Non Abelian T-duality and Holography.

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We study non-abelian T-duality as a supergravity solution technique and explore its application to holography. We consider well understood geometries which describe the strong coupling regime of minimally supersymmetric gauge theories in 3 and 4 dimensions. We then use non-abelian T-duality to generate new solutions in type-II supergravity and use these to define new gauge theories with interesting dynamics such as confinement. The work contains extensive field theoretic analysis of these new solutions.

We explore how the supersymmetry of the "seed" solutions is preserved under T-duality transformations by employing the powerful techniques of G-structures and generalised geometry. As well as giving a geometric description of non-abelian T-duality, this also enables us to extend the duality to cases with calibrated sources. We find that quite generically $SU(3)$-structures in 6d are mapped to $SU(2)$-structures. Further we find an intimate relationship between dynamic $SU(2)$-structures and confinement in these new solutions.

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Statement 1

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Statement 2

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Chapter 1

Introduction

The notion of duality within physics is of course quite old, going back to well-known examples like the Maxwell equations in vacuum. The true power of the idea became clear around 1940 with the Kramers-Wannier [1] duality of the Ising model. In more recent times dualities have continued to be a driver of theoretical progress with examples including Bosonisation [2], Montonen-Olive duality [3], S and T-dualities, Seiberg-Witten duality [4], Seiberg duality [5] and more general String dualities (U dualities). The duality conjectured by Maldacena [6], also called AdS/CFT or Gauge-Strings duality or simply holography, is arguably the most powerful, widely applicable and conceptually deep duality of all known at present. These dualities present common features: the degrees of freedom on both sides of the dual descriptions are in principle quite different; a strongly coupled (highly fluctuating) description of the system is characteristically mapped into a weakly coupled (semiclassical) one, in the same vein a phenomena that is 'local' in one set of variables becomes 'non-local' in the other (as exemplified by order-disorder operators and their typical 'uncertainty' relations), global symmetries are common to both dual descriptions, etc. In this thesis, we will mostly work with two dualities, the one conjectured by Maldacena and its extensions (see the papers [7, 8] for a sample of representative work and reviews) together with what is called 'non-Abelian T-duality' [9, 10].

In its original formation [6] the gauge-gravity correspondence was a conjecture between type-IIB supergravity on $AdS_5 \times S^5$ and strong coupling limit of $\mathcal{N} = 4$ super Yang-Mills with gauge group $SU(N_c)$. The latter has not only maximal supersymmetry in 4d, it is also conformal invariant (hence the name AdS-CFT). Many superconformal field theories are well understood and this is especially true of $\mathcal{N} = 4$ SYM. This is principally due to its abundance of symmetries which allows a vast array of powerful mathematical tools to be applied to it. However, many interesting physical processes which had been impervious to analytical study for many years are scale dependent, and of course supersymmetry has not, as yet, been observed. AdS-CFT provided such an analytical tool, however it clearly needed to be extended to non conformal gauge theories, with less supersymmetry, if it were to be a viable probe of many phenomenologically interesting strong coupling dynamics, such as confinement in QCD. These extensions necessitated a rebranding of the correspondence, hence the names gauge-string and holography.

Since the original work of Maldacena there has been significant progress in constructing geometries dual to gauge theories with minimal supersymmetry. Two important early examples are the Maldacena-Nunez [11] and Klebanov-Strassler solutions [12]. These both provide holographic descriptions of strongly coupled confining gauge theories that flow in the IR to $\mathcal{N} = 1$ SYM in 4d. In the UV, unlike $\mathcal{N} = 1$ SYM which is asymptotically free, they remain strongly couple which is a general feature of holography. None the less they have both shone a bright light on strong coupling dynamics of "realistic" gauge theories. It is fair to say however that there are in general considerable difficulties in constructing such supergravity solutions and *any* help in doing so is extremely useful.

One method of constructing new holographic duals which has borne much fruit is to use supergravity solution techniques. Indeed the Maldacena-Nunez solution itself was derived by lifting a gauged supergravity in $4d$ [13] to a full solution of type-IIB supergravity using [14, 15]. Further, it was later shown in [16] that U-duality could be used to map a deformation of the Maldacena-Nastase solution [17] to the Baryonic Branch of Klebanov-Strassler [18]. This is a particularly striking result because the "seed" solution is dual to a QFT with an irreverent operator insertion dominating the high energy dynamics. Klebanov-Strassler has no such operator and so U-duality provided a method of UV completing the original QFT.

Another method of constructing relevant supergravity solutions is to use the powerful techniques of generalised geometry and G-structures [19]. These provide a geometric description of the supersymmetry conditions for both backgrounds and probe branes [20] and originate from work relating to string compactifications (see [21] for a review). They have proved very useful in holography in resent years either as an aid to direct construction or as a framework in which to construct new solution generating techniques. One such technique, known as G-structure rotation [22], is actually equivalent to U-duality. In fact rotation/U-duality has lead to the construction of *many* new supergravity solutions in resent years, see for example [23, 22, 24, 25, 26, 27, 28, 29], and by now its effects on the gauge theory side is well understood.

The focus of this thesis will be to use non-abelian T-duality as a supergravity solution generating technique. The aim will be to generate new geometries dual to gauge theories with minimal supersymmetry in 3 and 4 dimensions.

As "seed" solutions we will consider backgrounds of Type II Supergravity that have a well understood (strongly coupled) field theory description. This will lead us to the construction of new solutions of ten-dimensional Supergravity and, as advocated in [30], we will use these new backgrounds to define new field theories at strong coupling. We will then study the effect of this generating technique on the field theory side.

Although the idea of generalising T-duality to non-Abelian isometry groups has rather old roots [9], it is only recently that it has been studied as full solution generating symmetry of supergravity [30, 31, 32, 33, 34, 35]. This is in part due to the fact that it was took some time to appreciated how to perform the duality in the presence of a non trivial RR sector [31]. Since then, most attention has been focused on dualising along $SU(2)$ isometries, because they are quite simple and it has been explicitly shown (for a quite general ansatz) that they are always a map between SuGra solutions [36]. This has already bore some quite interesting results, for example a new AdS_6 solution was generated in type-IIB [34], which promises to shed some light on CFT'S in 5-d (see also [37]). Attention has also been focused on performing $SU(2)$ T-dualities on type-IIB conifold solutions [30, 35].

The minimally supersymmetric solutions considered in this thesis have a well understood description in terms of G-structures. An important focus of this thesis will be how these are transformed under non-abelian T-duality. This will not only furnish us with information about the supersymmetric cycles and brane embeddings in the geometries the duality generates, but will also allow us to classify the geometries. We will see that quite generically non-abelian T-duality provides a map between a background supporting a common structure such as $SU(3)$ in 6d, to a rather more exotic one, particularly when the "seed" solution confines.

The outline of the thesis is as follows:

In chapter 2 we review the process of T-dualising a type-II supergravity solution along a isometry group G. Most attention shall be given to the cases where $G = U(1)$ or $SU(2)$. This is because all the new solutions generated in this thesis are generated by $SU(2)$ isometry non-abelian T-duality. The abelian case is explained principally to aid the understanding of T-dualityâs second most simple case, namely $SU(2)$. This chapter reviews established work in the literature in particular [10, 38, 39, 40, 30]

In chapter 3, based on [41], we show how the techniques of G-structures and generalised geometry can aid the understanding of non-abelian T-duality. We give a first example of how G-structures and calibrated sources in 6-d are transformed under the duality. This enables us to show solutions with smeared sources transform, which gives hints of how fundamental matter is effected by the duality.

In chapter 4, based on [42], we generate new solutions in massive type-IIA supergravity. Analysis indicates that these are dual to 3d Yang-Mills Chern-Simons like theories and some of their dynamics are studied. This is aided by a comparison to G_2-structure rotation. In addition we take the first steps toward extending the results of chapter 3 to G-structures in 7d.

In chapter 5, based on [43], we study the $SU(2)$-isometry T-dual to the Baryonic Branch of Klebanov-Strassler derived in [30]. We show that this this massive type-IIA solution supports what is called a dynamical $SU(2)$-structure (see appendix B), which is intimately tied up with confinement. We also perform a detailed field theoretic analysis determining how many observables are transformed under non-abelian T-duality.

In chapter 6, based on [44], we extended the ideas of the previous section to generate new solutions in type-IIB describing confining QFTs in 4d. We once more find that supersymmetry is preserved in the form of a dynamical $SU(2)$-structure and perform an extensive QFT analysis both before and after the duality.

Finally we summarise the results of the previous sections and comment on some future directions and possible limitations in chapter 7.

Chapter 2

(non-Abelian) T-duality: A Pedagogical Review

2.1 Abelian T-duality

We will start this chapter with a review of T-duality in the abelian case where the isometry group on which one dualised is $U(1)$.

T-duality has roots that dates back to the early 1980s [45]. Its most simple avatar can be explained in terms of the spectrum of mass states of a string propagating in $\mathbb{R}^{1,8} \times S^1$. Let the compact coordinate x^9 satisfy

$$x^9 \sim x^9 + 2\pi R. \tag{2.1.1}$$

The momentum of the string in the compact direction must be quantised in integer units as

$$p = \frac{n}{R}, \quad n \in \mathbb{Z}. \tag{2.1.2}$$

It is possible for the string to wrap the compact direction with the consequence that the world sheet coordinate σ need not be single valued. This may be stated as

$$x^9(\tau, \sigma + 2\pi) \sim x(\tau, \sigma) + 2\pi R m, \quad m \in \mathbb{Z}, \tag{2.1.3}$$

where m is the winding number of the string around x^9. It is possible to show that the mass spectrum of such a string is expressed in terms of n and m as

$$M^2 = \frac{n^2}{R^2} + \frac{m^2 R^2}{\alpha'} + \text{oscillator part}. \tag{2.1.4}$$

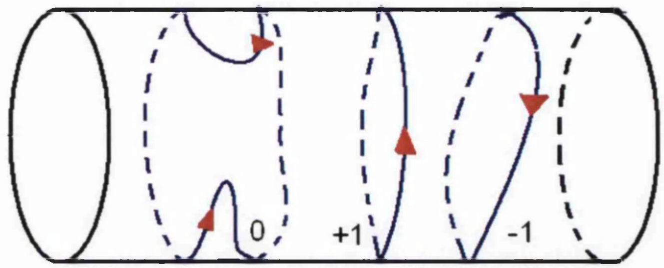

Figure 2.1: Strings propagating on a manifold with a compact direction. The Winding number m is indicated in each case.

This equation is invariant under the exchange

$$n \longleftrightarrow m, \qquad R \longleftrightarrow \frac{\alpha'^2}{R}, \tag{2.1.5}$$

which exchanges momentum and winding modes of the string but also inverts the radius of the circle. This has the rather striking consequence that a string moving on a circle of radius R has the same mass spectrum as one moving on a circle of radius α'/R. This equivalence of the string spectrum extends to string interactions, modulo some subtleties concerning the dilaton (see below), and goes by the name T-duality.

2.1.1 Buscher T-duality

There is a path integral derivation of T-duality due to Buscher [46, 47] that enables one to perform the duality on any geometry with a $U(1)$ isometry. This implies that the target space metric may be expressed as

$$ds^2 = G_{\mu\nu}(x)dx^\mu dx^\nu + 2G_{\mu\theta}(x)dx^\mu d\theta + R(x)^2 d\theta^2, \tag{2.1.6}$$

with an equivalent expressions for the NS two form and dilaton. This method acts on a sigma model defined in terms of the combination

$$\begin{aligned} E_{\mu\nu} &= G_{\mu\nu} + B_{\mu\nu}, \quad E_{\mu\nu} = G_{\mu\theta} + B_{\mu\theta}, \\ E_{\theta\nu} &= G_{\theta\nu} + B_{\theta\nu}, \quad E_{\theta\theta} = R^2. \end{aligned} \tag{2.1.7}$$

The action of the sigma model is given by

$$S[x,\theta] = \frac{1}{4\pi}\int d^2\sigma\Big[R^2\partial_+\theta\partial_-\theta + E_{\mu\theta}\partial_+x^\mu\partial_-\theta + E_{\theta\nu}\partial_+\theta\partial_-x^\nu + E_{\mu\nu}\partial_+x^\mu\partial_-x^\nu\Big] \tag{2.1.8}$$

where $\sigma^\alpha = (\sigma^+,\sigma^-)$, we have set $\alpha' = 1$ and the Euclidean partition function is $Z = \int \mathcal{D}\theta\mathcal{D}xe^{-S[x,\theta]}$. The $U(1)$ isometry manifests itself via an invariance of this action under the shift $\theta \to \theta + \lambda$. The first step of the Buscher "recipe" is to gauge the this symmetry by promoting derivatives to covariant ones

$$\partial_\pm\theta \to D_\pm\theta = \partial_\pm\theta + A_\pm, \tag{2.1.9}$$

where $A_\pm \to A_\pm - \partial_\pm\lambda$. In general this will change the theory the action describes, which is not what we want. We wish to describe T-duality, under which the partition function of the theory should be invariant. However we do not change the theory, at least locally, if the gauge fields are non dynamical which is ensured if they are pure gauge. This can be achieved by imposing a flat connection with a Lagrange multiplier term so that the new action reads

$$S[x,\theta,\tilde{\theta},A] = \frac{1}{4\pi}\int d^2\sigma\Big[R^2D_+\theta D_-\theta + E_{\mu\theta}\partial_+x^\mu D_-\theta + E_{\theta\nu}D_+\theta\partial_-x^\nu + E_{\mu\nu}\partial_+x^\mu\partial_-x^\nu\Big] + \frac{i\tilde{\theta}}{2\pi}\int F, \tag{2.1.10}$$

where $F = dA$ and $A = A_\alpha d\sigma^\alpha$ and the partition function is

$$Z = \int \mathcal{D}\theta\mathcal{D}x\mathcal{D}A\mathcal{D}\tilde{\theta}e^{-S[x,\theta,\tilde{\theta}A]} \tag{2.1.11}$$

Integrating out $\tilde{\theta}$ in the path integral leads to $F = 0$, which on a topologically trivial worldsheet imposes that $A_\pm$ is pure gauge. We are then free to set $A_\pm = 0$ and arrive back at the theory described by eq (2.1.13).

However a when the worldsheet is not topologically trivial $F = 0$ does not necessarily imply that $A_\pm$ is pure gauge. The gauge fields can have non trivial holonomy around cycles in the world sheet. These can only be made to be gauge trivial, but only when $0 \leq \tilde{\theta} \leq 2\pi$. In this case we can set $A_\pm = 0$ and the partition functions defined by eq (2.1.13) and eq (2.1.10) coincide.

So what of the T-dual solution? This is actually extracted from eq (2.1.11) in a surprisingly simple fashion, one simply integrates out θ instead of $\tilde{\theta}$. This can be achieved gauge fixing $\theta = 0$ and after an integration of the multiplier term by parts the action becomes

$$S[x,\tilde{\theta},A] = \frac{1}{4\pi}\int d^2\sigma\Big[R^2 A_+ A_- + E_{\mu\theta}\partial_+ x^\mu A_- + E_{\theta\nu}A_+\partial_- x^\nu \\ + E_{\mu\nu}\partial_+ x^\mu \partial_- x^\nu\Big] - \frac{i}{2\pi}\int \epsilon^{\alpha\beta}\partial_\alpha\tilde{\theta}A_\beta. \tag{2.1.12}$$

Finally we must integrate out the gauge field and we arrive at the T-dual sigma model

$$S[x,\tilde{\theta}] = \frac{1}{4\pi}\int d^2\sigma\Big[\frac{1}{R^2}\partial_+\tilde{\theta}\partial_-\tilde{\theta} - \frac{1}{R^2}E_{\mu\theta}\partial_+ x^\mu\partial_-\tilde{\theta} + \frac{1}{R^2}E_{\theta\nu}\partial_+\tilde{\theta}\partial_- x^\nu \\ + \big(E_{\mu\nu} - \frac{1}{R^2}E_{\mu\theta}E_{\nu\theta}\big)\partial_+ x^\mu\partial_- x^\nu\Big], \tag{2.1.13}$$

form which the dual metric and NS 2-form may be easily extracted as

$$\begin{aligned} &\hat{G}_{\tilde{\theta}\tilde{\theta}} = \frac{1}{R^2}, \quad \hat{G}_{\tilde{\theta}\nu} = -\frac{1}{R^2}B_{\theta\nu}, \quad \hat{B}_{\theta\nu} = -\frac{1}{R^2}G_{\theta,\nu}, \\ &\hat{G}_{\mu\nu} = G_{\mu\nu} + \frac{1}{R^2}(G_{\theta\mu}G_{\theta\nu} - B_{\theta\nu}B_{\theta\nu}) \\ &\hat{B}_{\mu\nu} = B_{\mu\nu} + \frac{1}{R^2}(G_{\theta\mu}B_{\theta\nu} - B_{\theta\nu}G_{\theta\nu}). \end{aligned} \tag{2.1.14}$$

Notice that the inversion of radius has been reproduced.

There is a subtlety that one must take account of if one wishes to embed this sigma model in type-IIB supergravity. What is described above is not quite the whole story, the dilaton receives a shift due to an anomaly at 1-loop and is given in in fact given in the T-dual theory by

$$e^{-2\hat{\Phi}} = R^2 e^{-2\Phi}. \tag{2.1.15}$$

We will close this subsection with a comment about the period of dual coordinates. It is the requirement that the partition functions associated with eq (2.1.13) and eq (2.1.10) coincide that enabled the period of the T-dual coordinate $\tilde{\theta}$ to be fixed. Despite many people working on the topic, no one has ever been able to come up with an equivalent worldsheet criterium for the case of non-abelian T-duality.

2.1.2 On Generating the RR Sector

It is actually possible to modify the approach of the previous subsection to include a non trivial RR-sector. This was done using a pure spinor approach in

[48]. However it is the indirect method of Hassan [39, 40] which will be describe here as this is the method we use in the non-abelian T-dual case.

If one expresses the metric of a type-II solution in terms of vielbeins e^a, then the T-dual solution has two sets of vielbeins $e^a_\pm$ given by

$$\hat{e}^a_\pm = e^a \Theta_\pm, \tag{2.1.16}$$

which couple naturally to either left or right movers. The matrices giving rise to the T-duality transformation of the vielbeins are

$$\begin{aligned} (\Theta_\pm) &= \begin{pmatrix} \mp\frac{1}{R^2} & -\frac{1}{R^2}(G \mp B)_{\theta\mu} \\ 0 & \mathbb{I} \end{pmatrix}, \\ (\Theta_\pm)^{-1} &= \begin{pmatrix} \mp R^2 & \pm(G \mp B)_{\theta\mu} \\ 0 & \mathbb{I} \end{pmatrix}. \end{aligned} \tag{2.1.17}$$

As both sets of vielbeins are describe the same T-dual geometry they must be related by a Lorentz transformation Λ,

$$\hat{e}^a_+ = \Lambda^a{}_b \hat{e}^b_-. \tag{2.1.18}$$

Using the matrics in eq (2.1.17) it is possible to show that

$$\Lambda^a{}_b = \delta^a_b - \frac{2}{R^2} e^a{}_\theta e_{\theta b}. \tag{2.1.19}$$

where $det\Lambda = -1$.

The critical realisation of Hassan was that one could use this Lorentz transformation to define an action on spinors Ω by demanding that

$$\Omega^{-1}\Gamma^a\Omega = \Lambda^a{}_b\Gamma^b. \tag{2.1.20}$$

This condition is solved for

$$\Omega = \Gamma^{(10)}\Gamma^\theta. \tag{2.1.21}$$

In order to construct the T-dual RR-sector on first needs to construct polyforms by summing the democratic formalism RR fluxes as

$$F_{IIB} = \sum_{n=0}^{4} F_{2n+1}, \qquad F_{IIA} = \sum_{n=0}^{5} F_{2n}. \tag{2.1.22}$$

These are then mapped to bispinors under the Clifford map which relates forms to spinors:

$$X = X_{a_1 a_2 \ldots} e^{a_1} \wedge e^{a_2} \wedge \ldots \quad \longleftrightarrow \quad \not{X} = X_{a_1 a_2 \ldots} \Gamma^{a_1 a_2 \cdots}. \tag{2.1.23}$$

The RR sector then transforms as

$$e^{\Phi_{IIA}} \not{F}_{IIA} = e^{\Phi_{IIB}} \not{F}_{IIB} \Omega^{-1}. \tag{2.1.24}$$

Notice that eq (2.1.21) contains a single gamma matrix Γ^{θ}. This means that a component of an n-form with a leg in θ will get sent to an $(n-1)$-form. A component with no leg in θ will be sent to an $(n+1)$-form. This is just one of many way that one can see that T-duality must map between Type-IIA and Type-IIB.

We will now proceed to give some general details of T-duality for non-abelian isometries.

2.2 Some Generalities of non-Abelian T-duality

In this section we present some useful details of non-Abelian T-duality (this is based on the review sections of [41, 42]), a comprehensive treatment may be found in [30] and further details, pertinent to the $SU(2)$ isometry case, in the following section.

The three step Buscher procedure of gauging a $U(1)$ isometry, enforcing a flat connection for the corresponding gauge field with a Lagrange multiplier, and integrating out these Lagrange multipliers provides a powerful way to construct a T-dual σ-model. This approach can be readily generalised to the case of non-Abelian isometries and provides a putative non-Abelian T-duality transformation. Unlike its Abelian counter part, this non-Abelian T-duality typically destroys the isometries dualised (though they may be recovered as non-local symmetries of the string σ-model). Due to global complications, it is thought that this non-Abelian dualisation is not a full symmetry of string (genus) perturbation theory however it remains valid as a solution generating symmetry of supergravity. In this regard its status is rather similar to fermionic T-duality [49], which has proven to be very useful in the context of the AdS-CFT correspondence in providing an explanation of the scattering amplitude/Wilson loop connection at strong coupling [50].

Let us first consider a bosonic string σ-model in a NS background. We will assume that this background admits some isometry group G and that background fields can be expressed in terms of left-invariant Maurer–Cartan forms, $L^i = -iTr(g^{-1}dg)$, for this group. That is to say the target space metric has a decomposition

$$ds^2 = G_{\mu\nu}(x)dx^\mu dx^\nu + 2G_{\mu i}(x)dx^\mu L^i + g_{ij}(x)L^i L^j$$
$$B = B_{\mu\nu}(x)dx^\mu \wedge dx^\nu + B_{\mu i}(x)dx^\mu \wedge L^i + \frac{1}{2}b_{ij}(x)L^i \wedge L^j \tag{2.2.1}$$

with corresponding expressions for the dilaton Φ. The non-linear σ-model is

$$S = \int d^2\left(\sigma Q_{\mu\nu}\partial_+ x^\mu \partial_- x^\nu + Q_{\mu i}\partial_+ x^\mu L^i_- + Q_{i\mu}L^i_+\partial_- x^\mu + E_{ij}L^i_+ L^j_-\right), \tag{2.2.2}$$

where

$$Q_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu}, \quad Q_{\mu i} = G_{\mu i} + B_{\mu i}, \quad E_{ij} = g_{ij} + b_{ij}, \tag{2.2.3}$$

and $L^i_\pm$ are the left-invariant forms pulled back to the world sheet. To obtain the dual σ-model one first gauges the isometry by making the replacement

$$\partial_\pm g \to D_\pm g = \partial_\pm g - A_\pm g, \tag{2.2.4}$$

in the Maurer–Cartan forms. Also, the addition of a Lagrange multiplier term $-i\mathrm{Tr}(vF_{+-})$ enforces a flat connection, where v is a vector with dimG components.

After the multiplier term is integrated by parts the gauge fields can be solved for which gives the T-dual sigma model. At this point there are actually twice as many coordinates needed as both the Euler angles and Lagrange multipliers are present, namely $(\theta, \varphi, \psi, .., v_1, v_2, v_3, ..)$ in total, the redundancy must be eliminated by choosing a gauge. This may be parametrised by the matrix [30]

$$D^T = Tr(t_i g t_j g^{-1}) \tag{2.2.5}$$

where where t_i are the generators of G. The T-dual coordinates are then given by

$$\tilde{v} = D^T v \tag{2.2.6}$$

where half of the 2dimG combination of angles and multipliers must be fixed such that dimG independent coordinates remain in $\tilde{v}$.

We then obtain the Lagrangian,

$$\tilde{\mathcal{L}} = Q_{\mu\nu}\partial_+ x^\mu \partial_- x^\nu + (\partial_+ v_i + \partial_+ x^\mu Q_{\mu i})(M_{ij})^{-1}(\partial_- v_j - Q_{j\mu}\partial_- x^\mu), \tag{2.2.7}$$

where

$$M_{ij} = E_{ij} + f_{ij}{}^k \hat{v}_k. \tag{2.2.8}$$

The Buscher rules defining the dual NS sector may be read off from this as

$$\begin{aligned}
&\tilde{Q}_{\mu\nu} = Q_{\mu\nu} - Q_{\mu i} M_{ij}^{-1} Q_{j\nu}, \quad \tilde{E}_{ij} = M_{ij}^{-1}, \\
&\tilde{Q}_{\mu i} = Q_{\mu j} M_{ji}^{-1}, \quad \tilde{Q}_{i\mu} = -M_{ij}^{-1} Q_{j\mu},
\end{aligned} \tag{2.2.9}$$

from which the dual metric and NS 2-form may be extracted as symmetric and anti symmetric components. As with Abelian T-duality the dilaton receives a shift from performing the above manipulations in a path integral given by

$$\hat{\Phi}(x, v) = \Phi(x) - \frac{1}{2} \ln(\det M) \tag{2.2.10}$$

Using the equations of motion, one can ascertain the following transformation rules for the world sheet derivatives

$$\begin{aligned}
&L_+^i = -(M^{-1})_{ji} \left(\partial_+ \hat{v}_j + Q_{\mu j} \partial_+ x^\mu \right) , \\
&L_-^i = M_{ij}^{-1} (\partial_- \hat{v}_j - Q_{j\mu} \partial_- x^\mu) , \\
&\partial_\pm x^\mu = \text{invariant} .
\end{aligned} \tag{2.2.11}$$

These relations provide a classical canonical equivalence between the two T-dual σ-models [51].

The consequence of this is that left and right movers couple to different sets of vielbeins for the T-dual geometry. Suppose that we define frame fields for the initial metric eq (2.2.1) by

$$ds^2 = \eta_{AB} e^A e^B + \sum_{i=1}^{\dim G} \delta_{ab} e^a e^b , \quad e^A = e^A_\mu dx^\mu , \quad e^a = \kappa^a_i L^i + \lambda^a_\mu dx^\mu . \tag{2.2.12}$$

Then by making use of the transformation rules in eq (2.2.11) one finds that after T-dualisation left and right movers couple to the vielbeins

$$\begin{aligned}
&\hat{e}^a_+ = -\kappa M^{-T} \left(dv + Q^T dx \right) + \lambda dx , \quad \hat{e}^A_+ = e^A \\
&\hat{e}^a_- = \kappa M^{-1} \left(dv - Q\, dx \right) + \lambda dx , \quad \hat{e}^A_- = e^A .
\end{aligned} \tag{2.2.13}$$

Both these frames fields define the T-dual target space metric obtained from eq (2.2.7) given by

$$\widehat{ds}^2 = \eta_{AB} e^A e^B + \sum_{i=1}^{\dim G} \delta_{ab} \hat{e}^a_+ \hat{e}^b_+ = \eta_{AB} e^A e^B + \sum_{i=1}^{\dim G} \delta_{ab} \hat{e}^a_- \hat{e}^b_- . \tag{2.2.14}$$

Since these frame fields define the same metric they must be related by a Lorentz transformation and indeed

$$\hat{e}_+ = \Lambda \hat{e}_- \,, \quad \Lambda = -\kappa M^{-T} M \kappa^{-1} \,. \tag{2.2.15}$$

We note that $\det \Lambda = (-1)^{\dim G}$, which implies that the dualisation of an odd dimensional isometry group maps between type IIA and IIB theories whereas the an even dimensional isometry group preserves the chirality. This Lorentz transformation induces an action on spinors defined by the invariance property of gamma matrices [1];

$$\Omega^{-1} \Gamma^a \Omega = \Lambda^a{}_b \Gamma^b \,. \tag{2.2.16}$$

We are particularly interested in performing this duality in supergravity backgrounds of relevance to the AdS/CFT correspondence which are typically supported by RR fluxes. Then one ought to, in principle, reconsider the above derivation in a formalism suitable of including RR fluxes. In the case of Abelian and Fermonic T-duality this has explicitly been done in the pure spinor approach [48, 52] and and a simple extrapolation of these results to this non-Abelian context leads to the following conclusion which can also be motivated from the considerations of [39, 40]. The dual RR fluxes are obtained by right multiplication by the above matrix Ω on the RR bispinor (this can be viewed equivalently as a Clifford multiplication on the RR poly form/pure spinor). Explicitly, the T-dual fluxes are given by

$$e^{\hat{\Phi}} \hat{\not{F}} = e^{\Phi} \not{F} \cdot \Omega^{-1} \,, \tag{2.2.17}$$

where the RR poly forms are defined by

$$\mathrm{IIB}: \; \mathbf{F} = \sum_{n=0}^{4} F_{2n+1} \,, \qquad \mathrm{IIA}: \; \mathbf{F} = \sum_{n=0}^{5} F_{2n} \,, \tag{2.2.18}$$

and the slashed notation in eq (2.2.17) indicates that we have converted these polyforms to bispinors by contraction with gamma matrices. Here we are working in the democratic formalism in which all ranks of fluxes are considered as independent and Hodge duality implemented by hand afterwards[2].

For many applications knowledge of the transformation laws for the gauge invariant field strengths is sufficient. However, in some applications we will also be interested on how the RR potentials themselves transform. We define

[1]Unfortunately, the existing notation in the literature means we have the same symbol Ω for the spinorial transformation matrix and for the $SU(3)$-structure three-form. We trust the reader will infer from the context which is meant.

[2]See the appendices for details of the conventions used.

potentials as

$$\text{IIB}: \ \mathbf{C} = \sum_{n=0}^{4} C_{2n} \,. \qquad \text{IIA}: \ \mathbf{C} = \sum_{n=0}^{4} C_{2n+1} \,. \tag{2.2.19}$$

related to the field strengths by

$$\text{IIB}: \ \mathbf{F} = (d - H\wedge)\mathbf{C} \,. \qquad \text{IIA}: \ \mathbf{F} = (d - H\wedge)\mathbf{C} + me^{B} \,, \tag{2.2.20}$$

in which m is the Romans mass parameter of type IIA. Actually we will need to be a bit more general than this when we consider the addition of sources, see appendix D.

The potentials so defined have a straightforward transformation rule:

$$e^{\hat{\Phi}}\hat{\not{C}} = e^{\Phi}\not{C} \cdot \Omega^{-1} \,, \tag{2.2.21}$$

We should comment briefly about a subtlety; the potentials in the equation above have to be chosen in such a way that the non-Abelian duality can be performed. This applies also in the case of usual Abelian T-duality, one must choose the potentials C_p so that they respect the same isometries as the fields F_{p+1}. In other words, the choice of potentials C_p should be compatible with the isometries. A less judicious choice of potentials would require composing the above transformation law with an appropriate gauge transformation that first brings the potential into the desired form (this is well explained in [53] for the NS two-form potential which need not have a vanishing Lie derivative under the isometry dualised but instead obey $\mathcal{L}_k b = d\xi$).

We conclude this section by remarking the status of supersymmetry under non-Abelian T-duality. Supersymmetry need not be preserved by T-duality (Abelian or not).[3] Whether (and how much) supersymmetry is preserved depends on how the Killing vectors about which we dualise act on the supersymmetry. The action of a vector on a spinor, which is only well defined when the vector is Killing, is given by [54, 55]

$$\mathcal{L}_k \epsilon = k^{\mu} D_{\mu}\epsilon + \frac{1}{4}\nabla_{\mu}k_{\nu}\gamma^{\mu\nu}\epsilon \,. \tag{2.2.22}$$

If, when acting on the Killing spinor of the initial geometry, this vanishes automatically for all the Killing vectors that generate the action of G then we anticipate supersymmetry to be preserved in its entirety. If on the other hand this vanishes only for some projected subset of Killing spinors then we expect only a corresponding projected amount of supersymmetry to be preserved in the T-

[3] In principle, supersymmetry can even be enhanced by T-duality but given that non-Abelian T-duality destroys isometry this seems rather unlikely in this case.

dual.[4] In this thesis we will consider the case of $\mathcal{N}=1$ supersymmetry which is invariant under the above action of G so that the non-Abelian duality should preserve supersymmetry. Suppose we start with ten-dimensional MW Killing spinors ϵ^1 and ϵ^2, then the Killing spinors in the T-dual will be given by

$$\hat{\epsilon}^1 = \epsilon^1\,, \quad \hat{\epsilon}^2 = \Omega \cdot \epsilon^2\,. \tag{2.2.23}$$

2.3 Details in the $SU(2)$ Isometry Case

In this section, which essentially reviews some salient results from [30], we will specialise to the T-duality transformations on $G = SU(2)$ isometries. It is this type of T-duality that will be considered in this thesis and so it is here that we fix the duality conventions used throughout.

Group Theory Conventions

We define the $SU(2)$ generators as

$$t^i = \frac{1}{\sqrt{2}}\tau_i, \tag{2.3.1}$$

where τ_i Pauli matrices which are given by

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.3.2}$$

One can show that the generators then obey the relations

$$Tr(t^i t^j) = \delta^{ij}, \quad [t^i, t^j] = i f^{ij}{}_k t^k = i\sqrt{2}\epsilon_{ijk} t^k. \tag{2.3.3}$$

An arbitrary element of $SU(2)$ may be defined in terms of Euler angles as

$$g = e^{i/2\varphi\tau_3}\, e^{i/2\theta\tau_2}\, e^{i/2\psi\tau_3}, \quad 0 \le \theta \le \pi,\ 0 \le \varphi \le 2\pi,\ 0 \le \psi \le 2\pi. \tag{2.3.4}$$

Finally the left invariant 1-forms are defined by

$$L^i = -iTr(t^i g^{-1} dg), \tag{2.3.5}$$

[4] In all examples so far considered this holds but an explicit detailed proof has not yet been given.

such that

$$dL^i = \frac{1}{2} f^i{}_{jk} L^j \wedge L^k. \tag{2.3.6}$$

It is the the g in eq (2.3.5) in which the gauge fixing procedure is performed. Specifically in these conventions the left invariant one forms are given by

$$\begin{aligned}
L_1 &= \frac{1}{\sqrt{2}} \left(-\sin\psi d\theta + \cos\psi \sin\theta d\varphi\right), \\
L_3 &= \frac{1}{\sqrt{2}} \left(\cos\psi d\theta + \sin\psi \sin\theta d\varphi\right), \\
L_3 &= \frac{1}{\sqrt{2}} \left(d\psi + \cos\theta d\varphi\right).
\end{aligned} \tag{2.3.7}$$

Explicit Form of the RR Transformation Matrix

We will now proceed to give some details necessary for an explicit calculation of the spinor transformation Ω which is used to derive the dual RR sector. Since our isometry group is 3 dimensional we can define a vector

$$b_i = \frac{1}{2} \epsilon_{ijk} b_{kj} \tag{2.3.8}$$

so the the transformation matrix M in eq (2.2.8) my be expressed as

$$M_{ij} = g_{ij} + \epsilon_{ijk}, \quad y_i = b_i + \hat{v}_i. \tag{2.3.9}$$

To derive Ω we must construct the explicit form of the Lorentz transformation of eq (2.2.15) which requires that we invert M. To this end we define an anti-symmetric density and vector on the group manifold as

$$\tilde{\epsilon}_{ijk} = \sqrt{detg}\epsilon_{ijk}, \quad z^i = \frac{y^i}{\sqrt{detg}} = \frac{y^i}{det\kappa}. \tag{2.3.10}$$

Indices of z^i can be raised and with g_{ij} and M can be written as

$$M_{ij} = g_{ij} + \tilde{\epsilon}_{ijk} z^k. \tag{2.3.11}$$

The inverse can now easily be shown to be

$$(M^{-1})^{ij} = \frac{1}{1+z^2} \left(g^{ij} + z^i z^j - \tilde{\epsilon}^{ij}{}_k z^k\right), \quad z^2 = z^i z_j. \tag{2.3.12}$$

We are now in a position to compute Λ which we remind the reader takes the form

$$\Lambda = -\kappa M^{-T} M \kappa^{-1} = -\kappa^{-T} M M^{-T} \kappa^{T}. \tag{2.3.13}$$

At this point it is convenient to introduce the flat coordinate

$$\zeta^a = \kappa^a{}_i z^i, \tag{2.3.14}$$

this will play an important role through out the thesis as well as here. First note that

$$(\kappa^{-T} M \kappa^{-1})_{ab} = \delta_{ab} + \epsilon_{abc}\zeta^c. \tag{2.3.15}$$

Inverting this expression leads to

$$(\kappa M^{-1} \kappa^T)^{ab} = \frac{1}{1+\zeta^2}(\delta^{ab} + \zeta^a\zeta^b - \epsilon_{abc}\zeta^c). \tag{2.3.16}$$

It is now possible to calculate Ω by demanding that eq (2.2.16) is satisfied, this leads to

$$\Omega = \Gamma^{(11)} \frac{-\Gamma_{123} + \zeta_a \Gamma^a}{\sqrt{1+\zeta^2}}. \tag{2.3.17}$$

Notice that this is rather more complicated than in the abelian case. It consists of a part with product of 3 gamma matrices and a part which is a sum of single matrices. This means that a form with n legs can be mapped to forms with $(n \pm 3)$ legs and $(n \pm 1)$, with $\pm$ depending on whether the component in question is parallel or orthogonal to the directions being dualised. Clearly, as with the abelian case this is consistent with a map between type-IIA and type-IIB. However, unlike the abelian case the duality will generically turn on a *many* fluxes.

Gauge fixing

In this thesis we will work with the gauge fixing $\theta = \varphi = v_1 = 0$. This implies that

$$D = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.3.18}$$

with the immediate consequence that

$$\hat{v} = (-\sin\psi v_2, \cos\psi v_2, v_2). \tag{2.3.19}$$

This is the same gauge fixing convention used in [30] and is motivated by the fact that it leaves the $U(1)$ R-symmetry of the 4-d gauge theories untouched, as this is parametrised by the ψ in the conifold solutions in type-IIB. It is less well motivated for the geometries dualised in chapters 4 and 6, where the R-symmetries have been less studied. However this this gauge fixing none the less aids in writing solutions compactly.

One could of course choose different gauge fixings, however these will all be locally diffeomorphic to the choice made here.

One should realise that the method laid out in this chapter is not the only one that may be used to generate the NS and RR sectors of a non-abelian T-dual solution. An alternative is to use a consistent truncation to 7-d and match the original solutions there [33]. Another interesting method is given by [56] where topological defects that generate the duality are constructed. This later method guarantees the Bianchi identities, but has not yet been shown to match the other methods in all cases. Finally this section has only discussed the dualising along $SU(2)$ isometries acting without isotropy, for more general isometries the interested reader is referred to [32].

Chapter 3

G-structures and a Geometric Description of non-Abelian T-duality

3.1 Introduction

This chapter, based on work done in collaboration with Barranco, Gaillard, Núñez and Thompson [41], presents the first time the techniques of G-structures and generalised geometry where used to gain a geometric view of non-abelian T-duality. It answers two questions important for better understanding how non-abelian T-duality is transforming the field theory the original geometry describes. How G-structures/pure spinors transform and how fundamental quarks transform.

The recent work of Itsios et al. [35, 30] considered the application of an $SU(2)$ non-abelian T-duality transformation in IIB supergravity backgrounds preserving $\mathcal{N} = 1$ supersymmetry. For instance applying an $SU(2)$ non-Abelian T-duality to the internal space of the Klebanov–Witten background ($AdS^5 \times T^{1,1}$) results in a solution of type IIA which retains the AdS_5 factor and has a lift to M-theory which corresponds to the geometries obtained in [57] from wrapping M5 branes on an S^2. In [30] similar dualisations were applied to non-conformal geometries (Klebanov–Tsetylin, Klebanov–Strassler and wrapped D5 models) resulting in a new class of smooth solutions of massive type IIA supergravity. The field theory interpretation of these massive IIA solutions is, as yet, undetermined however an analysis of the gravity solution indicates they retain rich RG dynamics displaying signatures of Seiberg duality, domain walls and confinement in the IR.

A common feature of the geometries obtained in [30] is that they retain four dimensional Poincaré invariance and it was argued that they should also retain

$\mathcal{N} = 1$ supersymmetry. The conditions for a solution of type II supergravity to possess these symmetries can be very elegantly stated using the language of G-structures [19]. The existence of a single four dimensional conserved spinor implies that on the six dimensional internal manifold M we have two spinors η^1 and η^2. If these spinors are proportional, the structure group of TM is reduced to $SU(3)$ and can be characterised by an invariant real two-form J and complex three-form Ω with $J \wedge \Omega = 0$ and $i\Omega \wedge \bar{\Omega} = \frac{4}{3}J^3$. If on the other hand the two spinors are nowhere parallel they each define a separate $SU(3)$-structure and together equip M with an $SU(2)$-structure consisting of a complex nowhere vanishing vector field $v + iw$, a real two-form j and a complex two-form ω.

These conditions can also be restated using the language of generalised complex geometry in which we consider the bundle $TM \oplus T^*M$. The algebraic conditions of supersymmetry imply that there exist two pure spinors $\Phi_\pm = \eta^{1\dagger}_+ \otimes \eta^{2\dagger}_\pm$. Using the Clifford map these pure spinors can be described as a formal sum of forms, for instance in the case of $SU(3)$-structure we identify $\Phi_+ = e^{-iJ}$ and $\Phi_- = \Omega$. The differential conditions of supersymmetry can be succinctly expressed in this language (as closure conditions for the annihilator space of these pure spinors under the H-twisted Courant bracket) and are schematically given by

$$d_H \Phi_1 = 0\,, \qquad d_H \Phi_2 = F_{RR}\,, \tag{3.1.1}$$

where $d_H = d + H\wedge$, F_{RR} denotes the RR fields and $\Phi_{1,2}$ are related to the pure spinors $\Phi_\pm$ depending on the type of supergravity in question.

This approach also makes clear the transformation rules under T-duality; these pure spinors essentially transform in the same way as Ramond fields. Indeed, in the case where M is Calabi-Yau, mirror symmetry serves to interchange the pure spinors $e^{iJ} \leftrightarrow \Omega$. The extension of this, *à la* Strominger, Yau and Zaslow, to general $SU(3)$-structures has been developed in [58].

The first purpose of this chapter is to study the effects of non-Abelian T-duality on these G-structures and thereby giving credence to the conjecture made in [30] that in general the result of the dualisation will be to take an $SU(3)$-structure background to one with $SU(2)$-structure. An heuristic reason for this can be found by looking at the abelian case following [53]. After T-duality, left and right movers couple to different set of frame fields for the same geometry call them $\hat{e}^i_+$ and $\hat{e}^i_-$. In the simplest case we can understand the effect of T-duality as a reflection on right movers so that in directions dualised $\hat{e}^i_+ = -\hat{e}^i_-$. The J and Ω of the starting $SU(3)$-structure gives rise, after dualisation, to a $\hat{J}$ and $\hat{\Omega}$ which may expressed in terms of either the left or right moving frame fields giving a corresponding $\hat{J}_\pm$ and $\hat{\Omega}_\pm$. Suppose that the expression for $\hat{J}$ is

$$\hat{J}_\pm = \hat{e}^1_\pm \wedge \hat{e}^2_\pm + \hat{e}^3_\pm \wedge \hat{e}^4_\pm + \hat{e}^5_\pm \wedge \hat{e}^6_\pm\,. \tag{3.1.2}$$

Consider the case where the dualised directions are 1 and 2, then $\hat{J}_+ = \hat{J}_-$ and in this case the T-dual also has $SU(3)$-structure. Now consider the dualisation of two directions that are not paired by the complex structure, say 1 and 3, in this case $\hat{J}_+ \neq \hat{J}_-$ and type changing has occurred; the $SU(3)$-structure gives rise to a T-dual $SU(2)$-structure. Since the non-Abelian T-dualisations performed in [30] involve three directions they can necessarily not respect the paring of the complex structure and so we anticipate them to be type changing. One goal of this chapter is to make this reasoning precise and to provide explicit examples where the T-dual $SU(2)$-structure can be obtained.

The second part of this chapter concerns a topic which at first sight might seem rather disconnected from the above discussion, namely the application of non-Abelian T-duality in the construction of new 'flavoured' solutions of supergravity. The string dual view on the addition of fundamental matter to the field theories has a rich history. Starting from the study of the 'quenched' dynamics of fundamental fields, equivalent to the addition of probe branes in the string backgrounds to the case in which flavour branes (sources) backreact and change the original geometries, various technical problems have been resolved. For reviews see [59], [60, 61].

In the case of backgrounds preserving some amount of SUSY, the first technical point to be addressed is to find SUSY embeddings for these sources or flavour branes. The embedding were initially found solving differential equations associated with the kappa-symmetry matrix. A more refined and efficient way of expressing the same conditions relies on G-structures and calibration forms. Indeed, the findings of papers like [17, 62, 28, 63, 29] among many others can be thought as examples of the generic formalism developed in [20], [64] and more explicitly laid-out in [65], [22].

A generic feature about these solutions encoding the dynamics of N_f fields transforming in the fundamental representation of the $SU(N_c)$ gauge group is that the string backgrounds should in principle represent sources localised on those SUSY-preserving submanifolds. The complications associated with the non-linear and coupled partial differential equations this problem requires, lead to the consideration of 'smeared' sources. The field theoretical effect of such simplification is the explicit breaking of $SU(N_f) \to U(1)^{N_f}$. The SUSY preserving way of implementing this smearing is also described by the G-structures classifying the original (unflavored) background, see [65], [22] for details.

Hence, there is a rich interplay between G-structures and the dynamics of SUSY sources in Supergravity. This is one of the themes of this chapter. Using the results established in the first part of this chapter we will be able to construct the non-Abelian T-dual of a flavoured background.

In section 3.2 we provide some more details on $SU(3)$ and $SU(2)$-structures and their transformation rules under non-Abelian T-duality. In section 3.3 we look at examples of the T-dual of the un-flavoured Klebanov-Witten model studied in [35, 30] and explicitly construct its $SU(2)$-structure. In section 3.4 we present the flavoured Klebanov-Witten model and its T-dual. Finally the chapter closes with some concluding remarks in section 3.5.

3.2 G-structures and their transformations

We now give a brief summary of the important details concerning G-structures. We follow the conventions of [20] except where indicated otherwise. We consider ten dimensional backgrounds consisting of a warped product of four dimensional Minkowski space and a six dimensional internal manifold M:

$$ds^2_{10} = e^{2A} ds^2_{1,3} + ds^2(M) \,. \tag{3.2.1}$$

Since we require $\mathcal{N} = 1$ supersymmetry there should exist a single four-dimensional conserved spinor. The ten-dimensional MW spinors of type II supergravity are decomposed as

$$\begin{aligned} \epsilon^1 &= \zeta_+ \otimes \eta^1_+ + \zeta_- \otimes \eta^1_- \,, \\ \epsilon^2 &= \zeta_+ \otimes \eta^2_\mp + \zeta_- \otimes \eta^2_\pm \,, \end{aligned} \tag{3.2.2}$$

where the upper sign in ϵ^2 corresponds to IIA and the lower to IIB— here $\pm$ denotes both four and six dimensional chiralities and we choose a basis such that $(\eta_+)^* = \eta_-$. From the internal spinors we define two $Cliff(6,6)$ pure spinors (or polyforms):

$$\Phi_\pm = \eta^1_+ \otimes (\eta^2_\pm)^\dagger. \tag{3.2.3}$$

We define the norms of the internal spinors $||\eta^1||^2 = |a|^2$ and $||\eta^2||^2 = |b|^2$. The dilatino and gravitino equations can be recast succinctly, for the type IIA case, as

$$\begin{aligned} e^{-2A+\Phi}(d + H\wedge)\Big[e^{2A-\Phi}\Phi_-\Big] &= dA \wedge \bar{\Phi}_- + \frac{e^\Phi}{8} i e^A \star_6 F_{IIA}, \\ (d + H\wedge)\Big[e^{2A-\Phi}\Phi_+\Big] &= 0 \,, \end{aligned} \tag{3.2.4}$$

The RR fluxes entering on the right hand side of this equation are defined for the type IIA case as,

$$F_{IIA} = F_0 + F_2 + F_4 + F_6. \tag{3.2.5}$$

Similar expressions hold in the case of Type IIB after exchanging $\Phi_+ \leftrightarrow \Phi_-$ and $F_{IIA} \leftrightarrow F_{IIB}$, see [20, 66] or appendix B for details.

Two important extreme cases are when the internal spinors are always parallel (corresponding to $SU(3)$-structure) and when they are nowhere parallel (that corresponds with $SU(2)$-structure). In the first case there is a single spinor of unit norm such that $\eta^1_+ = a\eta_+$, $\eta^2_+ = b\eta_+$ for $|a|^2 = |b|^2 = e^A$. The spinor bilinears then define a two form and a complex three form with components

$$J_{mn} = -\frac{i}{|a|^2}\eta_+^{1\dagger}\gamma_{mn}\eta_+^1\,, \qquad \Omega_{mnp} = -\frac{i}{a^2}\eta_-^{1\dagger}\gamma_{mnp}\eta_+^1\,. \tag{3.2.6}$$

These are normalised such that $J^3 = \frac{3i}{4}\Omega\wedge\bar{\Omega}$ and obey $J\wedge\Omega = 0$. The corresponding pure spinors are

$$SU(3)\ structure: \quad \Phi_+ = \frac{ab^*}{8}e^{-iJ}\,, \quad \Phi_- = -\frac{iab}{8}\Omega\,. \tag{3.2.7}$$

The second case when the spinors are nowhere parallel we have a non-vanish complex vector field defined by $\eta_+^1 = a\eta_+$, $\eta_+^2 = b(v^i + iw^i)\gamma_i\eta_-$. In this case one can show that the corresponding pure spinors have the form

$$SU(2)\ structure: \quad \Phi_+ = \frac{ab^*}{8}e^{-iv\wedge w}\wedge\omega\,, \quad \Phi_- = \frac{ab}{8}e^{-ij}\wedge(v+iw)\,. \tag{3.2.8}$$

We can express the forms v, w, ω and j directly in terms of the spinors (see for example [67]):

$$\begin{aligned} v_m - iw_m &= -\frac{1}{ab}\eta_-^{2\dagger}\gamma_m\eta_+^1\,,\\ \omega_{mn} &= \frac{i}{ab^*}\eta_+^{2\dagger}\gamma_{mn}\eta_+^1\,,\\ j_{mn} &= \frac{i}{2|b|^2}\eta_+^{2\dagger}\gamma_{mn}\eta_+^2 - \frac{i}{2|a|^2}\eta_+^{1\dagger}\gamma_{mn}\eta_+^1\,. \end{aligned} \tag{3.2.9}$$

To ascertain the non-Abelian T-dual of these structures one can work explicitly with the T-dual Killing spinors defined in eq (2.2.23) and construct from first principles the pure-spinors $\Phi_\pm$ defined above. Alternatively, for the spinor-phobic one can circumvent this by using the following transformation rules on the polyforms

$$\Phi_+^{SU(2)} = \Phi_-^{SU(3)}\Omega^{-1}\,, \quad \Psi_-^{SU(2)} = \Psi_+^{SU(3)}\Omega^{-1} \tag{3.2.10}$$

The D-brane generalised calibrations, follows from this as shown in appendix C.

Let us just remark at this stage that the condition of supersymmetry being preserved as detailed in eq (2.2.22) simply translates (using the Liebniz derivation property obeyed the Lorentz-Lie derivative [54, 55]) into the invariance of the pure-spinors under the regular Lie derivative acting on forms:

$$\mathcal{L}_k\epsilon = 0 \Rightarrow \mathcal{L}_k\Phi_\pm = 0\,. \tag{3.2.11}$$

For the case of the abelian T-duality one can show that this criteria does indeed ensure that supersymmetry is preserved after T-duality [53]. The essence of the proof is that up to terms proportional to this Lie derivative, the twisted differential d_H commutes with the Clifford multiplication rule (c.f. eq (3.2.10)) used to extract the T-dual pure spinors. Using this, one can infer that supersymmetry is preserved by the dualisation. Although we have not verified the details the situation here appears to be exactly analogous, indeed as we shall shortly see one can find a basis in which the non-Abelian T-duality essentially mimics the Abelian case.

