymmetric contributions $A_n^{N=4}$ and $A_n^{N=1\ chiral}$ and the non-Supersymmetric term $A_n^{[0]}$. The scalar loop is more complicated than the Supersymmetric terms. However, since a scalar does not carry any spin information this term is simpler than the gluon loop, so overall the substitution has simplified the calculation.

The contributions from the three multiplets, $A_n^{N=1\ vector}$, $A_n^{N=1\ chiral}$ and $A_n^{N=4}$, are not independent. Continuing our policy of counting particles that circulate within the loop, we can write,

$$(g + f) = (g + 4f + 3s) - 3(f + s). \tag{3.28}$$

Therefore the contributions from $A_n^{N=1\ vector}$, $A_n^{N=1\ chiral}$ and $A_n^{N=4}$ satisfy,

$$A_n^{N=1\ vector} = A_n^{N=4} - 3A_n^{N=1\ chiral}. \tag{3.29}$$

Provided the $N = 4$ amplitude is known, and this is usually the simplest non-trivial term to compute, we must only calculate one of the two possibilities for $N = 1$. Therefore, by using this decomposition we can reduce the number of independent calculations we must complete to determine a particular scattering amplitude.

Chapter 4

Analytic Properties of Amplitudes

Tree and loop level amplitudes have a number of factorisation properties that can be exploited to make calculations easier. Although they are traditionally used as consistency checks, the analytic properties of amplitudes can sometimes also be used to derive information about amplitudes and thus help to construct their general form.

4.1 Tree Level

For tree level amplitudes the main factorisation properties occur as kinematic invariants vanish, i.e. as $K_{ij}^2 \to 0$, where $K_{ij} \equiv (k_i + k_{i+1} + \ldots + k_j)$. We call such a property a pole. Poles in color ordered amplitudes can only occur in channels with cyclically adjacent momenta. There are essentially two types of pole that can occur, multi-particle poles and collinear poles. For channels corresponding to three or more momenta the pole is referred to as a multi-particle pole, whereas if there are only two momenta contributing to the channel then the pole is called a collinear pole.

4.1.1 Multi-particle Poles

As $K^2 \to 0$ we write a multi-particle pole at tree level as [3],

$$A_n^{\rm tree}(1,...,n) \sim \sum_\lambda A_{r+1}^{\rm tree}(1,...,r,K^\lambda)\frac{i}{K^2}A_{n-r+1}^{\rm tree}(r+1,...,n,K^{-\lambda}), \tag{4.1}$$

where the intermediate state has momentum K and helicity λ, with λ reversed in going from one side of the pole to the other due the convention of always writing the helicity as that of an outgoing particle.

MHV amplitudes do not have multi-particle poles, since for MHV amplitudes we are always restricted to having only three negative helicity states to distribute around the pole. Since $A_n^{\rm tree}(1^\pm,2^+,...,n^+) = 0$, this is not enough to prevent one of the factorised amplitudes vanishing. Therefore, for the special case of MHV amplitudes, only the collinear poles are present.

4.1.2 Collinear Limits

We write a general collinear pole at tree level as follows. When the momenta of two neighbouring legs become collinear we write the resultant pole as [3],

$$A_n^{\rm tree}(...,i^{\lambda_i},j^{\lambda_j},...) \xrightarrow{i\|j} \sum_{\lambda=\pm} {\rm Split}_{-\lambda}^{\rm tree}(z,i^{\lambda_i},j^{\lambda_j}), A_{n-1}^{\rm tree}(...,K^\lambda,...), \tag{4.2}$$

where the two collinear legs are denoted by i and j and ${\rm Split}^{\rm tree}$ denotes a splitting amplitude. The momentum of the intermediate state given by the null vector K, helicity λ, is simply the sum of the momenta of the collinear legs, i.e. $K = p_i + p_j$. The collinear limit is defined by $p_i = zK$ and $p_j = (1-Z)K$.

We can use the collinear limits of known five-point amplitudes to derive the splitting amplitudes in eq. (4.2). As an example [3], consider the $p_4 \to p_5$ collinear limit of the five-point amplitude $A_5^{\rm tree}(1^-,2^-,3^+,4^+,5^+)$.

$$A_5^{\rm tree}(1^-,2^-,3^+,4^+,5^+) = i\frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,5\rangle\langle 5\,1\rangle}$$

$$\begin{aligned}
&\stackrel{4\|5}{\longrightarrow} \frac{1}{\sqrt{z(1-z)}\,\langle 4\,5\rangle} \times i\frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,K\rangle\,\langle K\,1\rangle} \\
&= \text{Split}_-^{\text{tree}}(4^+,5^+) \times A_4^{\text{tree}}(1^-,2^-,3^+,K^+).
\end{aligned} \tag{4.3}$$

Thus using the known five-point result we have derived the splitting function $\text{Split}_-^{\text{tree}}(i^+,j^+)$. We can use other five-point amplitudes to derive the remaining $g \to gg$ splitting amplitudes, which are given by [5, 35, 36, 4],

$$\begin{aligned}
\text{Split}_-^{\text{tree}}(i^-,j^-) &= 0, \\
\text{Split}_-^{\text{tree}}(i^+,j^+) &= \frac{1}{\sqrt{z(1-z)}\,\langle i\,j\rangle}, \\
\text{Split}_+^{\text{tree}}(i^+,j^-) &= \frac{(1-z)^2}{\sqrt{z(1-z)}\,\langle i\,j\rangle}, \\
\text{Split}_-^{\text{tree}}(i^+,j^-) &= -\frac{z^2}{\sqrt{z(1-z)}\,[i\,j]}.
\end{aligned} \tag{4.4}$$

The splitting amplitudes for factorisations involving fermions are also easy to obtain from the limits of fermionic amplitudes derived from a SWI argument.

4.2 Soft Limits

The behaviour of QCD amplitudes in the soft limit, i.e. where a gluon momentum vector k_i goes to zero, is also well understood. For amplitudes at tree level we get,

$$A_n^{\text{tree}}(\ldots,a,i,b,\ldots) \stackrel{K_i\to 0}{\longrightarrow} \text{Soft}^{\text{tree}}(a,i,b)A_{n-1}^{\text{tree}}(\ldots,a,b,\ldots). \tag{4.5}$$

The soft factor,

$$\text{Soft}^{\text{tree}}(a, i, b) = \frac{\langle a\, b\rangle}{\langle a\, i\rangle \langle i\, b\rangle}, \tag{4.6}$$

depends on the neighbouring partons, a and b, of the soft gluon i. Since we are working with color ordered amplitudes now, to be precise this statement should read the color ordered neighbouring partons. In spite of this dependence on the partons a and b, the soft behaviour is independent of both the helicity of a and b and their particle type, i.e. whether they are gluons or quarks.

4.3 Loop Level

The factorisation structure of loop amplitudes is analogous to that of amplitudes at tree level. However, the splitting of amplitudes around poles is not as clean as in the tree level case. Thus in generalising the results for multi-particle poles and collinear limits from tree level to loop level we shall see that the expressions are not as simple as before, since momenta on each side of the pole are not as segregated as in the tree level case. The factorisation properties of loop level amplitudes have been used as consistency checks on calculations, see [37, 8, 38].

4.3.1 Multi-particle Poles

The multi-particle factorisation properties of loop amplitudes in a channel $(k_i+k_{i+1}+\cdots+k_{i+r-1})^2 = K^2 \to 0$ can be written as [38],

$$\begin{aligned}A_n^{\text{loop}} \overset{K^2\to 0}{\longrightarrow} \sum_{\lambda=\pm} \Big[& A_{r+1}^{\text{loop}}(k_i,...,k_{i+r-1},K^\lambda)\frac{1}{K^2}A_{n-r+1}^{\text{tree}}(K^{-\lambda},k_{i+r},...,k_{i-1}) \\ & +A_{r+1}^{\text{tree}}(k_i,...,k_{i+r-1},K^\lambda)\frac{1}{K^2}A_{n-r+1}^{\text{loop}}(K^{-\lambda},k_{i+r},...,k_{i-1}) \\ & +A_{r+1}^{\text{tree}}(k_i,...,k_{i+r-1},K^\lambda)\frac{1}{K^2}A_{n-r+1}^{\text{tree}}(K^{-\lambda},k_{i+r},...,k_{i-1})c_\Gamma F_n(K^2;k_1,...,k_n)\Big],\end{aligned} \tag{4.7}$$

where F_n is the one-loop factorisation function, and the factor c_Γ is given by,

$$c_\Gamma = \frac{1}{(4\pi)^{2-\epsilon}}\frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \quad . \tag{4.8}$$

The inclusion of a factorisation function reflects the property that for loop amplitudes, momenta from each side of the pole still interact. As such, F_n is a function of all momenta $k_i...k_n$. F_n contains both factorising and non-factorising pieces, i.e.

$$F_n = F_n^{\text{fact}} + F_n^{\text{non-fact}}. \tag{4.9}$$

For SUSY theories the non-factorising pieces vanish.

4.3.2 Collinear Limits

The behaviour of a one-loop amplitude at a collinear limit is analogous to that of a tree level amplitude. An n-point amplitude is reduced to a sum of $n-1$-point amplitudes multiplied by splitting functions when two external legs become collinear.

For loop level amplitudes, the summation includes lower point tree amplitudes multiplied by loop splitting functions, and lower point loop amplitudes multiplied by tree splitting functions. We write the collinear limits of a color ordered one-loop amplitude as [38],

$$\begin{aligned} A_{n;1}^{\text{loop}} \xrightarrow{i\|j} \sum_{\lambda=\pm} \Big(& \text{Split}_{-\lambda}^{\text{tree}}(i^{\lambda_i}, j^{\lambda_j}) A_{n-1;1}^{\text{loop}}(\ldots(i+j)^{\lambda}\ldots) \\ & + \text{Split}_{-\lambda}^{\text{loop}}(i^{\lambda_i}, j^{\lambda_j}) A_{n-1}^{\text{tree}}(\ldots(i+j)^{\lambda}\ldots)\Big), \end{aligned} \tag{4.10}$$

where we define the collinear limit by $p_i = zK$ and $p_j \to (1-z)K$ with $K = p_i + p_j$, as in the tree level case. The helicity and momentum of the intermediate state are given by λ and K respectively.

For Supersymmetric theories the loop splitting amplitudes $\text{Split}_{-\lambda}^{\text{loop}}(a^{\lambda_a}, b^{\lambda_b})$ are proportional to the tree splitting amplitudes,

$$\text{Split}_{-\lambda}^{\text{loop}}(a^{\lambda_a}, b^{\lambda_b}) = c_\Gamma \times \text{Split}_{-\lambda}^{\text{tree}}(a^{\lambda_a}, b^{\lambda_b}) \times r_S^{\text{SUSY}}(z, s_{ab}). \tag{4.11}$$

and for particular multiplets, i.e. $N = 1$ chiral the loop splitting amplitudes vanish. Thus we need only consider the first term in eq. (4.8). As is the case at tree level, the loop level splitting functions can be extracted from the collinear behaviour of known one-loop gluon amplitudes.

4.4 Unitarity

The factorisation properties of loop amplitudes are similar to those of tree amplitudes. However, at loop level there is an additional, distinctive analytic property. One-Loop amplitudes have cuts as well as collinear and multi-particle poles.

4.4.1 The Optical Theorem and Cutkosky's Rules

When expressed as a function of energy, a scattering amplitude has a branch cut on the positive real axis. The optical theorem states that the discontinuity across this represents the imaginary part of the scattering amplitude. Manipulating this theorem and applying the resultant technique of "cutting" amplitudes allows us to solve for certain amplitudes.

Unitarity of the S-matrix i.e. $S^{\dagger}S = 1$, leads directly to the optical theorem. From the unitarity constraint, we can see that the T-matrix, defined by $S = 1 + iT$, must obey the relation,

$$-i(T - T^{\dagger}) = T^{\dagger}T \tag{4.12}$$

From this we can derive the optical theorem, see [2] for a full treatment. This describes how the imaginary part of a scattering amplitude arises from a sum of contributions from all possible intermediate state particles.
We represent this diagrammatically as [2],

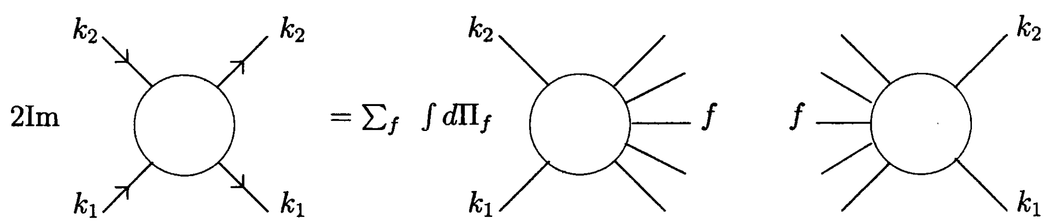


Figure 4.3.1: The Optical Theorem

where we must sum over all on-shell intermediate states f.

Cutkosky generalised this result to multi-loop diagrams [39]. In doing so he gave a set of cutting rules that allow us to determine the physical discontinuity across the branch cut of any Feynman diagram (see [2] for a more thorough description of how these rules are derived for Feynman Diagrams). Cutkosky's rules are,

- Cut through the diagram in every way that allows the cut propagators to be simultaneously put on-shell. Since the only imaginary contribution to a Feynman diagram comes from the $i\epsilon$ factor in the pole denominator, the diagram must contain on-shell particles, where $k^2 = m^2$, for there to be an imaginary part for $\mathcal{M}$, i.e. the $i\epsilon$ factor must become significant.

- For each of the allowed cuts, replace the factor of $1/(k^2 - m^2 + i\epsilon)$ appearing in the cut propagator with a factor of $-2\pi i\delta(k^2 - m^2)$ and then carry out the loop integral.

- Finally sum all of the possible cut contributions.

Cutkosky's method is completely general and can be applied to all orders in perturbation theory.

4.4.2 Performing a Unitarity Cut

When using these techniques there are some technical points to be aware of. In ref [8] the concept that amplitudes are cut constructible was introduced. At first sight the meaning of this term appears obvious, that one may calculate an amplitude from a knowledge of its cuts in all channels (see ref. [40] for a modern review). This means that if we calculate the cuts precisely and regularise them in the same fashion as the amplitude, then we can determine any amplitude. Specifically, if we regularise the amplitude by dimensional regularisation then, for consistency, in the cut $C_{i\ldots j}$ we should use tree amplitudes with external momenta defined in four dimensions, while the momenta crossing the cut should be defined in $4-2\epsilon$ dimensions. These are not normal tree amplitudes. Fortunately, for $N=4$ and $N=1$ Supersymmetric gauge theory amplitudes it is not necessary to evaluate the cuts in this precise manner. Bern, Dixon, Dunbar and Kosower [8] showed that in Supersymmetric theories unitarity cuts can be calculated using amplitudes where the cut legs lie in four dimensions. This means that the cut can be evaluated using the conventional four dimensional tree amplitudes. This is an enormous simplification as the expressions obtainable for on-shell tree amplitudes in four dimensions are relatively simple.

The unitarity cut of an amplitude is written as the product of two tree amplitudes, one on each side of the cut, with the loop integral replaced by an integral over the phase space of the particles crossing the cut, i.e.

$$C_{i\,\ldots\,j} \;\equiv\; \frac{i}{2}\int d\mathrm{LIPS}\Big[A^{\mathrm{tree}}(\ell_1,i,i+1,\;\ldots\;,j,\ell_2)\times A^{\mathrm{tree}}(-\ell_2,j+1,j+2,\;\ldots\;,i-1,-\ell_1)\Big] \tag{4.13}$$

The helicity of each intermediate particle is reversed upon crossing the cut due to our convention of taking the helicity as that of an outgoing particle. We must consider each intermediate helicity configuration that can contribute and sum these to reconstruct the full amplitude.

There are a number of reasons why it is simpler to consider these cuts rather than a direct loop calculation. For instance, we can simplify the required tree amplitudes before substituting them into eq. (4.13). Likewise, the tree amplitudes are quite simple, and can be considered as effectively Supersymmetric, since at tree level there is no difference between a fermion loop and a bosonic loop as there are no loops at all. Thus even if the full amplitude is not Supersymmetric the amplitudes we use in eq. (4.13) are. Finally, as we have insisted that the intermediate legs be defined as being on-shell, there are various associated properties of on-shell momenta that can be used when computing eq. (4.13).

However, it is not always possible to use the cuts of an amplitude to reconstruct it in full. In addition to the known scalar integral functions that appear in loop amplitudes, there may also be functions that are rational in the kinematic variables. These cannot be detected by the cutting technique. For Supersymmetric amplitudes this effect is not present, but must be considered if we want to calculate non-Supersymmetric amplitudes in full. It is possible, however, to remedy this situation by demanding that an amplitude possesses consistent collinear factorisation properties in all channels [3]. We can use these properties to isolate the rational terms and thus complete the calculation [37, 38].

4.4.3 Supersymmetric ρ Factors

As we have just discussed, the cutting technique we use to compute loop amplitudes involves the calculation of the product of MHV and $\overline{\text{MHV}}$ tree amplitudes. Recall that the purely gluonic MHV tree level amplitudes were given by Parke and Taylor [5] as,

$$A_n^{\text{MHV}} = i\frac{\langle s\,r\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\ldots\langle n\,1\rangle}, \tag{4.14}$$

where s and r are the negative helicity gluons. The gluonic $\overline{\text{MHV}}$ tree amplitudes are given by [5],

$$A_n^{\overline{\mathrm{MHV}}} = i\frac{[s\,r]^4}{[1\,2]\,[2\,3]\ldots[n\,1]}, \tag{4.15}$$

where s and r now label the only positive helicity gluons in the amplitude.

For MHV tree amplitudes the contributions from non-scalar particles can be related to that of the real scalar [11, 41].

If we introduce particles of helicity λ, in positions r^- and t^+, where $\lambda = 1$ for a gluon, $\lambda = 1/2$ for a fermion and $-1/2$ for an anti-fermion, and $\lambda = 0$ for a scalar, then the tree amplitude eq. (4.14) becomes [41],

$$A_n^{\mathrm{MHV}} = i\frac{\langle s\,r\rangle^{2+2\lambda}\,\langle s\,t\rangle^{2-2\lambda}}{\langle 1\,2\rangle\,\langle 2\,3\rangle\ldots\langle n\,1\rangle}, \tag{4.16}$$

In effect, we have multiplied the purely gluonic term by a factor of $[\langle s\,t\rangle\,/\,\langle s\,r\rangle]^{2-2\lambda}$. We use the scalar (i.e. $\lambda = 0$) MHV, and the equivalent scalar $\overline{\mathrm{MHV}}$, tree amplitudes when performing loop amplitude cuts. By using this approach, we can account for the different Supersymmetric multiplets by multiplying the resultant expression by an appropriate factor, called the ρ factor. For the chiral multiplet the contribution relative to the real scalar has a factor,

$$\rho^{\mathrm{N}=1} = -X + 2 - \frac{1}{X} = -\frac{(X-1)^2}{X} \tag{4.17}$$

where

$$X = \frac{[i^a\,\ell^a]\,\langle j^b\,\ell^a\rangle}{[i^a\,\ell^b]\,\langle j^b\,\ell^b\rangle}. \tag{4.18}$$

where i and j are external legs of opposite helicity a and b that define either side of the cut as MHV or $\overline{\text{MHV}}$, and where the terms ℓ represent the momenta crossing the cut.

For example, consider the cut,

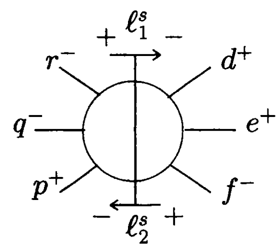


Figure 3.4.1: An Example Unitarity Cut

where ℓ_1 and ℓ_2 are considered to be real scalar particles. The $(\ell_2^-, p^+, q^-, r^-, \ell_1^+)$ side of the cut is $\overline{\text{MHV}}$, and is given by

$$A^{\overline{\text{MHV}}} = \frac{[p\,\ell_1]^2\,[p\,\ell_2]^2}{[\ell_2\,p]\,[p\,q]\,[q\,r]\,[r\,\ell_1]\,[\ell_1\,\ell_2]} \tag{4.19}$$

where p^+ is the external leg that has defined the cut as $\overline{\text{MHV}}$. The $(\ell_1^-, d^+, e^+, f^-, \ell_2^+)$ side of the cut is MHV, which we write as

$$A^{\text{MHV}} = \frac{\langle f\,\ell_1\rangle^2\,\langle f\,\ell_2\rangle^2}{\langle \ell_1\,d\rangle\,\langle d\,e\rangle\,\langle e\,f\rangle\,\langle f\,\ell_2\rangle\,\langle \ell_2\,\ell_1\rangle} \tag{4.20}$$

where f^- is the external leg that has defined the cut as MHV. Therefore, the X factor for this cut is given by

$$X = \frac{[p\,\ell_2]\,\langle f\,\ell_2\rangle}{[p\,\ell_1]\,\langle f\,\ell_1\rangle}, \tag{4.21}$$

which is obtained by following the procedure just described.

The $N = 4$ multiplet has a factor of

$$\rho^{\mathrm{N=4}} = X^2 - 4X + 6 - 4\frac{1}{X} + \frac{1}{X^2} = \frac{(X-1)^4}{X^2}. \tag{4.22}$$

where X is the same as in the $N = 1$ case. Therefore, the ρ factors for the $N = 4$ and $N = 1$ multiplets are simply related as,

$$\rho^{\mathrm{N=4}} = (\rho^{\mathrm{N=1}})^2. \tag{4.23}$$

4.5 Loop Integral Reduction Techniques

Loop calculations with many external legs are very complicated. In particular, most of the complications arise when one is actually carrying out the loop integral. Thankfully there exist a number of techniques that can be used to simplify these integrals sufficiently to allow the computations to be carried out much more easily. Although these techniques all differ in their application in some way, they share the same foundation, as discussed below.

For a one-loop n-point calculation the general integral in $4-2\epsilon$ dimensions is written as [3],

$$I_n[P(\ell^\mu)] = \int \frac{d^{4-2\epsilon}\ell}{(2\pi)^{4-2\epsilon}} \frac{P(\ell^\mu)}{\ell^2(\ell-k_1)^2 \ldots (\ell-k_1-k_2-\ldots-k_{n-1})^2} \tag{4.24}$$

where the momenta k_i through leg i is defined to be flowing out of the loop. The function $P(\ell^\mu)$ is polynomial in the loop momentum.

For integrals of more than five external points, i.e. $n \geq 5$, we have at least four independent momenta, $p_1 = k_1$, $p_2 = k_1 + k_2$, $p_3 = k_1 + k_2 + k_3$, $p_4 = k_1 + k_2 + k_3 + k_4$. Following the method of [42], as discussed in [3], we expand the loop momentum ℓ^μ in terms of a set of axial momenta v_i^μ, written as,

$$\begin{aligned} v_1^\mu &= \epsilon(\mu,2,3,4) \qquad v_2^\mu = \epsilon(1,\mu,3,4) \qquad v_3^\mu = \epsilon(1,2,\mu,4) \qquad v_4^\mu = \epsilon(1,2,3,\mu) \\ v_i \cdot p_j &= \epsilon(1,2,3,4)\delta_{ij} \end{aligned} \tag{4.25}$$

where we have taken the convention [42],

$$\epsilon(1,2,3,4) = \epsilon^{\mu\nu\rho\sigma} p_{1\mu} p_{2\nu} p_{3\rho} p_{4\sigma}. \tag{4.26}$$

We write the expansion as,

$$\ell^\mu = \frac{1}{\epsilon(1,2,3,4)} \sum_{i=1}^{4} v_i^\mu \ell.p_i \tag{4.27}$$

We now want to write ℓ^μ in terms of the propagator denominators of the general loop integral equation, eq. (4.24). We can achieve this by expanding the right hand side of eq. (4.27). In doing so we also pick up a term that is independent of the loop momentum. We get,

$$\ell^\mu = \frac{1}{2\epsilon(1,2,3,4)} \sum_{i=1}^{4} v_i^\mu [\ell^2 - (\ell - p_i)^2 + p_i^2]. \tag{4.28}$$

Substituting eq. (4.28) into the degree p polynomial $P(\ell^\mu)$ in the general integral equation, eq. (4.24), the terms dependent on the propagator denominators now cancel top and bottom. This has the desired effect of reducing the original n point loop integral with polynomials of order p into an $(n-1)$ point integral with polynomials of degree $p-1$, plus scalar n-point integrals that are derived from the additional terms in the expansion that were independent of the loop momentum. If we repeat this procedure iteratively we can reduce n-point integrals all the way down to four-point box integrals plus the additional scalar pieces. Writing the loop momentum polynomial $P(\ell^\mu)$ in terms of the propagator denominators in this way is a general technique that allows loop integrals to be reduced without the need for excessive algebra.

Other similar techniques [42, 43, 34, 45] allow us to write the scalar integrals for $I_n[1]$ as a linear combination of integrals $I_{n-1}[1]$. Likewise we can write a scalar pentagon integral as a sum of box integrals, which can be reduced further either through a standard Passarino-Veltman reduction [46], or by using techniques like the one discussed above [47].

Each of these techniques shares the same fundamental approach, that the degree of the loop momentum polynomial $P(\ell^\mu)$ is reduced by one in each step. The result of such analysis is the principle that amplitudes can be expanded in terms of integral functions representing scalar boxes, triangles and bubbles, i.e.

$$A = \sum \hat{c}_i I_4 + \sum \hat{d}_i I_3 + \sum \hat{e}_i I_2 + \text{rational} \tag{4.29}$$

where I_4, I_3 and I_2 are the scalar box, triangle and bubble integral functions.

One-loop amplitudes depend on the particles circulating within the loop. For Supersymmetric amplitudes there are generically cancellations between the bosons and fermions in the loop. For Supersymmetric Yang-Mills theories these cancellations lead to considerable simplifications in the loop momentum integrals [48]. In general, theories with more Supersymmetry have a more restricted set of integral functions.

For $N = 4$ theories the series only contains scalar box functions, I_4, and hence is entirely determined by the box coefficients $\hat{c}_i$ [8]. For $N = 1$ Super Yang-Mills we have to consider the box functions together with scalar triangle and bubble functions, I_3 and I_2 [9]. For theories without Supersymmetry the amplitude may also contain rational pieces.

With this expansion, the computation of one-loop amplitudes is simplified by carefully considering the integral functions I_i that may appear, and by realising that the full amplitude is a linear combination of such functions with rational coefficients.

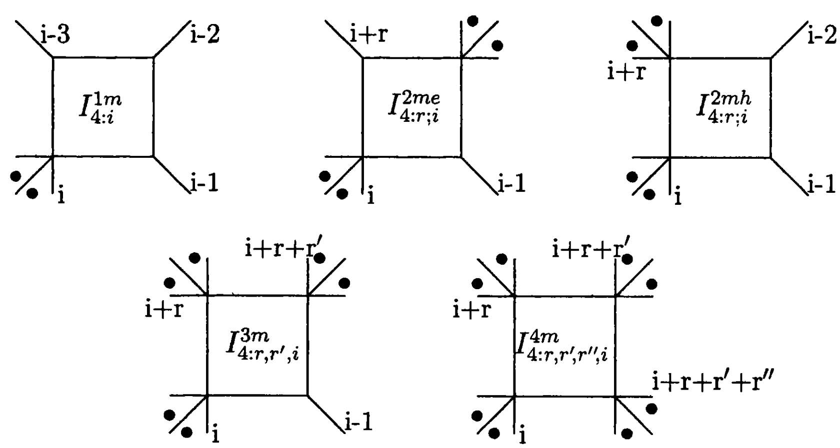
4.6 Basis of Functions

As we have seen, one-loop amplitudes can, in general, be decomposed in terms of a set of basis functions, I_i, with coefficients that are rational in terms of spinor products, and where the set of functions that appear in the summation can contain scalar boxes, I_4, scalar triangles, I_3, and scalar bubbles, I_2. Ultimately, there is a choice as to which basis of functions to use. As most of the work in this thesis is concerned with calculating box coefficients we shall focus on these as we discuss the various bases we can chose to work with.

In general, we can organise box functions according to the number of legs with non-null input momenta and the relative labelling of legs. Specifically we have,

$$I_{4:i}^{1\mathrm{m}} \quad I_{4:r;i}^{2\mathrm{me}} \quad I_{4:r;i}^{2\mathrm{m}h} \quad I_{4:r,r',i}^{3\mathrm{m}} \quad I_{4:r,r',r'',i}^{4\mathrm{m}} \tag{4.30}$$

with the labelling as indicated,



where the indices r, r' and r'' represent the number of legs at a particular corner.

We can consider three choices of basis, each of which has advantages in certain circumstances:

- $D = 4$ scalar box integrals.
- $D = 6$ scalar box integrals.
- $D = 4$ scalar box F-functions.

The natural choice would be to use the $D = 4$ scalar box integrals, but $D = 6$ scalar box integrals have certain practical advantages. For instance, determining the collinear limits of $D = 6$ scalar box integrals is particularly simple as they are IR finite. In reference [45], it was shown that the relationship between the $D = 4$ boxes and $D = 6$ boxes involves triangle functions and an overall factor. We can write this relationship as (see [45] for a full derivation and complete definitions),

$$I_4^{D=4} = \frac{1}{2N_4}\left[\sum_i \alpha_i \gamma_i I_3^{(i)} + (-1+2\epsilon)\hat{\Delta}_4 I_4^{D=6}\right]. \tag{4.31}$$

where the α_i are Feynman parameters (see [45, 49]), the integration variables in a Feynman parameterised integral. The variables γ_i are given by [45, 44],

$$\gamma_i = \sum_{j=1}^{n} \eta_{ij} \alpha j = \frac{1}{2}\frac{\partial \hat{\Delta}_n}{\partial \alpha_i}|_{\rho_{ij} \text{ fixed}} \tag{4.32}$$

where the η_{ij} are proportional to the inverse matrix ρ [45],

$$\eta_{ij} = N_n \rho_{ij}^{-1} \tag{4.33}$$

with ρ_{ij} defined by [45],

$$S_{ij} = \frac{\rho_{ij}}{\alpha_i \alpha_j} \tag{4.34}$$

where the symmetric matrix S_{ij} is given by $S_{ij} = -1/2(k_i + ... + k_{j-1})$, $i \neq j$, $S_{ii} = 0$. The $\hat{\Delta}_4$ are rational functions of the momentum invariants [45],

$$
\begin{aligned}
\frac{\hat{\Delta}^{1m}_{4:i}}{2N_4} &= -2\left(\frac{t^{[2]}_{i-3} + t^{[2]}_{i-2} - t^{[n-3]}_{i}}{t^{[2]}_{i-3}t^{[2]}_{i-2}}\right) = 2\frac{(k_{i-1}+k_{i-3})^2}{(k_{i-3}+k_{i-2})^2(k_{i-2}+k_{i-1})^2} \\
\frac{\hat{\Delta}^{2mh}_{4:r;i}}{2N_4} &= -2\left(\frac{(t^{[r+1]}_{i-1} - t^{[r]}_{i})(t^{[r+1]}_{i-1} - t^{[n-r-2]}_{i+r}) + t^{[r+1]}_{i-1}t^{[2]}_{i-2})}{t^{[2]}_{i-2}(t^{[r+1]}_{i-1})^2}\right) \\
&= \frac{\mathrm{tr}(\not{k}_{i-1}\not{P}_{i-1...i+r-1}\not{k}_{i-2}\not{P}_{i-1...i+r-1})}{(k_{i-2}+k_{i-1})^2(P^2_{i-1...i+r-1})^2} \\
\frac{\hat{\Delta}^{2me}_{4:r;i}}{2N_4} &= -2\left(\frac{t^{[r+1]}_{i-1} + t^{[r+1]}_{i} - t^{[r]}_{i} - t^{[n-r-2]}_{i+r+1}}{t^{[r+1]}_{i-1}t^{[r+1]}_{i} - t^{[r]}_{i}t^{[n-r-2]}_{i+r+1}}\right)
\end{aligned}
\tag{4.35}
$$

where $t^{[p]}_a \equiv (k_a + k_{a+1} + \cdots + k_{a+p-1})^2 = P^2_{a...a+p-1}$ and $P_{i...j} = k_i + k_{i+1} \ldots + k_j$.

The four dimensional boxes have dimension -2. It is convenient to define dimension zero F-functions by removing the momentum prefactors of the $D = 4$ scalar boxes [9],

$$
I_4^{D=4} = \frac{1}{K}F_4 \,.
\tag{4.36}
$$

where the prefactors for each type of box coefficient are given by [9],

$$
\begin{aligned}
I^{1m}_{4:i} &= -2r_\Gamma \frac{F^{1m}_{n:i}}{t^{[2]}_{i-3}t^{[2]}_{i-2}} \\
I^{2me}_{4:r,i} &= -2r_\Gamma \frac{F^{2me}_{n:r,i}}{t^{[r+1]}_{i-1}t^{[r+1]}_{i} - t^{[r]}_{i}t^{[n-r-2]}_{i+r+1}} \\
I^{2mh}_{4:r,i} &= -2r_\Gamma \frac{F^{2mh}_{n:r,i}}{t^{[2]}_{i-2}t^{[r+1]}_{i-1}}
\end{aligned}
$$

$$I^{3m}_{4:r,r',i} = -2r_\Gamma \frac{F^{3m}_{n:r,r',i}}{t^{[r+1]}_{i-1} t^{[r+r']}_i - t^{[r]}_i t^{[n-r-r'-1]}_{i+r+r'}}$$
$$I^{2m}_{4:r,r',r'',i} = -2r_\Gamma \frac{F^{4m}_{n:r,r',r'',i}}{t^{[r+r']}_i t^{[r'+r'']}_{i+r}} \tag{4.37}$$

It is the coefficients of these F-functions which the twistor inspired techniques act on, such as the collinearity and co-planarity operators [7, 50, 51].

For the box functions it is easy to switch between bases since,

$$A|_{\text{boxes}} = \sum_i c_i^{D=4} I_i^{D=4} = \sum_i c_i^{D=6} I_i^{D=6} = \sum_i c_i^F F_i, \tag{4.38}$$

Therefore the coefficients must satisfy,

$$c_i^{D=4} = \frac{c_i^{D=6}}{(-\hat{\Delta}_4/2N_4)} = c_i^F K. \tag{4.39}$$

Chapter 5

A Gauge Theory – String Theory Duality

As we discussed in the Introduction, many QCD amplitudes are more simple than we would expect. Parke and Taylor [5], and later Berends and Giele [6], showed that for a tree level gluon scattering amplitude to be non-vanishing, it must have at least two gluons of helicity opposite to the rest. Such an amplitude is called a MHV amplitude, and is given by,

$$A_n^{\mathrm{MHV}} \equiv A_n^{\mathrm{tree}}(1^+, 2^+, \dots, s^-, ..., r^-, ..., n^+) = i\frac{\langle s\, r\rangle^4}{\langle 1\, 2\rangle\, \langle 2\, 3\rangle \dots \langle n\, 1\rangle}, \tag{5.1}$$

where we use the notation described in chapter 3. The negative helicity gluons are labelled by s and r, and momentum conservation is implicit.

Such simplicity naturally leads us to question whether there is some underlying structure that is not yet apparent. This led Witten [1] to propose that there is a duality between $N = 4$ Super Yang-Mills Gauge Theory and a Topological String Theory. Instead of working with traditional space coordinates and momentum variables, to uncover this structure Witten transformed the amplitudes given in eq. (5.1) into the twistor space of Penrose [52]. Exposing more of the structure of tree amplitudes in this way has given rise to new, more efficient ways to compute amplitudes.

We take this opportunity to study Witten's transform, see [1] and [29] for a full analysis. In effect, we make a half Fourier transform. We exchange each left-handed spinor $\tilde{\lambda}_i$ for the conjugate Fourier variable μ_i, and do nothing to the right-handed spinors λ_i. We define the transformation with the exchanging variables written as [29],

$$\tilde{\lambda}_{\dot{\alpha}} = i\frac{\partial}{\partial \mu^{\dot{\alpha}}}, \qquad \mu^{\dot{\alpha}} = i\frac{\partial}{\partial \tilde{\lambda}_{\dot{\alpha}}}. \tag{5.2}$$

We work in signature $(+ \; + \; - \; -)$, as the transformation is made by a simple Fourier transform in this signature . From Quantum Mechanics we know we can write such a transformation between functions of $\tilde{\lambda}$ and functions of μ as [1],

$$\tilde{f}(\mu) = \int \frac{d^2\tilde{\lambda}}{(2\pi)^2} \exp(i\mu^{\dot{a}}\tilde{\lambda}_{\dot{a}}) f(\tilde{\lambda}). \tag{5.3}$$

By making this Fourier transform for each particle in the momentum space scattering amplitude $\hat{A}(\lambda_i, \tilde{\lambda}_i)$ we derive the twistor space scattering amplitude $\tilde{A}(\lambda_i, \mu_i)$, i.e. we write the transformation as [1],

$$A(\lambda_i, \tilde{\lambda}_i) \rightarrow A(\lambda_i, \mu_i) \equiv \int \prod_{i=1}^{n} d\tilde{\lambda}_i \exp(i\mu_i \tilde{\lambda}_i) A(\lambda_i, \tilde{\lambda}_i). \tag{5.4}$$

Having done the Fourier transform, points ρ_i in twistor space now label each external particle in the scattering process. We now write the scattering amplitude as a function of the ρ_i. Witten recognised that the scattering amplitudes will only be non-zero if the points have geometric support in twistor space. In particular, he suggested that they were supported on an algebraic curve in twistor space where the curve has degree d, given by [1],

$$d = q - 1 + l, \tag{5.5}$$

where l is the number of loops involved in the process and q is the number of negative helicity gluons.

Following the method of [29], we can consider such a transformation for the most basic tree amplitudes we know, the Parke Taylor MHV amplitudes.
As the major term in eq. (5.1) contains only angle brackets, and thus depends only on the right-handed spinors λ_i, the only dependence on left-handed spinors $\tilde{\lambda}_i$ comes from the delta function implicit in eq. (5.1) that represents the conservation of momentum. Explicitly we can write this δ function as [29],

$$\delta^4(k) = \int d^4x \, \exp[ik \cdot x] \tag{5.6}$$

We can write a massless momentum vector as the product of a right-handed and left-handed spinor [29] using the positive energy projector $u(k)\bar{u}(k) = \not{k}$, i.e.

$$(\not{k}_i)_{\alpha\dot{\alpha}} = (\lambda_i)_\alpha (\tilde{\lambda}_i)_{\dot{\alpha}}. \tag{5.7}$$

Therefore, the momentum conserving delta function can be rewritten as [29],

$$\delta^4\Big(\sum_{i=1}^{n} k_i\Big) = \int d^4x \exp\Big[ix^{\alpha\dot{\alpha}} \sum_{i=1}^{n} (\lambda_i)_\alpha (\tilde{\lambda}_i)_{\dot{\alpha}}\Big]. \tag{5.8}$$

With this technique the MHV amplitudes, when transformed into twistor space, can be written as [29],

$$\begin{aligned} A_n^{\mathrm{MHV}}(\lambda_i, \mu_i) &= \int \prod_{i=1}^{n} d\tilde{\lambda}_i \exp[i\mu_i\tilde{\lambda}_i] \int d^4x \; A_n^{\mathrm{MHV}}(\lambda_i) \exp[ix\lambda_i\tilde{\lambda}_i] \\ &= \int d^4x \; A_n^{\mathrm{MHV}}(\lambda_i) \int \prod_{i=1}^{n} d\tilde{\lambda}_i \exp[i(\mu_i + x\lambda_i)\tilde{\lambda}_i] \\ &= A_n^{\mathrm{MHV}}(\lambda_i) \int d^4x \prod_{i=1}^{n} \delta(\mu_i + x\lambda_i). \end{aligned} \tag{5.9}$$

Witten's interpretation of this result was that the amplitude must be supported on a line in twistor space as a result of the constraints imposed by the product of δ functions, as shown in part (a) of the figure below.

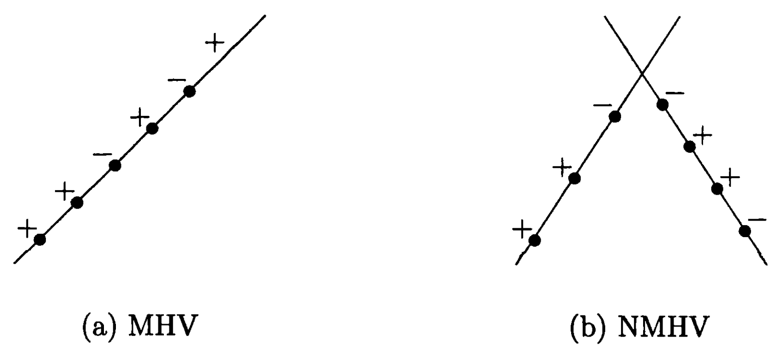


Figure 5.1.1: The twistor space structure of Tree Amplitudes

Calculating more complicated amplitudes from these twistor transforms is particularly difficult. Now we are aware of the existence of this structure in twistor space, we can propose an alternative approach. Let us allow a particular amplitude to have support in twistor space on a curve described by the polynomial equation

$P_i \equiv P(\lambda_i, \lambda_2, \mu^{\dot{1}}, \mu^{\dot{2}}) = 0$. As $\mu_i = i\partial/\partial\tilde{\lambda}_i$, transforming back into spinor space turns P_i into a differential operator.