In the following sections, we will consider two examples that will make clear various points discussed above. The first case-study will be the non-Abelian T-dual of the Klebanov-Witten system as presented in [35, 30]. We will explicitly show the $SU(2)$-structure of the solution (and hence its SUSY preservation). We will then consider the background obtained by adding fundamental fields (quarks) to the Klebanov-Witten field theory [68] (conversely, we will consider the addition of source-branes to the Klebanov-Witten background). With the essential help of the $SU(2)$-structure formalism described above, we will find the non-Abelian T-dual of this configuration.

3.3 Example 1: Un-flavoured Klebanov–Witten and its T-Dual

In this section we shall examine the T-dual of the Klebanov–Witten geometry and explicitly demonstrate its $SU(2)$-structure.

The theory living on D3 branes placed at the tip of the conifold was studied by Klebanov and Witten in [69]. The gauge theory describing the low energy dynamics of the branes is an $\mathcal{N}=1$ superconformal field theory with product gauge group $SU(N)\times SU(N)$. It can be described by a two node quiver and has two sets of bi-fundamental matter fields A_i in the $(N,\bar{N})$ representation of the gauge group and B^m in the $(\bar{N},N)$. The indices i and m correspond to two sets of $SU(2)$ global symmetries. The super potential for the matter fields is given by

$$W=\frac{\lambda}{2}\epsilon^{ij}\epsilon_{mn}Tr\left(A_iB^mA_jB^n\right)\ . \tag{3.3.1}$$

This gauge theory is dual to string theory on $AdS^5\times T^{(1,1)}$ with N units of RR flux supporting the geometry:

$$\begin{aligned} ds^2&=\frac{r^2}{L^2}dy_{1,3}^2+\frac{L^2}{r^2}dr^2+L^2ds^2(T^{(1,1)})\ ,\\ F_{(5)}&=\frac{4}{g_sL}\left(\mathrm{vol(AdS_5)}-\mathrm{L^5vol(T^{(1,1)})}\right)\ . \end{aligned} \tag{3.3.2}$$

We will work with the following frame fields for this geometry

$$
\begin{aligned}
&e^{y^\mu} = \frac{r}{L}dy^\mu\,, \quad e^r = \frac{L}{r}dr\,, \quad e^\varphi = \lambda_1 \sin\theta d\varphi\,, \quad e^\theta = \lambda_1 d\theta\,, \\
&e^1 = \lambda_1\sigma_1\,, \quad e^2 = \lambda_1\sigma_2\,, \quad e^3 = \lambda\left(\sigma_3 + \cos\theta d\varphi\right)\,,
\end{aligned}
\tag{3.3.3}
$$

in which $\lambda_1^2 = \frac{L^2}{6}$ and $\lambda^2 = \frac{L^2}{9}$ and we have introduced $SU(2)$ left invariant one-forms parametrised by Euler angles:

$$
\begin{aligned}
&\sigma_1 = (-\sin\psi d\tilde{\theta} + \cos\psi\sin\tilde{\theta}d\tilde{\varphi})\,, \\
&\sigma_2 = (\cos\psi d\tilde{\theta} + \sin\psi\sin\tilde{\theta}d\tilde{\varphi})\,, \quad \sigma_3 = (\cos\tilde{\theta}d\tilde{\varphi} + d\psi)\,.
\end{aligned}
\tag{3.3.4}
$$

For reference we state the ten-dimensional spinors of KW in this basis are given by

$$
\epsilon_1 = \sqrt{\frac{r}{L}}\Big(\zeta_+\otimes\eta_+ + \zeta_-\otimes\eta_-\Big)\,, \quad \epsilon_2 = \sqrt{\frac{r}{L}}\Big(i\,\zeta_+\otimes\eta_+ - i\,\zeta_-\otimes\eta_-\Big)\,. \tag{3.3.5}
$$

The chiralities in these expressions are defined with respect to four and six-dimensional chirality matrices

$$
\gamma_{(4)} = i\,\gamma^{y^0y^1y^2y^3}\,, \quad \gamma_{(6)} = -i\,\gamma^{\varphi\theta 123r}\,, \tag{3.3.6}
$$

such that under the ten-dimensional chirality operator is $\Gamma_{(10)} = \gamma_{(4)}\otimes\gamma_{(6)}$ both ϵ_1 and ϵ_2 are positive. In addition the spinor η_+ is constant and normalised such that $\eta_+^\dagger\eta_+ = 1$. Supersymmetry imposes the following projections on the spinor (as above $\eta_+ = (\eta_-)^*$),

$$
\gamma^{r3}\eta_+ = \gamma^{12}\eta_+ = \gamma^{\varphi\theta}\eta_+ = -i\,\eta_+\,. \tag{3.3.7}
$$

Using this expressions, we can determine the $SU(3)$-structure of KW in this basis to be

$$
\begin{aligned}
&J = e^{\theta\varphi} - e^{12} + e^{3r}\,, \\
&\Omega = (e^2 + i\,e^1)\wedge(e^\theta + i\,e^\varphi)\wedge(e^3 + i\,e^r)\,.
\end{aligned}
\tag{3.3.8}
$$

The non-Abelian T-dual of this geometry with respect to the $SU(2)$ global symmetry defined by the σ_i was constructed in [35, 30]. The result is an $\mathcal{N} = 1$

supersymmetric solution of type IIA whose NS sector is given by[1]

$$
\begin{aligned}
d\hat{s}^2 =& ds^2_{\mathrm{AdS}_5} + \lambda_1^2(d\theta^2 + \sin^2\theta d\varphi^2) + \frac{\lambda_1^2\lambda^2}{\Delta}x_1^2\hat{\sigma}_3^2 \\
&+ \frac{1}{\Delta}\Big((x_1^2 + \lambda^2\lambda_1^2)dx_1^2 + (x_2^2 + \lambda_1^4)dx_2^2 + 2x_1x_2dx_1dx_2\Big) , \\
\hat{B} =& -\frac{\lambda^2}{\Delta}\Big[x_1x_2dx_1 + (x_2^2 + \lambda_1^4)dx_2\Big] \wedge \hat{\sigma}_3 , \\
e^{-2\hat{\Phi}} =& \frac{8}{g_s^2}\Delta ,
\end{aligned}
\tag{3.3.9}
$$

where $\hat{\sigma}_3 = d\psi + \cos\theta d\varphi$ and

$$
\Delta \equiv \lambda_1^2x_1^2 + \lambda^2(x_2^2 + \lambda_1^4) . \tag{3.3.10}
$$

The metric evidently has an $SU(2) \times U(1)_\psi$ isometry and for a fixed value of (x_1, x_2) the remaining directions give a squashed three sphere. This geometry is supported by two and four form RR fluxes which may be computed using eq (2.2.17) and whose explicit form can be found in [30] or eq (3.4.17) once the limits of footnote 9 are taken. We remark in passing that the lift of this geometry to eleven dimensions has an interpretation in terms of recently discovered $\mathcal{N} = 1$ SCFT's obtained from wrapping M5 branes on a Riemann surface (of genus zero in this case) [57].

One can establish the left and right moving T-dual frames for this geometry along the lines of eq (2.2.13). The frames in the AdS direction are unaltered as are e^θ and e^φ. In the directions dualised we find new frame fields $\hat{e}^i_\pm$ for $i = 1\ldots3$. The plus and minus T-dual frames are related by a Lorentz transformation which, as described in chapter 2, induces a transformation on spinors given by [2],

$$
\Omega = \frac{\Gamma_{(10)}}{\sqrt{\Delta}}\Big(-\lambda_1^2\lambda\Gamma^{123} + \lambda_1x_1\cos\psi\,\Gamma^1 + \lambda_1x_1\sin\psi\,\Gamma^2 + \lambda x_2\Gamma^3\Big) . \tag{3.3.11}
$$

This defines the Killing spinors of the T-dual to be

$$
\hat{\epsilon}_1 = \epsilon_1 , \qquad \hat{\epsilon}_2 = \Omega \cdot \epsilon_2 . \tag{3.3.12}
$$

Implementing the four-six decomposition one finds from eq (3.3.5) using eq

[1]We have set $L = 1$ which may be restored by appropriate rescaling.

[2]The careful reader will not confuse this matrix Ω and its inverse Ω^{-1} with the complex three-form defining an $SU(3)$-structure, that appears for example in eq (3.3.8).

(3.3.7) that

$$\hat{\epsilon}_1 = \sqrt{\frac{r}{L}}\Big(\zeta_+ \otimes \eta_+ + \zeta_- \otimes \eta_-\Big)\,,$$
$$\hat{\epsilon}_2 = \sqrt{\frac{r}{L}}\Big(\zeta_+ \otimes \hat{\eta}^2_- + \zeta_- \otimes \hat{\eta}^2_+\Big)\,, \tag{3.3.13}$$

where

$$\hat{\eta}^2_- = -\frac{i}{\sqrt{\Delta}}\Big(\lambda_1^2\lambda\gamma^r + \lambda_1 x_1 \cos\psi\,\gamma^1 + \lambda_1 x_1 \sin\psi\,\gamma^2 + \lambda x_2 \gamma^3\Big)\eta_+\,,$$
$$\hat{\eta}^2_+ = (\hat{\eta}^2_-)^*. \tag{3.3.14}$$

It is clear that in this basis, the T-dual Killing spinors depend not only on the radial coordinate but also on the T-dual coordinates x_1, x_2. It is helpful work in a different basis in which this new spinor can be expressed as simply as possible. In addition, we would like the new vielbein basis to preserve the geometric structure defined by η_+, because ϵ_1 is invariant under the non-Abelian T-duality. To do so we perform a rotation to a new basis $\tilde{e} = R\hat{e}$ (ordered as $r, \varphi, \theta, 1, 2, 3$) with the rotation matrix

$$R = \frac{1}{\sqrt{1+\zeta.\zeta}}\begin{pmatrix} 1 & 0 & 0 & \zeta^1 & \zeta^2 & \zeta^3 \\ 0 & \sqrt{1+\zeta.\zeta} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{1+\zeta.\zeta} & 0 & 0 & 0 \\ -\zeta^1 & 0 & 0 & 1 & -\zeta^3 & \zeta^2 \\ -\zeta^2 & 0 & 0 & \zeta^3 & 1 & -\zeta^1 \\ -\zeta^3 & 0 & 0 & -\zeta^2 & \zeta^1 & 1 \end{pmatrix} \tag{3.3.15}$$

with,

$$\zeta^1 = \frac{x_1 \cos\psi}{\lambda\lambda_1}\,, \quad \zeta^2 = \frac{x_1 \sin\psi}{\lambda\lambda_1}\,, \quad \zeta^3 = \frac{x_2}{\lambda_1^2}\,. \tag{3.3.16}$$

Notice that these parameters are reflecting the structure of the spinor transformation matrix Ω. The rotated vielbeins are given, in coordinate frame, by:

$$\tilde{e}^r = \frac{\lambda\lambda_1^2 dr - r(x_1 dx_1 + x_2 dx_2)}{r\sqrt{\Delta}}, \quad \tilde{e}^\varphi = \lambda_1 \sin\theta d\varphi,$$
$$\tilde{e}^1 = \lambda_1 \frac{r\lambda(x_1 \sin\psi\hat{\sigma}_3 - \cos\psi\, dx_1) - x_1\cos\psi dr}{r\sqrt{\Delta}}, \quad \tilde{e}^\theta = \lambda_1 d\theta, \tag{3.3.17}$$
$$\tilde{e}^2 = -\lambda_1 \frac{r\lambda(x_1 \cos\psi\hat{\sigma}_3 + \sin\psi dx_1) + x_1 \sin\psi dr}{r\sqrt{\Delta}}, \quad \tilde{e}^3 = -\frac{\lambda x_2 dr + \lambda_1^2 r\, dx_2}{r\sqrt{\Delta}}.$$

Then in this new basis (in which the gamma matrices are of course also rotated $\tilde{\gamma} = R\gamma$), we can easily show that,

$$\begin{aligned}
\tilde{\epsilon}_1 &= \sqrt{\frac{r}{L}}\Big(\zeta_+ \otimes \eta_+ + \zeta_- \otimes \eta_-\Big) \,, \\
\tilde{\epsilon}_2 &= \sqrt{\frac{r}{L}}\Big(\zeta_+ \otimes \tilde{\eta}^2_- + \zeta_- \otimes \tilde{\eta}^2_+\Big) \,,
\end{aligned} \tag{3.3.18}$$

with $\tilde{\eta}^2_+ = (\tilde{\eta}^2_-)^*$ and,

$$\tilde{\eta}^2_- = -i\,\tilde{\gamma}^r \eta_+ \,. \tag{3.3.19}$$

Note that, as required for type IIA supergravity, the new spinors have opposite chirality. With this simple relation between $\tilde{\eta}^2_-$ and η_+, we clearly see that they are never parallel, hence we have an $SU(2)$-structure. Because we were careful about the definition of our new vielbein basis, the projections on η_+ are not modified,

$$\tilde{\gamma}^{r3}\eta_+ = \tilde{\gamma}^{12}\eta_+ = \tilde{\gamma}^{\varphi\theta}\eta_+ = -i\,\eta_+ \,, \tag{3.3.20}$$

but the projections obeyed by $\tilde{\eta}^2_-$ are different

$$-\,\tilde{\gamma}^{r3}\tilde{\eta}^2_- = \tilde{\gamma}^{12}\tilde{\eta}^2_- = \tilde{\gamma}^{\varphi\theta}\tilde{\eta}^2_- = -i\,\tilde{\eta}^2_- \,. \tag{3.3.21}$$

The Killing spinors define two different $SU(3)$-structures

$$\begin{aligned}
J^1 &= \tilde{e}^{\theta\varphi} + \tilde{e}^{21} - \tilde{e}^{3r} \,, \\
\Omega^1 &= (\tilde{e}^2 + i\,\tilde{e}^1) \wedge (\tilde{e}^\theta + i\,\tilde{e}^\varphi) \wedge (-\tilde{e}^3 + i\,\tilde{e}^r) \,, \\
J^2 &= \tilde{e}^{\theta\varphi} + \tilde{e}^{21} + \tilde{e}^{3r} \,, \\
\Omega^2 &= (\tilde{e}^2 + i\,\tilde{e}^1) \wedge (\tilde{e}^\theta + i\,\tilde{e}^\varphi) \wedge (-\tilde{e}^3 - i\,\tilde{e}^r) \,,
\end{aligned} \tag{3.3.22}$$

whose intersection is the $SU(2)$-structure given by

$$\begin{aligned}
v + iw &= -\tilde{e}^3 + i\tilde{e}^r \,, \\
j &= \tilde{e}^{\theta\varphi} + \tilde{e}^{21} \,, \\
\omega &= (\tilde{e}^2 + i\,\tilde{e}^1) \wedge (\tilde{e}^\theta + i\,\tilde{e}^\varphi) \,.
\end{aligned} \tag{3.3.23}$$

An explicit check shows that these do indeed satisfy the dilatino and gravitino equations that follow from eq (3.2.4).

Note that it makes sense to mix e^r with e^1, e^2 and e^3 when performing the rotation of eq (3.3.15) because the geometric structure links e^r and e^3 in the projection $\gamma^{r3}\eta_+ = -i\,\eta_+$. Actually the choice of this rotation appears clearer when considering that, because of the geometric structure, the transformation of the spinor ϵ_2 can be written very easily as $\Omega\epsilon_2 = -\tilde{\Gamma}^r\epsilon_2$. It is in this new basis that the transformation closely resembles the T-duality of the abelian case.

3.4 Example 2: Flavoured Klebanov Witten and its T-Dual.

An important step if one is to try and use the AdS/CFT paradigm to understand QCD-like dynamics is to incorporate fundamental flavours (quarks) into the gauge theory and corresponding gravity descriptions. A first step in this direction is to add a finite number N_f of fundamental flavours which in the IIB set-up is typically achieved by the inclusion of a finite number of flavour $D7$ branes. This is the probe or quenched limit; the colour D3 branes generate the geometry but the flavour branes do not back-react and only minimise their world volume (DBI) action without deforming the geometry. Remarkably one can even work beyond this quenched approximation by allowing a large number of flavour branes ($N_f \sim N_c$) in which case the $D7$ branes deform the geometry, see [60, 61] for reviews.

In the case at hand we will consider adding N_f $D7$ branes to the KW geometry in such a way that supersymmetry is preserved. We first describe the gauge theory engineered from the D3-D7 system in the conifold. We consider $D7$ branes parallel to the D3 stack in the Minkowski direction with the remaining four direction embedded holomorphically and non-compactly in the conifold. The strings that run between the $D7$ and the $D3$ give rise to massless flavours. To avoid gauge anomalies on the field theory side of the description and supergravity tadpoles on the string side of it, one must include two branches of $D7$ branes giving rise to fundamental chiral superfields for each gauge group ($q, \tilde{q}$ in the $(N, 1)$ and $(\bar{N}, 1)$ and $Q, \tilde{Q}$ in the $(1, N)$ and $(1, \bar{N})$). The super potential for this theory is given by [68],

$$W = \frac{\lambda}{2}\epsilon^{ij}\epsilon_{mn} Tr\left(A_i B^m A_j B^n\right) + h_1 \tilde{q}^a A_1 Q_a + h_2 \tilde{Q}^a B_1 q_a \,. \tag{3.4.1}$$

Notice that the $SU(2)$ global symmetries are explicitly broken by the embedding of the $D7$ branes - this symmetry will be recovered by smearing the sources, when we go beyond the probe limit. The addition of flavours implies that the theory looses conformality; a positive beta function is generated and *a priori* one expects a Landau pole in the UV.

We now turn to the gravity description. By considering the κ-symmetry projectors one can determine that the supersymmetric embeddings of D7 branes in the KW background to lie along two branches (the y^μ denote the Minkowski directions) [68],

$$\xi = (y^\mu, r, \psi, S^2), \quad \tilde{\xi} = (y^\mu, r, \psi, \tilde{S}^2) \tag{3.4.2}$$

where S^2 and $\tilde{S}^2$ are the 2-spheres parametrised by θ, φ and $\tilde{\theta}$, $\tilde{\varphi}$ respectively. To avoid the D7 charge tadpole we must include N_f branes on both branches. One can write an action for the whole system consisting of supergravity together with DBI and WZ terms of the D7 branes (in string frame)

$$
\begin{aligned}
S_{DBI} &= -T_{D7}\sum_{Nf}\int_{\xi} d^8\sigma e^{-\Phi}\sqrt{|P[g]|} - T_{D7}\sum_{Nf}\int_{\tilde{\xi}} d^8\sigma e^{-\Phi}\sqrt{|P[g]|}\,, \\
S_{WZ} &= T_7\sum_{Nf}\int P[C_8]\,,
\end{aligned}
\tag{3.4.3}
$$

where P indicates the pull back to the appropriate cycle, sometimes also denoted below as $g\Big|_{\xi}$. We do not activate the gauge field on the brane itself and since there is no NS two-form in this geometry the WZ-term is simple. Now we consider the case where the number of flavour branes goes to infinity in which case they can be smeared. In other words we consider that each stack is distributed homogeneously across the two sphere it does not wrap.[3] In a field theory perspective the $U(N_f)$ flavours symmetries are broken to their maximal torus. The supergravity effect can be encoded by introducing a *smearing form*:

$$
\Xi_2 = -\frac{N_f}{4\pi}\left(\sin\theta d\theta\wedge d\varphi + \sin\tilde{\theta}d\tilde{\theta}\wedge d\tilde{\varphi}\right)\ . \tag{3.4.4}
$$

The smearing procedure essentially boils down to replacing the DBI and WZ contributions of eq (3.4.3) with

$$
\begin{aligned}
S_{DBI} &\to -T_{D7}\sum_{Nf}\int d^{10}x e^{-\Phi}\left(\sin\tilde{\theta}\sqrt{|P[g]|} + \sin\theta\sqrt{|P[g]|}\right)\,, \\
S_{WZ} &\to T_7\sum_{Nf}\int \Xi_2\wedge C_8\,.
\end{aligned}
\tag{3.4.5}
$$

Once consequence of this smearing is that the Bianchi identities are modified

$$
dF_1 = \Xi_2\,,\qquad dF_5 = 0\,. \tag{3.4.6}
$$

The $D7$ brane back-reaction is accommodated by the following ansatz (as above

[3]This smearing procedure overcomes the bound on the number of D7 branes that comes form looking the deficit angle of the D7 solution so N_f may indeed be taken large.

we work in string frame)

$$ds^2 = \frac{e^{\frac{\Phi}{2}}}{\sqrt{h}} dy_{1,3}^2 + e^{\frac{\Phi}{2}}\sqrt{h}\Big(dr^2 + \lambda_1^2 e^{2g}(\sin^2\theta d\varphi^2 + d\theta^2) +$$

$$\lambda_2^2 e^{2g}(\sigma_1^2 + \sigma_2^2) + \lambda^2 e^{2f}(\sigma_3 + \cos\theta d\varphi)^2\Big)\,, \qquad (3.4.7)$$

$$F_1 = \frac{N_f}{4\pi}(\sigma_3 + \cos\theta d\varphi)\,, \qquad F_5 = (1+\star)dt\wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge K dr\,,$$

where the warp factors f, g, h and the dilaton Φ are functions of the radial variable r and $\lambda_1^2 = \lambda_2^2 = 1/6$, $\lambda^2 = 1/9$ and as a consequence of the Bianchi-identities $Kh^2e^{4g+f} = 27\pi N_c$. [4] The σ_i's are $SU(2)$ left invariant 1-forms defined in eq (3.3.4). A convenient basis of vielbeins is given by:

$$\begin{aligned}
e^{y^\mu} &= e^{\Phi/4}h^{-1/4}dy^\mu\,, & e^r &= e^{\Phi/4}h^{1/4}dr\,,\\
e^{\varphi} &= \lambda_1 e^{g+\Phi/4}h^{1/4}\sin\theta d\varphi\,, & e^{\theta} &= \lambda_1 e^{g+\Phi/4}h^{1/4}d\theta\,,\\
e^1 &= \lambda_1 e^{g+\Phi/4}h^{1/4}\sigma_1\,, & e^2 &= \lambda_1 e^{g+\Phi/4}h^{1/4}\sigma_2\,,\\
e^3 &= \lambda h^{1/4}e^{f+\Phi/4}(\sigma_3 + \cos\theta d\varphi)\,. & &
\end{aligned} \qquad (3.4.8)$$

Like the unflavoured version, this solution supports an $SU(3)$-structure:

$$\begin{aligned}
J &= -\left(e^{r3} + e^{\varphi\theta} + e^{12}\right) = -\frac{4\pi\sqrt{h}}{3N_f}e^{\frac{\Phi}{2}}\left(\frac{1}{2}e^{2g}\Xi_2 + e^f dr\wedge F_1\right)\\
\Omega &= (e^2 + ie^1)\wedge(e^\theta + ie^\varphi)\wedge(e^3 + ie^r).
\end{aligned} \qquad (3.4.9)$$

With these and the structure conditions for $SU(3)$ it is possible to derive a set of first order BPS equations for the various functions introduced thus far:

$$\begin{aligned}
f' &= e^{-f}(3 - 2e^{2f-2g}) - \frac{3N_f}{8\pi}e^{\Phi - f}\,, \quad g' = e^{f-2g}\,,\\
h' &= -27\pi N_c e^{-f-4g}\,, \quad \Phi' = \frac{3N_f}{4\pi}e^{\Phi-f}\,.
\end{aligned} \qquad (3.4.10)$$

The RR potentials can be expressed in terms of these the $SU(3)$-structure forms

[4] The unflavoured Klebanov-Witten can be recovered with the following:

$$y^\mu \to \frac{1}{\sqrt{g_s}}y^\mu\,,\ N_f = 0\,,\ h = \frac{L^4}{g_s r^4}\,,\ e^{2f} = e^{2g} = r^2\,,\ K = \frac{4r^3 g_s}{L^4}\,,\ e^\Phi = g_s\,.$$

as:

$$C_8 = -\frac{1}{2}e^{-\Phi}\left(\frac{e^{\Phi}}{h}vol_4\right)\wedge J\wedge J\,,\qquad C_4 = e^{-\Phi}\left(\frac{e^{\Phi}}{h}vol_4\right)\,. \tag{3.4.11}$$

where $F_9 = \star F_1$. The reason why we did not cancel both factors of the dilaton is just for comparison with formulas below.

Finally for the brane embedding to be supersymmetric it must obey the calibration condition:

$$\sqrt{-g_{\xi}}d^8\xi = -\frac{1}{2}\left(\frac{e^{\Phi}}{h}vol_4\right)\wedge J\wedge J\Big|_{\xi}\,, \tag{3.4.12}$$

where $\hat{g}_{\xi}$ is the induced metric on ξ whilst $\Big|_{\xi}$ indicates the pull back onto ξ, and similarly for $\tilde{\xi}$. This allows the DBI and WZ actions of the smeared brane embedding to be expressed as:

$$S_{DBI} = \frac{1}{2}\int_{\mathcal{M}_{10}} e^{-\Phi}\left(\frac{e^{\Phi}}{h}vol_4\right)\wedge J\wedge J\wedge\Xi_2\,,\qquad S_{WZ} = \int_{\mathcal{M}_{10}} C_8\wedge\Xi_2\,, \tag{3.4.13}$$

from which it is immediate that $S_{DBI} + S_{WZ} = 0$, as required by SUSY. As the sources are calibrated the dilaton equation of motion, Einstein's equations and the flux equation for H are all satisfied once the Bianchi identities are imposed. This is proved for any $SU(3)\times SU(3)$-structure background in [64].

We will now find the non-Abelian T-dual of this system involving metric, fluxes and sources. The interest of this problem is two-fold. On the one hand, it teaches us the effect of the non-Abelian duality on the Born-Infeld-Wess-Zumino Action. On the other hand, it will tell us how to find the new smearing forms. Both these points give clues to a generic procedure.

3.4.1 The T-dual

We perform the non-Abelian T-duality along the $SU(2)$ directions as before. To compactly display the results it is convenient to perform a supplementary rotation as detailed in equation (3.21) of [30]. We find the frame fields for the T-dual metric to be,

$$
\begin{aligned}
\hat{e}^1 &= -\frac{\lambda_1}{\Delta}e^{g+\frac{\Phi}{4}}h^{1/4}\left((\lambda_1^2\lambda^2 h e^{2f+2g+\Phi}+x_1^2)dx_1+x_1x_2(dx_2+\lambda^2\sqrt{h}e^{2f+\frac{\Phi}{2}}\hat{\sigma}_3)\right)\,,\\
\hat{e}^2 &= \frac{\lambda_1}{\Delta}e^{g+\frac{3}{4}\Phi}h^{3/4}\left(\lambda^2 x_2 e^{2f}dx_1-\lambda_1^2 x_1 e^{2g}(dx_2+\lambda^2\sqrt{h}e^{2f+\frac{\Phi}{2}}\hat{\sigma}_3)\right)\,, \qquad (3.4.14)\\
\hat{e}^3 &= -\frac{\lambda}{\Delta}e^{f+\frac{\Phi}{4}}h^{1/4}\left(x_1x_2dx_1+(\lambda_1^4 h e^{4g+\Phi}+x_2^2)dx_2-\lambda_1^2\sqrt{h}x_1^2e^{2g+\frac{\Phi}{2}}\hat{\sigma}_3\right).
\end{aligned}
$$

where we recall $\hat{\sigma}_3=\cos\theta d\phi+d\psi$ and

$$
\Delta=\sqrt{h}e^{\frac{\Phi}{2}}\left(\lambda_1^4\lambda^2 h e^{2f+4g+\Phi}+\lambda_1^2x_1^2e^{2g}+\lambda^2x_2^2e^{2f}\right)\,. \qquad (3.4.15)
$$

The T-dual NS sector is then given by

$$
\begin{aligned}
d\hat{s}^2 &= (e^{y_\mu})^2+(e^r)^2+(e^\varphi)^2+(e^\theta)^2+(\hat{e}^1)^2+(\hat{e}^2)^2+(\hat{e}^3)^2\,,\\
\hat{B} &= \frac{\lambda e^{f-g}x_2}{\lambda_1x_1}\hat{e}^{13}+\frac{\lambda\lambda_1e^{f+g+\frac{\Phi}{2}}\sqrt{h}}{x_1}\hat{e}^{23}\,, \qquad (3.4.16)\\
H &= d\hat{B}\,,\\
e^{-2\hat{\Phi}} &= 8\Delta e^{-2\Phi}\,.
\end{aligned}
$$

This geometry is supported by RR fluxes, obtained using the general formula eq (2.2.17),

$$
\begin{aligned}
F_0 &= \frac{N_f}{\sqrt{2}\pi}x_2\,,\\
F_2 &= \frac{\lambda_1e^{g-f-\frac{\Phi}{2}}}{\sqrt{2}\lambda\pi}(4\pi\lambda_1\lambda^2Ke^{2f+g}h^{3/2}e^{\varphi\theta}+\lambda\lambda_1N_fe^{f+g+\Phi}\sqrt{h}\hat{e}^{12} \qquad (3.4.17)\\
&\qquad -x_1N_fe^{\frac{\Phi}{2}}\hat{e}^{13})\\
F_4 &= -2\sqrt{2}e^{-\Phi}hKe^{\varphi\theta}\wedge\left(\lambda x_2e^f\hat{e}^{12}+\lambda_1x_1e^g\hat{e}^{23}\right)\,.
\end{aligned}
$$

Although there is an F_0, it is possible that one should not regard this as a solution of Massive IIA – the would-be mass parameter is neither constant nor quantised— but rather, as we shall discuss, this should be thought of as a solution to Type IIA in the presence of $D8$ sources. Now since the original Bianchi identities were not satisfied (due to $D7$ source) one would not expect these new fluxes in eq (3.4.17), to obey standard Bianchi identities after the non-Abelian

T-duality. Indeed, one finds T-dual smearing forms enter the game

$$
\begin{aligned}
dF_0 &= \Xi_1 \\
dF_2 - F_0 H &= \Xi_1 \wedge B + \Xi_3 \\
dF_4 - H \wedge F_2 &= \frac{1}{2}\Xi_1 \wedge B \wedge B + B \wedge \Xi_3
\end{aligned}
\tag{3.4.18}
$$

We find a rather nice result, the T-dual smearing forms which can be calculated directly as

$$
\begin{aligned}
\Xi_1 &= -\frac{N_f e^{-g-\frac{\Phi}{4}}}{\sqrt{2}\pi\lambda_1 h^{1/4}}\left(x_1\hat{e}^2 + \lambda\lambda_1\sqrt{h}e^{f+g+\frac{\Phi}{2}}\hat{e}^3\right) = \frac{N_f}{\sqrt{2}\pi}dx_2 \\
\Xi_3 &= \frac{N_f e^{-2g-\frac{\Phi}{4}}}{\pi h^{1/4}}e^{\varphi\theta} \wedge \left(\sqrt{3}x_1 e^g \hat{e}^1 + \sqrt{2}x_2 e^f \hat{e}^3\right) \\
&= \frac{N_f}{\sqrt{2}\pi}\sin\theta\,(x_1 d\theta \wedge d\varphi \wedge dx_1 + x_2 d\theta \wedge d\varphi \wedge dx_2)
\end{aligned}
\tag{3.4.19}
$$

These may be obtained equally using a transformation rule much like that of the RR fields

$$
e^{\Phi}\not{\Xi}_2\Omega^{-1} = e^{\hat{\Phi}}\hat{\not{\Xi}}_B \tag{3.4.20}
$$

where $\hat{\Xi}_B = e^B \wedge (\Xi_1 + \Xi_3)$. The active smearing forms indicate sources for both $D6$ and $D8$ branes.

3.4.2 A Nice Subtlety

There is a subtlety here. A naive reasoning would lead us to believe that when the non-Abelian T-dual is applied to D7 sources, it will generate charge for $D8, D6, D4$ branes, whilst in eq (3.4.19) we only have $D8, D6$ charges, since Ξ_5, the smearing form for D4 charges is absent in eq (3.4.18). Below, we will solve this apparent contradiction.

If we consider the Bianchi identity of the RR polyform

$$
dF - H \wedge F = \hat{\Xi} \wedge e^B\,, \tag{3.4.21}
$$

it is clear that since the LHS of this equation is gauge invariant the RHS must also be. Throughout this note we have set to zero gauge fields on the world volume however one should remember that they occur in conjunction with the NS two from in the gauge invariant combination $\mathcal{F} = B + 2\pi\alpha' dA$. Then the most conservative view is that performing a gauge transformation on the NS B-field simply activates appropriate compensating world volume gauge field. There is however another point of view which is to keep the world volume gauge fields turned off and instead compensate for a B-field transformation with an appropriate redefinition of the smearing form $\hat{\Xi}$. This is best not thought of as a gauge transformation but rather as a mapping. In this picture the transformation of the NS potential, $B \to B + \Delta B$, mediates a redistribution of source charge between the D4 and D6 branes. The reason to prefer this second viewpoint is that turning on a one form gauge field on the brane would break either the $SU(N_f)$ or the $U(1)^{N_f}$ symmetry.

To explain this second viewpoint, we consider the transformation $B \to B' = B + \Delta B$. Such a transformation must be supplemented by a transformation of the smearing polyform $\hat{\Xi} \to \hat{\Xi}'$ so that the Bianchi identity of the RR polyform is unchanged. This requires that

$$\hat{\Xi}' \wedge e^{B'} = \hat{\Xi} \wedge e^{B} \, . \tag{3.4.22}$$

As an example consider a transformation for which $\Xi_1 \wedge \Delta B = 0$ then we still have

$$dF_0 = \Xi_1, \quad dF_2 - HF_0 = \Xi_3 + B \wedge \Xi_1 \, . \tag{3.4.23}$$

The final Bianchi identity of the RR sector then becomes:

$$dF_4 - H \wedge F_2 = \Xi_5 + B \wedge \Xi_3 + \frac{1}{2} B \wedge B \wedge \Xi_1 \tag{3.4.24}$$

where $\Xi_5 = \Delta B \wedge \Xi_3$. So we generate an explicit source for D4 branes under such a transformation. Clearly there are always source D8 branes but wether we have explicit source D6's or source D6 and D4's is a gauge dependent statement. We do not believe it is possible to find a gauge in which we only have explicit D8 sources. This appears to be related to the fact that the original type IIB D7 brane embedding has two branches. This may seem rather mysterious, however one should understand that the total DBI and WZ actions of the source branes depend only on the sources through the gauge invariant quantity $\Xi \wedge e^B$. The higher potentials in the WZ action, C_5, C_7 and C_9, are gauge invariant as consequence of the $SU(2)$ SUSY conditions (see appendix C for details on this). So, it is only the 'portion' of the sources that are viewed as being explicit rather than induced that changes, the equations of motion, the Bianchi identities and the total Maxwell charge are all invariants.

In summary, we advocated a picture in which gauge transformations mediate a redistribution of the source charge between the D4 and D6 branes. This could be thought of as an 'inverse' of the Myers effect.

To emphasize these points above, we can consider their Page charges [70] defined as

$$
\begin{aligned}
Q^{D6}_{page} &= \int_{\mathcal{M}_2} (F_2 - F_0 B) \\
Q^{D4}_{page} &= \int_{\mathcal{M}_4} (F_4 - B\wedge F_2 + \frac{1}{2} F_0 B\wedge B)
\end{aligned}
\tag{3.4.25}
$$

the Maxwell charges are invariant under a shift in the B-field described above. While the shift of the Page charges is given by:

$$
\begin{aligned}
\Delta Q^{D6}_{page} &= \int_{\mathcal{M}_2} F_0 \Delta B \\
\Delta Q^{D4}_{page} &= \int_{\mathcal{M}_4} (-\Delta B\wedge (F_2 - F_0 B) + \frac{1}{2} F_0 \Delta B\wedge \Delta B)
\end{aligned}
\tag{3.4.26}
$$

As these these integrals are defined over compact manifolds these quantities are invariant for small gauge transformations. The integrands are exacts so the integrals are zero. It is of course a generic feature of Page charges that they are only defined up to quantised shifts under large gauge transformations[5]. This is generally interpreted in the literature as a Seiberg duality in the dual gauge theory as in [68].

3.4.3 Potentials, $SU(2)$-structure and Calibration.

We may use the formula for the T-dual RR potential eq (2.2.21) to find the RR potentials. These are given in coordinate frame by (for alternative expressions see below),

$$
\begin{aligned}
C_5 &= e^{-\hat{\Phi}}\left(\frac{e^{\Phi}}{h} vol_4\right)\wedge\left(\frac{\lambda\lambda_1^2 e^{f+2g+\Phi} h dr - (x_1 dx_1 + x_2 dx_2)}{\sqrt{\Delta}}\right), \\
C_7 &= e^{-\hat{\Phi}}\left(\frac{e^{\Phi}}{h} vol_4\right)\wedge\left(\frac{\lambda_1^2 e^{2g+\Phi} h \sin\theta d\theta\wedge d\varphi\wedge(\lambda e^f x_2 dr + \lambda_1^2 e^{2g} dx_2)}{\sqrt{\Delta}} + \right. \\
&\quad \frac{\lambda\lambda_1^2 x_1 e^{f+2g+\frac{3\Phi}{2}} h^{\frac{3}{2}}}{\Delta^{3/2}}\Big[\lambda_1^2 e^{2g}(x_1 dr\wedge dx_2 + \lambda e^f dx_1\wedge dx_2) - \\
&\quad \left.\lambda^2 e^{2f} x_2 dr\wedge dx_1\Big]\wedge\hat{\sigma}_3\right),
\end{aligned}
\tag{3.4.27}
$$

$$
\begin{aligned}
C_9 &= e^{-\hat{\Phi}}\left(\frac{e^{\Phi}}{h} vol_4\right)\wedge\left(\lambda\lambda_1^4 e^{f+4g+\frac{3\Phi}{2}} x_1 h^{\frac{3}{2}} \sin\theta d\theta\wedge d\varphi\wedge\hat{\sigma}_3\right)\wedge \\
&\quad \left(\frac{(h\lambda^2\lambda_1^2 e^{2f+2g+\Phi} + x_1^2) dr\wedge dx_1 + x_1 x_2 dr\wedge dx_2 + \lambda e^f x_2 dx_1\wedge dx_2}{\Delta^{3/2}}\right).
\end{aligned}
$$

This background is again of $SU(2)$-structure where the defining forms $v + iw, j, \omega$ are the same as in the un-flavoured case —see eqs (3.3.23)— the only

[5] Large gauge transformations are topological in nature and always induce quantised shifts.

difference being that the parameters entering the rotation matrix used in eq (3.3.15) become

$$\zeta^1 = \frac{e^{-f-g-\frac{\Phi}{2}} x_1 \cos\psi}{\lambda\lambda_1\sqrt{h}}; \quad \zeta^2 = \frac{e^{-f-g-\frac{\Phi}{2}} x_1 \sin\psi}{\lambda\lambda_1\sqrt{h}}; \quad \zeta^3 = \frac{e^{-2g-\frac{\Phi}{2}} x_2}{\lambda_1^2\sqrt{h}}. \tag{3.4.28}$$

This rotation leads to the following simple vielbeins for the dual geometry

$$\begin{aligned}
\tilde{e}^r &= \frac{h\lambda\lambda_1^2 e^{f+2g} dr - (x_1 dx_1 + x_2 dx_2)}{\sqrt{\Delta}}, \\
\tilde{e}^\varphi &= h^{\frac{1}{4}}\lambda_1 e^{g+\frac{\Phi}{4}} \sin\theta d\varphi, \quad \tilde{e}^\theta = h^{\frac{1}{4}}\lambda_1 e^{g+\frac{\Phi}{4}} d\theta\,, \\
\tilde{e}^1 &= \sqrt{h}\lambda_1 e^{g+\Phi/2} \frac{-x_1 \cos\psi\,, dr - e^f\lambda(\cos\psi\, dx_1 - x_1 \sin\psi\, \hat{\sigma}_3)}{\sqrt{\Delta}}, \\
\tilde{e}^2 &= -\sqrt{h}\lambda_1 e^{g+\Phi/2} \frac{x_1 \sin\psi\, dr + e^f\lambda(\sin\psi\, dx_1 + x_1 \cos\psi\, \hat{\sigma}_3)}{\sqrt{\Delta}}, \\
\tilde{e}^3 &= -\sqrt{h} e^{\frac{\Phi}{2}} \frac{\lambda e^f x_2 dr + \lambda_1^2 e^{2g} dx_2}{\sqrt{\Delta}}.
\end{aligned} \tag{3.4.29}$$

This whole background is indeed a solution to the combined (massive)-IIA supergravity plus DBI plus WZ action (the details are explicit in appendix D).

$$S = S_{\text{Massive IIA}} + S_{DBI} + S_{WZ}\,. \tag{3.4.30}$$

In the gauge in which the B-field is given by eq (3.4.16) and there are no explicit D4 sources, the appropriate WZ terms are given by,

$$\begin{aligned}
S_{WZ} &= S_{WZ}^{D8} + S_{WZ}^{D6} \\
S_{WZ}^{D6} &= \int_{M_{10}} \Big(C_7 - B\wedge C_5 \Big) \wedge \Xi_3 \\
S_{WZ}^{D8} &= -\int_{M_{10}} \Big(C_9 - B\wedge C_7 + \frac{1}{2} B\wedge B\wedge C_5 \Big) \wedge \Xi_1
\end{aligned} \tag{3.4.31}$$

whilst the DBI action, expressed in terms of the D8 and D6 calibrations –c.f. eq (3.4.13)– is given by,

$$\begin{aligned}
S_{DBI} &= S_{DBI}^{D8} + S_{DBI}^{D6}, \\
S_{DBI}^{D6} &= -\int_{M_{10}} e^{-\Phi} \left(\frac{e^{\Phi}}{h} vol_4 \right) \wedge \Big(v_1 \wedge j_2 - w_1 \wedge B \Big) \wedge \Xi_3, \\
S_{DBI}^{D8} &= -\int_{M_{10}} e^{-\Phi} \left(\frac{e^{\Phi}}{h} vol_4 \right) \wedge \left(\frac{1}{2} w_1 \wedge j_2 \wedge j_2 + v_1 \wedge j_2 \wedge B - \frac{1}{2} w_1 \wedge B \wedge B \right) \wedge \Xi_1.
\end{aligned} \tag{3.4.32}$$

Operating with the SU(2) structure we can recast the RR potentials as

$$\begin{aligned} C_5 &= e^{-\hat{\Phi}}(\frac{e^{\Phi}}{h}vol_4) \wedge w_1 \\ C_7 &= e^{-\hat{\Phi}}(\frac{e^{\Phi}}{h}vol_4) \wedge j_2 \wedge v_1 \\ C_9 &= -\frac{1}{2}e^{-\hat{\Phi}}(\frac{e^{\Phi}}{h}vol_4) \wedge j_2 \wedge j_2 \wedge w_1. \end{aligned} \tag{3.4.33}$$

This makes it clear that on shell, as is required by sypersymmetry, $S_{DBI} + S_{WZ} = 0$. This reflects the fact that the branes are calibrated, a fact that we now discuss in some detail.

3.4.4 Analysis of the dualised geometry

One is often interested, particularly in the context of the AdS/CFT correspondence, in the possibility that D-branes may wrap certain sub-manifolds of the geometry in a way the preserves supersymmetry. One approach to check whether a brane embedding is supersymmetric is to look carefully at the κ-symmetry projectors. An alternative approach is the use of calibrations. We recall that a calibration ϖ is a closed l-form that bounds the volume of any oriented l dimensional submanifold Σ by,

$$d^l\sigma\sqrt{\det g|_\Sigma} \geq \varpi|_\Sigma \,. \tag{3.4.34}$$

A submanifold is said to calibrated when this bound is saturated and it follows that such a calibrated cycle will have the minimal volume within its homology class. Of course in the geometries described above we have both NS and RR fluxes and this simple calibration is not enough to establish supersymmetric D-brane confirguations. For this one needs a *generalised calibration,* ϖ which is a $d_H = d + H\wedge$ closed polyform such that for any D-brane with world volume field strength $\mathcal{F} = B|_\Sigma + 2\pi\alpha' dA$ wrapping an internal cycle Σ, one has

$$\mathcal{E} \geq \varpi|_\Sigma \wedge e^{\mathcal{F}} \,, \tag{3.4.35}$$

where $\mathcal{E}$ is the energy density of the D-brane. When this bound is saturated the D-brane minimises its energy and is supersymmetric. $SU(3) \times SU(3)$ backgrounds admit a rich structure of supersymmetric cycles and the poly forms $\Phi_\pm$ (or rather the appropriate imaginary parts) serve as generalised calibrations as detailed by Martucci and Smyth in [20].

For the case of $SU(2)$-structure backgrounds with non-trivial NS 3 form the calibrations for odd cycles are given by (and here we assume no gauge field on the brane world volumes)[6]

$$\Psi_{Cal\ odd} = -8h^{\frac{1}{4}}e^{-\frac{\Phi}{4}}\mathrm{Im}(\Phi_-) \wedge e^B \,, \tag{3.4.36}$$

while those for the even cycles by,

$$\Psi_{Cal\ even} = -8h^{\frac{1}{4}}e^{-\frac{\Phi}{4}}\mathrm{Im}(\Phi_+) \wedge e^B \,, \tag{3.4.37}$$

where the pure spinors are given by eq (3.2.8) for $|ab| = e^A = \frac{e^{\frac{\Phi}{4}}}{h^{\frac{1}{4}}}$. Specifically this gives:

$$\begin{aligned}
\mathcal{C}_1 &= -w_1 \\
\mathcal{C}_2 &= -\mathrm{Re}(\omega_2) \\
\mathcal{C}_3 &= v_1 \wedge j_2 - w_1 \wedge B \\
\mathcal{C}_4 &= -v_1 \wedge w_1 \wedge \mathrm{Im}(\omega_2) - B \wedge \mathrm{Re}(\omega_2) \\
\mathcal{C}_5 &= \frac{1}{2}w_1 \wedge j_2 \wedge j_2 + v_1 \wedge j_2 \wedge B - \frac{1}{2}w_1 \wedge B \wedge B \\
\mathcal{C}_6 &= -v_1 \wedge w_1 \wedge \mathrm{Im}(\omega_2) \wedge B - \frac{1}{2}\mathrm{Re}(\omega_2) \wedge B \wedge B.
\end{aligned} \tag{3.4.38}$$

A cycle in the internal space is SUSY if it satisfies the calibration condition

$$\sqrt{g_i + B}d^i\xi = \mathcal{C}_i. \tag{3.4.39}$$

One can explicitly check that spacetime filling D4, D6 and D8 branes wrapping the following cycles are indeed supersymmetric:

$$\begin{aligned}
\Sigma_{D4} &= (y^\mu, r) \quad \text{with} \quad x_1 = x_2 = 0 \\
\Sigma_{D6} &= (y^\mu, r, \psi, x_1) \quad \text{with} \quad x_1^2 + x_2^2 = const. \\
\Sigma_{D8} &= (y^\mu, r, \psi, \theta, \varphi, x_1).
\end{aligned} \tag{3.4.40}$$

some further SUSY cycles can be found in section 5.5, however an exhaustive list is left for future work.

[6]Here we calibrations for cycles defined on in internal space, an additional warp factor is required if the sub manifold under consideration includes the space time directions as in eq (C.14)

3.5 Discussion and Conclusions

In this chapter we have clarified the action of non-Abelian T-duality in the context of backgrounds possessing $SU(3) \times SU(3)$ structure and $\mathcal{N} = 1$ supersymmetry.

We saw that rather generically the effect of performing a dualisation along an $SU(2)$ isometry group is to map an $SU(3)$ structure background to an $SU(2)$ structure background. Such geometries remain an interesting sector of compactifications which are much less well explored than their type-IIB $SU(3)$ structure cousins. This work then opens the door to constructing a rich class of such geometries. Indeed although we have illustrated this with the Klebanov Witten geometry everything we have said holds true for the wide variety of $N = 1$ backgrounds presented in [30] (details and extensions of this appear in chapters 5 and 6). A particularly noteworthy direction is to consider the dualisation of more general toric Calabi-Yau geometries [71].

One feature of the geometries presented above was that they possess *static* $SU(2)$ structure (that is the pure spinors are of type (2,1) everywhere). An interesting question from the point of view of generalised complex geometry is whether backgrounds with a dynamic $SU(2)$ structure can be found using these techniques.

Chapter 4

Dualising a 3-d QFT and a Comparison to G_2-Structure Rotation

4.1 Introduction

This chapter is based on [42] which I completed alone. The purpose of the original work was two fold. First to perform a $SU(2)$-isometry dualisation on a geometry that gives a holographic description of a confining gauge theory in 3-d. Secondly to compare the solution generated by non-abelian T-duality to that generated by U-duality, which is better understood. The starting point in both cases is (a deformation of) the Maldacena-Nastase solution [72].

The Maldacena-Nastase solution consists of wrapped D5-branes wrapping a 3-cycle in a manifold that supports a G_2-structure. The field theory living on the world volume of these branes is only effectively 3-d in the IR and so the geometry is only a good description of the low energy dynamics of SYM in 3-d. A UV completion is provided by another solution generating technique, G_2-structure rotation [24], which is analogous to U-duality. This acts on the S-dual of a deformation of the Maldacena-Nastase solution [73] and maps it to a geometry supporting D2 and fractional D2 branes that asymptotes to $AdS_4 \times Y$ in the UV and is the G_2 analogue of the Baryonic Branch [18] of Klebanov-Strassler [12]. The compact metric Y has finite volume in the UV and so the fractional branes which wrap cycles in Y remain effectively 3-d in the whole geometry. A gauge theory analysis of the G_2-structure rotated solution was performed in [29] which suggested that the dual QFT was likely a confining 2 node quiver with a Chern-Simons term that dominates the IR dynamics and a conformal fixed point in the UV. This clearly presents an improvement on the UV behaviour of the geometry and coupled with the possibility of a duality cascade by analogy with Klebanov-Strassler, indicates that the dual field theory is potentially very interesting.

In this chapter a new type-IIA SuGra solution, dual to a confining gauge theory in 3-d, is generated by performing a T-duality along one of the $SU(2)$ isometries of the deformed Maldacena-Nastase solution [73]. This new solution preserves $\mathcal{N} = 1$ SUSY in the form of a dynamic $SU(3)$-structure in 7-d [74, 75]. Some details of the G_2-structure rotation [24] are also given so that the two solutions may be compared. This also includes a new proposal for the Chern-Simons term of the G_2-structure rotation in term of a probe D8 brane with the level given by the D6 Page charge. Such a proposal was absent from the literature until [42]. The gauge theory dual to the T-dual geometry is analysed and compared to that of the original wrapped D5 brane and G_2-structure rotated solutions. A more specific outline is as follows:

In section 4.2 the deformation of the Maldacena-Nastase solution is briefly reviewed. Details of the metric and RR sector ansatz and G_2-structure SUSY equations are all presented. SUSY preserving semi-analytic solutions of the ansatz are given that are characterised by either a UV constant or UV linear dilaton. And finally some cycles and charges that will be of relevance to the field theoretic description are introduced.

Section 4.3 gives some details of the solution generating technique G_2-structure rotation and presents the result of applying this to the deformed Maldacena-Nastase solution. Some details of the rotated G_2-structure SUSY conditions are given and cycles and Page charges, that will be of interest in penultimate section, are introduced.

Section 4.4 is where the new results begin, the reader familiar with the salient features of the Maldacena-Nastase solution and its G_2-structure rotation may wish to start here. The section begins with a brief review of non-abelian T-duality on $SU(2)$-isometries before the dual geometry is presented in as concise way as possible. After this attention is turned to the generalised geometric description of the dual solution. It is shown that the dual structure is dynamic $SU(3)$ in 7-d which is characterised by a non-constant angle between the two 10-d MW Killing spinors of type-IIA. Finally some cycles and Page charges are introduced that will be important in the field theory analysis.

Section 4.5 contains a field theory analysis of each of the solutions presented in the previous sections. The analysis of the deformed Maldacena-Nastase and its G_2-structure rotation is mostly a review of what can be found in [72, 29, 73, 63] although additional clarifications are made. In particular further details of a Seiberg like duality in the G_2-structure rotated solution are given and how this effects the Chern-Simons level, the proposal for which is new. The analysis of the T-dual geometry suggests that it is dual to a confining Chern-Simons like gauge theory that is potentially a 3-node quiver.

The chapter is finally closed with concluding remarks and an outlook in section 4.6

4.2 Wrapped D5 Branes on Σ^3

The Maldacena-Nastase solution [72] is a solution of type-IIB (first presented in [76]) that consists of D5 branes wrapping a 3-cycle in a G_2-structure manifold and is dual in the IR to $\mathcal{N} = 1$ SYM in 3-d. The purpose of this section is to review its deformation due to Canoura, Merlatti and Ramallo [73], as this constitutes more general ansatz to a set of wrapped D5 brane solutions [73, 24, 63, 29, 77] which contain the Maldacena-Nastase solution as a special case [1].