Therefore if, when we apply the differential operator P_i to our original amplitude in spinor space, the result vanishes, then we have shown that the amplitude must have been supported on the curve in twistor space described by P_i.

Specifically we apply two particular differential operators to amplitudes in spinor space to investigate their structure in twistor space. If an amplitude is annihilated by the operator,

$$[F_{ijk}, \eta] = \langle i\, j\rangle \left[\frac{\partial}{\partial\tilde{\lambda}_k}, \eta\right] + \langle j\, k\rangle \left[\frac{\partial}{\partial\tilde{\lambda}_i}, \eta\right] + \langle k\, i\rangle \left[\frac{\partial}{\partial\tilde{\lambda}_j}, \eta\right], \tag{5.10}$$

where η is some arbitrary spinor and where the square brackets indicate spinor products rather than commutators, then the points i, j, and k are said to be collinear, i.e. they lie on a line in twistor space.

Similarly, annihilation by the operator,

$$\begin{aligned} K_{ijkl} \;=\; & \frac{1}{4}\Big[\langle ij\rangle\epsilon^{\dot{a}\dot{b}}\frac{\partial}{\partial\tilde{\lambda}_k^{\dot{a}}}\frac{\partial}{\partial\tilde{\lambda}_l^{\dot{b}}} - \langle ik\rangle\epsilon^{\dot{a}\dot{b}}\frac{\partial}{\partial\tilde{\lambda}_j^{\dot{a}}}\frac{\partial}{\partial\tilde{\lambda}_l^{\dot{b}}} + \langle il\rangle\epsilon^{\dot{a}\dot{b}}\frac{\partial}{\partial\tilde{\lambda}_j^{\dot{a}}}\frac{\partial}{\partial\tilde{\lambda}_k^{\dot{b}}} \\ & +\langle jk\rangle\epsilon^{\dot{a}\dot{b}}\frac{\partial}{\partial\tilde{\lambda}_i^{\dot{a}}}\frac{\partial}{\partial\tilde{\lambda}_l^{\dot{b}}} + \langle jl\rangle\epsilon^{\dot{a}\dot{b}}\frac{\partial}{\partial\tilde{\lambda}_k^{\dot{a}}}\frac{\partial}{\partial\tilde{\lambda}_i^{\dot{b}}} - \langle kl\rangle\epsilon^{\dot{a}\dot{b}}\frac{\partial}{\partial\tilde{\lambda}_j^{\dot{a}}}\frac{\partial}{\partial\tilde{\lambda}_i^{\dot{b}}}\Big] \end{aligned} \tag{5.11}$$

indicates co-planarity of the points i, j, k and l, i.e. they define a plane in twistor space.

Cachazo, Svrcek and Witten [1, 7, 13] used these operators to develop an understanding of the twistor space structure of tree level amplitudes. They found that n gluon scattering amplitudes, where s gluons are of negative helicity, were supported in twistor space on a series of lines. They were able to extend this approach beyond the MHV case we have already discussed. For amplitudes with three negative helicity gluons, i.e. $s = 3$, which we call next to MHV (NMHV), they found a sum of terms.

Each term is supported on a pair of intersecting lines, where the distribution of points can vary between the two lines for each separate term. This structure is shown in (b) of Figure 5.1.1.

5.1 CSW Construction

The twistor structure proposed by Witten offers a revealing insight into the fundamental structure of tree amplitudes. However, it cannot be used to calculate numerical values for these amplitudes. Further study and consideration did, however, prompt Cachazo, Svrcek and Witten [7] to propose a set of rules that could fulfil such a purpose. The CSW rules describe how to calculate an amplitude using a series of CSW diagrams, where the entire amplitude can be constructed by calculating the sum of all CSW diagrams allowed for a particular process. Such an approach is clearly similar to that of Feynman diagrams and Feynman rules, but ultimately proves to be much more efficient.

Diagrammatically, we must denote the helicity of all gluons, internal as well as external, in a CSW diagram as + or −. Each vertex in a CSW diagram must be attached to two negative helicity gluons. An arbitrary number of positive helicity gluons may be attached to the vertex but there must be exactly two negative helicity gluons. These vertices are given analytically by the MHV tree amplitudes eq. (1.2), with an additional condition that we must take intermediate legs as off shell. To define this off shell continuation, we recall that we can write a momentum vector p_μ in bispinor notation $p_{a\dot{a}}$. As discussed previously, this can be factorised into the right-handed and left-handed spinors λ_a and $\tilde{\lambda}_{\dot{a}}$, i.e.

$$p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}} \tag{5.12}$$

Each particle in a physical amplitude, like the MHV tree amplitudes used for the vertices, is considered on shell, and thus has a light-like momentum vector which we can rewrite as in eq. (5.12).

To make the off shell continuation we take an arbitrary left-handed spinor $\eta^{\dot{a}}$. For the internal lines, which are off shell, a tree amplitude will remain invariant under rescaling of the λ's. Thus we can write the right-handed spinor of the momentum being taken off shell as $\lambda_a = p_{a\dot{a}}\eta^{\dot{a}}$. Using this approach we can define the off shell continuation we need to make to use the MHV tree amplitudes as vertices. We take an arbitrary left-handed spinor $\eta^{\dot{a}}$. For an internal line of momentum $p_{a\dot{a}}$ we then define the right-handed spinor as,

$$\lambda_a = p_{a\dot{a}}\eta^{\dot{a}}. \tag{5.13}$$

We rewrite all of the internal, off shell, lines in the same way using the same arbitrary spinor η to define λ_a. With this definition of λ we can now take the MHV tree amplitudes as the vertices in the CSW diagrams. Since we are using the MHV tree amplitudes as our building blocks, this approach works provided the number of gluons at each vertex (internal and external) is three or more. Conventionally we take each gluon, for both internal and external lines, at a vertex to be incoming and assign helicity as such. The helicity for outgoing gluons is simply reversed We use a factor of $1/p^2$ to describe the propagation of a gluon of momentum p along an internal line. Since we have taken the convention that incoming and outgoing gluons must have opposite helicity, we must have positive helicity at one end of the propagator and negative helicity at the other.

With this approach, we are able to generate tree amplitudes beyond MHV level using just the simple form for the MHV tree amplitudes, given in eq. (1.2). For example, a NNMHV amplitude can be considered as three MHV vertices sewn together as we have described. This is shown in Figure 5.2.1.

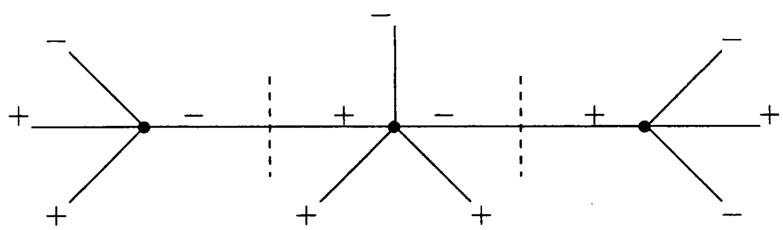


Figure 5.2.1: An Example of a NNMHV diagram, constructed using a MHV vertex approach.

Since we are using MHV tree amplitudes as vertices, the twistor structure discussed is manifest in this approach, with each MHV vertex in the diagram generating a particular twistor structure.

Although this approach is apparently similar to that of Feynman diagrams, it has proved remarkably more efficient. In particular, many Feynman diagrams are characterised by the same CSW diagram, so there are considerably fewer CSW diagrams to consider. Likewise, the algebra that must be completed to calculate a CSW diagram is simpler than that required for a standard Feynman diagram. Thus CSW construction is a very efficient way to calculate amplitudes, and has been applied to a number of processes, see [53, 54, 55].

5.1.1 A Simple Example

To illustrate the simplicity of the algebra required to compute a CSW diagram, let us consider a simple example [7]. We study the four-point amplitude $(+\ -\ -\ -)$. As we have already stated, Parke and Taylor showed that this amplitude vanishes in Yang-Mills theory. Thus, if we use the CSW approach to calculate this amplitude we should also find that the full amplitude equals zero. There are two diagrams we must consider, as shown in figure 5.2.2.

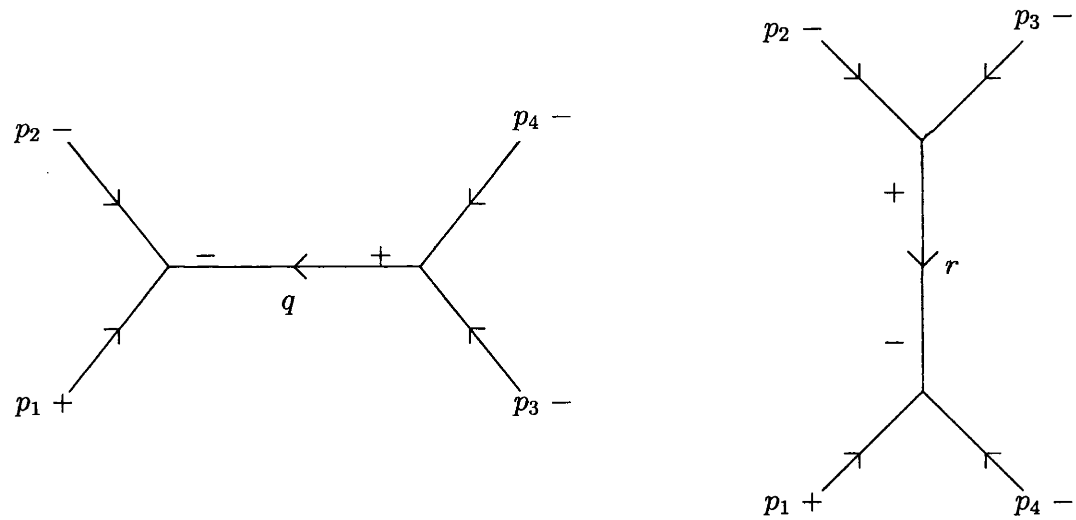


Figure 5.2.2: CSW diagrams that contribute to a $(+\ -\ -\ -)$ amplitude.

Let us begin by considering the first diagram. The momentum of the internal line is given by $q = -p_1 - p_2 = p_3 + p_4$. Our notation is to write the right-handed and left-handed spinors corresponding to this momentum as λ_q and $\tilde{\lambda}_q$.

Following the CSW approach just outlined, we introduce an arbitrary negative spinor $\eta^{\dot{a}}$ and write the right-handed spinor of the internal momentum as $\lambda_{qa} = q_{a\dot{a}}\eta^{\dot{a}}$. Using the abbreviation that $\phi_i = \tilde{\lambda}_{i\dot{a}}\eta^{\dot{a}}$ [7], we can rewrite λ_{qa} as,

$$\lambda_{qa} = -\lambda_{1a}\phi_1 - \lambda_{2a}\phi_2 = \lambda_{3a}\phi_3 + \lambda_{4a}\phi_4, \tag{5.14}$$

Writing the propagator as $1/q^2$ and using eq. (1.2) for the vertices, we can write the amplitude corresponding to this first diagram as,

$$\frac{\langle\lambda_2,\lambda_q\rangle^3}{\langle\lambda_q,\lambda_1\rangle\langle\lambda_1,\lambda_2\rangle}\frac{1}{q^2}\frac{\langle\lambda_3,\lambda_4\rangle^3}{\langle\lambda_4,\lambda_q\rangle\langle\lambda_q,\lambda_3\rangle}. \tag{5.15}$$

From eq. (5.14), we have the identities,

$$\begin{aligned}\langle\lambda_2,\lambda_q\rangle &= -\langle 2\,1\rangle\,\phi_1,\\ \langle\lambda_q,\lambda_1\rangle &= -\langle 2\,1\rangle\,\phi_2,\\ \langle\lambda_4,\lambda_q\rangle &= \langle 4\,3\rangle\,\phi_3,\\ \langle\lambda_q,\lambda_3\rangle &= \langle 4\,3\rangle\,\phi_4.\end{aligned} \tag{5.16}$$

Substituting these into eq. (5.15), we get that,

$$\frac{\phi_1^3}{\phi_2\phi_3\phi_4}\frac{\langle 2\,1\rangle^3}{\langle 2\,1\rangle\,\langle 1\,2\rangle}\frac{1}{q^2}\frac{\langle 3\,4\rangle^3}{\langle 4\,3\rangle\,\langle 4\,3\rangle}. \tag{5.17}$$

For the intermediate internal momentum q we know that $q^2 = (p_1+p_2)^2$. Since p_1 and p_2 are on shell momenta, $p_1^2 = p_2^2 = 0$, and thus $(p_1+p_2)^2 = 2p_1\cdot p_2$. Using the notation set out in chapter 3, this can be rewritten in terms of spinor variables as $2p_1\cdot p_2 = \langle 1\,2\rangle\,[1\,2]$. Therefore, eq. (5.17) can be rewritten as,

$$-\frac{\phi_1^3}{\phi_2\phi_3\phi_4}\frac{\langle 3\,4\rangle}{[2\,1]}. \tag{5.18}$$

where we have used the fact that $\langle i\,j\rangle = -\langle j\,i\rangle$ as discussed in chapter 3. The second diagram in Figure 5.2.2 similarly gives,

$$-\frac{\phi_1^3}{\phi_2\phi_3\phi_4}\frac{\langle 3\,2\rangle}{[4\,1]}. \tag{5.19}$$

This can be seen by simply swapping particles 2 and 4 in the first evaluation. From momentum conservation we have that,

$$\sum_{i=1}^{4}\langle 3\,i\rangle\,[i\,1] = 0. \tag{5.20}$$

Since $\langle i\,i\rangle = 0$ and $[i\,i] = 0$, we can write this as,

$$\sum_{i=1}^{4}\langle 3\,i\rangle\,[i\,1] = \langle 3\,2\rangle\,[2\,1] + \langle 3\,4\rangle\,[4\,1] = 0. \tag{5.21}$$

Thus the sum of the two terms eq. (5.18) and eq. (5.19) vanishes, and the CSW construction for this process is in agreement with the expected result.

5.2 Generalised Unitarity

As we discussed in chapter 4, an amplitude can be expanded into a sum of known scalar integral functions multiplied by unknown rational coefficients [8]. For $N = 4$ Super Yang-Mills theory, only scalar box integral functions can appear in this summation. Since the scalar box integrals are known, expanding the theory into this sum of integral functions reduces the problem of calculating the entire amplitude to simply finding the rational coefficients that appear in the summation.

We have also discussed how Supersymmetric one-loop gluonic amplitudes can be entirely determined from a knowledge of their unitarity cuts, i.e. they are said to be cut constructible [8, 9]. However, this technique suffers from the complication that more than one, and often several, scalar box integrals can have the same unitarity cut. Thus when we undertake the analysis, we are left trying to identify many unknown coefficients mixed together.

Britto, Cachazo and Feng [56] recently proposed a different way of computing the scalar box integral coefficients, using an enhanced version of the unitarity cut method discussed in chapter 4. They observed that box integral coefficients can be obtained from generalised unitarity cuts by analytically continuing the massless corners of quadruple cuts to signature $(- \; - \; + \; +)$, see [57, 58, 51] for detailed discussion of generalised unitarity cuts.

We can review the principle behind their work here, see [56] for a complete description. The problem is that many box integral functions can share the same unitarity cut. However, Britto, Cachazo and Feng noted that the leading singularity [57] of any scalar box integral function is unique. From this observation they realised that analysing the discontinuity associated with the leading singularity would isolate a particular integral function and allow the coefficient to be calculated uniquely. To obtain the discontinuity of the leading singularity in a general Feynman graph, we must cut all of the propagators. For a scalar box integral function there are four propagtors, and thus we must perform a quadruple cut.

Of the distinct classes of box coefficients we discussed in chapter 4, Britto, Cachazo and Feng found that they could apply a quadruple cut quite simply to the $N = 4$ four mass box to determine it's coefficient. However, in Minkowski signature they could not apply this technique to any of the other classes of box integral function. The reason for this was simple. The distinctive feature of a four mass box coefficient compared to all other box coefficients is that it contains no massless legs. For the other box integrals there is at least one massless vertex. Thus when we make the unitarity cut and rewrite this vertex as a tree amplitude we are left with a three gluon amplitude. In Minkowski space three gluon amplitudes do not exist. Recall that with the bispinor notation $p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$, the inner product can be written $2p \cdot q = \langle \lambda_p \, \lambda_q \rangle \left[\tilde{\lambda}_p \, \tilde{\lambda}_q \right]$, as we have discussed already. At tree level we can write a three gluon amplitude with helicity $(+ \; + \; -)$ as [56],

$$A_3^{\text{tree}}(p^+, q^+, r^-) = \frac{\left[\tilde{\lambda}_p \, \tilde{\lambda}_q \right]^3}{\left[\tilde{\lambda}_r \, \tilde{\lambda}_p \right] \left[\tilde{\lambda}_q \, \tilde{\lambda}_r \right]}, \tag{5.22}$$

Likewise, we can write the three gluon amplitude with helicity $(- \; - \; +)$ as [56],

$$A_3^{\text{tree}}(p^-, q^-, r^+) = \frac{\langle \lambda_p \, \lambda_q \rangle^3}{\langle \lambda_r \, \lambda_p \rangle \langle \lambda_q \, \lambda_r \rangle}. \tag{5.23}$$

We know that λ_p and $\tilde{\lambda}_p$ for real momenta in Minkowski space are not independent, but are related by $\tilde{\lambda}_p = \pm \bar{\lambda}_p$. Thus, since $2p \cdot q = \langle \lambda_p \, \lambda_q \rangle \left[\tilde{\lambda}_p \, \tilde{\lambda}_q \right]$, both $\langle \lambda_p \, \lambda_q \rangle$ and $\left[\tilde{\lambda}_p \, \tilde{\lambda}_q \right]$ must be zero for the scalar product $2p \cdot q = 0$. Therefore in Minkowski space both eq. (5.22) and eq. (5.23) cannot exist, since $\langle \lambda_p \, \lambda_q \rangle = \left[\tilde{\lambda}_p \, \tilde{\lambda}_q \right] = 0$ explicitly.

Therefore, if we work with Minkowski signature $(- \; + \; + \; +)$, then we must use more general cuts, such as triple cuts (cut three propagators), to evaluate the box coefficients with massless vertices. Once again we are stuck with the disadvantage that these more general cuts do not uniquely isolate any one coefficient. Although we have simplified the problem, we have not achieved our goal.

Britto, Cachazo and Feng realised that this could be resolved if we work in signature $(--++)$, that is to say that the three gluon vertices causing the problem do not vanish in this signature. If we reconsider our previous analysis, we note that for real momenta in signature $(--++)$, λ_p and $\tilde{\lambda}_p$ are now independent. There must now be two solutions that satisfy vanishing inner product, $2p\cdot q = \langle \lambda_p\, \lambda_q\rangle \left[\tilde{\lambda}_p\, \tilde{\lambda}_q\right] = 0$. Either $\left[\tilde{\lambda}_p\, \tilde{\lambda}_q\right] = 0$ or $\langle \lambda_p\, \lambda_q\rangle = 0$.

Conservation of momentum requires that $p\cdot q = p\cdot r = q\cdot r = 0$. Thus if $\left[\tilde{\lambda}_p\, \tilde{\lambda}_q\right] = 0$ then we can say that $\left[\tilde{\lambda}_p\, \tilde{\lambda}_r\right] = 0$ and $\left[\tilde{\lambda}_r\, \tilde{\lambda}_q\right] = 0$ must also be true and all three $\tilde{\lambda}$'s are related to each other. Likewise, if $\langle \lambda_p\, \lambda_q\rangle = 0$, then so do $\langle \lambda_p\, \lambda_r\rangle$ and $\langle \lambda_r\, \lambda_q\rangle$ and the λ's are all related as well.

Thus it is clear that the three gluon tree amplitudes with helicity $(+\ +\ -)$ appearing in eq. (5.22) will not vanish in this signature if we take the λ's to be related to each other, and similarly the three gluon tree amplitudes with helicity $(-\ -\ +)$ appearing in eq. (5.23) will also not vanish if we take the $\tilde{\lambda}$'s to be related to each other

With this continuation to signature $(-\ -\ +\ +)$ Britto, Cachazo and Feng were able to avoid the problem of vanishing three gluon vertices. Furthermore, they showed that the coefficients calculated in signature $(-\ -\ +\ +)$ could still be used in Minkowski space even though they were determined in a different signature, since the final term depends only on the spinor invariants λ and $\tilde{\lambda}$.

Their technique gives scalar box integral coefficients as a product of four tree amplitudes, i.e. specifically, for the scalar box integral function shown in Figure 5.3.1,

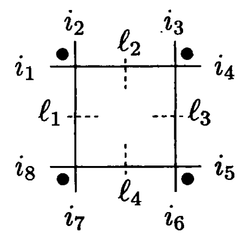


Figure 5.3.1: A Quadruple Cut on a one-loop scalar box integral.

the coefficient is given by the product of four tree amplitudes where the cut legs satisfy on-shell conditions,

$$\hat{c} = \frac{1}{2}\sum_{S}\Big(A^{\text{tree}}(\ell_1, i_1, ..., i_2, \ell_2) \times A^{\text{tree}}(\ell_2, i_3, ..., i_4, \ell_3) \\ \times A^{\text{tree}}(\ell_3, i_5, ..., i_6, \ell_4) \times A^{\text{tree}}(\ell_4, i_7, ..., i_8, \ell_1)\Big), \tag{5.24}$$

where S indicates the set of helicity configurations and particle types of the legs ℓ_j giving a non-vanishing product of tree amplitudes. The analytic continuation allows this to be evaluated even when one or more of the tree amplitudes in eq. (5.24) is a three-point amplitude which would vanish in Minkowski signature. The sum is over all allowed intermediate configurations and particle types [56].

5.2.1 A Simple Example

Let us illustrate a simple application of this technique. We perform a quadruple cut on the box coefficient shown in Figure 5.3.2.

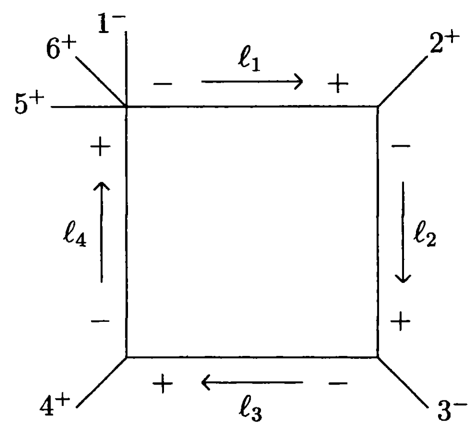


Figure 5.3.2: Example box coefficient.

We consider a scalar particle circulating in the loop. The quadruple cut for other internal particles circulating in the loop can be found by multiplying by the appropriate ρ factor, as we discussed in chapter 4. We denote the momentum of the i^{th} external gluonic particle as k_i, and the internal scalar propagators as ℓ_i. Following Britto, Cachazo and Feng, we can write the Quadruple cut, Q, for this particular box as the product of four tree amplitudes, i.e.

$$Q = A^{\text{tree}}(\ell_4^+, 5^+, 6^+, 1^-, \ell_1^-) \times A^{\text{tree}}(\ell_1^+, 2^+, \ell_2^-) \times A^{\text{tree}}(\ell_2^+, 3^-, \ell_3^-) \times A^{\text{tree}}(\ell_3^+, 4^+, \ell_4^-) \tag{5.25}$$

Substituting Parke Taylor amplitudes for the tree amplitudes at each corner we find that,

$$
\begin{aligned}
Q &= \frac{\langle 1\,\ell_1\rangle^2\,\langle 1\,\ell_4\rangle^2}{\langle \ell_4\,5\rangle\,\langle 5\,6\rangle\,\langle 6\,1\rangle\,\langle 1\,\ell_1\rangle\,\langle \ell_1\,\ell_4\rangle} \times \frac{[2\,\ell_1]^2\,[2\,\ell_2]^2}{[\ell_1\,2]\,[2\,\ell_2]\,[\ell_2\,\ell_1]} \\
&\quad \times \frac{\langle 3\,\ell_2\rangle^2\,\langle 3\,\ell_3\rangle^2}{\langle \ell_2\,3\rangle\,\langle 3\,\ell_3\rangle\,\langle \ell_3\,\ell_2\rangle} \times \frac{[4\,\ell_3]^2\,[4\,\ell_4]^2}{[\ell_3\,4]\,[4\,\ell_4]\,[\ell_4\,\ell_3]} \\
&= -\frac{\langle 1\,\ell_1\rangle\,\langle 1\,\ell_4\rangle^2\,[2\,\ell_1]\,[2\,\ell_2]\,\langle 3\,\ell_2\rangle\,\langle 3\,\ell_3\rangle\,[4\,\ell_3]\,[4\,\ell_4]}{\langle \ell_4\,5\rangle\,\langle 5\,6\rangle\,\langle 6\,1\rangle\,\langle \ell_1\,\ell_4\rangle\,[\ell_2\,\ell_1]\,\langle \ell_3\,\ell_2\rangle\,[\ell_4\,\ell_3]}.
\end{aligned}
\tag{5.26}
$$

Momentum conservation at each corner requires that,

$$
\begin{aligned}
\ell_2 &= \ell_1 - k_2, \\
\ell_3 &= \ell_2 - k_3, \\
\ell_4 &= \ell_3 - k_4, \\
\ell_1 &= \ell_4 - P_{561}.
\end{aligned}
\tag{5.27}
$$

Total momentum conservation also implies that,

$$
k_2 + k_3 + k_4 + p_{561} = 0.
\tag{5.28}
$$

Since the k's and ℓ's are null, the massless corners immediately give the constraints,

$$
\begin{aligned}
\ell_1 \cdot k_2 = {}& \ell_2 \cdot k_2 = {}& \ell_1 \cdot \ell_2 = 0, \\
\ell_2 \cdot k_3 = {}& \ell_3 \cdot k_3 = {}& \ell_2 \cdot \ell_3 = 0, \\
\ell_3 \cdot k_4 = {}& \ell_4 \cdot k_4 = {}& \ell_3 \cdot \ell_4 = 0.
\end{aligned}
\tag{5.29}
$$

As we discussed previously, we know that $2p \cdot q = \langle \lambda_p \, \lambda_q \rangle \left[\tilde{\lambda}_p \, \tilde{\lambda}_q \right] = 0$. Working in signature $(+ \; + \; - \; -)$, we can set the λ_i and $\tilde{\lambda}_i$ independently, so we need only set one bracket to zero for the vanishing spinor inner product to hold.

For the three-point corner $(\ell_1^+, 2^+, \ell_2^-)$ we set,

$$\langle \lambda_2 \, \lambda_{\ell_1} \rangle = \langle \lambda_2 \, \lambda_{\ell_2} \rangle = \langle \lambda_{\ell_1} \, \lambda_{\ell_2} \rangle = 0, \tag{5.30}$$

which is implemented by,

$$\lambda_{\ell_1} = \alpha \lambda_2 \qquad\qquad \lambda_{\ell_2} = \beta \lambda_2. \tag{5.31}$$

where α and β are arbitrary complex parameters.

For the three-point corner $(\ell_2^+, 3^-, \ell_3^-)$ we set,

$$\left[\tilde{\lambda}_{\ell_2} \, \tilde{\lambda}_3 \right] = \left[\tilde{\lambda}_{\ell_3} \, \tilde{\lambda}_3 \right] = \left[\tilde{\lambda}_{\ell_3} \, \tilde{\lambda}_{\ell_2} \right] = 0, \tag{5.32}$$

which is implemented by,

$$\tilde{\lambda}_{\ell_2} = \gamma \tilde{\lambda}_3 \qquad\qquad \tilde{\lambda}_{\ell_3} = \delta \tilde{\lambda}_3. \tag{5.33}$$

where γ and δ are arbitrary complex parameters.

For the three-point corner $(\ell_3^+, 4^+, \ell_4^-)$ we set,

$$\langle \lambda_4 \, \lambda_{\ell_3} \rangle = \langle \lambda_4 \, \lambda_{\ell_4} \rangle = \langle \lambda_{\ell_3} \, \lambda_{\ell_4} \rangle = 0, \tag{5.34}$$

which is implemented by,

$$\lambda_{\ell_3} = \mu\lambda_4 \qquad\qquad \lambda_{\ell_4} = \nu\lambda_2. \tag{5.35}$$

where μ and ν are arbitrary complex parameters.

Furthermore, we can find the internal propagator spinors we do not know in terms of the others by rewriting momentum constraints in bispinor notation. We can write $\ell_4 = \ell_3 - k_4$ as,

$$\lambda_{\ell_4}\tilde{\lambda}_{\ell_4} = \lambda_{\ell_3}\tilde{\lambda}_{\ell_3} - \lambda_4\tilde{\lambda}_4. \tag{5.36}$$

Substituting for spinors λ_{ℓ_4}, λ_{ℓ_3} and $\tilde{\lambda}_{\ell_3}$ we get,

$$\begin{aligned} \nu\lambda_4\tilde{\lambda}_{\ell_4} &= \mu\delta\lambda_4\tilde{\lambda}_3 - \lambda_4\tilde{\lambda}_4 \\ \rightarrow \tilde{\lambda}_{\ell_4} &= \frac{\mu\delta\tilde{\lambda}_3 - \tilde{\lambda}_4}{\nu}. \end{aligned} \tag{5.37}$$

Similarly, we can write $\ell_1 = \ell_2 + k_2$ as,

$$\lambda_{\ell_1}\tilde{\lambda}_{\ell_1} = \lambda_{\ell_2}\tilde{\lambda}_{\ell_2} + \lambda_2\tilde{\lambda}_2. \tag{5.38}$$

Substituting for spinors λ_{ℓ_1}, λ_{ℓ_2} and $\tilde{\lambda}_{\ell_2}$ we get,

$$\alpha\lambda_2\tilde{\lambda}_{\ell_1} = \beta\gamma\lambda_2\tilde{\lambda}_3 + \lambda_2\tilde{\lambda}_2$$

$$\rightarrow \tilde{\lambda}_{\ell_1} = \frac{\beta\gamma\tilde{\lambda}_3 + \tilde{\lambda}_2}{\alpha}. \tag{5.39}$$

We can find the form of the arbitrary spinor parameter products $\beta\gamma$ and $\mu\delta$ from momentum conservation constraints as follows. Since the ℓ's and k's are null, from $\ell_1 = \ell_3 + P_{23}$ we know that $2\ell_3 \cdot P_{23} = -P_{23}^2$. We can rewrite this as,

$$\langle \ell_3\, 2\rangle\, [\ell_3\, 2] + \langle \ell_3\, 3\rangle\, [\ell_3\, 3] = -P_{23}^2. \tag{5.40}$$

Substituting for the spinors we know, this becomes,

$$\mu\, \langle 4\, 2\rangle\, \delta\, [3\, 2] = -P_{23}^2 \tag{5.41}$$

where we have used the fact that $[j\, j] = 0$ to remove the second term. Expanding the momentum P_{23}^2 as discussed in chapter 3 we find that,

$$\begin{aligned} \mu\delta\, \langle 4\, 2\rangle\, [3\, 2] &= -\langle 2\, 3\rangle\, [2\, 3] \\ \rightarrow \mu\delta &= \frac{\langle 2\, 3\rangle}{\langle 4\, 2\rangle}. \end{aligned} \tag{5.42}$$

Similarly, from $\ell_4 = \ell_1 + P_{561} = \ell_1 - P_{234}$ we know that $2\ell_1 \cdot P_{234} = P_{234}^2$. We can rewrite this as,

$$\langle \ell_1\, 2\rangle\, [\ell_1\, 2] + \langle \ell_1\, 3\rangle\, [\ell_1\, 3] + \langle \ell_1\, 4\rangle\, [\ell_1\, 4] = P_{234}^2. \tag{5.43}$$

Substituting for the spinors we know, this becomes,

$$\alpha \langle 2\,3\rangle \frac{1}{\alpha}[2\,3] + \alpha \langle 2\,4\rangle \left(\frac{\beta\gamma}{\alpha}[3\,4] + \frac{1}{\alpha}[2\,4]\right) = P_{234}^2 \tag{5.44}$$

where we have used the fact that $\langle i\,i\rangle = 0$ to remove the first term. Expanding the momentum P_{234}^2 we find that,

$$\begin{aligned} \langle 2\,3\rangle [2\,3] + \beta\gamma \langle 2\,4\rangle [3\,4] + \langle 2\,4\rangle [2\,4] &= \langle 2\,3\rangle [2\,3] + \langle 3\,4\rangle [3\,4] + \langle 2\,4\rangle [2\,4] \\ \rightarrow \beta\gamma &= \frac{\langle 3\,4\rangle}{\langle 2\,4\rangle}. \end{aligned} \tag{5.45}$$

Therefore, the quadruple cut eq. (5.26) becomes,

$$\begin{aligned} Q &= \frac{\alpha^2\nu^3\beta^2\gamma^2\mu^2\delta^2 \langle 1\,2\rangle \langle 1\,4\rangle^2 [2\,3] [2\,3] \langle 3\,2\rangle \langle 3\,4\rangle [4\,3] [4\,3]}{\nu^2\alpha\gamma\beta\mu\delta \langle 4\,5\rangle \langle 5\,6\rangle \langle 6\,1\rangle \langle 2\,4\rangle [3\,2] \langle 4\,2\rangle [4\,3]} \\ &= \beta\gamma\mu\delta \frac{\langle 1\,2\rangle \langle 1\,4\rangle^2 [2\,3] \langle 3\,2\rangle \langle 3\,4\rangle [4\,3]}{\langle 4\,5\rangle \langle 5\,6\rangle \langle 6\,1\rangle \langle 2\,4\rangle^2} \\ &= \frac{\langle 1\,2\rangle \langle 1\,4\rangle^2 \langle 2\,3\rangle^2 \langle 3\,4\rangle^2 [2\,3] [4\,3]}{\langle 4\,5\rangle \langle 5\,6\rangle \langle 6\,1\rangle \langle 2\,4\rangle^4}. \end{aligned} \tag{5.46}$$

To complete the calculation we multiply this term by the appropriate Supersymmetric ρ factor. We must also sum over the different internal helicity configurations.

5.3 The Twistor Structure of One-loop Amplitudes and the "Holomorphic Anomaly" of Unitarity Cuts.

As the techniques of CSW construction began to be applied to different loop level processes it became possible to develop an understanding of the twistor structure of one-loop amplitudes [13]. This is a much more complicated process as loop amplitudes and generically more complicated than amplitudes at tree level. A further complication was introduced with the discovery of a "holomorphic anomaly" [59].

Brandhuber, Spence and Travaglini [12, 60] explicitly showed that the CSW approach of using MHV vertices could be used to calculate the one-loop $N = 4$ MHV amplitudes. This was in contrast to the observation of Cachazo, Svercek and Witten that application of the collinear operator F to one-loop MHV amplitudes did not result in their annihilation, even though naively it should have. More specifically, let us consider the operator $F_{i,i+1,i+2}$ acting on the cut $C_{i,\dots,j}$ of a one-loop MHV amplitude. We can write this cut as,

$$C_{i,\dots,j} \equiv \frac{i}{2}\int d\text{LIPS}\Big[A^{\text{MHV-tree}}(\ell_1, i, i+1, \dots, j, \ell_2) \\ \times A^{\text{MHV-tree}}(-\ell_2, j+1, j+2, \dots, i-1, -\ell_1)\Big]. \tag{5.47}$$

Since $F_{i\,i+1\,i+2}$ annihilates both tree amplitudes on either side of the cut, we might naively expect it to annihilate the whole expression for the cut. However Brandhuber, Spence and Travaglini showed it explicitly does not.

This apparent paradox was resolved by Witten [61] by observing that when a differential operator acts within the loop-momentum integral it yields a δ function. This "holomorphic anomaly" of the unitarity cut produces a rational function as a result even though the tree amplitudes within the cut are localised on lines in twistor space [61, 62, 50]. As a spin-off of this resolution, it was observed that acting with

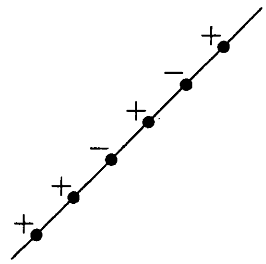
F_{ijk} upon both the cut and the imaginary part of the amplitude, and demanding consistency via the optical theorem, leads to algebraic equations for the coefficients of the integral functions which appear in the amplitude. These algebraic equations can be used to compute an entire amplitude [50].

Due to the added complications imposed by the "holomorphic anomaly", it has proved easier to consider a simpler approach when examining the twistor structure of one-loop amplitudes. As discussed in chapter 4, we can expand one-loop amplitudes into a series of scalar integral functions; boxes, triangles and bubbles (plus rational pieces for non Supersymmetric theories). We can then investigate the twistor structure of the coefficients of these integral functions to build up an idea of the geometric description of one-loop amplitudes in twistor space.

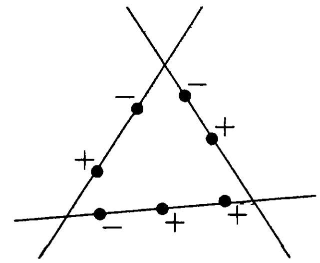
As there are no triangle or bubble functions present in $N = 4$ Super Yang-Mills theory, this is the simplest theory to begin examining. We need only consider the coefficients of the box integral functions [8], which can be determined using the methods discussed previously, see [59, 51, 56, 63, 64, 8, 65]. The twistor structure of one-loop amplitudes in $N = 4$ Super Yang-Mills has been extensively investigated [51, 63, 66].

For MHV amplitudes only the two mass scalar box integrals appear [29]. The coefficient for this class of integral function is simply an MHV tree amplitude, as described in eq. (1.2). Therefore, the coefficient of this integral function has collinear support in twistor space, that is all of the points lie on a line in twistor space, as shown in (a) of Figure 5.3.1.

For NMHV one-loop amplitudes, the picture is more complicated as the simplest box coefficients that exist are from the class known as three mass boxes [29]. The twistor structure for these coefficients involves three lines that intersect, and thus describe a plane in twistor space, as shown in (b) of Figure 5.3.1. These coefficients are therefore said to have coplanar support in twistor space.



(a) MHV



(b) NMHV

Figure 5.3.1: The twistor space structure of the Box Coefficients of $N = 4$ One-Loop Amplitudes.

The picture described for the twistor space structure of one-loop amplitudes in $N = 4$ Super Yang-Mills theory would appear to mimic that of tree level, i.e. the coefficients of box integral functions can be represented on a series of lines in twistor space.

Part of the challenge undertaken in this research was to develop the twistor space picture for theories with $N < 4$ Supersymmetries, where we must consider the twistor space structure of the coefficients of the triangle and bubble integral functions that appear, in addition to the box integral functions, in such theories.

Chapter 6

$N < 4$ One-Loop Gluonic Amplitudes

In this chapter we focus on extending techniques that have been successfully used with $N = 4$ purely gluonic one-loop amplitudes to theories with $N < 4$ Supersymmetries and examine the effectiveness of these techniques in theories with less Supersymmetry. Where appropriate we also discuss the twistor structure of amplitudes calculated in theories with less than four Supersymmetries and consider whether these exhibit similar properties to amplitudes in $N = 4$ theories.

We begin by considering the "holomorphic anomaly" of unitarity cuts. Witten noted that at $N = 4$ differential operators acting within the loop momentum integral yield delta functions, and thus suggested the existence of a "holomorphic anomaly" in $N = 4$ theories. Indeed he noted that the existence of such a feature could be used to derive algebraic equations for the coefficients of integral functions which appear in an amplitude. In the first section of this chapter we extend this analysis to examine how the "holomorphic anomaly" acts upon the cuts of $N = 1$ Supersymmetric one-loop amplitudes, focusing on a six-gluon non-MHV amplitude which had been previously calculated by other collaborators involved in this work. We also examine the usefulness of the "holomorphic anomaly" as a calculational tool to evaluate amplitudes, and compare the $N = 1$ case to the $N = 4$ case examined by Witten.

In the second section of this chapter we examine the many fascinating geometric features that appear in the twistor space realisation of gauge theory amplitudes. It has been observed that the coefficients of integral functions contained in an amplitude exhibit interesting structure in twistor space, particularly the coefficients of the I_4 integral functions. In theories with $N = 4$ Supersymmetries it has been shown that these I_4 coefficients for next to MHV amplitudes have planar support in twistor space, behaviour that is analogous to that of tree amplitudes. In this section we investigate whether similar behaviour exists for theories with $N < 4$ Supersymmetries by computing the I_4 coefficients for all six-point $N = 1$ amplitudes and examining their twistor space structure. We explicitly determine the twistor space description for the coefficients of next to MHV $N = 1$ amplitudes and discuss whether this behaviour implies a continuation of the twistor structure exhibited at $N = 4$. We also extend this analysis to include certain classes of n point $N = 1$ amplitudes and discuss their twistor space structure.