The string frame metric is given by

$$ds^2_{\rm str} = e^{\phi}\left(dx^2_{1,2} + ds^2_7\right), \tag{4.2.1}$$

where the internal part of the metric, ds^2_7, describes a manifold supporting a G_2-structure and is given by

$$ds^2_7 = N_c\left[e^{2g}dr^2 + \frac{e^{2h}}{4}(\sigma^i)^2 + \frac{e^{2g}}{4}(\omega^i - \frac{1}{2}(1+w)\sigma^i)^2\right], \tag{4.2.2}$$

The functions g, h, w and the dilaton ϕ all depend on the holographic coordinate r only. σ^i and ω^i are 2 sets of $SU(2)$ left invariant 1-forms which satisfy the following differential relations:

$$d\sigma^i = -\frac{1}{2}\epsilon_{ijk}\sigma^j \wedge \sigma^k, \quad d\omega^i = -\frac{1}{2}\epsilon_{ijk}\omega^j \wedge \omega^k. \tag{4.2.3}$$

These can be represented by introducing 3 angles for σ^i, $(\theta_1, \phi_1, \psi_1)$ and a further 3 for ω^i, $(\theta_2, \phi_2, \psi_2)$ such that:

$$\begin{aligned}
\sigma^1 =& \ \cos\psi_1 d\theta_1 + \sin\psi_1 \sin\theta_1 d\phi_1, \\
\sigma^2 =& \ -\sin\psi_1 d\theta_1 + \cos\psi_1 \sin\theta_1 d\phi_1, \\
\sigma^3 =& \ d\psi_1 + \cos\theta_1 d\phi_1,
\end{aligned} \tag{4.2.4}$$

and similarly for ω^i. The angles are defined over the ranges: $0 \leq \theta_{1,2} \leq \pi$,

[1] Actually further modification of the RR 3-form is required to include sources, which is the main focus of the majority of these references.

$0 \leq \phi_{1,2} < 2\pi$ and $0 \leq \psi_{1,2} < 4\pi$. The solution has a non-trivial RR 3-form:

$$\begin{aligned} F_3 = \; & \tfrac{N_c}{4}\Big[\left(\sigma^1 \wedge \sigma^2 \wedge \sigma^3 - \omega^1 \wedge \omega^2 \wedge \omega^3\right) + \tfrac{\gamma'}{2} dr \wedge \sigma^i \wedge \omega^i + \\ & \tfrac{1}{4}\epsilon_{ijk}\left((1+\gamma)\,\sigma^i \wedge \sigma^j \wedge \omega^k - (1+\gamma)\,\omega^i \wedge \omega^j \wedge \sigma^k\right)\Big], \end{aligned} \tag{4.2.5}$$

which satisfies the simple Bianchi identity

$$dF_3 = 0. \tag{4.2.6}$$

This solution preserves $\mathcal{N} = 1$ SUSY in 3-d, which is 2 real supercharges. This can be expressed in terms of the following differential constraints on an associative 3-form Φ_3:

$$\begin{aligned} & d\left(e^{2A-\phi}\right) = 0, \\ & \Phi_3 \wedge d\Phi_3 = 0, \\ & d\left(e^{2A-\phi} \star_7 \Phi_3\right) = 0, \\ & d\left(e^{3A-\phi}\Phi_3\right) + e^{3A} \star_7 F_3 = 0, \end{aligned} \tag{4.2.7}$$

where $A = \phi/2$. Generically the 3-form Φ_3 may be expressed in terms of an auxiliary $SU(3)$-structure as [78]:

$$\Phi_3 = e^r \wedge J + Re\Omega_{hol}, \quad \star_7\Phi_3 = \frac{1}{2} J \wedge J + Im\Omega_{hol} \wedge e^r. \tag{4.2.8}$$

A convenient set of vielbeins for the metric are given by

$$\begin{aligned} & e^{x_i} = e^{\phi/2} dx_i, \quad e^r = \sqrt{N_c} e^{\phi/2+g} dr, \quad e^i = \sqrt{N_c}\tfrac{e^{\phi/2+h}}{2}\sigma^i, \\ & e^{\hat{i}} = \sqrt{N_c}\tfrac{e^{\phi/2+g}}{2}\left(\omega^i - \tfrac{1}{2}(1+w)\sigma^i\right) \end{aligned} \tag{4.2.9}$$

with respect to which the auxiliary $SU(3)$-structure of the deformed Maldacena-Nastase solution can be expressed as [79, 24]

$$\begin{aligned} & J = e^{\hat{1}} \wedge e^1 + e^{\hat{2}} \wedge e^2 + e^{\hat{3}} \wedge e^3, \\ & \Omega_{hol} = ie^{i\alpha}(e^{\hat{1}} + ie^1) \wedge (e^{\hat{2}} + ie^2) \wedge (e^{\hat{3}} + ie^3,) \end{aligned} \tag{4.2.10}$$

where α depends on r. Plugging eq (4.2.10) into eq (4.2.7) gives rise to a set of first order differential equations that are presented, for example, in appendix A of [29][2]. The solution of these equations is only known semi-analytically in the IR and UV, but it is possible to numerically interpolate between these two sets

[2]There a flavour brane profile $P(r)$ is also considered which should be set to zero when there are no sources.

of series solutions.

In the IR where $r \sim 0$ the solution is regular and is given by

$$\begin{aligned} e^{2g} &= g_0 + \frac{(g_0-1)(9g_0+5)}{12g_0}r^2 + O(r^4), \\ e^{2h} &= g_0 r^2 - \frac{3g_0^2 - 4g_0 + 4}{18g_0}r^4 + O(r^6), \\ w &= 1 - \frac{3g_0 - 2}{3g_0}r^2 + O(r^4), \\ \gamma &= 1 - \frac{1}{3}r^2 + O(r^4), \\ \phi &= \phi_0 + \frac{7}{24g_0^2}r^2. \end{aligned} \tag{4.2.11}$$

Notice that $g_0 = 1$ seems to be special, the solution to e^{2g} truncates and $\gamma = w$, indeed this persists to all orders in r. This is the Maldacena-Nastase solution, its UV expansion about $r \sim \infty$ is characterised by an asymptotically linear dilaton

$$\begin{aligned} e^{2g} &= 1, \\ e^{2h} &= 2r + h_0 + \frac{1}{8r} + \frac{1-2h_0}{32r^2} + O(r^{-3}), \\ w &= \frac{1}{4r} + \frac{5-4h_0}{32r^2} + O(r^{-3}), \\ \gamma &= w, \\ \phi &= \phi_\infty + r - \frac{3}{8}\log r - \frac{3h_0}{16r} + O(r^{-2}), \end{aligned} \tag{4.2.12}$$

where the constant needs to be fine tuned to the value $h_0 = -\frac{3}{2}$ so that the IR and UV numerically matched.

When $g_0 > 1$ the solution is a deformation of Maldacena-Nastase characterised by an asymptotically constant dilaton

$$\begin{aligned} e^{2g} &= ce^{4r/3} - 1 + \frac{33}{4c}e^{-4r/3} + O(e^{-8r/3}), \\ e^{2h} &= \frac{3c}{4}e^{4r/3} + \frac{9}{4} - \frac{77}{16c}e^{-4r/3} + O(e^{-8r/3}), \\ w &= \frac{2}{c}e^{-4r/3} + O(e^{-8r/3}), \\ \gamma &= \frac{1}{3} + O(e^{-8r/3}), \\ \phi &= \phi_\infty + \frac{2}{c^2}e^{-8r/3} + O(e^{-2r}). \end{aligned} \tag{4.2.13}$$

At higher orders polynomial terms also appear and a sub expansion in odd powers of $e^{-2r/3}$ has been set to zero [29]. The UV constant c must be tuned for specific choices of the IR constant g_0 such that the series solutions may be smoothly connected numerically.

It is possible to show that the flux equation of motion

$$d \star F_3 = 0 \tag{4.2.14}$$

is satisfied once eq (4.2.7) is imposed and likewise are Einstein's equations and the dilaton equation of motion. The last line of eq (4.2.7) gives a definition of the potential C_6 such that $dC_6 = F_7$:

$$C_6 = e^{3A} Vol_3 \wedge \Phi_3 \tag{4.2.15}$$

There are several important 3-cycles in the geometry which are related to gauge theory observables that shall be discussed at length in section 4.5 , these are:

$$S^3 = \{\sigma^i | \omega^i = 0\}, \quad \tilde{S}^3 = \{\omega^i | \sigma^i = 0\}, \quad \Sigma^3 = \{\sigma^i = \omega^i\}. \tag{4.2.16}$$

The induced metrics on S^3 and $\tilde{S}^3$ are non-vanishing in the whole geometry and are thus suitable for defining flux quantisation. The integrals of F_3 on these cycles give respectively

$$\int_{S^3} F_3 = -\int_{\tilde{S}^3} F_3 = 2\kappa_{10}^2 T_5 N_c. \tag{4.2.17}$$

once one sets $2\kappa_{10}^2 = (2\pi)^7$ and $T_p = \frac{1}{(2\pi)^p}$. The pullback of F_3 onto Σ^3 is zero, while the induced metric

$$ds^2_{\Sigma^3} = \frac{e^{\phi} N_c}{4} \left[e^{2h} + \frac{e^{2g}}{4} (w-1)^2 \right] (\sigma^i)^2 \tag{4.2.18}$$

vanishes in the IR and blows up in the UV. This is the 3-cycle on which the D5 branes are wrapped, their world volume becomes 3-dimensional in the IR as the cycle shrinks to zero but the background remains non-singular because F_3 vanishes on Σ^3.

4.3 Solution Generating Technique I: G_2-Structure Rotation

In [24] a solution generating technique was found by Gaillard and Martelli that maps any unwarped type-IIA G_2-structure solution with asymptotically constant dilaton and NS 3-form flux H to a more exotic G_2-structure solution with a non-trivial RR sector. This method of solution generating is referred to as Rotation, as it acts on the space of Killing spinors thus, but can also be viewed as a U-duality[3].

If one dimensionally reduces the M-theory solution of [75] one is left with a G_2-structure solution in type-IIA. Its string frame metric is:

$$ds^2_{str} = e^{2\Delta+2\hat{\phi}/3}(dx^2_{1,2} + d\hat{s}^2_7), \tag{4.3.1}$$

where $\hat{\phi}$ is the dilaton and $d\hat{s}^2_7$ is any G_2-structure manifold. The condition that $\mathcal{N} = 1$ SUSY is preserved can be expressed as the following differential relations between the fluxes, the 3-form $\hat{\Phi}_3$ and a phase ζ

$$\begin{aligned}
&\hat{\Phi}_3 \wedge d\hat{\Phi}_3 = 0,\\
&d(e^{6\Delta} \star_7 \hat{\Phi}_3) = 0,\\
&d(e^{2\Delta+2\hat{\phi}/3}\cos\zeta) = 0,\\
&2d\zeta - e^{-3\Delta}\cos\zeta d(e^{3\Delta}\sin\zeta) = 0,\\
&\tfrac{1}{\cos^2\zeta}e^{-4\Delta+2\hat{\phi}/3} \star_7 d(e^{6\Delta}\cos\zeta\hat{\Phi}_3) = H_3,\\
&Vol_3 \wedge d(e^{3\Delta}\sin\zeta) - \tfrac{\sin\zeta}{\cos^2\zeta}e^{-3\Delta}d(e^{6\Delta}\cos\zeta\hat{\Phi}_3) = F_4.
\end{aligned} \tag{4.3.2}$$

The central observation of Gaillard and Martelli was that if one sets $\zeta = 0$ eq (4.3.2) truncates to the S-dual of eq (4.2.7), that is:

$$\begin{aligned}
&\Phi_3^{(0)} \wedge d\Phi_3^{(0)} = 0,\\
&d(e^{-2\phi^{(0)}} \star_7 \Phi_3^{(0)}) = 0,\\
&e^{2\phi^{(0)}}d(e^{-2\phi^{(0)}}\Phi_3^{(0)}) + \star_7 H_3 = 0,
\end{aligned} \tag{4.3.3}$$

[3]for closely related work on $SU(3)$-structure rotations in IIB see, for example, [22, 26, 27, 28] and for U-duality [16, 25]

and the metric is simply

$$ds^2_{(0)} = dx^2_{1,2} + ds_7^{(0)2}. \tag{4.3.4}$$

Any solution of this simplified system will also be a solution of eq (4.3.2) when the following identifications are made:

$$\begin{aligned} \hat{\Phi}_3 &= \left(\tfrac{\cos\zeta}{\kappa_1}\right)^3 \Phi_3^{(0)}, \quad e^{2\hat{\phi}} = \tfrac{\cos\zeta}{\kappa_1} e^{2\phi^{(0)}}, \\ e^{3\Delta} &= \left(\tfrac{\kappa_1}{\cos\zeta}\right)^2 e^{-\phi^{(0)}}, \quad d\hat{s}_7^2 = \left(\tfrac{\cos\zeta}{\kappa_1}\right)^2 ds_7^{(0)2}, \quad \sin\zeta = \kappa_2 e^{-\phi^{(0)}}, \end{aligned} \tag{4.3.5}$$

where κ_1 and κ_2 are integration constants and $\phi^{(0)}$ must be bounded to satisfy the last equation.

It is possible to perform a rotation of the deformed Maldacena-Nastase solution, detailed in the last section, once an S-duality has been performed on it. This sends

$$F_3 \to H_3, \quad \phi \to \phi^0 = -\phi, \quad ds^2_{str} \to e^{-\phi} ds^2_{str} \tag{4.3.6}$$

so that the resulting metric is unwarped. Specifically it is the solution with UV given by eq (4.2.13) that is suitable for this as the dilaton is bounded. The 3-form, $\Phi_3^{(0)}$ is still given by eq (4.2.8) but with the auxiliary $SU(3)$-structure of eq (4.2.10) with no dilaton factor

$$\hat{e}^a = e^{-\phi/2} e^a. \tag{4.3.7}$$

As the solution is now in the common type-II NS sector it can be viewed as a type-IIA theory, as required by the rotation. The interested reader is referred to [24] for further details of the solution generating algorithm.

The rotated solution has a warped metric and modified dilaton, which after fixing the integration constants and rescaling the field theory coordinates may be expressed as:

$$\begin{aligned} ds^2_{str} &= \tfrac{1}{c\sqrt{H}} dx^2_{1,2} + \sqrt{H} ds_7^2, \\ e^{2\hat{\phi}} &= c\sqrt{H} e^{2(\phi^{(0)}-\phi_\infty)}, \\ H &= 1 - e^{-2(\phi^{(0)}-\phi_\infty)}, \end{aligned} \tag{4.3.8}$$

where $\hat{\phi}$ is the new dilaton, ϕ_∞ is the UV value of $\phi^{(0)}$ and c is a constant which appears in the UV series solutions to the BPS equations [29], but which will not play an important role here. The metric in string frame tends in the UV towards $AdS_4 \times Y$ where Y is the metric at the base of a G_2-cone, however the dilaton is not constant, $e^{2\hat{\phi}} \sim e^{-8r/3}$, and so the solution does not enjoy full conformal symmetry.

The NS 3-form is unchanged but an RR 4-form has been generated:

$$
\begin{aligned}
H_3 = \ & \tfrac{N_c}{4}\Big[\left(\sigma^1\wedge\sigma^2\wedge\sigma^3 - \omega^1\wedge\omega^2\wedge\omega^3\right) + \tfrac{\gamma'}{2} dr\wedge\sigma^i\wedge\omega^i + \\
& \qquad \tfrac{1}{4}\epsilon_{ijk}\left((1+\gamma)\,\sigma^i\wedge\sigma^j\wedge\omega^k - (1+\gamma)\,\omega^i\wedge\omega^j\wedge\sigma^k\right)\Big], \qquad (4.3.9) \\
F_4 = \ & -\tfrac{1}{c^2} Vol_3\wedge dH^{-1} + \tfrac{\sqrt{N_c}}{\sqrt{c}} e^{2(\phi_\infty-\phi^{(0)})} \star_7 H_3,
\end{aligned}
$$

These obey the Bianchi identities

$$
dH_3 = 0, \quad dF_4 = 0, \qquad (4.3.10)
$$

and flux equations of motion

$$
\begin{aligned}
& d\star F_4 + H_3\wedge F_4 = 0, \\
& d(e^{-2\hat{\phi}}\star H_3) - \tfrac{1}{2} F_4\wedge F_4 = 0.
\end{aligned} \qquad (4.3.11)
$$

One can use eq (4.3.2) to define a canonical potential C_3 such that $dC_3 = F_4$

$$
C_3 = \frac{1}{c^2} Vol_3\wedge dH^{-1} + \frac{1}{\sqrt{c}} e^{2(\phi_\infty-\phi^{(0)})}\Phi_3^{(0)}. \qquad (4.3.12)
$$

In [29] some cycles of interest were identified. Those that give flux quantisation are:

$$
\hat{\Sigma}^3 = \Sigma^3 = \{\sigma^i = \omega^i\}, \quad \hat{\Sigma}^6 = \{\sigma^1,\sigma^2,\sigma^3,\omega^1,\omega^2,\omega^3\}. \qquad (4.3.13)
$$

The Maxwell and Page charges [70] coincide for the NS5 brane (as they did for the D5 brane in the previous section). However the flux equation of motion for F_4 implies that this is not so for the D2 brane and it is only the Page charge that is quantised for this object. Define $F_6 = -\star F_4$, then the following charges are quantised.

$$
\begin{aligned}
Q_{NS5} &= -\frac{1}{4\pi^2}\int_{\hat{S}^3} H_3 = N_c, \\
Q_{D2} &= -\frac{1}{(2\pi)^5}\int_{\hat{\Sigma}^6} (F_6 + H_3\wedge C_3) = 0 \ mod\ N_c.
\end{aligned} \qquad (4.3.14)
$$

Actually substituting eq (4.3.12) into the definition of Q_{D2} gives zero, but C_3 is not a gauge invariant quantity and this gives rise to non-zero integers under the large gauge transformation. Let $Q_{D2} = M_c$ then consider the large gauge transformation $C_3 \to C_3 + \Delta C_3$ where

$$\Delta C_3 = -\frac{\pi}{4}\left[\sigma^1 \wedge \sigma^2 \wedge \sigma^3 + \omega^1 \wedge \omega^2 \wedge \omega^3\right]. \tag{4.3.15}$$

This will shift the Page charge as $Q_{D2} \to M_c - N_c$.

Another cycle with interesting properties is the 2-cycle at constant r

$$\hat{\Sigma}^2 = \{\theta_1 = \theta_2, \varphi_1 = \varphi_2 | \psi_1 = \psi_2 = \text{constant}\}. \tag{4.3.16}$$

On this cycle F_4 vanish and the induced metric

$$ds^2_{\hat{\Sigma}^2} = \frac{N_c}{4}\sqrt{H}\big(e^{2h} + \frac{e^{2g}}{4}(w-1)^2\big)\left(d\theta_1^2 + d\varphi_1^2\right), \tag{4.3.17}$$

has vanishing volume in the IR and constant volume in the UV.

4.4 Solution Generating Technique II: non-Abelian T-duality

The purpose of this section is to present the first non-abelian T-dual of a background with minimal SUSY in 3-d, specifically a dual of deformed Maldacena-Nastase along an $SU(2)$ isometry.

4.4.1 non-Abelian T-dual of wrapped D5 branes on Σ^3

In this section a non-abelian T-duality transformation is performed on the wrapped D5 brane solution of section 4.2. It acts along the $SU(2)$ isometry parametrised by ω^i and gauge fixing is imposed such that:

$$v_1 = \theta_2 = \phi_2 = 0. \tag{4.4.1}$$

The dual NS sector is by,

$$\begin{aligned} e^{-2\Phi} &= \det M e^{-2\phi}, \\ \det M &= \frac{1}{8}N_c^3 e^{6g+3\phi} + N_c e^{2g+\phi} v_2^2 + N_c e^{2g+\phi} v_3^2, \end{aligned} \tag{4.4.2}$$

for the dual dilaton Φ,

$$
\begin{aligned}
B_2 = & \frac{(w+1)N_c e^{2g+\phi}\left(N_c^2 e^{4g+2\phi}+8v_3^2\right)\left(\hat{\sigma}^1\wedge dv_2 - dv_3\wedge\sigma^3\right)}{16\sqrt{2}\det M} - \\
& \frac{v_2(w+1)N_c e^{2g+\phi}\left(v_3\hat{\sigma}^1\wedge dv_3 - v_3 dv_2\wedge\sigma^3\right)}{2\sqrt{2}\det M} + \\
& \frac{v_2(w+1)N_c^3 e^{6g+3\phi}\hat{\sigma}^2\wedge d\psi_2}{16\sqrt{2}\det M} - \\
& \frac{(v_2-v_3)(v_2+v_3)(w+1)N_c e^{2g+\phi}\hat{\sigma}^1\wedge dv_2}{2\sqrt{2}\det M} + \\
& \frac{v_2 N_c e^{2g+\phi}\left(v_2 dv_3\wedge d\psi_2 - v_3 dv_2\wedge d\psi_2\right)}{\sqrt{2}\det M} - \\
& \frac{(w+1)^2 N_c^3 e^{6g+3\phi}\left(v_2\hat{\sigma}^2\wedge\sigma^3 + v_3\hat{\sigma}^1\wedge\hat{\sigma}^2\right)}{32\sqrt{2}\det M},
\end{aligned}
\tag{4.4.3}
$$

for the NS two-form potential and

$$
\begin{aligned}
d\hat{s}^2 = & \; e^{\phi}\left[dx_{1,2}^2 + e^{2g}dr^2 + \frac{e^{2h}}{4}\left((\hat{\sigma}^1)^2+(\hat{\sigma}^2)^2+(\sigma^3)^2\right)\right] + \\
& \frac{1}{4\det M}\Bigg[v_2 v_3(w+1)N_c^2 e^{4g+2\phi}\hat{\sigma}^1\left(d\psi_2 - \frac{1}{2}(w+1)\sigma^3\right) + \\
& \left(N_c^2 e^{4g+2\phi}\left(dv_2^2+dv_3^2\right) + 8\left(v_2 dv_2 + v_3 dv_3\right)^2\right) + \\
& (w+1)N_c^2 e^{4g+2\phi}\hat{\sigma}^2\left(v_3 dv_2 - v_2 dv_3\right) + \\
& v_2^2 N_c^2 e^{4g+2\phi}\left(d\psi_2 - \frac{1}{2}(w+1)\sigma^3\right)^2 + \\
& \frac{1}{4}v_3^2(w+1)^2 N_c^2 e^{4g+2\phi}(\hat{\sigma}^1)^2 + \\
& \frac{1}{4}\left(v_2^2+v_3^2\right)(w+1)^2 N_c^2 e^{4g+2\phi}(\hat{\sigma}^2)^2\Bigg],
\end{aligned}
\tag{4.4.4}
$$

for the dual metric. The new hatted 1-forms are simply a rotation in σ^1,σ^2:

$$
\hat{\sigma}^1 = \cos\psi_2\sigma^1 - \sin\psi_2\sigma^2, \quad \hat{\sigma}^2 = \sin\psi_2\sigma^1 + \cos\psi_2\sigma_2, \tag{4.4.5}
$$

that enables compact expressions.

The RR sector of the solution is rich including a quantised F_0 meaning that the solution is massive type-IIA. In order to express them compactly it is helpful to introduce the following basis of spectator vielbeins

$$\begin{aligned} e^{x^i} &= e^{\phi/2}dx^i,\ e^r = \sqrt{N_c}e^{\phi/2+2g}dr, \\ e^{1,2} &= \frac{\sqrt{N_c}e^{\phi/2+h}}{2}\hat{\sigma}^{1,2},\ e^3 = \frac{\sqrt{N_c}e^{\phi/2+h}}{2}\sigma^3 \end{aligned} \tag{4.4.6}$$

with dual vielbeins given by

$$\begin{aligned} e^{\hat{1}} &= \frac{N_c^{3/2}e^{3g+\frac{3\phi}{2}}}{16\det M}\Bigg[4\left(\hat{\sigma}^2\left(v_2^2+v_3^2\right)(w+1)+2\left(v_3dv_2-v_2dv_3\right)\right)- \\ &\qquad \sqrt{2}N_ce^{2g+\phi}\left(\hat{\sigma}^1v_3(w+1)+2v_2\left(d\psi_2-\frac{1}{2}(w+1)\sigma^3\right)\right)\Bigg], \\ e^{\hat{2}} &= \frac{\sqrt{N_c}e^{g+\frac{\phi}{2}}}{16\det M}\Bigg[-2\sqrt{2}N_c^2e^{4g+2\phi}\left(\hat{\sigma}^2v_3(w+1)+dv_2\right)-16\sqrt{2}v_2\left(v_2dv_2+v_3dv_3\right)- \\ &\qquad 4v_3N_ce^{2g+\phi}\left(\hat{\sigma}^1v_3(w+1)+2v_2\left(d\psi_2-\frac{1}{2}(w+1)\sigma^3\right)\right)\Bigg], \\ e^{\hat{3}} &= \frac{\sqrt{N_c}e^{g+\frac{\phi}{2}}}{16\det M}\Bigg[\sqrt{2}N_c^2e^{4g+2\phi}\left(\hat{\sigma}^2v_2(w+1)-2dv_3\right)-16\sqrt{2}v_3\left(v_2dv_2+v_3dv_3\right)+ \\ &\qquad 4v_2N_ce^{2g+\phi}\left(\hat{\sigma}^1v_3(w+1)+2v_2\left(d\psi_2-\frac{1}{2}(w+1)\sigma^3\right)\right)\Bigg], \end{aligned} \tag{4.4.7}$$

which is a rotation of the rather complicated vielbeins generated by the duality procedure. The fluxes are then:

$$
\begin{aligned}
F_0 &= \frac{N_c}{\sqrt{2}}, \\
F_2 &= -\frac{1}{4}e^{-2(g+h)-\phi}\Big[\sqrt{2}\left(-e^{3\hat{3}}+e^{2\hat{2}}+e^{1\hat{1}}\right)N_c(w-\gamma)e^{3g+h+\phi} - \\
&\quad 8e^{2h}\left(v_3e^{\hat{1}\hat{2}}-v_2e^{\hat{2}\hat{3}}\right)-2e^{2g}U\left(e^{23}v_2+e^{12}v_3\right)+ \\
&\quad 2e^{g+h}\left(\gamma'\left(v_3e^{r3}+v_2e^{r1}\right)+2(w-\gamma)\left(v_3\left(e^{1\hat{2}}-e^{2\hat{1}}\right)-v_2\left(e^{3\hat{2}}+e^{2\hat{3}}\right)\right)\right)\Big], \\
F_4 &= -\frac{(w-1)Ve^{3g-3h}N_c}{8\sqrt{2}}e^{tx^1x^2r}+ \\
&\quad \left[\frac{1}{4}V(w-1)(v_3e^{3\hat{3}}-v_2e^{e\hat{2}})-\frac{1}{8}Ue^{-2h-\phi}\left(\sqrt{2}N_ce^{\hat{1}\hat{2}}e^{2g+\phi}-4v_2e^{\hat{1}\hat{3}}\right)\right]\wedge e^{12}+ \\
&\quad \frac{1}{8}Ue^{-2h-\phi}\left[\sqrt{2}N_ce^{\hat{1}\hat{3}}e^{2g+\phi}+4v_2e^{\hat{1}\hat{2}}+4v_3e^{\hat{2}\hat{3}}\right]\wedge e^{13}+ \\
&\quad \frac{\gamma'}{8}e^{-(g+h+\phi)}e^r\wedge\Big[4v_3(e^{1\hat{1}\hat{3}}+e^{2\hat{2}\hat{3}})+4v_2e^{1\hat{1}\hat{2}}-e^{\hat{1}\hat{2}\hat{3}}- \\
&\quad \sqrt{2}N_c(e^{1\hat{2}\hat{3}}-e^{2\hat{1}\hat{3}}-e^{3\hat{1}\hat{2}})e^{2g+\phi}\Big]+\frac{1}{8}Ue^{-2h-\phi}e^{23\hat{3}}\left(\sqrt{2}e^{\hat{2}}N_ce^{2g+\phi}-4e^{\hat{1}}v_3\right)+ \\
&\quad e^{-(g+h+\phi)}(w-\gamma)\left(e^1v_2+e^3v_3\right)e^{\hat{1}\hat{2}\hat{3}},
\end{aligned}
\tag{4.4.8}
$$

where the functions

$$V = 4+w^2-3w\gamma+w-3\gamma, \quad U = 1+w^2-2w\gamma, \tag{4.4.9}$$

were introduced for convenience. Its interesting to see that F_4 has legs on the field theory directions. This tells us that, like the non-abelian T-dual of wrapped D5 branes on S^2 [30], the RR sector contains magnetic fluxes of D8, D6 and D4 branes, but here there is also an electric flux of D2 branes.

It is simple to check that the RR fluxes automatically satisfy the Bianchi identities

$$dF_0 = dF_2 - F_0H = dF_4 - H_3\wedge F_2 = 0. \tag{4.4.10}$$

The flux equations of motion

$$d\star F_2 + H_3\wedge\star F_4 = d\star F_4 + H_3\wedge F_4 = 0, \tag{4.4.11}$$

as well as Einsteins equations and the dilaton equation of motion all follow once eq (4.2.7) is imposed. One may also confirm that NS flux obeys

$$d\left(e^{-2\tilde{\Phi}} \star H_3\right) = F_0 \star F_2 + F_2 \wedge \star F_4 + \frac{1}{2} F_4 \wedge F_4. \tag{4.4.12}$$

4.4.2 Supersymmetry

In this section the issue of how much SUSY the non-abelian T-dual background preserves is addressed. There is a simple criterium which determines this, which is detailed in [31, 30]. One needs to consider the Kosmann derivative along each of the Killing vectors k associated with the isometry on which one wishes to dualise. This acts on the Killing spinors of the initial solution ϵ and is given by

$$\mathcal{L}_k \epsilon = k^\mu D_\mu \epsilon + \frac{1}{4} \nabla_\mu k_\nu \Gamma^{\mu\nu} \epsilon, \tag{4.4.13}$$

D_μ is the spinor covariant derivative, while ∇_μ is the ordinary covariant derivative of the geometry. The Kosmann derivative should vanish along the isometry of the dualisation. Each additional projective constraints that needed to be imposed to ensure this reduces the SUSY of the non-abelian T-dual by half. If no new constraints are required then all the SUSY of the original background is preserved.

The Killing vectors associated with the relevant $SU(2)$ isometry of the metric eq (4.2.1) are,

$$\begin{aligned}
k^{(1)} &= -\cos\varphi_2 \partial_{\theta_2} + \cot\theta_2 \sin\varphi_2 \partial_{\varphi_2} - \csc\theta_2 \sin\varphi_2 \partial_{\psi_2}, \\
k^{(2)} &= -\sin\varphi_2 \partial_{\theta_2} - \cot\theta_2 \cos\varphi_2 \partial_{\varphi_2} + \csc\theta_2 \cos\varphi_2 \partial_{\psi_2}, \\
k^{(3)} &= -\partial_{\varphi_2},
\end{aligned} \tag{4.4.14}$$

and it is possible to show (using Mathematica) that

$$\mathcal{L}_{k^{(1)}} \epsilon = \mathcal{L}_{k^{(2)}} \epsilon = \mathcal{L}_{k^{(3)}} \epsilon = 0, \tag{4.4.15}$$

where ϵ only depends on r (appendix B of [73] provides further details). Thus one expects the non-abelian T-dual of wrapped D5 branes on Σ^3 to preserve $\mathcal{N} = 1$ SUSY.

A more explicit check that SUSY is preserved was provided by [41], which shows how non-abelian T-duality acts on the g-structure of the original geometry. The original geometry supports a G_2-structure, which is characterised by parallel ϵ_1 and ϵ_2 in $\epsilon = (\epsilon_1, \epsilon_2)^T$. The SUSY conditions of the G_2 may be written in terms 2 real 7-d bispinors [78]

$$\Phi_+ = 1 - \star_7 \Phi_3, \quad \Phi_- = -\Phi_3 + Vol_7, \tag{4.4.16}$$

that obey the conditions [79, 80]

$$
\begin{aligned}
&< \Psi_1, F >= 0, \\
&(d - H_3\wedge)(e^{2A-\phi}\Phi_\pm) = 0, \\
&(d - H_3\wedge)(e^{3A-\phi}\Phi_\mp) + e^{3A} \star_7 \lambda(F) = 0,
\end{aligned}
\tag{4.4.17}
$$

where the upper signs are taken in type-IIB and lower in type-IIA. These are the conditions for $\mathcal{N} = 1$ for a generic $G_2 \times G_2$ structure manifold, where e^{2A} is the warp factor of the Minkowski directions, $\lambda(X_p) = (-1)^{\frac{p(p-1)}{2}} X_p$ and $< X, Y >$ is the Mukai pairing which selects the 7-d part of $X \wedge \lambda(Y)$. As the original solution is in type-IIB one should identify $\Psi_{1,2} = \Psi_{+,-}$, with the opposite identification made in type-IIA. The relevant observation of [41] is that non-abelian T-duality acts on the bispinors of the geometry as

$$
\begin{aligned}
\tilde{\Psi}_+ &= \Psi_- \Omega^{-1}, \\
\tilde{\Psi}_- &= \Psi_+ \Omega^{-1}.
\end{aligned}
\tag{4.4.18}
$$

It is possible to show (in Mathematica) that eq (4.4.17) is satisfied with $\Psi_{1,2} = \tilde{\Psi}_{-,+}$ and F and H_3 given by eq (4.4.8) and eq (4.4.3) respectively, which shows that $\mathcal{N} = 1$ SUSY is preserved. The action of non-abelian T-dual on the 10-d MW Killing spinors is:

$$
\tilde{\epsilon}_1 = \epsilon_1, \quad \tilde{\epsilon}_2 = \Omega.\epsilon_2, \tag{4.4.19}
$$

which is a rotation. Since the G_2-structure of the original geometry requires that the spinors are parallel, the dual structure must be something more exotic. To identify the structure of the dual geometry it is sufficient to calculate how the Ω matrix transforms the G_2-structure spinors. There exists a basis[4] such that the projections the original killing spinor obeys are given by

$$
\Gamma_{r\hat{1}\hat{2}\hat{3}}\epsilon = \epsilon, \quad \Gamma_{1\hat{1}}\epsilon = \Gamma_{2\hat{2}}\epsilon = \Gamma_{3\hat{3}}\epsilon. \tag{4.4.20}
$$

One may decompose the 10-d geometry into a 3+7 split using an auxiliary 2-d space so that the gamma matrices are given by

$$
\begin{aligned}
\Gamma_\mu &= \gamma_\mu \otimes \sigma_3 \otimes \mathbb{1}, \\
\Gamma_a &= \mathbb{I} \otimes \sigma_1 \otimes \gamma_a, \\
\Gamma^{(10)} &= -\mathbb{I} \otimes \sigma_2 \otimes \mathbb{I},
\end{aligned}
\tag{4.4.21}
$$

[4] This is a rotation of the basis of eq (4.2.9) such that $e^1 \to \cos\alpha e^1 - \sin\alpha e^{\hat{1}}$ and $e^{\hat{1}} \to \cos\alpha e^{\hat{1}} + \sin\alpha e^1$ with all other vielbeins unchanged.

where $\mu = 0, 1, 3$ and $a = 1, 2, 3, \hat{1}, \hat{2}, \hat{3}$. The killing spinor may be decomposed such that they have positive chirality as

$$\epsilon_{1,2} = \xi \otimes \begin{pmatrix} 1 \\ -i \end{pmatrix} \otimes \chi, \tag{4.4.22}$$

where ξ and χ are spinors in 3-d and 7-d respectively. In such conventions the 3 form associated to G_2 is then given by $\Phi_{abc} = -i\bar{\chi}\gamma_{abc}\chi$. It is possible to show that in this decomposition the T-dual Killing spinors are given by

$$\tilde{\epsilon}_1 = \xi \otimes \begin{pmatrix} 1 \\ -i \end{pmatrix} \otimes \tilde{\chi}_1, \quad \tilde{\epsilon}_2 = \xi \otimes \begin{pmatrix} 1 \\ i \end{pmatrix} \otimes \tilde{\chi}_2, \tag{4.4.23}$$

which have the correct chirality for type-IIA. Using the projections of eq (4.4.20) the 7-d spinors may be massaged into the form

$$\tilde{\chi}_1 = \chi, \quad \tilde{\chi}_2 = \frac{\sin\alpha}{\sqrt{1+\zeta^2}}\chi + \sqrt{\frac{\cos^2\alpha + \zeta^2}{1+\zeta^2}}\chi^{\perp}, \tag{4.4.24}$$

where $\zeta^2 = \zeta_a\zeta^a$ ans ζ_a is given by eq (2.3.14). As the notation suggests $\tilde{\chi}_2$ is a sum of parts which are parallel and orthogonal to $\tilde{\chi}_1$. The orthogonal complement to χ is

$$\chi^{\perp} = iK\chi, \quad K = \frac{\cos\alpha\gamma_r + \zeta_1\cos\alpha\gamma_{\hat{1}} + \zeta_2\gamma_{\hat{2}} + \zeta_3\gamma_{\hat{3}} + \zeta_1\sin\alpha\gamma_{\hat{1}}}{\sqrt{\cos^2\alpha + \zeta^2}}, \tag{4.4.25}$$

where K defines the 1-form associated with an $SU(3)$-structure in 7-d when contracted with the vielbeins of eq (4.4.7) [5]. Specifically the structure is what should be called dynamical $SU(3)$, by analogy to dynamical $SU(2)$-structures in 6-d [81]. This is because the coefficients of χ and $\chi^{\perp}$ are not constant through out the space, in fact $\sin\alpha \to 0$ as $r \to \infty$ so the structure becomes orthogonal $SU(3)$ in the UV, but through out the whole space the coefficients change. The details of this calculation and presentation of all the forms associated with $SU(3)$ shall be left for forthcoming work.

4.4.3 Cycles and Charges

The T-dual geometry supports many fluxes and so contains several different branes. Since the internal space is 7-d, the possible quantised charges are given

[5]The contraction should be performed once the vielbeins have been rotated to the canonical frame of footnote 4.

by

$$
\begin{aligned}
Q_{D6} &= \frac{1}{2\pi}\int_{\tilde{\Sigma}^2} F_2 - F_0 B_2,\\
Q_{D4} &= \frac{1}{8\pi^3}\int_{\tilde{\Sigma}^4} F_4 - B_2\wedge F_2 + \frac{F_0}{2} B_2\wedge B_2,\\
Q_{D2} &= -\frac{1}{32\pi^5}\int_{\tilde{\Sigma}^6} F_6 - B_2\wedge F_4 + \frac{1}{2} B_2\wedge B_2\wedge F_2 - \frac{F_0}{6} B_2\wedge B_2\wedge B_2.
\end{aligned}
\tag{4.4.26}
$$

Sensible cycles over which to define these quantities are

$$
\begin{aligned}
\tilde{\Sigma}^2 &= \{\theta_1, \varphi_1 | v_2 = v_3 = \psi_1 = \psi_2 = 0\},\\
\tilde{\Sigma}^4 &= \{\theta_1, \varphi_1, \psi_1, v_2 | v_3 = 0, \psi_2 = constant\},\\
\tilde{\Sigma}^6 &= \{\theta_1, \varphi_1, \psi_1, v_2, v_3, \psi_2\}.
\end{aligned}
\tag{4.4.27}
$$

Actually $\tilde{\Sigma}^2$ shrinks to zero in the IR, but as F_2 and B_2 vanish on this cycle this will not cause a singularity in the geometry as a non-zero Page charge must be pure gauge in origin. On these cycles eq (4.4.26) takes the simple form:

$$
\begin{aligned}
Q_{D6} &= 0 \text{ up to large gauge transformations},\\
Q_{D4} &= \frac{1}{8\pi^3}\int_{\tilde{\Sigma}^4} \frac{N_c}{2\sqrt{2}} v_2 \sin\theta_1 d\theta \wedge d\varphi_1 \wedge d\psi_1 \wedge dv_2,\\
Q_{D2} &= -\frac{1}{32\pi^5}\int_{\tilde{\Sigma}^6} \frac{N_c}{4} v_2 \sin\theta_1 d\theta_1 \wedge d\varphi_1 \wedge dv_1 \wedge dv_2 \wedge d\psi_2.
\end{aligned}
\tag{4.4.28}
$$

to make further progress one needs to fix the periods of the dual coordinates v_2, v_3. A rigorous prescription for doing this is absent form the literature, but it is at least reasonable to assume that they are compact. Here the periodicities shall be chosen such that

$$
\int v_2 dv_2 = \pi, \quad \int dv_3 = \sqrt{2}\pi, \tag{4.4.29}
$$

with these choices the D2 and D4 Page charges coincide with the Romans mass,

$$
Q_{D4} = Q_{D2} = F_0 = \frac{N_c}{\sqrt{2}}. \tag{4.4.30}
$$

Since these charges contain explicit B_2 terms in their definitions they can experience quantised shifts under large gauge transformations $B_2 \to B_2 + \Delta B_2$. For example

$$
\Delta B_2 = -\frac{n}{2}\sin\theta d\theta_1 \wedge d\varphi_1, \quad \Delta Q_{D6} = n\frac{N_c}{\sqrt{2}}, \quad \Delta Q_{D4} = \Delta Q_{D2} = 0. \tag{4.4.31}
$$

Finally there are 2-cycles on which the induced metric takes a particularly simple form. On $\tilde{\Sigma}^2$ the induced metric is given by

$$ds^2_{\tilde{\Sigma}^2} = N_c \frac{e^{2h+\phi}}{4}(d\theta_1^2 + d\varphi_1^2), \tag{4.4.32}$$

this cycle vanishes in the IR, blows up in the UV and F_2 and B_2 vanish on it. The second is the 3-sphere $S^3 = (\theta_1, \varphi_1, \psi_1)$ on which the metric is

$$ds^2_{S^3} = N_c \frac{e^{2h+\phi}}{4}(d\theta_1^2 + d\varphi_1^2 + 2\cos\theta_1 d\varphi_1 d\psi_1 + d\psi_1^2), \tag{4.4.33}$$

which has the same asymptotic behaviour as the previous cycle.

4.5 Probe Analysis and Comparison of the Gauge Theories

In this section some field theory observables shall be studied via a probe brane analysis. To begin, the results of [73] and [29] shall be reviewed to study the field theories dual to the wrapped D5-brane solution and its G_2-structure rotation respectively. A new proposal for the Chern-Simons level of G_2-structure rotated solution will be made before the non-abelian T-dual solution is considered.

The analysis of this section will rely on two important observations. They give a prescription for defining gauge couplings and Chern-Simons levels from probe branes.

Gauge Coupling

A gauge coupling may be defined in terms of the DBI action of a probe Dp brane wrapping an n-cycle, where the embedding of this brane is $\xi = (t, x^1, x^2, \Sigma^n)$. In the following it shall be assumed that the induced metric can be expressed as

$$ds^2_{Dp} = e^{2A} dx^2_{1,2} + ds^2_{\Sigma^n}, \tag{4.5.1}$$

where its components $\hat{G}_{M,N}$ are decomposed as $M = (\mu, a)$ for $\mu = 0, 1, 2$ and $a = 1, ..., p-2$. In addition the only non-zero part of the world volume gauge field F and pull back of B_2 will be

$$F_{\mu\nu}, \quad \hat{B}_{ab}, \tag{4.5.2}$$

and the dilaton shall be ϕ. The DBI action of this probe Dp brane may be factorised into $\mathbb{R}^{1,2}$ and Σ^{p-2} parts

$$
\begin{aligned}
S_{DBI}^{Dp} &= -T_{Dp} \int_{M_{p+1}} d^{p+1}\xi e^{-\phi} \sqrt{-\det\left(\hat{G}_{MN} + \hat{B}_{NM} + 2\pi\alpha' F_{MN}\right)} \\
&= -T_{Dp} \int_{\mathbb{R}^{1,2}} d^3x \sqrt{-\det(\hat{G}_{\mu\nu} + 2\pi\alpha' F_{\mu\nu})} \int_{\Sigma^{p-2}} d^{(p-2)}\Sigma e^{-\phi} \sqrt{\det(\hat{G}_{ab} + \hat{B}_{ab})}.
\end{aligned} \tag{4.5.3}
$$

One may then expand the $\mathbb{R}^{1,2}$ determinant for small values of α', which leads to

$$
\begin{aligned}
\sqrt{-\det(\hat{G} + 2\pi\alpha' F)} &= e^{3A} \sqrt{-\det(I + 2\pi\alpha' \hat{G}^{-1} F)} \\
&= e^{3A} \left[1 - \frac{(2\pi\alpha')^2}{4} e^{-4A} tr(\eta^{-1}F\eta^{-1}F) + O(\alpha')^4\right],
\end{aligned} \tag{4.5.4}
$$

where indices have been suppressed. $tr(\eta^{-1}F\eta^{-1}F) \equiv F_{\mu\nu}F^{\mu\nu}$ is the standard object appearing the in YM action

$$
S_{YM} = \frac{1}{4g^2} \int d^3x F_{\mu\nu} F^{\mu\nu} \tag{4.5.5}
$$

and so one may relate the α'^2 term in the expansion of eq (4.5.3) to this coupling which leads to the identification

$$
\frac{1}{g^2} = T_{Dp} \int_{\Sigma^{p-2}} d^{(p-2)}\Sigma e^{-\phi - A} \sqrt{\det(\hat{G}_{ab} + \hat{B}_{ab})}, \tag{4.5.6}
$$

A second way one can define a gauge coupling is with a Euclidean Dp brane. The DBI action of such a brane will wrap a compact (p+1)-cycle Σ^{p+1} in the internal space and is given by

$$
S_{Euclid}^{Dp} = T_p \int_{\Sigma^p} d^{(p+1)}\Sigma e^{-\phi} \sqrt{\det\left(\hat{G}_{ab} + \hat{B}_{ab}\right)}. \tag{4.5.7}
$$

This can be identified with the action of an instanton

$$
e^{-S_{inst}} = e^{-\frac{8\pi^2}{g^2}}. \tag{4.5.8}
$$

Thus for a Euclidean Dp brane can give a gauge coupling

$$
\frac{8\pi^2}{g^2} = T_p \int_{\Sigma^p} d^{(p+1)}\Sigma e^{-\phi} \sqrt{\det\left(\hat{G}_{ab} + \hat{B}_{ab}\right)}. \tag{4.5.9}
$$

In 4-d one would also include the WZ term to define a Θ angle. However the ideas behind this do not necessarily extend to 3-d so this term will be ignored here.

Chern-Simons level

A Chern-Simons level can be be extracted from the WZ action of a Dp brane with embedding $\xi = (t, x^1, x^2, \Sigma^n)$ in a similar fashion. The exact prescription is dependent on what conventions are being used. For instance in conventions such that the RR ployform may be expressed as

$$F_{poly} = dC - H_3 \wedge C + F_0 e^{B_2} \tag{4.5.10}$$

where C is the sum over the potentials of type-IIA or type-IIB (additionally F_0 should be taken to be zero in this case). Define also the Page charge of D(10-p) brane on a $(p-2)$-cycle $\Sigma^{(p-2)}$ to be

$$Q^{D(10-p)} = \frac{s_{(10-p)}}{2\kappa_{10}^2 T_{(10-p)}} \int_{\Sigma^{(p-2)}} \left(F_{poly} \wedge e^{-B} - F_0\right), \tag{4.5.11}$$

where $-F_0$ ensures that the D8 brane Page charge, which is really the Romans mass is not included. The orientation of the cycle is parametrised by $s_{(10-p)} = \pm 1$. Finally the WZ action of a Dp-brane shall be given by

$$S_{WZ}^{Dp} = s'_p T_p \int_{M_{p+1}} C \wedge e^{-\hat{B}_2 - 2\pi\alpha' F} \tag{4.5.12}$$

where the action is allowed to come with an overall positive or negative sign, i,e. $s'_p = \pm 1$. A Chern-Simons term in a gauge theory is given by the action

$$S_{CS} = -\frac{k}{4\pi} \int d^3x \mathcal{L}_{CS} = -\frac{k}{4\pi} \int d^3x tr(dA \wedge A + \frac{2}{3} A \wedge A \wedge A) \tag{4.5.13}$$

where A is a gauge field with field strength $F = dA + A \wedge A$ and so $d\mathcal{L}_{CS} = F \wedge F$. The order $F \wedge F$ term in eq (4.5.12) may be manipulated by adding an exact to give a Chern-Simons term,

$$\begin{aligned} d\left[C \wedge e^{-B_2} \wedge \mathcal{L}_{CS}\right] \pm C \wedge e^{-B_2} \wedge F \wedge F &= \left(dC - H_3 \wedge C\right) \wedge e^{-B_2} \wedge \mathcal{L}_{CS} \\ &= \left(F_{poly} \wedge e^{-B_2} - F_0\right) \wedge \mathcal{L}_{CS}. \end{aligned} \tag{4.5.14}$$

where the $+/-$ sign is for IIA/IIB. The Chern-Simons term is then given by

$$\begin{aligned} S_{CS} &= s'_p \frac{(2\pi\alpha')^2}{2} T_p \int_{\Sigma^{(p-2)}} \left(F_{poly} \wedge e^{-B_2} - F_0\right) \int_{\mathbb{R}^{1,2}} \mathcal{L}_{CS} \\ &= -\frac{Q^{D(10-p)}}{4\pi} \int_{\mathbb{R}^{1,2}} \mathcal{L}_{CS} \end{aligned} \tag{4.5.15}$$

where $s'_p s_{10-p} = -1$ and $2\kappa_{10}^2 T_p T_{10-p} = (2\pi)^{-3}$ have been used. This shows quite generally that the WZ action of a Dp brane contains a Chern-Simons coupling of level Q_{10-p}. Actually this is not the whole story as the true Chern-Simons level can experience an additional shift when all the p-dimensional KK modes are integrated out. Extra care must be taken when different conventions are used, indeed this is the case in the G_2-structure rotated solution, where further details will be given.

4.5.1 Wrapped D5 Branes and $\mathcal{N} = 1$ SYM with Gauge Group $SU(N_c)_{\frac{N_c}{2}}$

The solution of wrapped D5 Branes on Σ^3 is dual in the IR to $\mathcal{N} = 1$ SYM in 3 dimensions. It contains N_c color branes as can be seen from the flux quantisation condition

$$-\frac{1}{4\pi^2} \int_{\tilde{S}^3} F_3 = N_c \tag{4.5.16}$$

so the gauge group is $SU(N_c)$. The geometry only gives a good holographic description of a field theory in 3-d in the IR where $r \sim 0$. This is because Σ^3 vanishes in the IR and the QFT living on the wrapped D5 branes is effectively 3 dimensional there, however in the UV the cycle blows up and the world volume is explicitly 6 dimensional.

A suitable definition of the gauge coupling is given by a probe D5-brane extended along Minkowski and wrapping Σ^3. Once a gauge field F with legs in the Minkowski directions is turned on, the action of such a brane is given by

$$S_{probe} = T_5 \int_{\mathbb{R}^{1,2} \times \Sigma^3} d^3x d\Sigma^3 e^{-\Phi} \sqrt{-\det(G_{ind} + 2\pi\alpha' F)}, \tag{4.5.17}$$

there is no WZ contribution as $F_3 = 0$ on Σ^3. At this stage it will be instructive to reintroduce g_s and α' so that the induced metric is given by

$$ds^2_{ind} = e^{\Phi} \left[dx^2_{1,2} + \frac{\alpha' g_s N_c}{4} (e^{2h} + \frac{e^{2g}}{4} (w-1)^2)(\sigma^i)^2 \right]. \tag{4.5.18}$$

This and the fact that $(2\pi)^2 \alpha'^3 g_s T_5 = 1$ gives the following α' expansion of the

DBI action,

$$S_{probe} \sim \frac{\sqrt{g_s N_c} N_c}{16\pi^3 \alpha'^{3/2}} e^{2\phi} \left(e^{2h} + \frac{e^{2h}}{4}(w-1)^2\right)^{3/2} \int d^3x \left[2\pi^2 e^{-2\phi} F_{\mu\nu} F^{\mu\nu} \alpha'^2\right] \quad (4.5.19)$$

where indices are contracted with the Minkowski metric. One can then identify F^2 term with the Yang-Mills action and make the identification

$$\frac{2\pi}{g^2 N_c} = \sqrt{g_s N_c \alpha'} \left(e^{2h} + \frac{e^{2g}}{4}(w-1)^2\right)^{3/2}. \quad (4.5.20)$$

which gives a coupling of mass dimension 1, as it should have in 3-d. The RHS of eq (4.5.20) blows up in the UV and vanishes in the UV, which is consistent with the asymptotic freedom and confinement on expects of SYM in 3-d. The second of these is further supported by a Wilson loop calculation as in [73, 63], which gives an area law with string tension $\sigma = \frac{1}{2\pi\alpha'} e^{\phi_0}$.

One can also calculate the Chern-Simons level from a probe brane. Consider a D5-brane extended along Minkowski and wrapping S^3, the WZ action of such a brane is

$$S_{WZ} = T_5 \int_{\mathbb{R}^{1,2} \times S^3} \left(C_6 + (2\pi\alpha')^2 C_2 \wedge F \wedge F\right), \quad (4.5.21)$$

where F is once more a world volume gauge field with legs in the field theory directions. Integrating the second term in this action by parts gives the Chern-Simons action [72]

$$-\frac{1}{16\pi^3} \int_{S^3} F_3 \int d^3x tr\left(dA + \frac{2}{3} A \wedge A \wedge A\right) = -\frac{k_6}{4\pi} \int d^3x \mathcal{L}_{CS}, \quad (4.5.22)$$

where $k_6 = N_c$. There is no g_s or α' factors because they cancel with those in F_3 once they are reimposed. The k_6 here is to distinguish this object from the true CS level k which gets an extra contribution when one integrates out all the 6-d Kaluza Klein modes[6] . The Chern-Simons level is then

$$k = k_6 - \frac{N_c}{2} = \frac{N_c}{2} \quad (4.5.23)$$

which is half integer as one expects due to the parity anomaly in 3-d.

The wrapped D5-brane solution has two distinct UV solutions characterised by an asymptotically linear and constant dilaton, i.e. eq (4.2.12) and eq (4.2.13). The field theoretic interpretation for this is that the constant dilaton solutions have an irrelevant operator insertion in their Lagrangian.

[6]Specifically this it is integrating the massive fermions that generates the shift [72].

4.5.2 The G_2-structure Rotation and a 2-node Quiver Chern-Simons Theory

In [29] much of the gauge theory analysis of the G_2-structure rotated solution was performed. It was concluded that the gauge group was that of a 2 node quiver of the form $SU(r_l) \times SU(r_s)$, where $r_l > r_s$, by analogy with the Baryonic Branch of Klebanov-Strassler. The objects that must give rise to the ranks of this product group are the Page charges of the D2 and NS5 branes. As explained at length in [62], under a Seiberg duality the ranks of the gauge groups transform as $r'_l = r_s$ and $r'_s = 2r_s - r_l$. It is possible to see this manifestly in the supergravity solution if one defines

$$Q_{NS5} = r_l - r_s, \qquad Q_{D2} = r_s. \tag{4.5.24}$$

The Page charges on the LHS of these equalities transform under the large gauge transformation of eq (4.3.15) in precisely the same way as the ranks on the RHS do under a Seiberg duality. Thus large gauge transformations are equivalent to Seiberg duality. This along with the fact that there is a running integral of C_3 at infinity [29] is very suggestive of a duality cascade, once more by analogy with Klebanov-Strassler. It is reasonable then to propose that in the UV, where $Q_{D2} = M_c$, the gauge group is $SU(N_c + M_c) \times SU(M_c)$ and this then cascades down in ranks as one flow towards the IR terminating at $SU(N_c)$ as Klebanov-Strassler does.

It is possible to define two couplings for this quiver[7], g_1 and g_2 in the same spirit as in the previous section. A probe D4 brane with $(t, x^1, x^2, \hat{\Sigma}^2)$, with $\hat{\Sigma}^2$ as in eq (4.3.16) defines a coupling

$$\frac{4\pi^2}{g_1^2 N_c} = \sqrt{\alpha'} e^{\phi_\infty - \phi^0} \sqrt{H} \big(e^{2h} + \frac{1}{4} e^{2g}(w-1)^2\big). \tag{4.5.25}$$

while a probe D2 brane extended in Minkowski can be used to define the coupling

$$\frac{1}{g_2^2} = \frac{\sqrt{\alpha'}}{g_s} e^{\phi_\infty - \phi^0} \tag{4.5.26}$$

where both of these couplings have mass dimension 1 as they should. The LHS of eq (4.5.25) vanishes at $r \sim 0$ and becomes constant as $r \to \infty$. This indicates that the coupling g_1 is consistent with confinement in the IR and dilation invariance in the UV. On the other hand the LHS of eq (4.5.26) interpolates between a smaller and larger constant between the IR and the UV respectively. In [29] the difference in the IR behaviour of g_2 was interpreted as a signal of

[7] In [29] a third coupling is also proposed in terms of a D2 instanton, however this is probably not a good definition because the the WZ term is not quantised.

a confining Chern-Simons term dominating the gauge theory dynamics there. This can be understood because in a YM-CS like theory the level k induces an effective mass for the gauge field, $g_{YM}^2|k|$, which causes the Yang-Mills coupling to freeze at a constant value in the IR. Further evidence of confinement is given by Wilson loop calculations which obey an area law with string tension $\sigma = (c\sqrt{1-e^{2(\phi_\infty-\phi_0)}})^{-1}$. However a proposal for the Chern-Simons term which is claimed to be dominating the dynamics in the IR has been absent from the literature until now, the expression is provided below.

Indeed consider a probe D8 brane with embedding $(t, x^1, x^2, \hat{\Sigma}^6)$ on which a world volume gauge field is turned on with support in the Minkowski direction, the order $F \wedge F$ term of the WZ action of such a brane is

$$S_{CS} = -\frac{(2\pi\alpha')^2 T_8}{2} \int_{\mathbb{R}^{1,2}\times\Sigma^6} (C_5 + B_2 \wedge C_3) \wedge F \wedge F. \tag{4.5.27}$$

The integrand can be manipulated by adding an exact

$$(C_5 + B_2 \wedge C_3) \wedge F \wedge F + d\big[(C_5 + B_2 \wedge C_3) \wedge \mathcal{L}_{CS}\big] = (F_6 + H_3 \wedge C_3) \wedge \mathcal{L}_{CS}, \tag{4.5.28}$$

where $F_6 = dC_5 + B_2 \wedge F_4$, $F_4 = dC_3$ and $F \wedge F = d\mathcal{L}_{CS}$ have been used. Plugging this back into eq (4.5.27) and taking note of the definition of the D2 Page charge in eq (4.3.14) gives

$$S_{CS} = -\frac{Q_{D2}}{4\pi} \int_{\mathbb{R}^{1,2}} \mathcal{L}_{CS} \tag{4.5.29}$$

thus if one takes into account the definitions of the ranks in eq (4.5.24), a Chern-Simons level can be defined which is equal to the rank of smaller group

$$\hat{k} = r_s. \tag{4.5.30}$$

Of course it is possible that the level will experience a shift when one integrates out the 8-d KK modes, therefore this result should be viewed as correct up to the possible effect of this subtlety. Clearly $\hat{k}$ is not a fixed number from the perspective of supergravity, it shifts under large gauge transformations of C_3, however it is always quantised as a Chern-Simons level must be. In eq (4.5.29) only the positive orientation of $\hat{\Sigma}^6$ is considered, indeed it is possible to define another with the negative orientation, ie $k_1 = -k_2 = k$. This is what happens in the ABJM [82] where the $AdS_4 \times S^7/\mathbb{Z}_k$ geometry in M-theory is dual to the 2 node quiver $SU(N)_{-k} \times SU(N)_k$. This suggests that the quiver of the rotated solution could be $SU(r_l)_{-r_s} \times SU(r_s)_{r_s}$ by analogy with ABJM. If this is correct the effect of the Seiberg-like duality of eq (4.5.24) on the field theory is such that

$$G = SU(N_c + M_c)_{-\hat{k}} \times SU(M_c)_{\hat{k}} \ , \quad \hat{k} = M_c \tag{4.5.31}$$

becomes

$$G' = SU(M_c - N_c)_{-\hat{k}'} \times SU(M_c)_{\hat{k}'} \ , \quad \hat{k}' = M_c - N_c \tag{4.5.32}$$

and so clearly any cascade in the ranks of the groups must be associated with a corresponding cascade in the Chern-Simons levels.

The G_2-structure rotation acts on the SuGra solution that is dual to a QFT with an irrelevant operator that dominates the UV. The rotation induces additional warping on the metric by the function H which makes the new metric asymptotically AdS_4, this means that the rotated solution no longer contains this operator. These warp factors also pre-multiply the internal space ds_7^2 and ensure that in the UV this remains finite. The field theory lives on the world volume of D2 and fractional D2 branes which do not unwrap like the D5 branes of the original solution, so the rotated solution gives a good holographic description of a 3-d gauge theory throughout the whole space. In this sense the rotation procedure can be seen as providing a UV completion to the original QFT dual to the wrapped D5 solution with asymptotically constant dilaton.

4.5.3 The non-Abelian T-dual: Probe Analysis

The geometry of the non-abelian T-dual of the wrapped D5 solution supports all possible fluxes. This fact and comparison to the rotated solution suggest that the field theory is a type of quiver. As discussed in section 4.4.3, it is possible to define several quantised charges once the periods of the dual coordinates v_2 and v_3 are fixed. Note however that the charges defined in eq (4.4.30) have a common $\sqrt{2}$ factor. This is an artefact of the conventions used in the dualisation procedure, it has no deep meaning and so it makes sense to make the redefinition

$$\tilde{N}_c = \frac{N_c}{\sqrt{2}} \tag{4.5.33}$$

where $\tilde{N}_c$ should now be thought of as integer valued. The Page charges supported by the dual geometry are then

$$Q_{D6} = 0 \; mod \; \tilde{N}_c, \quad Q_{D4} = Q_{D2} = \tilde{N}_c. \tag{4.5.34}$$

This is enough charges to potentially define a 3 node quiver, but the identification of this quiver will not be pursued here, instead this section will focus only on probing the dynamics of the dual gauge theory. These probe calculation in the T-dual solution will be more complicated than the previous examples, for that reason the units as well as multiplicative constants in the couplings will be suppressed.

Like the rotated solution it is possible to define a coupling via a D2 brane parallel to the field theory coordinates. The dilaton, which is expressed in eq (4.4.2), depends on the dual coordinates that shall be set to constant values on the world volume of the brane. The simplest choice is that $v_2 = v_3 = 0$ for which the coupling is given by

$$\frac{1}{\tilde{g}_1^2} \sim e^{3g} = \begin{cases} \left(1\,,\;\; g_0^{3/2}\right) & r \sim 0 \\ \left(1\,,\;\; c^{3/2}e^{2r}\right) & r \to \infty \end{cases} \tag{4.5.35}$$

where the brackets correspond to asymptotically $\big($Linear, Constant$\big)$ dilaton solutions. The coupling is constant for the linear dilaton solution however this is clearly not a sign of conformal invariance as the non-compact dual metric is not AdS_4 and the dilaton is not constant. The coupling for solutions with constant dilaton in the UV are more interesting, asymptotically it is free and increases as one flows towards the IR finally freezing at a constant at $r = 0$.

Another way to define a gauge coupling is a probe D4 brane with embedding $(t, x^1, x^2, \tilde{\Sigma}^2)$, where the induced metric is given by eq (4.4.32). B_2 as defined in eq (4.4.3) vanishes on this cycle, however non-vanishing contributions can be induced by large gauge transformations as in eq (4.4.31). Generically the F^2 contribution to the DBI action gives a coupling of the form

$$\begin{aligned} \frac{1}{\tilde{g}_2^2} &= T_{D4}\int_{\tilde{\Sigma}^2} e^{-3\phi/2}\sqrt{\det\left(\hat{G}_{ab} + \hat{B}_{ab}\right)} \\ &\sim \frac{e^{3g}}{\pi}\int_0^{\pi} d\theta_1 \sqrt{e^{4h+2\phi}\tilde{N}_c^2 + 2n^2\sin^2\theta_1} \\ &= \frac{2N_c}{\pi}e^{3g+2h+\phi}E\left(\frac{-n^2}{e^{4h+2\phi}\tilde{N}_c^2}\right) \end{aligned} \tag{4.5.36}$$

where $\hat{B} = n/2\sin\theta_1 d\theta_1 \wedge d\varphi_1$, to include the effect of large gauge transformations. The function E is a complete elliptic integral, which as a statement is not very illuminating. When $n = 0$ there is no gauge transformation and $E(0) = \pi/2$ so the coupling is simply

$$\frac{1}{\tilde{g}_2^2} \sim e^{3g+2h+\phi} = \begin{cases} \left(e^{\phi_0}r^2\,,\;\; g_0^{5/2}e^{\phi_0}r^2\right) & r \sim 0 \\ \left(2e^{\phi_\infty}r\,,\;\; \frac{3c^{5/2}e^{\phi_\infty}}{4}e^{10r/3}\right) & r \to \infty \end{cases} \tag{4.5.37}$$

where the brackets correspond to asymptotically (Linear, Constant) dilaton solutions. This coupling is consistent with confinement in the IR and asymptotic

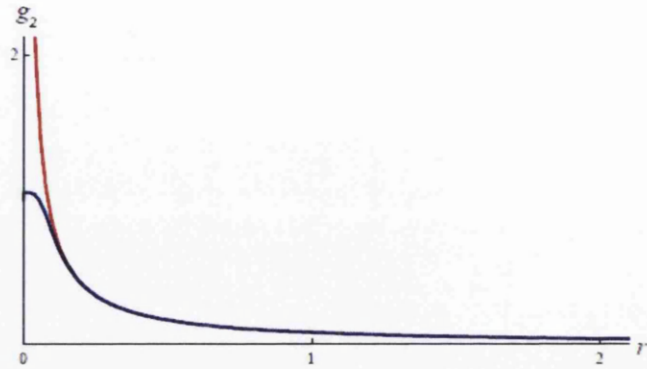

Figure 4.1: Plot of the coupling $\tilde{g}_2$. Blue shows $n = 1$ where the coupling freezes in the IR, whilst red shows $n = 0$ where the coupling blows up in the IR.

freedom in the UV, in fact it is much like the coupling of the original background in eq (4.5.20). For non-zero values of n the UV behaviour is unchanged because $e^{4h+2\phi}$ becomes large and the elliptic integral is well approximated by

$$E(\frac{-n^2}{e^{4h+2\phi}\tilde{N}_c^2}) \sim \frac{\pi}{2} + \frac{e^{-4h-2\phi}n^2}{4\tilde{N}_c^2}, \tag{4.5.38}$$

where the second term vanishes as $r \to \infty$. The IR behaviour changes quite dramatically under large gauge transformations because for $r \sim 0$, $\sqrt{e^{4h+2\phi}N_c^2 + 2n^2 \sin^2\theta_1} \sim \sqrt{2}N_c n \sin\theta_1$ and so the coupling tends to

$$\frac{1}{\tilde{g}_2^2}\bigg|_{r\sim 0} \sim \frac{2\sqrt{2}|n|g_0^{3/2}}{\pi} \tag{4.5.39}$$

so the effect of the large gauge transformation is to freeze the coupling in the IR making $n = 0$ a special case. The coupling $\tilde{g}_2$ is plotted in the whole space in figure 4.5.3

A third coupling may be defined in terms of Euclidean $D2$ on $\tilde{S}^3$. B_2 vanishes on this cycle up to the same large gauge transformations as before and the induced metric is given by eq (4.4.33) which leads to the coupling

$$\frac{1}{\tilde{g}_3^2} \sim e^{3g+h+\phi}\sqrt{2n^2+e^{4h+2\phi}\tilde{N}_c^2} \sim \begin{cases} (e^{2\phi_0}r^3\,,\; e^{2\phi_0}(g_0 r)^3) & r\sim 0,\; n=0 \\ (\,e^{2\phi_\infty}r^{3/2}\,,\;\; c^3 e^{4r}) & r\to\infty \\ g_0^2|n|\sqrt{r} & r\sim 0,\; n\neq 0 \end{cases} \tag{4.5.40}$$

this coupling is consistent with a strong coupling in the UV and asymptotic freedom in the IR. The effect of the large gauge transformation is less pronounced than it was for $\tilde{g}_2$ the behaviour in the IR is modified such that the power law changes, but the RHS of eq (4.5.40) still tends to zero in the IR.

The confining behaviour of the $\tilde{g}_3$ coupling should come as no surprise, the field theory and holographic directions are the same in both the original and dual geometries and so the conclusion of confinement from the Wilson loop studies of [73, 63] transfer to this solution also. That all the coupling exhibit asymptotic freedom is tied up with the fact that the bad UV behaviour of the original geometry, fractional branes unwrapping in the UV, is not being fixed by the T-duality, the irrelevant operator of the asymptotically constant dilaton solution will also still be present. As the original wrapped D5 brane solution is dual to a gauge theory with a Chern-Simons term it is reasonable to expect that the non-abelian T-dual geometry will be dual to a theory that also contains this type of term. That the couplings $\tilde{g}_1$ and (after a large gauge transformation) $\tilde{g}_2$ freeze out in the IR is certainly suggestive of a Chern-Simons term (or terms) that dominate the physics there.

At the beginning of this section it was shown that it is possible to define a Chern-Simons level for each Page charge in the geometry. For the non-abelian T-dual solution this gives 3 possible definitions

- Probe D8 brane on $(t, x^1, x^2, \tilde{\Sigma}_6)$ gives $k_1 = Q_{D2} = \tilde{N}_c$

- Probe D6 brane on $(t, x^1, x^2, \tilde{\Sigma}_4)$ gives $k_2 = Q_{D4} = \tilde{N}_c$

- Probe D4 brane on $(t, x^1, x^2, \tilde{\Sigma}_2)$ gives $k_3 = Q_{D6} = n\tilde{N}_c$

up to possible shifts from integrating out all the massive KK modes. The n in the definition of k_3 comes from the large gauge transformation.

The Chern-Simons level defined in the wrapped D5-brane solution is calculated on the 3-cycle $S^3 = (\sigma^1, \sigma^2, \sigma^3)$. This cycle is orthogonal to the directions on which the dualisation is performed and so must be mapped to a 4-cycle and 6-cycle. This accounts for k_1 and k_2 and suggests that the D8 and D6 branes may be probing the same gauge theory object. k_3 is unambiguously distinct, it is zero when B_2 is defined as in eq (4.4.3) but is shifted by large gauge transformation which is analogous to the Chern-Simons level of the G_2-structure rotated solution. It is interesting to see that when $k_3 = 0$ the couplings $\tilde{g}_2$ and $\tilde{g}_3$ behave quite differently than when it is not. Most pronounced is the effect on g_2 that exhibits typical confining behaviour when $n = 0$ but freezes in the IR becoming constant otherwise. This can be interpreted as a clear example of the effect a non-zero Chern-Simons term can have on a Yang-Mills coupling, see the discussion below eq (4.5.26).

Comments on Large Gauge Transformations and the Range of v_2, v_3

Although we have discussed the possibility of performing large gauge transformations on B_2 of the form $\Delta B_2 = -n/2 \sin\theta_1 d\theta_1 \wedge d\varphi_1$, it should be pointed out that n cannot take arbitrary values. String theory requires that

$$b_0 = -\frac{1}{4\pi}\int_{S^2} B_2 \in [0,1] \tag{4.5.41}$$

which restricts the value n can take. On $S^2 = \tilde{\Sigma}^2$ (see eq (4.4.27)), after performing n large gauge transformations one has

$$b_0 = \frac{1}{4\pi^2}\left[2n\pi + \frac{e^{4g+2\phi}N_c^2\pi v_3(1+w)^2}{\sqrt{2}(e^{4g+2\phi}N_c^2 + 8(v_2^2+v_3^2))}\right]. \tag{4.5.42}$$

In the UV it is possible to show that $b_0 \leq \frac{1}{4} + \frac{n}{2\pi}$ where the upper bounds of v_2, v_3 in eq (4.4.29) have been assumed so that

$$v_2 \in [0,\pi], \quad v_3 \in [0,\sqrt{2}\pi] \tag{4.5.43}$$

and we take $e^{4g+2\phi} \gg 8(v_2^2+v_3^2)$. This implies that $n = 0,1,3,4$[8] are allowed, indeed a numerical study shows that up to $g_0 \lesssim 4$ eq (4.5.41) is satisfied for all r. However as g_0 increases it is possible to perform less gauge transformations and still satisfy eq (4.5.42) until $g_0 \lesssim 12$ where it is impossible to perform any such transformations.

As further evidence of the range of dual coordinates taken in eq (4.5.43) consider the following. For $g_0 \gtrsim 4$, b_0 takes its maximum value at $r = 0$, in the IR. When one considers $n = 0$ and plots the relationship between g_0 and b_0^{IR} one finds that when (v_2, v_3) take the upper bounds in eq (4.5.43), $b_0 \to 1$as g_0 increases. Taking an upper bound greater than eq (4.5.43) causes b_0 to fall outside the bound of eq (4.5.42) for large g_0, see figure 4.5.3. This indicates that v_2, v_3 need to be finitely bounded for the solution to remain sensible and that the maximum range they may take is precisely as in eq (4.5.43)

4.6 Concluding Remarks

In this chapter the results of applying two solution generating techniques to the wrapped D5 solution of Maldacena and Nastase [72] (and its deformation [73]) were studied, with the aim of better understanding the dual gauge theories that are generated.

[8]We are excluding the possibility of negative n as we do not wish to encounter negative ranks for the gauge groups.

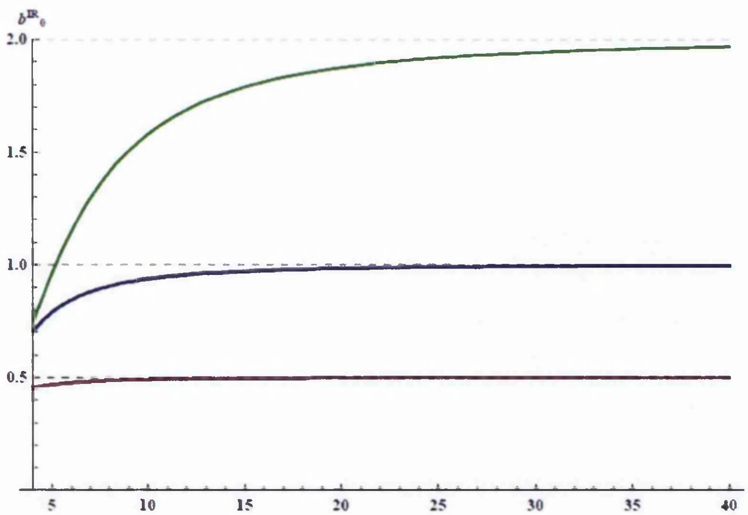

Figure 4.2: Graphs of the IR value of b_0 as a function of $g_0 \geq 4$ for $v_2 = m\pi$, $v_3 = \sqrt{2}m\pi$ and $n = 0$. Purple is $m = 1/2$, blue $m = 1$ and green $m = 2$. Notice that each graph asymptotes to m as g_0 increases.

The first technique, G_2-structure rotation [24], which is equivalent to U-duality, has an action on the field theories that is already quite well understood, partially due to explicit calculation [24, 29, 63, 77] and partially by analogy to its 6-d $SU(3)$-structure equivalent [22, 26, 27, 28, 16, 25]. The rotation acts on the wrapped D5 solution with asymptotically constant dilaton which is dual to a $\mathcal{N} = 1$ SYM-CS in 3-d with an irrelevant operator insertion. After the rotation this operator is removed and the metric is asymptotically $AdS_4 \times Y$, where Y has finite volume. It was shown in [24, 29] that the rotated geometry is dual to a 2-node quiver that very likely exhibits a duality cascade like the Baryonic Branch [18] of Klebanov-Strassler [12], due to similarities between the two solutions. One way in which the solutions differ is that the G_2-structure rotated, being a holographic description of a 3-d QFT, can contain a Chern-Simons term. Evidence for this was given in [29] where, through a probe brane calculation, a YM coupling was shown to freeze in the IR. This was interpreted as a signal of a Chern-Simons term that was dominating the IR, but no proposal for the level of this theory was given. This is resolved in section 4.5.2 where it is shown that a probe D8 brane wrapping the whole compact part of the rotated G_2-manifold, gives rise to a Chern-Simons level which is equal to the D2 Page charge. Thus the putative duality cascade must be accompanied by a cascade in the Chern-Simons level, which is very interesting and deserves further study. This will be left for future study as the main purpose of introducing the rotated solution was to aid, by comparison, the understanding of the main focus of this chapter.

A non-abelian T-duality [9, 31, 32, 33] was performed on the $SU(2)$ isometry parametrised by the left invariant 1-forms ω^i of the deformed Maldacena-

Nastase solution. The result of this is a rather complicated solution in massive type-IIA with all possible RR forms turned on. As the duality does not change the directions orthogonal to the isometry, it does not improve the asymptotic behaviour of the field theory directions and holographic coordinate as the rotation does, this of course was to be expected. It was possible to explicitly show that under the T-duality the G_2-structure of the original solution is mapped to a dynamic $SU(3)$-structure in 7-d [74, 75]. This is the analogue of result of [41] where it was shown that the 6-d $SU(3)$-structure of Klebanov-Witten [69] is mapped to a static $SU(2)$-structure for the T-dual solution [35, 30]. Indeed the structure of the dual geometry considered here becomes static $SU(3)$ in a limit in which, like Klebanov-Witten, there is no rotation in the projections of the original background (see appendix A of [73] for details of the projections).

A rigorous prescription for fixing the periodicities of the dual coordinates is lacking. The view taken in this chapter was that the coordinates were at least likely to be compact. If this were not the case it would only be possible to define a D6 brane Page charge and this seems strange given the rich variety of fluxes. Periods were chosen for the dual coordinates such that Page charges for D2 and D4 branes could also be defined and such that these charges were equal. It is important to realise however that it should be possible to fix the periods of the dual coordinates by some requirement on the global properties of the dual manifold and that such a prescription may not match the choice made here. At any rate, it is unlikely that the specific choice would drastically change the salient features of the manifold so a probe analysis of the geometry was performed with periodicity thus fixed to gain some insight into the possible dual QFT.

That it is possible to define 3 Page charges suggests, by analogy with the rotated solution, that the dual gauge theory may be a 3 node quiver. However, unlike the rotated solution, it is possible in this case to define as many Chern-Simons terms as there are charges. The most interesting of these is the Chern-Simons term with level that coincided with the D6 Page charge k_3. This experiences shifts under large gauge transformations of B_2 in much the same way as is true for the level in the rotated solution (albeit with that shift mediated by large gauge transformations of C_3). A new feature in the T-dual solution is that the large gauge transformation actually changes the IR behaviour of two of the couplings that can be defined, this is most pronounced for the coupling $\tilde{g}_2$. When $k_3 = 0$ the coupling exhibits typical confining behaviour, it tends to infinity as when flows towards the IR. However when $k_3 \neq 0$ the coupling freezes in the IR, a sign that a confining Chern-Simons term is now dominating the dynamics this coupling is sensitive to. This constitutes a very clean example of such behaviour, and it is nice to see a familiar dynamical effect in this complicated SuGra solution. It is probable that this solution also experiences some shift in the ranks of gauge groups of the QFT and that it can perhaps be identified with a Seiberg/level-rank like duality, this will be left for future work.

Chapter 5

On the Dual of the Baryonic Branch of Klebanov-Strassler

5.1 Introduction

This chapter is based on collaborative work done with Gaillard, Núñez and Thompson [43]. The system on which this work was focused is the Baryonic Branch of the Klebanov-Strassler field theory [12], [83], [18]. This is perhaps, among the minimally SUSY examples known at the moment, the one that better passed test of the correspondence between geometry and (strongly coupled) field theoretical aspects. Besides, the Baryonic Branch field theory and geometry unifies the original Klebanov-Strassler system and the system of five branes wrapping a two cycle inside the resolved conifold [11, 84]. Field theoretically, this unification can be thought as a Higgs-like mechanism and a particular limit where an accidental symmetry appears. See the papers [16, 25, 26] for different geometric and physical aspects of this connection.

In this Chapter, we will perform an $SU(2)$ non-Abelian T-duality on the Baryonic Branch geometry. This is a geometry described by an $SU(3)$-structure. All features of the geometry are characterised by a couple of forms J_2, Ω_3 that also encode many aspects of the strongly coupled dual field theory. Using non-abelian T-duality, we will obtain a new background in Massive Type IIA Supergravity. The G-structure will change to what is called $SU(2)$-structure, characterised by forms j_2, w_1, v_1, ω_2. The $SU(2)$-structure will transition from being *static* in the large radius region of the geometry, corresponding to high energies in the dual field theory, to being *dynamical* once the small radius region of the geometry is considered. Hence, the phenomena of confinement and symmetry breaking are given a geometric description by the change in $SU(2)$-structure from static to dynamical.

The action of non-Abelian T-duality on the G-structures has been studied in many backgrounds which we take the opportunity to summarise in the table below.[1]

Seed Solution	Seed Structure	Dual Structure
Klebanov-Witten	$SU(3)$	Orthogonal $SU(2)$
Klebanov-Tseytlin	$SU(3)$	Orthogonal $SU(2)$
$Y^{p,q}$	$SU(3)$	Orthogonal $SU(2)$
Klebanov-Strassler	$SU(3)$	Dynamical $SU(2)$
KS Baryonic Branch	$SU(3)$	Dynamical $SU(2)$
Wrapped D5's on S^2	$SU(3)$	Dynamical $SU(2)$
Wrapped D6's on S^3	$SU(3)$	Dynamical $SU(2)$
Wrapped D5's on S^3	G_2	Dynamical $SU(3)$

The contents of this chapter are organised as follows. In section 5.2 we will briefly summarise the original background and field theory corresponding to the Baryonic Branch of the Klebanov-Strassler. This is the *seed* background/field theory pair on which we will apply an $SU(2)$ isometry T-duality. In section 5.3 the new solution is presented explicitly.

In section 5.4, we will organise all the previous information using the language of G-structures. This will lead to a compact way of writing things, that can be very useful for other studies. We will study how the dynamical or static character of the G-structure depends on the field theoretic low energy dynamics captured by the original solution. In section 5.5, we will discuss different aspects of the field theory dual to our new backgrounds. We close the chapter with a list of possible future problems and conclusions.

5.2 Generalities on the Baryonic Branch

The Klebanov-Strassler field theory is a two-group quiver with bifundamental matter, charged under a global symmetry of the form $SU(2) \times SU(2) \times U(1)_R \times U(1)_B$. The ranks of the gauge groups are $(N, N+M)$ and the bifundamental matter A_1, A_2, B_1, B_2 self-interact via a superpotential of the form $\mathcal{W} \sim ABAB$. For a very clear explanation of many of the details of this quantum field theory, see [85], [86]. One detail that will be crucial to our present work is the fact that the so called 'duality cascade', a succesion of Seiberg dualities, ends in a situation where the quantum field theory may choose to develop VEVs for the Baryon and anti-Baryon operators.

[1]The details of the case of $Y^{p,q}$ are to appear in [71] and a detailed study of the D6 branes on S^3 appears in chapter 6 in [44].

In the last step of the duality cascade the gauge group is $SU(M) \times SU(2M)$. This theory has mesons $\mathcal{M} = (A_a)^{\alpha}_i (B_b)^i_{\beta}$ and also baryonic operators [83]

$$\begin{aligned} \mathcal{B} = \ & \epsilon_{\alpha_1 \ldots\ldots \alpha_{2M}} (A_1)^{\alpha_1}_1 (A_1)^{\alpha_2}_2 \ldots\ldots (A_1)^{\alpha_{M-1}}_{M-1} (A_1)^{\alpha_M}_M \\ & \times (A_2)^{\alpha_{M+1}}_1 (A_2)^{\alpha_{M+2}}_2 \times \ldots\ldots (A_2)^{\alpha_{2M-1}}_{M-1} (A_2)^{\alpha_{2M}}_M \end{aligned} \tag{5.2.1}$$

and similar for $\tilde{\mathcal{B}}$ made out of $(B_i)^a_l$ fields. One can see that both baryons and anti-baryons are neutral under $SU(2) \times SU(2)$ transformations.

The moduli space consists of two branches - the *mesonic* and the *baryonic* [86]. On the mesonic branch the baryons are zero ($\mathcal{B} = \tilde{\mathcal{B}} = 0$) and the mesons satisfy $\det \mathcal{M} = \Lambda^{4M}$. The non-perturbative contribution to the superpotential means that the associated moduli space can be identified with a symmetric product of the deformed conifold. On the Baryonic Branch the mesons are zero ($\mathcal{M} = 0$) but the baryons acquire expectation values,

$$\mathcal{B} = i\xi\Lambda^{2M}, \quad \tilde{\mathcal{B}} = \frac{i}{\xi}\Lambda^{2M}, \tag{5.2.2}$$

where Λ is the strong coupling scale of the group $SU(2M)$. Notice that both VEVs are equal only if $\xi = 1$. This corresponds to a $\mathbb{Z}_2$-symmetric point, represented by the exact solution in [12].

On this Baryonic Branch the $U(1)_B$ symmetry is spontaneously broken and the associated massless (pseudo-scalar) Goldstone mode corresponds to the phase of ξ. By supersymmetry this Goldstone lives in a chiral multiplet and comes along with scalar partner, the saxion, which corresponds to changing the modulus of ξ. As discussed in [86], the VEV of the operator,

$$\mathcal{U} = Tr[A_i A_i^{\dagger} - B_j B_j^{\dagger}], \tag{5.2.3}$$

which contains the $U(1)_B$ current J_μ as its $\theta\sigma^\mu\bar{\theta}$ component, encodes the motion along the Baryonic Branch (the different values of ξ) according to

$$\langle \mathcal{U} \rangle \sim M\Lambda^2 \ln|\xi|. \tag{5.2.4}$$

Let us focus on the situation where the field theory chooses to move to the purely Baryonic branch. In this case, there is a smooth solution of the equations of motion of type-IIB supergravity, that describes the strong dynamics of this field theory, including the spontaneous breaking of the $U(1)_B$ symmetry [83], [18]. In the notation that we will adopt in this work, such background can be written compactly by introducing the (string frame) vielbein basis,

$$
\begin{aligned}
e^{x^i} &= e^{\frac{\Phi}{2}}\hat{h}^{-\frac{1}{4}}dx^i\,, \quad e^{\rho} = e^{\frac{\Phi}{2}+k}\hat{h}^{\frac{1}{4}}d\rho\,, \quad e^{\theta} = e^{\frac{\Phi}{2}+h}\hat{h}^{\frac{1}{4}}d\theta\,, \quad e^{\varphi} = e^{\frac{\Phi}{2}+h}\hat{h}^{\frac{1}{4}}\sin\theta\, d\varphi\,, \\
e^1 &= \frac{1}{2}e^{\frac{\Phi}{2}+g}\hat{h}^{\frac{1}{4}}(\tilde{\omega}_1 + a\, d\theta)\,, \qquad e^2 = \frac{1}{2}e^{\frac{\Phi}{2}+g}\hat{h}^{\frac{1}{4}}(\tilde{\omega}_2 - a\,\sin\theta\, d\varphi)\,, \\
e^3 &= \frac{1}{2}e^{\frac{\Phi}{2}+k}\hat{h}^{\frac{1}{4}}(\tilde{\omega}_3 + \cos\theta\, d\varphi)\,,
\end{aligned}
\tag{5.2.5}
$$

where $\tilde{\omega}_i$ are the left invariant forms of $SU(2)$. The metric, RR and NSNS fields are

$$
\begin{aligned}
& ds^2 = \sum_{i=1}^{10}(e^i)^2\,, \\
& F_3 = \frac{e^{-\frac{3}{2}\Phi}}{\hat{h}^{3/4}}\Big[f_1 e^{123} + f_2 e^{\theta\varphi 3} + f_3(e^{\theta 23} + e^{\varphi 13}) + f_4(e^{\rho 1\theta} + e^{\rho\varphi 2})\Big]\,, \\
& B_2 = \kappa\,\frac{e^{\Phi}}{\hat{h}^{1/2}}\Big[e^{\rho 3} - \cos\alpha(e^{\theta\varphi} + e^{12}) - \sin\alpha(e^{\theta 2} + e^{\varphi 1})\Big]\,, \\
& H_3 = -\kappa\,\frac{e^{\frac{1}{2}\Phi}}{\hat{h}^{3/4}}\Big[-f_1 e^{\theta\varphi\rho} - f_2 e^{\rho 12} - f_3(e^{\theta 2\rho} + e^{\varphi 1\rho}) + f_4(e^{1\theta 3} + e^{\varphi 23})\Big]\,, \\
& C_4 = -\kappa\,\frac{e^{2\Phi}}{\hat{h}} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3\,, \\
& F_5 = \kappa\, e^{-\frac{5}{2}\Phi - k}\hat{h}^{\frac{3}{4}}\partial_\rho\left(\frac{e^{2\Phi}}{\hat{h}}\right)\Big[e^{\theta\varphi 123} - e^{x^0x^1x^2x^3\rho}\Big]\,.
\end{aligned}
\tag{5.2.6}
$$

We have defined

$$
\cos\alpha = \frac{\cosh(2\rho) - a}{\sinh(2\rho)}\,, \quad \sin\alpha = -\frac{2e^{h-g}}{\sinh(2\rho)}\,, \qquad \hat{h} = 1 - \kappa^2 e^{2\Phi}\,, \tag{5.2.7}
$$

where κ is a constant that we will choose to be $\kappa = e^{-\Phi(\infty)}$. The functions are,

$$
\begin{aligned}
f_1 &= -2N_c e^{-k-2g}\,, \qquad & f_2 &= \frac{N_c}{2}e^{-k-2h}(a^2 - 2ab + 1)\,, \\
f_3 &= N_c e^{-k-h-g}(a-b)\,, \qquad & f_4 &= \frac{N_c}{2}e^{-k-h-g}b'\,.
\end{aligned}
\tag{5.2.8}
$$

The system has a radial coordinate ρ, on which (a, b, Φ, g, h, k) depend, and we have set $\alpha' g_s = 1$. The background is then determined by solving the equations of motion for the functions (a, b, Φ, g, h, k). A system of BPS equations is derived. These non-linear and coupled first-order equations can be arranged in a convenient form, by rewriting the functions of the background in terms of a combination of them, that decouples the equations (as explained in [87]-[88]). We will not go over these in this thesis. Enough will be for us to state that the whole dynamics of the string background is controlled by a single function $P(\rho)$, subject to a second order non-linear and ordinary differential equation. This function $P(\rho)$ can be determined numerically and has IR and UV behaviors

$$\begin{aligned} UV: \quad P &= e^{4\rho/3}\left[c_+ + \ldots\right], \qquad \rho \to \infty, \\ IR: \quad P &= h_1\rho + \mathcal{O}(\rho^3), \qquad \rho \to 0. \end{aligned} \tag{5.2.9}$$

There is only one independent parameter, $c_+ > 0$ (the constant h_1 is determined by c_+) and it is this parameter that can be identified with the Baryonic expectation value

$$\mathcal{U} \sim \frac{1}{c_+}. \tag{5.2.10}$$

It is convenient to define a dimensionless quantity $\lambda = 2^{2/3}c_+\epsilon^{-4/3}$ where ϵ may be identified with the conifold deformation. See the paper [28] for a good account of the logic and technical details.

5.2.1 $SU(3)$ structure of the Baryonic Branch

The Supergravity background above is characterised by what is called an $SU(3)$ structure. That is, there exists a couple of forms $\hat{J}_2$ and $\hat{\Omega}_3$, in terms of which the BPS equations, the fluxes and various other quantities characterising the space can be written.

The observation of [22], is that the forms $\hat{J}, \hat{\Omega}$, describing the full Baryonic Branch can be obtained from the simpler ones describing a set of D5 branes wrapping the two cycle of the resolved conifold. We will not repeat the details of the derivation here, but we quote the results to the extent that we will find useful.

In general, an $SU(3)$ structure solution can be described by the following pure spinors in type-IIB [20],

$$\Psi_+ = -e^{i\zeta(r)}\frac{e^A}{8}e^{-i\hat{J}}, \quad \Psi_- = -i\frac{e^A}{8}\hat{\Omega}_{hol}, \tag{5.2.11}$$

where e^{2A} is the warp factor of the metric. Let us define

$$e^{i\zeta(r)} = \mathcal{C} + i\mathcal{S} \tag{5.2.12}$$

where $\mathcal{C}^2 + \mathcal{S}^2 = 1$. It is possible to show that for zero axion field, that is $F_1 = 0$, SUSY requires the following equalities to hold (these are the BPS equations previously mentioned)

$$d\big(e^{-\Phi}\mathcal{S}\big) = 0, \quad d\big(e^{2A-\Phi}\mathcal{C}\big) = 0,$$
$$d\big(e^{3A-\Phi}\hat{\Omega}_{hol}\big) = 0, \quad d\big(e^{4A-2\Phi}\hat{J}\wedge\hat{J}\big) = 0. \tag{5.2.13}$$

The fluxes are determined as

$$B_2 = \frac{\mathcal{S}}{\mathcal{C}}\hat{J}, \quad \frac{1}{\mathcal{C}^2}d\big(e^{2A}\hat{J}\big) = e^{4A}\star_6 F_3, \quad d\big(e^{4A-\Phi}\mathcal{S}\big) = -e^{4A}\star_6 F_5. \tag{5.2.14}$$

The system of N_c D5 branes wrapped on the resolved conifold is supported by just F_3 flux and is a solution to these equations when $\mathcal{S} = 0$. The (string-frame) frame fields that describe this geometry can be obtained from those of eq (5.2.5) by setting $\hat{h} = 1$. In terms of these, the J_2, Ω_3 (denoted without hats to distinguish them from those of the Baryonic branch) are given by

$$J = e^{r3} + (\cos\alpha e^{\varphi} + \sin\alpha e^2)\wedge e^{\theta} + (\cos\alpha e^2 - \sin\alpha e^{\varphi})\wedge e^1, \tag{5.2.15}$$
$$\Omega_{hol} = \big(e^r + ie^3\big)\wedge\big((\cos\alpha e^{\varphi} + \sin\alpha e^2) + ie^{\theta}\big)\wedge\big((-\sin\alpha e^{\varphi} + \cos\alpha e^2) + ie^1\big),$$

which obey the relations $J\wedge\Omega_{hol} = 0, \quad J\wedge J\wedge J = \frac{3i}{4}\Omega_{hol}\wedge\bar{\Omega}_{hol}$. The BPS equations for the functions h, g, k, a, b, Φ and the RR three-form flux, are

$$d(J\wedge J) = 0, \quad d(e^{\Phi/2}\Omega_{hol}) = 0$$
$$d(e^{\Phi}J) + e^{2\Phi}\star_6 F_3 = 0. \tag{5.2.16}$$

Then the results of [22] show that the $\hat{J}, \hat{\Omega}$ of the full Baryonic Branch solution are obtained by introducing a non-zero phase or rotation parameter[2] $\zeta(r)$ into (5.2.11) and defining:

$$\hat{J} = \mathcal{C}J, \quad \hat{\Omega}_{hol} = \mathcal{C}^{3/2}\Omega_{hol}, \; e^{2A} = \frac{e^{\Phi}}{\sqrt{\mathcal{C}}}, \quad \mathcal{S} = e^{\Phi-\Phi_\infty}, \tag{5.2.17}$$

where e^{2A} is the warp factor of the Baryonic Branch solution. For further details on the geometry and physics implied by this 'scaling of forms', we refer the reader to the original papers [16, 22, 25, 26].

[2]This parameter can also be understood in terms of the boost parameter that enters in the duality chain that relates the wrapped brane geometries to the Baryonic Branch [16, 25, 26].

5.2.2 A useful gauge transformation

Let us comment on a small subtlety that will be important in what follows. The above rotation argument makes it quite clear that by sending $\zeta \to 0$, the geometry becomes that of the wrapped D5 branes. On the other hand taking $\zeta \to \frac{\pi}{2}$ accompanied with $\lambda \to 0$, the geometry becomes that given by Klebanov and Strassler i.e. the $\mathbb{Z}_2$ point of the Baryonic branch. Taking this limit is slightly delicate. One finds that $\sin\zeta \to 1$ and $\cos\zeta \to \frac{1}{\lambda}h_{KS}$ where h_{KS} is the Klebanov-Strassler warp factor. Expanding the functions (a, b, Φ, g, h, k) in the large λ limit and rescaling Minkowski coordinates $x_i \to x_i\lambda^{-1}$ one finds that leading term of the metric is independent of λ and reproduces the KS geometry. The limit applied on the NS two form is less trivial, in fact its expansion in inverse powers of λ is

$$B_2 = \lambda \frac{\epsilon^2 \sinh(2\rho)}{2\sqrt{3}\kappa P_1\sqrt{P_1'}} d(P_1(\tilde{\omega}_3 + \cos\theta d\varphi) - B_{KS} + \mathcal{O}(\lambda^{-1}) \,. \tag{5.2.18}$$

However the form of P_1 (the leading contribution of $P(\rho)$ in this expansion) ensures that the pre-factor on the first term in this expression reduces to a constant and one recovers the Klebanov-Strassler NS two form *modulo a pure gauge term.*

In fact it is going to suit our purposes to perform a similar gauge transformation across the whole Baryonic Branch eq (5.2.6). We do this by defining

$$B_2 \to B_2 + d(\mathcal{Z}(\subset)(\tilde{\omega}_3 + \cos\theta d\varphi)), \;\; \mathcal{Z} = -\frac{1}{2}\int_0^\rho e^{2k(\rho')+\Phi(\rho')}S(\rho')d\rho' \tag{5.2.19}$$

In the KS limit this reduces to exactly the gauge transformation required in (5.2.18) and it has the effect of removing certain mixing between the angular directions and the radial direction in the NS two-form.[3] This will greatly simplify matters upon performing a duality transformation.

5.3 Non-Abelian duality on the Baryonic Branch

In this section, we will present the result for the non-Abelian T-duality when applied to one of the $SU(2)$ isometries of the Baryonic Branch background in eq (5.2.5)-(5.2.6). We extend the results of [30] in which the NS sector was established but full details of the RR sector were not provided.[4] We will perform

[3] This transformation leaves unchanged the gauge coupling defined through the integral of B_2 however it is non-vanishing at infinity and so one should exercise appropriate caution.

[4] The results of [30] lead at first sight to a geometry that has a mixing between angular and radial directions. This is however a gauge artifact as will be made clear in appendix F. By making the gauge transformation eq (5.2.19) to the seed geometry, as we do here, one removes this

the transformation described in [30] to the coordinates $(\tilde{\theta}, \tilde{\varphi}, \psi)$, present in the left-invariant forms of $SU(2)$, $\tilde{\omega}_i, i = 1, 2, 3$ of eq (5.2.5). We will choose a gauge where the new coordinates after the duality will be (v_2, v_3, ψ). We present the results here and refer the reader to the appendix F for details.

We will start by specifying the vielbeins. The components

$$e^{x^i} = e^{\frac{\Phi}{2}} \hat{h}^{-\frac{1}{4}} dx^i , \quad e^{\rho} = e^{\frac{\Phi}{2}+k} \hat{h}^{\frac{1}{4}} d\rho \tag{5.3.2}$$

do not change. The vielbeins in the (θ, φ) directions are also unchanged by the duality however we find it useful to introduce a rotation in $(e^{\theta}, e^{\varphi})$ such that the dual solution has no explicit ψ dependence.

$$e^{\hat{\theta}} = \sqrt{C} e^{h+\Phi/2} \omega_1 , \quad e^{\hat{\varphi}} = \sqrt{C} e^{h+\Phi/2} \omega_2 , \tag{5.3.3}$$

where we have introduced left invariant $SU(2)$ forms for the angles $\{\theta, \phi, \psi\}$. The vielbeins in the directions $\hat{1}, \hat{2}, \hat{3}$ and NS 2-form potential can be compactly written in terms of the quantities defined as,

$$\begin{aligned} \mathcal{H} &= \frac{2\sqrt{2} v_3 + 4\mathcal{Z} + e^{2g+\Phi} \mathcal{S} \cos\alpha}{2\sqrt{2}} , \\ \mathcal{Z} &= -\frac{1}{2} \int_0^{\rho} \mathcal{S} e^{\Phi+2k} d\rho' , \\ \mu_1 &= a e^g \cos\alpha + 2e^h \sin\alpha . \end{aligned} \tag{5.3.4}$$

The function $\mathcal{Z}$ was introduced as a gauge transformation to the seed solution

mixing. Alternatively one can perform the following coordinate transformation to the solution presented in [30] to obtain the solution presented here:

$$v_3^{there} \to v_3^{here} + \sqrt{2}\mathcal{Z} , \tag{5.3.1}$$

already in eq (5.2.19). With these, we have

$$
\begin{aligned}
e^{\hat{1}} &= \frac{e^{g+\Phi/2}}{8\mathcal{W}}\sqrt{\mathcal{C}}\Big[4e^{2k+\Phi}\mathcal{C}\mathcal{H}(a\mathcal{H}\omega_1 - v_2\omega_3) - \sqrt{2}e^{2(g+k+\Phi)}\mathcal{C}^2(dv_2 + a\mathcal{H}\omega_2) \\
&\quad -8\sqrt{2}v_2(v_2 dv_2 + \mathcal{H}dv_3) + \\
&\quad \frac{1}{2}\mu_1\mathcal{S}e^{g+\Phi}(8v_2^2\omega_2 + e^{2k+\Phi}\mathcal{C}(e^{2g+\Phi}\mathcal{C}\omega_2 - 2\sqrt{2}\mathcal{H}\omega_1))\Big]\,, \\
e^{\hat{2}} &= \frac{e^{g+3\Phi/2+g}}{8\mathcal{W}}\mathcal{C}^{3/2}\Big[4e^{2g}v_2(dv_3 - av_2\omega_2) - 4\mathcal{H}e^{2k}(dv_2 + a\mathcal{H}\omega_2) \\
&\quad -\sqrt{2}\mathcal{C}e^{2k+2g+\Phi}(a\mathcal{H}\omega_1 - v_2\omega_3) + \\
&\quad \frac{1}{2}\mu_1\mathcal{S}e^{g+2k+\Phi}(e^{2g+\Phi}\mathcal{C}\omega_1 + 2\sqrt{2}\mathcal{H}\omega_2)\Big]\,, \\
e^{\hat{3}} &= \frac{e^{k+\Phi/2}}{8\mathcal{W}}\sqrt{\mathcal{C}}\Big[4\mathcal{C}v_2e^{4g+\Phi}(v_2\omega_3 - a\mathcal{H}\omega_1) - \sqrt{2}\mathcal{C}^2(dv_3 - v_2a\omega_2) \\
&\quad -8\sqrt{2}\mathcal{H}(v_2dv_2 + \mathcal{H}dv_3) + e^{g+\Phi}\mu_1v_2\mathcal{S}(\sqrt{2}\mathcal{C}e^{2g+\Phi}\omega_1 + 4\mathcal{H}\omega_2)\Big]\,.
\end{aligned}
\tag{5.3.5}
$$

We will then have a metric that in terms of these vielbeins reads, $ds^2_{st} = \sum_{i=1}^{10}(e^i)^2$.
In terms of these vielbeins, the NS two-form B_2 reads,

$$
\begin{aligned}
\widehat{B}_2 = &-\frac{1}{4v_2}\left(2e^{-h}a(e^g v_2 e^{\hat{\theta}\hat{1}} + e^k\mathcal{H}e^{\hat{\theta}\hat{3}}) - 4e^{k-g}\mathcal{H}e^{\hat{1}\hat{3}} + \sqrt{2}\mathcal{C}e^{g+k+\Phi}e^{\hat{2}\hat{3}}\right) + \\
&\frac{\mathcal{S}}{\mathcal{C}}\Big[\frac{\mathcal{H}e^k}{2v_2}(2e^{-g}e^{\hat{1}\hat{3}} - ae^{-h}e^{\hat{\theta}\hat{3}}) + \frac{e^{g+k+\Phi-h}}{4\sqrt{2}v_2}\mathcal{C}(\mu_1e^{\hat{\theta}\hat{3}} - 2e^he^{\hat{2}\hat{3}}) - \\
&\frac{e^{-h}}{2}(2e^{-h-\Phi}\frac{\mathcal{Z}}{\mathcal{S}} + 2e^h\cos\alpha - ae^g\sin\alpha)e^{\hat{\theta}\hat{\varphi}} - \frac{e^{-h}}{2}(ae^ge^{\hat{\theta}\hat{1}} + \mu_1e^{\hat{\theta}\hat{2}})\Big].
\end{aligned}
\tag{5.3.6}
$$

The dual dilaton is given by

$$
\widehat{\Phi} = \Phi - \frac{1}{2}\ln\mathcal{W}\,, \quad \mathcal{W} = \frac{\mathcal{C}}{8}\left(e^{4g+2k+3\Phi}\mathcal{C}^2 + 8e^{2g+\Phi}v_2^2 + 8e^{2k+\Phi}\mathcal{H}^2\right)\,. \tag{5.3.7}
$$

And the RR sector is given by,

$$
\begin{aligned}
F_0 &= \frac{N_c}{\sqrt{2}}\,, \\
F_2 &= -\frac{e^{-\Phi}}{4\mathcal{C}}N_c\Big[2e^{-2h}(1+a^2-2ab)\mathcal{H}e^{\hat\theta\hat\varphi}+e^{-g-h-k}\mathcal{C}(a-b)\Big(\sqrt{2}e^{2g+k+\Phi}(e^{\hat\theta\hat 1}-e^{\hat\varphi\hat 2})+ \\
&\qquad 4e^k\mathcal{H}(e^{\hat\theta\hat 2}-e^{\hat\varphi\hat 1})-4v_2e^g e^{\hat\varphi\hat 3}\Big)-8e^{-2g}\mathcal{H}e^{\hat 1\hat 2}-8e^{-g-k}v_2e^{\hat 2\hat 3}-2e^{-h-k}v_2e^{r\hat\theta}\Big]- \\
&\quad \frac{\mathcal{S}e^{g-h}}{\sqrt{2}\mathcal{C}\sin\alpha}\Big(N_cb+a(e^{2g}\cos^2\alpha-N_c)+e^{g+h}\sin 2\alpha\Big)e^{\hat\theta\hat\varphi}\,, \qquad (5.3.8) \\
F_4 &= \frac{e^{-g-h-k-\Phi}}{8\mathcal{C}}N_c\Big[\mathcal{C}(1+a^2-2ab)e^{\hat\theta\hat\varphi}\wedge(\sqrt{w}e^{2g+k+\Phi-h}e^{\hat 1\hat 2}+4e^{2g-h}e^{\hat 1\hat 3}) \\
&\qquad \mathcal{C}b'e^{r\hat\theta}\wedge(4e^k\mathcal{H}e^{\hat 1\hat 3}-\sqrt{2}e^{2g+k+\Phi}e^{\hat 2\hat 3})-8e^g v_2(a-b)e^{\hat\theta\hat 1\hat 2\hat 3} \\
&\qquad e^{r\hat\varphi}\wedge(4e^g v_2 e^{\hat 1\hat 2}-b'e^k(\sqrt{2}e^{2g+\Phi}e^{\hat 1\hat 3}+4\mathcal{H}e^{\hat 2\hat 3}))\Big]- \\
&\quad \frac{2\mathcal{S}e^{-g-h-k-\Phi}}{\mathcal{C}^2\sin\alpha}\Big(a(e^{2g}\cos^2\alpha-N_c)+(N_cb+e^{g+h}\sin 2\alpha)\Big)\Big(\mathcal{H}e^k e^{\hat\theta\hat\varphi\hat 1\hat 2}+v_2e^g e^{\hat\theta\hat\varphi\hat 2\hat 3}\Big)\,.
\end{aligned}
$$

Warning on potentially confusing nomenclature: The N_c appearing in the above originated as the number of $D5$ branes wrapping the resolved conifold which was then rotated to give the Baryonic Branch and then T-dualised to this solution. Prior to T-duality, N_c corresponds to the $D5$ charge which is also commonly denoted by M (which we will also use in section 5.5 when we specialised to the Klebanov-Tseytlin geometry). We hope the reader will not get overly confused by this point.