Finally, in the third section of this chapter we continue to study the twistor space structure for amplitudes in theories with $N < 4$ Supersymmetries. Although Witten's proposed relationship between twistor string theory and perturbative field theory has been observed at $N = 4$, it is as yet unresolved as to what degree this relationship extends to theories with less or indeed no Supersymmetry. It therefore seems reasonable to continue gathering information by studying the properties of amplitudes in such theories until a direct connection is uncovered. By focusing on the I_4 integral functions that appear in specific example amplitudes, and exploiting the generalised unitarity technique of Britto, Cachazo and Feng by using quadruple cuts, we compute the coefficients of these functions, and examine whether these amplitudes obey the same collinearity and coplanarity conditions as $N = 4$ I_4 coefficients, i.e. are the collinearity and coplanarity conditions independent of the Supersymmetry. We demonstrate by means of a relatively simple proof that the $N = 4$, $N = 1$ and $N = 0$ cases for amplitudes that are "MHV-deconstructible" are inherently related, and as such one must only demonstrate that the expected twistor space properties are exhibited in two of the above cases to conclude that the third case must also satisfy these properties. We further exploit the approach of Britto, Cachazo and Feng by using triple cuts to determine the coefficients of I_3 and I_2 integral functions and present the full expression for an example one-loop amplitude.

6.1 $N = 1$ Supersymmetric One-Loop Amplitudes and the "Holomorphic Anomaly" of Unitarity Cuts

Recently, it has been shown that the "holomorphic anomaly" of unitarity cuts can be used as a tool in determining the one-loop amplitudes in $N = 4$ Super Yang-Mills theory [13]. It is interesting to examine whether this method can be applied to more general cases. In this section we present results for a non-MHV $N = 1$ Supersymmetric one-loop amplitude. We show that the "holomorphic anomaly" of each unitarity cut correctly reproduces the action on the amplitude's imaginary part of the differential operators corresponding to collinearity in twistor space. Furthermore, we show that the use of the "holomorphic anomaly" to evaluate the amplitude requires the solution of differential equations, rather than the algebraic equations found in [13].

The amplitude $A^{N=4}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$ has been calculated [9], so we choose to examine the $N = 1$ chiral matter multiplet contribution.

6.1.1 The Six-point Amplitude $A^{N=1 \text{ chiral}}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$

For a six-point Yang-Mills amplitude there are a relatively small number of independent colour-ordered helicity configurations. The non-vanishing Supersymmetric amplitudes are either MHV, the conjugate of MHV (Googly), or have three negative and three positive helicities.

The MHV amplitudes are rather special cases and indeed the "holomorphic anomaly" of the three particle cuts of $A^{N=1,\text{chiral}}(1^-, 2^-, 3^+, 4^+, 5^+, 6^+)$ is zero and is a rather uninteresting case. Consequently, we consider an amplitude with three negative helicities. There are three possible such color ordered configurations: $A(---+++)$, $A(--+-++)$ and $A(-+-+-+)$. We shall consider the effect of the "holomorphic anomaly" on the first of these.

This amplitude is fairly simple in that it contains no box integral functions [14], but only L_0 and K_0 functions, which are defined by [9],

$$\begin{aligned}
\mathrm{K}_0[s] &= \frac{1}{\epsilon(1-2\epsilon)}(-s)^{-\epsilon} = \frac{1}{\epsilon} - \ln(-s) + 2 + \mathcal{O}(\epsilon) , \\
\mathrm{L}_0[r] &= \frac{\ln(r)}{1-r} + \mathcal{O}(\epsilon) .
\end{aligned} \tag{6.1}$$

The function $\mathrm{K}_0[s]$ is simply proportional to the scalar bubble function. The function $\mathrm{L}_0[r]$ has several representations; it can be expressed as a linear combination of bubble functions or as a Feynman parameter integral for a two-mass triangle integral, see [9].

Written in full, the amplitude is

$$\begin{aligned}
A_6^{N=1\ \mathrm{chiral}}(1^-,2^-,3^-,4^+,5^+,6^+) &= a_1 K_0[s_{61}] + a_2 K_0[s_{34}] - \frac{i}{2}\Bigg[b_1 \frac{L_0[s_{345}/s_{61}]}{s_{61}} \\
&\quad + b_2 \frac{L_0[s_{234}/s_{34}]}{s_{34}} + b_3 \frac{L_0[s_{234}/s_{61}]}{s_{61}} + b_4 \frac{L_0[s_{345}/s_{34}]}{s_{34}}\Bigg]
\end{aligned} \tag{6.2}$$

where the coefficients are

$$a_1 = a_2 = \frac{1}{2} A_6^{\mathrm{tree}}(1^-,2^-,3^-,4^+,5^+,6^+), \tag{6.3}$$

and

$$b_1 = \frac{\langle 6|\not{P}|3\rangle^2 \langle 6^+|(\not{2}\not{P}\not{P} - \not{P}\not{2}\not{P})|3^+\rangle}{\langle 2|\not{P}|5\rangle\,[6\,1]\,[1\,2]\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,P^2}, \qquad P = P_{345} \equiv k_3 + k_4 + k_5,$$

$$
\begin{aligned}
b_2 &= \frac{\langle 4|\not{P}|1\rangle^2\langle 4^+|(\not{P}\not{2}\not{P}-\not{2}\not{P}\not{P})|1^+\rangle}{\langle 2|\not{P}|5\rangle\,[2\,3]\,[3\,4]\,\langle 5\,6\rangle\,\langle 6\,1\rangle\,P^2}, & P &= P_{234} \equiv k_2+k_3+k_4,\\
b_3 &= \frac{\langle 4|\not{P}|1\rangle^2\langle 4^+|(\not{P}\not{P}\not{5}-\not{P}\not{5}\not{P})|1^+\rangle}{\langle 2|\not{P}|5\rangle\,[2\,3]\,[3\,4]\,\langle 5\,6\rangle\,\langle 6\,1\rangle\,P^2}, & P &= P_{234},\\
b_4 &= \frac{\langle 6|\not{P}|3\rangle^2\langle 6^+|(\not{P}\not{5}\not{P}-\not{P}\not{P}\not{5})|3^+\rangle}{\langle 2|\not{P}|5\rangle\,[6\,1]\,[1\,2]\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,P^2}, & P &= P_{345}.
\end{aligned}
\tag{6.4}
$$

This amplitude was constructed in previous, unpublished work by D. C. Dunbar and L. J. Dixon by calculating the three-particle cuts together with an analysis of the infra-red poles. The six-point tree amplitudes appearing in the coefficients a_i were calculated in [67].

The amplitudes we calculate in this thesis have an overall factor in dimensional regularisation of $(\mu^2)^\epsilon c_\Gamma$, where

$$
c_\Gamma = \frac{1}{(4\pi)^{2-\epsilon}}\frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)},
\tag{6.5}
$$

Throughout this thesis we shall not explicitly include this.

We define $s_{ij} \equiv [i\,j]\,\langle j\,i\rangle$, $s_{ijk} \equiv P^2_{ijk} \equiv [i\,j]\,\langle j\,i\rangle + [j\,k]\,\langle k\,j\rangle + [k\,i]\,\langle i\,k\rangle \equiv (k_i + k_j + k_k)^2$ and $\langle a|\not{b}|c\rangle \equiv \langle a^+|\not{b}|c^+\rangle \equiv [ab]\langle bc\rangle$, where $\langle i\,j\rangle$ and $[i\,j]$ are the usual spinor helicity inner products, as discussed in chapter 3.

6.1.2 The "Holomorphic Anomaly" of the Unitarity Cuts

The amplitude we are considering has three potential three-particle cuts: $s_{123} > 0$, $s_{234} > 0$ and $s_{345} > 0$. The first of these vanishes identically for $N = 1$ chirral: $A_5^{\rm tree}(\ell_1^{h_1}, 1^-, 2^-, 3^-, \ell_2^{h_2}) = 0$ unless $h_1 = h_2 = 1$, which requires the states crossing the cut to be gluons, not fermions or scalars. The two non-vanishing cuts are not independent but may be obtained from one another by the symmetry $1 \leftrightarrow 3, 4 \leftrightarrow 6$.

In order to examine the "holomorphic anomaly", we compute the action of F_{561} on the cut C_{561} (which is equal to C_{234}). The cut for $s_{561} > 0$ (the imaginary part, or 1/2 the discontinuity) is defined as

$$C_{561} = \frac{i}{2}\int d\text{LIPS}\sum_h A_5^{\text{tree}}(\ell_1^h, 5^+, 6^+, 1^-, \ell_2^{-h}) A_5^{\text{tree}}((-\ell_2)^h, 2^-, 3^-, 4^+, (-\ell_1)^{-h}), \tag{6.6}$$

where $\ell_1 + \ell_2 = P_{234} \equiv P$ and $h \in \{-1/2, 0, 1/2\}$. Writing out all amplitudes in this expression and summing over the Supersymmetric multiplet we obtain

$$C_{561} = \frac{i}{2}\int d\text{LIPS} \frac{\langle 1\,\ell_1\rangle^2 \langle 1\,\ell_2\rangle^2}{\langle 5\,6\rangle \langle 6\,1\rangle \langle 1\,\ell_2\rangle \langle \ell_2\,\ell_1\rangle \langle \ell_1\,5\rangle} \times \frac{[4\,\ell_1]^2\,[4\,\ell_2]^2}{[2\,3]\,[3\,4]\,[4\,\ell_1]\,[\ell_1\,\ell_2]\,[\ell_2\,2]} \times \rho_{N=1}. \tag{6.7}$$

The factor $\rho_{N=1}$ may be obtained using Supersymmetric Ward identities [11], giving

$$\rho_{N=1} = \frac{\langle 4|\not{P}|1\rangle^2}{\langle 1\,\ell_1\rangle\,[\ell_1\,4]\,\langle 1\,\ell_2\rangle\,[\ell_2\,4]}. \tag{6.8}$$

Simplifying the expression, we can write C_{561} in a compact form

$$C_{561} = i\frac{K}{2}\int d\text{LIPS}\frac{[4\,\ell_2]\,\langle 1\,\ell_1\rangle}{[2\,\ell_2]\,\langle 5\,\ell_1\rangle}, \tag{6.9}$$

where we define K as

$$K = \frac{\langle 4|\not{P}_{234}|1\rangle^2}{[2\,3]\,[3\,4]\,\langle 5\,6\rangle\,\langle 6\,1\rangle\,s_{234}}. \tag{6.10}$$

Next we act with the collinear operator $[F_{561}, \eta]$ on this expression. We only pick up the contribution from the term with $\partial/\partial\tilde{\lambda}_{5\dot{a}}$, so that

$$[F_{561},\eta]C_{561} = \frac{iK}{2}\int d\mathrm{LIPS}\frac{[4\,\ell_2]\,\langle 1\,\ell_1\rangle}{[2\,\ell_2]}\,\langle 6\,1\rangle\left[\frac{\partial}{\partial\tilde\lambda_5},\eta\right]\frac{1}{\langle 5\,\ell_1\rangle}\,. \tag{6.11}$$

The parametrisation of the Lorentz-invariant phase-space measure dLIPS is the same as that employed in [7, 62], i.e.

$$\begin{aligned}\int d\mathrm{LIPS}(\bullet) &\equiv \int d^4\ell_1\,\delta^{(+)}(\ell_1^2)\int d^4\ell_2\delta^{(+)}(\ell_2^2)\delta^{(4)}(\ell_1+\ell_2-P)(\bullet)\\ &= \int_0^\infty tdt\int\langle\lambda_{\ell_1},d\lambda_{\ell_1}\rangle[\tilde\lambda_{\ell_1},d\tilde\lambda_{\ell_1}]\int d^4\ell_2\delta^{(+)}(\ell_2^2)\delta^{(4)}(\ell_1+\ell_2-P)(\bullet),\end{aligned} \tag{6.12}$$

and we change coordinates, $\lambda\to\lambda'$ and $\tilde\lambda\to t\tilde\lambda'$, then drop the primes. The integral becomes

$$\begin{aligned}[F_{561},\eta]C_{561} &= i\frac{K}{2}\int_0^\infty tdt\int\langle\lambda_{\ell_1},d\lambda_{\ell_1}\rangle[\tilde\lambda_{\ell_1},d\tilde\lambda_{\ell_1}]\\ &\times\int d^4\ell_2\delta^{(+)}(\ell_2^2)\delta^{(4)}(\ell_1+\ell_2-P)\frac{[4\,\ell_2]\,\langle 1\,\ell_1\rangle\,\langle 6\,1\rangle}{[2\,\ell_2]}\left[\frac{\partial}{\partial\tilde\lambda_5},\eta\right]\frac{1}{\langle 5\,\ell_1\rangle}.\end{aligned} \tag{6.13}$$

We now carefully follow the prescription of Cachazo [50]. We use the identity [50],

$$\left[\frac{\partial}{\partial\tilde\lambda_5},\eta\right]\frac{1}{\langle\ell_1\,5\rangle} = -\left[\frac{\partial}{\partial\tilde\lambda_{\ell_1}},\eta\right]\frac{1}{\langle\ell_1\,5\rangle}, \tag{6.14}$$

which can be rewritten using the Schouten identity $[A\,B]\,[C\,D] = [A\,C]\,[B\,D] - [A\,D]\,[B\,C]$, so that

$$[\tilde\lambda_{\ell_1},d\tilde\lambda_{\ell_1}]\left[\frac{\partial}{\partial\tilde\lambda_{\ell_1}},\eta\right] = \left[\tilde\lambda_{\ell_1},\frac{\partial}{\partial\tilde\lambda_{\ell_1}}\right][d\tilde\lambda_{\ell_1},\eta] - [\tilde\lambda_{\ell_1},\eta]\left[d\tilde\lambda_{\ell_1},\frac{\partial}{\partial\tilde\lambda_{\ell_1}}\right], \tag{6.15}$$

where the first term does not contribute to the integral [50]. Hence inside the integral we can now write [50],

$$[\tilde{\lambda}_{\ell_1}, d\tilde{\lambda}_{\ell_1}]\left[\frac{\partial}{\partial\tilde{\lambda}_5},\eta\right]\frac{1}{\langle \ell_1\,5\rangle} = [\tilde{\lambda}_{\ell_1},\eta]\left[d\tilde{\lambda}_{\ell_1},\frac{\partial}{\partial\tilde{\lambda}_{\ell_1}}\right]\frac{1}{\langle \ell_1\,5\rangle} = [\tilde{\lambda}_{\ell_1},\eta]2\pi\bar{\delta}(\langle\lambda_{\ell_1},\lambda_5\rangle), \tag{6.16}$$

and the integral becomes

$$\begin{aligned}[F_{561},\eta]C_{561} &= -i\pi K\int_0^\infty t dt \int \langle \lambda_{\ell_1}, d\lambda_{\ell_1}\rangle \\ &\times \int d^4\ell_2 \delta^{(+)}(\ell_2^2)\delta^{(4)}(\ell_1+\ell_2-P)\frac{[4\,\ell_2]\,\langle 1\,\ell_1\rangle\,\langle 6\,1\rangle\,[\tilde{\lambda}_{\ell_1},\eta]}{[2\,\ell_2]}\bar{\delta}(\langle\lambda_{\ell_1},\lambda_5\rangle).\end{aligned} \tag{6.17}$$

The δ function in $\langle\lambda_{\ell_1},\lambda_5\rangle$ reduces the integral to

$$[F_{561},\eta]C_{561} = -i\pi K\int_0^\infty t dt \int d^4\ell_2\delta^{(+)}(\ell_2^2)\delta^{(4)}(\ell_1+\ell_2-P)\frac{[4\,\ell_2]\,\langle \ell_2\,a\rangle\,\langle 6\,1\rangle\,\langle 1\,5\rangle\,[5,\eta]}{[2\,\ell_2]\,\langle \ell_2\,a\rangle}. \tag{6.18}$$

We introduce a factor of $\langle \ell_2\,a\rangle\,/\,\langle \ell_2\,a\rangle$, which makes applying the δ function in ℓ_2 more transparent. Doing the integral in ℓ_2 using $\delta^{(4)}(\ell_1+\ell_2-P)$ we end up with

$$[F_{561},\eta]C_{561} = -i\pi K\int_0^\infty t dt\delta^{(+)}(\ell_2^2)\frac{[4\,\ell_2]\,\langle \ell_2\,a\rangle\,\langle 6\,1\rangle\,\langle 1\,5\rangle\,[5,\eta]}{[2\,\ell_2]\,\langle \ell_2\,a\rangle}, \tag{6.19}$$

where $\ell_2^\mu = P^\mu - tk_5^\mu$, and hence $\ell_2^2 = P^2 - 2tk_5\cdot P$, where $t = \frac{P^2}{2k_5\cdot P}$. Doing the t-integral yields

$$\begin{aligned}[F_{561},\eta]C_{561} &= i\pi K\frac{\langle 6\,1\rangle\,\langle 1\,5\rangle\,[5,\eta]P^2}{(2k_5\cdot P)^2}\frac{(2k_5\cdot P)\langle 4|\not{P}|a\rangle - P^2\langle 4|\not{5}|a\rangle}{(2k_5\cdot P)\langle 2|\not{P}|a\rangle - P^2\langle 2|\not{5}|a\rangle} \\ &= i\pi\frac{\langle 4|\not{P}|1\rangle^2}{[2\,3]\,[3\,4]\,\langle 5\,6\rangle}\frac{\langle 1\,5\rangle\,[5,\eta]}{(2k_5\cdot P)^2}\frac{\langle 4|\not{P}|5\rangle}{\langle 2|\not{P}|5\rangle},\end{aligned} \tag{6.20}$$

after reinstating the definition of K and choosing $a = 5$, for example.

From the optical theorem, the cut C_{561} is equal to the imaginary part of the amplitude in the kinematic region $s_{561} > 0$ [39]. For our amplitude eq. (6.2), using $\mathrm{Im}\ln(-s)|_{s>0} = -\pi$, this is

$$-\frac{1}{\pi}\mathrm{Im}A_{s_{561}>0} = -\frac{i}{2}\left[\frac{b_3}{2k_5\cdot P} - \frac{b_2}{2k_2\cdot P}\right]. \tag{6.21}$$

Operating on eq. (6.21) with the collinear operator $[F_{561},\eta]$ we have

$$[F_{561},\eta]\left(-\frac{1}{\pi}\mathrm{Im}A_{s_{561}>0}\right) = -\frac{i}{2}\left[\frac{[F_{561},\eta](b_3)}{2k_5\cdot P} - \frac{[F_{561},\eta](b_2)}{2k_2\cdot P} - \frac{b_3[F_{561},\eta](2k_5\cdot P)}{(2k_5\cdot P)^2}\right]. \tag{6.22}$$

Using the solutions for b_2 and b_3, eq. (6.4), we have

$$b_3 = \frac{\langle 4|\not{P}|1\rangle^2}{\langle 2|\not{P}|5\rangle}\frac{\langle 4^+|\not{P}\not{P}\not{5} - \not{P}\not{5}\not{P}|1^+\rangle}{[2\,3]\,[3\,4]\,\langle 5\,6\rangle\,\langle 6\,1\rangle\,P^2} = \frac{K}{\langle 2|\not{P}|5\rangle}\langle 4^+|\not{P}\not{P}\not{5} - \not{P}\not{5}\not{P}|1^+\rangle \equiv K'\hat{b}_3, \tag{6.23}$$

where

$$K' \equiv \frac{K}{\langle 2|\not{P}|5\rangle} \tag{6.24}$$

is annihilated by F_{561}, and where

$$\hat{b_3} \equiv 2P^2\langle 4|\not{5}|1\rangle - (2k_5\cdot P)\langle 4|\not{P}|1\rangle\,. \tag{6.25}$$

Also,

$$b_2 = \frac{\langle 4|\not{P}|1\rangle^2 \langle 4^+|(\not{P}\not{2}\not{P} - \not{2}\not{P}\not{P})|1^+\rangle}{\langle 2|\not{P}|5\rangle\,[2\,3]\,[3\,4]\,\langle 5\,6\rangle\,\langle 6\,1\rangle\, P^2} = \frac{K}{\langle 2|\not{P}|5\rangle}\langle 4^+|(\not{P}\not{2}\not{P} - \not{2}\not{P}\not{P})|1^+\rangle \equiv K'\hat{b}_2, \tag{6.26}$$

where

$$\hat{b_2} \equiv -2P^2\langle 4|\not{2}|1\rangle + (2k_2\cdot P)\langle 4|\not{P}|1\rangle\,. \tag{6.27}$$

Using $[F_{561},\eta](2k_5\cdot P) = \langle\eta|\not{P}|5\rangle\langle 16\rangle$, we have

$$\begin{aligned}[F_{561},\eta]\hat{b}_3 &= 2P^2\langle 51\rangle[F_{561},\eta][45] - \langle 4|\not{P}|1\rangle[F_{561},\eta](2k_5\cdot P)\\ &= -2P^2[\eta,4]\,\langle 1\,6\rangle\,\langle 5\,1\rangle - \langle\eta|\not{P}|5\rangle\,\langle 1\,6\rangle\,\langle 4|\not{P}|1\rangle\,,\end{aligned} \tag{6.28}$$

and

$$[F_{561},\eta]\hat{b}_2 = 0. \tag{6.29}$$

Inserting eq. (6.28) and eq. (6.29) into eq. (6.22), we find,

$$\begin{aligned}-\frac{1}{\pi}[F_{561},\eta]\mathrm{Im}A &= -\frac{i}{2}\frac{K'}{2k_5\cdot P}\Big(-2P^2[\eta,4]\,\langle 1\,6\rangle\,\langle 5\,1\rangle - \langle\eta|\not{P}|5\rangle\,\langle 1\,6\rangle\,\langle 4|\not{P}|1\rangle\\ &\qquad - \frac{2P^2\langle 4|\not{5}|1\rangle\langle\eta|\not{P}|5\rangle\,\langle 1\,6\rangle}{2k_5\cdot P} + \langle\eta|\not{P}|5\rangle\,\langle 1\,6\rangle\,\langle 4|\not{P}|1\rangle\Big)\\ &= -\frac{i}{2}\frac{2K'P^2\langle 16\rangle\langle 15\rangle}{(2k_5\cdot P)^2}\Big[[\eta,4](2k_5\cdot P) + \langle\eta|\not{P}|5\rangle\,[4\,5]\Big]\,.\end{aligned} \tag{6.30}$$

Combining $[\eta,4][P,5] - [\eta,P][45] = [\eta,5][P,4]$ using the Schouten identity, we obtain

$$\begin{aligned} -\frac{1}{\pi}[F_{561},\eta]\mathrm{Im}A &= -i\frac{K'P^2\langle 1\,6\rangle\langle 1\,5\rangle\langle 5,P\rangle}{(2k_5\cdot P)^2}\Big[[\eta,5][P,4]\Big] \\ &= -i\frac{\langle 4|\not{P}|1\rangle^2\langle 1\,5\rangle}{[2\,3]\,[3\,4]\,\langle 5\,6\rangle}\frac{[5,\eta]}{(2k_5\cdot P)^2}\frac{\langle 4|\not{P}|5\rangle}{\langle 2|\not{P}|5\rangle}, \end{aligned} \tag{6.31}$$

which matches the expression in eq. (6.20). Thus we have shown that the "holomorphic anomaly" of the unitarity cuts correctly reproduces the action of F_{ijk} upon the imaginary part of the amplitude.

6.1.3 Reconstructing Amplitudes from Differential Equations

In $N = 4$ one-loop amplitudes, appropriate collinear operators F_{ijk} annihilate the coefficients of the scalar box integral functions which span the amplitude [50]. This has the implication that the coefficients may be reconstructed by solving algebraic equations resulting from the action of the F_{ijk} operator upon the cuts equation. For $N = 1$ we have a more delicate situation as the collinear operator F_{ijk} in this case acts non-trivially on the coefficients b_i in the amplitude. This means that to reconstruct the amplitude we will generally have to solve differential equations for the coefficients b_i. In this section we explore the possibility of reconstructing the amplitude using the "holomorphic anomaly" of the cuts. In general $N = 1$ amplitudes contain integral functions derived from box, triangle and bubble integrals. As for the $N = 4$ case, we expect that the appropriate F_{ijk} operators should annihilate the coefficients of the box integral functions. However, F_{ijk} need not annihilate the coefficients of bubble and triangle functions. Instead, the action of F_{ijk} produces differential equations which these coefficients must satisfy.

To clarify the situation, consider the amplitude $A^{N=1\ \mathrm{chiral}}(1^-,2^-,3^-,4^+,5^+,6^+)$ which contains only triangle and bubble integrals. Consider the action of F_{561} on the C_{561} cutting equation,

$$[F_{561},\eta]\mathrm{Im}A_{s_{561}>0} = \quad [F_{561},\eta]C_{561}. \tag{6.32}$$

Expanding the amplitude into the series of known scalar integrals multiplied by rational coefficients, and keeping only those coefficients which have non-vanishing cuts in this channel, namely b_2 and b_3 in eq. (6.2), we have

$$\frac{i\pi}{2}[F_{561},\eta]\left(\frac{b_3}{2k_5\cdot P}-\frac{b_2}{2k_2\cdot P}\right) = \quad [F_{561},\eta]C_{561}. \tag{6.33}$$

The right-hand side of this equation is a rational function of λ_i and $\tilde{\lambda}_j$, determined via the "holomorphic anomaly" to be the expression given in eq. (6.20). In eq. (6.33) the functions multiplying the b_i are rational functions — in contrast to the $N=4$ situation where logarithms appear. Although the left-hand side is required to be rational this does not imply that F_{ijk} annihilate the b_i. The b_i must satisfy the linear differential equation

$$\frac{i\pi}{2}\left[\frac{[F_{561},\eta]b_3}{2k_5\cdot P}-\frac{b_3[F_{561},\eta](2k_5\cdot P)}{(2k_5\cdot P)^2}-\frac{[F_{561},\eta]b_2}{2k_2\cdot P}\right]=[F_{561},\eta]C_{561}\,. \tag{6.34}$$

We can also act with the operator

$$\langle\bar{F}_{ijk},\bar{\eta}\rangle = [i\,j]\left\langle\frac{\partial}{\partial\lambda_k},\bar{\eta}\right\rangle + [j\,k]\left\langle\frac{\partial}{\partial\lambda_i},\bar{\eta}\right\rangle + [k\,i]\left\langle\frac{\partial}{\partial\lambda_j},\bar{\eta}\right\rangle, \tag{6.35}$$

which produces an "anti-holomorphic anomaly" upon the same cut to yield

$$\frac{i\pi}{2}\left[\frac{\langle\bar{F}_{234},\bar{\eta}\rangle b_2}{2k_2\cdot P}-\frac{b_2\langle\bar{F}_{234},\bar{\eta}\rangle(2k_2\cdot P)}{(2k_2\cdot P)^2}-\frac{\langle\bar{F}_{234},\bar{\eta}\rangle b_3}{2k_5\cdot P}\right]=\langle\bar{F}_{234},\bar{\eta}\rangle C_{561}\,. \tag{6.36}$$

As a function of $\tilde{\lambda}_5$, $\tilde{\lambda}_6$, and $\tilde{\lambda}_1$, we find explicitly that $[F_{561},\eta]C_{561}$ is a function of $\tilde{\lambda}_5$ only. Similarly $\langle\bar{F}_{234},\bar{\eta}\rangle C_{561}$ is a function of λ_2 only. The coefficients b_2 and b_3 are related by the symmetry of the amplitude to satisfy $b_2(123456) = \bar{b}_3(456123)$. Also note that $\langle\bar{F}_{234},\bar{\eta}\rangle[F_{561},\eta]C_{561} = 0$. This motivates us to separate the equations, by assuming that $\langle\bar{F}_{234},\bar{\eta}\rangle b_3 = 0$ and $[F_{561},\eta]b_2 = 0$, to obtain the equation for b_3,

$$\frac{i\pi}{2}\left[\frac{[F_{561},\eta]b_3}{2k_5\cdot P} - \frac{b_3[F_{561},\eta](2k_5\cdot P)}{(2k_5\cdot P)^2}\right] = [F_{561},\eta]C_{561} \tag{6.37}$$

(with the equation for b_2 obtained by relabelling). To solve this equation, it is convenient to define

$$b_3 = K'\hat{b}_3 \tag{6.38}$$

as in eq. (6.23). Note that K' is independent of $\tilde{\lambda}_i$, $i = 5,6,1$. Since eq. (6.38) is independent of $\tilde{\lambda}_i$, $i = 6,1$, we deduce that b_3 depends only on $\tilde{\lambda}_5$. The right-hand side of eq. (6.38), from eq. (6.20),

$$\frac{[F_{561},\eta]C_{561}}{K'} = i\pi\frac{P^2\langle 16\rangle\langle 15\rangle\langle 5,P\rangle}{(2k_5\cdot P)^2}\Big[[\eta,5][P,4]\Big], \tag{6.39}$$

is of the form $[X,5]$. So we make a trial solution for $\hat{b}_3$

$$\hat{b}_3 = [5,\mathcal{C}], \tag{6.40}$$

which implies

$$\begin{aligned}\frac{[F_{561},\eta]\hat{b}_3}{(2k_5\cdot P)} - \frac{\hat{b}_3[F_{561},\eta](2k_5\cdot P)}{(2k_5\cdot P)^2} &= -\frac{[5,P]\langle P,5\rangle[\eta,\mathcal{C}]\,\langle 6\,1\rangle}{(2k_5\cdot P)^2} + \frac{[5,\mathcal{C}]\,\langle 6\,1\rangle\,[\eta,P]\langle P,5\rangle}{(2k_5\cdot P)^2}\\ &= \frac{\langle 1\,6\rangle\,\langle 5,P\rangle}{(2k_5\cdot P)^2}\Big[[\eta,5][P,\mathcal{C}]\Big].\end{aligned} \tag{6.41}$$

Thus eq. (6.38) is solved by

$$C_{\dot{a}} = 2P^2\langle 15\rangle\tilde{\lambda}_{4\dot{a}}\,, \tag{6.42}$$

giving

$$\hat{b}_3 = 2P^2\langle 4|\not{3}|1\rangle \tag{6.43}$$

as a specific solution to eq. (6.38). However, this solution is not unique, as

$$\hat{b}_3 = 2P^2\langle 4|\not{3}|1\rangle + (2k_5\cdot P)\times A \tag{6.44}$$

is also a solution, for any rational function A not involving $\tilde{\lambda}_i$, $i = 5, 6, 1$. To also satisfy $\langle \bar{F}_{234}, \bar{\eta}\rangle(b_3/(2k_5\cdot P)) = 0$, we must have;

$$\langle \bar{F}_{234}, \bar{\eta}\rangle A = 0. \tag{6.45}$$

This relation is not sufficient to fix A. Indeed, any function of $P_{a\dot{a}} = \sum_{i=5,6,1}(\lambda_i)_a(\tilde{\lambda}_i)_{\dot{a}}$ will satisfy eq. (6.45). We have used the action of all F_{ijk} functions which give rational functions acting upon the cut. The information in other cut channels is equivalent to this cut by relabelling. Thus we are led to conclude that the action of the F_{ijk} operators upon the cuts does not uniquely fix the coefficients without the input of further information. Examples of the constraints that $\hat{b}_3$ must satisfy are: dimensionality, spinor weight, collinear limits, multi-particle poles, *etc.* For example, the coefficient $\hat{b}_3$ must have dimension 2 and the spinor weight of $+1$ with respect to leg 4, -1 with respect to leg 1, and 0 for other legs. (Spinor weight is an additive assignment of $+r$ for each $(\tilde{\lambda}_i)^r$ and $-r$ for each $(\lambda_i)^r$ in a product of terms.) The simplest solution to this condition is a quartic polynomial in the $\tilde{\lambda}_i, \lambda_i$, linear in $\tilde{\lambda}_4$ and λ_1, with others appearing in the combination $\tilde{\lambda}_i\lambda_i$. The differential equation then forces a solution of the form

$$\hat{b}_3 = 2P^2\langle 4|\not{5}|1\rangle + \alpha(2k_5 \cdot P)\langle 4|\not{P}|1\rangle \,. \tag{6.46}$$

The arbitrary coefficient α can be fixed to be -1 by considering the collinear limit $2-3$.

Thus we have demonstrated how the action of the "holomorphic anomaly" on the cuts can be used to provide information about $N=1$ Supersymmetric amplitudes. In general, we obtain differential equations; hence fixing the coefficients unambiguously does require the input of suitable physical information, such as the collinear limits.

6.1.4 A term in $A^{N=1 \text{ chiral}}(1^-, 2^-, 3^-, 4^+, \ldots, n^+)$

As a further example let us consider the n-point amplitude $A^{N=1 \text{ chiral}}(1^-, 2^-, 3^-, 4^+, \ldots, n^+)$ and deduce some of its integral function coefficients. Consider the cut analogous to the previous case $C_{5\cdots n1}$ which is

$$C_{5\cdots n1} = \frac{iK}{2}\int d\text{LIPS}\frac{[4\,\ell_2]\,\langle 1\,\ell_1\rangle}{[2\,\ell_2]\,\langle 5\,\ell_1\rangle}\,, \tag{6.47}$$

where now

$$K = \frac{\langle 4|\not{P}_{234}|1\rangle^2}{[2\,3]\,[3\,4]\,\langle 5\,6\rangle\,\langle 6\,7\rangle\cdots\langle n\,1\rangle\,s_{234}} \tag{6.48}$$

Notice that on the cut the integrand is

$$\frac{[4\,\ell_2]\,\langle 1\,\ell_1\rangle}{[2\,\ell_2]\,\langle 5\,\ell_1\rangle} = \frac{\langle 4^+|\,\not{\ell}_2|2^+\rangle\langle 5^+|\,\not{\ell}_1|1^+\rangle}{\langle 2^+|\,\not{\ell}_2|2^+\rangle\langle 5^+|\,\not{\ell}_1|5^+\rangle} = -\frac{\langle 4^+|\,\not{\ell}_2\,\not{2}\,\not{P}_{234}\,\not{5}\,\not{\ell}_1|1^+\rangle}{\langle 2^+|\,\not{P}_{234}|5^+\rangle(\ell_2-k_2)^2(\ell_1+k_5)^2}. \tag{6.49}$$

The two propagators in eq. (6.49), plus the two cut propagators, make up a cut box integral. However, in the numerator of eq. (6.49), we can anti-commute $\not{\ell}_2$ and $\not{\ell}_1$ toward each other, to get

$$\begin{aligned}\frac{[4\,\ell_2]\,\langle 1\,\ell_1\rangle}{[2\,\ell_2]\,\langle 5\,\ell_1\rangle} &= \frac{\langle 4^+|\,\not{P}_{234}\,\not{5}\,\not{\ell}_1|1^+\rangle}{\langle 2^+|\,\not{P}_{234}|5^+\rangle(\ell_1+k_5)^2} + \frac{\langle 4^+|\,\not{2}\,\not{\ell}_2\,\not{P}_{234}|1^+\rangle}{\langle 2^+|\,\not{P}_{234}|5^+\rangle(\ell_2-k_2)^2} \\ &\quad -\frac{\langle 4^+|\,\not{2}\,\not{\ell}_2(\not{\ell}_1+\not{\ell}_2)\,\not{\ell}_1\,\not{5}|1^+\rangle}{\langle 2^+|\,\not{P}_{234}|5^+\rangle(\ell_2-k_2)^2(\ell_1+k_5)^2},\end{aligned} \tag{6.50}$$

where we have used $P_{234}=\ell_1+\ell_2$ in the last term, making it clear that it vanishes. Thus the cut reduces to a sum of two cut linear triangles, or in other words,

$$\begin{aligned}A^{N=1\text{ chiral}}(1^-,2^-,3^-,4^+,5^+,\ldots,n^+) &= -\frac{i}{2}\left[b_2\frac{L_0[s_{234}/s_{34}]}{s_{34}} + b_3\frac{L_0[s_{234}/s_{6\ldots1}]}{s_{6\ldots1}}\right] \\ &\quad + \text{terms not contributing to the cut,}\end{aligned} \tag{6.51}$$

where $s_{6\ldots1}\equiv(k_6+k_7+\cdots+k_n+k_1)^2$. Acting upon $C_{5\cdots n1}$ as before with $\langle\bar{F}_{234},\bar{\eta}\rangle$, we obtain

$$\langle\bar{F}_{234},\bar{\eta}\rangle C_{5\cdots n1} = -i\pi K\frac{P^2\,[2\,4]\,[3\,4]\,\langle 2,\bar{\eta}\rangle}{(2k_2\cdot P)^2}\frac{\langle 2|\not{P}|1\rangle}{\langle 2|\not{P}|5\rangle}. \tag{6.52}$$

Applying exactly the same steps as before we have a trial solution

$$\hat{b}_2 = -2P^2\langle 4|\not{2}|1\rangle + \alpha(2k_2 \cdot P)\langle 4|\not{P}|1\rangle\,, \tag{6.53}$$

where we can fix $\alpha = 1$ using collinear limits.

6.2 Twistor Space Structure of the Box Coefficients of $N = 1$ One-Loop Amplitudes

We examine the coefficients of box functions in $N = 1$ Supersymmetric one-loop amplitudes, presenting the box coefficients for all six-point $N = 1$ amplitudes and certain n-point example coefficients. We also examine the twistor structure of $N = 1$ one-loop amplitudes and show that the box coefficients for "next-to MHV" amplitudes have coplanar support in twistor space.

6.2.1 Box Coefficients of The Six-point $N = 1$ Amplitudes

We can organise the six-point amplitudes according to the number of negative helicities; amplitudes with zero, one, five or six vanish in any Supersymmetric theory. The amplitudes with two negative helicities are the MHV amplitudes, which were computed previously [9], while those with four are the "Googly" MHV amplitudes which are obtained by conjugation of the MHV amplitudes. Here we present the remaining box coefficients and examine the twistor structure of all the six-point amplitudes.

The two independent types of six-point amplitude have rather different box structures. The MHV amplitudes contain "two-mass easy" and single mass boxes, whereas the amplitudes with three negative helicities contain "two-mass hard" and single mass boxes. This feature does not extend to higher point functions.

MHV Amplitudes

There are three independent MHV amplitudes. In terms of the $D = 6$ boxes the box parts of these amplitudes are [9],

$$
\begin{aligned}
A(1^-,2^-,3^+,4^+,5^+,6^+)|_{\text{box}} &= 0 \\
A(1^-,2^+,3^-,4^+,5^+,6^+)|_{\text{box}} &= b_1^{D=6} I_{4:3}^{2\text{m}\,e} + b_2^{D=6} I_{4:5}^{1\text{m}} + b_3^{D=6} I_{4:3}^{1\text{m}} \\
A(1^-,2^+,3^+,4^-,5^+,6^+)|_{\text{box}} &= c_1^{D=6} I_{4:1}^{2\text{m}\,e} + c_2^{D=6} I_{4:3}^{2\text{m}\,e} + c_3^{D=6} I_{4:6}^{1\text{m}} + c_4^{D=6} I_{4:3}^{1\text{m}}
\end{aligned}
\tag{6.54}
$$

where,

$$
\begin{aligned}
b_1^{D=6} &= A_{13}^{\text{tree MHV}} \frac{\text{tr}_+(1325)\,\text{tr}_+(1352)}{s_{13}^2 s_{25}} \qquad b_2^{D=6} = A_{13}^{\text{tree MHV}} \frac{\text{tr}_+(1324)\,\text{tr}_+(1342)}{s_{13}^2 s_{24}} \\
b_3^{D=6} &= A_{13}^{\text{tree MHV}} \frac{\text{tr}_+(1326)\,\text{tr}_+(1362)}{s_{13}^2 s_{26}}
\end{aligned}
\tag{6.55}
$$

$$
\begin{aligned}
c_1^{D=6} &= A_{14}^{\text{tree MHV}} \frac{\text{tr}_+(1436)\,\text{tr}_+(1463)}{s_{14}^2 s_{36}} \qquad c_2^{D=6} = A_{14}^{\text{tree MHV}} \frac{\text{tr}_+(1425)\,\text{tr}_+(1452)}{s_{14}^2 s_{25}} \\
c_3^{D=6} &= A_{14}^{\text{tree MHV}} \frac{\text{tr}_+(1435)\,\text{tr}_+(1453)}{s_{14}^2 s_{35}} \qquad c_4^{D=6} = A_{14}^{\text{tree MHV}} \frac{\text{tr}_+(1426)\,\text{tr}_+(1462)}{s_{14}^2 s_{26}}
\end{aligned}
\tag{6.56}
$$

where $\text{tr}_+(\text{abcd}) = [\text{a}\,\text{b}]\,\langle \text{b}\,\text{c}\rangle\,[\text{c}\,\text{d}]\,\langle \text{d}\,\text{a}\rangle$. If we examine the coefficients of the F-functions we have, for example,

$$
b_1^F = A_{13}^{\text{tree MHV}} \times \frac{\text{tr}_+(1325)\,\text{tr}_+(1352)}{s_{13}^2 s_{25}^2} = A_{13}^{\text{tree MHV}} \times \frac{\langle 3\,2\rangle\,\langle 1\,5\rangle\,\langle 3\,5\rangle\,\langle 2\,1\rangle}{\langle 1\,3\rangle^2\,\langle 2\,5\rangle^2}, \tag{6.57}
$$

which is a holomorphic function (*i.e.* a function of λ alone).