5.3.1 UV asymptotic behaviour

Using the semi-analytic UV expansions that can be found, for example, in [28] it is possible to calculate the UV behaviour of the dual metric. The dual vielbeins at leading order in the UV are given by

$$
e^{\hat 1,\hat 2}=(-)^{1,2}\frac{c_+e^{-2\rho/3}(24\rho-3)^{1/4}}{2^{3/4}\sqrt{N_c}(1-2\rho)}\omega_{1,2},\quad e^{\hat 3}=-\frac{2^{3/4}3^{1/4}}{\sqrt{N_c}(8\rho-1)^{1/4}}dv_3. \qquad (5.3.9)
$$

Thus the dual 3-manifold shrinks as one flows towards the UV, in line with our expectations from abelian T-duality, where big circles are mapped to small circles.

One may worry that this vanishing manifold is a signal of a singularity in the UV, however, an explicit check shows that the curvature invariants: Ricci scalar, $R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu\nu\lambda\kappa}R^{\mu\nu\lambda\kappa}$ are finite. In other words, both the g_s and the α' expansions are under control and the background is trustable in the far UV. Notice that there is a one-cycle, labelled by the coordinate ψ in ω_3, that shrinks to zero size in the large-ρ regime. This implies that strings wrapping this cycle will become light and will enter the spectrum of the dual QFT at high energies.

The dual dilaton is defined as $e^{2\hat{\Phi}} = \frac{e^{2\Phi}}{\mathcal{W}}$ where

$$\mathcal{W} = 3c_+ N_c \sqrt{12\rho - \frac{3}{2}e^{8\rho/3}} \tag{5.3.10}$$

asymptotically, and so the dilaton is UV vanishing.

5.3.2 IR asymptotic behaviour

Let us now study the small radius regime of the metric, corresponding with the low energy regime of the dual QFT. Things are a bit less-simple. At leading order, terms in the metric depend explicitly of the original IR-parameters of the Baryonic Branch solution, but they also depend on the values of the v_2, v_3 coordinates. The dual vielbeins in the IR tend to

$$\begin{aligned}
e^{\hat{1}} &= -\frac{32e^{\Phi_0/2}\sqrt{\mathcal{F}}h_1^{3/2}}{\mathcal{G}}\Big(v_3(dv_2+v_2\omega_3)+v_2(v_2\omega_2-\frac{1}{2\sqrt{2}}dv_3)-v_3^2(\omega_1-\omega_2)\Big)\\
e^{\hat{2}} &= -\frac{2e^{\Phi_0/2}\sqrt{\mathcal{F}}\sqrt{h_1}}{\mathcal{G}}\Big(\sqrt{2}v_3\mathcal{F}e^{3\Phi_0}\omega_1-\sqrt{2}v_2\mathcal{F}e^{\Phi_0}\omega_3+\\
&\qquad 16h_1(v_3dv_2-v_2dv_3+(v_2^2+v_3^2)\omega_2)\Big)\\
e^{\hat{3}} &= -\frac{2e^{-\Phi_0/2}\sqrt{\frac{h_1}{\mathcal{F}}}}{\mathcal{G}}\Big(\sqrt{2}\mathcal{F}^2e^{2\Phi_0}(\frac{1}{2\sqrt{2}}dv_3-v_2\omega_2)-16h_1v_2\mathcal{F}(v_2\omega_3-v_3\omega_1)+\\
&\qquad \sqrt{2}128h_1^2v_3(v_2dv_2+v_3dv_3)\Big)
\end{aligned} \tag{5.3.11}$$

where we have defined

$$\mathcal{F}^2 = 4(2)^{3/2}(h_1^{5/2}-2\sqrt{2}e^{\Phi_0}h_1),\ \mathcal{G} = e^{2\Phi_0}\mathcal{F}^2+128h_1^2(v_2^2+v_3^2). \tag{5.3.12}$$

for convenience. The function $\mathcal{W}$ tends to

$$\frac{\mathcal{F}e^{\Phi_0}}{512h_1}\Big(\mathcal{F}^2e^{2\Phi_0}+128(v_2^2+v_3^2)\Big) \tag{5.3.13}$$

Here again, it happens that the dilaton is bounded and the Ricci scalar and

Ricci and Riemann tensors squared are finite. This was expected, as we are performing a duality transformation on a space that in the small-ρ regime was of finite size (the S^3 in the deformed conifold). Dualities typically invert 'sizes' (or couplings). This example is not an exception. One may start with a background solution where Supergravity is a good approximation and obtain that in the far IR the new generated solution is still a supergravity background we can trust.

A point that we want to emphasize again is that in the far IR, the parameter that was labelling the different 'positions' on the Baryonic branch (that is the different baryonic VEVs) still appears in the small-radius expansion above. There is a still a one-parameter family of solutions. Indeed, notice the dependence on the integration constants $e^{\Phi(0)}$ and h_1 as defined in [22], both related to the number parametrising the Baryonic branch.

5.4 $SU(2)$ Structure of the background

We will now study the associated G-structure with this solution. Again, we will postpone details to the appendix F. The geometry supports two pure spinors given by

$$\begin{aligned} \Phi_+ &= \frac{e^A}{8} e^{i\theta_+} e^{-iv\wedge w}\left(k_\| e^{-ij} - ik_\perp \omega\right), \\ \Phi_- &= \frac{ie^A}{8} e^{i\theta_-}(v+iw)\wedge\left(k_\perp e^{-ij} + ik_\| \omega\right). \end{aligned} \tag{5.4.1}$$

In the case at hand we find

$$\begin{aligned} & e^{2A} = \frac{e^\Phi}{c} \\ & \theta_+ = 0, \quad \theta_- = \zeta(r) \\ & k_\| = \frac{\sin\alpha}{\sqrt{1+\zeta.\zeta}} \qquad k_\perp = \sqrt{\frac{\cos^2\alpha+\zeta.\zeta}{1+\zeta.\zeta}} \\ & z = w - iv = \frac{1}{\sqrt{\cos^2\alpha+\zeta.\zeta}}\left(\sqrt{\Delta}\tilde{e}^3 + \zeta_2\sin\alpha\tilde{e}^\theta + i(\sqrt{\Delta}\tilde{e}^\rho + \zeta_2\sin\alpha\tilde{e}^\varphi)\right) \\ & j = \tilde{e}^{\rho 3} + \tilde{e}^{\varphi\theta} + \tilde{e}^{21} - v\wedge w \\ & \omega = \frac{i}{\sqrt{\cos^2\alpha+\zeta.\zeta}}\left(\sqrt{\Delta}(\tilde{e}^\varphi + i\tilde{e}^\theta) - \zeta_2\sin\alpha(\tilde{e}^\rho + i\tilde{e}^3)\right)\wedge(\tilde{e}^2 + i\tilde{e}^1). \end{aligned} \tag{5.4.2}$$

Here the frames $\tilde{e}$ are obtained by a rotation, given by eq (E.19), of those in eq (5.3.5) and the parameters Δ, ζ_i which enter into this rotation are specified by eq (F.15).

There are various immediate things to observe. If we move to the large radius region of the geometry, the functions $\sin\alpha(\rho) \sim a(\rho) \sim b(\rho) \to 0$. The formulas simplify and we obtain, among other things that $k_{||} \to 0$. This implies that, as happens in chapter 3, the two pure spinors are 'perpendicular' in the large radius regime of the solution and the $SU(2)$-structure is *static*. Similar behaviour was found in [42], where a dynamical $SU(3)$- structure in 7-d becomes orthogonal in the UV. This changes as we evolve to the small radius regime of the background, the $SU(2)$-structure is said to become *dynamical*. In section 5.5, we will discuss the physical effects that are associated with a change in the $SU(2)$-structure, from *static* in the far UV to *dynamic* in the IR.

5.5 Correspondence with Field Theory

In this section, we will connect our previous geometrical studies with aspects of the quantum field theory that our background is dual to. As it was anticipated in the paper [30], we believe that the field theory dual to our massive IIA background should be a non-conformal version of the Sicilian gauge theories presented in [89, 57] or the linear quiver field theories studied in [90]. There are certain things that can be inferred immediately, like for example the confining character of the QFT. This follows from the fact that the calculation of the Wilson loop will proceed exactly as in the case of the Baryonic Branch field theory. Indeed, the $R^{1,3} \times \rho$ part of the geometry is unchanged, hence, the Wilson loop will give the same result as before the non-Abelian T-duality. Nevertheless, many calculations done with the Klebanov-Strassler/Baryonic Branch background involved the 'internal' five dimensional space. The purpose of this section will be to learn how some of those calculations for field theory observables change (or not) for the new geometries in massive IIA.

The idea that will guide us is that for a given correlation function or related QFT observable, that in the original background was calculated in a way that is 'independent' of the $SU(2)$ isometry used to perform the non-Abelian duality, will give the same result in the transformed background. We can think about those operators or correlators as 'uncharged' under the $SU(2)$ symmetry in question. Ideas of this sort already worked in other solution generating techniques, like T-s-T dualities. Similar ideas also appeared in large N_c (planar) equivalences between parent-daughter theories. The physics of the common or 'uncharged' sector goes through to the new field theory. The rest of the chapter deals with observables that are, in principle 'charged' under the $SU(2)$ symmetry.

In the paper [30] it was shown that the cascade of Seiberg dualities–defined geometrically as a large gauge transformation of the NS two form and its effect on Page charges, persisted in the massive IIA background. In chapter 3, we started to geometrise some of the field theory effects corresponding to the Klebanov-Witten non-Abelian T-dual. In the rest of this section, we will focus our attention on the relation between the dynamical character of the $SU(2)$-structure and the field theoretical phenomena of confinement and discrete R-symmetry breaking. We will show how the presence of domain Walls with an induced Chern-Simons dynamics on their world-volume follows as a consequence of the confinement and the dynamical character of the $SU(2)$-structure. Then, we will make clear that the symmetry associated with changes in the ψ-direction is related with an anomalous $U(1)_R$ R-symmetry in the field theory. We will define an instantonic object using an euclidean D0 brane; this will lead us to a possible definition for a Θ-angle and gauge coupling. We will find that this coupling has a non-conventional running in the far UV. We will then move into studying different aspects of the 'baryonic branch', also present in our new backgrounds. We will find that a given fluctuation of the RR background fields can be put in correspondence with a global continuous symmetry that the IR dynamics breaks spontaneously. We will find the associated Goldstone boson and an expression for the conformal dimension of such a baryonic operator.

5.5.1 Dynamic SU(2): A pathway to confinement

In this section, we will make more concrete the relation between the QFT phenomena of confinement and the dynamical character of the $SU(2)$-structure. The first observation is that the 'parallel projection' between both spinors, represented by $k_{\|}$ in eq (F.3), is proportional to the quantity $\sin\alpha$. This quantity is related to the background functions as can be read from appendix B of the paper [17],

$$\sin\alpha(\rho) = \frac{4ae^{h-g}}{\sqrt{a^2 + 2a^2(4e^{2h-2g}) + (4e^{2h-2g}+1)^2}}. \tag{5.5.1}$$

This is compatible with the expression in eq (5.2.7) after following the algebra in appendix B of the paper [17].

The presence of the functions $a(\rho), b(\rho)$ in the Baryonic Branch solution–see eqs (5.2.5)-(5.2.8)—are responsible for the de-singularisation of the space (the appearance of a finite size S^3) and the IR minimization of the dilaton and warp factor. These have as a consequence the linear law, $E_{QQ} = \sigma L_{QQ}$ for large distance separations between the quark-antiquark pair. In other words, the functions $a(\rho), b(\rho)$ and their effects on the warp factor and dilaton 'produce' confinement. In the same vein, at the level of the metric, the presence of $a(\rho)$ implies the breaking of the symmetry $\psi \to \psi + \epsilon$ into $\psi \to \psi + 2\pi$. This is the remaining $\mathbb{Z}_2$ symmetry after the spontaneous discrete R-symmetry breaking. So, we see clearly that confinement and spontaneous R-symmetry breaking go hand-in-hand with the function $a(\rho)$. Hence, these phenomena in the dual QFT are closely related to the presence of $k_{\|}$, which as we made clear is related to the dynamical character of the $SU(2)$-structure. In the papers [91, 92], the point was made that the functions $a(\rho), b(\rho)$ were directly related with the gaugino condensate. This suggests that in our massive IIA picture, there exists a relation of the form $< \lambda\lambda > \sim k_{\parallel}$.

5.5.2 A comment on domain walls

It was proposed in [30], that domain wall objects were realised in the Non-Abelian T-dual of the geometries we are considering, as D2 branes that extend on $R^{1,2}$. Indeed, the induced metric, action and tension of a $(2+1)$-dimensional object are,

$$
\begin{aligned}
ds^2_{ind} &= e^{\Phi}\hat{h}^{-1/2}(-dt^2 + dx_1^2 + dx_2^2),\\
S_{BI} &= -T_{D2}\int d^3x e^{\Phi/2}\hat{h}^{-3/4}, \quad T_{DW} = T_{D2}e^{\Phi/2}\hat{h}^{-3/4}|_{\rho=0}.
\end{aligned}
$$

If we also turn on a gauge field in the world-volume of this D2 brane, a Chern-Simons-Ma [illegible] α' on this D-brane,

$$
\begin{aligned}
S_{BIWZ} = &- T_{D2}\int d^{2+1}x e^{\Phi/2}\hat{h}^{-3/4}\sqrt{1-\alpha' F_{\mu\nu}F^{\mu\nu}}\\
&+ T_{D2}\int d^{2+1}x F_0 A_1 \wedge F_2. \qquad \overline{F^{\mu\nu}}\\
&+ T_{D2}\int d^{2+1}x F_0 A_1 \wedge F_2.
\end{aligned}
\tag{5.5.2}
$$

We have used that a new WZ-like term appears in Massive IIA as explained in [93]. The Chern-Simons term is quantised, being proportional to $T_{D2}N_c$.[5]

In the type-IIB Baryonic Branch solution(s), domain walls were realised by D5-branes extended on $R^{1,2}$ and the three-sphere $\tilde{S}^3 = [\tilde{\theta}, \tilde{\varphi}, \psi]$. Once a gauge field is turned on, a Chern-Simons terms was induced, proportional to $T_{D5}\int_{\tilde{S}^3} F_3$. Naively, we can think that both objects are 'connected' by the non-abelian T-duality, under which the directions on $\tilde{S}^3$ disappear and we are left with a D2 brane as described above.

Supersymmetry gives support to this. Indeed, around eq.(6.19) of the paper [20], we are presented with the calibration form for a domain-wall like object, which is given by the real part of the pure spinor Ψ_+. Using that $|a|^2 = e^A = e^{\Phi/2}\hat{h}^{-1/4}$, we obtain that the BI action equals the calibration form. Notice also that this selects the $k_{\parallel}$ component of the pure spinor.

As it was shown in the paper [30], once the R-symmetry is broken in the type-IIB set-up, the non-abelian T-duality maps these backgrounds to their partners in Massive IIA. In a minimally SUSY quantum field theory, the presence of

[5] Note that it is the presence of an F_0 that allows D2 branes to be interpreted in this way, by way of comparison in [94] the relevant branes with Chern-Simons dynamics are D4 branes with a bulk F_2 turned on.

domain walls is tied up with confinement and the spontaneous breaking of the $\mathbb{Z}_{2N_c}$-symmetry. As we emphasized, these phenomena are related to the 'dynamical' character of the $SU(2)$-structure, hence to the presence of the $k_{||}$ part of the pure spinor.

5.5.3 The fate of the $U(1)_R$ anomaly

In the backgrounds presented in [30] and those of this chapter it is somewhat natural to expect that the coordinate ψ is singled out as being related to an R-symmetry of any putative field theory dual. That this is true is by no means obvious, after all in the technical process of dualisation the fact that we retained the coordinate ψ was purely a result of a judicious gauge choice. Here we provide evidence that this is indeed the correct identification and furthermore that this $U(1)$ is afflicted with an anomaly, breaking it down to a discrete subgroup.

A robust understanding of how ∂_ψ plays the role of the R-symmetry in the holographic dual was given in [95] with several important details of the supergravity solution clarified in [96]. The essential point of [95] is to introduce a bulk 5d gauge field that gauges this $U(1)_\psi$ by making the replacement $d\psi \to \chi = d\psi - 2A$ in the metric. This must be supplemented with an appropriate ansatz for the fluxes. In the case of the Klebanov-Witten background one finds that the resultant gauge field is massless and is the dual fluctuation to the global $U(1)_R$ of the gauge theory. However, in the non-conformal cases, the correct ansatz for the fluxes actually yields a massive gauge field (the mass here comes from a Stückelberg rather than Brout-Englert-Higgs mechanism).

Let us begin our discussion with the non-abelian T-dual of the Klebanov-Witten backgound. The NS sector of the geometry is given by

$$
\begin{aligned}
ds^2 &= ds^2_{AdS_5} + \frac{1}{6} ds^2_{S^2} + \frac{6v_2^2}{\Delta}\sigma_{\hat{3}}^2 + \\
&\quad \frac{6}{\Delta}\left[(1+27v_2^2)dv_2^2 + 54 v_2 v_3 dv_2 dv_3 + \frac{3}{4}\left(\Delta - 54v_2^2\right)dv_3^2\right] , \\
B_2 &= \frac{18\sqrt{2}}{\Delta} v_2 v_3 \sigma_{\hat{3}} \wedge dv_2 + \frac{(\Delta - 54 v_2^2)}{\sqrt{2}\Delta}\sigma_{\hat{3}} \wedge dv_3 , \\
e^{2\Phi} &= 81\Delta^{-1} = 81\left(2 + 54v_2^2 + 36v_3^2\right)^{-1} ,
\end{aligned}
$$

where $\sigma_{\hat{3}} = d\psi + \cos\theta d\phi$. This metric is supported by RR two and four form fluxes. The $U(1)$ acting as ∂_ψ can be gauged by making the replacement $\sigma_{\hat{3}} \to \tilde{\chi} = \sigma_{\hat{3}} - 2A$ in the NS sector above. The potentials corresponding to the correct modification of the RR forms that support this fluctuation are given by

$$
\begin{aligned}
C_1 &= -\frac{2\sqrt{2}}{27}\left(\cos\theta d\phi + A\right) , \\
C_3 &= -\frac{2}{27}v_3\tilde{\chi}\wedge(\tilde{\omega}_2 - dA) + \frac{2}{9}v_3 \star_5 dA ,
\end{aligned}
\tag{5.5.3}
$$

where we introduce the volume form on the S^2, $\tilde{\omega}_2 = \sin\theta d\theta d\phi$ and $\star_5$ is the Hodge dual in the AdS_5 directions. This solves the linearised equations of motions, linearised Einstein equations and Bianchi identities provided that the gauge field obeys the equation $d\star_5 dA$. This, together with the fact that the Killing spinors of the geometry are charged under $U(1)_\psi$ identifies this as the dual to the R-symmetry. Upon substitution of this ansatz into the action one finds all the gauge field dependance gives a field strength squared contribution,

$$
\delta S = f(v_2, v_3) F_{\mu\nu}F^{\mu\nu} \tag{5.5.4}
$$

for some function $f(v_2, v_3)$ of the internal coordinates that will be integrated over in a reduction to a five-dimensional theory.

Now we turn to the non-conformal geometry obtained by transformation of the Klebanov-Tseytlin geometry (since we are only interested in the UV behaviour we will not need the full Klebanov-Strassler or Baryonic branch). The NS sector, with the $U(1)_\psi$ gauged, is given by

$$
\begin{aligned}
ds^2 &= h^{\frac{1}{2}}dr^2 + h^{-\frac{1}{2}}ds^2_{R^{1,3}} + \frac{r^2h^{\frac{1}{2}}}{6}ds^2_{S^2} + \frac{6r^4hv_2^2}{\Delta}\tilde{\chi}^2 \\
&\quad + \frac{6}{\Delta}\left[(r^4h + 27v_2^2)dv_2^2 + 54v_2\mathcal{V}_3dv_2dv_3 + \frac{3}{4}\left(\frac{\Delta}{r^2h^{\frac{1}{2}}} - 54v_2^2\right)dv_3^2\right] \\
B_2 &= \frac{18\sqrt{2}}{\Delta}v_2\mathcal{V}_3\tilde{\chi}\wedge dv_2 + \frac{\left(\Delta - 54r^2h^{\frac{1}{2}}v_2^2\right)}{\sqrt{2}\Delta}\tilde{\chi}\wedge dv_3 + \frac{r^5h'(r)}{54M}\tilde{\omega}_2 \\
e^{2\Phi} &= 81\Delta^{-1} = 81\left(2r^4h + 54v_2^2 + 36\mathcal{V}_3^2\right)^{-1} .
\end{aligned}
\tag{5.5.5}
$$

Here $h(r)$ is the usual Klebanov-Tseytlin warp factor and $\mathcal{V}_3 = v_3 + \frac{r^5h'(r)}{27\sqrt{2}M}$. Without the gauging this is a solution of massive IIA with Romans mass proportional to M. By examining how the non-abelian T-duality transformation acts on the ansatz given by Krasnitz in [96], we can determine a suitable ansatz for the fluxes:

$$
\begin{aligned}
C_1 &= -\frac{M}{2}v_3\cos\theta d\phi + \frac{M}{2}\psi dv_3 - 2\sqrt{2}K_1 - \sqrt{2}C_0\left(\mathcal{V}_3 dv_3 + v_2 dv_2\right) \\
C_3 &= 2\mathcal{V}_3 K_3 - \frac{M\sqrt{2}}{4}\psi\tilde{\omega}_2\wedge(v_2dv_2 + v_3dv_3) \\
&\quad + \frac{2\sqrt{2}}{M}f(r)C_0\tilde{\omega}_2\wedge(v_2dv_2+\mathcal{V}_3dv_3) - 2v_3\tilde{\chi}\wedge dK_1 - 4v_3\tilde{\omega}_2\wedge K_1 + \Theta_3
\end{aligned}
\tag{5.5.6}
$$

The remaining term in the three-form potential is given implicitly by[6]

$$
d\Theta_3 = \frac{1}{\sqrt{2}}Mh^{\frac{1}{4}}\star_5\left(C_0dr + \frac{2}{3}rW\right) + \frac{3M}{\sqrt{2}}dr\wedge K_3\,. \tag{5.5.7}
$$

Here W is a gauge invariant 1-form that combines the gauge field A with a Stückelberg scalar scalar $W = A - d\lambda$ though for practical purposes we follow [96] and chose a gauge in which $W = A$. This is a solution to the linearised flux equations and Bianchi identities provided the fields introduced obey the constraints on the ansatz required in [96]:

$$
\begin{aligned}
K_3 &= -\frac{3}{rh^{\frac{1}{4}}}\star_5 dK_1\,, \\
dK_3 &= \frac{24}{r^3h^{\frac{3}{4}}}\star_5\left(K_1 + f(r)W\right)\,, \\
0 &= \frac{1}{3}\partial_r\left(rh^{-1}W_r\right) + \frac{r}{3}\partial_iW_i + \frac{1}{2}\partial_r(h^{-1}C_0) - \frac{36}{r^4h^2}\left((K_1)_r + f(r)W_r\right)\,, \\
0 &= \frac{1}{54}\partial_r\left(r^5\partial_rC_0\right) + \frac{r^5h}{54}\partial_i\partial_iC_0 - \frac{M^2}{2h}W_r - \frac{3M^2}{4hr}C_0\,.
\end{aligned}
\tag{5.5.8}
$$

Here $\star_5$ is the Hodge dual with respect to the metric $ds_5^2 = h^{\frac{1}{2}}dr^2 + h^{-\frac{1}{2}}ds^2_{R^{1,3}}$. In [96] it was shown how these eqs (5.5.8) can be diagonalised by defining

$$
W^1 = W - \frac{54}{hr^4}K_1\,,\quad W^2 = W + \frac{27}{hr^4}K_1\,. \tag{5.5.9}
$$

The mode W^1 corresponds to a massive gauge field whose mass as a result of the spontaneous (anomalous) breaking of R-symmetry. The mass of this mode is given by [96]:

$$
m^2 = \frac{4}{\alpha'(3\pi)^{\frac{3}{2}}}\frac{(g_sM)^2}{(\lambda N)^{\frac{3}{2}}} \tag{5.5.10}
$$

The interpretation is identical here and we conclude therefore that the $U(1)_R$ symmetry is anomalously broken.

[6]The exterior derivative of right hand side of this expression vanishes on the eqs (5.5.8).

Dependence on ψ in the potentials and D0 brane instantons

To understand this breaking as an anomaly it is informative to look at the forms of the RR potentials. For the non-Abelian T-dual of the Klebanov-Witten we have following potentials

$$
\begin{aligned}
C_1 &= \frac{N\pi}{\sqrt{2}} \cos\theta d\phi\,, \\
C_3 &= -\frac{N\pi v_3}{2} \sin\theta d\theta \wedge d\phi \wedge d\psi\,.
\end{aligned}
\tag{5.5.11}
$$

For the dual of the Klebanov-Tseytlin (which has Romans mass proportional to M) we have

$$
\begin{aligned}
C_1 &= \frac{M}{2} v_3 \cos\theta d\phi - \frac{M}{2}\psi dv_3\,, \\
C_3 &= -\frac{\sqrt{2}M}{8}\left(v_2^2 + v_3^2\right) \sin\theta d\theta \wedge d\phi \wedge d\psi\,.
\end{aligned}
\tag{5.5.12}
$$

Note how the dependence on ψ in C_1 is quite different in the potentials in the conformal and non-conformal cases.

Let us now consider D0 branes. These D0 branes will move in the v_3 direction, leaving all other coordinates fixed, in particular we will choose $v_2 = 0$. We can then calculate using eq (5.5.5) the induced metric for this D0 brane, relevant gauge potential and its BIWZ action, that will read

$$
\begin{aligned}
& ds^2_{ind} = g_{v_3 v_3} dv_3^2 = \frac{9}{2r^2 h^{1/2}} dv_3^2, \quad C_1 = -\frac{M}{2}\psi dv_3, \\
& S_{BIWZ} = -T_{D0}\int dv_3 e^{-\Phi}\sqrt{g_{v_3 v_3}} + \\
& \qquad T_{D0}\int C_1 = T_{D0}\int dv_3 \sqrt{\frac{r^2 h^{1/2}}{9} + \frac{2V_3^2}{r^2 h^{1/2}}} - T_{D0}\frac{M\psi}{2}\int dv_3.
\end{aligned}
\tag{5.5.13}
$$

We use now that $T_{D0} = \frac{1}{g_s\sqrt{\alpha'}}$. Also, we call $\sqrt{\alpha'} L_{v_3} = \int dv_3$, the dimensionless length of the v_3 direction.

We will equate the BIWZ action of this euclidean D0 brane with the gauge coupling and the Θ angle imposing that $S_{BIWZ} = \frac{8\pi^2}{g^2} + i\Theta$. In other words, we consider this D0 brane to be an instanton in the dual gauge theory.

Analysing the WZ term, we have that (like above, we choose $g_s = 1$),

$$
S_{WZ} = \frac{M}{2}\psi L_{v_3} = \Theta. \tag{5.5.14}
$$

Using that the theta angle should be periodic, we can impose that the allowed changes in the angle ψ get selected to be

$$\frac{M}{2}(\psi + \Delta\psi)L_{v_3} = \Theta + 2k\pi \tag{5.5.15}$$

which implies that

$$\Delta\psi = \frac{4k\pi}{ML_{v_3}} \,. \tag{5.5.16}$$

So, we see that there is a breaking of the global continuous symmetry into a discrete one. The residual discrete symmetry is determined by the domain of the coordinate v_3. *In the case in which we would like to impose this discrete symmetry to be the same as before the non-Abelian duality we should impose that* $L_{v_3} = 2$. Indeed, one of the major challenges with understanding non-abelian T-duality is to identify the periodicities of the coordinates of the T-dual geometry. Here we see a direct link between a field theory property (the anomaly) and the global properties of the geometry.

Let us look at the BI term. We have that the gauge coupling associated to this is

$$\frac{8\pi^2}{g^2} = T_{D0}\int dv_3 \Big[r^2 h^{1/2} + \frac{2}{r^2 h^{1/2}}(v_3 + \frac{r^5 h'}{27\sqrt{2}M})^2\Big]^{1/2}. \tag{5.5.17}$$

We can perform the integral explicitly, but it is perhaps more illuminating to look at the large radius limit of the expression above. After all, we are doing this calculation in the non-Abelian dual of the Klebanov-Tseytlin solution, we should only trust the result in the far UV. We have then, considering the leading term in the large-r expansion,

$$\frac{1}{g^2} \sim (\log r)^{3/2} \tag{5.5.18}$$

this reproduces a result obtained by other means in [30].

5.5.4 The fate of $U(1)_B$

The Klebanov-Witten $SU(N) \times SU(N)$ conformal field theory coming from D3 branes at the tip of the conifold has a $U(1)$ baryonic number symmetry acting as $A_i \to e^{i\alpha}A_i$, $B_j \to e^{-i\alpha}B_j$. In the gravity dual this number current gives rise to a massless AdS_5 gauge field

$$\delta C_4 = \omega_3 \wedge \mathcal{A}\,, \tag{5.5.19}$$

where ω_3 is the usual closed three form on $T^{1,1}$. The non-abelian T-dual of the $AdS_5 \times T^{1,1}$ geometry was obtained in [35]. In the T-dual geometry, this $U(1)_B$ mode translates into a perturbation, which solves the linearised supergravity equations of motion, given by

$$\begin{aligned} \delta C_1 &= \frac{1}{9}\mathcal{A}\,, \\ \delta C_3 &= W_2 \wedge \mathcal{A} + \frac{1}{9} u du \wedge \mathcal{F} + \frac{\sqrt{2}}{6} u dv_3 \wedge \star_4 \mathcal{F}\,. \end{aligned} \tag{5.5.20}$$

The final two terms in δC_3 come from the a contribution from δC_6 under the T-duality transformation[7]. Although the two-form W_2 has a simple form

$$W_2 = \frac{v_3}{9} d\sigma_3 + \frac{\sqrt{2} v_2 e^{2\hat{\phi}}}{81} \sigma_3 \wedge (2 v_3 dv_2 - 3 v_2 dv_3) \tag{5.5.21}$$

it can not easily be written in terms of the invariant tensors that define the $SU(2)$ structure of the geometry.

The existence of this mode is suggestive that the field theory duals corresponding to the conformal geometries constructed in [35] have a global $U(1)$ symmetry in addition to the preserved $U(1)_R$. In fact, the geometry T-dual to the Klebanov Witten is closely related to those proposed in [57] as the gravity duals to $\mathcal{N} = 1$ SCFT's formed by wrapping M5 branes on Riemann surfaces (which in this case is genus zero giving rise to many subtleties). These SCFT's do indeed have $U(1)_R \times U(1)_F$ Abelian global symmetries which are seen geometrically as isometries of the corresponding eleven-dimensional supergravity solution. Upon reduction to ten-dimension one of these U(1)'s gets de-geometrised corresponding to the above gauge field $\delta C_1 = \mathcal{A}$.

In this paper our main focus has been the cascading field theory where at the last step of the cascade when the gauge group is $SU(M) \times SU(2M)$ the baryons acquire expectation values,

$$\mathcal{B} = i\xi \Lambda^{2M}, \quad \tilde{\mathcal{B}} = \frac{i}{\xi}\Lambda^{2M}\,. \tag{5.5.22}$$

On this Baryonic Branch the $U(1)_B$ symmetry is spontaneously broken. To see this from the gravity perspective it is sufficient to work with the Klebanov-Strassler geometry corresponding to the field theory at the $\mathbb{Z}_2$ symmetric point of the Baryonic branch. As shown in [83], there is a massless glue ball corresponding to a Goldstone mode associated with changing the phase of ξ which

[7]For the $AdS_5 \times T^{1,1}$ we use $ds^2_{AdS} = du^2 + e^{2u}(\eta_{ij} dx^i dx^j)$.

is given by[8]

$$
\begin{aligned}
&\delta H = 0\,, \\
&\delta F_3 = f_1 \star_4 da - d(f_2(\tau) da \wedge g^5)\,, \\
&\delta F_5 = f_1 \left(\star_4 da - \frac{\epsilon^{\frac{4}{3}}}{6K^2(\tau)} h(\tau) da \wedge d\tau \wedge g^5 \right) \wedge B_2\,.
\end{aligned}
\tag{5.5.23}
$$

The linearised supergravity equations are solved when the pseudo-scalar is a harmonic function in $\mathbb{R}^{3,1}$ and the function $f_2(\tau)$ obeys a second order differential equation admitting a normalisable solution.

The non-abelian T-dual geometries considered also admits a similar mode, which can be obtained simply by performing a T-dualisation of the ansatz for the scalar modes in the seed IIB solutions. The T-dual of the Klebanov-Strassler geometry was obtained explicitly in [30]. Performing a dualisation of the ansatz eq (5.5.23) gives rise to a perturbation δF_2 and δF_4. This perturbation solves the supergravity equations of motion when f_2 obeys the same differential equation as for the ansatz eq (5.5.23). The expressions for F_2 and F_4 are not particularly enlightening though for completeness let us provide a few details. Here we display the results in the UV regime where the geometry is given by eq (5.5.5). The corresponding deformations to the potentials are given by

$$
\begin{aligned}
\delta C_1 &= (2v_3 f_2(r) + f_3(r)) da \\
\delta C_3 &= \left[f_4(r) - \frac{f_1}{\sqrt{2}} \left(v_2^2 + (v_3 - \frac{N\pi}{\sqrt{2}M})^2 \right) \right] \star_4 da - \\
&\frac{f_2}{\sqrt{2}} da \wedge \sigma_3 \wedge d(v_2^2 + v_3^2) - \frac{f_3}{\sqrt{2}} da \wedge \sigma_3 \wedge dv_3 + da \wedge \sin\theta d\theta \wedge d\phi \left(f_5 - \frac{v_3}{\sqrt{2}} f_3 \right)
\end{aligned}
\tag{5.5.24}
$$

The extra functions introduced above are completely determined by f_1 and f_2 according to

$$
\begin{aligned}
&f_1' = 0\,, \quad 2r^4 f_2'' = -6r^3 f_2' + 16 r^2 f_2 + 27 M^2 f_1 \log r/r_0\,, \\
&f_3' = \frac{1}{6}\left(-3\sqrt{2} r f_1 h(r) \log r/r_0 - 2T(r) f_2' \right)\,, \quad f_4' = \frac{2\sqrt{2}}{3} r f_2\,, \\
&f_5' = \frac{1}{108}\left(-2\sqrt{2} r^5 f_1 h(r) = 18 M r f_1 h(r) T(r) \log r/r_0 - 3\sqrt{2} T(r)^2 f_2' \right)\,,
\end{aligned}
\tag{5.5.25}
$$

where $T(r) = \frac{9}{\sqrt{2}} M \log r/r_0$ and $h(r) = \frac{27}{32r^4}\left(3M^2 + 8N\pi + 12M^2 \log r/r_0\right)$.

The existence of this mode suggests a spontaneously broken global $U(1)$ in

[8] Here and elsewhere we use the standard notation for the deformed conifold and Klebanov Strassler geometry which can be found e.g. in appendix of [83]. For the KS we stick with the notation τ as the radial coordinate but will use r elsewhere.

the field theories dual to the geometries obtained in section 5.3. In the conformal case, the unbroken $U(1)$ becomes geometrised upon lifting to M-theory whereas these non-conformal backgrounds are solutions of *massive* IIA and so can not be lifted. This further underlines the expectation that a $U(1)$ is broken.

In the same multiplet as the pseudo-scalar Goldstone is a scalar perturbation corresponding to changing the magnitude of ξ. In the same vein as above, one could deduce the fate of this scalar perturbation under the T-duality transformation; it will give a similar, albeit complicated, perturbation in the dual IIA background. Since the full Baryonic Branch geometry found in [18] can be thought of as exponentiating such transformations to give arbitrary values of the Baryonic VEV, implicitly in the geometries presented in section 5.3 we have already done just that.

5.5.5 The fate of the baryon condensate

In Klebanov-Witten theory the closest analogy to a baryon vertex - the object to which N *external* quarks can attach [97] - would be a D5 brane wrapping the $T^{1,1}$ space with world volume coordinates $\{x_0, \theta_1, \phi_1, \theta_2, \phi_2, \psi\}$ [98]. The primary reason for this identification follows the argument made in [97]; since we have

$$\int_{T^{1,1}} F_5 \propto N\,, \tag{5.5.26}$$

the WZ term induces a charge to the world volume $U(1)$ gauge field $\mathcal{A}$ via the coupling

$$\int_{\mathbb{R}\times T^{(1,1)}} \mathcal{A} \wedge F_5\,. \tag{5.5.27}$$

This introduces N units of charge which must be canceled by some other source to give zero net charge in a closed universe. This cancelation is achieved by N elementary strings stretching from the boundary to the brane whose end points are external quarks. A perhaps naive approach would be to suggest in the IIA geometry dual to the Klebanov-Witten theory a similar role could be played by a D2 brane wrapping the S^2 with world volume coordinates $\{x_0, \theta, \phi\}$. Indeed, since in the case of T-dual to Klebanov-Witten we have $C_1 \propto \cos\theta d\phi$ the WZ coupling $\mathcal{F} \wedge C_1$ produces a charge contribution for the gauge field that could be cancelled with external quarks just as in the Klebanov-Witten scenario. It would be of some interest to study the baryon vertex in the massive IIA backgrounds.[9]

This baryon vertex should however be distinguished from the configuration representing the actual baryon condensate - which should be supersymmetric,

[9]Before duality in the cascading theories this is a D3 brane and it seems quite possible that D0 branes might play this role of the baryon vertex in the cascading massive IIA geometries. We thank O. Aharony and J. Sonnenschein for this suggestion.

gauge invariant and not require BIon spikes. The configuration that describes the baryon condensate is a Euclidean D5 brane wrapping the $T^{1,1}$ and the radial directions [98]. This D5 has D3 branes dissolved within [99] which are traded for a world volume gauge field. Following the logic applied to the baryon vertex one might anticipate that in the IIA geometries presented here, the role of the condensate is played by a wrapped Euclidean D2 brane on the $S^2 \times \mathbb{R}$ with a world volume gauge field.

To determine the existence of such a configuration, rather than calculate the kappa symmetry projectors, we will harness the power of the G-structure and the calibration techniques of [20]. The condition for a supersymmetric Euclidean p brane on a cycle Σ is essentially the same as that of a Lorentzian $p+4$ brane that is spacetime filling in the Minkowski directions. This condition is given by

$$e^{-\phi}\sqrt{-\det(g|_\Sigma + \mathcal{F})}\, d^p\sigma = 8e^{3A-\phi} Im\Phi \wedge e^{-\mathcal{F}}|_\Sigma \tag{5.5.28}$$

where the world volume field strength is $\mathcal{F} = B|_\Sigma + 2\pi\alpha' d\mathcal{A}$ and the pure spinor entering the calibration form is given $\Phi = \Psi_+$ for IIB and $\Phi = \Psi_-$ for IIA.

Before looking at this question in the context of the full Baryonic Branch let us address it in the conformal case in which we would still anticipate a supersymmetric configuration to exist. In the Klebanov-Witten theory the E5 configuration of a brane extended along $\Sigma = \{r, \theta_1, \phi_1, \theta_2, \phi_2, \psi\}$ with a world volume gauge field

$$\mathcal{A} = \frac{1}{3}\zeta(r)\,(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)\ , \tag{5.5.29}$$

obeys the calibration condition eq (5.5.28) provided that

$$\zeta\zeta' = \frac{1}{4} - \zeta^2\,, \tag{5.5.30}$$

which of course can be readily integrated.

In the IIA non-Abelian T-dual of the Klebanov-Witten geometry we find an E2 configuration extended along $\Sigma = \{r, \theta, \phi\}$ at the point $v_2 = 0$ but with a non-trivial embedding $v_3 = f(r)$. We search for a supersymmetric configuration solving the calibration condition eq (5.5.28) when supported by a gauge field

$$\mathcal{A} = \frac{1}{\sqrt{2}}\alpha(r)\cos\theta d\phi\,. \tag{5.5.31}$$

From the calibration condition one finds firstly that the embedding $f(r)$ and the gauge field should differ only by a constant c_0. The gauge field should then obey an equation

$$\alpha'(r) = \frac{1 - 18c_0\alpha - 18\alpha^2}{9(c_0 + 2\alpha)} \tag{5.5.32}$$

which can also be readily solved and one notices that when $c_0 = 0$, this equation has the same form as eq (5.5.30) governing the configuration in IIB.

Let us move on to the KT geometry working in the exact logarithmic solution[10]. First we recapitulate the calculation for the baryon condensate in the IIB background. Using the calibration technique one readily finds the E5 configuration is the same but with the gauge field equation of motion eq (5.5.30) modified to be

$$\zeta'(r) = \frac{2r^4 h(r) + T(r)^2 - 8\zeta(r)^2}{8r\zeta(r)}\,, \tag{5.5.33}$$

where $T(r) = \frac{9}{\sqrt{2}} M \log r/r_0$ and $h(r) = \frac{27}{32r^4}\left(3M^2 + 8N\pi + 12M^2 \log r/r_0\right)$. This equation may be integrated to yield

$$\zeta(r) = \frac{9M}{8r\sqrt{2}}\left(c + 3r^2 - 4r^2\log(r) + 8r^2\log(r)^2\right)^{\frac{1}{2}}\,, \tag{5.5.34}$$

where c is a constant of integration which we now set to zero since its contributions are in any case sub-leading. Inserting this into to the DBI action one finds, changing variables to $t = \log r$,

$$S_{E5} = \tau_5 vol(T^{1,1}) \int^{t_{uv}} dt \frac{27M^3}{64\sqrt{2}}(1 + 2t^2 + 8t^3)(3 - 4t + 8t^2)^{\frac{1}{2}}\,. \tag{5.5.35}$$

In [98], $e^{-S_{E5}}$ was identified with the bulk field dual to the baryonic condensate. Using the standard asymptotic expansion the field theory scaling dimension can be extracted (at least in the large t regime) as

$$\Delta(r) = \frac{dS_{E5}}{d\log r} = \frac{27}{16\pi^2} M^3 (\log r)^2 + \mathcal{O}(\log r)\,, \tag{5.5.36}$$

reproducing exactly the result of [98] notable for the scaling dimension dependence on the energy scale of the baryons as anticipated from the field theory.

In the non-abelian T-dual the situation is already rather involved. We search for an E2 configuration extended along $\Sigma = \{r, \theta, \phi\}$ at the point $v_2 = 0$ and now with $v_3 = \chi(r)$ and an ansatz for the gauge field

$$\mathcal{A} = \frac{1}{\sqrt{2}}\tilde{\zeta}(r)\cos\theta d\phi\,. \tag{5.5.37}$$

We take the square of the calibration equation eq (5.5.28) and first consider

[10]This is considerably simpler than the deformed conifold of the KS and reproduces all the main features of the calculation in [98] with the conformal dimension of the condensate agreeing to leading order. Using the calibration technique we checked that the resultant gauge field equation of motion agrees exactly with that of [98].

terms proportional to $\cos^2\theta$. From these one finds a first equation relating the gauge field and the embedding in v_3:

$$\tilde{\zeta}'(r) = \chi'(r)\,. \tag{5.5.38}$$

We let c_0 be the additive constant between $\tilde{\zeta}$ and χ. Then from the remaining terms in eq (5.5.28) one finds a differential equation for the gauge field

$$r\tilde{\zeta}'(r) = \frac{1}{18(c_0+2\tilde{\zeta})}\left(2r^4h(r) - 6c_0T + T^2 - 36c_0\tilde{\zeta} - 36\tilde{\zeta}^2\right)\,. \tag{5.5.39}$$

Changing variable to $t = \log r$ one can solve this equation on the exact logarithmic solution:

$$\begin{aligned}\tilde{\zeta}(r) = -\frac{c_0}{2} \pm \frac{r^{-3/2}}{8}\big[&64rc + r^3(16c_0^2 + 3M(8\sqrt{2}c_0 + 9M- \\ &4(4\sqrt{2}c_0 + 3M)\log r + 24M\log r^2))\big]^{\frac{1}{2}}\end{aligned} \tag{5.5.40}$$

here c is an integration constant giving sub-leading contributions that we hence ignore.

Using eq (5.5.39) we find that the DBI action is given by

$$S_{DBI} = \kappa\int \frac{dr}{r}\frac{1}{648}\frac{2r^4h + (T+6\tilde{\zeta})^2}{c_0+2\tilde{\zeta}}\left(2r^4h + (T - 6(c_0+\tilde{\zeta}))^2\right)\,. \tag{5.5.41}$$

If we expand out asymptotically we find that

$$S_{DBI} \sim \kappa\int^{t_{UV}} dt\frac{27M^3t^2}{8\sqrt{2}} + \frac{9M^2t}{32}\left(3\sqrt{2}M - 4c_0 + 8\sqrt{2}\frac{N}{M}\pi\right) + \mathcal{O}(t^0)\,, \tag{5.5.42}$$

which suggests an operator with a scaling dimension

$$\Delta = \frac{27\kappa M^3}{8\sqrt{2}}(\log r)^2 \tag{5.5.43}$$

where $\kappa = T_{D2}\mathrm{vol}(S^2) = \frac{1}{ß}$. It would be interesting to pursue this line of reasoning further by extracting the value of the condensate across the Baryonic branch. This is technically rather involved and we do not pursue this here.

5.6 Conclusions and Future Directions

In this chapter we have examined a new family of solutions of massive IIA supergravity. These new backgrounds were obtained by performing a non-abelian T-duality on the geometry that describes the non-perturbative physics of the baryonic-branch of the Klebanov-Strassler field theory. We have explored the transition from $SU(3)$ structure, characterising the 'seed' backgrounds to the dynamical $SU(2)$-structure that describes the resulting massive IIA solutions. We made clear–at least for the type of backgrounds studied here– that the dynamical character of the $SU(2)$-structure is directly related to the phenomena of confinement and symmetry breaking. We believe that all these new features have not been discussed in previous literature, in a context as clear and unifying as the one presented here.

The new backgrounds discussed in this chapter display a host of interesting non-perturbative phenomena that 'define' the dual field theory. Some of these are,

- The non-conformality of the geometry is enabled by a non-zero Romans' mass.
- Whilst the UV geometries proposed in [30] are characterized by *static* $SU(2)$ structure [41] the full IR complete geometry of this chapter has *dynamic* $SU(2)$ structure.
- The transition to dynamic $SU(2)$ structure gives a geometric realization of confinement and permits supersymmetric D2 branes that act as domain walls in the IR. This realises geometrically the relation between confinement, the spontaneous breaking of a discrete R-symmetry and the presence of domain walls.
- The $U(1)_R$ symmetry is realized by the vector ∂_ψ and the corresponding fluctuation, which is a massless gauge field in the conformal case, acquires a mass indicating an anomalous breaking.
- Euclidean 'instantonic' branes reproduce this anomaly of the R-symmetry and at the same time suggest a non-conventional running for a suitably defined gauge coupling.
- A further $U(1)$ (baryonic) symmetry is broken. In the conformal case of [30] this symmetry is unbroken and is realized geometrically by the M-theory circle. In our backgrounds, once conformality is broken by the addition of fractional branes, the symmetry is no longer geometrical as we are now in a massive IIA context. The $U(1)_B$ symmetry is spontaneously broken and we identified a corresponding massless glueball (the associated Goldstone boson).

- We give evidence that this $U(1)_B$ may be thought of as baryonic and that a baryonic condensate is given by a Euclidean D2 brane wrapping a two-cycle in the geometry.

Although we do not yet have a complete understanding of the field theory dual to this new geometry, the results of this chapter together with those in [30] suggest that it may be a non-conformal and cascading version of the Sicilian theories of [89, 57] or the linear quivers of [90].

We would like to close this chapter on a forward looking note. We suggest that the features mentioned above may be prototypical of a wider class of holographic duals. The theories in [89, 57] and also the IIA linear quivers of [90], present a wide new class of interesting examples of $\mathcal{N} = 1$ SCFTs. We anticipate that by a modification of these theories (this chapter suggests that the modification will involve adding D8 branes in IIA) one can obtain a variety of non-conformal gauge theories. Some of the non-perturbative features of these new field theories should be the ones we are describing in this chapter.

Aside from this and on a more geometrical note, we believe the backgrounds presented in this chapter may serve as a prototype for new dynamical $SU(2)$ solutions of massive IIA supergravity that will be the corresponding string duals to the new field theories described above. This is, of course, in the same vein as the route from the conformal geometry of Klebanov-Witten to the non-conformal geometry of Klebanov-Strassler.

Chapter 6

Generating new type-IIB solutions

6.1 Introduction

This chapter is based on work performed in collaboration with Caceres and Núñez [44]. It broadens the scope of the previous chapters by generating, via non-abelian T-duality, a new type-IIB solution dual to a 4-d QFT with minimal supersymmetry.

To begin we will consider the case in which $D6$ branes wrap a calibrated three-cycle inside the deformed conifold. Extensions of this case to different numbers of dimensions, a different number of preserved supercharges, etc; have been studied. In particular, if these configurations in type-IIA string theory are lifted to eleven dimensions, the configurations become purely geometric, leading to the associated seven-dimensional spaces possessing G_2 holonomy. This line of research [100, 101, 102, 103, 104, 105, 106], was quite fertile, especially on the mathematical side where it lead to the construction of new metrics with G_2 holonomy. However, it did not give as many physically interesting result as its type-IIB counterparts [107, 12, 11]. In this work we present a family of those 'old' G_2 metrics, reduce the system to type-IIA and study some of its physical implications, making clear the reasons for which they failed to capture some of the phenomena their type-IIB counterpart were able to calculate.

It was in parallel with these 'physically motivated' discoveries that the powerful line of research involving G-structures began to grow [74, 66]. In particular, in these four dimensional and SUSY preserving examples, it is possible to encode *all* the information about the background (BPS equations, metric, fluxes, calibrated sub-manifolds, etc), in a set of forms defined on the space 'external' to the Minkowski coordinates. Furthermore, the $SU(2)$ and $SU(3)$ structures typical of these backgrounds, their associated pure spinors and forms encode in subtle ways quite common operations in QFT [16, 22, 25, 108, 27, 26, 28].

In this chapter we complement the above mentioned study of the type-IIA backgrounds associated with the wrapped D6 branes and their precise description in terms of G-structures.

We also perform an $SU(2)$ non-Abelian T-duality isometry on the geometry. We generate new type-IIB solutions that preserve four supercharges; hence it is dual to a minimally SUSY 4-d QFT . We describe the result of the non-Abelian T-duality in terms of the generated G-structure. We believe, ours is one of the first few examples of *dynamical* $SU(2)$-structure in type-IIB. We will use the word 'dynamical' to denote the fact that the quantities $k_\perp, k_\parallel$ defined in eq (6.3.20), are point dependent, changing value through out the internal manifold. We will propose a relation between the 'dynamical' character of the $SU(2)$-structure and the phenomena of confinement in the dual QFT.

The structure of this chapter is the following. In section 6.2—that contains a fair amount of review but also various original pieces, we will summarise the eleven-dimensional and type-IIA supergravity solutions that will act as the 'seed backgrounds' for our non-Abelian T-duality generating technique. Their G-structure will be carefully discussed. We will also present the explicit numerical solutions to the BPS equations and clarify their asymptotics. In section 6.3, the action of non-Abelian T-duality on the type-IIA backgrounds, the new generated solutions in type-IIB and a discussion of their G-structure will be spelled-out in detail. Different dual field theory aspects of the original and of the generated solution will be described in section 6.4. Finally, we close the paper in section 6.5 with some global remarks and propose topics to be investigated. In appendix G gives details of the delicate numerical study performed on this solution and complements the presentation.

6.2 Presentation of the Background.

We will start with the pure metric configuration in eleven-dimensions found in [104], [102]. We consider the family called $\mathcal{D}_7$. The notation we will adopt is that of [102]. We will have two sets of left invariant forms of $SU(2)$,

$$\begin{aligned}
\sigma_1 &= \cos\psi_1\, d\theta + \sin\psi_1\, \sin\theta\, d\phi, & \Sigma_1 &= \cos\psi_2\, d\tilde{\theta} + \sin\psi_2\, \sin\tilde{\theta}\, d\tilde{\phi} \\
\sigma_2 &= -\sin\psi_1\, d\theta + \cos\psi_1\, \sin\theta\, d\phi, & \Sigma_2 &= -\sin\psi_2\, d\tilde{\theta} + \cos\psi_2\, \sin\tilde{\theta}\, d\tilde{\phi} \\
\sigma_3 &= d\psi_1 + \cos\theta\, d\phi, & \Sigma_3 &= d\psi_2 + \cos\tilde{\theta}\, d\psi_2
\end{aligned} \tag{6.2.1}$$

which satisfy the $SU(2)$ algebras

$$d\sigma_1 = -\sigma_2 \wedge \sigma_3 \ + \text{cyclic perms.}, \quad d\Sigma_1 = -\Sigma_2 \wedge \Sigma_3 \ + \text{cyclic perms.} \tag{6.2.2}$$

The eleven dimensional metric is of the form $ds^2_{11} = dx^2_{1,3} + ds^2_7$, with

$$\begin{aligned} ds^2_7 =& dr^2 + a^2\left[(\Sigma_1 + g\,\sigma_1)^2 + (\Sigma_2 + g\,\sigma_2)^2\right] \\ &+ b^2\left(\sigma_1^2 + \sigma_2^2\right) + c^2(\Sigma_3 + g_3\,\sigma_3)^2 + f^2\sigma_3^2\,, \end{aligned} \tag{6.2.3}$$

where a, b, c, f, g and g_3 are functions only of the radial variable r. The six functions are not all independent, the relations

$$g(r) = \frac{-a(r)f(r)}{2b(r)c(r)}, \quad g_3(r) = -1 + 2g(r)^2. \tag{6.2.4}$$

are necessary for the BPS system

$$\begin{aligned} \dot{a} &= -\frac{c}{2a} + \frac{a^5 f^2}{8b^4c^3}, \qquad & \dot{b} &= -\frac{c}{2b} - \frac{a^2(a^2 - 3c^2)f^2}{8b^3c^3}, \\ \dot{c} &= -1 + \frac{c^2}{2a^2} + \frac{c^2}{2b^2} - \frac{3a^2f^2}{8b^4}, \qquad & \dot{f} &= -\frac{a^4f^3}{4b^4c^3}, \end{aligned} \tag{6.2.5}$$

to satisfy the equations of motion. We have checked that these equations imply that the eleven dimensional metric satisfies $R_{\mu\nu} = 0$.

6.2.1 The Type-IIA Version

For our purposes, we need the type-IIA version of the configuration presented above and we need to pick a $U(1)$ isometry to reduce on. The relevant $U(1)$ isometry is generated by the Killing vector $\partial_{\psi_1}\partial_{\psi_2}$. Having this in mind we rewrite the metric in a way which makes the isometry manifest,

$$\begin{aligned} ds^2_{11} =& dx^2_{1,3} + dr^2 + b^2\left[(\sigma_1)^2 + (\sigma_2)^2\right] + a^2\left[(\Sigma_1 + g\sigma_1)^2 + (\Sigma_2 + g\sigma_2)^2\right] \\ &+ \frac{f^2c^2}{f^2 + (1+g_3)^2c^2}(\sigma_3 - \Sigma_3)^2 \\ &+ \frac{1}{4}\left[f^2 + (1+g_3)^2c^2\right]\left[\sigma_3 + \Sigma_3 + \frac{f^2 - c^2(1-g_3^2)}{f^2 + (1+g_3)^2c^2}(\sigma_3 - \Sigma_3)\right]^2 \end{aligned} \tag{6.2.6}$$

Note that in this metric nothing depends on the combination $(\psi_2 + \psi_1)$. Now Kaluza-Klein reduction simply amounts to dropping the last line in eq (6.2.6) which has been written as a complete square for that purpose. In particular we can now read off the dilaton and the RR one-form gauge field,

$$
\begin{aligned}
e^{\phi} =& 2^{-3/2}\left[f^2+(1+g_3)^2c^2\right]^{3/4}, \\
A_1 =& \frac{f^2-c^2(1-g_3^2)}{f^2+(1+g_3)^2c^2}\left(\sigma_3-\Sigma_3\right)+\cos\theta d\phi+\cos\tilde{\theta}d\tilde{\phi}\,.
\end{aligned}
\tag{6.2.7}
$$

The ten-dimensional metric in string frame is given by

$$
\begin{aligned}
ds^2_{IIA} \;=\; & \frac{1}{2}\Big\{dx^2_{1,3}+b^2\left[(\sigma_1)^2+(\sigma_2)^2\right]+a^2\left[(\Sigma_1+g\sigma_1)^2+(\Sigma_2+g\sigma_2)^2\right] \\
& +\frac{f^2c^2}{f^2+(1+g_3)^2c^2}\left(\sigma_3-\Sigma_3\right)^2+dr^2\Big\}\times\left[f^2+(1+g_3)^2c^2\right]^{1/2}
\end{aligned}
\tag{6.2.8}
$$

Notice that the metric depends explicitly on $\psi=\psi_2-\psi_1$ and not on the coordinate on which we reduced, $\psi_+=\psi_2+\psi_1$. It is then advantageous to introduce a third set of one-forms:

$$
\begin{aligned}
&\tilde{\omega}_1=\cos\psi\, d\tilde{\theta}+\sin\psi\,\sin\tilde{\theta}\,d\tilde{\varphi},\quad \tilde{\omega}_2=-\sin\psi\, d\tilde{\theta}+\cos\psi\,\sin\tilde{\theta}\,d\tilde{\varphi}, \\
&\tilde{\omega}_3=d\psi+\cos\tilde{\theta}d\tilde{\varphi}.
\end{aligned}
\tag{6.2.9}
$$

Upon rescaling the Minkowski part of the space by a constant μ and reinstating the factors of α', g_s, the full metric, dilaton and RR field strength are [1],

$$
\begin{aligned}
ds^2_{IIA,st} =& \alpha' g_s N e^{2A}\Big[\frac{\mu}{\alpha' g_s N}dx^2_{1,3}+dr^2+b^2(d\theta^2+\sin^2\theta d\varphi^2)+ \\
& a^2(\tilde{\omega}^1+gd\theta)^2+a^2(\tilde{\omega}^2+g\sin\theta d\varphi)^2+h^2(\tilde{\omega}^3-\cos\theta d\varphi)^2\Big]
\end{aligned}
\tag{6.2.10}
$$

$$
h^2=\frac{c^2f^2}{f^2+c^2(1+g_3)^2},\quad e^{4/3\phi}=\frac{c^2f^2}{4(g_sN)^{2/3}h^2},\quad e^{4A}=\frac{c^2f^2}{4h^2}
$$

$$
\frac{F_2}{\sqrt{\alpha'}g_sN}=-(1+K)\sin\theta d\theta\wedge d\varphi+(K-1)\tilde{\omega}^1\wedge\tilde{\omega}^2-K'dr\wedge(\tilde{\omega}^3-\cos\theta d\varphi).
$$

where

$$
K(r)=\frac{f^2-c^2(1-g_3^2)}{f^2+c^2(1+g_3)^2}.
$$

---

[1] One can send $ds^2_6\to A_1 ds^2_6$, $F_2\to A_2F_2$, $e^{-4\phi/3}\to A_3e^{-4\phi/3}$ and still have a solution of IIA supergravity, preserving $\mathcal{N}=1$ SUSY provided $A_1^2A_3^3=A_2^4$. We choose $A_1=\alpha' g_sN$, $A_2=\sqrt{\alpha'}g_sN$ and $A_3=(g_sN)^{2/3}$, so that the dilaton is independent of α'. The parameter μ is just a scaling the $R^{1,3}$ coordinates.

Note that F_2 contains two components with no 'legs' on the radial coordinate r:

$$F_2\Big|_{r=r_0} = -Ng_s\sqrt{\alpha'}\Big[(K+1)\sin\theta d\theta\wedge d\varphi - (K-1)\sin\tilde{\theta}d\tilde{\theta}\wedge d\tilde{\varphi}\Big]. \quad (6.2.11)$$

Thus, we only have flux quantisation on cycles for which the $K(r)$ parts mutually cancel. For example on $\Sigma_2 = [\tilde{\theta} = \theta, \tilde{\varphi} = \varphi]$, ψ =constant, which is a SUSY cycle in the IR, we have

$$F_2\Big|_{\Sigma_2} = -2g_sN\sqrt{\alpha'}\sin\theta d\theta\wedge d\varphi. \quad (6.2.12)$$

As we will see below, under the non-abelian T-duality, these two terms in F_2 will not be mapped to the same dual flux. We require that the flux on Σ_2 is quantised in the usual fashion

$$-\int F_2 = 2\kappa_{10}T_6N. \quad (6.2.13)$$

To achieve this we use,

$$T_p = \frac{1}{(2\pi)^p\alpha'^{\frac{p+1}{2}}g_s}, \quad 2\kappa_{10}^2 = 4(2\pi)^7\alpha'^4g_s^2. \quad (6.2.14)$$

So that we may associate the charge of the D6 branes N with an $SU(N)$ gauge group in the dual QFT.

6.2.2 G-Structures: from G_2 to $SU(3)$

We derive the G-structures and SUSY conditions at each step going from M-theory to type-IIA. For clarity in presentation, in this section $g_s = \alpha' = N = 1$.

As is shown in [102], the M-theory background obeys the condition of G_2 holonomy. Hence, following [102], but in notation suggestive of dimensional reduction, we introduce a set of vielbeins for the 7d internal space as defined in eq (6.2.6),

$$\begin{aligned} &\hat{e}^r = dr, \quad \hat{e}^\theta = b\sigma_1, \quad \hat{e}^\varphi = b\sigma_1, \quad \hat{e}^z = e^{2\phi/3}(dz + A_1) \\ &\hat{e}^1 = a(\Sigma_1 + g\sigma_1), \quad \hat{e}^2 = a(\Sigma_2 + g\sigma_2), \quad \hat{e}^3 = h(\Sigma_3 - \sigma_3). \end{aligned} \quad (6.2.15)$$

Here we have used the definitions introduced in previous sections (the reason for the cluttered by tildes definition will become clear shortly). The following three-form can be constructed from the projections on the SUSY spinor, needed to derive the BPS system [102],

$$\begin{aligned}\tilde{\Phi}_3 =& \hat{e}^r \wedge (\hat{\hat{e}}^{1\theta} + \hat{\hat{e}}^{2\varphi} + e^{3z}) + (\hat{\hat{e}}^{12} - \hat{\hat{e}}^{\theta\varphi}) \wedge (\alpha\hat{e}^3 + \beta\hat{e}^z) \\ &+ (\hat{\hat{e}}^{1\varphi} - \hat{\hat{e}}^{2\theta}) \wedge (\alpha\hat{e}^3 - \beta\hat{e}^z)\end{aligned} \tag{6.2.16}$$

where

$$\alpha(r) = \frac{ag}{\sqrt{b^2 + a^2 g^2}}, \quad \beta(r) = \frac{b}{\sqrt{b^2 + a^2 g^2}}, \quad \alpha^2 + \beta^2 = 1. \tag{6.2.17}$$

It is then simple to show that the three-form obeys

$$d\tilde{\Phi}_3 = 0, \quad d \star_7 \tilde{\Phi}_3 = 0, \tag{6.2.18}$$

once the BPS eqs (6.2.5) are imposed. We would now like to dimensionally reduce the G_2 SUSY conditions to find the corresponding conditions in type-IIA. Fortunately, this was done in full generality in [109] and in a rather similar scenario in [110]. The corresponding conditions are those of an $SU(3)$-structure. All one needs is to convert eq (6.2.16) to Scherk-Schwarz gauge then follow the prescription of [109]. This is achieved through a rotation in both the $\hat{\hat{e}}^\theta, \hat{\hat{e}}^\varphi$ and $\hat{\hat{e}}^1, \hat{\hat{e}}^1$ planes such that:

$$\begin{aligned}\hat{e}^\theta &= \cos\psi \hat{\hat{e}}^\theta - \sin\psi \hat{\hat{e}}^\varphi = b d\theta \\ \hat{e}^\varphi &= \sin\psi \hat{\hat{e}}^\theta + \cos\psi \hat{\hat{e}}^\varphi = b \sin\theta d\varphi \\ \hat{e}^1 &= \cos\psi \hat{\hat{e}}^1 - \sin\psi \hat{\hat{e}}^2 = a(\omega^1 + g d\theta) \\ \hat{e}^2 &= \sin\psi \hat{\hat{e}}^1 + \cos\psi \hat{\hat{e}}^2 = a(\omega^2 + g \sin\theta d\varphi).\end{aligned} \tag{6.2.19}$$

The corresponding three-form, Φ_3 is the same as eq (6.2.16) with $\hat{\hat{e}} \to \hat{e}$ and is obviously still both closed and co-closed. The vielbeins of the new 6-d internal space can be neatly expressed as

$$\begin{aligned}&e^r = e^{\Phi/3} dr, \quad e^\theta = e^{\Phi/3} b d\theta, \quad e^\varphi = e^{\Phi/3} b \sin\theta d\varphi \\ &e^1 = e^{\Phi/3} a(\tilde{\omega}^1 + g d\theta), \quad e^2 = e^{\Phi/3} a(\tilde{\omega}^2 + g \sin\theta d\varphi), \quad e^3 = e^{\Phi/3} h(\tilde{\omega}^3 - \cos\theta d\varphi),\end{aligned} \tag{6.2.20}$$

while the 11-D vielbeins are of the form $\hat{e}^A = (e^a, \hat{e}^z)$. The $SU(3)$ structure is then given in terms of the 3-form by:

$$J_{ab} = \Phi_{abz}, \qquad (\Omega_{hol})_{abc} = \Phi_{abc} - i(\star_6 \Phi)_{abc}. \tag{6.2.21}$$

which amounts in this case to

$$
\begin{aligned}
J = & \quad -e^{3r} + (\alpha e^{2} + \beta e^{\varphi}) \wedge e^{\theta} + e^{1} \wedge (-\alpha e^{\varphi} + \beta e^{2}) \\
\Omega_{hol} = & \quad (-e^{3} + ie^{r}) \wedge ((\alpha e^{2} + \beta e^{\varphi}) + ie^{\theta}) \wedge (e^{1} + i(-\alpha e^{\varphi} + \beta e^{2})).
\end{aligned}
\tag{6.2.22}
$$

These can be used to construct two pure-spinors,

$$
\Psi_{+} = \frac{e^{A}}{8} e^{-iJ}, \quad \Psi_{-} = \frac{e^{A}}{8} \Omega_{hol}, \tag{6.2.23}
$$

that can be shown to satisfy the pure spinors SUSY conditions

$$
\begin{aligned}
& d(e^{2A-\phi}\Psi_{+}) = 0 \\
& d(e^{2A-\phi}\Psi_{-}) = e^{2A-\phi} dA \wedge \bar{\Psi}_{-} + i\tfrac{e^{3A}}{8} \star_{6} F_{2},
\end{aligned}
\tag{6.2.24}
$$

which, collecting forms of equal size, gives

$$
\begin{aligned}
& dJ = 0 \\
& d(e^{3A-\phi}) = 0 \\
& d(e^{2A-\phi} Re\Omega_{hol}) = 0 \\
& d(e^{4A-\phi} Im\Omega_{hol}) - e^{4A} \star_{6} F_{2} = 0
\end{aligned}
\tag{6.2.25}
$$

these relations are all satisfied once eqs (6.2.5) are taken into account. We will choose $3A = \phi$. Also, notice that $F_4 = 0$ for backgrounds of $SU(3)$-structure.

Potential and Calibrations.

It is useful to derive an expression for the seven form C_7 that acts as a potential for F_8, i.e. $F_8 = \star F_2 = dC_7$. One finds,

$$
C_7 = e^{4A-\phi} vol_4 \wedge Im\Omega_{hol}. \tag{6.2.26}
$$

The calibration form of space-time filling D branes is given by [66],

$$
\Psi_{cal} = -8e^{3A-\phi} \, (Im\Psi_{-}) = -e^{4A-\phi} Im\Omega_{hol}. \tag{6.2.27}
$$

Clearly we have $vol_4 \wedge \Psi_{cal} + C_7 = 0$ so any space-time filling D6 brane wrapping a 3-cycle Σ^3 such that the calibration condition

$$e^{4A-\phi}\sqrt{detG_{\Sigma^3}} = e^{4A-\phi} Im\Omega_{hol}\Big|_{\Sigma^3} \tag{6.2.28}$$

is satisfied will be SUSY[2]. The same condition must be satisfied for any odd cycle and so the only non vanishing odd cycles are 3-cycles (if B_2 were turned on we could also have 5-cycles). A similar calculation shows that potential even SUSY cycles are Σ^2, Σ^6 such that (these are calibrated by $Im\Psi_+$),

$$\sqrt{detG_{\Sigma^2}} = J\Big|_{\Sigma^2}, \qquad \sqrt{detG_{\Sigma^2}} = -\frac{1}{6}J \wedge J \wedge J\Big|_{\Sigma^6}. \tag{6.2.29}$$

All the information above, only relies on the backgrounds in eq (6.2.3), (6.2.10) and their BPS equations (6.2.5). We will now describe some solutions to this system of first order, ordinary and non-linear equations.

6.2.3 Explicit Solutions

Let us first describe a couple of known exact solutions. There is a simple solution to eqs (6.2.5) given by,

$$a(r) = c(r) = -\frac{r}{3}, \quad b(r) = f(r) = \frac{r}{2\sqrt{3}}. \tag{6.2.30}$$

This solution corresponds to a $R^{1,4} \times \mathcal{M}_7$ space with metric eq (6.2.3),

$$\begin{aligned} ds_{11}^2 =& dx_{1,3}^2 + dr^2 + \frac{r^2}{9}\left[(\Sigma_1 - \frac{1}{2}\sigma_1)^2 + (\Sigma_2 - \frac{1}{2}\sigma_2)^2\right] \\ &+ \frac{r^2}{12}(\sigma_1^2 + \sigma_2^2) + \frac{r^2}{9}(\Sigma_3 - \frac{1}{2}\sigma_3)^2 + \frac{r^2}{12}\sigma_3^2. \end{aligned} \tag{6.2.31}$$

When reduced to ten dimensions the resulting IIA dilaton behaves as $e^{4\phi/3} \sim r^2$. This solution present a singularity at $r = 0$ and the need to lift this background to M-theory for large values of the radial coordinate, to avoid strong coupling in IIA. This solution is the 'unresolved' version of the one written in—for example– eqs.(3.16)-(3.17) of [111]. In that case, we will have

$$dr = \frac{d\rho}{\sqrt{1 - \frac{a^3}{\rho^3}}}, \quad b^2 = f^2 = \frac{\rho^2}{12}, \quad a^2 = c^2 = \frac{\rho^2}{9}(1 - \frac{a^3}{\rho^3}). \tag{6.2.32}$$

[2]We work in conventions where the DBI and WZ actions have a relative sign difference

This solution avoids the singularity by ending the space at $\rho = a$. Still, the behaviour of the dilaton is such that it the type-IIA description is strongly coupled for large values of the radial coordinate r. To avoid this last issue and to have a background fully contained in type-IIA, we will describe new solutions that are both non-singular and with bounded dilaton. These new solutions, turn out to not be known in exact form, but semi-analytically, that is as series expansions for large and small values of r, complemented with a careful numerical interpolation. We will study them below.

6.2.4 Semi-analytical solutions.

Since our goal is to work with backgrounds in type-IIA in which we can trust a holographic description, we will be mostly interested in solutions with bounded dilaton and everywhere finite Ricci and Riemann invariants. The asymptotic large radius $r \to \infty$, form of these solutions is,

$$
\begin{aligned}
a(r) =& \frac{r}{\sqrt{6}} - \frac{\sqrt{3}q_1 R_1}{\sqrt{2}} + \frac{21\sqrt{3}{R_1}^2}{\sqrt{2}\,16r} + \frac{63\sqrt{3}q_1{R_1}^3}{\sqrt{2}\,16r^2} + \\
& \frac{9\sqrt{3}\left(672{q_1}^2+221\right){R_1}^4}{\sqrt{2}\,512r^3} + \frac{81\sqrt{3}q_1\left(224{q_1}^2+221\right){R_1}^5}{\sqrt{2}\,512r^4} + \\
& \frac{\sqrt{3}\left(2048h_1 + 1377\left(768{q_1}^4+1632{q_1}^2+137\right){R_1}^6\right)}{\sqrt{2}\,8192r^5} + \dots \\
b(r) =& \frac{r}{\sqrt{6}} - \frac{\sqrt{3}q_1 R_1}{\sqrt{2}} - \frac{3\sqrt{3}R_1^2}{4\sqrt{2}r} - \frac{9\sqrt{3}q_1R_1^3}{4\sqrt{2}r^2} - \\
& \frac{9\sqrt{3}(37+96q_1^2)R1^4}{128\sqrt{2}r^3} - \frac{81\sqrt{3}q_1(37+32q_1^2)R_1^5}{128\sqrt{2}\,r^4} + \\
& \frac{\sqrt{3}(512h_1 - 81(133+1920q_1^2+960q_1^4)R_1^6)}{2048\sqrt{2}\,r^5} + \dots \\
c(r) =& -\frac{r}{3} + q_1R_1 - \frac{9R_1^2}{8r} - \frac{27q_1R_1^3}{8r^2} - \\
& \frac{9(17+36q_1^2)R_1^4}{32r^3} - \frac{81q_1(17+12q_1^2)R_1^5}{32r^4} + \frac{h_1}{r^5} + \dots \\
f(r) =& R_1 - \frac{27{R_1}^3}{8r^2} - \frac{81q_1{R_1}^4}{4r^3} - \frac{243{R_1}^5\left(12{q_1}^2+1\right)}{32r^4} - \\
& \frac{729{R_1}^6\left(4{q_1}^3+q_1\right)}{8r^5} + \dots
\end{aligned} \tag{6.2.33}
$$

where q_1, R_1 and h_1 are constants.

Close to $r \to 0$ one has

$$\begin{aligned}
a(r) &= \frac{r}{2} - \frac{(q_0^2+2)r^3}{288R_0^2} - \frac{(-74-29q_0^2+31q_0^4)r^5}{69120R_0^4} + \cdots , \\
b(r) &= R_0 - \frac{(q_0^2-2)r^2}{16R_0} - \frac{(13-21q_0^2+11q_0^4)r^4}{1152R_0^3} + \cdots , \\
c(r) &= -\frac{r}{2} - \frac{(5q_0^2-8)r^3}{288R_0^2} - \frac{(232-353q_0^2+157q_0^4)r^5}{34560R_0^4} + \cdots , \\
f(r) &= q_0R_0 + \frac{q_0^3r^2}{16R_0} + \frac{q_0^3(-14+11q_0^2)r^4}{1152R_0^3} + \cdots , \\
g(r) &= \frac{q_0}{2} + \frac{q_0(q_0^2-1)}{24R_0^2}r^2 + \cdots , \\
g_3(r) &= \frac{q_0^2-2}{2} + \frac{q_0^2(q_0^2-1)}{12R_0^2}r^2 + \cdots .
\end{aligned} \tag{6.2.34}$$

Note that $a(r)$ and $c(r)$ collapse in the IR and the other two functions do not. The constants q_0 and R_0 determine the IR behaviour. Similarly, q_1, R_1 and h_1 are the UV parameters. Not for every set of q_0, R_0, q_1, R_1, h_1 there will exist a solution that interpolates between eqs (6.2.33) and (6.2.34). For example, as seen in Figure 6.1, if we numerically integrate forward from the IR, not every value of R_0, q_0 leads to a stabilized dilaton. Similarly if we integrate back from the UV using eq (6.2.33) as boundary conditions we do not necessarily get to an IR like that in eq (6.2.34). Nevertheless, it is possible to show numerically that solutions interpolating between the behaviour of eqs (6.2.33) and (6.2.34) do exist.

In Figure 6.2 we present representatives of such solutions. To obtain these numerical solutions we shoot from the IR and minimize the mismatch between this forward integrated solution and the required UV behaviour. This minimization procedure determines the UV parameters (see appendix G for more details). Also, we have defined some other functions in terms of the above, their expansions read, for $r \to \infty$

$$\begin{aligned}
e^{4A} &= (g_sN)^{3/2}e^{4\phi/3} \\
e^{4A} &= \tfrac{R_1^2}{4} - \tfrac{9R_1^4}{8r^2} + \ldots \\
h^2 &= \tfrac{r^2}{9} - \tfrac{2}{3}r\,(q_1R_1) + \tfrac{1}{2}\left(2q_1^2+1\right)R_1^2 + \ldots \\
K &= \tfrac{6561R_1^4}{64r^4} + \ldots \\
g &= \tfrac{3R_1}{2r} + \tfrac{9q_1R_1^2}{2r^2} + \tfrac{27\left(16q_1^2-1\right)R_1^3}{32r^3} + \ldots \\
g_3 &= -1 + \tfrac{9R_1^2}{2r^2} + \tfrac{27q_1R_1^3}{r^3} + \tfrac{81\left(24q_1^2-1\right)R_1^4}{16r^4} + \ldots
\end{aligned} \tag{6.2.35}$$

and for $r \to 0$, we have

$$\begin{aligned}
e^{4A} =& \ \tfrac{1}{4} q_0^2 R_0^2 + \tfrac{3}{64} q_0^4 r^2 + \frac{q_0^4(37q_0^2-40)r^4}{3072R_0^2} + \ldots \\
h^2 =& \ \tfrac{r^2}{4} + \frac{(q_0^2-16)r^4}{576R_0^2} + \ldots \\
K =& \ 1 - \tfrac{r^2}{4R_0^2} + \frac{(40-7q_0^2)r^4}{576R_0^4} + \ldots \\
g =& \ \tfrac{q_0}{2} + \frac{q_0(q_0^2-1)r^2}{24R_0^2} + \frac{q_0(91q_0^4-179q_0^2+88)r^4}{13824R_0^4} + \ldots \\
g_3 =& \ \tfrac{1}{2}\left(q_0^2-2\right) + \frac{q_0^2(q_0^2-1)r^2}{12R_0^2} + \frac{q_0^2(115q_0^4-227q_0^2+112)r^4}{6912R_0^4} + \ldots
\end{aligned} \tag{6.2.36}$$

The numerical solutions presented in Figures 6.2 satisfy $R_0 q_0 = 2$. This corresponds to choosing the normalization of the dilaton such that $(g_s N)^{3/2} e^{4\phi_0/3} = 1$, where ϕ_0 is the value of the dilaton at $r = 0$. Also, since we want solutions with monotonically increasing dilaton, we require comparing eq (6.2.35) with eq (6.2.36), that $R_1^2 > q_0^2 R_0^2$.

Asymptotic behaviour

After reducing to ten dimensions the simple exact solution mentioned above leads to a background with metric easily obtained from eq (6.2.31), dilaton $e^{\frac{4\phi}{3}} = \frac{r^2}{36(g_s N)^{2/3}}$ and $F_2 = -\sqrt{\alpha'} g_s N(\sin\theta d\theta \wedge d\varphi + \tilde{\omega}^1 \wedge \tilde{\omega}^2)$. Notice that the space is not asymptotically $T^{1,1}$ for the exact solutions. On the other hand, the numerical solutions with stabilized dilaton behave in the UV as,

$$\begin{aligned}
ds^2_{IIA,st} = \alpha' g_s N \frac{R_1}{2} \Bigg[& \frac{\mu dx_{1,3}^2}{\alpha' g_s N} + dr^2 + \\
& r^2 \Big(\frac{1}{6}(d\theta^2 + \sin^2\theta d\varphi^2 + (\tilde{\omega}^1)^2 + (\tilde{\omega}^2)^2) + \frac{1}{9}(\tilde{\omega}^3 - \cos\theta d\varphi)^2 \Big) + \ldots \Bigg]
\end{aligned} \tag{6.2.37}$$

with

$$\begin{aligned}
(g_s N)^{2/3} e^{\frac{4\phi}{3}} &= \frac{R_1^2}{4} - \frac{9R_1^4}{8r^2} - \frac{27 q_1 R_1^5}{4r^3} \cdots \\
F_2 &= -\sqrt{\alpha'} g_s N[(1 - \frac{81R_1^2}{8r^2} + \ldots) \sin\theta d\theta \wedge d\varphi + \\
& (1 + \frac{81R_1^2}{8r^2} + \ldots)\tilde{\omega}^1 \wedge \tilde{\omega}^2 - (\frac{81R_1^2}{4r^3} + \ldots) dr \wedge (\tilde{\omega}^3 - \cos\theta d\varphi)] \\
& \sim -\sqrt{\alpha'} g_s N(\sin\theta d\theta \wedge d\varphi + \tilde{\omega}^1 \wedge \tilde{\omega}^2)
\end{aligned} \tag{6.2.38}$$

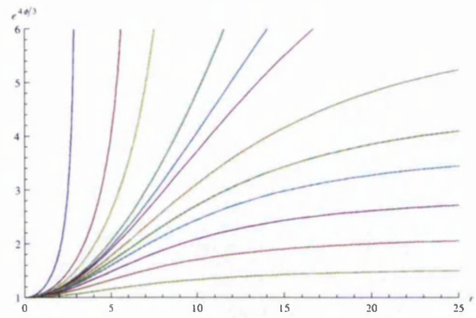

Figure 6.1: $e^{\frac{4\phi}{3}}$ for different values of q_0 and R_0. We keep $q_0 R_0 = 2$ fixed which amounts to fixing the normalization of the dilaton in the IR.

In the UV the five dimensional internal space is $T^{1,1}$. Thus, the space is asymptotically $R^{1,3} \times CY_6$ with a constant dilaton and constant F_2. This 'flat space' asymptotics is characteristic of duals to QFTs whose UV behaviour is controlled by an irrelevant operator—this will come back when dealing with the QFT analysis. Somehow the field theory is taken out of the 'decoupling limit'. On the other hand, in the IR the metric, dilaton and RR form asymptote to,

$$
\begin{aligned}
ds^2_{IIA,str} &= \frac{q_0 R_0 \alpha' g_s N}{2}\left[\mu dx^2_{1,3} + dr^2 + R_0^2(d\theta^2 + \sin^2\theta d\varphi^2) + \frac{r^2}{4} d\Omega_3\right] + \ldots \\
d\Omega_3 &= (\tilde{\omega}_1 + d\theta)^2 + (\tilde{\omega}_2 + \sin\theta d\varphi)^2 + (\tilde{\omega}_3 - \cos\theta d\varphi)^2, \\
(g_s N)^{2/3} e^{4\phi/3} &= \frac{q_0^2 R_0^2}{4} + \frac{3 q_0^4}{64} r^2 + \ldots \\
F_2 &= -2\sqrt{\alpha'} g_s N \sin\theta d\theta \wedge d\varphi + \ldots
\end{aligned} \tag{6.2.39}
$$

The material discussed in this section is not all original; we have rewritten some of it to ease the analysis of the next section. However, we should point out that the semi-analytic solutions with stabilized dilaton and no singularities (though have been discussed in [104] and [102]) are found explicitly. The explicit delicate numerics are delt with here with further details in appendix G . These solutions will play an important role in the next sections.

6.3 Non-Abelian T-duality.

In this section, we will present completely original material. We will construct a *new* solution in type-IIB supergravity preserving four supercharges. Imposing smoothness on this new solution will restrict the range of the new coordinates. This background will have $SU(2)$-dynamical structure. We believe this type of solution is new in the literature. The technique we will use to construct this

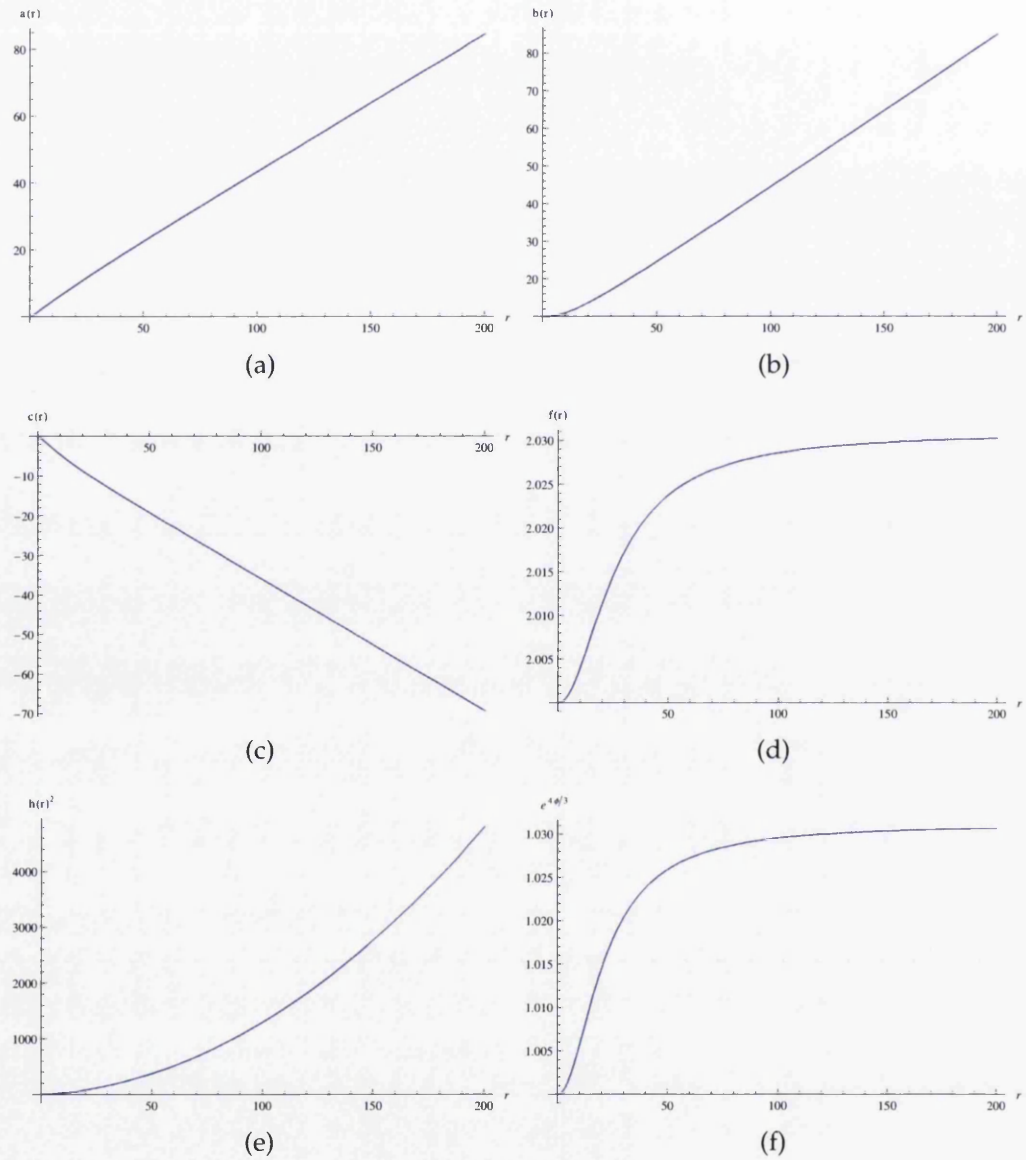

Figure 6.2: A numerical solution for $a(r), b(r), c(r)$ and $f(r)$ obtained by forward integration of the BPS equations with eq (6.2.34) as boundary conditions, $R_0 = 10$, and $q_0 = 1/5$. After the minimization procedure explained in the appendix G we find that for the UV parameters $q_1 = 1.31946$, $R_1 = -2.03087$, $h_1 = 1.9733$ this solution has the required UV behaviour eq (6.2.33). We also plot $h(r)^2$ and $e^{4\phi/3}$ defined in eq (6.2.10)

new background is once more non-Abelian T-duality, detailed in chapter 2

We will straightforwardly present the new background in type-IIB supergravity. Following the conventions of Section 2 of the paper [30] and starting from the background in eq (6.2.10) we perform a non-Abelian T-duality transformation on the $SU(2)$ isometry parametrised by (θ, φ, ψ) and gauge fix such that $\theta = \varphi = v_1 = 0$, so that the solution generated still depends on the angles $(\tilde{\theta}, \tilde{\varphi}, \psi)$ and on the new coordinates (v_2, v_3). We remind the reader that $\tilde{\omega}^i$,

$$\begin{aligned}&\tilde{\omega}_1 = \cos\psi\, d\tilde{\theta} + \sin\psi\, \sin\tilde{\theta}\, d\tilde{\varphi}, \quad \tilde{\omega}_2 = -\sin\psi\, d\tilde{\theta} + \cos\psi\, \sin\tilde{\theta}\, d\tilde{\varphi},\\ &\tilde{\omega}_3 = d\psi + \cos\tilde{\theta} d\tilde{\varphi}.\end{aligned} \tag{6.3.1}$$

In the process of doing this non-Abelian T-duality, we generate an entirely new NS and RR sector and type-IIB metric. The T-dual metric is given by (we take $g_s = \alpha' = \mu = 1$)

$$\begin{aligned}ds^2_{IIB,st} =& e^{2A}\left[dx^2_{1,3} + Ndr^2 + N\hat{a}^2(d\tilde{\theta}^2 + \sin^2\tilde{\theta} d\tilde{\varphi}^2)\right] +\\ &\frac{1}{\det M}\left[2(v_3 dv_2 + v_3 dv_3)^2 + 4N^2 e^{4A}\hat{b}\Big(\hat{b}^2(dv_3 + \hat{c}v_2\tilde{\omega}_2)^2 + \right.\\ &\left. h^2(\hat{c}^2 v_3^2\tilde{\omega}_1^2 + (dv_2 - \hat{c}v_3\tilde{\omega}_2)^2) + 2\hat{c}v_2 v_3\tilde{\omega}_1\tilde{\omega}_3 + v_2^2\tilde{\omega}_3)\Big)\right]\end{aligned} \tag{6.3.2}$$

where

$$\det M = 4e^{2A}N\left(2e^{4A}N^2\hat{b}^4h^2 + \hat{b}^2v_2^2 + h^2v_3^2\right), \tag{6.3.3}$$

which also appears in the definition of the dual dilaton

$$e^{-2\Phi} = \det M e^{-2\phi}, \tag{6.3.4}$$

and we have introduced the following functions for convenience of presentation

$$\hat{a} = \frac{ab}{\sqrt{b^2 + a^2g^2}}, \quad \hat{b} = \sqrt{b^2 + a^2g^2}, \quad \hat{c} = \frac{a^2g}{b^2 + a^2g^2}. \tag{6.3.5}$$

The many and complicated forms that this background supports can be expressed in a relatively compact manner through a judicious choice of dual vielbein basis $\hat{e}^a$, namely

$$e^{x^\mu} = e^A dx^\mu, \quad e^{r^\mu} = e^A\sqrt{N}dx^\mu, \quad e^{1,2} = e^A\sqrt{N}\hat{a}\tilde{\omega}_{1,2}$$

$$\begin{aligned}
e^{\hat{1}} &= \frac{2\sqrt{N}e^A\hat{b}}{\det M}\Big[-\sqrt{2}v_2(v_2dv_2+v_3dv_3)-2\sqrt{2}e^{4A}N^2\hat{b}^2h^2(dv_2-\hat{c}v_3\tilde{\omega}_2)+\\
&\qquad 2e^{2A}Nh^2v_3(v_3\hat{c}\tilde{\omega}_1+v_2\tilde{\omega}_3)\Big]\\
e^{\hat{2}} &= \frac{4e^{3A}N^{3/2}\hat{b}}{\det M}\Big[v_2\hat{b}^2(dv_3+\hat{c}v_2\tilde{\omega}_2)+h^2(\hat{c}v_3^2\tilde{\omega}_2-v_3dv_2)-\\
&\qquad 2\sqrt{2}e^{2A}Nh^2\hat{b}^2(\hat{c}v_3\tilde{\omega}_1+v_2\tilde{\omega}_3)\Big]\\
e^{\hat{3}} &= \frac{2e^A\sqrt{N}h}{\det M}\Big[-\sqrt{2}v_2(v_2dv_2+v_3dv_3)-2\sqrt{2}e^{4A}N^2\hat{b}^4(dv_3+\hat{c}v_2\tilde{\omega}_2)-\\
&\qquad 2e^{2A}N\hat{b}^2v_2(\hat{c}v_3\tilde{\omega}_1+v_2\tilde{\omega}_3)\Big].
\end{aligned} \tag{6.3.6}$$

With respect to this basis the NS two-form is given by [3]

$$B_2 = \frac{1}{\hat{a}\hat{b}v_2}\Big[\hat{a}hv_2e^{\hat{1}\hat{3}} - \hat{b}\hat{c}hv_3e^{1\hat{3}} - \hat{b}^2\big(\hat{c}v_2e^{1\hat{1}} + \sqrt{2}e^{2A}N\hat{a}he^{\hat{2}\hat{3}}\big)\Big] \tag{6.3.7}$$

The RR sector is given by,

$$\begin{aligned}
F_1 &= \frac{2e^{-A}\sqrt{N}(K+1)}{\hat{a}\hat{b}}\Big[\hat{b}\big(\hat{c}v_2e^2+\sqrt{2}e^{2A}N\hat{a}he^{\hat{3}}\big)-\hat{a}v_2e^{\hat{2}}\Big]-2e^{-A}\sqrt{N}K'v_3e^r,\\
F_3 &= \frac{2(K+1)e^{-A}\sqrt{N}}{\hat{a}\hat{b}^2}\Big[\hat{b}^2\hat{c}v_2e^{1\hat{1}\hat{2}}+\hat{b}\hat{c}hv_3(e^{2\hat{1}\hat{3}}-e^{1\hat{2}\hat{3}})-\\
&\qquad \sqrt{2}e^{2A}N\hat{b}^3\hat{c}(e^{1\hat{1}\hat{3}}+e^{2\hat{2}\hat{3}})+\hat{a}hv_3e^{\hat{1}\hat{2}\hat{3}}\Big]+\\
&\quad \frac{2e^{-A}\sqrt{N}\hat{b}K'}{h}\Big[\sqrt{2}e^{2A}N\hat{b}he^{r\hat{1}\hat{2}}+v_2e^{r\hat{1}\hat{3}}\Big]+\frac{2e^{-A}\sqrt{N}U}{\hat{a}}\Big[\hat{b}v_2e^{12\hat{1}}+hv_3e^{12\hat{3}}\Big],\\
F_5 &= \frac{2\sqrt{2}e^AN^{3/2}\hat{b}hU}{\hat{a}^2}\Big[e^{tx^1x^2x^3}-e^{12\hat{1}\hat{2}\hat{3}}\Big],
\end{aligned} \tag{6.3.8}$$

[3]Note that the procedure of [30] actually gives the NS two from up to an exact $B_{2,eq(6.3.7)} = B_{2,NATD} + \frac{1}{\sqrt{2}}d\psi \wedge dv_3$. The choice we make is merely more simple in vielbein basis.

where

$$U = \hat{c}(K+1) - (K-1), \tag{6.3.9}$$

has been defined for convenience. We also note that the potential such that $F_1 = dC_0$ is actually very simple, namely $C_0 = -2N(K+1)v_3$. We have checked using Mathematica that this background solves the Einstein, dilaton, Maxwell and Bianchi equations of type-IIB, once the eqs (6.2.5) are imposed.

Notice, that like in the paper [112], our background's warp factors and dilaton depend on more than one coordinate– (r, v_2, v_3)– in our case.

6.3.1 Asymptotics

In the IR the new 3 manifold that is generated has induced metric

$$d^2s_3 = \frac{1}{2Nq_0R_0^3v_2^2}\left(2v_2^2dv_2^2 + 4v_2v_3dv_2dv_3 + (N^2q_0^2R_0^6 + 2v_3)dv_3\right) + ... \tag{6.3.10}$$

The form of this metric suggests that $v_2 = 0$ produces a singularity and indeed calculating the curvature invariants in the IR are all inversely proportional to some power of v_2. For instance

$$R = \frac{q_0^2(2N^2R_0^6 - 15v_2) + 4v_3^2}{2Nq_0R_0^3v_2^2} + ... \tag{6.3.11}$$

because of this we choose to restrict the range of the coordinate $v_2 > 0$ to ensure our solution is non singular. This is a physical requisite on a coordinate, that the process of non-Abelian duality gives no information on. It should be interesting to determine if there is any geometrical obstruction to such restriction.

The appearance of this possible-singular behaviour at $v_2 = 0$ is due to the fact that we are T-dualising on a manifold (θ, φ, ψ) with a shrinking fibre ψ. See eq (6.2.10) together with eq (6.2.36). Since the non-Abelian T-duality (at least at the supergravity level as we are doing it) does not restrict the range of the coordinates, we have *chosen* this restriction $v_2 > 0$. Recent developments on the sigma model side of the formalism [113] may illuminate these issues, but still more work on the topic is needed. It may be that the restriction $v_2 > 0$ is not feasible and/or generates a manifold with a boundary. In that case, our solution would present a singularity at $v_2 = 0$. Physical observables would be trustable as long as they do not 'sit' on the point $v_2 = 0$.

In the UV the 3 manifold has induced metric

$$d^2s_3 = \frac{3}{NR_1r^2}\left(2dv_2^2 + 3dv_3^2 + 2v_2(d\psi + \cos\psi d\tilde{\theta})^2\right) + ... \tag{6.3.12}$$

Although this is vanishing, in line with our expectations from dualising a manifold which blows up, all the curvature invariants remain finite. Related to this is the fact that, whilst the induced metric g_3 vanishes, the string volume $e^{-\Phi}\sqrt{\det g_3}$ is finite.