Amplitudes with three minus helicities

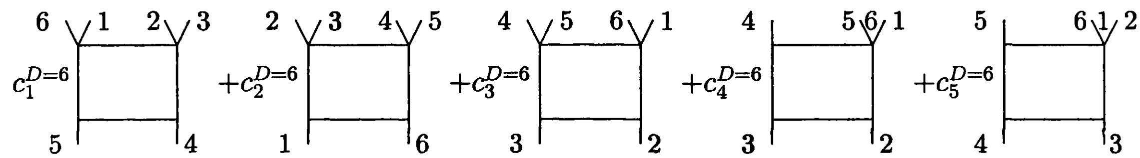
There are also three independent amplitudes with three minus helicities: $A(1^-,2^-,3^-,4^+,5^+,6^+)$, $A(1^-,2^-,3^+,4^-,5^+,6^+)$ and $A(1^-,2^+,3^-,4^+,5^-,6^+)$. Of these, the first consists only of triangle and bubble integrals [14] so we have a trivial box structure,

$$A(1^-,2^-,3^-,4^+,5^+,6^+)|_{\text{box}} = \quad 0. \tag{6.58}$$

The next amplitude, $A(1^-,2^-,3^+,4^-,5^+,6^+)$, does have a non-trivial box structure, which we express in terms of $D=6$ boxes as,

$$A(1^-,2^-,3^+,4^-,5^+,6^+)|_{\text{box}} = c_1^{D=6} I_{4:6}^{\text{2m}\,h} + c_2^{D=6} I_{4:2}^{\text{2m}\,h} + c_3^{D=6} I_{4:4}^{\text{2m}\,h} + c_4^{D=6} I_{4:5}^{\text{1m}} + c_5^{D=6} I_{4:6}^{\text{1m}}, \tag{6.59}$$

where the integral boxes are,



and we have computed the coefficients to be,

$$\begin{aligned}
c_1^{D=6} &= i\frac{(\langle 3|\not{P}|1\rangle)^2\langle 5|\not{P}|4\rangle\langle 3|\not{P}|5\rangle}{\langle 4|\not{P}|5\rangle\langle 2|\not{P}|5\rangle}\frac{\langle 5\,1\rangle}{[2\,3]\,\langle 5\,6\rangle\,\langle 6\,1\rangle\,P^2}, \qquad P=P_{234},\\
c_2^{D=6} &= i\frac{(\langle 3|\not{P}|4\rangle)^2\langle 6|\not{P}|1\rangle}{\langle 1|\not{P}|6\rangle}\frac{[3\,1]\,\langle 6\,4\rangle}{[1\,2]\,[2\,3]\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,P^2}, \qquad P=P_{123},\\
c_3^{D=6} &= i\frac{(\langle 6|\not{P}|4\rangle)^2\langle 2|\not{P}|4\rangle\langle 3|\not{P}|2\rangle}{\langle 2|\not{P}|3\rangle\langle 2|\not{P}|5\rangle}\frac{[6\,2]}{\langle 4\,5\rangle\,[6\,1]\,[1\,2]\,P^2}, \qquad P=P_{345},\\
c_4^{D=6} &= i\frac{(\langle 3|\not{P}|1\rangle)^2\langle 2|\not{P}|1\rangle}{\langle 2|\not{P}|5\rangle}\frac{\langle 2\,4\rangle}{\langle 5\,6\rangle\,\langle 6\,1\rangle\,P^2\,[2\,4]}, \qquad P=P_{234},\\
c_5^{D=6} &= i\frac{(\langle 6|\not{P}|4\rangle)^2\langle 6|\not{P}|5\rangle}{\langle 2|\not{P}|5\rangle}\frac{[3\,5]}{[6\,1]\,[1\,2]\,P^2\,\langle 3\,5\rangle}, \qquad P=P_{345},
\end{aligned} \tag{6.60}$$

where $\langle a|\not{K}|c\rangle \equiv \langle a^+|\not{K}|c^+\rangle$.

The remaining amplitude, $A(1^-,2^+,3^-,4^+,5^-,6^+)$, contains all six one-mass and all six "two mass-hard" boxes,

$$
\begin{aligned}
A(1^-,2^+,3^-,4^+,5^-,6^+)_{\text{box}} \;=\;& a_1^{D=6} I_{4:4}^{\text{1m}} + a_2^{D=6} I_{4:5}^{\text{1m}} + a_3^{D=6} I_{4:6}^{\text{1m}} + a_4^{D=6} I_{4:1}^{\text{1m}} \\
&+ a_5^{D=6} I_{4:2}^{\text{1m}} + a_6^{D=6} I_{4:3}^{\text{1m}} + b_1^{D=6} I_{4:3}^{\text{2m}\,h} + b_2^{D=6} I_{4:4}^{\text{2m}\,h} \\
&+ b_3^{D=6} I_{4:5}^{\text{2m}\,h} + b_4^{D=6} I_{4:6}^{\text{2m}\,h} + b_5^{D=6} I_{4:1}^{\text{2m}\,h} + b_6^{D=6} I_{4:2}^{\text{2m}\,h}.
\end{aligned}
\tag{6.61}
$$

Fortunately these are not all independent and symmetry demands relationships amongst the $a_i^{D=6}$'s,

$$
\begin{aligned}
a_3^{D=6}(123456) &= a_1^{D=6}(345612), & a_5^{D=6}(123456) &= a_1^{D=6}(561234),\\
a_4^{D=6}(123456) &= a_2^{D=6}(345612), & a_6^{D=6}(123456) &= a_2^{D=6}(561234),\\
a_2^{D=6}(123456) &= \bar{a}_1^{D=6}(234561), & a_1^{D=6}(123456) &= a_1^{D=6}(321654),
\end{aligned}
\tag{6.62}
$$

where $\bar{a}_1^{D=6}$ denotes $a_1^{D=6}$ with $\langle i\,j\rangle \leftrightarrow [i\,j]$. Thus there is a single independent $a_i^{D=6}$. Similarly we can use symmetry to generate all the $b_i^{D=6}$'s from $b_1^{D=6}$. We have computed the expressions for $a_1^{D=6}$ and $b_1^{D=6}$ to be,

$$
\begin{aligned}
a_1^{D=6} &= i\frac{\langle 2|\not{P}|5\rangle^2\langle 1|\not{P}|5\rangle\langle 3|\not{P}|5\rangle}{\langle 3|\not{P}|6\rangle\langle 1|\not{P}|4\rangle P^2}\frac{\langle 3\,1\rangle}{[1\,3]\,\langle 4\,5\rangle\,\langle 5\,6\rangle}, \quad P = P_{123},\\
b_1^{D=6} &= i\frac{\langle 2|\not{P}|5\rangle^2\langle 3|\not{P}|5\rangle\langle 2|\not{P}|4\rangle\langle 4|\not{P}|3\rangle}{\langle 3|\not{P}|6\rangle\langle 1|\not{P}|4\rangle\langle 3|\not{P}|4\rangle P^2}\frac{1}{[1\,2]\,\langle 5\,6\rangle}, \quad P = P_{123}.
\end{aligned}
\tag{6.63}
$$

Googly MHV Amplitudes

The Googly MHV amplitudes can be obtained from the MHV amplitudes by conjugation. These amplitudes are useful for testing hypotheses regarding amplitudes containing four minus helicities. For example we have,

$$
A(1^+,2^-,3^+,4^-,5^-,6^-)|_{\text{box}} = b_1^{D=6\prime} I_{4:3}^{\text{2m}\,e} + b_2^{D=6\prime} I_{4:5}^{\text{1m}} + b_3^{D=6\prime} I_{4:3}^{\text{1m}},
\tag{6.64}
$$

with $b_i^{D=6\prime} = \overline{b}_i^{D=6}$. The coefficients of the F-functions are anti-holomorphic functions, *e.g.*

$$b_1^{F\prime} = \bar{A}_{24}^{\rm tree} \times \frac{[3\,2]\,[1\,5]\,[3\,5]\,[2\,1]}{[1\,3]^2\,[2\,5]^2}. \tag{6.65}$$

6.2.2 Higher Point Box Coefficients

In this section we evaluate some sample box coefficients for certain n-point amplitudes. This will enable us to examine whether the twistor space structure of the six-point amplitudes extends to higher point amplitudes.

For higher point amplitudes the number of helicity configurations grows quite rapidly with increasing numbers of legs. As our first example we will consider the specific amplitude,

$$A^{N=1\ \rm chiral}(1^- 2^- \cdots j^+ (j+1)^- 5^+ \cdots n^+)\ . \tag{6.66}$$

We calculate the $123\cdots j$-cut of this amplitude, *i.e.*,

$$\begin{aligned} C_{123\cdots j} = \frac{i}{2}\int d\mathrm{LIPS}\ \ \textstyle\sum_{h\in\{-1/2,0,1/2\}}\ \ & A^{\rm tree}(\ell_1^h, 1^-, 2^-, \cdots, j^+, \ell_2^{-h}) \\ & A^{\rm tree}((-\ell_2)^h, (j+1)^-, \cdots, n^+, (-\ell_1)^{-h}), \end{aligned} \tag{6.67}$$

The sum is over the particles in the $N = 1$ chiral multiplet. The two tree amplitudes are a MHV amplitude and a MHV-Googly amplitude. For MHV amplitudes the different tree amplitudes for different particle types are related by Supersymmetric ρ factors, as we discussed in section 4.3.3. We obtain,

$$C_{123\cdots j} = \frac{i}{2}\int d\text{LIPS}A^{\text{tree MHV}}(\ell_1^s, 1^-, 2^-, \cdots, j^+, \ell_2^s) \times A^{\text{tree MHV Googly}}((-\ell_2)^s, (j+1)^-, \cdots, n^+, (-\ell_1)^s) \times \rho^{N=1} \tag{6.68}$$

where,

$$\rho^{N=1} = -x + 2 - \frac{1}{x} = -\frac{(x-1)^2}{x}, \quad \text{with} \quad x = \frac{[j\,\ell_2]\,\langle j+1\,\ell_2\rangle}{[j\,\ell_1]\,\langle j+1\,\ell_1\rangle} \tag{6.69}$$

so that,

$$\rho^{N=1} = -\frac{[j\,\ell_1]\,\langle j+1\,\ell_1\rangle}{[j\,\ell_2]\,\langle j+1\,\ell_2\rangle}\left(\frac{[j\,\ell_2]\,\langle j+1\,\ell_2\rangle}{[j\,\ell_1]\,\langle j+1\,\ell_1\rangle} - 1\right)^2 = -\frac{\langle j|P_{123\cdot j}|j+1\rangle^2}{[j\,\ell_1]\,\langle j+1\,\ell_1\rangle\,[j\,\ell_2]\,\langle j+1\,\ell_2\rangle} \tag{6.70}$$

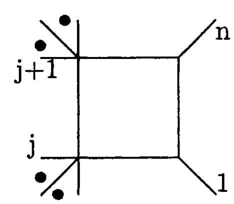
This gives the integrand above as,

$$\begin{aligned}
&\frac{[j\,\ell_1]^2\,[j\,\ell_2]^2}{[1\,2]\,[2\,3]\cdots[j-1\,j]\,[j\,\ell_2]\,[\ell_2\,\ell_1]\,[\ell_1\,1]} \\
&\times\frac{\langle j+1\,\ell_1\rangle^2\,\langle j+1\,\ell_2\rangle^2}{\langle j+1\,j+2\rangle\,\langle j+2\,j+3\rangle\cdots\langle n-1\,n\rangle\,\langle n\,\ell_1\rangle\,\langle \ell_1\,\ell_2\rangle\,\langle \ell_2\,j+1\rangle} \\
&\times\frac{\langle j|P_{123\cdots j}|j+1\rangle^2}{[j\,\ell_1]\,\langle j+1\,\ell_1\rangle\,[j\,\ell_2]\,\langle j+1\,\ell_2\rangle} \\
&= \frac{\langle j|P_{123\cdots j}|j+1\rangle^2}{[1\,2]\,[2\,3]\cdots[j-1\,j]\,\langle j+1\,j+2\rangle\,\langle j+2\,j+3\rangle\cdots\langle n-1\,n\rangle\,P^2_{123\cdots j}} \times \frac{[j\,\ell_1]}{[\ell_1\,1]} \\
&\times\frac{\langle j+1\,\ell_1\rangle}{\langle n\,\ell_1\rangle} \\
&= \frac{\langle j|P_{123\cdots j}|j+1\rangle^2}{[1\,2]\,[2\,3]\cdots[j-1\,j]\,\langle j+1\,j+2\rangle\,\langle j+2\,j+3\rangle\cdots\langle n-1\,n\rangle\,P^2_{123}} \\
&\times\frac{[j\,\ell_1]\,\langle \ell_1\,1\rangle\,\langle j+1\,\ell_1\rangle\,[\ell_1\,n]}{(\ell_1-k_1)^2(\ell_1+k_n)^2}.
\end{aligned} \tag{6.71}$$

This corresponds to the cut of a box integral with integrand quadratic in the loop momentum, *i.e.*,

$$C_{123\cdots j} = \frac{\langle j|P_{123\cdots j}|j+1\rangle^2}{[1\,2]\cdots[j-1\,j]\,\langle j+1\,j+2\rangle\,\langle j+2\,j+3\rangle\cdots\langle n-1\,n\rangle\,P^2_{123\cdots j}} \times \left(I_2^{2mh}[[j\,\ell_1]\,\langle \ell_1\,1\rangle\,\langle j+1\,\ell_1\rangle\,[\ell_1\,n]]\right)_{\text{cut}}. \quad (6.72)$$

The specific box integral is the "two mass-hard" depicted below,



with a non-trivial (quadratic in loop momenta) numerator.

Rewriting the numerator,

$$\langle j|\ell\!\!/_1|1\rangle\langle n|\ell\!\!/_1|j+1\rangle = \frac{\langle j^+|\ell\!\!/_1|1^+\rangle\langle 1^+|P|n^+\rangle\langle n|\ell\!\!/_1|j+1\rangle}{\langle 1^+|P|n^+\rangle} = \frac{\langle j|\ell\!\!/_1 1\!\!/ P\!\!\!/ k\!\!\!/_n \ell\!\!/_1|j+1\rangle}{\langle 1|P\!\!\!/|n\rangle}, \quad (6.73)$$

and commuting the cut momenta toward $P\!\!\!/ = \ell\!\!/_1 - \ell\!\!/_2$,

$$\begin{aligned} \ell\!\!/_1 1\!\!/ P\!\!\!/ k\!\!\!/_n \ell\!\!/_1 &= (2\ell_1\cdot k_1)P\!\!\!/ k\!\!\!/_n \ell\!\!/_1 - 1\!\!/ \ell\!\!/_1 P\!\!\!/ k\!\!\!/_n \ell\!\!/_1 \\ &= (2\ell_1\cdot 1)P\!\!\!/ k\!\!\!/_n \ell\!\!/_1 - (2\ell_1\cdot k_n)1\!\!/ \ell\!\!/_1 P\!\!\!/ + 1\!\!/ \ell\!\!/_1 P\!\!\!/ \ell\!\!/_1 k\!\!\!/_n \\ &= -(\ell_1-k_1)^2 P\!\!\!/ k\!\!\!/_n \ell\!\!/_1 - (\ell_1+k_n)^2 k\!\!\!/_1 \ell\!\!/_1 P\!\!\!/ + (2\ell_1\cdot P)k\!\!\!/_1 \ell\!\!/_1 k\!\!\!/_n. \end{aligned} \quad (6.74)$$

In this expression the first two terms cancel a propagator yielding triangle integrals - which we discard for the present purposes - and the third term can be rearranged as $(2\ell_1 \cdot P) = -(\ell_1 - P)^2 + \ell_1^2 + P^2 = -\ell_2^2 + \ell_1^2 + P^2 \equiv P^2$ discarding momenta null on the cut. The remaining expression is a box with linear integrand which can be evaluated and the result expressed as a $D = 6$ scalar box function,

$$C_{123\cdots j} = \frac{\langle j|P_{123\cdots j}|(j+1)\rangle^2 \langle n|\not{P}|1\rangle\,[1\,j]\,\langle j+1\,n\rangle}{\langle 1|\not{P}|n\rangle\,[1\,2]\,[2\,3]\cdots[j-1\,j]\,\langle j+1\,j+2\rangle\,\langle j+2\,j+3\rangle\cdots\langle n-1\,n\rangle\,P_{123\cdots j}^2} \times \left(I_4^{2mh,D=6}\right)_{\text{cut}} \tag{6.75}$$

so we deduce, using the arguments of the previous section, that the coefficient of the box is

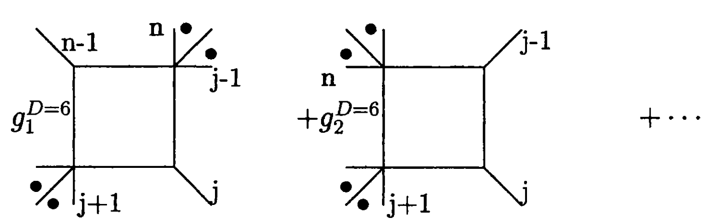
$$f_1^{D=6} = i\frac{\langle j|\not{P}|(j+1)\rangle^2 \langle n|\not{P}|1\rangle\,[1\,j]\,\langle j+1\,n\rangle}{\langle 1|\not{P}|n\rangle\,[1\,2]\,[2\,3]\cdots[j-1\,j]\,\langle j+1\,j+2\rangle\,\langle j+2\,j+3\rangle\cdots\langle n-1\,n\rangle\,P^2}, \tag{6.76}$$

where $P = P_{123\cdots j}$. This is a generalisation of the coefficient c_2 within the six point amplitude $A(1^-, 2^-, 3^+, 4^-, 5^+, 6^+)$.

As a further example, by looking at the $C_{n\cdots j-1}$ cut we can deduce that the amplitude,

$$A^{N=1\text{ chiral}}(1^-2^-\cdots(j-1)^-j^+(j+1)^+\cdots k^-\cdots(n-1)^+n^+)\;, \tag{6.77}$$

(where legs 1 to $j-1$ and leg k have negative helicity and the remainder have positive helicity) contains boxes,



The first appearance of the two-mass easy box in non-MHV amplitudes occurs at seven-point amplitudes. The coefficients are

$$g_1^{D=6} = -i\frac{\langle n|K|k\rangle^2\langle n|K|n-1\rangle\,\langle k\,n-1\rangle\,[n-1\,j]\,\langle j\,k\rangle}{[n\,1]\,[1\cdots j-1]\,\langle j\,j+1\rangle\,\langle j+1\cdots n-1\rangle\langle j-1|K|n-1\rangle\,\langle n-1\,j\rangle\,P^2} \tag{6.78}$$

$$g_2^{D=6} = i\frac{\langle n|K|k\rangle^2\langle j-1|K|k\rangle\langle j|K|j-1\rangle\,[n\,j-1]\,\langle j\,k\rangle}{[n\,1]\,[1\cdots j-1]\,\langle j\,j+1\rangle\,\langle j+1\cdots n-1\rangle\langle j-1|K|j\rangle\langle j-1|K|n-1\rangle P^2} \tag{6.79}$$

Using symmetry arguments various other box coefficients can be obtained from these expressions by relabelling.

6.2.3 Twistor Structure

It was observed by Witten [1] that the twistor space properties of amplitudes expressed in terms of the helicity states $(\lambda_i, \tilde{\lambda}_i)$ can be investigated using particular differential operators. Specifically, if points i, j and k are collinear in twistor space, then the amplitude $A(i,\ j,\ k)$ is annihilated by the operator

$$[F_{ijk},\eta] = \langle i\,j\rangle\left[\frac{\partial}{\partial\tilde{\lambda}_k},\eta\right] + \langle j\,k\rangle\left[\frac{\partial}{\partial\tilde{\lambda}_i},\eta\right] + \langle k\,i\rangle\left[\frac{\partial}{\partial\tilde{\lambda}_j},\eta\right]\ , \tag{6.80}$$

where the square brackets indicate spinor products rather than commutators. Similarly, annihilation by the operator

$$\begin{aligned} K_{ijkl} = \frac{1}{4}\ \Big[\ & \langle ij\rangle\epsilon^{\dot{a}\dot{b}}\frac{\partial}{\partial\tilde{\lambda}_k^{\dot{a}}}\frac{\partial}{\partial\tilde{\lambda}_l^{\dot{b}}} - \langle ik\rangle\epsilon^{\dot{a}\dot{b}}\frac{\partial}{\partial\tilde{\lambda}_j^{\dot{a}}}\frac{\partial}{\partial\tilde{\lambda}_l^{\dot{b}}} + \langle il\rangle\epsilon^{\dot{a}\dot{b}}\frac{\partial}{\partial\tilde{\lambda}_j^{\dot{a}}}\frac{\partial}{\partial\tilde{\lambda}_k^{\dot{b}}} \\ & +\langle jk\rangle\epsilon^{\dot{a}\dot{b}}\frac{\partial}{\partial\tilde{\lambda}_i^{\dot{a}}}\frac{\partial}{\partial\tilde{\lambda}_l^{\dot{b}}} + \langle jl\rangle\epsilon^{\dot{a}\dot{b}}\frac{\partial}{\partial\tilde{\lambda}_k^{\dot{a}}}\frac{\partial}{\partial\tilde{\lambda}_i^{\dot{b}}} - \langle kl\rangle\epsilon^{\dot{a}\dot{b}}\frac{\partial}{\partial\tilde{\lambda}_j^{\dot{a}}}\frac{\partial}{\partial\tilde{\lambda}_i^{\dot{b}}}\Big], \end{aligned} \tag{6.81}$$

indicates co-planarity of points i, j, k and l in twistor space.

Here we will explore the twistor space structure of the box coefficients of the $N = 1$ amplitudes. At tree-level an important implication of the CSW-formalism is that the twistor space properties of amplitudes are completely determined by the number of minus legs. For this reason we organise the one-loop amplitudes according to the number of negative helicities. We have investigated the twistor space properties for all the possible 5-point box coefficients and all the 6-point box coefficients together with the n-point coefficients of the previous section. This was carried out by generating sets of on-shell kinematic points consisting of specific values of λ_i and $\tilde{\lambda}_i$ and determining the action of the operators at these points.

For the six-point amplitudes there are three different classes of amplitudes organised by the number of negative helicities: MHV-amplitudes, next-to-MHV amplitudes and Googly MHV-amplitudes. For the n-point amplitudes we have extended certain six-point amplitudes by adding extra plus legs to the MHV side of the cut and extra minus legs to the Googly side. This produces the following classes of n-point configurations: $(-\cdots-+\cdots+-+\cdots+)$ and $(-\cdots-+-+\cdots+)$.

For the MHV-amplitudes all helicity configurations for the box coefficients are holomorphic and are thus annihilated by any F_{ijk} and K_{ijkl} operator, as noted in [61]. The geometric picture of these configurations is simply a line in twistor space.

Now we consider next-to-MHV amplitudes with three minus helicities. By acting with the K_{ijkl} operators we find that the box coefficients are annihilated for any four-points,

$$K_{ijkl}\left[c^F_{\text{next to MHV}}\right] = 0, \tag{6.82}$$

indicating a geometric picture where all points lie in a plane in twistor space.

The line structure of the box coefficients can be deduced by acting with the F_{ijk} operators. In the cuts we have used to determine these coefficients, there is a MHV tree amplitude on one side of the cut (the "mostly plus side") and a Googly MHV tree amplitude on the other (the "mostly minus side"). The box coefficients calculated from each cut will be annihilated by F_{ijk} when i, j and k are any legs lying on the mostly plus side of that cut, indicating that these legs define points in twistor space that lie on a line. Similar behaviour was found for the box coefficients in $N = 4$ amplitudes [51, 66].

For the $q(> 3)$ minus configurations, the box coefficients are only annihilated by F_{ijk} operators where all three of the points lie on the MHV, mostly plus, side of the cut used to calculate them. These points will lie on a line in twistor space. Hence the box coefficients are annihilated by any K_{ijkl} operator where three or more of these points lie on the line. For generic points in twistor space, we have confirmed explicitly that only these K_{ijkl} operators annihilate the box coefficients. The geometric interpretation is thus of $n - q$-points lying on a line with no restriction on the positions of the remaining q-points. In general, if a box has a cut in the channel $C_{i...j}$ and $A^{\text{tree}}(i \ldots j)$ is a MHV tree amplitude, then the box coefficient is supported on configurations in twistor space where points $i \ldots j$ are collinear. If there are two or more such cuts, this would imply a support of two or more lines with the remaining points unrestricted. When any pair of these cuts have a common leg, the corresponding lines intersect at the common point.

We have presented explicitly the results for the $N = 1$ chiral multiplet. Since the $N = 1$ vector multiplet is a linear combination of this and the $N = 4$ multiplet, the box coefficients of the $N = 1$ vector multiplet will also have planar support for next to MHV amplitudes.

6.3 One-Loop Gluon Scattering Amplitudes in Theories with $N < 4$ Supersymmetries

In this section we use Generalised Unitarity techniques [56] to calculate the coefficients of box and triangle integral functions of one-loop gluon scattering amplitudes in gauge theories with $N < 4$ Supersymmetries. We show that the box coefficients in $N = 1$ and $N = 0$ theories inherit the same coplanar and collinear constraints as the corresponding $N = 4$ coefficients. We use triple cuts to determine the coefficients of the triangle integral functions and present, as an example, the full expression for the one-loop amplitude $A^{N=1}(1^-, 2^-, 3^-, 4^+, \ldots, n^+)$.

6.3.1 Relationships between the Box Coefficients of different Supersymmetric Multiplets

We first show that the box coefficients for the three matter contributions are not independent for a certain class of box functions that we refer to as MHV-deconstructible boxes, where the term MHV-deconstructible simply refers to a box integral fuctions that reduces to four MHV tree amplitudes when a BCF quadruple cut is applied. Ultimtely, we will prove that the $N = 0$ coefficient can be derived from the $N = 4$ and $N = 1$ coefficients. For MHV tree amplitudes the contributions from the non-scalar particles can be related to that of the real scalar via Supersymmetric Ward identities [11, 41] and are simply,

$$A^{\text{tree}}((\ell_1)^{\mp}, i_1, \ldots, i_2, (\ell_2)^{\pm}) = (x)^{\pm 2h} A^{\text{tree}}((\ell_1)^{s}, i_1, \ldots, i_2, (\ell_2)^{s}) \,, \qquad (6.83)$$

where $h = 1/2$ for fermions and $h = 1$ for gluons and $x = \langle l_1\, i_a \rangle \,/\, \langle l_2\, i_a \rangle$ with i_a being the negative helicity gluon leg. The contribution to the box coefficient will then be

$$(X)^{2h} \times \text{real scalar contribution} \,, \qquad (6.84)$$

where $X = x_1x_2x_3x_4$, and x_j is the factor from the j-th corner.

When we consider the contribution from a Supersymmetric multiplet to the loop amplitude, we must sum over particle types. For the chiral multiplet the contribution, relative to the real scalar, has a factor

$$\rho^{N=1} = -X + 2 - \frac{1}{X} = -\frac{(X-1)^2}{X}, \tag{6.85}$$

whilst for the $N = 4$ multiplet the factor is

$$\rho^{N=4} = X^2 - 4X + 6 - 4\frac{1}{X} + \frac{1}{X^2} = \frac{(X-1)^4}{X^2} = (\rho^{N=1})^2. \tag{6.86}$$

For $N = 4$ boxes we also have solutions where the two cut legs attached to a corner have the same helicity. Such tree amplitudes are only non-zero if the cut legs are gluons. We refer to such configurations as "singlet" contributions. It is the remaining "non-singlet" contributions which can be obtained from the scalar by applying a factor of $\rho^{N=4}$. We thus have

$$\hat{c}^{N=4\ non-singlet} = \rho^{N=4}\hat{c}^{real\ scalar}, \quad \hat{c}^{N=1\ chiral} = \rho^{N=1}\hat{c}^{real\ scalar}, \tag{6.87}$$

which given that $\rho^{N=4} = (\rho^{N=1})^2$ yields

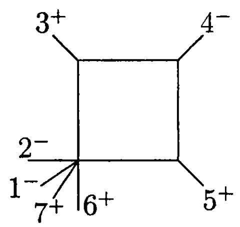
$$\hat{c}^{N=0} = 2\frac{(\hat{c}^{N=1\ chiral})^2}{\hat{c}^{N=4\ non-singlet}}. \tag{6.88}$$

This formula applies to any box which is MHV-deconstructible. It can be used to determine the $N = 0$ (or scalar) coefficient from the two Supersymmetric coefficients provided we have identified the non-singlet contribution in the $N = 4$ case.

Not all box coefficients are MHV-deconstructible. For example in the amplitude

$$A(1^-, 2^-, 3^+, 4^-, 5^+, 6^+, 7^+) \tag{6.89}$$

the box



will have a NMHV corner. The scalar tree amplitude at this corner is of the form

$$\frac{C_1}{K_{671}^2} + \frac{C_2}{K_{712}^2}, \tag{6.90}$$

where $K_{i\ldots j} \equiv (k_i + \ldots + k_j)$ and the amplitudes for other particles types [68, 69] are of the form

$$x_1^h \frac{C_1}{K_{671}^2} + x_2^h \frac{C_2}{K_{712}^2}, \tag{6.91}$$

which leads to box coefficients which are a sum of two terms

$$\hat{c} = \hat{c}_A + \hat{c}_B\,, \tag{6.92}$$

each of which satisfy eq. (6.88) individually,

$$\hat{c}_A^{N=0} = 2\frac{(\hat{c}_A^{N=1\ chiral})^2}{\hat{c}_A^{N=4\ non-singlet}} \quad \text{and} \quad \hat{c}_B^{N=0} = 2\frac{(\hat{c}_B^{N=1\ chiral})^2}{\hat{c}_B^{N=4\ non-singlet}}\,. \tag{6.93}$$

This formula has obvious generalisations to higher point box coefficients.

6.3.2 Example Box Coefficients

In this section we present some specific examples of "MHV-deconstructible" box coefficients. We use color ordered amplitudes throughout and only present the leading in color expression.

As we discussed in section 4.5, there is a choice of representations for the box integral functions. There are scalar box integral functions and F-functions which have zero mass dimension and are related to the former by the removal of the momentum prefactors [8],

$$I_4 = \frac{1}{K} F \,. \tag{6.94}$$

We denote the coefficients of the scalar box functions as $\hat{c}_i$ and those of the F-functions as c_i. Both the $\hat{c}_i$ and c_i satisfy the relations eq. (6.88).

In all cases we present the $N = 4$, $N = 1$ and $N = 0$ results. For the $N = 4$ case the results are generally already known [8, 9, 51, 71] whilst the six-point $N = 1$ box coefficients appear in [15].

MHV box coefficients

Consider the case of MHV amplitudes where all box coefficients are known and we may check the relationship eq. (6.88). In general, the box functions are "two-mass-easy" boxes and single mass boxes. The $N = 4$ non-singlet terms occur where there is a single negative helicity leg in each massive corner. The $N = 4$ amplitude was calculated in [8] and the $N = 1$ in [9] (the five-point amplitude appeared earlier in [34]) whilst the $N = 0$ coefficient was computed by Bedford, Brandhuber, Spence and Travaglini [72]. Denoting the two negative helicities as i and j and considering the box with two massless legs m_1 and m_2, the coefficients of the F-functions are

$$c^{N=4} \;=\; A^{\text{tree}} \times 1 \,,$$

$$\begin{aligned} c^{N=1} &= A^{\text{tree}} \times \frac{b^{ij}_{m_1 m_2}}{2}, \\ c^{N=0} &= A^{\text{tree}} \times \frac{(b^{ij}_{m_1 m_2})^2}{2}, \end{aligned} \tag{6.95}$$

where

$$b^{ij}_{m_1 m_2} = 2\frac{\langle i\, m_1\rangle\,\langle i\, m_2\rangle\,\langle j\, m_1\rangle\,\langle j\, m_2\rangle}{\langle i\, j\rangle^2\,\langle m_1\, m_2\rangle^2}, \tag{6.96}$$

and we use spinor inner-products, $\langle j\, l\rangle \equiv \langle j^-|l^+\rangle$, $[j\, l] \equiv \langle j^+|l^-\rangle$. Clearly these amplitudes satisfy the relation eq. (6.88).

Six-point NMHV box coefficients

All boxes for the six-point amplitudes are MHV-deconstructible and the box coefficients are known for both $N=4$ and $N=1$ [9, 15], so we can apply eq. (6.88) to generate the coefficients of the scalar boxes. The amplitudes with all-positive helicity legs and those with one-negative helicity leg are non-zero in non-Supersymmetric theories, however these amplitudes are rational functions with no scalar box contributions. Thus, the two independent amplitudes with non-vanishing box coefficients are the MHV case (or $\overline{\text{MHV}}$), which was covered in the previous section, and the NMHV case with three negative helicities.

There are three independent amplitudes with three negative helicity legs: $A(1^-,2^-,3^-,4^+,5^+,6^+)$, $A(1^-,2^-,3^+,4^-,5^+,6^+)$ and $A(1^-,2^+,3^-,4^+,5^-,6^+)$. Of these, the first has vanishing box coefficients for $N=1$ and $N=0$ [14],

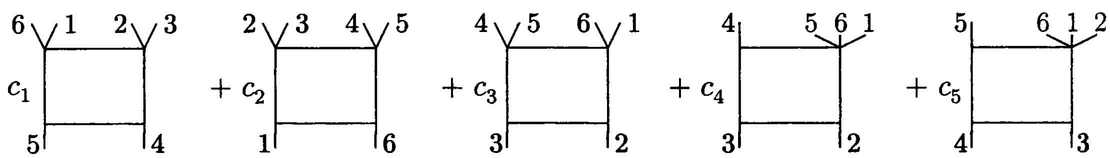
$$A^{N=0,1}(1^-,2^-,3^-,4^+,5^+,6^+)|_{\text{box}} = 0. \tag{6.97}$$

The $N=4$ amplitude only has singlet contributions in this case.

The second amplitude, $A(1^-,2^-,3^+,4^-,5^+,6^+)$, does have a non-trivial box structure,

$$A(1^-,2^-,3^+,4^-,5^+,6^+)|_{\rm box} = c_1 F^{\rm 2mh}_{4:4} + c_2 F^{\rm 2mh}_{4:6} + c_3 F^{\rm 2mh}_{4:2} + c_4 F^{\rm 1m}_{4:2} + c_5 F^{\rm 1m}_{4:3}, \tag{6.98}$$

which is depicted



Of these coefficients, only three are truly independent, since under flipping, conjugation and relabelling,

$$c_1 \leftrightarrow c_3\,, \qquad c_4 \leftrightarrow c_5\,. \tag{6.99}$$

Explicitly the independent box coefficients are,

$$\begin{aligned}
c_1^{N=4,\,non-singlet} &= i\frac{\langle 3^+|\not{K}|1^+\rangle^4}{[2\,3]\,[3\,4]\,\langle 5\,6\rangle\,\langle 6\,1\rangle\,\langle 2^+|\not{K}|5^+\rangle\langle 4^+|\not{K}|1^+\rangle K^2}\,,\\
c_1^{N=1\,chiral} &= i\frac{\langle 5\,1\rangle\,\langle 3^+|\not{K}|1^+\rangle^2\langle 3^+|\not{K}|5^+\rangle}{[2\,3]\,\langle 5\,6\rangle\,\langle 6\,1\rangle\,\langle 2^+|\not{K}|5^+\rangle\langle 4^+|\not{K}|5^+\rangle^2}, \qquad K = K_{234},\\
c_1^{N=0} &= 2i\frac{\langle 1\,5\rangle^2\,[3\,4]\,\langle 3^+|\not{K}|5^+\rangle^2\langle 4^+|\not{K}|1^+\rangle K^2}{[2\,3]\,\langle 5\,6\rangle\,\langle 6\,1\rangle\,\langle 2^+|\not{K}|5^+\rangle\langle 4^+|\not{K}|5^+\rangle^4}\,,
\end{aligned} \tag{6.100}$$

$$\begin{aligned}
c_2^{N=4,\,non-singlet} &= i\frac{\langle 3^+|\not{K}|4^+\rangle^4}{[1\,2]\,[2\,3]\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 1^+|\not{K}|4^+\rangle\langle 3^+|\not{K}|6^+\rangle K^2}\,,\\
c_2^{N=1\,chiral} &= i\frac{[3\,1]\,\langle 6\,4\rangle\,\langle 3^+|\not{K}|4^+\rangle^2}{[1\,2]\,[2\,3]\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 1^+|\not{K}|6^+\rangle^2}, \qquad K = K_{123},\\
c_2^{N=0} &= 2i\frac{[3\,1]^2\,\langle 6\,4\rangle^2\,\langle 1^+|\not{K}|4^+\rangle\langle 3^+|\not{K}|6^+\rangle K^2}{[1\,2]\,[2\,3]\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 1^+|\not{K}|6^+\rangle^4}\,,
\end{aligned} \tag{6.101}$$

$$\begin{aligned}
c_5^{N=4,\,non-singlet} &= i\frac{\langle 6^+|\not{K}|4^+\rangle^4}{[6\,1]\,[1\,2]\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 6^+|\not{K}|3^+\rangle\langle 2^+|\not{K}|5^+\rangle K^2},\\
c_5^{N=1\,chiral} &= i\frac{\langle 6^+|\not{K}|4^+\rangle^2\langle 6^+|\not{K}|5^+\rangle}{[6\,1]\,[1\,2]\,\langle 3\,5\rangle^2\,\langle 2^+|\not{K}|5^+\rangle K^2}, \qquad K=K_{345},\\
c_5^{N=0} &= 2i\frac{\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 6^+|\not{K}|5^+\rangle^2\langle 6^+|\not{K}|3^+\rangle^2}{\langle 3\,5\rangle^4\,[6\,1]\,[1\,2]\,\langle 2^+|\not{K}|5^+\rangle K^2}.
\end{aligned} \tag{6.102}$$

The remaining amplitude, $A(1^-,2^+,3^-,4^+,5^-,6^+)$, contains all six one mass and all six "two mass-hard" boxes,

$$A(1^-,2^+,3^-,4^+,5^-,6^+)_{\rm box} = \sum_{i=1}^{6} a_i I_{4:i}^{1\rm m} + \sum_{i=1}^{6} b_i I_{4:i}^{2{\rm m}\,h} \tag{6.103}$$

These are not all independent and symmetry demands relationships amongst the a_i's,

$$\begin{aligned}
a_3(123456) &= a_1(345612), \quad a_5(123456) = a_1(561234),\\
a_4(123456) &= a_2(345612), \quad a_6(123456) = a_2(561234),\\
a_2(123456) &= \bar{a}_1(234561), \quad a_1(123456) = a_1(321654),
\end{aligned} \tag{6.104}$$

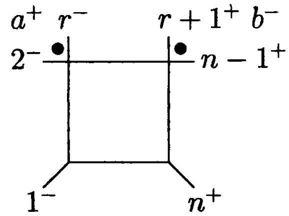
where $\bar{a}_1$ denotes a_1 with $\langle i\,j\rangle \leftrightarrow [i\,j]$. Thus there is a single independent a_i. Similarly we can use symmetry to generate all the b_i's from b_2. The expressions for a_1 and b_2 are,

$$\begin{aligned}
a_1^{N=4,\,non-singlet} &= i\frac{\langle 2^+|\not{K}|5^+\rangle^4}{[1\,2]\,[2\,3]\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 1^+|\not{K}|4^+\rangle\langle 3^+|\not{K}|6^+\rangle K^2},\\
a_1^{N=1,\,chiral} &= i\frac{\langle 2^+|\not{K}|5^+\rangle^2\langle 1^+|\not{K}|5^+\rangle\langle 3^+|\not{K}|5^+\rangle}{[1\,3]^2\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 1^+|\not{K}|4^+\rangle\langle 3^+|\not{K}|6^+\rangle K^2}, \qquad K=K_{123},\\
a_1^{N=0} &= 2i\frac{[1\,2]\,[2\,3]\,\langle 1^+|\not{K}|5^+\rangle^2\langle 3^+|\not{K}|5^+\rangle^2}{[1\,3]^4\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 1^+|\not{K}|4^+\rangle\langle 3^+|\not{K}|6^+\rangle K^2},
\end{aligned} \tag{6.105}$$

$$\begin{aligned}
b_2^{N=4,\, non-singlet} &= i\frac{\langle 2^+|\not{K}|5^+\rangle^4}{[1\,2]\,[2\,3]\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 1^+|\not{K}|4^+\rangle\langle 3^+|\not{K}|6^+\rangle K^2}\,, \\
b_2^{N=1,\, chiral} &= i\frac{\langle 2^+|\not{K}|5^+\rangle^2\langle 3^+|\not{K}|5^+\rangle\langle 2^+|\not{K}|4^+\rangle}{[1\,2]\,\langle 5\,6\rangle\,\langle 3^+|\not{K}|6^+\rangle\langle 1^+|\not{K}|4^+\rangle\langle 3^+|\not{K}|4^+\rangle^2}\,, \qquad K = K_{123}. \\
b_2^{N=0} &= 2i\frac{[2\,3]\,\langle 4\,5\rangle\,\langle 3^+|\not{K}|5^+\rangle^2\langle 2^+|\not{K}|4^+\rangle^2 K^2}{[1\,2]\,\langle 5\,6\rangle\,\langle 3^+|\not{K}|6^+\rangle\langle 1^+|\not{K}|4^+\rangle\langle 3^+|\not{K}|4^+\rangle^4}\,. \qquad (6.106)
\end{aligned}$$

Two Mass-Hard Box

As an n-point example, we can consider the coefficient of the following box function,



which has two massless corners, a corner with a single external positive helicity leg and a corner with a single external negative helicity leg. This box is thus MHV-deconstructible and can be computed using quadruple cuts and the technique of Britto, Cachazo and Feng [56].

Solving for the box coefficients we find

$$\rho_{N=1} = -\frac{\langle 1^+|\not{K}|n^+\rangle^2 \langle a^+|\not{K}|b^+\rangle^2}{K^2\,[a\,1]\,\langle n\,b\rangle\,(K^2\,[a\,1]\,\langle n\,b\rangle - \langle 1^+|\not{K}|n^+\rangle\langle a^+|\not{K}|b^+\rangle)}, \tag{6.107}$$

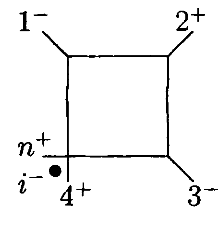
where $K = K_{1\dots r}$ and the box coefficients are given by,

$$\begin{aligned}
c^{N=4\,non-singlet} &= i\frac{s_{n1}\langle a^+|\not{K}|b^+\rangle^4}{[1\,2]\dots[r-1\,r]\,\langle r+1\,r+2\rangle\dots\langle n-1\,n\rangle\,\langle 1^+|\not{K}|r+1^+\rangle\langle r^+|\not{K}|n^+\rangle},\\
c^{N=1\,chiral} &= i\frac{[a\,1]\,\langle b\,n\rangle\,\left(K^2\,[a\,1]\,\langle n\,b\rangle - \langle 1^+|\not{K}|n^+\rangle\langle a^+|\not{K}|b^+\rangle\right)s_{n1}(K^2)\langle a^+|\not{K}|b^+\rangle^2}{[1\,2]\,..\,[r-1\,r]\,\langle r+1\,r+2\rangle\,..\,\langle n-1\,n\rangle\,\langle 1^+|\not{K}|n^+\rangle^2\langle 1^+|\not{K}|r+1^+\rangle\langle r^+|\not{K}|n^+\rangle},\\
c^{N=0} &= 2i\frac{[a\,1]^2\,\langle b\,n\rangle^2\left(K^2\,[a\,1]\,\langle n\,b\rangle - \langle 1^+|\not{K}|n^+\rangle\langle a^+|\not{K}|b^+\rangle\right)^2 s_{n1}(K^2)^2}{[1\,2]\dots[r-1\,r]\,\langle r+1\,r+2\rangle\dots\langle n-1\,n\rangle\,\langle 1^+|\not{K}|n^+\rangle^4\langle 1^+|\not{K}|r+1^+\rangle\langle r^+|\not{K}|n^+\rangle}.
\end{aligned} \tag{6.108}$$

The One Mass Boxes

For a one mass box, adjacent massless legs must have opposite helicity [56] to yield a non-vanishing result. Using parity we need only consider the case where the massive corner is mostly positive. The case where exactly two of the massless legs have positive helicity is just the MHV case considered previously.

The remaining case where exactly two of the massless legs have negative helicity is a contribution to the NMHV amplitudes. Specifically we have the one mass scalar box:



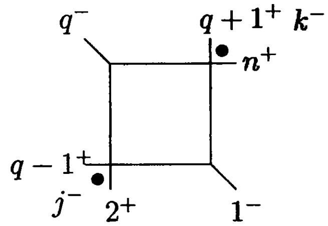
Using the quadruple cuts we can determine the coefficients in the three cases,

$$\begin{aligned}
c^{N=4,\ non-singlet} &= i\frac{\langle 2^+|\not{K}|i^+\rangle^4}{[1\,2]\,[2\,3]\,\langle 4\,5\rangle \ldots \langle i\,i+1\rangle \ldots \langle n-1\,n\rangle\,\langle 1^+|\not{K}|4^+\rangle\langle 3^+|\not{K}|n^+\rangle K^2},\\
c^{N=1\ chiral} &= i\frac{\langle 2^+|\not{K}|i^+\rangle^2\langle 1^+|\not{K}|i^+\rangle\langle 3^+|\not{K}|i^+\rangle}{[1\,3]^2\,\langle 4\,5\rangle \ldots \langle i\,i+1\rangle \ldots \langle n-1\,n\rangle\,\langle 1^+|\not{K}|4^+\rangle\langle 3^+|\not{K}|n^+\rangle K^2},\\
c^{N=0} &= 2i\frac{[1\,2]\,[2\,3]\,\langle 1^+|\not{K}|i^+\rangle^2\langle 3^+|\not{K}|i^+\rangle^2}{[1\,3]^4\,\langle 4\,5\rangle \ldots \langle i\,i+1\rangle \ldots \langle n-1\,n\rangle\,\langle 1^+|\not{K}|4^+\rangle^2\langle 3^+|\not{K}|n^+\rangle^2 K^2}.
\end{aligned} \tag{6.109}$$

with $K = K_{123}$.

The Two Mass-Easy Boxes

In the case of two mass easy boxes, there are no solutions to the kinematic constraints if the massless legs have opposite helicity, so $c^{N=0}$, $c^{N=1,\ chiral}$ and $c^{N=4,\ non-singlet}$ vanish for such configurations. As an example of a non-vanishing two mass easy box we consider the box below, which has a single negative helicity leg at each corner.



Setting, $K_2 = k_2 + k_3 + .. + k_j + .. + k_{q-1}$ and $K_4 = k_{q+1} + .. + k_k + .. + k_n$, we find,

$$
\begin{aligned}
c^{N=4,\, non-singlet} &= \frac{i}{\mathcal{D}} \langle j^- | K_2 K_4 | k^- \rangle^4 , \\
c^{N=1\, chiral} &= -\frac{i}{\mathcal{D}} \frac{\langle q^+ | \not{K}_2 | j^+ \rangle \langle 1^+ | \not{K}_2 | j^+ \rangle \langle 1^+ | \not{K}_4 | k^+ \rangle \langle q^+ | \not{K}_4 | k^+ \rangle \langle j^- | K_2 K_4 | k^+ \rangle^2}{[1\, q]^2} , \\
c^{N=0} &= 2\frac{i}{\mathcal{D}} \frac{\langle q^+ | \not{K}_2 | j^+ \rangle^2 \langle 1^+ | \not{K}_2 | j^+ \rangle^2 \langle 1^+ | \not{K}_4 | k^+ \rangle^2 \langle q^+ | \not{K}_4 | k^+ \rangle^2}{[1\, q]^4} ,
\end{aligned}
\tag{6.110}
$$

where,

$$
\begin{aligned}
\mathcal{D} = \; K_2^2 K_4^2 \; & \langle q^+ | \not{K}_2 | 2^+ \rangle \langle 1^+ | \not{K}_2 | q-1^+ \rangle \langle 1^+ | \not{K}_4 | q+1^+ \rangle \langle q^+ | \not{K}_4 | n^+ \rangle \\
& \times \langle 2\, 3 \rangle \langle 3\, 4 \rangle .. \langle q-2\, q-1 \rangle \langle q+1\, q+2 \rangle \langle q+2\, q+3 \rangle .. \langle n-1\, n \rangle .
\end{aligned}
\tag{6.111}
$$

6.3.3 Twistor Related Properties of Box Coefficients

The results for the twistor structure of the box coefficients are relatively simple. We find that the box coefficients within the MHV amplitudes have collinear support in twistor space

$$
F_{ijk} c^{N=4\, MHV} = F_{ijk} c^{N=1\, MHV} = F_{ijk} c^{N=0\, MHV} = 0 , \tag{6.112}
$$

while box coefficients within NMHV amplitudes have coplanar support

$$K_{ijkl} c^{N=4\,NMHV} = K_{ijkl} c^{N=1\,NMHV} = K_{ijkl} c^{N=0\,NMHV} = 0\,, \tag{6.113}$$

in twistor space. The coplanarity of the box coefficients for the $N = 4$ amplitudes was shown in [51, 66]. It was verified for the $N = 1$ box coefficients in [15].

In the generic NMHV case, where we have a three mass box, the legs will have support upon three intersecting lines in twistor space, with the legs at each massive corner being collinear. The geometric picture of this is identical to that of $N = 4$ [71].

6.3.4 Triangles from Triple Cuts

To obtain the coefficients of triangle integral functions we consider triple cuts [58]. This corresponds to inserting three $\delta(\ell_i^2)$ functions into the four dimensional integrals. Specifically we consider,

$$\begin{aligned}&\int d^4\ell_1 d^4\ell_2 d^4\ell_3 \delta^4(\ell_1 - \ell_2 - K_1)\delta^4(\ell_2 - \ell_3 - K_2)\delta(\ell_1^2)\delta(\ell_2^2)\delta(\ell_3^2)\\ &\times A^{\rm tree}(\ell_1, k_1, \ldots, k_r, \ell_2) A^{\rm tree}(-\ell_2, k_{r+1}, \ldots, k_{r'}, \ell_3) A^{\rm tree}(-\ell_3, k_{r'+1}, \ldots, k_n, -\ell_1)\,.\end{aligned} \tag{6.114}$$

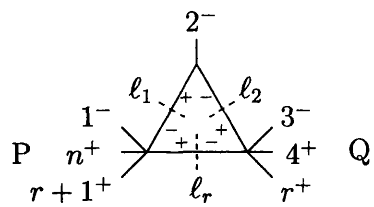
Both triangle functions and box functions contribute to this triple cut. As a strategy, one can first determine the box coefficients from quadruple cuts and then subtract these from the triple cut to obtain the triangle coefficients. Unlike the quadruple cuts case, the three $\delta(\ell_i^2)$ functions do not freeze the integral, so we must carry out manipulations within the cut integral to recognise the coefficient.

As an example application of triple cuts, consider the amplitude

$$A^{N=1}(1^-, 2^-, 3^-, 4^+, 5^+, \ldots, n^+)\,. \tag{6.115}$$

This amplitude is particularly amenable in that it contains no box integral functions. This can be seen by examining the integrals in a two-particle cut [14] or, fairly obviously, by observing that there are no solutions to the quadruple cuts.

Consider the following triple cut:



with the momenta on the two massive legs being $P \equiv k_{r+1} + \ldots + k_n + k_1$ and $Q \equiv k_3 + k_4 + \ldots + k_r$. Within the cut integral, where the cut legs are scalars, the product of the three tree amplitudes is

$$\frac{\langle 1\,\ell_1\rangle^2 \langle 1\,\ell_r\rangle^2}{\langle r+1\,r+2\rangle \ldots \langle n\,1\rangle \langle 1\,\ell_1\rangle \langle \ell_1\,\ell_r\rangle \langle \ell_r\,r+1\rangle} \times \frac{\langle 3\,\ell_2\rangle^2 \langle 3\,\ell_r\rangle^2}{\langle 3\,4\rangle \ldots \langle r-1\,r\rangle \langle r\,\ell_r\rangle \langle \ell_r\,\ell_2\rangle \langle \ell_2\,3\rangle}$$
$$\times \frac{\langle 2\,\ell_1\rangle \langle 2\,\ell_2\rangle}{\langle \ell_1\,\ell_2\rangle}\,. \qquad (6.116)$$

To obtain the contribution from the $N = 1$ multiplet we must multiply this by $\rho^{\,N=1}$ within the integral. Using

$$\frac{1}{\langle \ell_1\,\ell_r\rangle} = \frac{[\ell_1\,\ell_r]}{P^2}\,, \quad \frac{1}{\langle \ell_2\,\ell_r\rangle} = \frac{[\ell_2\,\ell_r]}{Q^2}\,, \quad \text{and} \quad \frac{1}{\langle r\,\ell_r\rangle} = \frac{[\ell_r\,2]}{\langle r\,\ell_r\rangle\,[\ell_r\,2]} = \frac{[\ell_r\,2]}{\langle 2^+|\not{P}|r^+\rangle}, \qquad (6.117)$$

this product can be rearranged to give

$$\frac{F[\ell_i] \times \rho^{\,N=1}}{\langle 2^+|P|r^+\rangle\langle 2^+|P|r+1^+\rangle \langle 3\,4\rangle \ldots \langle r-1\,r\rangle \langle r+1\,r+2\rangle \ldots \langle n\,1\rangle\, P^2 Q^2 \langle \ell_1\,\ell_2\rangle}, \qquad (6.118)$$

where much of the denominator can now be taken outside the cut integral and

$$F[\ell_i] = \langle 1\,\ell_1\rangle\,\langle 1\,\ell_r\rangle^2\,\langle 3\,\ell_2\rangle\,\langle 3\,\ell_r\rangle^2\,\langle 2\,\ell_1\rangle\,\langle 2\,\ell_2\rangle\,[\ell_2\,\ell_r]\,[\ell_1\,\ell_r]\,[2\,\ell_r]^2\,. \tag{6.119}$$

When combining the different particles' contributions we have

$$X = \frac{\langle 1\,\ell_1\rangle}{\langle 1\,\ell_r\rangle}\times\frac{\langle 2\,\ell_2\rangle}{\langle 2\,\ell_1\rangle}\times\frac{\langle 3\,\ell_r\rangle}{\langle 3\,\ell_2\rangle},\quad \text{so that}\ \rho^{N=1} = \frac{\big(\langle 1\,\ell_1\rangle\,\langle 2\,\ell_2\rangle\,\langle 3\,\ell_r\rangle - \langle 1\,\ell_r\rangle\,\langle 2\,\ell_1\rangle\,\langle 3\,\ell_2\rangle\big)^2}{\langle 1\,\ell_1\rangle\,\langle 1\,\ell_r\rangle\,\langle 2\,\ell_2\rangle\,\langle 2\,\ell_1\rangle\,\langle 3\,\ell_r\rangle\,\langle 3\,\ell_2\rangle}. \tag{6.120}$$

Thus the loop momentum dependent part of the integrand is

$$\frac{F[\ell_i]\rho^{N=1}}{\langle \ell_1\,\ell_2\rangle} = \frac{\langle 1\,\ell_r\rangle\,\langle 3\,\ell_r\rangle\,[\ell_2\,\ell_r]\,[\ell_1\,\ell_r]\,[2\,\ell_r]^2\,\big(\langle 1\,\ell_1\rangle\,\langle 2\,\ell_2\rangle\,\langle 3\,\ell_r\rangle - \langle 1\,\ell_r\rangle\,\langle 2\,\ell_1\rangle\,\langle 3\,\ell_2\rangle\big)^2}{\langle \ell_1\,\ell_2\rangle}. \tag{6.121}$$

To evaluate this we use the identity

$$\big(\langle 1\,\ell_1\rangle\,\langle 2\,\ell_2\rangle\,\langle 3\,\ell_r\rangle - \langle 1\,\ell_r\rangle\,\langle 2\,\ell_1\rangle\,\langle 3\,\ell_2\rangle\big) = (\langle 3^-|QP|1^+\rangle)\frac{\langle \ell_1\,\ell_2\rangle}{[2\,\ell_r]}\,, \tag{6.122}$$

which is valid due to the momentum constraints. The part of the integrand which still depends on the loop momentum can be rearranged

$$\begin{aligned}\langle 1\,\ell_r\rangle\,\langle 3\,\ell_r\rangle\,[\ell_2\,\ell_r]\,[\ell_1\,\ell_r]\,\langle \ell_1\,\ell_2\rangle &= \langle 1\,\ell_r\rangle\,[\ell_r\,\ell_1]\,\langle \ell_1\,\ell_2\rangle\,[\ell_2\,\ell_r]\,\langle \ell_r\,3\rangle \\ &= \langle 1^-|\not\ell_r\not\ell_1\not\ell_2\not\ell_r|3^+\rangle = \langle 1^-|\not P\not\ell_1\not\ell_2\not Q|3^+\rangle\,,\end{aligned} \tag{6.123}$$

using $\not{\ell}_r = \not{\ell}_1 + \not{P}$, and $\not{\ell}_r = \not{\ell}_2 - \not{Q}$. Finally we can reduce this to a linear function by using $\not{\ell}_1 = \not{\ell}_2 + \not{k}_2$,

$$\frac{1}{2}\langle 1^-|\not{P}\Big(\not{k}_2\not{\ell}_2 - \not{\ell}_1\not{k}_2\Big)\not{Q}|3^+\rangle\,, \tag{6.124}$$

where we chose to perform the algebra in such a way as to reflect the symmetry of the diagram: this facilitates the identification of the triangle coefficients. To solve this triangle we first Feynman parameterise and make a shift of momenta

$$\ell_1^\mu \longrightarrow \ell_1^{\mu\prime} - k_2^\mu a_3 - (k_2+Q)^\mu a_{r+1} \quad \ell_2^\mu \longrightarrow \ell_1^{\mu\prime} - k_2^\mu a_3 - (k_2+Q)^\mu a_{r+1} - k_2^\mu\,. \tag{6.125}$$

leading to

$$\frac{1}{2}\langle 1^-|\not{P}\Big(\not{k}_2\not{Q} - \not{Q}\not{k}_2\Big)\not{Q}|3^+\rangle \times a_{r+1}\,. \tag{6.126}$$

Finally, the Feynman parameter integral $I[a_{r+1}]$ can be expressed in terms of the L_0 functions

$$I[a_{r+1}] = \frac{\mathrm{L}_0[P^2/Q^2]}{Q^2}\,. \tag{6.127}$$

where we use the integral functions defined in eq. (6.1).

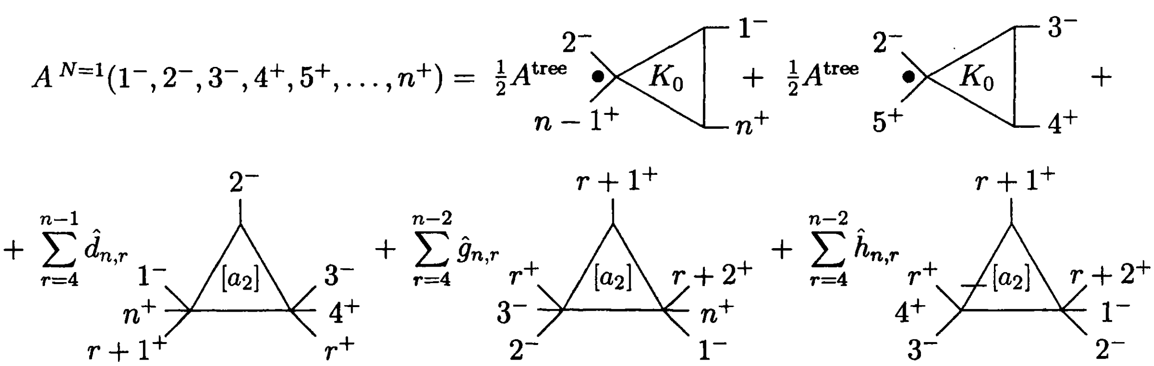
From the triple cut we can now identify the coefficient of the L_0 triangle function as,

$$\frac{(\langle 3^-|QP|1^+\rangle)^2\langle 3^-|(Q(2P-P2)P)|1+\rangle}{\langle 2^+|P|r^+\rangle\langle 2^+|P|r+1^+\rangle\,\langle 3\,4\rangle\ldots\langle r-1\,r\rangle\,\langle r+1\,r+2\rangle\ldots\langle n\,1\rangle\,P^2Q^2}\,. \tag{6.128}$$

Similarly, we can determine all the triangle functions present in the amplitude using triplet cuts, obtaining the expression for the full amplitude

$$
\begin{aligned}
A^{N=1}(1^-,2^-,3^-,4^+,5^+,\cdots,n^+) \;=\; & \frac{A^{\rm tree}}{2}\left(\mathrm{K}_0(s_{n1})+\mathrm{K}_0(s_{34})\right) - \frac{i}{2}\sum_{r=4}^{n-1}\hat{d}_{n,r}\frac{\mathrm{L}_0[t_3^{[r-2]}/t_2^{[r-1]}]}{t_2^{[r-1]}} \\
& -\frac{i}{2}\sum_{r=4}^{n-2}\hat{g}_{n,r}\frac{\mathrm{L}_0[t_2^{[r-1]}/t_2^{[r]}]}{t_2^{[r]}} - \frac{i}{2}\sum_{r=4}^{n-2}\hat{h}_{n,r}\frac{\mathrm{L}_0[t_3^{[r-2]}/t_3^{[r-1]}]}{t_3^{[r-1]}}\,,
\end{aligned}
\tag{6.129}
$$

which can be depicted in the following way,



where,

$$
\begin{aligned}
\hat{d}_{n,r} &= \frac{(\langle 3^-|K_{r-3}\overline{K}_{r-3}|1^+\rangle)^2\langle 3^-|K_{r-3}(k_2\overline{K}_{r-3}-\overline{K}_{r-3}k_2)\overline{K}_{r-3})|1^+\rangle}{\langle 2^+|\overline{K}_{r-3}|r^+\rangle\langle 2^+|\overline{K}_{r-3}|r+1^+\rangle\,\langle 3\,4\rangle\ldots\langle r-1\,r\rangle\,\langle r+1\,r+2\rangle\ldots\langle n\,1\rangle\,\overline{K}_{r-3}^2K_{r-3}^2}\,, \\
\hat{g}_{n,r} &= \sum_{i=1}^{r-3}\frac{\langle 3^-|K_i\overline{K}_i|1^+\rangle^2\langle 3^-|K_i\overline{K}_i(k_{r+1}\overline{K}_{r-3}-\overline{K}_{r-3}k_{r+1})|1^+\rangle\,\langle i+3\,i+4\rangle}{\langle 2^+|\not{K}_i|i+3^+\rangle\langle 2^+|\not{K}_i|i+4^+\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\ldots\langle n\,1\rangle\,K_i^2\overline{K}_i^2}\,, \\
\hat{h}_{n,r} &= \hat{g}_{n,n-r+2}\big|_{(123..n)\to(321n..4)}\,,
\end{aligned}
\tag{6.130}
$$

with $K_i = k_3+k_4+..+k_{i+3}$ and $\overline{K}_i = k_2+k_3+..+k_{i+3}$. We have checked that this expression satisfies the correct collinear limits.

We find that application of the Collinear and Coplanar operators to these triangle coefficients exposes no obvious twistor structure. This is consistent with the general understanding of the twistor structure of triangle coefficients. They satisfy differential equations which suggests there is no simple interpretation with respect to their twistor structure. This leads us to the conclusion that the twistor space structure exhibited in $N = 4$ loop amplitudes extends only to the box coefficients of $N = 1$ one-loop amplitudes.

Chapter 7

$N = 4$ Fermionic Amplitudes

On-shell Supersymmetric Ward Identities (SWI) impose powerful constraints on amplitudes in gauge theories, giving algebraic relations between amplitudes with the same helicity configuration but different external particle types. These constraints apply at any order in perturbation theory. From a Feynman diagram perspective, these relationships are most naturally employed to obtain purely gluonic amplitudes from amplitudes involving fermions. Motivated by the recent advances in calculating purely gluonic amplitudes, we reverse this process and generate amplitudes involving fermions from the purely gluonic ones.

In particular, in this chapter we focus on NMHV one-loop amplitudes. As we discussed in chapter 2, application of the SWI for these NMHV amplitudes results in a system that has rank 2. Thus it would appear that we cannot solve a SWI set for NHMV amplitudes unambiguously. However, we show how the SWI can be solved in a natural way to obtain amplitudes with two gluinos in terms of the purely gluonic case.

We first apply this to six-point tree amplitudes where we can compare the results to known computations. Secondly we determine the one-loop six-point NMHV amplitudes in $N = 4$ Supersymmetric Yang-Mills theory which involve two gluinos. More generally there also exist SWI which involve amplitudes with two gluinos, four gluinos, two scalars and two gluinos plus a scalar. We explicitly determine the two scalar amplitudes. The SWI then give the remaining amplitudes directly in terms of already known amplitudes.

We then extend this principle of applying SWI to NMHV amplitudes to include n-point $N = 4$ NMHV one-loop amplitudes, where we exploit the fact that one-loop NMHV amplitudes in $N = 4$ gauge theory can be expressed in terms of MHV-deconstructible diagrams and so can be evaluated using quadruple cuts and known MHV tree amplitudes. We use the SWI to minimise the number of independent diagrams that must be computed explicitly. We use these techniques to determine a set of conversion factors that relate two-gluino box coefficients to purely gluonic ones. Analysis of quadruple cuts is then used to show how these factors can be compounded to give two-scalar and scalar-gluino-gluino box coefficients. Amplitudes involving more external fermions and scalars then follow from the appropriate SWI.

7.1 SWI and NMHV Amplitudes involving Gluinos

We show how Supersymmetric Ward Identities can be used to obtain amplitudes involving gluinos or adjoint scalars from purely gluonic amplitudes. We obtain results for all one-loop six-point NMHV amplitudes in $N = 4$ Super Yang-Mills theory which involve two gluinos or two scalar particles. Additionally, more general cases are also discussed.

7.1.1 Six-point NMHV Tree Amplitudes

In this section we demonstrate how to generate tree amplitudes involving two gluinos from purely gluonic tree amplitudes and then compare these to the known expressions [73, 74] which themselves agree with the Feynman diagram computations [75]. For color ordered gluonic tree amplitudes there are three independent NMHV helicity configurations. When we consider amplitudes with two fermions and four gluons there are considerably more depending on the position of the two fermions. We restrict ourselves to only consider adjoint fermions (gluinos).

We first consider amplitudes derived from the gluonic amplitude [67],

$$\begin{aligned} A_6^{\rm tree}(g_1^-, g_2^-, g_3^-, g_4^+, g_5^+, g_6^+) &= \frac{i\langle 4|K_{234}|1\rangle^3}{t_{234}\,[2\,3]\,[3\,4]\,\langle 5\,6\rangle\,\langle 6\,1\rangle\,\langle 2|K_{234}|5\rangle} \\ &+ \frac{i\langle 6|K_{345}|3\rangle^3}{t_{612}\,[6\,1]\,[1\,2]\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 2|K_{345}|5\rangle}, \end{aligned} \tag{7.1}$$

where, $\langle A|K_{abc}|B\rangle \equiv \langle A^+|\not{k}_a + \not{k}_b + \not{k}_c|B^+\rangle = [A\,a]\,\langle a\,B\rangle + [A\,b]\,\langle b\,B\rangle + [A\,c]\,\langle c\,B\rangle$. The amplitudes involving two fermions which are related to this purely gluonic amplitude can be obtained by conjugation, relabelling and flipping (i.e. $A(1234556) = A(654321)$) from the following four,

$$A_6^{\rm tree}(\Lambda_1^-, g_2^-, g_3^-, \bar{\Lambda}_4^+, g_5^+, g_6^+), \qquad A_6^{\rm tree}(g_1^-, \Lambda_2^-, g_3^-, \bar{\Lambda}_4^+, g_5^+, g_6^+), \tag{7.2}$$

$$A_6^{\rm tree}(g_1^-, g_2^-, \Lambda_3^-, \bar{\Lambda}_4^+, g_5^+, g_6^+), \qquad A_6^{\rm tree}(g_1^-, \Lambda_2^-, g_3^-, g_4^+, \bar{\Lambda}_5^+, g_6^+)\,. \tag{7.3}$$

The SWI relating the first three of these amplitudes to the gluonic amplitude is given by,

$$\begin{aligned} 0 &= \langle 1\,\eta\rangle\, A_n(\Lambda_1^-, g_2^-, g_3^-, \bar{\Lambda}_4^+, g_5^+, g_6^+) + \langle 2\,\eta\rangle\, A_n(g_1^-, \Lambda_2^-, g_3^-, \bar{\Lambda}_4^+, g_5^+, g_6^+) \\ &+ \langle 3\,\eta\rangle\, A_n(g_1^-, g_2^-, \Lambda_3^-, \bar{\Lambda}_4^+, g_5^+, g_6^+) - \langle 4\,\eta\rangle\, A_n(g_1^-, g_2^-, g_3^-, g_4^+, g_5^+, g_6^+). \end{aligned} \tag{7.4}$$

As we discussed in chapter 2, this SWI set has rank 2 and hence, in principle, is not sufficient to determine the fermionic amplitudes in terms of the gluonic. However, when we utilise their inherent symmetries, we can unambiguously determine these fermionic amplitudes. The basic idea is to look for identities of the form,

$$A\,\langle 1\,\eta\rangle + B\,\langle 2\,\eta\rangle + C\,\langle 3\,\eta\rangle - D\,\langle 4\,\eta\rangle = 0\,, \tag{7.5}$$

where the form of D is motived by the terms in the numerator of the compact expressions for the gluonic tree amplitudes eq. (7.1). We shall search for solutions where A, B and C are polynomial in the spinor invariants $\langle i\,j\rangle$ and $[i\,j]$, so that the gluino amplitudes are free from spurious singularities and poles.

Equation (7.1) contains two terms which we examine individually. Writing the second term as $\langle 6|K_{612}|3\rangle X$ and focusing on the the $\langle 6|K_{612}|3\rangle$ factor, the Schouten identity yields,

$$\begin{aligned}\langle 6|K_{612}|3\rangle\,\langle 4\,\eta\rangle &= -\langle 6|K_{612}|\eta\rangle\,\langle 3\,4\rangle + \langle 6|K_{612}|4\rangle\,\langle 3\,\eta\rangle \\ &= \langle 6|K_{612}|4\rangle\,\langle 3\,\eta\rangle - [6\,1]\,\langle 3\,4\rangle\,\langle 1\,\eta\rangle - [6\,2]\,\langle 3\,4\rangle\,\langle 2\,\eta\rangle\,. \qquad (7.6)\end{aligned}$$

This implies that the following are solutions of the SWI eq. (7.4),

$$\begin{aligned}A_6^{\rm tree}(\Lambda_1^-,g_2^-,g_3^-,\bar\Lambda_4^+,g_5^+,g_6^+) &= -[6\,1]\,\langle 3\,4\rangle\,X \\ A_6^{\rm tree}(g_1^-,\Lambda_2^-,g_3^-,\bar\Lambda_4^+,g_5^+,g_6^+) &= -[6\,2]\,\langle 3\,4\rangle\,X \\ A_6^{\rm tree}(g_1^-,g_2^-,\Lambda_3^-,\bar\Lambda_4^+,g_5^+,g_6^+) &= \langle 6|K_{612}|4\rangle X \\ A_6^{\rm tree}(g_1^-,g_2^-,g_3^-,g_4^+,g_5^+,g_6^+) &= \langle 6|K_{612}|3\rangle X\,. \qquad (7.7)\end{aligned}$$

Similarly, writing the first term as $\langle 4|K_{234}|1\rangle Y$ we find,

$$\langle 4|K_{234}|1\rangle\,\langle 4\,\eta\rangle = \langle 1|K_{234}4|\eta\rangle = t_{234}\,\langle 1\,\eta\rangle - \langle 2|K_{234}|1\rangle\,\langle 2\,\eta\rangle - \langle 3|K_{234}|1\rangle\,\langle 3\,\eta\rangle\,, \qquad (7.8)$$

which suggests a second solution to the SWI of the form,

$$A_6^{\rm tree}(\Lambda_1^-,g_2^-,g_3^-,\bar\Lambda_4^+,g_5^+,g_6^+) = t_{234}Y$$

$$\begin{aligned}
A_6^{\rm tree}(g_1^-, \Lambda_2^-, g_3^-, \bar\Lambda_4^+, g_5^+, g_6^+) &= -\langle 2|K_{234}|1\rangle Y \\
A_6^{\rm tree}(g_1^-, g_2^-, \Lambda_3^-, \bar\Lambda_4^+, g_5^+, g_6^+) &= -\langle 3|K_{234}|1\rangle Y \\
A_6^{\rm tree}(g_1^-, g_2^-, g_3^-, g_4^+, g_5^+, g_6^+) &= \langle 4|K_{234}|1\rangle Y\,.
\end{aligned} \tag{7.9}$$

The two gluino tree amplitudes are thus,

$$\begin{aligned}
A_6^{\rm tree}(g_1^-, g_2^-, \Lambda_3^-, \bar\Lambda_4^+, g_5^+, g_6^+) &= -\frac{i\langle 4|K_{234}|1\rangle^2 \langle 3|K_{234}|1\rangle}{t_{234}\,[2\,3]\,[3\,4]\,\langle 5\,6\rangle\,\langle 6\,1\rangle\,\langle 2|K_{234}|5\rangle} \\
&+ \frac{i\langle 6|K_{612}|3\rangle^2 \langle 6|K_{612}|4\rangle}{t_{612}\,[6\,1]\,[1\,2]\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 2|K_{612}|5\rangle} \\
A_6^{\rm tree}(g_1^-, \Lambda_2^-, g_3^-, \bar\Lambda_4^+, g_5^+, g_6^+) &= -\frac{i\langle 4|K_{234}|1\rangle^2 \langle 2|K_{234}|1\rangle}{t_{234}\,[2\,3]\,[3\,4]\,\langle 5\,6\rangle\,\langle 6\,1\rangle\,\langle 2|K_{234}|5\rangle} \\
&+ \frac{i\langle 6|K_{612}|3\rangle^2\,[2\,6]\,\langle 3\,4\rangle}{t_{612}\,[6\,1]\,[1\,2]\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 2|K_{612}|5\rangle} \\
A_6^{\rm tree}(\Lambda_1^-, g_2^-, g_3^-, \bar\Lambda_4^+, g_5^+, g_6^+) &= \frac{i\langle 4|K_{234}|1\rangle^2 t_{234}}{t_{234}\,[2\,3]\,[3\,4]\,\langle 5\,6\rangle\,\langle 6\,1\rangle\,\langle 2|K_{234}|5\rangle} \\
&+ \frac{i\langle 6|K_{612}|3\rangle^2\,[1\,6]\,\langle 3\,4\rangle}{t_{612}\,[6\,1]\,[1\,2]\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 2|K_{612}|5\rangle}\,.
\end{aligned} \tag{7.10}$$

In principle there is some ambiguity in these solutions since the coefficients of $\langle 6|K_{612}|3\rangle$ and $\langle 4|K_{234}|1\rangle$ are not unique, i.e.,

$$\begin{aligned}
\langle 6|K_{612}|3\rangle X + \langle 4|K_{234}|1\rangle Y &= \langle 6|K_{612}|3\rangle \left(X + \frac{Z}{\langle 6|K_{612}|3\rangle} \right) \\
&+ \langle 4|K_{234}|1\rangle \left(Y - \frac{Z}{\langle 4|K_{234}|1\rangle} \right).
\end{aligned} \tag{7.11}$$

However, by taking X and Y to be the values that appear in the gluon amplitudes we do not introduce any of the unphysical singularities/poles that arise in the general ($Z \neq 0$) case. The remaining amplitude, $A_6^{\rm tree}(g_1^-, \Lambda_2^-, g_3^-, g_4^+, \bar\Lambda_5^+, g_6^+)$, can be obtained from the SWI,

$$\begin{aligned} 0 &= \langle 1\,\eta\rangle\, A_6^{\rm tree}(\Lambda_1^-, g_2^-, g_3^-, g_4^+, \bar\Lambda_5^+, g_6^+) + \langle 2\,\eta\rangle\, A_6^{\rm tree}(g_1^-, \Lambda_2^-, g_3^-, g_4^+, \bar\Lambda_5^+, g_6^+) \\ &+ \langle 3\,\eta\rangle\, A_6^{\rm tree}(g_1^-, g_2^-, \Lambda_3^-, g_4^+, \bar\Lambda_5^+, g_6^+) - \langle 5\,\eta\rangle\, A_6^{\rm tree}(g_1^-, g_2^-, g_3^-, g_4^+, g_5^+, g_6^+), \end{aligned} \tag{7.12}$$

which is obtained by acting with Q on $A_6^{\rm tree}(g_1^-, g_2^-, g_3^-, g_4^+, \bar\Lambda_5^+, g_6^+)$. Here we use the identities,

$$\begin{aligned} \langle 6|K_{612}|3\rangle\,\langle 5\,\eta\rangle &= \langle 6|K_{612}|5\rangle\,\langle 3\,\eta\rangle - [6\,1]\,\langle 3\,5\rangle\,\langle 1\,\eta\rangle - [6\,2]\,\langle 3\,5\rangle\,\langle 2\,\eta\rangle\,, \\ \langle 4|K_{234}|1\rangle\,\langle 5\,\eta\rangle &= \langle 4|K_{234}|5\rangle\,\langle 1\,\eta\rangle - [4\,2]\,\langle 1\,5\rangle\,\langle 2\,\eta\rangle - [4\,3]\,\langle 1\,5\rangle\,\langle 3\,\eta\rangle\,, \end{aligned} \tag{7.13}$$

to obtain,

$$\begin{aligned} A_6^{\rm tree}(g_1^-, \Lambda_2^-, g_3^-, g_4^+, \bar\Lambda_5^+, g_6^+) &= \frac{-i\langle 4|K_{234}|1\rangle^2\,[4\,2]\,\langle 1\,5\rangle}{t_{234}\,[2\,3]\,[3\,4]\,\langle 5\,6\rangle\,\langle 6\,1\rangle\,\langle 2|K_{234}|5\rangle} \\ &- \frac{i\langle 6|K_{612}|3\rangle^2\,[6\,2]\,\langle 3\,5\rangle}{t_{612}\,[6\,1]\,[1\,2]\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 2|K_{612}|5\rangle}. \end{aligned} \tag{7.14}$$

This SWI also yields consistent but independent expressions for two of the amplitudes found previously. For example,

$$\begin{aligned} A_6^{\rm tree}(\Lambda_1^-, g_2^-, g_3^-, g_4^+, \bar\Lambda_5^+, g_6^+) &= \frac{i\langle 4|K_{234}|1\rangle^2\langle 4|K_{234}|5\rangle}{t_{234}\,[2\,3]\,[3\,4]\,\langle 5\,6\rangle\,\langle 6\,1\rangle\,\langle 2|K_{234}|5\rangle} \\ &- \frac{i\langle 6|K_{612}|3\rangle^2\,[6\,1]\,\langle 3\,5\rangle}{t_{612}\,[6\,1]\,[1\,2]\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 2|K_{612}|5\rangle}. \end{aligned} \tag{7.15}$$

The expressions eq. (7.10) and eq. (7.15) satisfy the consistency check,

$$A_6^{\rm tree}(\Lambda_1^-, g_2^-, g_3^-, g_4^+, \bar\Lambda_5^+, g_6^+) = \left[A_6^{\rm tree}(g_1^-, \Lambda_2^-, g_3^-, \bar\Lambda_4^+, g_5^+, g_6^+)\right]\Big|_{j\to j+3}. \tag{7.16}$$

Thus we have a self-consistent set of six-point, two gluino tree amplitudes for the helicity configuration $(- - - + ++)$.