Finally, let us quote the asymptotics of the dilaton of type-IIB. For small values of r, we have

$$e^{\Phi} = \frac{q_0}{4Nv_2} - \frac{r^2\left(q_0\left(N^2 q_0^2 R_0^6 + 2\left(v_3^2 - \left(q_0^2 - 1\right)v_2^2\right)\right)\right)}{64\left(NR_0^2 v_2^3\right)} + ... \tag{6.3.13}$$

while the dual dilaton for $r \to \infty$ is,

$$e^{\Phi} = \frac{9}{\sqrt{2}N^2 r^3} + \frac{81 q_1 R_1}{\sqrt{2}N^2 r^4} + \frac{243\sqrt{2}q_1^2 R_1^2}{N^2 r^5} + ... \tag{6.3.14}$$

6.3.2 G-structure.

The seed type-IIA solution of section 6.2.1 exhibits confinement and supports an $SU(3)$ structure as discussed in section 6.2.2. The results of [43] suggest that the T-dual solution should support a dynamical $SU(2)$-structure, defined by a point dependent rotation between the two 6-d internal killing spinors. This is indeed the case, we will present the structure here and refer the reader to appendix D of [43] for the details of the calculation[4] To express the structure succinctly it is useful to introduce a new set of vielbeins, which are a rotation of

[4]Actually it is the isometry defined by $(\tilde{\theta}, \tilde{\varphi}, \psi)$ that is dualised in appendix D of [43], but this calculation is completely analogous to our's. Our result is non-singular in the radial coordinate r.

eq (6.3.6),

$$
\begin{aligned}
e^r &= e^A\sqrt{N}dr\,, \quad \breve{e}^\theta = e^A\sqrt{N}\hat{a}\,d\tilde{\theta}\,, \quad \beta e^\varphi + \alpha e^2 = e^A\sqrt{N}\hat{a}\sin\tilde{\theta}d\tilde{\varphi}\,,\\
e^{1'} &= \frac{2e^A\sqrt{N}\hat{b}}{\det M}\Big[-2\sqrt{2}e^{4A}N^2\hat{b}^2h^2\big(\cos\psi dv_2 - \hat{c}v_3(\sin\psi\tilde{\omega}_1 + \cos\psi\tilde{\omega}_2) - v_2\sin\psi\tilde{\omega}_3\big)\\
&\quad - \sqrt{2}v_2\cos\psi(v_2dv_2 + v_3dv_3) + 2e^{2A}N\Big(-\hat{b}^2v_2\sin\psi(dv_3 + \hat{c}v_2\tilde{\omega}_3)\\
&\quad + h^2v_3\big(\sin\psi dv_2 + \hat{c}v_3(\cos\psi\tilde{\omega}_1 - \sin\psi\tilde{\omega}_2) + v_2\cos\psi\tilde{\omega}_3\big)\Big)\Big]\\
\alpha e^\varphi - \beta e^{2'} &= \frac{2e^A\sqrt{N}\hat{b}}{\det M}\Big[-2\sqrt{2}e^{4A}N^2\hat{b}^2h^2\Big(\cos\psi dv_2\\
&\quad - \hat{c}v_3(\sin\psi\tilde{\omega}_1 + \cos\psi\tilde{\omega}_2) - v_2\sin\psi\tilde{\omega}_3\Big)\\
&\quad - \sqrt{2}v_2\cos\psi(v_2dv_2 + v_3dv_3) + 2e^{2A}N\Big(-\hat{b}^2v_2\sin\psi(dv_3 + \hat{c}v_2\tilde{\omega}_3)\\
&\quad + h^2v_3\big(\sin\psi dv_2 + \hat{c}v_3(\cos\psi\tilde{\omega}_1 - \sin\psi\tilde{\omega}_2) + v_2\cos\psi\tilde{\omega}_3\big)\Big)\Big]\\
e^{3'} &= \frac{2e^A\sqrt{N}h}{\det M}\Big[-\sqrt{2}v_2(v_2dv_2 + v_3dv_3) - 2\sqrt{2}e^{4A}N^2\hat{b}^4(dv_3 + \hat{c}v_2\tilde{\omega}_2)\\
&\quad - 2e^{2A}N\hat{b}^2v_2(\hat{c}v_3\tilde{\omega}_1 + v_2\tilde{\omega}_3)\Big]\,. \qquad (6.3.15)
\end{aligned}
$$

One then takes these vielbeins ordered as $(r\theta\varphi 1'2'3')$ and rotates to define another basis of vielbeins as

$$\tilde{e} = R.e'. \qquad (6.3.16)$$

The matrix with which this rotation is performed is

$$R = \frac{1}{\sqrt{\Delta}}\begin{pmatrix} \beta & 0 & 0 & \zeta_1 & -\zeta_2\beta & \zeta_3\\ 0 & \sqrt{\Delta} & 0 & 0 & 0 & 0\\ 0 & 0 & \sqrt{\Delta} & 0 & 0 & 0\\ -\zeta_1 & 0 & 0 & \beta & \zeta_3 & \zeta_2\beta\\ \zeta_2\beta & 0 & 0 & -\zeta_3 & \beta & \zeta_1\\ -\zeta_3 & 0 & 0 & -\zeta^2\beta & -\zeta^1 & \beta \end{pmatrix} \qquad (6.3.17)$$

where

$$\Delta = \beta^2 + \zeta_1^2 + \zeta_2^2\beta^2 + \zeta_3^2 \qquad (6.3.18)$$

and

$$\zeta_1 = -\frac{e^{-2A}v_2 \cos\psi}{\sqrt{2}N\hat{b}h}, \quad \zeta_2 = -\frac{e^{-2A}v_2 \sin\psi}{\sqrt{2}N\hat{b}h}, \quad \zeta_3 = -\frac{e^{-2A}v_3}{\sqrt{2}N\hat{b}^2}. \tag{6.3.19}$$

Let us now express the forms of the geometric structure, following the conventions of [114] we have

$$\begin{aligned}
k_{\parallel} &= \frac{\alpha}{\sqrt{1+\zeta.\zeta}} \qquad k_{\perp} = \sqrt{\frac{\beta^2+\zeta.\zeta}{1+\zeta.\zeta}} \\
z = w - iv &= \frac{1}{\sqrt{\beta^2+\zeta.\zeta}}(\sqrt{\Delta}\tilde{e}^r + \zeta_2\alpha\tilde{e}^\varphi - i(\sqrt{\Delta}\tilde{e}^3 + \zeta_2\alpha\tilde{e}^\theta)) \\
j &= \tilde{e}^{r3} + \tilde{e}^{\varphi\theta} + \tilde{e}^{21} - v\wedge w \\
\omega &= \frac{-i}{\sqrt{\beta^2+\zeta.\zeta}}(\sqrt{\Delta}(\tilde{e}^\varphi + i\tilde{e}^\theta) - \zeta_2\alpha(\tilde{e}^r + i\tilde{e}^3)) \wedge (\tilde{e}^2 + i\tilde{e}^1).
\end{aligned} \tag{6.3.20}$$

In terms of those forms, we can define two 6-d pure spinors as:

$$\begin{aligned}
\Phi_+ &= \frac{ie^A}{8}e^{-iv\wedge w}(k_{\parallel}e^{-ij} - ik_{\perp}\omega) \\
\Phi_- &= \frac{ie^A}{8}(v+iw)\wedge(k_{\perp}e^{-ij} + ik_{\parallel}\omega)
\end{aligned} \tag{6.3.21}$$

Notice that because $k_{\parallel}$ is point dependent we have a dynamical $SU(2)$-structure. To have a good idea of the dynamical character of the $SU(2)$-structure, we can expand the quantities $k_{\parallel}, k_{\perp}$. For the solution in eq (6.2.32), we have for $\rho \to \infty$,

$$k_{\perp} = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}a^3}{16\rho^3} + ..., \qquad k_{\parallel} = -\frac{1}{2} + \frac{3a^3}{16\rho^3} + ... \tag{6.3.22}$$

While for $\rho \to a$ we have,

$$k_{\perp} = 1 - \frac{(a^4N^2)(\rho-a)^2}{1728v_2^2} + ..., \qquad k_{\parallel} = -\frac{(a^2\mathrm{Nc})(\rho-a)}{12\left(\sqrt{6}v_2\right)} + ... \tag{6.3.23}$$

On the other hand for the semi-analytic solutions we have,

$$k_{\perp} = 1 - \frac{9R_1^2}{8r^2} + ..., \qquad k_{\parallel} = -\frac{3R_1}{2r} + ... \tag{6.3.24}$$

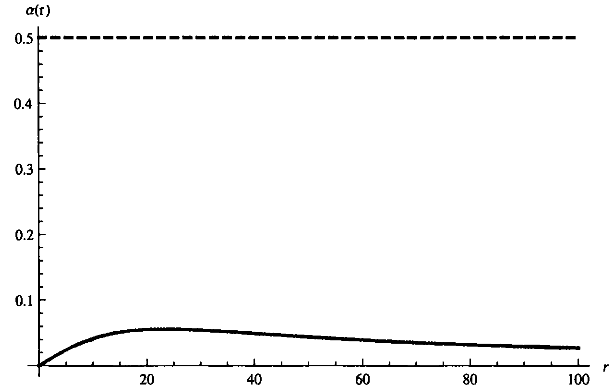

Figure 6.3: Solid line: $\alpha(r)$ for the numerical solution with $R_0 = 10$, $q_0 = 1/5$. Dashed line: $\alpha(r)$ for the exact solution of eq (6.2.30), $\alpha_{exact}(r) = \frac{1}{2}$

for the large radius expansion and

$$k_\perp = 1 - \frac{r^4\,(q_0^4 R_0^2)}{256 v_2^2} + ..., \qquad k_\parallel = -\frac{r^2\,(q_0^2 R_0)}{8\left(\sqrt{2} v_2\right)} + ... \tag{6.3.25}$$

for the case of $r \to 0$. These expansions make clear the dynamical character of the structure. Also very descriptive is the quantity $\alpha(r)$ shown in Figure 6.3.2.

It is interesting to notice that for the non-Abelian T-dual of the exact and singular solution in eq (6.2.30), the $SU(2)$-structure is not dynamical. It is precisely the deformation of the space, displayed by the non-singular solution or the semi-analytical ones that makes the structure dynamical. This may be related with the phenomena of 'confinement' and 'symmetry breaking' that occur in the dual field theory.

The calibration forms of SUSY cycles in the 6-d internal space are defined by

$$\Upsilon_n = -8e^{-\Phi}\mathrm{Im}\left(\Psi_\pm\right) \wedge e^{-B_2}\Big|_n \tag{6.3.26}$$

where on the left hand side it should be understood that we restrict to the part with n-legs and the even/odd calibrations are given by $\Psi_\pm$ respectively. In the bibliography, one can find examples of SuGra solutions with $SU(2)$-dynamical structure [115, 116, 117], however these remain rare.

6.3.3 SUSY Cycles

Here we present a list of supersymmetric sub-manifolds, that while not exhaustive, gives at least some indication of the types of SUSY cycles this type-IIB solution supports. Attention shall be restricted to cycles with no legs in the

r-direction.

One-cycles

These may be defined by imposing $v_2(v_3)$ with all other coordinates constant. The DBI action is given by

$$\begin{aligned} S^1_{DBI} =& T_1 \int dv_3 \mathcal{L}_{DBI} \\ =& T_1 \int dv_3 e^{-\phi} \sqrt{2e^{4A} N^2 (\hat{b}^4 + \hat{b}^2 h^2 v_2') + (v_3 + v_2 v_2')^2} \end{aligned} \tag{6.3.27}$$

and the behaviour of the integrand in the IR and and UV is

$$\mathcal{L}^1_{DBI} = \begin{cases} 4\sqrt{\frac{N(n^2 q_0^2 R_0^6 + 2(v_3 + v_2 v_2')^2)}{2q_0^3 R_0^3}} + ... & \text{as } r \to 0 \\ \frac{1}{3}\sqrt{\frac{2}{3}}\sqrt{\frac{N^3(3+2v_2')}{R_1}} r^2 + ... & \text{as } r \to \infty \end{cases} \tag{6.3.28}$$

A one-cycle is SUSY when $\mathcal{L}^1_{DBI} dv_3 = \Psi_1$ on that cycle. This may be used to fix $v_2(v_3)$. The calibration 1-form on $\{v_3 | v_2 = v_2(v_3)\}$ is given by

$$\Psi_1 = \frac{4e^{6A-\phi} N b \hat{b}^2}{4b^2 + \frac{a^4 f^2}{b^2 c^2}} dv_3 = \begin{cases} \frac{R_0^2 (N q_0 R_0)^{3/2}}{\sqrt{2}} dv_3 + ... & \text{as } r \to 0 \\ \frac{(NR_1)^{3/2}}{6\sqrt{2}} r^2 dv_3 + ... & \text{as } r \to \infty \end{cases} \tag{6.3.29}$$

It is a simple matter to show that a 1-cycle which is SUSY in the UV is given by

$$v_2 = \frac{1}{4}\sqrt{\frac{3}{2}}\sqrt{R_1^4 - 16 v_3} + C \tag{6.3.30}$$

where C is any real constant and a real solution requires $|R_1| \geq 2$ which is consistent with the numerical solutions presented in Section 6.2.4. Whilst there is a one cycle which is SUSY in the IR whenever

$$v_2^2 = \frac{1}{4}\left(N q_0 R_0^3 \sqrt{2 R_0^4 q_0^4 - 32 v_3} - 4 v_3^2 + 4C^2 \right) \tag{6.3.31}$$

where C is a different real constant. Notice that this simplifies to $v_2^2 + v_3^2 = C^2$ when $R_0 q_0 = 2$ and then the cycle defines a circle, a similar cycle was defined for a flavour D6 brane in [41].

Two-cycles

There are some cycles which preserve SUSY for large values of r. One of them is given by $(\tilde{\theta}, v_3)$ such that $\psi = 0$ and $v_2 = \frac{2\sqrt{6}}{R_1^4-16} v_3$ [5]. For this cycle the DBI action is obtained by integrating

$$e^{-\Phi}\sqrt{g+B_2}\Big|_{\Sigma_2} = B\sqrt{v_3^2 + C} \tag{6.3.32}$$

where

$$\begin{aligned} B &= e^{A-\phi}\frac{8+R_1^4}{R_1^4-16}\sqrt{2N(\hat{a}^2+\hat{b}^2\hat{c}^2)} \\ C &= \frac{2e^{4A}N^2(R_1^4-16)\hat{b}^2(24\hat{b}^2\hat{c}^2h^2+\hat{a}^2((R_1^4-16)\hat{b}^2+24h^2))}{(r_1^4+8)^2(\hat{a}^2+\hat{b}^2\hat{c}^2)} \end{aligned} \tag{6.3.33}$$

One can integrate this to get the volume of the cycle to behave as

$$\int dv_3 e^{-\Phi}\sqrt{g+B_2}\Big|_{\Sigma_2} = \begin{cases} \frac{N^2R_1^2\Delta v_3}{3\sqrt{6}\sqrt{R_1^4-16}} r^3 + ..., & r\to\infty \\ (\mathcal{F}(v_{3a}) - \mathcal{F}(v_{3b}))r + ..., & r\to 0 \end{cases} \tag{6.3.34}$$

where

$$\begin{aligned} \mathcal{F}(v_{3a}) &= \frac{N\left(R_1^4+8\right)\left(v_3\sqrt{v_3^2+\Lambda}+\Lambda\log\left(\sqrt{v_3^2+\Lambda}+v_3\right)\right)}{\sqrt{2}q_0R_0\left(R_1^4-16\right)}, \\ \Lambda &= \frac{N^2q_0^2R_0^6\left(R_1^4-16\right)^2}{2\left(R_1^4+8\right)^2}. \end{aligned} \tag{6.3.35}$$

The behaviour is similar for the exact solution, although that is not SUSY on this cycle. In all cases the cycle blows up in the UV and contracts to zero in the IR.

We do not report about calibrated three-cycles or higher.

6.4 Comments on the Quantum Field Theory.

In this section, we will study some aspects of the four dimensional QFTs dual to the background we presented in eq (6.2.10). Comparisons with a suitable analysis for the solution after the non-abelian T-duality written in eqs (6.3.2)-(6.3.8), will be made when possible.

We emphasize that the field theory dual to the type-IIA backgrounds is char-

[5] Or equivalently $(\tilde{\varphi}, v_3)$ such that $\tilde{\theta} = \psi = \pi/2$, $v_2 = \frac{2\sqrt{6}}{R_1^4-16} v_3$

acteristically non-local or 'higher-dimensional'. This should not come as a surprise, as it was already observed in [7], full decoupling of the gravity modes is not achieved for the case of flat D6 branes. We will make this point via the study of some observables that will be sensitive to the high energy properties of the QFT. We will analyse Wilson loops, with emphasis on its UV behaviour. We will then study the entanglement entropy and central charge. Both observables will present signs of non-locality. We will also discuss the behaviour of Wilson, 't Hooft loops, domain walls and gauge couplings, when studied as IR effects. These observables are well-behaved for the solutions presented in this work. In other words, the dual QFT to our background in eq (6.2.10) or our new background in eqs (6.3.2)-(6.3.8)—together with the solutions in section 6.2.3, behave as QFTs that at low energies show signs of the expected four dimensional behaviour, like confinement and symmetry breaking, but need to be defined with a UV-cut off, or need a UV-completion.

Various properties are 'inherited' (in a sense that will become clear) by the new type-IIB solution that we have constructed. We will finally calculate the Page charges of this new solution. We will propose a possible quiver suggested by these charges.

It will be clear by analysing the backgrounds that the initial QFT, corresponding to the compactified D6 branes has global symmetries given by $SU(2) \times SU(2)$, while the QFT dual to the type-IIB background will only have $SU(2)$. This reduction of global symmetries (isometries, for the dual backgrounds) is characteristic of non-Abelian T-duality.

6.4.1 Some useful sub-manifolds

It will be useful for the analysis below, to define some sub-manifolds of the metric in eq (6.2.10). We can define then

$$\Sigma_3 = [\theta, \varphi, \psi], \quad \tilde{\Sigma}_3 = [\tilde{\theta}, \tilde{\varphi}, \psi], \quad \hat{\Sigma}_3 = [\theta = \tilde{\theta}, \varphi = \tilde{\varphi}, \psi]. \tag{6.4.1}$$

The volume element of each of these cycles is,

$$\begin{aligned}
\sqrt{\det g_{\Sigma_3}} &= 16\pi^2 (\alpha' g_s N)^{3/2} e^{\Phi} h (b^2 + a^2 g^2), \\
\sqrt{\det g_{\tilde{\Sigma}_3}} &= 16\pi^2 (\alpha' g_s N)^{3/2} e^{\Phi} h a^2, \\
\sqrt{\det g_{\hat{\Sigma}_3}} &= 16\pi^2 (\alpha' g_s N)^{3/2} e^{\Phi} h (b^2 + a^2 (g^2 + 1)).
\end{aligned} \tag{6.4.2}$$

We can see using the IR expansions that each of these cycles vanish at $r \to 0$ and diverge as $r \to \infty$ for the explicit solutions presented in section 6.2.1.

If we consider the three-cycles after the non-Abelian T-duality, we have the submanifold defined by the coordinates $(\tilde{\theta}, \tilde{\varphi}, v_2)$. This cycle is not calibrated.

6.4.2 Wilson and 't Hooft loops.

The type-IIA background in eq (6.2.10), ends in a smooth way, with finite values for the combinations $F^2 = g_{tt}g_{xx}$, $G^2 = g_{tt}g_{xx}$. This might suggest that the system confines as usual. But there are some subtleties. Indeed, when calculating the Wilson loop with the prescription of hanging a fundamental string from a brane very far away in the UV of the geometry, we are assuming that this string will end on the D-brane satisfying the boundary condition of ending 'perpendicularly' to the brane. This is discussed, for example in [118]. Following the formalism in [118], the boundary condition boils to defining $V_{eff} = \frac{F}{G}\sqrt{F^2 - F_0^2}$ and imposing that for large values of the radial coordinate V_{eff} diverges. In our present case, $F^2 = G^2 = (\alpha' g_s N)^2 e^{\frac{4}{3}\phi}$ (we choose $\mu = 1$). The value of

$$V_{eff} \sim \sqrt{e^{4\phi/3} - e^{4\phi_0/3}}\alpha' g_s N$$

is a finite constant for the semi-analytic solutions. This suggests, that the QFT needs to be UV-completed or be supplemented by a hard UV-cutoff which in turn suggests that the QFT is afflicted by the presence of an irrelevant operator. Conversely, one can consider the case in which the dilaton diverges at infinity, as described by eq (6.2.32). In that case, the UV-boundary conditions are satisfied, but one will find that there is a minimal length-separation for the quark-antiquark pair. For r_* close to the boundary $L_{QQ}(r_*)$ is finite, instead of vanishing. This indicates the presence of a minimal length in the dual QFT. Hence, some form of non-locality. In summary, regardless the solution we choose, the high energy behaviour of the dual field theory seems to be not the expected one for a 4-dimensional QFT.

Once assumed a UV-cutoff, the Wilson loop can be calculated. The QCD string tension is finite (suggesting confinement) and given by,

$$\sigma = \frac{1}{2\pi\alpha'}\sqrt{g_{tt}g_{xx}}|_{IR} = \frac{1}{2\pi\alpha'}e^{2A(0)} = \frac{(q_0 R_0)^2}{4\pi\alpha'} .$$

The components of the metric that enter this particular Wilson loop calculation are g_{tt}, g_{xx}, g_{rr}. These components are not changed by the non-Abelian T-duality. We should then expect that the comments above should be valid also for the QFT dual to the background in eq (6.3.2).

In contact with the discussion on the dynamical character of the $SU(2)$-structure, notice that this is a consequence of the deformation of the space associated with the confining behavior. Relations of this kind have been reported in [43].

't Hooft loops.

In a very similar way as described above, we could wrap a D4 brane on any of the three-cycles in eq (6.4.1) and extend the brane on $[t, x_1]$, to form a magnetic string-like object. We propose that this object computes the 't Hooft loop in the QFT. On the type-IIA side, let us consider the different three-manifolds in eq (6.4.1), we will have that the effective tension of the 't Hooft string-like object is

$$\frac{T_{eff,\Sigma_3}}{16\pi^2 T_{D4}(\alpha' g_s N)^{3/2}} = e^{5A-\Phi} h(b^2 + a^2 g^2)|_{r=0}.$$
$$\frac{T_{eff,\tilde{\Sigma}_3}}{16\pi^2 T_{D4}(\alpha' g_s N)^{3/2}} = e^{5A-\Phi} ha^2|_{r=0}.$$
$$\frac{T_{eff,\hat{\Sigma}_3}}{16\pi^2 T_{D4}(\alpha' g_s N)^{3/2}} = e^{5A-\Phi} h(b^2 + a^2(g^2+1))|_{r=0}.$$

Notice that all these present a vanishing tension–hence screening– of the monopole-antimonopole pair. Again, the behavior of this low energy observable is in line with the expected.

We can define a screened magnetic string in the type-IIB picture. To do so, we will use the two cycle described below eq (6.3.32) and wrap a D3 brane on it, also extending the brane on the two directions (t, x_1). For the effective tension we will get,

$$\frac{T_{eff}}{T_{D3}} = e^{A-\Phi} \int d\theta d\varphi \sqrt{\det[g+B]_{\Sigma_2}}|_{r=0}. \tag{6.4.3}$$

We observe using the asymptotics associated with this cycle a tensionless magnetic string or conversely, a 'screened' force between a pair of monopoles, as expected. Let us move to study another IR-observable.

6.4.3 Domain Walls

In our type-IIA geometry of eq (6.2.10), there is a natural two-cycle defined by

$$\Sigma_2 = [\theta = \tilde{\theta}, \varphi = \tilde{\varphi}],$$

for some fixed value of the angle $\psi = \psi_0$, which is SUSY in the IR.

The objects of potential interest to represent domain Walls, are D4 branes that wrap the two-cycle above and that extend on the Minkowski directions (t, x_1, x_2). If this object has finite tension, then it may act as a domain Wall, separating different vacua. Let us study the object in more detail.

The induced metric (for constant radial coordinate and constant angle ψ_0)

is,

$$
\begin{aligned}
ds^2_{ind,st} =& e^{2A}\Big[\mu dx^2_{1,2} + \\
& \alpha' g_s N\Big(b^2 + a^2(g^2 + 1 + 2g\cos\psi_0)\Big)(d\theta^2 + \sin^2\theta d\varphi^2)\Big].
\end{aligned}
\tag{6.4.4}
$$

So, the action of the object (choosing $\mu = 1$) is,

$$
\begin{aligned}
S =& -T_{eff}\int d^3x, \\
T_{eff} =& 16\pi^2 e^{5A-\phi}(\alpha' g_s N)T_{D4}\Big(b^2 + a^2(g^2 + 1 + 2g\cos\psi_0)\Big)|_{r=0}.
\end{aligned}
\tag{6.4.5}
$$

We can use the IR expansions of eq (6.2.34), to check that this object has a constant tension in the far IR of the geometry. If we follow the logic presented in [94] and add a gauge field (a_1, with curvature $f_2 = da_1$) on the Minkowski part of the world volume of the brane This will create a Wess-Zumino term of the form

$$
S_{WZ} = T_{D4}\int C_1 \wedge f_2 \wedge f_2 = -T_{D4}\int d\theta d\varphi F_2 \int d^3x f_2 \wedge a_1. \tag{6.4.6}
$$

Using that on the particular cycle $F_2 = -2N\sin\theta d\theta \wedge d\varphi$, we have induced a Chern-Simons term. These domain walls, should separate vacua coming from the breaking of some global (discrete) symmetry, see [119].

After the non-Abelian T-duality, we can define domain Walls by using the calibrated one-cycle defined around eq (6.3.28) and extend a D3 brane on the (t, x_1, x_2) directions, also wrapping the one-cycle parametrised by v_3. We will have a simple induced metric

$$
ds^2_{D3} = e^{2A}(-dt^2 + dx_1^2 + dx_2^2) + d\Sigma_1^2. \tag{6.4.7}
$$

The Action and effective tension of this object will be given by,

$$
\begin{aligned}
S_{D3} &= -T_{D3}e^{3A-\Phi}\sqrt{\det g_{\Sigma_1}}\int dv_3 \int d^{2+1}x, \\
T_{eff} &= T_{D3}\int dv_3 e^{3A-\Phi}\sqrt{\det g_{\Sigma_1}}|_{r=0}.
\end{aligned}
\tag{6.4.8}
$$

Notice that imposing that the domain Wall has a finite tension implies a finite range of values (or periodicity) for the coordinate v_3. Here again, like when we restricted the range of v_2 to avoid singularities— see around eq (6.3.11), we find that a 'physical' requirement implies conditions on the range of coordinates. These conditions are not imposed by non-Abelian T-duality when thought as a solution generating technique in supergravity.

We can also turn on a gauge field $\mathcal{A}$ with curvature $\mathcal{F}_2$ on the $R^{1,2}$ directions. The Wess-Zumino term will read

$$S_{WZ} = \left(T_{D3}\int_{v_3} F_1|_{r=0}\right)\int d^{2+1}x\mathcal{A}_1\wedge\mathcal{F}_2 = \kappa\int d^{2+1}x\mathcal{A}_1\wedge\mathcal{F}_2. \qquad (6.4.9)$$

Using that the Ramond form $C_0 = 2N(K+1)v_3$—see below eq (6.3.9)— implies that the 'charge' of the domain Wall (or the coefficient of the Chern-Simons term induced on it) is

$$\kappa = 2N(K(0)+1)\oint dv_3. \quad K(0) = 1 \qquad (6.4.10)$$

Let us move now to the definition of a gauge coupling.

6.4.4 A gauge coupling

We can define the gauge coupling of the QFT, by wrapping a D6 brane on any of the three-cycles in eq (6.4.1). We turn on a gauge field on the brane (for the argument, it is enough to turn on just F_{tx_1}), and we also turn on a pure gauge C_3-field of the form

$$C_3 = \frac{k}{16\pi^2}\sin\tilde{\theta}d\tilde{\theta}\wedge d\tilde{\varphi}\wedge d\psi,$$

we will have, for the cycle $\tilde{\Sigma}_3$ in eq (6.4.1) [6] that the induced metric and Born-Infeld-Wess-Zumino-action are,

$$\begin{aligned}
ds^2_{\tilde{\Sigma}_3} &= e^{2\phi/3}\left[\mu dx^2_{1,3} + \alpha' g_s N\left(a^2(\omega_1^2+\omega_2^2) + h^2\omega_3^2\right)\right],\\
S_{BIWZ} &= -T_{D6}\int e^{-\phi}\sqrt{-\det[g_{ab}+2\pi\alpha' F_{ab}]} + \\
&\quad T_{D6}\int C_7 + C_3\wedge F_2\wedge F_2.\\
S_{BIWZ} &\sim -T_{D6}(\alpha' g_s N_c)^{3/2}16\pi^2\int e^{\frac{4\phi}{3}}\mu^2 h a^2\left(1-\frac{1}{2\mu^2}e^{-4\phi/3}4\pi^2\alpha'^2 F_{\mu\nu}F^{\mu\nu}\right)\\
&\quad + T_{D6}k\int F_2\wedge F_2 + T_{D6}\int C_7, \qquad (6.4.11)
\end{aligned}$$

where the last contraction $F_{\mu\nu}F^{\mu\nu}$ is in Minkowski space and we have expanded for small field strengths (equivalently for small values of α'). This leaves us with a gauge coupling of the form,

$$\frac{1}{g^2_{YM}N_c} = (g_s N)^{1/2}\frac{a^2 h}{2\pi^4}. \qquad (6.4.12)$$

[6] We found that this cycle fails to be calibrated, in far UV, by a factor of 1/2.

with asymptotic behaviour as $r \to \infty$,

$$\frac{1}{g_{YM}^2 N_c} \sim \frac{(g_s N)^{1/2}}{2\pi^4} \left(\frac{r^3}{18} + \cdots \right) \tag{6.4.13}$$

and as $r \to 0$

$$\frac{1}{g_{YM}^2 N_c} \sim \frac{(g_s N)^{1/2}}{2\pi^4} \left(\frac{r^3}{8} + \cdots \right) \tag{6.4.14}$$

Notice that there is no effect of the rescaling by μ. This is expected, because this defines a a four-dimensional gauge coupling, that should be classically invariant under dilations.

We can run this calculation for the other three-cycles defined in eq (6.4.1) and get analogous expressions. All these expressions present a divergent gauge coupling in the IR—in the solution of eq (6.2.32) it diverges at $\rho = a$— while vanishing in the far UV. This should not be taken as a sign that the QFT is weakly coupled in the far UV. Indeed, these QFTs contain also superpotential couplings that make the whole system strongly interacting. This is in agreement with the dual spacetimes being weakly curved and trustable in the far UV.

After the non-Abelian T-duality, we can define a gauge coupling in the type-IIB dual by using D5 branes; extend them on $R^{1,3}$ and wrapping the calibrated two cycle defined below eq (6.3.32). We should also turn on a gauge field on the $R^{1,3}$ directions and also consider the projection of the NS B_2 field on the two-cycle. We find that this gauge coupling reads,

$$\frac{1}{g^2} = 4\pi^2 \alpha'^2 T_{D5} e^{-\Phi} \int \sqrt{\det[g_{\Sigma_2} + B_2]} \tag{6.4.15}$$

Using the asymptotics associated with the cycle above, we see that this gauge coupling 'confines' in the IR and vanishes in the far UV. The Wess-Zumino term for this D5 brane should define the Θ-angle.

In summary, we see that these observables, behave in the far IR as expected for a confining four dimensional QFT. Nevertheless, the Wilson loop indicates the need for a UV-completion. Below, we will briefly discuss another observable showing the same need for UV-completion.

6.4.5 Central Charge and Entanglement Entropy

A couple of quantities that characterise nicely the QFT dual to a geometry are the central charge and entanglement entropy of the QFT. These quantities have been studied in many different papers. Let us quote a couple of original references [120], [121].

We will follow the systematic treatment summarised in [122]. Consider a metric of the form,

$$ds_{st}^2 = \alpha\beta dr^2 + \alpha dx_{1,d}^2 + g_{ij}dy^i dy^j, \tag{6.4.16}$$

we can compute the following quantities in our generic background of eq (6.2.10)

$$\begin{aligned}
&V_{int} = \int d^{8-d}y\sqrt{\det[g_{ij}]} = (4\pi)^3 b^2 a^2 h(\alpha' g_s N)^{5/2} e^{5\phi/3},\\
&\alpha = \mu e^{2A}, \quad \beta = \frac{\alpha' g_s N}{\mu}, \quad d = 3,\\
&H = e^{-4\phi} V_{int}^2 \alpha^d = (4\pi)^6 \mu^3 b^4 a^4 h^2 (\alpha' g_s N)^5 e^{16A-4\phi},\\
&ds_5^2 = \kappa[dx_{1,3}^2 + dr^2], \quad \kappa^3 = H
\end{aligned} \tag{6.4.17}$$

This implies that the central charge is given by,

$$c \sim 27N^{3/2}\frac{H^{7/2}}{(H')^3}. \tag{6.4.18}$$

The UV and IR behaviour of the central charge for the solution with stabilized dilaton is

$$\begin{aligned}
&\log(c) \sim -8\log(1/r) + \cdots, && r \to \infty\\
&\log(c) \sim 6\log(r) + \cdots, && r \to 0
\end{aligned} \tag{6.4.19}$$

For comparison, we note that the central charge of the exact solution is, in the UV, $\log(c_{exact}) \sim \log(\frac{r^9}{2239488\sqrt{3}})$. In Figure 6.4.5 we plot the central charge for a numerical solution with stabilized dilaton and for the exact solution with linear dilaton. If we calculate the central charge after the non-abelian T-duality using the background of eq (6.3.2), we follow [122] and write the relevant quantities are,

$$\alpha = e^{2A}\mu, \quad \beta = \frac{\alpha' g_s N_c}{\mu}, \quad V = \int d\theta d\varphi d\psi dv_1 dv_2 e^{-2\hat{\Phi}}\sqrt{g_{int}}. \tag{6.4.20}$$

and the we will have

$$H = V^2\alpha^3, \quad c \sim \frac{H^{7/2}}{(H')^3}.$$

Following the algebra, one gets

$$c_{new} = \pi\mathcal{N} c_{old},$$

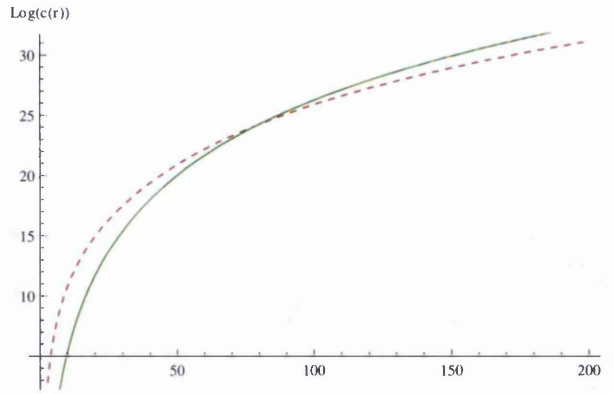

Figure 6.4: The central charge for a numerical solution with stabilized dilaton (red dashed curve) and for the exact solution with linear dilaton (green curve).

where $\mathcal{N}$ is an radius (energy) independent factor. Then, the central charges of the original and T-dual solutions differ by a constant with no much dynamical content. This can be traced to the invariance under NATD of the quantity $\sqrt{g_{initial}}e^{-\phi_{initial}}$, being equal, up to a Fadeev-Popov like factor to the same quantity in the dual background. This is explained in [30]. The Fadeev-Popov factor is associated with the scale independent number $\mathcal{N}$ above. This central charge and the entanglement entropy described below are two observables whose behavior is 'inherited' by the non-Abelian T-dualised background QFT pair.

Entanglement Entropy.

We now turn to the entanglement entropy. Consider a boundary region $\mathbb{R}^{d-1} \times \mathcal{I}_L$ where $\mathcal{I}_L$ is a line segment of length L. We calculate the entanglement entropy following [122] and obtain,

$$L(r_*) = 2\sqrt{H(r_*)N}\int_{r_*}^{\infty} \frac{dr}{\sqrt{H(r) - H(r_*)}}, \tag{6.4.21}$$

$$S_{conn} - S_{disc} \sim \int_{r_*}^{\infty} dr\sqrt{H}\Big[\frac{\sqrt{H}}{\sqrt{H - H(r_*)}} - 1\Big] - \int_{r_0}^{r_*} dr\sqrt{H}. \tag{6.4.22}$$

Evaluating eq (6.4.21) using the numerical solutions with stabilized dilaton found in Section 6.2.3 we can show that $L(r_*)$ grows indefinitely and has not a maximum value. The non-existence of a maximum and hence the absence of double-valuedness for $L(\rho_*)$, suggests the absence of a first order phase transition in the entanglement entropy. This falls within the description of [123] for the entanglement entropy of non-local QFTs. Same behaviour will present the background of eq (6.3.2).

A tricky point that should not confuse the diligent reader is that if a UV cutoff is imposed on the geometry, numerically a double valuedness of $L(r_*)$ is obtained and correspondingly, a first order transition in the entanglement entropy will be observed. But a more detailed analysis will show that changing the position of the cutoff, moves also the position of the maximum of the separation $L(r_*)$ and the maximum of the phase transition. Hence, this is a cutoff effect and should perhaps be taken as non-physical. The resolution is that a cutoff in the radial direction is needed to solve some stability problems in the configurations that compute the Entanglement Entropy. At the same time a Volume-law for the divergent part of the Entanglement Entropy will take place. A more detailed analysis of these issues appears in [124].

6.4.6 Page Charges

Finally, we will study some global quantities in the QFT that are defined using the background of eqs ((6.3.2))-((6.3.9)). Following [70] we write some given currents at constant radial position,

$$\begin{aligned} \star\mathcal{J}_{D7}^{Page} &= dF_1, \\ \star\mathcal{J}_{D5}^{Page} &= d(F_3 - B_2 \wedge F_1) \\ \star\mathcal{J}_{D3}^{Page} &= d(F_5 - B_2 \wedge F_3 + \tfrac{1}{2}B_2 \wedge B_2 \wedge F_1). \end{aligned} \tag{6.4.23}$$

In terms of these we can define three Page charges,

$$\begin{aligned} Q_{D7}^{page} &= \frac{1}{2\kappa_{10}^2 T_{D7}} \int_{V_2} \star\mathcal{J}_{D7}^{Page}, \quad Q_{D5}^{page} = \frac{1}{2\kappa_{10}^2 T_{D5}} \int_{V_4} \star\mathcal{J}_{D5}^{Page}, \\ Q_{D3}^{page} &= \frac{1}{2\kappa_{10}^2 T_{D3}} \int_{V_6} \star\mathcal{J}_{D3}^{Page}. \end{aligned} \tag{6.4.24}$$

where V_{9-p} is the transverse space of the corresponding Dp brane. Using Stokes theorem these may be expressed as integrals over three compact spaces. Notice that we demand that v_2 and v_3 are compact to have these charges well-defined. Let us propose the following cycles at constant radius,

$$\Sigma_1 = (v_3), \;\; \Sigma_3 = (\tilde{\theta}, \tilde{\varphi}, v_2 = v_3), \;\; \Sigma_5 = (\tilde{\theta}, \tilde{\varphi}, v_2, v_3, \psi) \tag{6.4.25}$$

Then the Page charges in our conversions are expressed as,

$$\begin{aligned} Q_{D7} &= \frac{1}{4}\int F_1, \quad Q_{D5} = \frac{1}{16\pi^2}\int F_3 - B_2 \wedge F_1 \\ Q_{D3} &= \frac{1}{64\pi^4}\int F_5 - B_2 \wedge F_3 - \frac{1}{2}B_2 \wedge B_2 \wedge F_1. \end{aligned}$$

We then get that the relevant quantities are,

$$
\begin{aligned}
&F_1 = -2N(K+1)dv_3 \\
&F_3 - B_2 \wedge F_1 = \sqrt{2}N(K-1)\sin\tilde{\theta}\big(v_2 dv_2 + v_3 dv_3\big) \wedge d\tilde{\theta} \wedge d\tilde{\varphi} \\
&F_5 - B_2 \wedge F_3 + \tfrac{1}{2}B_2 \wedge B_2 \wedge F_1 = 0.
\end{aligned}
\tag{6.4.26}
$$

Performing explicitly the integrals, we get

$$
Q_{D7} = -N\hat{A}(K+1), \quad Q_{D5} = N(K-1)\hat{B}, \quad Q_{D3} = 0. \tag{6.4.27}
$$

Importantly, we have imposed that the range of the coordinates v_2, v_3 is finite. We have defined them as periodic with periodicity of the coordinate v_3 being $\hat{A}$ and that for v_2 being $\hat{B}$, according to,

$$
\hat{A} = \frac{1}{2}\int dv_3, \quad \hat{B} = \frac{1}{\sqrt{2}\pi}\int v_2 dv_2 \tag{6.4.28}
$$

The integrals are performed over the range of those variables v_2, v_3. We can form the combination,

$$
Q_{int} = -(Q_{D7} + Q_{D5}) \tag{6.4.29}
$$

If we impose that the periods $\hat{A}, \hat{B}$ are equal and integer, we have defined a quantised quantity Q_{int}. This together with Q_{D3}, suggest a situation reminiscent of the Klebanov-Strassler QFT, with two gauge groups and one of the Page charges (that associated with D3 branes), vanishing.

This suggests that we are dealing with a two-node quiver, plus some bifundamental matter. It is certainly not the KS-field theory. We leave for future studies to describe the precise matter content and interactions of the bifundamental matter.

6.5 Conclusion

Let us start by briefly summarising what we have done in this chapter. We started with backgrounds in M-theory, reduced them to type-IIA, wrote the conditions for these backgrounds to preserve minimal SUSY in four dimensions (this was material already present in the bibliography). The first piece of new material consisted in explicitly solving the differential equations with a careful numerical integration that used as boundary conditions the asymptotic solutions, obtained analytically by solving (asymptotically) the BPS system. This is why we called our solutions 'semi-analytical'. We then studied the transition between G_2 structure (in eleven dimensions) to $SU(3)$ structure in type-IIA. We constructed explicit expressions for the potential and calibration forms.

Then, we performed a non-Abelian T-duality transformation on this type-IIA background. We obtained a family of backgrounds in type-IIB with all Ramond and Neveu-Schwarz forms turned on. This is a *new* family of solutions. We established its $SU(2)$-dynamical structure, pure spinors, calibration forms and found some calibrated cycles. Restrictions on the range of the T-dual coordinates were imposed, by requiring the smoothness of the generated space and the good behaviour of field theoretical observables.

After that, we moved into the study of the correspondence between the family of type-IIA solutions and its dual QFT, also extending the study of various observables to the QFT's dual to the new family of IIB backgrounds. In this line, we made clear that the QFTs are non-local and in the need of a UV-completion (this is especially clear from the behaviour of the Wilson loop and central charges at high energies). On the other hand, observables relevant to the IR dynamics show the expected four-dimensional behaviour. Finally, based on global charges, we loosely proposed a possible two-nodes quiver describing the QFT dual to the new type-IIB background. Notice that in the logic we are advocating, the background is *defining the QFT* via its observables at strong coupling.

A couple of points emerged as especially interesting from the previous study. If we impose that some physical observables of the QFT dual to our new background behave as expected, this in turn imposes constraints on the new coordinates 'after the duality'. We also restricted the range of one of the dual coordinates v_2 in order to avoid singularities. This is not free of ambiguities, unlike the restriction imposed on v_3 to be periodic, such that the domain wall charge is quantised.

These new coordinates originally play the role of Lagrange multipliers in the sigma model Action. Working at the genus-zero level in the sigma model gives no information on the periodicity (or possible non-compact nature), of such new coordinates. It is quite nice to find some conditions imposing the good-behaviour of the dual QFT.

It is also quite interesting to have found an $SU(2)$-dynamical structure in type-IIB for a solution preserving four supercharges. It is our understanding that such backgrounds are not easy to come by. The technique presented here provides a way of generating these and other backgrounds with similar features.

Chapter 7

Concluding Remarks

Let us now summarise the results of this thesis. We used non-abelian T-duality as a supergravity solution generating technique. Or goal was to construct new type-II solutions that describe the strong coupling regime of minimally supersymmetric gauge theories. Our starting point was existing solutions with well understood gauge theory descriptions that exhibit interesting dynamics such as confinement, duality cascades and chiral symmetry breaking. The geometries of each of the solutions considered has an $SU(2)$ isometry. We performed a non-abelian T-duality transformation on this isometry and generated new type-II solutions which we used to define new strongly coupled gauge theories.

In chapter 3 we considered a solution originally generated in [30] by acting on Klebanov-Witten with non-abelian T-duality. We used this as a testing ground in which to implement, for the first time, the powerful techniques of generalised geometry and G-structures within the context of non-abelian T-duality. We found that the 6d $SU(3)$ structure of Klebanov-Witten was mapped to an (orthogonal) $SU(2)$-structure, in a way that indicated a general rule mapping $SU(3)$-structures. Equipped with geometrical information about how SUSY is preserved it was possible to ascertain how calibrated sources transform under non-abelian T-duality. We used this information to generate a new flavoured solution in massive type-IIA, which indicated how the $SU(2)$ isometry T-duality acts on fundamental quarks in the field theoretic description.

Chapter 4 considered how non-abelian T-duality acted on the holographic dual of $\mathcal{N} = 1$ SYM-CS in 3d and its deformations. As an aid to this we compared this new solution to that generated when G_2-structure rotation is applied to the same "seed" solution. We found, rather stinkingly, that both generated solutions described field theory's containing Chern-Simons levels that could be shifted by large gauge transformations of B_2. Both were also confining in such a way that it was evident that a Chern-Simons term was determining (at least some of) the IR dynamics. This pointed evidenced by the fact that certain couplings began to "freeze" as one flowed towards the IR, tending to a constant.

However, despite some "cosmetic" similarities the solution of G_2-structure rotation and non-abelian T-duality describe quite different gauge theories. The most sticking difference being that the AdS_4 asymptotics of the former provides a UV completion of the seed solution, while the latter does not.

Chapters 5 and 6 dealt with the dualisation of confining conifold solutions in type-IIB and type-IIA respectively. We found in both cases that the structure of the T-dual solutions supported a dynamical $SU(2)$. The dynamical structure seems to be closely related to the confining behaviour of the gauge theories. In particular, the angle between internal spinors only changes noticeably in the confining regime. The structures tend quickly to static $SU(2)$ as one flows towards the UV. In addition to this we performed an extensive field theoretic analysis of the gauge theories generated by the T-duality transformation. We ascertain how many field theoretic observables are transformed, giving credence to the proposal that field theory observables not charged under the global $SU(2)$ on which the duality is performed should remain present in the T-dual solution.

Extending our results on the transformation of G-structures to other dimensions seems like a fruitful avenue of further research. We make the first steps towards this for the case of 7d structures in chapter 4 where we show that G_2 is mapped to dynamical $SU(3)$ in the case considered there. This needs to be fleshed out though and other examples considered. It would be particularly interesting to know to what extent the interplay between confinement and dynamical structures persist across diverse dimensions and examples. We have shown in the thesis that non-abelian T-duality provides a prescriptive method of generating solutions with G-structures that are quite rare. One may hope to learn from this something about the general construction of such solutions.

The general outlook for non-abelian T-duality as a SuGra solution generating technique with applications to holography seems good. This thesis contains concrete examples where interesting dynamics are generated by the duality. Given the wide range of backgrounds that have an $SU(2)$-isometry, there must be much more of interest that can be generated in a similar way to what is shown here. It would however be desirable to have greater general understanding of the effect on the gauge theories before one actually performs the dualisation, like one does for G-structure rotation. It would be interesting to add flavours to the dual background considered in chapters 4, 5 and 6 in the spirit of chapter 3, where it is shown that one may flavour the dual solution by simply dualising the original solution with smeared flavours added. This would surely work and it would be particularly interesting to see the effect this had on the Chern-Simons levels of chapter 4. Flavour is added to the Maldacena-Nastase solution in [63, 73]. It would also be interesting to dualise the G_2-structure rotated solution.

The main confusing aspect of non-abelian T-duality is whether the dual coordinates v_i are compact or not. We cannot rely on a direct worldsheet method to determine this as is possible in the abelian case. There imposing that holonomy's on non trivial topologies are gauge trivial fixes the period of the dual coordinate when the original coordinate is compact. In this thesis we have used arguments motivated by holography to fix the periods of the dual coordinates. We required that Page charges are quantised which leads to a restricted range for dual coordinates. As further evidence for finite limits on the dual coordinates we find in chapter 3 that the flux of B_2 over S^2 is only constrained to lie within $[0,1]$ up to a finite upper bound on the the dual coordinates. This isplies that out side this region our supergravity solutions are not well defined globally.

At this point we should make some comments about something that was largely sidestepped in this thesis. What does restricting the range of v_i actually mean? First off we could assume that $v_i \in \mathbb{R}$ however this raises some problems motivated by AdS-CFT considerations. Firstly if our T-dual geometry has an AdS factor we expect it to describe a strongly coupled CFT. However, as pointed out in [37], $v_i \in \mathbb{R}$ would lead to a CFT containing operators with dimension proportional to a continuous parameter. If we restrict to $v_i \in [v_{ai}, v_{bi}]$ the T-dual geometries do indeed have discrete spectrum of fluctuations that may be identified with a discrete spectrum of conformal dimensions, but this restriction raises some question on the supergravity side. Requiring $v \in [0, v_{max}]$ say, means that the geometry is terminating at at at a regular point v_{max}. This indicates that we should have localised delta function sources to satisfy the equations of motion at v_{max}.

So what are we to make of this? One attractive explanation is that there exists a regular and well defined solution which the non-abelian T-dual is approximating within some range of the coordinates v_i [125]. Recently this idea was given support by [113] where non-abelian T-dual sigma models arise as the end point of a whole line of integrable deformations of exact CFTs. There a WZW model is added to the "seed" sigma model and a continuous parameter interpolates between this and the T-dual solution . For generic values of this parameter all coordinates remain compact. But when it comes infinitely large one can effectively rescale the compact coordinates and zoom into the manifold. Thus the apparent non-compactness of non-abelian T-dual variables would appear to be much the same as if we were to use planer coordinates to describe a sphere, which only works locally. If one can embed such a construction in type-II supergravity, the problem of the range of the dual coordinates is resolved. Finding such an embedding is clearly the most urgent avenue of future research.

Chapter 8

Appendices

A Supergravity Conventions without Sources

We work in string where the action of type-IIB in the absence of sources is given by The action of type-IIB without sources is given in string frame by

$$
\begin{aligned}
S_{IIB} = \int_{M_{10}} \sqrt{g} \left[e^{-2\Phi} \left(R + 4(\partial\Phi)^2 - \frac{H^2}{12} \right) - \frac{1}{2} \left(F_1^2 + \frac{F_3^2}{3!} + \frac{1}{2}\frac{F_5^2}{5!} \right) \right] \\
- \frac{1}{2} \left(C_4 \wedge H \wedge dC_2 \right).
\end{aligned}
\tag{A.1}
$$

While the equivalent action in massive type-IIA is

$$
\begin{aligned}
S_{\text{MassIIA}} = \int_{M_{10}} \sqrt{g} \left[e^{-2\Phi} \left(R + 4(\partial\Phi)^2 - \frac{H^2}{12} \right) - \frac{1}{2} \left(F_0^2 + \frac{F_2^2}{2} + \frac{F_4^2}{4!} \right) \right] \\
- \frac{1}{2} \left(dC_3 \wedge dC_3 \wedge B + \frac{F_0}{3} dC_3 \wedge B^3 + \frac{F_0^2}{20} B^5 \right).
\end{aligned}
\tag{A.2}
$$

The 10-d hodge dual is defined such that

$$
F_n = (-1)^{int[n/2]} \star F_{10-n}. \tag{A.3}
$$

where F_n are the RR fluxes of either type-IIA or type-IIB supergravity. The fluxes may be used to define a polyform F such that

$$
F = \begin{cases} F_0 + F_2 + F_4 + F_6 + F_8 + F_{10} & \text{for type-IIA} \\ F_1 + F_3 + F_5 + F_7 + F_9 & \text{for type-IIB} \end{cases}. \tag{A.4}
$$

In terms of the polyform the Bianchi identities may be expressed as

$$(d - H\wedge)F = 0, \quad dH = 0. \tag{A.5}$$

It is easy to show this is satisfied with the definition

$$F = (d - H\wedge)C + F_0 e^{B_2} \tag{A.6}$$

where C is a polyform constructed from the RR potentials in the same fashion as above and F_0 should be taken to be zero in type-IIB. The flux equations of motion are expressed as

$$(d + H\wedge) \star F = 0, \quad d(e^{-2\Phi} \star H) = \frac{1}{2}\sum_n F_n \wedge \star F_n. \tag{A.7}$$

where the sum needs to me taken over the appropriate RR fluxes of type-IIA/IIB.

The dilaton must obey the equation of motion

$$d \star d\Phi + \star\frac{R}{4} - d\Phi \wedge \star d\Phi - \frac{1}{8}H \wedge \star H = 0, \tag{A.8}$$

while Einstein's equations are in type-IIA by

$$\begin{aligned} R_{\mu\nu} &= -2D_\mu D_\nu \hat{\Phi} + \frac{1}{4}H^2_{\mu\nu} + \\ &e^{2\Phi}\left[\frac{1}{2}(F_2^2)_{\mu\nu} + \frac{1}{12}(F_4^2)_{\mu\nu} - \frac{1}{4}g_{\mu\nu}(F_0^2 + \frac{1}{2}F_2^2 + \frac{1}{4!}F_4^2)\right], \end{aligned} \tag{A.9}$$

with the equation in type-IIB given by

$$\begin{aligned} R_{\mu\nu} &= -2D_\mu D_\nu \hat{\Phi} + \frac{1}{4}H^2_{\mu\nu} + \\ &e^{2\Phi}\left[\frac{1}{2}(F_1^2)_{\mu\nu} + \frac{1}{4}(F_3^2)_{\mu\nu} + \frac{1}{96}(F_5^2)_{\mu\nu} - \frac{1}{4}g_{\mu\nu}(F_1^2 + \frac{1}{3!}F_3^2)\right]. \end{aligned} \tag{A.10}$$

In section D, we will give expressions for the fluxes and their Bianchi identities in the presence of sources.