Next we consider the helicity configuration $(- - + - ++)$ and obtain two gluino amplitudes from the gluonic amplitude [67],

$$
\begin{aligned}
A_6^{\rm tree}(g_1^-, g_2^-, g_3^+, g_4^-, g_5^+, g_6^+) &= \frac{i\,\langle 1\,2\rangle^3\,[5\,6]^3}{t_{123}\,\langle 2\,3\rangle\,[4\,5]\,\langle 4|K_{123}|1\rangle\langle 6|K_{123}|3\rangle} \\
&+ \frac{i\langle 3|K_{234}|1\rangle^4}{t_{234}\,[2\,3]\,[3\,4]\,\langle 5\,6\rangle\,\langle 6\,1\rangle\,\langle 2|K_{234}|5\rangle\langle 4|K_{234}|1\rangle} \\
&+ \frac{i\langle 6|K_{612}|4\rangle^4}{t_{345}\,[6\,1]\,[1\,2]\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 6|K_{612}|3\rangle\langle 2|K_{612}|5\rangle}.
\end{aligned}
\tag{7.17}
$$

Six amplitudes involving two gluinos are needed to generate all possibilities by relabelling, conjugation and flipping,

$$
\begin{array}{ll}
A_6^{\rm tree}(\Lambda_1^-, g_2^-, \bar\Lambda_3^+, g_4^-, g_5^+, g_6^+), & A_6^{\rm tree}(g_1^-, \Lambda_2^-, \bar\Lambda_3^+, g_4^-, g_5^+, g_6^+), \\
A_6^{\rm tree}(g_1^-, g_2^-, \bar\Lambda_3^+, \Lambda_4^-, g_5^+, g_6^+), & A_6^{\rm tree}(\Lambda_1^-, g_2^-, g_3^+, g_4^-, \bar\Lambda_5^+, g_6^+), \\
A_6^{\rm tree}(g_1^-, \Lambda_2^-, g_3^+, g_4^-, \bar\Lambda_5^+, g_6^+), & A_6^{\rm tree}(\Lambda_1^-, g_2^-, g_3^+, g_4^-, g_5^+, \bar\Lambda_6^+).
\end{array}
\tag{7.18}
$$

These are related to the gluonic amplitude via the three SWI,

$$
\begin{aligned}
0 &= \langle 1\,\eta\rangle\, A_6^{\rm tree}(\Lambda_1^-, g_2^-, \bar\Lambda_3^+, g_4^-, g_5^+, g_6^+) + \langle 2\,\eta\rangle\, A_6^{\rm tree}(g_1^-, \Lambda_2^-, \bar\Lambda_3^+, g_4^-, g_5^+, g_6^+) \\
&+ \langle 4\,\eta\rangle\, A_6^{\rm tree}(g_1^-, g_2^-, \bar\Lambda_3^+, \Lambda_4^-, g_5^+, g_6^+) - \langle 3\,\eta\rangle\, A_6^{\rm tree}(g_1^-, g_2^-, g_3^+, g_4^-, g_5^+, g_6^+), \\
0 &= \langle 1\,\eta\rangle\, A_6^{\rm tree}(\Lambda_1^-, g_2^-, g_3^+, g_4^-, \bar\Lambda_5^+, g_6^+) + \langle 2\,\eta\rangle\, A_6^{\rm tree}(g_1^-, \Lambda_2^-, g_3^+, g_4^-, \bar\Lambda_5^+, g_6^+) \\
&+ \langle 4\,\eta\rangle\, A_6^{\rm tree}(g_1^-, g_2^-, g_3^+, \Lambda_4^-, \bar\Lambda_5^+, g_6^+) - \langle 5\,\eta\rangle\, A_6^{\rm tree}(g_1^-, g_2^-, g_3^+, g_4^-, g_5^+, g_6^+), \\
0 &= \langle 1\,\eta\rangle\, A_6^{\rm tree}(\Lambda_1^-, g_2^-, g_3^+, g_4^-, g_5^+, \bar\Lambda_6^+) + \langle 2\,\eta\rangle\, A_6^{\rm tree}(g_1^-, \Lambda_2^-, g_3^+, g_4^-, g_5^+, \bar\Lambda_6^+) \\
&+ \langle 4\,\eta\rangle\, A_6^{\rm tree}(g_1^-, g_2^-, g_3^+, \Lambda_4^-, g_5^+, \bar\Lambda_6^+) - \langle 6\,\eta\rangle\, A_6^{\rm tree}(g_1^-, g_2^-, g_3^+, g_4^-, g_5^+, g_6^+).
\end{aligned}
\tag{7.19}
$$

To solve the first of these, as before, we find two independent identities,

$$\begin{aligned}
\langle 6|K_{612}|4\rangle \langle 3\,\eta\rangle &= \langle 6|K_{612}|3\rangle \langle 4\,\eta\rangle - [1\,6]\,\langle 3\,4\rangle \langle 1\,\eta\rangle - [2\,6]\,\langle 3\,4\rangle \langle 2\,\eta\rangle\,,\\
\langle 3\,\eta\rangle \langle 3|K_{234}|1\rangle &= t_{234}\,\langle 1\,\eta\rangle - \langle 2|K_{234}|1\rangle \langle 2\,\eta\rangle - \langle 4|K_{234}|1\rangle \langle 4\,\eta\rangle\,,
\end{aligned} \tag{7.20}$$

which give the following solutions to the SWI,

$$\begin{aligned}
A_6^{\text{tree}}(\Lambda_1^-, g_2^-, \bar{\Lambda}_3^+, g_4^-, g_5^+, g_6^+) &= [6\,1]\,\langle 3\,4\rangle\,X + t_{234}Y,\\
A_6^{\text{tree}}(g_1^-, \Lambda_2^-, \bar{\Lambda}_3^+, g_4^-, g_5^+, g_6^+) &= [6\,2]\,\langle 3\,4\rangle\,X - \langle 2|K_{234}|1\rangle Y,\\
A_6^{\text{tree}}(g_1^-, g_2^-, g_3^+, g_4^-, g_5^+, g_6^+) &= \langle 6|K_{612}|4\rangle X + \langle 3|K_{234}|1\rangle Y,\\
A_6^{\text{tree}}(g_1^-, g_2^-, \bar{\Lambda}_3^+, \Lambda_4^-, g_5^+, g_6^+) &= \langle 6|K_{612}|3\rangle X - \langle 4|K_{234}|1\rangle Y.
\end{aligned} \tag{7.21}$$

We could rewrite the purely gluonic tree amplitude in the form $\langle 6|K_{612}|4\rangle X + \langle 3|K_{234}|1\rangle Y$ by using the identity,

$$\frac{\langle 1\,2\rangle\,[5\,6]}{\langle 4|K_{234}|1\rangle\langle 6|K_{612}|3\rangle} = -\frac{\langle 3|K_{234}|1\rangle}{\langle 4|K_{234}|1\rangle\langle 2|K_{234}|5\rangle} + \frac{\langle 6|K_{612}|4\rangle}{\langle 6|K_{612}|3\rangle\langle 2|K_{612}|5\rangle}. \tag{7.22}$$

However, it is more convenient and in line with our philosophy of not generating extra poles to use the Schouten identity to produce,

$$\langle 3\,\eta\rangle\,\langle 1\,2\rangle\,[5\,6] = \langle 1\,\eta\rangle\,\langle 3\,2\rangle\,[5\,6] + \langle 2\,\eta\rangle\,\langle 1\,3\rangle\,[5\,6]\,. \tag{7.23}$$

Whether we rearrange to use two identities or use three, we obtain the same solutions,

$$A_6^{\text{tree}}(\Lambda_1^-, g_2^-, \bar{\Lambda}_3^+, g_4^-, g_5^+, g_6^+) = \frac{-i\,\langle 1\,2\rangle^2\,\langle 2\,3\rangle\,[5\,6]^3}{t_{123}\,\langle 2\,3\rangle\,[4\,5]\,\langle 4|K_{123}|1\rangle\langle 6|K_{123}|3\rangle}$$

$$
\begin{aligned}
& + \frac{i\langle 3|K_{234}|1\rangle^3 t_{234}}{t_{234}\,[2\,3]\,[3\,4]\,\langle 5\,6\rangle\,\langle 6\,1\rangle\,\langle 2|K_{234}|5\rangle\langle 4|K_{234}|1\rangle} \\
& + \frac{i\langle 6|K_{612}|4\rangle^3\,[6\,1]\,\langle 3\,4\rangle}{t_{345}\,[6\,1]\,[1\,2]\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 6|K_{612}|3\rangle\langle 2|K_{612}|5\rangle} \\
A_6^{\rm tree}(g_1^-,\Lambda_2^-,\bar\Lambda_3^+,g_4^-,g_5^+,g_6^+) = & \frac{i\,\langle 1\,2\rangle^2\,\langle 1\,3\rangle\,[5\,6]^3}{t_{123}\,\langle 2\,3\rangle\,[4\,5]\,\langle 4|K_{123}|1\rangle\langle 6|K_{123}|3\rangle} \\
& + \frac{-i\langle 3|K_{234}|1\rangle^3\langle 2|K_{234}|1\rangle}{t_{234}\,[2\,3]\,[3\,4]\,\langle 5\,6\rangle\,\langle 6\,1\rangle\,\langle 2|K_{234}|5\rangle\langle 4|K_{234}|1\rangle} \\
& + \frac{i\langle 6|K_{612}|4\rangle^3\,[6\,2]\,\langle 3\,4\rangle}{t_{345}\,[6\,1]\,[1\,2]\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 6|K_{612}|3\rangle\langle 2|K_{612}|5\rangle} \\
A_6^{\rm tree}(g_1^-,g_2^-,\bar\Lambda_3^+,\Lambda_4^-,g_5^+,g_6^+) = & \frac{-i\langle 3|K_{234}|1\rangle^3\langle 4|K_{234}|1\rangle}{t_{234}\,[2\,3]\,[3\,4]\,\langle 5\,6\rangle\,\langle 6\,1\rangle\,\langle 2|K_{234}|5\rangle\langle 4|K_{234}|1\rangle} \\
& + \frac{i\langle 6|K_{612}|4\rangle^3\langle 6|K_{612}|3\rangle}{t_{345}\,[6\,1]\,[1\,2]\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 6|K_{612}|3\rangle\langle 2|K_{612}|5\rangle}
\end{aligned}
\tag{7.24}
$$

The remaining two amplitudes can be obtained similarly.

For the final gluonic configuration,

$$
\begin{aligned}
A_6^{\rm tree}(g_1^-,g_2^+,g_3^-,g_4^+,g_5^-,g_6^+) = & \frac{i\langle 2|K_{123}|5\rangle^4}{t_{123}\,[1\,2]\,[2\,3]\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 1|K_{123}|4\rangle\langle 3|K_{123}|6\rangle} \\
& + \frac{i\langle 6|K_{234}|3\rangle^4}{t_{234}\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,[5\,6]\,[6\,1]\,\langle 5|K_{234}|2\rangle\langle 1|K_{234}|4\rangle} \\
& + \frac{i\langle 4|K_{345}|1\rangle^4}{t_{345}\,\langle 6\,1\rangle\,\langle 1\,2\rangle\,[3\,4]\,[4\,5]\,\langle 3|K_{345}|6\rangle\langle 5|K_{345}|2\rangle},
\end{aligned}
\tag{7.25}
$$

there are two independent amplitudes involving two gluinos,

$$
A_6^{\rm tree}(\Lambda_1^-,\bar\Lambda_2^+,g_3^-,g_4^+,g_5^-,g_6^+)\quad, A_6^{\rm tree}(g_1^-,\bar\Lambda_2^+,g_3^-,g_4^+,\Lambda_5^-,g_6^+),
\tag{7.26}
$$

which we can obtain from the SWI,

$$
0 \;= \langle 1\,\eta\rangle\, A_6^{\rm tree}(\Lambda_1^-,\bar\Lambda_2^+,g_3^-,g_4^+,g_5^-,g_6^+) \;+ \langle 3\,\eta\rangle\, A_6^{\rm tree}(g_1^-,\bar\Lambda_2^+,\Lambda_3^-,g_4^+,g_5^-,g_6^+)
$$

$$- \langle 2\,\eta\rangle\, A_6^{\rm tree}(g_1^-, g_2^+, g_3^-, g_4^+, g_5^-, g_6^+) \quad + \langle 5\,\eta\rangle\, A_6^{\rm tree}(g_1^-, \bar\Lambda_2^+, g_3^-, g_4^+, \Lambda_5^-, g_6^+). \tag{7.27}$$

We solve this using the identities,

$$\begin{aligned}
\langle 2|K_{123}|5\rangle\,\langle 2\,\eta\rangle &= t_{123}\,\langle 5\,\eta\rangle - \langle 1|K_{123}|5\rangle\,\langle 1\,\eta\rangle - \langle 3|K_{123}|5\rangle\,\langle 3\,\eta\rangle \\
\langle 6|K_{234}|3\rangle\,\langle 2\,\eta\rangle &= \langle 6|K_{234}|2\rangle\,\langle 3\,\eta\rangle + \langle 2\,3\rangle\,[5\,6]\,\langle 5\,\eta\rangle - \langle 2\,3\rangle\,[6\,1]\,\langle 1\,\eta\rangle \\
\langle 4|K_{345}|1\rangle\,\langle 2\,\eta\rangle &= \langle 4|K_{345}|2\rangle\,\langle 1\,\eta\rangle + [3\,4]\,\langle 1\,2\rangle\,\langle 3\,\eta\rangle - [4\,5]\,\langle 1\,2\rangle\,\langle 5\,\eta\rangle\,,
\end{aligned} \tag{7.28}$$

giving the tree amplitudes,

$$\begin{aligned}
A_6^{\rm tree}(\Lambda_1^-, \bar\Lambda_2^+, g_3^-, g_4^+, g_5^-, g_6^+) &= \frac{-i\langle 2|K_{123}|5\rangle^3\langle 1|K_{123}|5\rangle}{t_{123}\,[1\,2]\,[2\,3]\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 1|K_{123}|4\rangle\langle 3|K_{123}|6\rangle} \\
&+ \frac{-i\langle 6|K_{234}|3\rangle^3\,\langle 2\,3\rangle\,[6\,1]}{t_{234}\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,[5\,6]\,[6\,1]\,\langle 5|K_{234}|2\rangle\langle 1|K_{234}|4\rangle} \\
&+ \frac{i\langle 4|K_{345}|1\rangle^3\langle 4|K_{345}|2\rangle}{t_{345}\,\langle 6\,1\rangle\,\langle 1\,2\rangle\,[3\,4]\,[4\,5]\,\langle 3|K_{345}|6\rangle\langle 5|K_{345}|2\rangle} \\
A_6^{\rm tree}(g_1^-, \bar\Lambda_2^+, \Lambda_3^-, g_4^+, g_5^-, g_6^+) &= \frac{-i\langle 2|K_{123}|5\rangle^3\langle 3|K_{123}|5\rangle}{t_{123}\,[1\,2]\,[2\,3]\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 1|K_{123}|4\rangle\langle 3|K_{123}|6\rangle} \\
&+ \frac{i\langle 6|K_{234}|3\rangle^3\langle 6|K_{234}|2\rangle}{t_{234}\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,[5\,6]\,[6\,1]\,\langle 5|K_{234}|2\rangle\langle 1|K_{234}|4\rangle} \\
&+ \frac{i\langle 4|K_{345}|1\rangle^3\,[3\,4]\,\langle 1\,2\rangle}{t_{345}\,\langle 6\,1\rangle\,\langle 1\,2\rangle\,[3\,4]\,[4\,5]\,\langle 3|K_{345}|6\rangle\langle 5|K_{345}|2\rangle} \\
A_6^{\rm tree}(g_1^-, \bar\Lambda_2^+, g_3^-, g_4^+, \Lambda_5^-, g_6^+) &= \frac{i\langle 2|K_{123}|5\rangle^3 t_{123}}{t_{123}\,[1\,2]\,[2\,3]\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 1|K_{123}|4\rangle\langle 3|K_{123}|6\rangle} \\
&+ \frac{i\langle 6|K_{234}|3\rangle^3\,\langle 2\,3\rangle\,[5\,6]}{t_{234}\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,[5\,6]\,[6\,1]\,\langle 5|K_{234}|2\rangle\langle 1|K_{234}|4\rangle} \\
&+ \frac{-i\langle 4|K_{345}|1\rangle^3\,[4\,5]\,\langle 1\,2\rangle}{t_{345}\,\langle 6\,1\rangle\,\langle 1\,2\rangle\,[3\,4]\,[4\,5]\,\langle 3|K_{345}|6\rangle\langle 5|K_{345}|2\rangle}
\end{aligned} \tag{7.29}$$

The six-point two-quark amplitudes have been computed previously [75] and can be obtained in compact expressions using recursion relations [73]. Our results for adjacent gluinos match these exactly - demonstrating that, at tree level, by respecting the symmetries and factorisation structures of the amplitudes we can use the SWI to generate the correct results.

7.1.2 Six-point One-loop NMHV Amplitudes with Two Gluinos

The SWI apply to all orders in perturbation theory, so we can apply our technique to one-loop amplitudes. Furthermore, $N = 4$ one-loop amplitudes can be expressed as sums of box integrals with rational coefficients [8]. Since the box integrals are an independent set of functions the SWI for these amplitudes will apply box by box.

For the six-point, one-loop, NMHV amplitudes the only types of box contributing are the "two mass-hard" and one mass boxes. These appear in certain very specific combinations [9],

$$
\begin{aligned}
W_6^{(i)} &\equiv F^{1\mathrm{m}}_{n:i} + F^{1\mathrm{m}}_{n:i+3} + F^{2\mathrm{m}\,h}_{n:2;i+1} + F^{2\mathrm{m}\,h}_{n:2;i+4} \\
&= -\frac{1}{2\epsilon^2}\sum_{j=1}^{6}\left(\frac{\mu^2}{-s_{j,j+1}}\right)^{\epsilon} - \ln\left(\frac{-t_{i,i+1,i+2}}{-s_{i,i+1}}\right)\ln\left(\frac{-t_{i,i+1,i+2}}{-s_{i+1,i+2}}\right) \\
&- \ln\left(\frac{-t_{i,i+1,i+2}}{-s_{i+3,i+4}}\right)\ln\left(\frac{-t_{i,i+1,i+2}}{-s_{i+4,i+5}}\right) + \ln\left(\frac{-t_{i,i+1,i+2}}{-s_{i+2,i+3}}\right)\ln\left(\frac{-t_{i,i+1,i+2}}{-s_{i+5,i}}\right) \\
&+ \frac{1}{2}\ln\left(\frac{-s_{i,i+1}}{-s_{i+3,i+4}}\right)\ln\left(\frac{-s_{i+1,i+2}}{-s_{i+4,i+5}}\right) + \frac{1}{2}\ln\left(\frac{-s_{i-1,i}}{-s_{i,i+1}}\right)\ln\left(\frac{-s_{i+1,i+2}}{-s_{i+2,i+3}}\right) \\
&+ \frac{1}{2}\ln\left(\frac{-s_{i+2,i+3}}{-s_{i+3,i+4}}\right)\ln\left(\frac{-s_{i+4,i+5}}{-s_{i+5,i}}\right) + \frac{\pi^2}{3}\,. \qquad (7.30)
\end{aligned}
$$

There are only three independent $W_6^{(i)}$ since $W_6^{(i+3)} = W_6^{(i)}$. The $W_6^{(i)}$ have certain features that will extend to amplitudes involving fermions. In particular, it was shown in [76] that the IR divergences of the W_6, represented by the pole terms in ϵ, are such that the loop amplitudes we calculate should be related to the corresponding tree amplitudes via the simple relation [76],

$$A_6^{\text{One loop}} = c_\Gamma \left[\frac{-1}{\epsilon^2} \sum_{j=1}^{6} \left(\frac{\mu^2}{-s_{j,j+1}} \right)^{\epsilon} \right] A_6^{\text{Tree}}, \tag{7.31}$$

This leads to the sum of the coefficients of the $W_6^{(i)}$ being proportional to the tree amplitude [9].

The first set of amplitudes we shall consider are based on the gluonic amplitude,

$$A_6^{N=4}(1^-,2^-,3^-,4^+,5^+,6^+) \ = \ c_\Gamma \left[B_1 \, W_6^{(1)} + B_2 \, W_6^{(2)} + B_3 \, W_6^{(3)} \right], \tag{7.32}$$

where,

$$\begin{aligned}
B_1 &= B_0 \equiv i \frac{(t_{123})^3}{[1\,2]\,[2\,3]\,\langle 4\,5\rangle\,\langle 5\,6\rangle \ \langle 1|K_{123}|4\rangle \langle 3|K_{123}|6\rangle}, \\
B_2 &= \left(\frac{\langle 4|K_{234}|1\rangle}{t_{234}} \right)^4 B_+ + \left(\frac{\langle 2\,3\rangle\,[5\,6]}{t_{234}} \right)^4 B_+^\dagger, \\
B_3 &= \left(\frac{\langle 6|K_{345}|3\rangle}{t_{345}} \right)^4 B_- + \left(\frac{\langle 1\,2\rangle\,[4\,5]}{t_{345}} \right)^4 B_-^\dagger,
\end{aligned} \tag{7.33}$$

and,

$$B_+ = B_0|_{j\to j+1} \ , \quad B_- = B_0|_{j\to j-1} \ , \tag{7.34}$$

where the operation † implies $[i\,j] \leftrightarrow \langle j\,i\rangle$. This amplitude has two symmetries,

$$\begin{aligned}
S_1 \ &: A_6^{N=4}(1^-,2^-,3^-,4^+,5^+,6^+) \ = \ \left[A_6^{N=4}(1^-,2^-,3^-,4^+,5^+,6^+) \right]^\dagger_{j\to j+3}, \\
S_2 \ &: A_6^{N=4}(1^-,2^-,3^-,4^+,5^+,6^+) \ = \ \left[A_6^{N=4}(1^-,2^-,3^-,4^+,5^+,6^+) \right]^\dagger_{j\to 6-j},
\end{aligned} \tag{7.35}$$

which impose constraints on the coefficients. Under S_1, $W_i \to W_i$ so we have,

$$S_1 : B_i \longrightarrow B_i \tag{7.36}$$

whereas under S_2, $W_1 \to W_1$ and $W_2 \leftrightarrow W_3$ so that

$$S_2 : B_1 \longrightarrow B_1, \quad B_2 \leftrightarrow B_3. \tag{7.37}$$

The coefficients clearly satisfy these conditions when we note that B_0 itself satisfies,

$$S_1 : B_0 \longrightarrow B_0, \quad S_2 : B_0 \longrightarrow B_0. \tag{7.38}$$

Applying S_i to the gluino amplitudes provides a set of consistency conditions that enable us to resolve the ambiguities that arise in solving the SWI.

As for the tree amplitudes, we can generate all the possible two-gluino amplitudes from a minimal set of four by conjugation, relabelling and flipping. These gluino amplitudes have a subset of the invariances of the gluonic amplitudes. Specifically, $A(g_1^-, \Lambda_2^-, g_3^-, g_4^+, \bar{\Lambda}_5^+, g_6^+)$ is invariant under S_1 and S_2, while $A(\Lambda_1^-, g_2^-, g_3^-, \bar{\Lambda}_4^+, g_5^+, g_6^+)$ is only invariant under S_1, $A(g_1^-, g_2^-, \Lambda_3^-, \bar{\Lambda}_4^+, g_5^+, g_6^+)$ is only invariant under S_2 and $A(g_1^-, \Lambda_2^-, g_3^-, \bar{\Lambda}_4^+, g_5^+, g_6^+)$ is invariant under neither.

For this helicity configuration the SWI are,

$$\begin{aligned} 0 \;=\; & \langle 1\,\eta\rangle\, A_6(\Lambda_1^-, g_2^-, g_3^-, \bar{\Lambda}_4^+, g_5^+, g_6^+) + \langle 2\,\eta\rangle\, A_6(g_1^-, \Lambda_2^-, g_3^-, \bar{\Lambda}_4^+, g_5^+, g_6^+) \\ & + \langle 3\,\eta\rangle\, A_6(g_1^-, g_2^-, \Lambda_3^-, \bar{\Lambda}_4^+, g_5^+, g_6^+) - \langle 4\,\eta\rangle\, A_6(g_1^-, g_2^-, g_3^-, g_4^+, g_5^+, g_k)\,. \end{aligned} \tag{7.39}$$

and,

$$\begin{aligned} 0 \;=\; & \langle 1\,\eta\rangle\, A_6(\Lambda_1^-, g_2^-, g_3^-, g_4^+, \bar{\Lambda}_5^+, g_6^+) + \langle 2\,\eta\rangle\, A_6(g_1^-, \Lambda_2^-, g_3^-, g_4^+, \bar{\Lambda}_5^+, g_6^+) \\ + \; & \langle 3\,\eta\rangle\, A_6(g_1^-, g_2^-, \Lambda_3^-, g_4^+, \bar{\Lambda}_5^+, g_6^+) - \langle 5\,\eta\rangle\, A_6(g_1^-, g_2^-, g_3^-, g_4^+, g_5^+, g_6^+), \end{aligned} \tag{7.40}$$

To solve for B_1 we need identities involving $\langle 4\,\eta\rangle$ and $\langle 5\,\eta\rangle$. These are,

$$\begin{aligned} t_{123}\,\langle 4\,\eta\rangle &= \langle 1|K_{123}|4\rangle\,\langle 1\,\eta\rangle + \langle 2|K_{123}|4\rangle\,\langle 2\,\eta\rangle + \langle 3|K_{123}|4\rangle\,\langle 3\,\eta\rangle\,, \\ t_{123}\,\langle 5\,\eta\rangle &= \langle 1|K_{123}|5\rangle\,\langle 1\,\eta\rangle + \langle 2|K_{123}|5\rangle\,\langle 2\,\eta\rangle + \langle 3|K_{123}|5\rangle\,\langle 3\,\eta\rangle\,. \end{aligned} \tag{7.41}$$

We can check that these equations are consistent with the symmetries S_i, if we have solutions,

$$\begin{aligned} A\,\langle 4\,\eta\rangle &= B\,\langle 1\,\eta\rangle + C\,\langle 2\,\eta\rangle + D\,\langle 3\,\eta\rangle\,, \\ A'\,\langle 5\,\eta\rangle &= B'\,\langle 1\,\eta\rangle + C'\,\langle 2\,\eta\rangle + D'\,\langle 3\,\eta\rangle\,, \end{aligned} \tag{7.42}$$

then we must have,

$$S_1 : (B/A) \to (B/A), \quad S_2 : (D/A) \to (D/A), \quad S_i : (C'/A) \to (C'/A). \tag{7.43}$$

The coefficients in eq. (7.41) clearly satisfy these constraints. Thus we have solutions,

$$\begin{aligned} B_1(\Lambda_1^-, g_2^-, g_3^-, \bar{\Lambda}_4^+, g_5^+, g_6^+) &= \frac{i(t_{123})^2\langle 1|K_{123}|4\rangle}{[1\,2]\,[2\,3]\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 1|K_{123}|4\rangle\langle 3|K_{123}|6\rangle}, \\ B_1(g_1^-, \Lambda_2^-, g_3^-, \bar{\Lambda}_4^+, g_5^+, g_6^+) &= \frac{i(t_{123})^2\langle 2|K_{123}|4\rangle}{[1\,2]\,[2\,3]\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 1|K_{123}|4\rangle\langle 3|K_{123}|6\rangle}, \\ B_1(g_1^-, g_2^-, \Lambda_3^-, \bar{\Lambda}_4^+, g_5^+, g_6^+) &= \frac{i(t_{123})^2\langle 3|K_{123}|4\rangle}{[1\,2]\,[2\,3]\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 1|K_{123}|4\rangle\langle 3|K_{123}|6\rangle}, \\ B_1(g_1^-, \Lambda_2^-, g_3^-, g_4^+, \bar{\Lambda}_5^+, g_6^+) &= \frac{i(t_{123})^2\langle 2|K_{123}|5\rangle}{[1\,2]\,[2\,3]\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 1|K_{123}|4\rangle\langle 3|K_{123}|6\rangle}. \end{aligned} \tag{7.44}$$

To solve for the first three B_2's we use the identities,

$$\begin{aligned}\langle 4|K_{234}|1\rangle\,\langle 4\,\eta\rangle &= t_{234}\,\langle 1\,\eta\rangle - \langle 2|K_{234}|1\rangle\,\langle 2\,\eta\rangle - \langle 3|K_{234}|1\rangle\,\langle 3\,\eta\rangle\,,\\ \langle 2\,3\rangle\,\langle 4\,\eta\rangle &= \langle 4\,3\rangle\,\langle 2\,\eta\rangle + \langle 2\,4\rangle\,\langle 3\,\eta\rangle\,.\end{aligned} \tag{7.45}$$

Which give solutions,

$$\begin{aligned}B_2(\Lambda_1^-, g_2^-, g_3^-, \bar\Lambda_4^+, g_5^+, g_6^+) &= \left(\frac{\langle 4|K_{234}|1\rangle^3}{t_{234}^3}\right) B_+,\\ B_2(g_1^-, \Lambda_2^-, g_3^-, \bar\Lambda_4^+, g_5^+, g_6^+) &= \left(\frac{-\langle 4|K_{234}|1\rangle^3\langle 2|K_{234}|1\rangle}{t_{234}^4}\right) B_+ + \left(\frac{\langle 2\,3\rangle^3\,\langle 4\,3\rangle\,[5\,6]^4}{t_{234}^4}\right) B_+^\dagger,\\ B_2(g_1^-, g_2^-, \Lambda_3^-, \bar\Lambda_4^+, g_5^+, g_6^+) &= \left(\frac{-\langle 4|K_{234}|1\rangle^3\langle 3|K_{234}|1\rangle}{t_{234}^4}\right) B_+ + \left(\frac{\langle 2\,3\rangle^3\,\langle 2\,4\rangle\,[5\,6]^4}{t_{234}^4}\right) B_+^\dagger.\end{aligned} \tag{7.46}$$

The absence of a second term from the first coefficient is consistent with the observation that this box coefficient does not have a singlet term when we consider two-particle cuts in the t_{234} channel. (This observation would naturally lead us to an identity that does not involve $\langle 1\,\eta\rangle$)

For the final B_2 box coefficient there are three identities we might use,

$$\begin{aligned}\langle 4|K_{234}|1\rangle\,\langle 5\,\eta\rangle &= \langle 4|K_{234}|5\rangle\,\langle 1\,\eta\rangle - [4\,2]\,\langle 1\,5\rangle\,\langle 2\,\eta\rangle - [4\,3]\,\langle 1\,5\rangle\,\langle 3\,\eta\rangle\,,\\ \langle 2\,3\rangle\,[5\,6]\,\langle 5\,\eta\rangle &= -\langle 2\,3\rangle\,[1\,6]\,\langle 1\,\eta\rangle + \langle 6|K_{234}|3\rangle\,\langle 2\,\eta\rangle - \langle 6|K_{234}|2\rangle\,\langle 3\,\eta\rangle\,,\\ \langle 2\,3\rangle\,\langle 5\,\eta\rangle &= \langle 5\,3\rangle\,\langle 2\,\eta\rangle + \langle 2\,5\rangle\,\langle 3\,\eta\rangle\,.\end{aligned} \tag{7.47}$$

Of these, only the first two have the correct behaviour under S_i. Using these identities we find,

$$B_2(g_1^-, \Lambda_2^-, g_3^-, g_4^+, \bar\Lambda_5^+, g_6^+) = \frac{-\langle 4|K_{234}|1\rangle^3\,[4\,2]\,\langle 1\,5\rangle}{t_{234}^4} B_+ + \frac{\langle 2\,3\rangle^3\,[5\,6]^3\,\langle 6|K_{234}|3\rangle}{t_{234}^4} B_+^\dagger, \tag{7.48}$$

which has the appropriate symmetries. This pair of identities also lead to the same forms for the other B_2 coefficients obtained previously.

For the B_3 coefficients, the identities,

$$\begin{aligned}
\langle 6|K_{345}|3\rangle \langle 4\,\eta\rangle &= [6\,1] \langle 3\,4\rangle \langle 1\,\eta\rangle + [6\,2] \langle 3\,4\rangle \langle 2\,\eta\rangle + \langle 6|K_{345}|4\rangle \langle 3\,\eta\rangle\,, \\
\langle 1\,2\rangle [4\,5] \langle 4\,\eta\rangle &= -\langle 5|K_{345}|2\rangle \langle 1\,\eta\rangle + \langle 5|K_{345}|1\rangle \langle 2\,\eta\rangle - \langle 1\,2\rangle [3\,5] \langle 3\,\eta\rangle\,, \\
\langle 6|K_{345}|3\rangle \langle 5\,\eta\rangle &= +[6\,1] \langle 3\,5\rangle \langle 1\,\eta\rangle + [6\,2] \langle 3\,5\rangle \langle 2\,\eta\rangle + \langle 6|K_{345}|5\rangle \langle 3\,\eta\rangle\,, \\
\langle 1\,2\rangle [4\,5] \langle 5\,\eta\rangle &= -\langle 1\,2\rangle [4\,3] \langle 3\,\eta\rangle + \langle 4|K_{345}|2\rangle \langle 1\,\eta\rangle - \langle 4|K_{345}|1\rangle \langle 2\,\eta\rangle\,,
\end{aligned} \tag{7.49}$$

give the following solutions with the correct symmetries under S_1,

$$\begin{aligned}
B_3(\Lambda_1^-, g_2^-, g_3^-, \bar{\Lambda}_4^+, g_5^+, g_6^+) &= \left(\frac{\langle 6|K_{345}|3\rangle^3 \langle 3\,4\rangle [6\,1]}{t_{345}^4}\right) B_- + \left(\frac{-\langle 1\,2\rangle^3 [4\,5]^3 \langle 5|K_{345}|2\rangle}{t_{345}^4}\right) B_-^\dagger\,, \\
B_3(g_1^-, \Lambda_2^-, g_3^-, \bar{\Lambda}_4^+, g_5^+, g_6^+) &= \left(\frac{\langle 6|K_{345}|3\rangle^3 \langle 3\,4\rangle [6\,2]}{t_{345}^4}\right) B_- + \left(\frac{\langle 1\,2\rangle^3 [4\,5]^3 \langle 5|K_{345}|1\rangle}{t_{345}^4}\right) B_-^\dagger\,, \\
B_3(g_1^-, g_2^-, \Lambda_3^-, \bar{\Lambda}_4^+, g_5^+, g_6^+) &= \left(\frac{\langle 6|K_{345}|3\rangle^3 \langle 6|K_{345}|4\rangle}{t_{345}^4}\right) B_- + \left(\frac{-\langle 1\,2\rangle^4 [4\,5]^3 [3\,5]}{t_{345}^4}\right) B_-^\dagger\,, \\
B_3(g_1^-, \Lambda_2^-, g_3^-, g_4^+, \bar{\Lambda}_5^+, g_6^+) &= \left(\frac{\langle 6|K_{345}|3\rangle^3 [6\,2] \langle 3\,5\rangle}{t_{345}^4}\right) B_- + \left(\frac{-\langle 1\,2\rangle^3 [4\,5]^3 \langle 4|K_{345}|1\rangle}{t_{345}^4}\right) B_-^\dagger\,.
\end{aligned} \tag{7.50}$$

Comparing these with the B_2 coefficients we see that the S_2 symmetry is also satisfied.

We can obtain the gluino amplitudes with helicity configurations $(--+-++)$ and $(-+-+-+)$ in a similar manner, i.e. by finding polynomial solutions to the SWI based on the gluonic amplitudes that respect the symmetries of the amplitudes. We have verified numerically that these expressions agree with those obtained using quadruple cuts. These coefficients are collected in section 7.1.5.

There are straightforward relationships between the box coefficients and tree amplitudes. As we discussed in section 7.1.2, for the amplitude to have the correct IR behaviour the box coefficients must satisfy [9],

$$B_1 + B_2 + B_3 = 2A^{\text{tree}} \ . \tag{7.51}$$

We have checked numerically that this is true by comparison with the tree amplitudes of section 7.1.1.

The twistor structure of the box coefficients is also rather simple. All the box coefficients satisfy coplanarity constraints,

$$K_{abcd} B_i = 0 \ . \tag{7.52}$$

In fact this is satisfied by each of the terms within B_i individually.

7.1.3 Amplitudes with more than Two Fermions

We can use the SWI to obtain amplitudes involving four or more gluinos of the same flavour from those involving two gluinos. In the six-point case the tree amplitudes involving four and six fermions have been computed directly [77, 27] and also using recursion relations [78].

If we consider n-point NMHV amplitudes with negative helicities on legs m_i, applying the $N = 1$ Supersymmetry operator to,

$$A_n(g_1^+, \dots, g_{m_1}^-, \dots g_{m_2}^-, \dots \Lambda_{m_3}^-, \dots \bar{\Lambda}_r^+ \dots \bar{\Lambda}_s^+ \dots, g_n^+), \tag{7.53}$$

gives the SWI,

$$0 = \langle m_1 \, \eta \rangle \, A_n^{m_1, m_3; r, s} + \langle m_2 \, \eta \rangle \, A_n^{m_2, m_3; r, s} - \langle r \, \eta \rangle \, A_n^{m_3; s} - \langle s \, \eta \rangle \, A_n^{m_3; r} \tag{7.54}$$

where we define,

$$\begin{aligned} A_n^{m_2,m_3;r,s} &\equiv A_n(g_1^+,\ldots,g_{m_1}^-,\ldots\Lambda_{m_2}^-,\ldots,\Lambda_{m_3}^-,\ldots\bar{\Lambda}_r^+\ldots\bar{\Lambda}_s^+\ldots,g_n^+), \\ A_n^{m_3;r} &\equiv A_n(g_1^+,\ldots,g_{m_1}^-,\ldots g_{m_2}^-,\ldots,\Lambda_{m_3}^-,\ldots\bar{\Lambda}_r^+\ldots g_s^+\ldots,g_n^+). \end{aligned} \quad (7.55)$$

This rank two system can be used to solve for the four fermion amplitudes in terms of the amplitudes with two fermions. For example choosing $\eta = m_1$ gives,

$$A_n^{m_2,m_3;r,s} = \frac{\langle r\, m_1\rangle}{\langle m_2\, m_1\rangle} A_n^{m_3;r} + \frac{\langle s\, m_1\rangle}{\langle m_2\, m_1\rangle} A_n^{m_3;s}. \quad (7.56)$$

Since we have used the $N = 1$ SWI, all of the fermions in this amplitude have the same flavour.

To obtain amplitudes with six gluinos we apply the Supersymmetry operator to,

$$A_n(g_1^+,\ldots,g_{m_1}^-,\ldots\Lambda_{m_2}^-,\Lambda_{m_3}^-,\ldots\bar{\Lambda}_r^+\ldots\bar{\Lambda}_s^+\ldots,\bar{\Lambda}_t^+\ldots g_n^+), \quad (7.57)$$

giving the SWI,

$$0 = \langle m_1\, \eta\rangle\, A_n^{m_1,m_2,m_3;r,s,t} - \langle r\, \eta\rangle\, A_n^{m_2,m_3;s,t} - \langle s\, \eta\rangle\, A_n^{m_2,m_3;r,t} - \langle t\, \eta\rangle\, A_n^{m_2,m_3;r,s}, \quad (7.58)$$

which allows us to express the six fermion amplitude in terms of four fermion amplitudes. For example, choosing $\eta = r$,

$$A_n^{m_1,m_2,m_3;r,s,t} = \frac{\langle s\, r\rangle}{\langle m_1\, r\rangle} A_n^{m_2,m_3;r,t} + \frac{\langle t\, r\rangle}{\langle m_1\, r\rangle} A_n^{m_2,m_3;r,s}. \quad (7.59)$$

Again the fermions are all of the same flavour. These relations are exact to all orders in perturbation theory in any Supersymmetric theory.

For amplitudes involving two fermion flavours we must be precise about which theory we are describing and in particular whether our theory contains scalars. Supersymmetric amplitudes with two flavours of fermions must include at least one scalar. For $N \geq 2$ (and indeed for $N = 1$ with adjoint matter) the fermions have Yukawa couplings to the scalars which simultaneously change both the flavour and the helicity of the fermions. Such Yukawa couplings do not contribute to tree amplitudes with two gluinos, but they can contribute to amplitudes with four gluinos of two different flavours.

In $N = 2$ we can generate a SWI by applying Q_2 to,

$$A_n^{N=2}(g_1^+, \ldots, g_{m_1}^-, \ldots g_{m_2}^-, \ldots, \Lambda_{m_3}^{1-}, \ldots \bar{\Lambda}_r^{1+} \ldots \bar{\Lambda}_s^{2+} \ldots, g_n^+). \tag{7.60}$$

We obtain,

$$\begin{aligned} 0 &= \langle m_1\,\eta\rangle\, A_n^{N=2}(g_1^+, \ldots, \Lambda_{m_1}^{2-}, \ldots g_{m_2}^-, \ldots, \Lambda_{m_3}^{1-}, \ldots \bar{\Lambda}_r^{1+} \ldots \bar{\Lambda}_s^{2+} \ldots, g_n^+) \\ &+ \langle m_2\,\eta\rangle\, A_n^{N=2}(g_1^+, \ldots, g_{m_1}^-, \ldots \Lambda_{m_2}^{2-}, \ldots, \Lambda_{m_3}^{1-}, \ldots \bar{\Lambda}_r^{1+} \ldots \bar{\Lambda}_s^{2+} \ldots, g_n^+) \\ &- i\,\langle m_3\,\eta\rangle\, A_n^{N=2}(g_1^+, \ldots, g_{m_1}^-, \ldots g_{m_2}^-, \ldots, \phi_{m_3}^-, \ldots \bar{\Lambda}_r^{1+} \ldots \bar{\Lambda}_s^{2+} \ldots, g_n^+) \\ &- \langle s\,\eta\rangle\, A_n^{N=2}(g_1^+, \ldots, g_{m_1}^-, \ldots g_{m_2}^-, \ldots, \Lambda_{m_3}^{1-}, \ldots \bar{\Lambda}_r^{1+} \ldots g_s^+ \ldots, g_n^+), \end{aligned} \tag{7.61}$$

which can be used to determine the two flavour, four fermion amplitude in terms of a two fermion amplitude we have already calculated and a scalar-fermion-fermion amplitude which we discuss next.

7.1.4 Amplitudes Involving Scalars

As noted above, for $\mathcal{N} \geq 2$ the fermions have Yukawa couplings to the scalars which simultaneously change both the flavour and the helicity of the fermion. At tree level, this vertex implies that amplitudes of the form,

$$A_n^{\text{tree}}(\phi^-, \Lambda^{1+}, \Lambda^{2+}, g^{\pm}, \ldots, g^{\pm}), \tag{7.62}$$

need not vanish. These amplitudes will appear in the SWI and must not be discarded.

In an $N = 2$ theory there are two flavours of gluino, Λ^i. Acting with Q_2 on,

$$A_n^{N=2}(\Lambda_1^{1-}, g_2^-, g_3^-, \phi_4^+, g_5^+, \ldots, g_n^+), \tag{7.63}$$

gives,

$$\begin{aligned} 0 &= -i \langle 1\,\eta\rangle\, A_n^{N=2}(\phi_1^-, g_2^-, g_3^-, \phi_4^+, g_5^+, \ldots, g_n^+) \\ &+ \langle 2\,\eta\rangle\, A_n^{N=2}(\Lambda_1^{1-}, \Lambda_2^{2-}, g_3^-, \phi_4^+, g_5^+, \ldots, g_n^+) \\ &+ \langle 3\,\eta\rangle\, A_n^{N=2}(\Lambda_1^{1-}, g_2^-, \Lambda_3^{2-}, \phi_4^+, g_5^+, \ldots, g_n^+) \\ &+ i \langle 4\,\eta\rangle\, A_n^{N=2}(\Lambda_1^{1-}, g_2^-, g_3^-, \bar{\Lambda}_4^{1+}, g_5^+, \ldots, g_n^+). \end{aligned} \tag{7.64}$$

To solve this we need to find polynomial expressions of the form,

$$0 = iA \langle 1\,\eta\rangle + B \langle 2\,\eta\rangle + C \langle 3\,\eta\rangle - iD \langle 4\,\eta\rangle\,. \tag{7.65}$$

Given such solutions, there will be relationships between the individual terms of the two gluino and two scalar amplitudes of the form,

$$A_n^{term}(\phi_1^-, g_2^-, g_3^-, \phi_4^+, g_5^+, \ldots, g_n^+) = \left(\frac{A}{D}\right) A_n^{term}(\Lambda_1^{1-}, g_2^-, g_3^-, \bar{\Lambda}_4^{1+}, g_5^+, \ldots, g_n). \tag{7.66}$$

If the appropriate solutions to eq. (7.65) are the same as those used to obtain the two-gluino amplitudes in section 7.1.2, then the scalar terms will be of the form,

$$A_n^{term}(\phi_1^-, g_2^-, g_3^-, \phi_4^+, g_5^+, \ldots, g_n^+) = \left(\frac{A}{D}\right)^2 A_n^{term}(g_1^-, g_2^-, g_3^-, g_4^+, g_5^+, \ldots, g_n). \tag{7.67}$$

For gluonic amplitudes of the form,

$$A_n^{gluon} = \sum_i X_i, \tag{7.68}$$

we might expect amplitudes containing a pair of particles of spin h to have the form,

$$A_n^{h-\text{pair}} = \sum_i (a_i)^{2-2h} X_i, \tag{7.69}$$

where $h = 1$ for gluons, $h = 1/2$ for fermions and $h = 0$ for scalars. Such structures are apparent in tree amplitudes as can be seen in the results of [73, 74]. For example, we can generalise our two gluino tree amplitude for the helicity configuration $(---+++)$ to give,

$$\begin{aligned} A_6^{N=2}(H_1^-, g_2^-, g_3^-, H_4^+, g_5^+, g_6^+) &= \left(\frac{t_{234}}{\langle 4|K_{234}|1\rangle}\right)^{2-2h} \frac{i\langle 4|K_{234}|1\rangle^3}{t_{234}\,[2\,3]\,[3\,4]\,\langle 5\,6\rangle\,\langle 6\,1\rangle\,\langle 2|K_{234}|5\rangle} \\ &+ \left(\frac{[1\,6]\,\langle 3\,4\rangle}{\langle 6|K_{612}|3\rangle}\right)^{2-2h} \frac{i\langle 6|K_{612}|3\rangle^3}{t_{612}\,[6\,1]\,[1\,2]\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 2|K_{612}|5\rangle}, \end{aligned} \tag{7.70}$$

where H represents a gluon for $h = 1$, a gluino for $h = 1/2$, a scalar for $h = 0$ and an anti-gluino for $h = -1/2$. Such formulae are extremely useful when computing one-loop amplitudes using cuts (see for example [15, 74]).

This behaviour extends to the coefficients of the one-loop box functions and we give expressions for the box functions for two scalars in section 7.1.5. We have checked numerically for a representative sample that the box coefficients thus obtained match those obtained via quadruple cuts.

Once we have the two gluino and two scalar amplitudes, the SWI eq. (7.64) gives amplitudes such as,

$$A_n(\Lambda_1^{1-}, \Lambda_2^{2-}, g_3^-, \phi_4^+, g_5^+, g_6^+), \tag{7.71}$$

directly. Given these amplitudes, the two flavour, four gluino amplitudes can be obtained directly from eq. (7.61).

7.1.5 Summary of One-Loop Two Gluino and Two Scalar Six-Point Amplitudes in $N = 4$ Supersymmetric Yang-Mills theory

The amplitudes for the $N = 4$ theory are all of the form,

$$A_6^{N=4}(1,2,3,4,5,6) = c_\Gamma \left[c_1\, W_6^{(1)} + c_2\, W_6^{(2)} + c_3\, W_6^{(3)} \right], \tag{7.72}$$

with the coefficients c_i depending on the helicity and type of the six particles. This combination of box functions is given explicitly in eq. (7.30). The amplitudes will have one particle denoted H and a second denoted by $\bar{H}$. Again, H will denote either a scalar or $\Lambda^{\pm}$. The amplitudes are obtained using the specific values of h as defined in table 1.

H	$\bar{H}$	h
g^-	g^+	1
Λ^-	$\bar{\Lambda}^+$	1/2
ϕ^-	ϕ^+	0
Λ^+	$\bar{\Lambda}^-$	-1/2

Table 7.1: The values of h for the choices of external particle H.

We express the box coefficients in terms of B_0 and $B_\pm$ and their conjugates where,

$$B_0 = i\frac{(t_{123})^3}{[1\,2]\,[2\,3]\,\langle 4\,5\rangle\,\langle 5\,6\rangle\ \langle 1^+|\not{K}|4^+\rangle\langle 3^+|\not{K}|6^+\rangle}, \tag{7.73}$$

and

$$B_+ = B_0|_{j\to j+1}\ ,\quad B_- = B_0|_{j\to j-1}. \tag{7.74}$$

For amplitudes with helicity configuration $(---+++)$ we denote the c_i in the purely gluonic case by B_i,

$$A_6^{N=4}(1^-,2^-,3^-,4^+,5^+,6^+) \;=\; c_\Gamma\left[B_1\,W_6^{(1)} + B_2\,W_6^{(2)} + B_3\,W_6^{(3)}\right], \tag{7.75}$$

where,

$$\begin{aligned}
B_1 &= B_0,\\
B_2 &\equiv B_2^A + B_2^B = \left(\frac{\langle 4|K_{234}|1\rangle}{t_{234}}\right)^4 B_+ + \left(\frac{\langle 2\,3\rangle\,[5\,6]}{t_{234}}\right)^4 B_+^\dagger,\\
B_3 &\equiv B_3^A + B_3^B = \left(\frac{\langle 6|K_{345}|3\rangle}{t_{345}}\right)^4 B_- + \left(\frac{\langle 1\,2\rangle\,[4\,5]}{t_{345}}\right)^4 B_-^\dagger.
\end{aligned} \tag{7.76}$$

For ease of presentation we shall denote the box coefficients with fermions/scalars as B_i^{ab} when legs a and b are the H and $\bar{H}$ particles. The solutions for the B_i^{ab} for gluinos were derived in section 7.1.2 and we present them here again in a form that also gives the two scalar amplitudes. For the four independent configurations, $(ab) = (14)$, (24), (34) or (25), we find,

$$\begin{aligned}
B_1^{14} &= \left(\frac{\langle 1|K_{123}|4\rangle}{t_{123}}\right)^{2-2h} B_0, \\
B_1^{24} &= \left(\frac{\langle 2|K_{123}|4\rangle}{t_{123}}\right)^{2-2h} B_0, \\
B_1^{34} &= \left(\frac{\langle 3|K_{123}|4\rangle}{t_{123}}\right)^{2-2h} B_0, \\
B_1^{25} &= \left(\frac{\langle 2|K_{123}|5\rangle}{t_{123}}\right)^{2-2h} B_0,
\end{aligned} \tag{7.77}$$

$$\begin{aligned}
B_2^{14} &= \left(\frac{t_{234}}{\langle 4|K_{234}|1\rangle}\right)^{2-2h} B_2^A, \\
B_2^{24} &= \left(\frac{-\langle 2|K_{234}|1\rangle}{\langle 4|K_{234}|1\rangle}\right)^{2-2h} B_2^A + \left(\frac{\langle 4\,3\rangle}{\langle 2\,3\rangle}\right)^{2-2h} B_2^B, \\
B_2^{34} &= \left(\frac{-\langle 3|K_{234}|1\rangle}{\langle 4|K_{234}|1\rangle}\right)^{2-2h} B_2^A + \left(\frac{\langle 2\,4\rangle}{\langle 2\,3\rangle}\right)^{2-2h} B_2^B, \\
B_2^{25} &= \left(\frac{-\,[4\,2]\,\langle 1\,5\rangle}{\langle 4|K_{234}|1\rangle}\right)^{2-2h} B_2^A + \left(\frac{\langle 6|K_{234}|3\rangle}{\langle 2\,3\rangle\,[5\,6]}\right)^{2-2h} B_2^B,
\end{aligned} \tag{7.78}$$

$$\begin{aligned}
B_3^{14} &= \left(\frac{\langle 3\,4\rangle\,[6\,1]}{\langle 6|K_{345}|3\rangle}\right)^{2-2h} B_3^A + \left(\frac{-\langle 5|K_{345}|2\rangle}{\langle 1\,2\rangle\,[4\,5]}\right)^{2-2h} B_3^B, \\
B_3^{24} &= \left(\frac{\langle 3\,4\rangle\,[6\,2]}{\langle 6|K_{345}|3\rangle}\right)^{2-2h} B_3^A + \left(\frac{\langle 5|K_{345}|1\rangle}{\langle 1\,2\rangle\,[4\,5]}\right)^{2-2h} B_3^B, \\
B_3^{34} &= \left(\frac{\langle 6|K_{345}|4\rangle}{\langle 6|K_{345}|3\rangle}\right)^{2-2h} B_3^A + \left(-\frac{[3\,5]}{[4\,5]}\right)^{2-2h} B_3^B,
\end{aligned}$$

$$B_3^{25} = \left(\frac{[6\,2]\,\langle 3\,5\rangle}{\langle 6|K_{345}|3\rangle}\right)^{2-2h} B_3^A + \left(\frac{-\langle 4|K_{345}|1\rangle}{\langle 1\,2\rangle\,[4\,5]}\right)^{2-2h} B_3^B. \tag{7.79}$$

Next we have amplitudes with helicity structure $(--+-++)$. In the purely gluonic case, the amplitude is symmetric under,

$$S_1 : A_6^{N=4}(1,2,3,4,5,6) \longrightarrow [A_6^{N=4}(6,5,4,3,2,1)]^\dagger. \tag{7.80}$$

In this case we denote the coefficients of the $W_6^{(i)}$ by D_i. These are given by,

$$\begin{aligned} D_1 \equiv D_1^A + D_1^B &= \left(\frac{\langle 3|K_{123}|4\rangle}{t_{123}}\right)^4 B_0 + \left(\frac{\langle 1\,2\rangle\,[5\,6]}{t_{123}}\right)^4 B_0^\dagger, \\ D_2 \equiv D_2^A + D_2^B &= \left(\frac{\langle 3|K_{234}|1\rangle}{t_{234}}\right)^4 B_+ + \left(\frac{\langle 2\,4\rangle\,[5\,6]}{t_{234}}\right)^4 B_+^\dagger, \\ D_3 \equiv D_3^A + D_3^B &= \left(\frac{\langle 6|K_{345}|4\rangle}{t_{345}}\right)^4 B_- + \left(\frac{\langle 1\,2\rangle\,[3\,5]}{t_{345}}\right)^4 B_-^\dagger. \end{aligned} \tag{7.81}$$

As above, we denote the coefficients of amplitudes with particle a of type H and particle b of type $\bar{H}$ by D_i^{ab}. For this helicity configuration there are six independent possibilities,

$$(ab) = (13), (23), (43), (15), (25), (16). \tag{7.82}$$

These six box coefficients are constrained by the system of three SWI eq. (7.19). In solving these we must find solutions which satisfy,

$$S_1 \;:\; D_1^{ab} \longrightarrow D_1^{ab} \quad (ab) = (34), (25), (16),$$

$$S_1 \;:\; D_2^{ab} \leftrightarrow D_3^{ab} \quad (ab) = (34), (25), (16). \tag{7.83}$$

The identities that give amplitudes with the appropriate symmetries are,

$$\begin{aligned}
\langle 3|K_{123}|4\rangle \langle 3\,\eta\rangle &= t_{123} \langle 4\,\eta\rangle - \langle 1|K_{123}|4\rangle \langle 1\,\eta\rangle - \langle 2|K_{123}|4\rangle \langle 2\,\eta\rangle \,, \\
\langle 1\,2\rangle [5\,6] \langle 3\,\eta\rangle &= \langle 1\,3\rangle [5\,6] \langle 2\,\eta\rangle + \langle 3\,2\rangle [5\,6] \langle 1\,\eta\rangle \,, \\
\langle 3|K_{234}|1\rangle \langle 3\,\eta\rangle &= t_{234} \langle 1\,\eta\rangle - \langle 2|K_{234}|1\rangle \langle 2\,\eta\rangle - \langle 4|K_{234}|1\rangle \langle 4\,\eta\rangle \,, \\
\langle 2\,4\rangle [5\,6] \langle 3\,\eta\rangle &= \langle 3\,4\rangle [5\,6] \langle 2\,\eta\rangle + \langle 2\,3\rangle [5\,6] \langle 4\,\eta\rangle \,, \\
\langle 6|K_{345}|4\rangle \langle 3\,\eta\rangle &= \langle 4\,3\rangle [6\,1] \langle 1\,\eta\rangle + \langle 4\,3\rangle [6\,2] \langle 2\,\eta\rangle + \langle 6|K_{345}|3\rangle \langle 4\,\eta\rangle \,, \\
\langle 1\,2\rangle [3\,5] \langle 3\,\eta\rangle &= -\langle 5|K_{345}|2\rangle \langle 1\,\eta\rangle + \langle 5|K_{345}|1\rangle \langle 2\,\eta\rangle - \langle 1\,2\rangle [4\,5] \langle 4\,\eta\rangle \,,
\end{aligned} \tag{7.84}$$

$$\begin{aligned}
\langle 3|K_{123}|4\rangle \langle 5\,\eta\rangle &= \langle 3|K_{123}|5\rangle \langle 4\,\eta\rangle - [3\,1] \langle 4\,5\rangle \langle 1\,\eta\rangle - [3\,2] \langle 4\,5\rangle \langle 2\,\eta\rangle \,, \\
\langle 1\,2\rangle [5\,6] \langle 5\,\eta\rangle &= -\langle 1\,2\rangle [4\,6] \langle 4\,\eta\rangle - \langle 6|K_{123}|1\rangle \langle 2\,\eta\rangle + \langle 6|K_{123}|2\rangle \langle 1\,\eta\rangle \,, \\
\langle 3|K_{234}|1\rangle \langle 5\,\eta\rangle &= \langle 3|K_{234}|5\rangle \langle 1\,\eta\rangle - \langle 1\,5\rangle [3\,2] \langle 2\,\eta\rangle - \langle 1\,5\rangle [3\,4] \langle 4\,\eta\rangle \,, \\
\langle 2\,4\rangle [5\,6] \langle 5\,\eta\rangle &= -\langle 2\,4\rangle [1\,6] \langle 1\,\eta\rangle - \langle 6|K_{234}|2\rangle \langle 4\,\eta\rangle + \langle 6|K_{234}|4\rangle \langle 2\,\eta\rangle \,, \\
\langle 6|K_{345}|4\rangle \langle 5\,\eta\rangle &= \langle 6|K_{345}|5\rangle \langle 4\,\eta\rangle + \langle 4\,5\rangle [6\,1] \langle 1\,\eta\rangle + \langle 4\,5\rangle [6\,2] \langle 2\,\eta\rangle \,, \\
\langle 1\,2\rangle [3\,5] \langle 5\,\eta\rangle &= -\langle 1\,2\rangle [3\,4] \langle 4\,\eta\rangle - \langle 3|K_{345}|1\rangle \langle 2\,\eta\rangle + \langle 3|K_{345}|2\rangle \langle 1\,\eta\rangle \,,
\end{aligned} \tag{7.85}$$

$$\begin{aligned}
\langle 3|K_{123}|4\rangle \langle 6\,\eta\rangle &= \langle 3|K_{123}|6\rangle \langle 4\,\eta\rangle - [3\,1] \langle 4\,6\rangle \langle 1\,\eta\rangle - [3\,2] \langle 4\,6\rangle \langle 2\,\eta\rangle \,, \\
\langle 1\,2\rangle [5\,6] \langle 6\,\eta\rangle &= \langle 1\,2\rangle [4\,5] \langle 4\,\eta\rangle + \langle 5|K_{123}|1\rangle \langle 2\,\eta\rangle - \langle 5|K_{123}|2\rangle \langle 1\,\eta\rangle \,, \\
\langle 3|K_{234}|1\rangle \langle 6\,\eta\rangle &= \langle 3|K_{234}|6\rangle \langle 1\,\eta\rangle - \langle 1\,6\rangle [3\,2] \langle 2\,\eta\rangle - \langle 1\,6\rangle [3\,4] \langle 4\,\eta\rangle \,, \\
\langle 2\,4\rangle [5\,6] \langle 6\,\eta\rangle &= \langle 2\,4\rangle [1\,5] \langle 1\,\eta\rangle + \langle 5|K_{234}|2\rangle \langle 4\,\eta\rangle - \langle 5|K_{234}|4\rangle \langle 2\,\eta\rangle \,,
\end{aligned}$$

$$
\begin{aligned}
\langle 6|K_{345}|4\rangle \langle 6\,\eta\rangle &= -t_{345}\langle 4\,\eta\rangle - \langle 1|K_{345}|4\rangle \langle 1\,\eta\rangle - \langle 2|K_{345}|4\rangle \langle 2\,\eta\rangle\,, \\
\langle 1\,2\rangle [3\,5]\langle 6\,\eta\rangle &= -[3\,5]\langle 6\,1\rangle \langle 2\,\eta\rangle - [3\,5]\langle 2\,6\rangle \langle 1\,\eta\rangle\,.
\end{aligned}
\tag{7.86}
$$

The box coefficients are then given by,

$$
\begin{aligned}
D_1^{13} &= \left(-\frac{\langle 1|K_{123}|4\rangle}{\langle 3|K_{123}|4\rangle}\right)^{2-2h} D_1^A + \left(\frac{\langle 3\,2\rangle}{\langle 1\,2\rangle}\right)^{2-2h} D_1^B\,, \\
D_1^{23} &= \left(-\frac{\langle 2|K_{123}|4\rangle}{\langle 3|K_{123}|4\rangle}\right)^{2-2h} D_1^A + \left(\frac{\langle 1\,3\rangle}{\langle 1\,2\rangle}\right)^{2-2h} D_1^B\,, \\
D_1^{43} &= \left(\frac{t_{123}}{\langle 3|K_{123}|4\rangle}\right)^{2-2h} D_1^A\,, \\
D_1^{15} &= \left(\frac{\langle 4\,5\rangle [1\,3]}{\langle 3|K_{123}|4\rangle}\right)^{2-2h} D_1^A + \left(\frac{\langle 6|K_{123}|2\rangle}{\langle 1\,2\rangle [5\,6]}\right)^{2-2h} D_1^B\,, \\
D_1^{25} &= \left(\frac{\langle 4\,5\rangle [2\,3]}{\langle 3|K_{123}|4\rangle}\right)^{2-2h} D_1^A + \left(-\frac{\langle 6|K_{123}|1\rangle}{\langle 1\,2\rangle [5\,6]}\right)^{2-2h} D_1^B\,, \\
D_1^{16} &= \left(-\frac{[3\,1]\langle 4\,6\rangle}{\langle 3|K_{123}|4\rangle}\right)^{2-2h} D_1^A + \left(\frac{-\langle 5|K_{123}|2\rangle}{\langle 1\,2\rangle [5\,6]}\right)^{2-2h} D_1^B\,,
\end{aligned}
\tag{7.87}
$$

$$
\begin{aligned}
D_2^{13} &= \left(\frac{t_{234}}{\langle 3|K_{234}|1\rangle}\right)^{2-2h} D_2^A\,, \\
D_2^{23} &= \left(-\frac{\langle 2|K_{234}|1\rangle}{\langle 3|K_{234}|1\rangle}\right)^{2-2h} D_2^A + \left(\frac{\langle 3\,4\rangle}{\langle 2\,4\rangle}\right)^{2-2h} D_2^B\,, \\
D_2^{43} &= \left(-\frac{\langle 4|K_{234}|1\rangle}{\langle 3|K_{234}|1\rangle}\right)^{2-2h} D_2^A + \left(\frac{\langle 2\,3\rangle}{\langle 2\,4\rangle}\right)^{2-2h} D_2^B\,, \\
D_2^{15} &= \left(\frac{\langle 3|K_{234}|5\rangle}{\langle 3|K_{234}|1\rangle}\right)^{2-2h} D_2^A + \left(-\frac{[1\,6]}{[5\,6]}\right)^{2-2h} D_2^B\,, \\
D_2^{25} &= \left(\frac{[2\,3]\langle 1\,5\rangle}{\langle 3|K_{234}|1\rangle}\right)^{2-2h} D_2^A + \left(\frac{\langle 6|K_{234}|4\rangle}{\langle 2\,4\rangle [5\,6]}\right)^{2-2h} D_2^B\,, \\
D_2^{16} &= \left(\frac{\langle 3|K_{234}|6\rangle}{\langle 3|K_{234}|1\rangle}\right)^{2-2h} D_2^A + \left(-\frac{[5\,1]}{[5\,6]}\right)^{2-2h} D_2^B\,,
\end{aligned}
\tag{7.88}
$$

$$
\begin{aligned}
D_3^{13} &= \left(-\frac{[6\,1]\,\langle 3\,4\rangle}{\langle 6|K_{345}|4\rangle}\right)^{2-2h} D_3^A + \left(-\frac{\langle 5|K_{345}|2\rangle}{\langle 1\,2\rangle\,[3\,5]}\right)^{2-2h} D_3^B,\\
D_3^{23} &= \left(-\frac{[6\,2]\,\langle 3\,4\rangle}{\langle 6|K_{345}|4\rangle}\right)^{2-2h} D_3^A + \left(\frac{\langle 5|K_{345}|1\rangle}{\langle 1\,2\rangle\,[3\,5]}\right)^{2-2h} D_3^B,\\
D_3^{43} &= \left(\frac{\langle 6|K_{345}|3\rangle}{\langle 6|K_{345}|4\rangle}\right)^{2-2h} D_3^A + \left(-\frac{[4\,5]}{[3\,5]}\right)^{2-2h} D_3^B,\\
D_3^{15} &= \left(\frac{[6\,1]\,\langle 4\,5\rangle}{\langle 6|K_{345}|4\rangle}\right)^{2-2h} D_3^A + \left(\frac{\langle 3|K_{345}|2\rangle}{\langle 1\,2\rangle\,[3\,5]}\right)^{2-2h} D_3^B,\\
D_3^{25} &= \left(\frac{[6\,2]\,\langle 4\,5\rangle}{\langle 6|K_{345}|4\rangle}\right)^{2-2h} D_3^A + \left(-\frac{\langle 3|K_{345}|1\rangle}{\langle 1\,2\rangle\,[3\,5]}\right)^{2-2h} D_3^B,\\
D_3^{16} &= \left(-\frac{\langle 1|K_{345}|4\rangle}{\langle 6|K_{345}|4\rangle}\right)^{2-2h} D_3^A + \left(\frac{\langle 6\,2\rangle}{\langle 1\,2\rangle}\right)^{2-2h} D_3^B.
\end{aligned}
\tag{7.89}
$$

Next we have amplitudes with helicity structure $(-+-+-+)$. The purely gluonic amplitude is symmetric under,

$$
\begin{aligned}
S_1 &: A_6^{N=4}(1,2,3,4,5,6) \longrightarrow A_6^{N=4}(1,2,3,4,5,6)|_{j\to j+2},\\
S_2 &: A_6^{N=4}(1,2,3,4,5,6) \longrightarrow [A_6^{N=4}(1,2,3,4,5,6)|_{j\to j+1}]^{\dagger}.
\end{aligned}
\tag{7.90}
$$

In this case we denote the coefficients of $W_6^{(i)}$ in the purely gluonic case by G_i. These are given by,

$$
\begin{aligned}
G_1 \equiv G_1^A + G_1^B &= \left(\frac{\langle 2|K_{123}|5\rangle}{t_{123}}\right)^4 B_0 + \left(\frac{\langle 1\,3\rangle\,[4\,6]}{t_{123}}\right)^4 B_0^{\dagger},\\
G_2 \equiv G_2^A + G_2^B &= \left(\frac{\langle 6|K_{234}|3\rangle}{t_{234}}\right)^4 B_+^{\dagger} + \left(\frac{\langle 5\,1\rangle\,[2\,4]}{t_{234}}\right)^4 B_+,\\
G_3 \equiv G_3^A + G_3^B &= \left(\frac{\langle 4|K_{345}|1\rangle}{t_{345}}\right)^4 B_-^{\dagger} + \left(\frac{\langle 3\,5\rangle\,[6\,2]}{t_{345}}\right)^4 B_-.
\end{aligned}
\tag{7.91}
$$

Although there are only two independent configurations with two gluinos in this case, we present results for all the two gluino amplitudes appearing in the SWI eq. (7.27). Amplitudes with the correct symmetries are produced by applying the following identities to the SWI,

$$\begin{aligned}
\langle 2|K_{123}|5\rangle \langle 2\,\eta\rangle &= t_{123} \langle 5\,\eta\rangle - \langle 1|K_{123}|5\rangle \langle 1\,\eta\rangle - \langle 3|K_{123}|5\rangle \langle 3\,\eta\rangle\,, \\
\langle 1\,3\rangle [4\,6] \langle 2\,\eta\rangle &= \langle 2\,3\rangle [4\,6] \langle 1\,\eta\rangle + \langle 1\,2\rangle [4\,6] \langle 3\,\eta\rangle\,, \\
\langle 6|K_{234}|3\rangle \langle 2\,\eta\rangle &= -[6\,1] \langle 2\,3\rangle \langle 1\,\eta\rangle + \langle 6|K_{234}|2\rangle \langle 3\,\eta\rangle + [5\,6] \langle 2\,3\rangle \langle 5\,\eta\rangle\,, \\
\langle 5\,1\rangle [2\,4] \langle 2\,\eta\rangle &= -\langle 5\,1\rangle [3\,4] \langle 3\,\eta\rangle + \langle 4|K_{234}|5\rangle \langle 1\,\eta\rangle - \langle 4|K_{234}|1\rangle \langle 5\,\eta\rangle\,, \\
\langle 4|K_{345}|1\rangle \langle 2\,\eta\rangle &= \langle 4|K_{345}|2\rangle \langle 1\,\eta\rangle + [3\,4] \langle 1\,2\rangle \langle 3\,\eta\rangle - [4\,5] \langle 1\,2\rangle \langle 5\,\eta\rangle\,, \\
\langle 3\,5\rangle [6\,2] \langle 2\,\eta\rangle &= -\langle 3\,5\rangle [6\,1] \langle 1\,\eta\rangle - \langle 6|K_{345}|5\rangle \langle 3\,\eta\rangle + \langle 6|K_{345}|3\rangle \langle 5\,\eta\rangle\,.
\end{aligned} \tag{7.92}$$

The amplitudes are,

$$\begin{aligned}
G_1^{12} &= \left(-\frac{\langle 1|K_{123}|5\rangle}{\langle 2|K_{123}|5\rangle}\right)^{2-2h} G_1^A + \left(\frac{\langle 2\,3\rangle}{\langle 1\,3\rangle}\right)^{2-2h} G_1^B\,, \\
G_1^{32} &= \left(-\frac{\langle 3|K_{123}|5\rangle}{\langle 2|K_{123}|5\rangle}\right)^{2-2h} G_1^A + \left(\frac{\langle 1\,2\rangle}{\langle 1\,3\rangle}\right)^{2-2h} G_1^B\,, \\
G_1^{52} &= \left(\frac{t_{123}}{\langle 2|K_{123}|5\rangle}\right)^{2-2h} G_1^A\,,
\end{aligned} \tag{7.93}$$

$$\begin{aligned}
G_2^{12} &= \left(-\frac{\langle 2\,3\rangle [6\,1]}{\langle 6|K_{234}|3\rangle}\right)^{2-2h} G_2^A + \left(\frac{\langle 4|K_{234}|5\rangle}{\langle 5\,1\rangle [2\,4]}\right)^{2-2h} G_2^B\,, \\
G_2^{32} &= \left(\frac{\langle 6|K_{234}|2\rangle}{\langle 6|K_{234}|3\rangle}\right)^{2-2h} G_2^A + \left(-\frac{[3\,4]}{[2\,4]}\right)^{2-2h} G_2^B\,, \\
G_2^{52} &= \left(\frac{\langle 2\,3\rangle [5\,6]}{\langle 6|K_{234}|3\rangle}\right)^{2-2h} G_2^A + \left(-\frac{\langle 4|K_{234}|1\rangle}{\langle 5\,1\rangle [2\,4]}\right)^{2-2h} G_2^B\,,
\end{aligned} \tag{7.94}$$

$$
\begin{aligned}
G_3^{12} &= \left(\frac{\langle 4|K_{345}|2\rangle}{\langle 4|K_{345}|1\rangle}\right)^{2-2h} G_3^A + \left(-\frac{[6\,1]}{[6\,2]}\right)^{2-2h} G_3^B, \\
G_3^{32} &= \left(\frac{[3\,4]\,\langle 1\,2\rangle}{\langle 4|K_{345}|1\rangle}\right)^{2-2h} G_3^A + \left(-\frac{\langle 6|K_{345}|5\rangle}{\langle 3\,5\rangle\,[6\,2]}\right)^{2-2h} G_3^B, \\
G_3^{52} &= \left(-\frac{[4\,5]\,\langle 1\,2\rangle}{\langle 4|K_{345}|1\rangle}\right)^{2-2h} G_3^A + \left(\frac{\langle 6|K_{345}|3\rangle}{\langle 3\,5\rangle\,[6\,2]}\right)^{2-2h} G_3^B.
\end{aligned}
\tag{7.95}
$$

The six-point box coefficients have been explicitly checked by numerically evaluating the quadruple cuts.

7.2 One-Loop NMHV Amplitudes involving Gluinos and Scalars in $N = 4$ Gauge Theory

We use Supersymmetric Ward Identities and quadruple cuts to generate n-point NMHV amplitudes involving gluinos and adjoint scalars from purely gluonic amplitudes. We present a set of factors that can be used to generate one-loop NMHV amplitudes involving gluinos or adjoint scalars in $N = 4$ Super Yang-Mills from the corresponding purely gluonic amplitude.

7.2.1 Summary of NMHV Gluonic Amplitudes

In this section we review the n-point one-loop NMHV gluonic amplitudes derived in [71]. Our gluino amplitudes will be derived from these.

In any one-loop NMHV box diagram there are seven legs with negative helicity - three external and four internal. As each massive corner of the box requires at least two legs with negative helicity to be non-zero, we can have at most three massive corners. Further, the three mass boxes have a particularly simple form, with three massive MHV corners and one massless $\overline{\text{MHV}}$ (or Googly) corner. Thus they are "MHV-deconstructible" - in that they can be determined using purely MHV vertexes and, in this case, quadruple cuts. The three mass box coefficient $c^{3m}(m_1, m_2, m_3; A, B, C, d)$ where A, B and C are the massive corners, d is the massless corner and m_1, m_2 and m_3 are the legs with negative helicity, is given by [71],

$$\begin{aligned} c^{3m}(m_1, m_2, m_3; A, B, C, d) &= \frac{[\mathcal{H}(m_1, m_2, m_3; A, B, C, d)]^4}{\langle 1\,2\rangle \langle 2\,3\rangle \ldots \langle n\,1\rangle\, K_B^2} \\ &\times \frac{\langle A_{-1}\, B_1\rangle}{\langle d^-|\not{K}_C \not{K}_B|A_{-1}^+\rangle \langle d^-|\not{K}_C \not{K}_B|B_1^+\rangle} \\ &\times \frac{\langle B_{-1}\, C_1\rangle}{\langle d^-|\not{K}_A \not{K}_B|B_{-1}^+\rangle \langle d^-|\not{K}_A \not{K}_B|C_1^+\rangle}, \end{aligned} \tag{7.96}$$

where A_1 denotes the first leg of corner A and A_{-1} the last. When leg d has positive helicity, $\mathcal{H}$ is given by,

$$\begin{aligned}
\mathcal{H} &= 0, && m_{1,2,3} \in A, \\
&= 0, && m_{1,2,3} \in B, \\
&= \langle m_1\, m_2 \rangle \langle d^- | \not{K}_C \not{K}_B | m_3^+ \rangle, && m_{1,2} \in A,\ m_3 \in B, \\
&= \langle m_2\, m_3 \rangle \langle d^- | \not{K}_C \not{K}_B | m_1^+ \rangle, && m_1 \in A,\ m_{2,3} \in B, \\
&= \langle m_1\, m_2 \rangle \langle d\, m_3 \rangle K_B^2, && m_{1,2} \in A,\ m_3 \in C, \\
&= \langle m_1\, m_2 \rangle \langle d^- | \not{K}_A \not{K}_B | m_3^+ \rangle \\
&+ \langle m_3\, m_2 \rangle \langle d^- | \not{K}_C \not{K}_B | m_1^+ \rangle, && m_1 \in A,\ m_2 \in B,\ m_3 \in C,
\end{aligned} \tag{7.97}$$

and when leg d has negative helicity, $d = m_3$, $\mathcal{H}$ is given by,

$$\begin{aligned}
\mathcal{H} &= 0, && m_{1,2} \in A, \\
&= \langle m_1\, m_2 \rangle \langle d^- | \not{K}_C \not{K}_B | d^+ \rangle, && m_{1,2} \in B, \\
&= \langle d\, m_1 \rangle \langle d^- | \not{K}_C \not{K}_B | m_2^+ \rangle, && m_1 \in A,\ m_2 \in B, \\
&= \langle d\, m_1 \rangle \langle d\, m_2 \rangle K_B^2, && m_1 \in A,\ m_2 \in C.
\end{aligned} \tag{7.98}$$

The two mass-hard boxes are also MHV-deconstructible. As boxes with adjacent massless corners of the same type vanish, each non-vanishing two mass hard box has a massless MHV corner, a massless Googly corner and two massive MHV corners. Unfortunately, two mass easy and one mass boxes are not all MHV-deconstructible. However, all three types of box can be generated from the three mass boxes using IR consistency arguments [71]. For the two mass hard boxes the result is,

$$c^{2mh}(A, B, c, d) = c^{3m}(A, B, \{c\}, d) + c^{3m}(A, B, c, \{d\}), \tag{7.99}$$

where lower case letters denote massless corners and {} indicates that the corner should be thought of as the massless limit of a massive corner. This relationship has a simple interpretation in terms of the box diagrams: for each internal helicity configuration, one of the massless corners of the two mass hard box will be MHV and can be thought of as the massless limit of a massive MHV corner. Summing over internal helicity configurations in general gives two terms. If one of the helicity configurations gives a vanishing contribution, the corresponding three mass box coefficient will also vanish.

The two mass easy boxes can also be expressed in terms of three mass boxes [71],

$$c^{2me}(A,b,C,d) = \sum_{\hat{X},\hat{Y},\hat{Z}} c^{3m}(b,\hat{X}(d),\hat{Y},\hat{Z}) + \sum_{\hat{X},\hat{Y},\hat{Z}} c^{3m}(d,\hat{X}(b),\hat{Y},\hat{Z}), \tag{7.100}$$

where the sum is over all clusters (maintaining cyclic ordering) where $\hat{X}(a)$ contains leg a and $\hat{Y}$ is massive. Finally, the one mass boxes are given by [71],

$$c^{1m}(A,b,c,d) = c^{2me}(A,b,\{c\},d) + c^{3m}(A,\{b\},c,\{d\}). \tag{7.101}$$

These relationships are based on the IR properties of the box integral functions and thus carry over directly to amplitudes involving gluinos and scalars.

7.2.2 Conversion Factors from Supersymmetric Ward Identities and Quadruple Cuts

We first consider amplitudes with a pair of external gluinos. These are related to purely gluonic amplitudes through the SWI obtained by acting with Supersymmetry generator Q on $A(g^-_{m_1}, g^-_{m_2}, g^-_{m_3}, \bar{\Lambda}^+_q, \ldots)$, where $\ldots$ represents a string of positive helicity gluons. The structure of the SWI is independent of the ordering of the legs, but there are different SWI for each distinct ordering. We will explicitly show the case where the first three legs have negative helicity. As the SWI apply box by box, we have,

$$\begin{aligned} \langle q\,\eta\rangle\, c(g^-_{m_1}, g^-_{m_2}, g^-_{m_3}, g^+_q, \ldots) &= \langle m_1\,\eta\rangle\, c(\Lambda^-_{m_1}, g^-_{m_2}, g^-_{m_3}, \bar\Lambda^+_q, \ldots) \\ &+ \langle m_2\,\eta\rangle\, c(g^-_{m_1}, \Lambda^-_{m_2}, g^-_{m_3}, \bar\Lambda^+_q, \ldots) \\ &+ \langle m_3\,\eta\rangle\, c(g^-_{m_1}, g^-_{m_2}, \Lambda^-_{m_3}, \bar\Lambda^+_q, \ldots), \end{aligned} \tag{7.102}$$

where c is a generic box coefficient.

The SWI eq. (7.102) has rank two, so it determines two of the box coefficients in terms of the other two. Our approach is to determine one of the two-gluino box coefficients using quadruple cuts and then use the SWI to generate the other two. As in the purely gluonic case, we can express all of our box coefficients as sums of the three mass ones, so we only need to evaluate the latter explicitly. As the three mass boxes are MHV-deconstructible, we can use Quadruple Cuts where we can determine any of the tree amplitudes we need using [41]. A generic box is shown in the figure below,

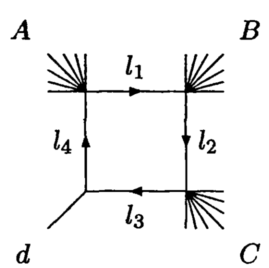


Figure 7.2.2.1: A Generic Three Mass Box Integral Function

The massless corner is a 3-point Googly vertex, so we have the following useful results,

$$l_3 = l_4 + d, \qquad |l_3^+\rangle \propto |d^+\rangle, \qquad |l_4^+\rangle \propto |d^+\rangle, \tag{7.103}$$

$$\langle dl_1\rangle[l_1 l_2]\langle l_2| = \langle d^-|\not{K}_A \not{K}_B|, \qquad \langle dl_2\rangle[l_2 l_1]\langle l_1| = \langle d^-|\not{K}_C \not{K}_B|, \tag{7.104}$$

which allow us to evaluate expressions that are homogeneous in l_i.

We label the purely gluonic boxes by the location of the negative helicity legs. The "AAB" box shown below has two negative helicity legs on corner A and one on corner B,

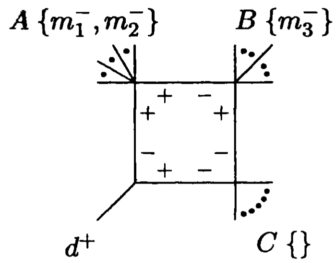


Figure 7.2.2.2: Example of an AAB Box Integral Function

When we relate this purely gluonic box to one with a pair of external gluinos, we must specify which g^+ to replace by $\bar{\Lambda}^+$ and which g^- to replace by Λ^-. Using m to label the external Λ^- leg, q to label the external $\bar{\Lambda}^+$ leg and $L(q)$ to denote its location, the conversion factor is given by $R^{\text{box label}}_{L(q)m}$, so that,

$$c^{xxx}(g^-, g^-, \Lambda^-_m, \bar{\Lambda}^+_q, \ldots) = R^{xxx}_{L(q)m} c^{xxx}(g^-, g^-, g^-_m, g^+_q, \ldots). \tag{7.105}$$

The AAB box shown is an example of a "singlet" box, where only gluons can circulate in the loop and there is a single contribution to the purely gluonic box coefficient. We can immediately see that there is no possible routing for a fermion, Λ^-, from corner B to corner A or d, so we have,

$$R^{AAB}_{Am_3} = R^{AAB}_{dm_3} = 0. \tag{7.106}$$

The remaining $R^{AAB}_{Am_i}$ and $R^{AAB}_{dm_i}$ then follow from the SWI. For $R^{AAB}_{Bm_i}$ and $R^{AAB}_{Cm_i}$ there are either one or two possible fermion routings and one box in each class must be calculated using quadruple cuts before the other two can be read off from the SWI. The conversion factors for the other singlet boxes can be similarly evaluated. The results of these calculations are presented in table 7.2.1.

The ABC boxes are "non-singlet" and any particle in the $N = 4$ multiplet can circulate in the loop. The purely gluonic box coefficients are obtained by summing over diagrams with all possible particles circulating in the loop. If the $\bar{\Lambda}^+$ and Λ^- attach to the same corner, any particle can still circulate in the loop. Each corner remains MHV, but care must be taken with the flavour structure of corners with four non-gluonic legs as these amplitudes are flavour dependent. In all cases the MHV tree amplitudes can be found using [41].

Our results are presented in table 7.2.1. For each type of box the conversion factors have a common denominator. The factor appearing in each denominator also appears in the numerator of the corresponding purely gluonic amplitude, where it is raised to the fourth power. Conversion factors are presented for all distinct cases. The factors not explicitly listed can be obtained by flipping (e.g. AAB boxes flip into BCC boxes). The denominator of each conversion factor is given next to the box name and the numerators are listed for each location of q and for each m.

AAB		$\langle d^- \vert \not{K}_C \not{K}_B \vert m_3^+ \rangle \langle m_1 m_2 \rangle$
A	m_1	$\langle d^- \vert \not{K}_C \not{K}_B \vert m_3^+ \rangle \langle q m_2 \rangle$
$/d$	m_2	$\langle d^- \vert \not{K}_C \not{K}_B \vert m_3^+ \rangle \langle m_1 q \rangle$
	m_3	0
B	m_1	$-\langle d^- \vert \not{K}_C \not{K}_B \vert m_2^+ \rangle \langle m_3 q \rangle$
	m_2	$-\langle d^- \vert \not{K}_C \not{K}_B \vert m_1^+ \rangle \langle q m_3 \rangle$
	m_3	$\langle d^- \vert \not{K}_C \not{K}_B \vert q^+ \rangle \langle m_1 m_2 \rangle$
C	m_1	$\langle d^- \vert \not{K}_A \not{K}_B \vert q^+ \rangle \langle m_2 m_3 \rangle - \langle d^- \vert \not{K}_C \not{K}_B \vert m_2^+ \rangle \langle m_3 q \rangle$
	m_2	$\langle d^- \vert \not{K}_A \not{K}_B \vert q^+ \rangle \langle m_3 m_1 \rangle - \langle d^- \vert \not{K}_C \not{K}_B \vert m_1^+ \rangle \langle q m_3 \rangle$
	m_3	$\langle d^- \vert \not{K}_A \not{K}_B \vert q^+ \rangle \langle m_1 m_2 \rangle + \langle d^- \vert \not{K}_C \not{K}_B \vert q^+ \rangle \langle m_1 m_2 \rangle$
		$= -K_B^2 \langle d q \rangle \langle m_1 m_2 \rangle$

AAC		$K_B^2\langle dm_3\rangle\langle m_1m_2\rangle$
A	m_1	$K_B^2\langle dm_3\rangle\langle qm_2\rangle$
$/d$	m_2	$K_B^2\langle dm_3\rangle\langle m_1q\rangle$
	m_3	0
B	m_1	$-\langle d^-\vert\not{K}_A\not{K}_B\vert m_3^+\rangle\langle qm_2\rangle + \langle d^-\vert\not{K}_C\not{K}_B\vert m_2^+\rangle\langle m_3q\rangle$
	m_2	$-\langle d^-\vert\not{K}_A\not{K}_B\vert m_3^+\rangle\langle m_1q\rangle + \langle d^-\vert\not{K}_C\not{K}_B\vert m_1^+\rangle\langle qm_3\rangle$
	m_3	$-\langle d^-\vert\not{K}_C\not{K}_B\vert q^+\rangle\langle m_1m_2\rangle$
C	m_1	$-K_B^2\langle dm_2\rangle\langle m_3q\rangle$
	m_2	$-K_B^2\langle dm_1\rangle\langle qm_3\rangle$
	m_3	$K_B^2\langle dq\rangle\langle m_1m_2\rangle$

ABB		$-\langle d^-\vert\not{K}_C\not{K}_B\vert m_1^+\rangle\langle m_2m_3\rangle$
A	m_1	$-\langle d^-\vert\not{K}_C\not{K}_B\vert q^+\rangle\langle m_2m_3\rangle$
$/d$	m_2	$\langle d^-\vert\not{K}_C\not{K}_B\vert m_3^+\rangle\langle m_1q\rangle$
	m_3	$\langle d^-\vert\not{K}_C\not{K}_B\vert m_2^+\rangle\langle qm_1\rangle$
B	m_1	0
	m_2	$-\langle d^-\vert\not{K}_C\not{K}_B\vert m_1^+\rangle\langle qm_3\rangle$
	m_3	$-\langle d^-\vert\not{K}_C\not{K}_B\vert m_1^+\rangle\langle m_2q\rangle$
C	m_1	$-\langle d^-\vert\not{K}_A\not{K}_B\vert q^+\rangle\langle m_2m_3\rangle$
	m_2	$\langle d^-\vert\not{K}_A\not{K}_B\vert q^+\rangle\langle m_3m_1\rangle - \langle d^-\vert\not{K}_C\not{K}_B\vert m_1^+\rangle\langle qm_3\rangle$
	m_3	$\langle d^-\vert\not{K}_A\not{K}_B\vert q^+\rangle\langle m_1m_2\rangle - \langle d^-\vert\not{K}_C\not{K}_B\vert m_1^+\rangle\langle m_2q\rangle$

ABd		$\langle d^-\vert\not{K}_C\not{K}_B\vert m_2^+\rangle\langle dm_1\rangle$
A	m_1	$\langle d^-\vert\not{K}_C\not{K}_B\vert m_2^+\rangle\langle dq\rangle$
	m_2	0
	d	$\langle d^-\vert\not{K}_C\not{K}_B\vert m_2^+\rangle\langle qm_1\rangle$
B	m_1	$-\langle d^-\vert\not{K}_C\not{K}_B\vert d^+\rangle\langle qm_2\rangle$
	m_2	$\langle d^-\vert\not{K}_C\not{K}_B\vert q^+\rangle\langle dm_1\rangle$
	d	$-\langle d^-\vert\not{K}_C\not{K}_B\vert m_1^+\rangle\langle m_2q\rangle$
C	m_1	$\langle d^-\vert\not{K}_A\not{K}_B\vert q^+\rangle\langle m_2d\rangle - \langle d^-\vert\not{K}_C\not{K}_B\vert d^+\rangle\langle qm_2\rangle$
	m_2	$-K_B^2\langle dq\rangle\langle dm_1\rangle$
	d	$\langle d^-\vert\not{K}_A\not{K}_B\vert q^+\rangle\langle m_1m_2\rangle - \langle d^-\vert\not{K}_C\not{K}_B\vert m_1^+\rangle\langle m_2q\rangle$

ACd		$K_B^2\langle dm_1\rangle\langle dm_2\rangle$
A	m_1	$K_B^2\langle dm_2\rangle\langle dq\rangle$
	m_2	0
	d	$K_B^2\langle dm_2\rangle\langle qm_1\rangle$
B	m_1	$\langle d^-\vert\not{K}_A\not{K}_B\vert q^+\rangle\langle m_2d\rangle$
	m_2	$-\langle d^-\vert\not{K}_C\not{K}_B\vert q^+\rangle\langle dm_1\rangle$
	d	$-\langle d^-\vert\not{K}_A\not{K}_B\vert m_2^+\rangle\langle qm_1\rangle + \langle d^-\vert\not{K}_C\not{K}_B\vert m_1^+\rangle\langle m_2q\rangle$
C	m_1	0
	m_2	$-K_B^2\langle dm_1\rangle\langle m_2q\rangle$
	d	$-K_B^2\langle dm_1\rangle\langle qd\rangle$

BBd		$\langle d^- \vert \not{K}_A \not{K}_B \vert d^+ \rangle \langle m_1 m_2 \rangle$
A	m_1	$\langle d^- \vert \not{K}_C \not{K}_B \vert m_2^+ \rangle \langle dq \rangle$
	m_2	$\langle d^- \vert \not{K}_C \not{K}_B \vert m_1^+ \rangle \langle qd \rangle$
	d	$-\langle d^- \vert \not{K}_C \not{K}_B \vert q^+ \rangle \langle m_1 m_2 \rangle$
B	m_1	$-\langle d^- \vert \not{K}_C \not{K}_B \vert d^+ \rangle \langle q m_2 \rangle$
	m_2	$-\langle d^- \vert \not{K}_C \not{K}_B \vert d^+ \rangle \langle m_1 q \rangle$
	d	0
C	m_1	$+\langle d^- \vert \not{K}_A \not{K}_B \vert m_2^+ \rangle \langle dq \rangle$
	m_2	$-\langle d^- \vert \not{K}_A \not{K}_B \vert m_1^+ \rangle \langle qd \rangle$
	d	$-\langle d^- \vert \not{K}_A \not{K}_B \vert q^+ \rangle \langle m_1 m_2 \rangle$

ABC		$\langle d^- \vert \not{K}_A \not{K}_B \vert m_3^+ \rangle \langle m_1 m_2 \rangle - \langle d^- \vert \not{K}_C \not{K}_B \vert m_1^+ \rangle \langle m_2 m_3 \rangle$
A	m_1	$\langle d^- \vert \not{K}_A \not{K}_B \vert m_3^+ \rangle \langle q m_2 \rangle - \langle d^- \vert \not{K}_C \not{K}_B \vert q^+ \rangle \langle m_2 m_3 \rangle$
	m_2	$-K_B^2 \langle d m_3 \rangle \langle m_1 q \rangle$
	m_3	$\langle d^- \vert \not{K}_C \not{K}_B \vert m_2^+ \rangle \langle q m_1 \rangle$
B	m_1	$\langle d^- \vert \not{K}_A \not{K}_B \vert m_3^+ \rangle \langle q m_2 \rangle$
	m_2	$\langle d^- \vert \not{K}_A \not{K}_B \vert m_3^+ \rangle \langle m_1 q \rangle - \langle d^- \vert \not{K}_C \not{K}_B \vert m_1^+ \rangle \langle q m_3 \rangle$
	m_3	$-\langle d^- \vert \not{K}_C \not{K}_B \vert m_1^+ \rangle \langle m_2 q \rangle$
C	m_1	$-\langle d^- \vert \not{K}_A \not{K}_B \vert m_2^+ \rangle \langle m_3 q \rangle$
	m_2	$K_B^2 \langle d m_1 \rangle \langle q m_3 \rangle$
	m_3	$\langle d^- \vert \not{K}_A \not{K}_B \vert q^+ \rangle \langle m_1 m_2 \rangle - \langle d^- \vert \not{K}_C \not{K}_B \vert m_1^+ \rangle \langle m_2 q \rangle$
d	m_1	$-\langle d^- \vert \not{K}_A \not{K}_B \vert m_2^+ \rangle \langle m_3 d \rangle$
	m_2	$K_B^2 \langle d m_1 \rangle \langle d m_3 \rangle$
	m_3	$\langle d^- \vert \not{K}_C \not{K}_B \vert m_2^+ \rangle \langle d m_1 \rangle$

Table 7.2.1: Numerators and Denominators For Conversion Factors R_{qm}^{xxx}

The general effect of applying one of these conversion factors is to replace the $\mathcal{H}^4$ factor in the purely gluonic box coefficient by $\mathcal{H}^3\tilde{\mathcal{H}}$, where $\tilde{\mathcal{H}}$ is the factor appearing in the "switched" purely gluonic box coefficient, where leg q is a negative helicity gluon and leg m is a positive helicity gluon. This is reminiscent of the behaviour of the MHV tree amplitudes, but in this case it appears at the level of the box coefficients.

So far we have only considered three mass boxes. As in the purely gluonic case, the two mass and one mass box coefficients for gluinos can be expressed as sums of three mass box coefficients. Given that the factors appearing in the SWI are simply determined by the momenta of the legs on which the Supersymmetry generator acts, we see that, when expressed in terms of three mass boxes, any SWI for (say) a two mass easy box is just a sum of the three mass SWI and thus trivially satisfied. We have explicitly calculated the n-point two mass-hard box coefficients with two gluinos using quadruple cuts and verified the consistency of the two approaches. We have also used the 6-point NMHV tree expressions of [17] to calculate both singlet and non-singlet example 8-point two mass-easy box coefficients using quadruple cuts. These results are also in agreement with those obtained by summing the appropriate three mass coefficients.

7.2.3 Beyond Two Fermion Amplitudes

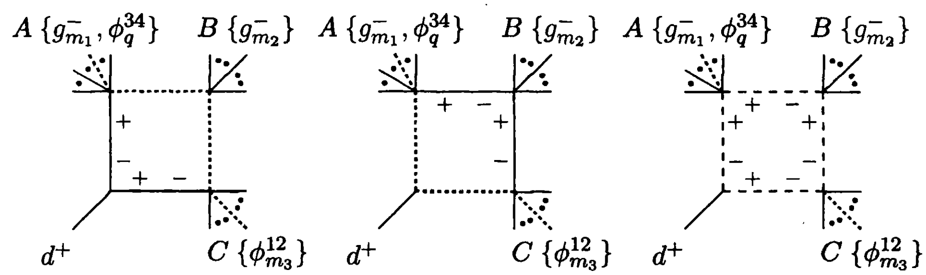
The conversion factors in table 7.2.1 can be compounded to generate amplitudes with arbitrary numbers of external adjoint scalars and fermions. The first step is to note that the box coefficient for a diagram involving two external scalars can be obtained by simply squaring the conversion factor for the corresponding two gluino diagram,

$$c^{xxx}(g^-, g^-, \phi_m^-, \phi_q^+, \ldots) = (R^{xxx}_{L(q)m})^2 c^{xxx}(g^-, g^-, g_m^-, g_q^+, \ldots). \tag{7.107}$$

For singlet two-gluino diagrams with only one possible route for the fermion, the corresponding two-scalar diagram is obtained from the two gluino by replacing the single fermion line with a single scalar line. As we only have MHV and Googly corners in the three mass boxes, this simply gives us the square of the factor relating the two gluino box coefficient to the gluonic. For two-gluino diagrams with two routes for the gluino, there are two-scalar diagrams where the scalar takes one of these two routes and additionally there is a diagram with a fermionic loop. The first two diagrams give factors which are the squares of the individual gluino factors, while explicit calculation shows that the last yields precisely the cross-term that arises when the sum of the gluino terms is squared. For the non-singlet diagrams, explicit calculation again shows that the two scalar box coefficients are also simply obtained by squaring the relevant conversion factor. Recalling that we can express all of our box coefficients in terms of the three mass ones, we see that all the two-scalar box coefficients are simply obtained by squaring the relevant factors in table 7.2.1.

To obtain SWI involving scalars we consider the action of a pair of Supersymmetry generators Q_1 and Q_2 that generate an $N = 2$ sub-algebra [11, 25].

The SWI then contain amplitudes involving two flavours of gluino and a single flavour scalar. In $N = 2$ terms it is natural to denote scalars as $\phi^+ \equiv \phi_{12}$ and $\phi^- \equiv \phi_{34}$. This notation is more compact than the full $N = 4$ flavour labelling, but care must be taken when counting the negative helicities required for a MHV vertex. In particular, replacing a g^+ by ϕ_{34} effectively introduces an extra negative helicity. This is important in understanding the two-scalar ABC boxes, as all three of the diagrams shown below contribute to this two-scalar box coefficient,



In these diagrams dotted lines represent scalars, while dashed lines represent fermions.

Next we consider the NMHV amplitudes with three non-gluonic external legs. These box coefficients are related to the purely gluonic ones by a pair of conversion factors,

$$c^{xxx}(g^-, g^-, \phi^-_m, \bar{\Lambda}^+_{q_1}, \bar{\Lambda}^+_{q_2} \ldots) = R^{xxx}_{L(q_1)m} R^{xxx}_{L(q_2)m} c^{xxx}(g^-, g^-, g^-_m, g^+_{q_1}, g^+_{q_2} \ldots), \tag{7.108}$$

$$c^{xxx}(g^-, \Lambda^-_{m_1}, \Lambda^-_{m_2}, \phi^+_q, \ldots) = R^{xxx}_{L(q)m_1} R^{xxx}_{L(q)m_2} c^{xxx}(g^-, g^-_{m_1}, g^-_{m_2}, g^+_q, \ldots). \tag{7.109}$$

For boxes with unique routings for the fermions, this result again follows directly from the form of the MHV amplitudes at each corner. For all boxes explicit calculation shows that the fermionic factors compound in this way.

Amplitudes involving four or more non-gluonic legs can now be generated directly from the appropriate SWI. We define the "level" of an amplitude to be the number of external fermions plus twice the number of external scalars. Amplitudes with odd level will vanish and we use these as the starting points for our SWI. For example, acting with Q_2 on the level 3 amplitude, $A(\Lambda_1^{1-}, g_2^-, g_3^-, \bar{\Lambda}_4^{1+}, \bar{\Lambda}_5^{2+}, \ldots)$, gives $A(\Lambda_1^{1-}, \Lambda_2^{2-}, g_3^-, \bar{\Lambda}_4^{1+}, \bar{\Lambda}_5^{2+}, \ldots)$ and $A(\Lambda_1^{1-}, g_2^-, \Lambda_3^{2-}, \bar{\Lambda}_4^{1+}, \bar{\Lambda}_5^{2+}, \ldots)$ in terms of known amplitudes. We can work systematically, level by level, to generate amplitudes with any number of external scalars and fermions.

Chapter 8

Conclusion

We have described techniques for efficient analytical calculation of scattering amplitudes in gauge theories, with particular reference to QCD. Soon, the Large Hadron Collider will become operational at CERN, giving the physics community an experimental handle on high energy regions of the Standard Model never before accessible. This will allow us to examine the predictions of the Standard Model in this region as well as probe for new physics, such as evidence of the Higgs scalar or manifestations of Supersymmetry. To detect the signals corresponding to the new physics we must be able to recognise, and subtract, the signals corresponding to the standard QCD processes that will obviously occur at a Hadron Collider. Many of the events we are interested in involve the production of multiple jets of final state particles. The traditional approach to calculating cross sections in perturbative field theory, Feynman diagrams, is not sufficient to calculate such processes as the technique is quickly rendered ineffective by the sheer number and complexity of the calculations required. Thus, a new approach must be found if we are to be able to make use of the LHC data.

This new approach has been developed over a number of years, and draws on a variety of techniques, some of which are new and some more traditional. By using tools such as the spinor helicity formalism, the color ordering of amplitudes, factorisation limits and unitarity cuts, Supersymmetric rearrangements and integral reduction techniques, we can greatly simplify calculations. However, by themselves these tools are not sufficient to complete the complex calculations required. In the last few years a number of new techniques have been developed that have led to significant progress.

The focus of this research was to examine these techniques in theories with less Supersymmetry than originally proposed, and to investigate the extent to which the twistor structure described by Witten extended to such theories. We focused on calculating one-loop amplitudes involving external gluonic particles in theories with $N = 1$ Supersymmetries. Primarily we calculated six-point one-loop amplitudes, but were able to extend our analysis to include n-point examples of certain helicity configurations. In later work, we also calculated one-loop amplitudes involving external fermionic particles in $N = 4$ theories

To begin with, we examined how the "holomorphic anomaly", which was discovered at $N = 4$, acts upon the cuts of $N = 1$ Supersymmetric one-loop amplitudes, as discussed in section 6.1. To do this we focused on the previously unpublished six-gluon non-MHV amplitude $A^{N=1 \text{ chiral}}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$, (that had been calculated by independent means). We showed that, when acting with the collinear differential operator on the cuts of an amplitude, in order to match the effect of the operator acting upon the imaginary part of the amplitude — as required by the optical theorem — the "holomorphic anomaly" must be taken into account. We found that as a calculational tool for evaluating amplitudes, application of the "holomorphic anomaly" gave differential equations for the coefficients of the integral functions, unlike the $N = 4$ case where algebraic equations were derived. Furthermore we found that, since the equations were differential, their general solution contained homogeneous parts which could be fixed by boundary conditions or physical constraints such as collinear limits. We used this principle to calculate some of the scalar integral function coefficients for the n-point amplitude $A^{N=1 \text{ chiral}}(1^-, 2^-, 3^-, 4^+, \ldots, n^+)$.

In section 6.2 we continued to investigate the twistor structure of the box coefficients of $N = 1$ one-loop amplitudes, as it had been observed that these coefficients in particular exhibit an interesting twistor space structure. For example in $N = 4$ gauge theory it had been shown that the box coefficients of next to MHV amplitudes have planar support in twistor space, behaviour that is analogous to that of tree amplitudes. We investigated whether similar behaviour exists for theories with $N < 4$ Supersymmetries. In doing so we calculated the box coefficients for all six-point $N = 1$ amplitudes and the n-point $N = 1$ amplitudes $A^{N=1 \text{ chiral}}(1^-, 2^-, \ldots, j^+, (j+1)^-, 5^+, \ldots, n^+)$ and $A^{N=1 \text{ chiral}}(1^-, 2^-, \ldots, (j-1)^-, j^+, (j+1)^+, \ldots, k^-, \ldots, (n-1)^+, n^+)$ and examined their twistor structure. We found that for next to MHV amplitudes these coefficients have planar support in twistor space, explicitly confirming that the structure of $N = 4$ box coefficients persists to $N = 1$.

In section 6.3 we continued to examine the twistor structure of amplitudes with $N < 4$ Supersymmetries. Although relations with twistor string theories had been observed for $N = 4$ Super Yang-Mills, it remained unresolved as to what degree theories with less or no Supersymmetry were related to a twistor string theory. Therefore, until a direct connection could be uncovered we felt it appropriate to continue to gather evidence by studying the properties of amplitudes. As we discussed in section 6.3, by computing some special examples, those that we describe as MHV-deconstructible, of box coefficients of the amplitudes $A^{N=0,1}(1^-, 2^-, 3^+, 4^-, 5^+, 6^+)$ and $A^{N=0,1}(1^-, 2^+, 3^-, 4^+, 5^-, 6^+)$, we observed that even for non-Supersymmetric theories (but still massless) the box coefficients still satisfy the same collinearity and coplanarity constraints as in $N = 4$ theories. As such these constraints can be viewed in one of two ways — as a consequence of the construction of box coefficients using unitarity techniques — or more significantly they may hint at an underlying structure compatible with the twistor description already seen at $N = 4$ and to a lesser extent at $N = 1$.

For $N = 4$ theories the amplitudes are completely determined from the box coefficients. Thus the properties of these coefficients determine entirely the properties of an amplitude. For theories with less Supersymmetry the amplitudes contain additional functional information that plays an equally important role. As an example of using unitarity constraints in such a context, in section 6.3.4 we presented the full structure of the simplest NMHV configuration for n-gluons in $N = 1$ Super Yang Mills, $A^{N=1}(1^-, 2^-, 3^-, 4^+, 5^+, \ldots, n^+)$ This amplitude was expressed entirely in terms of triangle functions. The coefficients of these functions were determined by carrying out triple cuts on the amplitude. We found that these coefficients did not have an obvious twistor property such as coplanarity, which suggests that it is only the box coefficients that inherit this structure in theories with less Supersymmetry.

The second part of the research presented here focused less on the twistor space properties of amplitudes and more on efficient techniques for calculating one-loop amplitudes directly. The recent progress in calculating purely gluonic one-loop amplitudes in compact forms stimulated our interest in using Supersymmetric Ward Identities (SWI), together with the inherent symmetries of an amplitude, to generate one-loop amplitudes where the external particles are gluinos or adjoint scalars. In particular, in section 7.1 we calculated all the six-point $N = 4$ NMHV one-loop amplitudes involving two gluinos or scalars, i.e. we considered all combinations of the helicity configurations $(- - - + + +)$, $(- - + - + +)$ and $(- + - + - +)$. The amplitudes with four or six gluinos (of a single flavour) were given as linear combinations of these two gluino amplitudes. Although the results we computed are specific to Supersymmetric theories with adjoint fermions, they do still reduce the amount of computation required to obtain results in non-Supersymmetric theories with fundamental quarks.

In section 7.2 we then looked to extend this analysis to include all n-point $N =$

4 NMHV one-loop amplitudes involving external gluinos. Again we exploited the property that one-loop NMHV amplitudes in $N = 4$ gauge theory can be expressed in terms of MHV-deconstructible diagrams. They can therefore be evaluated using quadruple cuts and known MHV tree amplitudes. We used the SWI to minimise the number of diagrams that had to be computed explicitly. We used this techniques to determine a set of conversion factors that relate two-gluino box coefficients to purely gluonic ones, and which are applicable to any number of external particles. We were also able to use our analysis of the quadruple cuts to show how these factors could be compounded to give two-scalar and scalar-gluino-gluino box coefficients. Finally we showed how amplitudes involving more external fermions/scalars would then follow from more SWI.

Although organising amplitudes in terms of helicity structure, particle type, color and Supersymmetry has helped enormously in understanding the structure of interactions in Yang-Mills theory, the list of simple amplitudes required to compute an experimental quantity is rather long. What we have shown is that SWI can be used to generate amplitudes without the need for explicit computation, and is thus a very helpful technique in this context.

Since Witten's original paper on the subject, a number of new techniques for calculating scattering amplitudes relevant to LHC physics have been developed. All of these were strongly motivated by the twistor description of amplitudes. However, it is important to point out that there is no one tool that we can use every time, but rather there are now a number of techniques we can use which, when combined, can complete calculations that were previously unattainable. This includes the traditional techniques we discussed previously, such as organising amplitudes in terms of spinor helicity and color ordering, Supersymmetric decompostions and loop integral reductions, as well as the new techniques developed more recently.

Thus far, the new techniques we have discussed have been used primarily to calculate tree amplitudes — that could have been calculated by existing numerical methods — and loop amplitudes in theories with Supersymmetry. Ultimately we want to calculate loop amplitudes in full QCD. As we have discussed, in gauge theories we can expand an amplitude as a sum of known scalar integral functions multiplied by rational coefficients. We have also seen that Supersymmetry restricts the different type of scalar integral function that can appear in this expansion. However, for non-Supersymmetric theories this expansion also contains rational pieces that we cannot calculate using the various methods discussed in this thesis. To reconstruct the full QCD amplitude we must calculate these rational pieces as well as the coefficients of the scalar integral functions. Thus there is much work left to be done. The techniques we have discussed in this thesis must be extended to $N = 0$ amplitudes before they offer direct phenomenological applications. However, extending this analysis to $N = 0$ amplitudes will be an extremely difficult and challenging task, as the scalar integral functions and their coefficients represent a smaller fraction of the information contained in an amplitude.

One twistor inspired method in particular has emerged as a promising technique for calculating loop amplitudes in full QCD. At the beginning of 2005, on-shell recursion relations were developed by Britto, Cachazo, Feng and Witten (BCFW) [79, 80]. The basic principle is that an amplitude can be represented as a sum of lower point amplitudes, evaluated on-shell, but for complex shifted values of the momenta. These relations are even more efficient than the CSW rules and lead to even more compact formulas for amplitudes. As the proof of recursion relations relies only on the factorisation properties of the amplitudes themselves and Cauchy's integral theorem, extending the recursion relations to more general processes is relatively simple. This includes applying similar techniques to one-loop amplitudes in QCD.

BCFW's method is particularly promising for completing the calculation of one-loop QCD amplitudes because it can be used to determine the rational pieces that appear in the expansion of an amplitude, once the coefficients of the scalar integral functions have been calculated by other means — for instance by using the techniques discussed previously in this thesis. Recursion relations have been used to calculate all of the one loop n-gluon QCD amplitudes with up to two adjacent negative helicity gluons, the rest positive [81, 82]. In addition, their has been more recent success in using this technique to calculate other full one-loop QCD amplitudes, see [83, 84].

Despite the obvious difficulties involved, theories without Supersymmetry are the most interesting. The techniques we have discussed, if generalised sufficiently to be applied to theories with no Supersymmetry, may in principle determine amplitudes in such theories. Although practical computations remain sparse at this point, the recent progress made in this regard is extremely promising. It remains a huge, yet fundamentally important challenge to develop techniques and perform calculations for theories without any Supersymmetry.

More generally, significant recent progress includes the following work. Mason, Skinner, Boels and Hull [85] have shown how to construct new actions for $N = 4$ super Yang-Mills theories. The results of their work indicate the existence of a theory in twistor space which is exactley equivalent to spacetime $N = 4$ super Yang-Mills. Bern, Dixon and Kosower [86] have led the way in extending the techniques that have been so sucessful at one-loop to higher loops. This work has shown strong links to $N = 4$ integrability structures. Finaly, both Mansfield and Morris have contributed to the development of a lagrangian derivation of the MHV rules [87]. This is particularly significant as it provides conclusive proof of a technique that worked extremely well practically but which lacked credibility in some quarters due to a lack of a formal theoretical proof.

All of these research groups are actively engaged in the study of ideas stimulated by Witten's original proposal of a gauge theory - string theory duality. That this original proposal has attracted such interest and continues to inspire research in ever more complex directions suggests that the unattainable calculations needed for LHC may soon be within our grasp.

Chapter 9

Appendix A

In this Appendix we describe the full notation discussed in section 3.1 and define the spinor algebra.

The massless Dirac equation

$$\not{p}\,\Psi(p) \equiv p^{\mu}\gamma_{\mu}\Psi(p) = 0 \qquad\qquad p^2 = 0 \tag{9.1}$$

has plane wave solutions of the form

$$\Psi = \psi(p)e^{-i.p.x},$$

where $\psi(p)$ represents a four dimensional Dirac spinor, p denotes the 4-momentum and x is defined as the 4-vector $(t, \mathbf{x})$. This has both positive and negative energy solutions, $\psi = u(p)$ and $\psi = v(p)$ respectively. We can define two helicity states by acting with the chiral projection operators

$$u_{\pm}(p) \quad = \quad \frac{1}{2}(1 \pm \gamma_5)u(p)$$

$$v_{\mp}(p) \quad = \quad \frac{1}{2}(1 \pm \gamma_5)v(p) \tag{9.2}$$

For amplitudes involving a large number of momenta labelled p_i, $i = 1, ..., n$, we can use the following notation, which we call the spinor helicity formalism,

$$\begin{aligned} u_{\pm}(p_i) &= v_{\mp}(p_i) \equiv |p_i^{\pm}\rangle \equiv |i^{\pm}\rangle \\ \overline{u_{\pm}(p_i)} &= \overline{v_{\mp}(p_i)} \equiv \langle p_i^{\pm}| \equiv \langle i^{\pm}| \end{aligned} \tag{9.3}$$

with normalisation,

$$\langle p^{\pm}|\gamma_{\mu}|p_{\pm}\rangle = 2p_{\mu}. \tag{9.4}$$

The basic spinor products are defined by

$$\begin{aligned} \langle i\, j\rangle &\equiv \langle i^{-}|j^{+}\rangle = \overline{u_{-}(p_i)}\; u_{+}(p_j) \\ [i\, j] &\equiv \langle i^{+}|j^{-}\rangle = \overline{u_{+}(p_i)}\; u_{-}(p_j). \end{aligned} \tag{9.5}$$

They are antisymmetric, i.e.

$$\langle i\, j\rangle = -\langle j\, i\rangle\,, \qquad [i\, j] = -[j\, i]\,, \qquad \langle i\, i\rangle = [i\, i] = 0. \tag{9.6}$$

Using the helicity projection operators and the properties of the Dirac algebra we can show that,

$$\langle p^{\pm}|k^{\pm}\rangle = 0. \tag{9.7}$$

Using eq. (9.4) and the identity

$$\not{p} = |p^{+}\rangle\langle p^{+}| + |p^{-}\rangle\langle p^{-}|, \tag{9.8}$$

we can write

$$\begin{aligned}\langle p^{+}|\ \not{k}\ |q^{+}\rangle &= [p\,k]\,\langle k\,q\rangle \\ \langle p^{-}|\ \not{k}\ |q^{-}\rangle &= \langle p\,k\rangle\,[k\,q]\,.\end{aligned} \tag{9.9}$$

We can define a number of useful identities for use with this notation, including,

$$|i^{\pm}\rangle\langle i^{\pm}| = \frac{1}{2}(1 \pm \gamma^{5})\,\not{p}_{i}, \tag{9.10}$$

found by applying the helicity projection operator eq. (9.2) to eq. (9.8). We can also define the following identities, which the spinor products must satisfy,

$$\begin{aligned}&\langle i\,j\rangle\,[j\,i] = \langle i^{-}|j^{+}\rangle\langle j^{+}|i^{-}\rangle = tr(\frac{1}{2}(1-\gamma^{5})\,\not{p}_{i}\,\not{p}_{j}) = 2p_{i}.p_{j} = s_{ij}. \\ &\langle i\,j\rangle\,[j\,k]\,\langle k\,l\rangle\,[l\,i] = tr_{-}(\ \not{i}\ \not{j}\ \not{k}\ \not{l}\ ) \\ &[i\,j]\,\langle j\,k\rangle\,[k\,l]\,\langle l\,i\rangle = tr_{+}(\ \not{i}\ \not{j}\ \not{k}\ \not{l}\ )\end{aligned} \tag{9.11}$$

We can make use of a particular application of the Fierz rearrangement theorem. For any general spinors $i,\ j,\ k,\ l$ satisfying our defined notation, we can expand the matrix $|j^+\rangle\langle i^+|$ into a linear sum of terms, i.e.,

$$2|j^+\rangle\langle i^+| = \langle i^+|\gamma^\mu|j^+\rangle\gamma_\mu\frac{1}{2}(1-\gamma^5) \tag{9.12}$$

Multiplying both sides of eq. (9.12) from the left by $\langle k^-|$ and from the right by $|l^-\rangle$ gives,

$$\langle i^+|\gamma^\mu|j^+\rangle\langle k^-|\gamma_\mu|l^+\rangle = 2\langle i^+|l^-\rangle\langle k^-|j^+\rangle. \tag{9.13}$$

We can also define an extremely useful identity called the Schouten identity,

$$\langle i^-|j^+\rangle\langle k^-|l^+\rangle = \langle i^-|l^+\rangle\langle k^-|j^+\rangle + \langle i^-|k^+\rangle\langle j^-|l^+\rangle. \tag{9.14}$$

Since the strong interactions do not violate charge conjugation the theory must be symmetric when we interchange particles for their corresponding antiparticle, i.e.,

$$\langle i^+|\gamma^\mu|j^+\rangle = \langle j^-|\gamma^\mu|i^-\rangle. \tag{9.15}$$

We can now extend our spinor representation and introduce the massless gauge boson polarisation vector with helicity ±1 by writing this as combinations of spinors $|p^\pm\rangle$,

$$\epsilon_\mu^+(p,q) = \frac{\langle q^-|\gamma_\mu|p^-\rangle}{\sqrt{2}\langle q^-|p^+\rangle} \qquad \epsilon_\mu^-(p,q) = -\frac{\langle q^+|\gamma_\mu|p^+\rangle}{\sqrt{2}\langle q^+|p^-\rangle} \tag{9.16}$$

where p is the vector boson momentum and q is an arbitrary null "reference momentum" that satisfies $q^2 = 0$ and $p.q \neq 0$. This reference momentum drops out of the final amplitudes, which are gauge invariant. The gluon helicities are denoted by the positive and negative labels on the polarisation vectors.

The polarisation vector for any q is transverse to p, i.e.

$$p.\epsilon_\mu^\pm(p,q) = 0. \tag{9.17}$$

The polarization vectors are normalised,

$$\epsilon_\mu^+\epsilon_\mu^+ = 0 \qquad \epsilon_\mu^+\epsilon_\mu^- = -1 \qquad \epsilon_\mu^\pm(\epsilon_\mu^\pm)^* = -1. \tag{9.18}$$

such that complex conjugation reverses the helicity, i.e. $(\epsilon_\mu^+)^* = \epsilon_\mu^-$.

The most powerful feature of this formalism can be seen if we consider changing the reference momentum, q. We can observe the corresponding shift in the polarization vector if we consider the difference between two polarization vectors with different reference momenta q and $q\prime$,

$$\begin{aligned}
\epsilon_\mu^+(p,q\prime) - \epsilon_\mu^+(p,q) &= \frac{\langle q\prime^-|\gamma_\mu|p^-\rangle}{\sqrt{2}\,\langle q\prime\; p\rangle} - \frac{\langle q^-|\gamma_\mu|p^-\rangle}{\sqrt{2}\,\langle q\, p\rangle} \\
&= -\frac{\langle q\prime^-|\gamma_\mu|p^-\rangle\,\langle p\, q\rangle + \langle q\prime\; p\rangle\,\langle p^+|\gamma_\mu|q^+\rangle}{\sqrt{2}\,\langle q\prime\; p\rangle\,\langle q\, p\rangle} \\
&= -\frac{\langle q\prime^-|\gamma_\mu \not{p}\,|q^+\rangle + \langle q\prime^-|\not{p}\,\gamma_\mu|q^+\rangle}{\sqrt{2}\,\langle q\prime\; p\rangle\,\langle q\, p\rangle} \\
&= -\frac{\langle q\prime^-|\{\gamma_\mu,\gamma_\nu\}|q^+\rangle}{\sqrt{2}\,\langle q\prime\; p\rangle\,\langle q\, p\rangle}p^\nu \\
&= -\frac{\sqrt{2}\,\langle q\prime\; q\rangle}{\langle q\prime\; p\rangle\,\langle q\, p\rangle}p_\mu.
\end{aligned} \tag{9.19}$$

Therefore, a change in the reference momentum produces a shift in the polarization vector proportional to p_μ, i.e.

$$\epsilon_\mu^+(p, q\prime\,) = \epsilon_\mu^+(p, q) - \frac{\sqrt{2}\,\langle q\prime\;\, q\rangle}{\langle q\prime\;\, p\rangle\,\langle q\, p\rangle} p_\mu. \tag{9.20}$$

This amounts to a gauge transformation. For each gluon momentum p_i we can choose a seperate reference momentum q_i. We must be careful and remember that as we are making a gauge choice we cannot change q_i while calculating a particular gauge invarient quantity, such as a partial amplitude. We can, however, make different choices for calculating different gauge invarient quantities, and drastically simplify calculations with a suitable choice of reference momentum q_i that makes some terms vanish. Using the identities defined, it can be shown that

$$\epsilon^+(p, q) \cdot \epsilon^+(P', q) = \epsilon^+(p, q) \cdot \epsilon^-(q, q') = 0 \tag{9.21}$$

This suggests that a particulalry convenient choice is to use the same reference momenta q for polarisation vectors of the same helicity, and for this to coincide with the external momenta of a polarisation vector with the opposite helicity. The remaining non-zero terms can also be written in the more compact spinor helicity notation.

Any amplitude involving massless external fermions and gluons can be expressed in terms of spinor products, defined for positive and negative energy solutions of the massless Dirac equation represented as massless spinors. We can compute a number of scattering amplitudes from each individual expression by using crossing symmetry, exchanging which momenta are incoming and which are outgoing. We must take care however. The helicity assigned to a particle is not independant of it's chirality and ultimately depends on whether particles are considered to be incoming or outgoing, i.e. a positive energy massless spinor has the same helicity sign as that of its chirality, while a negative energy massless spinor has the opposite helicity sign to its chirality. To avoid confusion we define the convention that all particles are labelled with an associated helicity when they are considered as outgoing. In such a convention, incoming particles of a particular helicity are now considered as outgoing with the helicity reversed.

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