B Pure Spinors

Here we follow the conventions of [117] except for a difference in the self duality condition of the RR section which leads to a few sign differences. We work in string frame and consider solution with metrics that can be expressed as

$$ds^2 = e^{2A} dx_{3,1}^2 + ds_6^2 \tag{B.1}$$

and preserve $\mathcal{N} = 1$ SUSY in 4-d with non trivial RR sector. This means that the internal space, with metric ds_6^2, must support an $SU(3) \times SU(3)$-structure [20]. We decompose the 10-d MW spinors into a $4+6$ split as

$$\epsilon^1 = \zeta_+ \otimes \eta_+^1 + \zeta_- \otimes \eta_-^1, \quad \epsilon^2 = \zeta_+ \otimes \eta_\mp^2 + \zeta_- \otimes \eta_\pm^2. \tag{B.2}$$

where in ϵ_2 the upper/lower signs should be taken in type-IIA/B, the $\pm$ indicates chirality of both 4-d and internal 6-d spinors and we choose a basis for the internal spinors such that $(\eta_+)^* = \eta_-$. It is possible to define two $Cliff(6,6)$ pure spinors on the internal space as

$$\Phi_\pm = \eta_+^1 \otimes (\eta_\pm^2)^\dagger \tag{B.3}$$

which may be identified with polyforms under the Clifford map. The internal spinors are decomposed as

$$\eta_+^1 = e^A e^{i\frac{\theta_+ + \theta_-}{2}} \eta_+, \quad \eta_+^2 = e^A e^{-i\frac{\theta_+ - \theta_-}{2}} (k_{||}\eta_+ + k_\perp \chi_+) \tag{B.4}$$

where $k_{||}^2 + k_\perp^2 = 1$, $\eta_+^\dagger \eta_+ = \chi_+^\dagger \chi_+ = 1$ and $\chi_+^\dagger \eta_+ = 0$. The $\mathcal{N} = 1$ SUSY conditions for such a $SU(3) \times SU(3)$-structure solution are given by the differential conditions

$$\begin{aligned} &(d - H\wedge)(e^{2A-\phi}\Phi_\pm) = 0 \\ &(d - H\wedge)(e^{2A-\phi}\Phi_\mp) = e^{2A-\phi} dA \wedge \bar{\Phi}_2 \mp \tfrac{1}{8} e^{3A} \star_6 i\lambda(\tilde{F}) \end{aligned} \tag{B.5}$$

where $\lambda(A_n) = (-1)^{\frac{n(n-1)}{2}} A_n$ and $\tilde{F}$ is the internal part of RR polyform in type-IIA/B where the RR forms are each decomposed such that

$$F_n = \tilde{F}_n \mp e^{4A} vol_4 \wedge \lambda(\star_6 \tilde{F}_{10-n}). \tag{B.6}$$

As before upper/lower signs correspond to type-IIA/B

Clearly in general η_+^2 is composed of a parts that is parallel and a part that is orthogonal to η_+^1. The $SU(3) \times SU(3)$-structure can categorised into 3 distinct cases depending on the values of the coefficients $k_\perp$ and $k_{||}$:

$SU(3)$-structure

When $k_\perp = 0$ the internal spinors are parallel and the pure spinors define an $SU(3)$-structure in 6-d such that

$$\begin{aligned}\Phi_+ &= -e^{i\theta_+}\frac{e^A}{8}e^{-iJ}, \\ \Phi_- &= -ie^{i\theta_-}\frac{e^A}{8}\Omega_{hol}\end{aligned} \tag{B.7}$$

where J and Ω_{hol} are the two and holomorphic three forms associated with $SU(3)$, they are defined as in terms of the 6-d gamma matrices as

$$\Omega^{(hol)}_{abc} = -i\eta_-^\dagger\gamma_{abc}\eta_+, \quad J_{ab} = -i\eta_+^\dagger\gamma_{ab}\eta_+, \tag{B.8}$$

and satisfy

$$J\wedge\Omega_{hol} = 0, \quad J\wedge J\wedge J = \frac{3i}{4}\Omega_{hol}\wedge\bar{\Omega}_{hol}. \tag{B.9}$$

Orthogonal $SU(2)$-structure

When $k_{||} = 0$ the internal spinors are orthogonal and the pure spinors define an orthogonal $SU(2)$-structure in 6-d such that

$$\begin{aligned}\Phi_+ &= -ie^{i\theta_+}\frac{e^A}{8}e^{-v\wedge w}\wedge\omega, \\ \Phi_- &= ie^{i\theta_-}\frac{e^A}{8}(v+iw)\wedge e^{-ij}\end{aligned} \tag{B.10}$$

where the $SU(2)$-structure one forms v, w and two forms j, ω are defined as

$$w_a - iv_a = \eta_-^\dagger\gamma_a\chi_+, \quad j_{ab} = -i\eta_+^\dagger\gamma_{ab}\eta_+ + i\chi_+^\dagger\gamma_{ab}\chi_+, \quad \omega_{ab} = \eta_-^\dagger\gamma_{ab}\chi_-. \tag{B.11}$$

and obey the relations

$$\begin{aligned}&j\wedge\omega = \omega\wedge\omega = \iota_{(w-iv)}(\omega) = \iota_{(w-iv)}(j) = 0 \\ &j\wedge j = \frac{1}{2}\omega\wedge\bar{\omega}.\end{aligned} \tag{B.12}$$

Intermediate and Dynamical $SU(2)$-structure

For intermediate $SU(2)$-structure $k_{||}$ and $k_\perp$ are non zero constants, this and the previous example are also referred to as static $SU(2)$-structure. For dynamical $SU(2)$-structure $k_{||}$ and $k_\perp$ are point dependent. For both these cases the pure spinors are given by

$$\begin{aligned}\Phi_+ &= \frac{e^A}{8}e^{i\theta_+}e^{-iv\wedge w}\left(k_{\|}e^{-ij} - ik_{\perp}\omega\right) \\ \Phi_- &= \frac{ie^A}{8}e^{i\theta_-}(v+iw)\wedge\left(k_{\perp}e^{-ij} + ik_{\|}\omega\right)\end{aligned}, \tag{B.13}$$

where eq (B.12) and eq (B.11) still hold.

In these conventions the SUSY conditions (here we consider type-IIA, details of type-IIB are given in appendix E) may be split up as follows:

$$\begin{aligned}&d\left[e^{3A-\hat{\Phi}}k_{\|}\right] = 0\\ &d\left[e^{3A-\hat{\Phi}}\left(k_{\|}(j+v\wedge w)+k_{\perp}\omega)\right)\right] - ie^{3A-\hat{\Phi}}k_{\|}H = 0\\ &d\left[e^{3A-\hat{\Phi}}\left(\tfrac{1}{2}k_{\|}(j+v\wedge w)^2 + k_{\perp}v\wedge w\wedge\omega\right)\right] -\\ &ie^{3A-\hat{\Phi}}H\wedge\left(k_{\|}(j+v\wedge w)+k_{\perp}\omega\right) = 0\end{aligned} \tag{B.14}$$

where the second of these gives a definition for H which can be combined with the first to give a definition of the NS potential, namely

$$B_2 = -\frac{k_{\perp}}{k_{\|}}Im\omega \tag{B.15}$$

this is not the same as the NS potential generated by non-abelian T-duality but must match it up to an exact.

The rest of the SUSY conditions are

$$\begin{aligned}&\star_6 F_6 = 0\\ &d\left[e^{4A-\hat{\Phi}}k_{\perp}w_-\right] = -e^{4A}\star_6 F_4\\ &d\left[e^{2A-\hat{\Phi}}k_{\perp}w_+\right] = 0\\ &d\left[e^{4A-\hat{\Phi}}\left(k_{\|}\omega_-\wedge w - k_{\|}\omega_+\wedge v + k_{\perp}w_+\wedge j\right)\right] + e^{4A-\hat{\Phi}}k_{\perp}H\wedge w_- = -e^{4A}\star_6 F_2\\ &d\left[e^{2A-\hat{\Phi}}\left(k_{\|}\omega_+\wedge w + k_{\|}\omega_-\wedge v - k_{\perp}w_-\wedge j\right)\right] + k_{\perp}e^{2A-\hat{\Phi}}H\wedge w_+ = 0\\ &d\left[\frac{1}{2}e^{4A-\hat{\Phi}}k_{\perp}j\wedge j\wedge w_-\right] - e^{4A-\hat{\Phi}}H\wedge\left(k_{\|}\omega_-\wedge w - k_{\|}\omega_+\wedge v + k_{\perp}w_+\wedge j\right)\\ &d\left[\frac{1}{2}e^{2A-\hat{\Phi}}k_{\perp}j\wedge j\wedge w_+\right] - e^{2A-\hat{\Phi}}H\wedge\left(k_{\|}\omega_+\wedge w + k_{\|}\omega_-\wedge v - k_{\perp}w_-\wedge j\right)\end{aligned} \tag{B.16}$$

where

$$
\begin{aligned}
w_+ &= \sin\theta_- v + \cos\theta_- w, \quad w_- = \sin\theta_- - \cos\theta_- v, \\
\omega_+ &= \sin\theta_- \mathrm{Re}\omega + \cos\theta_- \mathrm{Im}\omega, \quad \omega_- = \sin\theta_- \mathrm{Im}\omega - \cos\theta_- \mathrm{Re}\omega
\end{aligned}
\tag{B.17}
$$

from which it is possible to define the higher forms of the RR sector as:

$$
\begin{aligned}
F_6 &= dC_5 \\
F_8 &= dC_7 - H \wedge C_5 \\
F_{10} &= dC_9 - H \wedge C_7
\end{aligned}
\tag{B.18}
$$

where we assume $B_2 \wedge B_2 \wedge B_2 = 0$ as required by $\mathcal{N} = 1$ SUSY. The RR potentials are given by

$$
\begin{aligned}
C_5 &= e^{4A-\hat{\Phi}} vol_4 \wedge k_\perp (\sin\theta_- w - \cos\theta_- v) \\
C_7 &= -e^{4A-\hat{\Phi}} vol_4 \wedge \Big[k_\parallel (\sin\theta_- \mathrm{Im}\omega - \cos\theta_- \mathrm{Re}\omega) \wedge w - \\
&\quad k_\parallel (\sin\theta_- \mathrm{Re}\omega + \cos\theta_- \mathrm{Im}\omega) \wedge v + k_\perp (\sin\theta_- v + \cos\theta_- w) \wedge j \Big] \\
C_9 &= \frac{1}{2} e^{4A-\hat{\Phi}} vol_4 \wedge k_\perp j \wedge j \wedge (\cos\theta_- v - \sin\theta_- w)
\end{aligned}
\tag{B.19}
$$

The calibration is given by

$$
\Psi_{cal} = -8 e^{3A-\hat{\Phi}} \mathrm{Im}\Phi_- e^{-B_2} \tag{B.20}
$$

where $\pm$ depends on our conventions in the WZ action. That $S_{DBI} + S_{WZ} = 0$ is trivial because we have:

$$
C_5 + C_7 + C_9 = -8 vol_4 \wedge e^{3A-\hat{\Phi}} Im\Phi_- \tag{B.21}
$$

C On static $SU(2)$-Structures in 6-d

In this section, we give further details regarding the $SU(2)$ structure that are used through out the main body of this text. We sketch the derivation of the conditions that the $SU(2)$-structure must satisfy for $\mathcal{N} = 1$ SUSY in type-IIA. We will also use these to define potentials for the space-time filling RR-fluxes and the calibrations for space-time filling D4, D6 and D8 branes. We assume a string frame metric of the form:

$$ds^2 = e^{2A} dy_{1,3}^2 + ds_6^2 \tag{C.1}$$

with a dilaton Φ and a NS three form $H = dB$. We further assume that $\Phi(z), A(z)$ with z any coordinate in ds_6^2. Expanding out the $SU(2)$ pure spinors in eq (3.2.8) gives:

$$\begin{aligned}
\Phi_+ &= \frac{|ab|}{8}\Big[\omega_2 - i\omega_2 \wedge v_1 \wedge w_1 - \frac{1}{2}\omega_2 \wedge v_1 \wedge w_1 \wedge v_1 \wedge w_1\Big]. \\
\Phi_- &= \frac{|ab|}{8}(1 - ij_2 - \frac{1}{2}j_2 \wedge j_2) \wedge (v_1 + iw_1). \\
\bar{\Phi}_- &= \frac{|ab|}{8}\Big[v_1 - iw_1 + j_2 \wedge (w_1 + iv_1) - \frac{1}{2}j_2 \wedge j_2 \wedge (v_1 - iw_1)\Big].
\end{aligned} \tag{C.2}$$

Supersymmetry requires that $|a| = |b|$, we define:

$$|ab| = |a|^2 = e^A \tag{C.3}$$

Plugging eq (C.2) into eq (3.2.4), equating forms with equal number of legs and separating real and imaginary parts gives,

$$\begin{aligned}
&d\Big[e^{3A-\Phi}\omega_2\Big] = 0 \\
&d\Big[e^{3A-\Phi}\omega_2 \wedge v_1 \wedge w_1\Big] + ie^{3A-\Phi} H \wedge \omega_2 = 0.
\end{aligned} \tag{C.4}$$

For two forms,

$$\begin{aligned}
&d\Big[e^{3A-\Phi}v_1\Big] - e^{3A-\Phi} dA \wedge v_1 = 0. \\
&d\Big[e^{3A-\Phi}w_1\Big] + e^{3A-\Phi} dA \wedge w_1 = -e^{3A} \star_6 F_4.
\end{aligned} \tag{C.5}$$

For four forms,

$$\begin{aligned}
&-d\Big[e^{3A-\Phi}j_2 \wedge w_1\Big] - e^{3A-\Phi} H \wedge v_1 + e^{3A-\Phi} dA \wedge j_2 \wedge w_1 = 0, \\
&d\Big[e^{3A-\Phi}j_2 \wedge v_1\Big] - e^{3A-\Phi} H \wedge w_1 + e^{3A-\Phi} dA \wedge j_2 \wedge v_1 = e^{3A} \star_6 F_2,
\end{aligned} \tag{C.6}$$

while for the six-form,

$$
\begin{aligned}
&-\frac{1}{2}d\left[e^{3A-\Phi}j_2\wedge j_2\wedge v_1\right]+\\
&\qquad e^{3A-\Phi}H\wedge j_2\wedge w_1+\frac{1}{2}e^{3A-\Phi}dA\wedge j_2\wedge j_2\wedge v_1=0.\\
&\frac{1}{2}d\left[e^{3A-\Phi}j_2\wedge j_2\wedge w_1\right]+\\
&\qquad e^{3A-\Phi}H\wedge j_2\wedge v_1+\frac{1}{2}e^{3A-\Phi}dA\wedge j_2\wedge j_2\wedge w_1=e^{3A}\star_6 F_0.
\end{aligned}
\tag{C.7}
$$

Finally, we have for the zero-form

$$
\star_6 F_6 = 0 \tag{C.8}
$$

where the fluxes F_0, F_2 and F_4 are understood to have legs in the 6-d internal space only. These equations can be further simplified as follows:

$$
\begin{aligned}
&d\left[e^{3A-\Phi}\omega_2\right]=0\\
&\omega_2\wedge\left[d(v_1\wedge w_1)+iH\right]=0\\
&d\left[e^{2A-\Phi}v_1\right]=0\\
&d\left[e^{4A-\Phi}w_1\right]=-e^{4A}\star_6 F_4\\
&d\left[e^{2A-\Phi}j_2\wedge w_1\right]+e^{2A-\Phi}H\wedge v_1=0\\
&d\left[e^{4A-\Phi}j_2\wedge v_1\right]-e^{4A-\Phi}H\wedge w_1=e^{4A}\star_6 F_2\\
&\frac{1}{2}d\left[e^{2A-\Phi}j_2\wedge j_2\wedge v_1\right]-e^{2A-\Phi}H\wedge j_2\wedge w_1=0\\
&\frac{1}{2}d\left[e^{4A-\Phi}j_2\wedge j_2\wedge w_1\right]+e^{4A-\Phi}H\wedge j_2\wedge v_1=e^{4A}\star_6 F_0\\
&\star_6 F_6=0.
\end{aligned}
\tag{C.9}
$$

We clearly now have a definition of the Minkowski space-time filling RR-sector in terms of the $SU(2)$-structure:

$$
\begin{aligned}
F_6&=d\left[e^{4A-\Phi}vol_4\wedge w_1\right]\\
F_8&=d\left[e^{4A-\Phi}vol_4\wedge j_2\wedge v_1\right]-e^{4A-\Phi}H\wedge vol_4\wedge w_1\\
F_{10}&=-\frac{1}{2}d\left[e^{4A-\Phi}vol_4\wedge j_2\wedge j_2\wedge w_1\right]+e^{4A-\Phi}H\wedge vol_4\wedge j_2\wedge v_1,
\end{aligned}
\tag{C.10}
$$

where the remaining fluxes can be obtained from the duality condition $F_{2n} = (-)^n \star F_{10-2n}$. With these fluxes it is possible to derive expressions for the potentials associated with these fluxes. They take the most compact form when the space time filling part of the RR flux ployform is expressed as[1],

$$F_{Mink} = dC_{Mink} - H \wedge C_{Mink}. \tag{C.11}$$

We must have $-H \wedge C_3 + \frac{1}{3!}F_0 B^3 = 0$ for $\mathcal{N} = 1$ SUSY, otherwise the final line in eq (C.9) cannot hold. This allows the derivation of canonical potentials in terms of the $SU(2)$-structure,

$$\begin{aligned} C_5 &= e^{4A-\Phi} vol_4 \wedge w_1 \\ C_7 &= e^{4A-\Phi} vol_4 \wedge j_2 \wedge v_1 \\ C_9 &= -\frac{1}{2} e^{4A-\Phi} vol_4 \wedge j_2 \wedge j_2 \wedge w_1. \end{aligned} \tag{C.12}$$

The calibration for type-IIA space time filling D branes is defined as

$$\Psi_{cal} = -8e^{3A-\Phi}(\mathrm{Im}\,\Psi_-) \wedge e^B, \tag{C.13}$$

expanding this out and extracting the terms with an equal number of legs gives:

$$\begin{aligned} \Psi^{(1)}_{cal} &= -e^{4A-\Phi} w_1 \\ \Psi^{(3)}_{cal} &= e^{4A-\Phi}\Big(v_1 \wedge j_2 - w_1 \wedge B\Big) \\ \Psi^{(5)}_{cal} &= e^{4A-\Phi}\Big(\frac{1}{2} w_1 \wedge j_2 \wedge j_2 + v_1 \wedge j_2 \wedge B - \frac{1}{2} w_1 \wedge B \wedge B\Big). \end{aligned} \tag{C.14}$$

Which makes it clear that an $SU(2)$-structure in 6-d can potentially support Minkowski space time filling D4, D6 and D8 branes wrapping one, three, and five-cycles respectively.

D Some Details of the Flavoured $SU(3)$ and $SU(2)$-structure solutions

We will start analysing the case of the addition of flavors to the Klebanov-Witten field theory [68]. This will be explicitly dealt with using the language of $SU(3)$-structures. Then, we will extend the analysis to the background generated in section 3.4. This will require the full $SU(2)$-structure formalism, de-

[1] We are assuming B is defined only on the internal space so that $B^4 = 0$.

veloped above.

We consider the addition of Minkowski space time filling sources to an $SU(3)$-structure background in type-IIB. The action of type-IIB in string frame is modified as:

$$S = S_{IIB} + S_{DBI} + S_{WZ}. \tag{D.1}$$

With pure spinors defined as in eq (3.2.7) the calibration condition is given by:

$$\Psi_{Cal\ IIB} = -8e^{3A-\Phi}\Big(\mathrm{Im}\Phi_+\Big) = e^{-\Phi}\left(\frac{e^{\Phi}}{h}\right)\left(1 - \frac{1}{2}J \wedge J\right) \tag{D.2}$$

is compatible with source D3 and D7 branes We are assuming, as it is true for the Klebanov-Witten model with massless flavours, that $H = 0$. The combined DBI action of such a system will be given by:

$$\begin{aligned} S_{DBI} &= S^{D3}_{DBI} + S^{D7}_{DBI}, \\ S^{D3}_{DBI} &= -\int_{M_{10}} e^{-\Phi}\left(\frac{e^{\Phi}}{h}\right) vol_4 \wedge \Xi_6, \\ S^{D7}_{DBI} &= \frac{1}{2}\int_{M_{10}} e^{-\Phi}\left(\frac{e^{\Phi}}{h}\right) vol_4 \wedge J \wedge J \wedge \Xi_2. \end{aligned} \tag{D.3}$$

While the WZ terms will be given by:

$$\begin{aligned} S_{WZ} &= S^{D3}_{WZ} + S^{D7}_{WZ}, \\ S^{D3}_{WZ} &= -\int_{M_{10}} C_4 \wedge \Xi_6, \\ S^{D7}_{WZ} &= \int_{M_{10}} C_8 \wedge \Xi_2. \end{aligned} \tag{D.4}$$

The fluxes, in the presence of sources for the case of $B = 0$, should be defined as,

$$H = dB, \quad F_1 = dC_0, \quad F_3 = dC_2, \quad F_5 = dC_4 \tag{D.5}$$

and the Bianchi identities are modified as follows:

$$\begin{aligned} &dH = 0, \quad dF_1 = \Xi_2, \quad dF_3 - H \wedge F_1 = 0, \\ &dF_5 - H \wedge F_3 = \Xi_6. \end{aligned} \tag{D.6}$$

where which of the Ξ_i's are non zero is determined by the specific source brane content. The dual fluxes, related by the expression $F_{2n+1} = (-)^n \star F_{9-2n}$, are defined as:

$$\star F_5 = F_5, \quad F_7 = dC_6, \quad F_9 = dC_8 \tag{D.7}$$

and the fluxes have the following equations of motion:

$$d \star F_1 = 0, \quad d \star F_3 = 0 \tag{D.8}$$

for Klebanov-Witten with massless flavours we should set $\Xi_6 = 0$ and then the equation of motion of the dilaton and Einstein's equations can be shown to be satisfied also as in [60, 61].

D.1 Analysis of the Solution Generated

In chapter 3 we generated a flavoured type-IIA solution which supports an $SU(2)$-structure and non closed B. The action of (massive) type-IIA in string frame, is now modified,

$$S = S_{\text{Massive IIA}} + S_{DBI} + S_{WZ} \tag{D.9}$$

As shown around eq (C.14), an $SU(2)$-structure can in general support smeared source D4, D6 and D8 branes that extend in the Minkowski directions. The combined DBI and WZ actions of this system are given by:

$$\begin{aligned}
S_{DBI} &= S_{DBI}^{D8} + S_{DBI}^{D6} + S_{DBI}^{D4} \\
S_{DBI}^{D4} &= \int_{M_{10}} e^{-\hat{\Phi}} \left(\frac{e^{\Phi}}{h} vol_4 \right) \wedge w_1 \wedge \Xi_5, \\
S_{DBI}^{D6} &= - \int_{M_{10}} e^{-\hat{\Phi}} \left(\frac{e^{\Phi}}{h} vol_4 \right) \wedge \left(v_1 \wedge j_2 - w_1 \wedge B \right) \wedge \Xi_3, \\
S_{DBI}^{D8} &= - \int_{M_{10}} e^{-\hat{\Phi}} \left(\frac{e^{\Phi}}{h} vol_4 \right) \wedge \left(\frac{1}{2} w_1 \wedge j_2 \wedge j_2 + v_1 \wedge j_2 \wedge B - \frac{1}{2} w_1 \wedge B \wedge B \right) \wedge \Xi_1.
\end{aligned} \tag{D.10}$$

and

$$\begin{aligned}
S_{WZ} &= S_{WZ}^{D8} + S_{WZ}^{D6} + S_{WZ}^{D4}, \\
S_{WZ}^{D4} &= - \int_{M_{10}} C_5 \wedge \Xi_5, \\
S_{WZ}^{D6} &= \int_{M_{10}} \left(C_7 - B \wedge C_5 \right) \wedge \Xi_3, \\
S_{WZ}^{D8} &= - \int_{M_{10}} \left(C_9 - B \wedge C_7 + \frac{1}{2} B \wedge B \wedge C_5 \right) \wedge \Xi_1.
\end{aligned} \tag{D.11}$$

In the presence of such sources we should define the RR-potentials as:

$$F_0, \quad F_2 = dC_1 + F_0 B, \quad F_4 = dC_3 + B \wedge dC_1 + \frac{F_0}{2} B \wedge B; \tag{D.12}$$

this ensures that we have no ill defined potential terms appearing explicitly. We note that source D8 branes imply that F_0 will no longer be quantised. In general the Bianchi identities are given by,

$$\begin{aligned} dF_0 &= \Xi_1, \quad dF_2 - F_0 H = \Xi_3 + B \wedge \Xi_1; \\ dF_4 - H \wedge F_2 &= \Xi_5 + B \wedge \Xi_3 + \frac{1}{2} B \wedge B \wedge \Xi_1 \end{aligned} \tag{D.13}$$

The dual fluxes, related by the expression $F_{2n} = (-)^n \star F_{10-2n}$, are defined as:

$$\begin{aligned} F_6 &= dC_5, \quad F_8 = dC_7 - H \wedge C_5, \\ F_{10} &= dC_9 - H \wedge C_7. \end{aligned} \tag{D.14}$$

Here, we did not write the terms that are zero due to the $SU(2)$ SUSY conditions in 6-d. The flux equations of motion for the RR sector are given by:

$$d \star F_2 + H \wedge \star F_4 = 0, \quad d \star F_4 + H \wedge F_4 = 0, \tag{D.15}$$

while for the NS sector we find:

$$\begin{aligned} d\left(e^{-2\hat{\Phi}} \star H\right) &= F_0 \star F_2 + F_2 \wedge \star F_4 + \frac{1}{2} F_4 \wedge F_4 - \\ &\quad \frac{e^{\Phi - \hat{\Phi}}}{h} \left[vol_4 \wedge (w_1 \wedge B - v_1 \wedge j_2) \wedge \Xi_1 + vol_4 \wedge w_1 \wedge \Xi_3 \right], \end{aligned} \tag{D.16}$$

A careful calculation shows that the potentials do not enter into this equation explicitly [64]. We can express the variation of the dilaton as an integral for compactness,

$$S_{DBI} = -\int 8 e^{-2\hat{\Phi}} (d \star d\hat{\Phi} + \star \frac{R}{4} - d\hat{\Phi} \wedge \star d\hat{\Phi} - \frac{1}{8} H \wedge \star H) \tag{D.17}$$

It is useful at this stage to introduce the following notation,

$$\omega_{(p)} \lrcorner \lambda_{(p)} = \frac{1}{p!} \omega^{\mu_1 \dots \mu_p} \lambda_{\mu_1 \dots \mu_p} \tag{D.18}$$

where the following identity is helpful,

$$\int \omega_{(p)} \wedge \lambda_{(10-p)} = -\int \sqrt{-g}\lambda \lrcorner (\star\omega). \tag{D.19}$$

Then Einstein equations can be expressed in a gauge invariant fashion as:

$$\begin{aligned} R_{\mu\nu} =& -2D_\mu D_\nu \hat{\Phi} + \frac{1}{4}H^2_{\mu\nu} + \\ & e^{2\hat{\Phi}}\left[\frac{1}{2}(F_2^2)_{\mu\nu} + \frac{1}{12}(F_4^2)_{\mu\nu} - \frac{1}{4}g_{\mu\nu}(F_0^2 + \frac{1}{2}F_2^2 + \frac{1}{4!}F_4^2)\right] + \\ & \frac{e^{\Phi+\hat{\Phi}}}{h}\Bigg[\frac{1}{48}(\Xi_5 + \Xi_3 \wedge B + \frac{1}{2}B \wedge B \wedge \Xi_1)_{\mu\alpha_1\ldots\alpha_4} \star (vol_4 \wedge w_1)_\nu^{\alpha_1\ldots\alpha_4} \\ & - \frac{1}{4}(\Xi_3 + B \wedge \Xi_1)_{\mu\alpha_1\alpha_2} \star (vol_4 \wedge v_1 \wedge j_2)_\nu^{\alpha_1\alpha_2} \\ & - \frac{1}{4}\Xi_{1\,\mu} \star (vol_4 \wedge w_1 \wedge j_2 \wedge j_2)_\nu \\ & - \frac{1}{4}g_{\mu\nu}\Bigg((\Xi_5 + \Xi_3 \wedge B + \frac{1}{2}B \wedge B \wedge \Xi_1)\lrcorner \star (vol_4 \wedge w_1) \\ & - (\Xi_3 + B \wedge \Xi_1)\lrcorner \star (vol_4 \wedge v_1 \wedge j_2) \\ & - \frac{1}{2}\Xi_1 \lrcorner \star (vol_4 \wedge w_1 \wedge j_2 \wedge j_2)\Bigg)\Bigg] \end{aligned} \tag{D.20}$$

The eqs (D.13)-(D.20) are solved by the metric, fluxes and sources of section 3.4 once the BPS eqs (3.4.10) are imposed.

E Details of the non-Abelian T-duality on the D5 branes solution.

The purpose of this section is to give some details of the $SU(2)$ isometry T-dual of Wrapped D5 branes on S^2. This was first derived in [30], but in slightly different conventions and the G-structure was not found. This is the $\mathcal{C} = 1$, $\mathcal{S} = 0$ limit of the full Baryonic Branch dual solution, and as the procedure for find the the G-structure is the same in both case we hope that this more simple example will be instructive.

Solution of wrapped D5 branes on S^2 [17] has string frame metric given by

$$\begin{aligned} ds^2 =& e^{\Phi}\Big(dx^2_{1,3} + e^{2k}d\rho + e^{2h}(d\theta^2 + \sin^2\theta d\varphi^2) + \\ & \frac{e^{2g}}{4}((\tilde{\omega}_1 + ad\theta)^2 + (\tilde{\omega}_2 - a\sin\theta d\varphi)^2) + \frac{e^{2k}}{4}(\tilde{\omega}_3 + \cos d\varphi)^2\Big) \end{aligned} \tag{E.1}$$

where the functions a, b, g, h, k and the dilaton Φ only depend on the holographic coordinate r. The $\tilde{\omega}_i$ are $SU(2)$ left invariant 1-forms which can be parametrised as

$$\begin{aligned}
\tilde{\omega}_1 &= \cos\psi d\tilde{\theta} + \sin\psi \sin\tilde{\theta} d\tilde{\varphi}\,, \\
\tilde{\omega}_2 &= -\sin\psi d\tilde{\theta} + \cos\psi \sin\tilde{\theta} d\tilde{\varphi}\,,. \\
\tilde{\omega}_3 &= d\psi + \cos\tilde{\theta} d\tilde{\varphi}\,.
\end{aligned} \tag{E.2}$$

A convenient set of vielbeins is given by

$$\begin{aligned}
&e^{x^i} = e^{\frac{\Phi}{2}} dx^i\,, \quad e^{\rho} = e^{\frac{\Phi}{2}+k} d\rho\,, \quad e^{\theta} = e^{\frac{\Phi}{2}+h} d\theta\,, \quad e^{\varphi} = e^{\frac{\Phi}{2}+h} \sin\theta\, d\varphi\,, \\
&e^1 = \frac{1}{2} e^{\frac{\Phi}{2}+g} (\tilde{\omega}_1 + a\, d\theta)\,, \qquad e^2 = \frac{1}{2} e^{\frac{\Phi}{2}+g} (\tilde{\omega}_2 - a\, \sin\theta\, d\varphi)\,, \\
&e^3 = \frac{1}{2} e^{\frac{\Phi}{2}+k} (\tilde{\omega}_3 + \cos\theta\, d\varphi)\,.
\end{aligned} \tag{E.3}$$

with respect to which the non trivial RR flux F_3 may be expressed as

$$F_3 = e^{-\frac{3}{2}\Phi} \left[f_1 e^{123} + f_2 e^{\theta\varphi 3} + f_3 (e^{\theta 23} + e^{\varphi 13}) + f_4 (e^{\rho 1 \theta} + e^{\rho \varphi 2}) \right] \tag{E.4}$$

where the f_i are given by eq (5.2.8). In these conventions the projections the 10-d Killing spinor ϵ obeys are

$$\Gamma_{12}\epsilon = \Gamma_{\theta\varphi}\epsilon, \quad \Gamma_{r123}\epsilon = (\cos\alpha + \sin\alpha \Gamma_{\varphi 2})\epsilon, \quad i\epsilon^* = \epsilon\,, \tag{E.5}$$

with respect to the $4+6$ split we can define components of ϵ to be equal with positive chirality as

$$\epsilon_1 = \epsilon_2 = e^{A} (\zeta_+ \otimes \eta_+ + \zeta_- \otimes \eta_-) \tag{E.6}$$

where $2A = \Phi$. Once the usual decomposition of gamma matrices,

$$\Gamma_\mu = \hat{\gamma}_\mu \otimes \mathbb{I}, \quad \Gamma_a = \mathbb{I} \otimes \gamma_a \tag{E.7}$$

is performed it is a simple matter to derive the $SU(3)$-structure forms of eq (5.2.15) using eq (B.8), where we have chosen $i\gamma_{r\theta\varphi 123}\eta_+ = \eta_+$. To do this it is helpful to perform a rotation in e^φ, e^2 which will also be useful later

$$\begin{aligned}
\hat{e}^{\varphi} &= \cos\alpha e^{\varphi} + \sin\alpha e^2 \\
\hat{e}^2 &= -\sin\alpha e^{\varphi} + \cos\alpha e^2 \\
\hat{e}^a &= e^a \text{ for } a \neq \varphi, 2.
\end{aligned} \tag{E.8}$$

The rotated 6-d projections are then simply

$$\hat{\gamma}_{\varphi\theta}\eta_+ = \hat{\gamma}_{r3}\eta_+ = \hat{\gamma}_{21}\eta_+ = i\,\eta_+ \tag{E.9}$$

and the $SU(3)$-structure becomes canonical.

We want to T-dualise this wrapped D5-brane solution along the $SU(2)$ isometry parametrised by $\tilde{\omega}_i$. Section 2 and appendix B of [30] give all the details of the algorithm one must follow to do this and so we direct the interested reader there for details of the NS sector. For the RR sector we only give details that will be relevant for later calculations.

The duality will drastically change the vielbeins that contain the $SU(2)$ left invariant 1-forms e^1, e^2, e^3 and leave the others untouched. For the dual of the wrapped D5 brane solution gauge fixed such that the remaining dual coordinates are v_2, v_3 and ψ, the canonical vielbeins given by the procedure of [30] are

$$\begin{aligned}
e^{\hat{1}'} =& \frac{e^{g+\frac{\Phi}{2}}}{8W}\Big[e^{2k+\Phi}\Big(-\sqrt{2}e^{2g+\Phi}(\cos\psi(a\omega_2 v_3+dv_2)+\sin\psi(a\omega_1 v_3-\omega_3 v_2))-\\
&4v_3\sin\psi(a\omega_2 v_3+dv_2)+4v_3\cos\psi(a\omega_1 v_3-\omega_3 v_2)\Big)-\\
&4v_2 e^{2g+\Phi}\sin\psi(a\omega_2 v_2-dv_3)-8\sqrt{2}v_2\cos\psi(v_2dv_2+v_3dv_3)\Big] \qquad \text{(E.10)}\\
e^{\hat{2}'} =& \frac{e^{g+\frac{\Phi}{2}}}{8W}\Big[e^{2k+\Phi}\Big(\sqrt{2}e^{2g+\Phi}(\cos\psi(\omega_3 v_2-a\omega_1 v_3)+a\omega_2 v_3\sin\psi+dv_2\sin\psi)-\\
&4v_3(\cos\psi(a\omega_2 v_3+dv_2)+\sin\psi(a\omega_1 v_3-\omega_3 v_2))\Big)-\\
&4v_2 e^{2g+\Phi}\cos\psi(a\omega_2 v_2-dv_3)+8\sqrt{2}v_2\sin\psi(v_2dv_2+v_3dv_3)\Big]\\
e^{\hat{3}'} =& \frac{e^{k+\frac{\Phi}{2}}}{8W}\Big[\sqrt{2}e^{4g+2\Phi}(a\omega_2 v_2-dv_3)+4v_2 e^{2g+\Phi}(\omega_3 v_2-a\omega_1 v_3)-\\
&8\sqrt{2}v_3(v_2dv_2+v_3dv_3)\Big]
\end{aligned}$$

with the remaining vielbeins still given by eq (E.3), that is $e^{a'}=e^a$ for $a\neq 1,2,3$. The ω_i are defined as in eq (E.2) but with $\tilde{\theta}\to\theta$, $\tilde{\varphi}\to\varphi$. It is possible to remove all the explicit angular dependence from the dual solution by performing a rotation in the θ, φ directions such that

$$\begin{aligned}
e^{\hat{\theta}} &= e^{h+\Phi/2}\omega_1 = \cos\psi e^{\theta}+\sin\psi e^{\varphi}\\
e^{\hat{\varphi}} &= e^{h+\Phi/2}\omega_2 = -\sin\psi e^{\theta}+\cos\psi e^{\varphi}\,,
\end{aligned} \tag{E.11}$$

and an additional rotation in 1′,2′,3′ directions such that

$$
\begin{aligned}
e^{\hat{1}} &= \cos\psi e^{1'} - \sin\psi e^{2'} \\
e^{\hat{2}} &= \sin\psi e^{1'} + \cos\psi e^{2'} \\
e^{\hat{3}} &= e^{3'}.
\end{aligned}
\tag{E.12}
$$

Theses rotation make the expressions for the vielbeins and fluxes a lot more simple than they otherwise would be, they are given for the dual of the wrapped D5 solution as in section 5.3 but with $\mathcal{S} = 0$, $\mathcal{C} = 1$. However, it is the $e^{a'}$ vielbeins rather than the $e^{\hat{a}}$ ones that are more suited to calculating the G-structure of the dual solution.

It was shown explicitly in [41] that the 10-d MW Killing spinors transform under an $SU(2)$ isometry T-duality as

$$
\hat{\epsilon}_1 = \epsilon_1, \quad \hat{\epsilon}_2 = \Omega\epsilon_2. \tag{E.13}
$$

where Ω is given by

$$
\Omega = \Gamma^{(10)} \frac{-\Gamma_{123} + \sum_{a=1}^{3} \zeta_a \Gamma^a}{\sqrt{1+\zeta^2}} \tag{E.14}
$$

and for the wrapped D5 background we have

$$
\begin{aligned}
&\zeta^1 = 2\sqrt{2}e^{-g-k-\phi}v_2 \cos\psi, \quad \zeta^2 = -2\sqrt{2}e^{-g-k-\phi}v_2 \sin\psi, \\
&\zeta^3 = 2\sqrt{2}e^{-2g-\phi}v_3.
\end{aligned}
\tag{E.15}
$$

Starting from eq (E.10) we first rotate the vielbeins as in eq (E.8) so that the projections are canonical. The Ω matrix then becomes

$$
\Omega = \frac{\cos\alpha\hat{\Gamma}^{123} + \sin\alpha\hat{\Gamma}^{1\varphi 3} + \zeta_1\hat{\Gamma}^1 + \zeta_2\cos\alpha\hat{\Gamma}^2 + \zeta_2\sin\alpha\hat{\Gamma}^{\varphi} + \zeta_3\hat{\Gamma}^3}{\sqrt{1+\zeta.\zeta}} \tag{E.16}
$$

where we have used $\gamma^{1\varphi 3}\eta_+ = i\eta_-$. The new spinor $\hat{\epsilon}_2$ is:

$$
\hat{\epsilon}_2 = e^{\Phi/4}(\zeta_+ \otimes \hat{\eta}^2_- + \zeta_- \otimes \hat{\eta}^2_+) \tag{E.17}
$$

where

$$
\begin{aligned}
\hat{\eta}^2_- = &\frac{\cos\alpha\hat{\gamma}^r + \zeta_1\hat{\gamma}^1 + \zeta_2\cos\alpha\hat{\gamma}^2 + \zeta_3\hat{\gamma}^3 + \zeta_2\sin\alpha\hat{\gamma}^{\varphi}}{\sqrt{1+\zeta.\zeta}}\eta_+ + \\
&i\frac{\sin\alpha}{\sqrt{1+\zeta.\zeta}}\eta_-.
\end{aligned}
\tag{E.18}
$$

It is clear here that, as long as $\sin\alpha \neq 0$, we are in the dynamical $SU(2)$-structure case, because $\alpha = \alpha(r)$. In order to simplify the expressions we perform another transformation of the vielbein basis:

$$R = \frac{1}{\sqrt{\Delta}} \begin{pmatrix} \cos\alpha & 0 & 0 & \zeta_1 & \zeta_2\cos\alpha & \zeta_3 \\ 0 & \sqrt{\Delta} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\Delta} & 0 & 0 & 0 \\ -\zeta_1 & 0 & 0 & \cos\alpha & \zeta_3 & -\zeta_2\cos\alpha \\ -\zeta_2\cos\alpha & 0 & 0 & -\zeta_3 & \cos\alpha & \zeta_1 \\ -\zeta_3 & 0 & 0 & \zeta^2\cos\alpha & -\zeta^1 & \cos\alpha \end{pmatrix} \tag{E.19}$$

where

$$\Delta = \cos^2\alpha + \zeta_1^2 + \zeta_2^2\cos^2\alpha + \zeta_3^2 \tag{E.20}$$

We define a new basis:

$$\tilde{e} = R.\hat{e} \tag{E.21}$$

where the order is $r\theta\varphi 123$. In terms of this new basis, the spinor is:

$$\tilde{\eta}^2_- = \left(\frac{\sqrt{\Delta}\tilde{\gamma}^r + \zeta_2\sin\alpha\tilde{\gamma}^\varphi}{\sqrt{1+\zeta.\zeta}}\right)\eta_+ + i\frac{\sin\alpha}{\sqrt{1+\zeta.\zeta}}\eta_- \tag{E.22}$$

And the projections in this basis are still:

$$\tilde{\gamma}_{\varphi\theta}\eta_+ = \tilde{\gamma}_{r3}\eta_+ = \tilde{\gamma}_{21}\eta_+ = i\,\eta_+ \tag{E.23}$$

Let us now express the forms of the geometric structure, following the conventions of appendix ??.

$$\begin{aligned}
&e^{2A} = e^{\Phi} \\
&\theta_+ = 0 \qquad \theta_- = 0 \\
&k_\parallel = \frac{\sin\alpha}{\sqrt{1+\zeta.\zeta}} \qquad k_\perp = \sqrt{\frac{\cos^2\alpha + \zeta.\zeta}{1+\zeta.\zeta}} \\
&z = w - i\,v = \frac{1}{\sqrt{\cos^2\alpha+\zeta.\zeta}}\left(\sqrt{\Delta}\tilde{e}^3 + \zeta_2\sin\alpha\tilde{e}^\theta + i(\sqrt{\Delta}\tilde{e}^r + \zeta_2\sin\alpha\tilde{e}^\varphi)\right) \\
&j = \tilde{e}^{r3} + \tilde{e}^{\varphi\theta} + \tilde{e}^{21} - v\wedge w \\
&\omega = \frac{i}{\sqrt{\cos^2\alpha+\zeta.\zeta}}\left(\sqrt{\Delta}(\tilde{e}^\varphi + i\tilde{e}^\theta) - \zeta_2\sin\alpha(\tilde{e}^r + i\tilde{e}^3)\right)\wedge(\tilde{e}^2 + i\tilde{e}^1)
\end{aligned} \tag{E.24}$$

which is a dynamical $SU(2)$-structure.

F Details of the non-Abelian T-duality on the Baryonic Branch solution

In this section we give some details of the $SU(2)$ isometry T-dual of the Baryonic Branch of Klebanov-Strassler. This was originally derived in [30] with gauge fixing such that $v_1 = \varphi = \theta = 0$. The previous derivation indicated a departure in the T-dual from the log corrected AdS_5 asymptotics of the Baryonic branch. Let as begin by giving some details of original calculation in our current conventions

F.1 Dual of the Baryonic Branch without the shift in B_2

Once more we will start by specifying the dual vielbeins. The components

$$e^{x^i} = e^{\frac{\Phi}{2}}\hat{h}^{-\frac{1}{4}}dx^i\,, \quad e^{\rho} = e^{\frac{\Phi}{2}+k}\hat{h}^{\frac{1}{4}}d\rho \tag{F.1}$$

do not change. The vielbeins in the θ, φ are also unchanged by the duality however we find it useful to introduce a rotation in e^{θ}, e^{φ} such that the dual solution has no explicit ψ dependence.

$$e^{\hat{\theta}} = \sqrt{C}e^{h+\Phi/2}\omega_1, \quad e^{\hat{\varphi}} = \sqrt{C}e^{h+\Phi/2}\omega_2, \tag{F.2}$$

The vielbeins in the directions $\hat{1}, \hat{2}, \hat{3}$ can be compactly written in terms of the quantities defined as,

$$\begin{aligned}
\mathcal{V}_3 &= v_3 + \frac{e^{2g+\Phi}}{2\sqrt{2}}\mathcal{S}\cos\alpha\,,\\
\Lambda &= d\mathcal{V}_3 + \frac{e^{\Phi-2h}}{2\sqrt{2}}\mathcal{S}N_c\left(e^{2g} + 2e^{2h} - ae^{g}(be^{g} - 2e^{h}\cot\alpha)\right)d\rho\,,\\
\mu_1 &= ae^{g}\cos\alpha + 2e^{h}\sin\alpha\,,
\end{aligned} \tag{F.3}$$

With these, we have

$$\begin{aligned}
e^{\hat{1}} &= \frac{e^{g+\Phi/2}}{16\mathcal{W}}\sqrt{\mathcal{C}}\Big[e^{2k+\Phi}\Big(8\mathcal{V}_3(a\mathcal{V}_3\omega_1 - v_2\omega_3) - 2\sqrt{2}e^{2g+\Phi}\mathcal{C}(dv_2 + a\mathcal{V}_3\omega_2)\\
&\qquad - 2\sqrt{2}e^{g+\Phi}\mathcal{S}\mathcal{V}_3\mu_1\omega_1 + e^{3g+2\Phi}\mathcal{C}\mathcal{S}\mu_1\omega_2\Big)\\
&\qquad + 8v_2\big(e^{g+\Phi}v_2\mathcal{S}\mu_1\omega_2 - 2\sqrt{2}(\mathcal{V}_3\Lambda + v_2 dv_2)\big)\Big],\\
e^{\hat{2}} &= \frac{e^{g+\Phi/2}}{16\mathcal{W}}\mathcal{C}^{3/2}\Big[e^{2k}\Big(- 2\sqrt{2}e^{2g+\Phi}\mathcal{C}(a\mathcal{V}_3\omega_1 - v_2\omega_3) - 8\mathcal{V}_3(dv_2 + a\mathcal{V}_3\omega_2)\\
&\qquad + e^{3g+2\Phi}\mathcal{C}\mathcal{S}\mu_1\omega_1 + 2\sqrt{2}e^{g+\Phi}\mathcal{S}\mathcal{V}_3\mu_1\omega_2\Big)\\
&\qquad - 8e^{2g}v_2(-\Lambda + av_2\omega_2)\Big],\\
e^{\hat{3}} &= \frac{e^{k+\Phi/2}}{16\mathcal{W}}\sqrt{\mathcal{C}}\Big[e^{g+\Phi}v_2\Big(\sqrt{2}e^{2g+\Phi}\mathcal{C}(ae^{g}\mathcal{C}\omega_2 + \mathcal{S}\mu_1\omega_1)\\
&\qquad - 4e^{g}\mathcal{C}(a\mathcal{V}_3\omega_1 - v_2\omega_3) + 4\mathcal{S}\mathcal{V}_3\mu_1\omega_2\Big)\\
&\qquad - \sqrt{2}\Lambda(e^{4g+2\Phi}\mathcal{C}^2 + 8\mathcal{V}_3^2) - 8\sqrt{2}v_2\mathcal{V}_3 dv_2\Big]
\end{aligned} \tag{F.4}$$

where the rotation of eq (E.12) has been performed [2]. We will then have a metric that in terms of these vielbeins reads, $ds_{st}^2 = \sum_{i=1}^{10}(e^i)^2$. Notice that the quantity Λ in eq (5.3.4) will, when squared to construct the metric with the vielbeins above, imply the existence of crossed terms $g_{\rho v_3}$ and also the change of the asymptotic behaviour of $g_{\rho\rho}$ away from log corrected AdS_5.

In terms of these vielbeins, the NS two-form B_2 reads,

$$\begin{aligned}
\widehat{B}_2 = &-\frac{1}{4v_2}\Big(2e^{-h}a(e^{g}v_2e^{\hat{\theta}\hat{1}} + e^{k}\mathcal{V}_3e^{\hat{\theta}\hat{3}}) - 4e^{k-g}\mathcal{V}_3e^{\hat{1}\hat{3}} + \sqrt{2}\mathcal{C}e^{g+k+\Phi}e^{\hat{2}\hat{3}}\Big) +\\
&\frac{\mathcal{S}}{\mathcal{C}}\Big[\frac{\mathcal{V}_3e^{k}}{2v_2}(ae^{-h}e^{r\hat{\theta}} - 2e^{-g}e^{r\hat{1}}) + \frac{e^{g+k+\Phi-h}}{4\sqrt{2}V_2}\mathcal{C}(2e^{2h}e^{r\hat{2}} + \mu_1e^{\hat{\theta}\hat{1}}) -\\
&\qquad\frac{e^{-h}}{2}(2e^{h}\cos\alpha - ae^{g}\sin\alpha)e^{\hat{\theta}\hat{\varphi}} + e^{\rho\hat{3}} - \frac{e^{-h}}{2}\mu_1e^{\hat{\theta}\hat{2}}\Big].
\end{aligned} \tag{F.5}$$

[2] Actually this differs from [30] in orientation which can be compensated for via $\hat{1} \leftrightarrow \hat{2}$.

The dual dilaton is given by

$$\widehat{\Phi} = \Phi - \frac{1}{2}\ln \mathcal{W}\,, \quad \mathcal{W} = \mathcal{C}\left(\frac{1}{8}e^{4g+2k+3\Phi}\mathcal{C}^2 + e^{2g+\Phi}v_2^2 + e^{2k+\Phi}\mathcal{V}_3^2\right). \qquad \text{(F.6)}$$

And the RR sector is given by,

$$F_0 = \frac{N_c}{\sqrt{2}}\,,$$

$$\begin{aligned}
F_2 = &-\frac{e^{-\Phi}}{4}N_c\mathcal{C}\Big[2e^{-2h}(1+a^2-2ab)\mathcal{V}_3 e^{\hat{\theta}\hat{\varphi}} + e^{-g-h-k}\mathcal{C}(a-b)\Big(\sqrt{2}e^{2g+k+\Phi}(e^{\hat{\theta}\hat{1}} - e^{\hat{\varphi}\hat{2}}) + \\
&4e^k\mathcal{V}_3(e^{\hat{\theta}\hat{2}} - e^{\hat{\varphi}\hat{1}}) - 4v_2 e^g e^{\hat{\varphi}\hat{3}}\Big) - 8e^{-2g}\mathcal{V}_3 e^{\hat{1}\hat{2}} - 8e^{-g-k}v_2 e^{\hat{2}\hat{3}} - 2e^{-h-k}v_2 e^{r\hat{\theta}}\Big] - \\
&\frac{\mathcal{S}e^{g-h}}{\sqrt{2}\mathcal{C}\sin\alpha}\Big(N_c b + a(e^{2g}\cos^2\alpha - N_c) + e^{g+h}\sin 2\alpha\Big)e^{\hat{\theta}\hat{\varphi}}\,, \qquad \text{(F.7)}
\end{aligned}$$

$$\begin{aligned}
F_4 = &\frac{e^{-g-h-k-\Phi}}{8\mathcal{C}}N_c\Big[\mathcal{C}(1+a^2-2ab)e^{\hat{\theta}\hat{\varphi}}\wedge(\sqrt{w}e^{2g+k+\Phi-h}e^{\hat{1}\hat{2}} + 4e^{2g-h}e^{\hat{1}\hat{3}}) \\
&\mathcal{C}b'e^{r\hat{\theta}}\wedge(4e^k\mathcal{V}_3 e^{\hat{1}\hat{3}} - \sqrt{2}e^{2g+k+\Phi}e^{\hat{2}\hat{3}}) - 8e^g v_2(a-b)e^{\hat{\theta}\hat{1}\hat{2}\hat{3}} \\
&e^{r\hat{\varphi}}\wedge(4e^g v_2 e^{\hat{1}\hat{2}} - b'e^k(\sqrt{2}e^{2g+\Phi}e^{\hat{1}\hat{3}} + 4\mathcal{V}_3 e^{\hat{2}\hat{3}}))\Big] - \\
&\frac{2\mathcal{S}e^{-g-h-k-\Phi}}{\mathcal{C}^2\sin\alpha}\Big(a(e^{2g}\cos^2\alpha - N_c) + (N_c b + e^{g+h}\sin 2\alpha)\Big)\Big(\mathcal{V}_3 e^k e^{\hat{\theta}\hat{\varphi}\hat{1}\hat{2}} + v_2 e^g e^{\hat{\theta}\hat{\varphi}\hat{2}\hat{3}}\Big)\,.
\end{aligned}$$

We will now proceed to show that the bad asymptotic behaviour and off diagonal ρ terms of the metric are actually a gauge artefact.

F.2 The dual of the Baryonic Branch with the shift in B_2

The NS 2-from of the original solution contains the term

$$\tilde{B}_2 = -\frac{1}{2}e^{2k+\Phi}\mathcal{S}(\tilde{\omega}_3 + \cos\theta d\varphi)\wedge d\rho. \qquad \text{(F.8)}$$

It is this term, when dualised, that gives rise to the undesirable behaviour as this will contribute to the dual metric in both $g_{\rho\rho}$ and $g_{\rho v3}$ via the dual vielbeins $e^{\hat{i}}$ which will have legs in ρ. This happens because of the $d\rho\wedge\tilde{\omega}_i$ term in $\tilde{B}_2$ which is not a spectator under the duality transformation[3]. However, one is always free to add an exact to the NS potential as this will not change the fluxes

[3]See section 2 of [30] for details of how the initial B_2 enters into the definition of the dual vielbeins.

or metric of the original solution. Consider adding a closed form to the initial B_2

$$B_2 \to B_2 + d\big(\mathcal{Z}(r)(\tilde{\omega}_3 + \cos\theta d\varphi)\big) \tag{F.9}$$

This precisely cancels the effect of $\tilde{B}_2$ in the dual solution when $\mathcal{Z}' = -\frac{1}{2}\mathcal{S}e^{2k+\Phi}$ because

$$\begin{aligned}\tilde{B}_2 + d(\mathcal{Z}(r)\tilde{\omega}_3) = &-\mathcal{Z}(\tilde{\omega}_1 \wedge \tilde{\omega}_2 + \sin\theta d\theta \wedge d\varphi) \\ &+\frac{1}{2}(\mathcal{S}e^{2k+\Phi} + 2\mathcal{Z}')d\rho \wedge (\tilde{\omega}_3 + \cos\theta d\varphi).\end{aligned} \tag{F.10}$$

As there is no longer a $d\rho \wedge \tilde{\omega}_i$ term in the NS 2 form before dualisation, the dual vielbeins will have no legs in ρ and so there will no longer be a modification to $g_{\rho\rho}$ and $g_{\rho v_3}$. The trade off is that the function $\mathcal{Z}$ will now enter into the dual solution.

We now once more follow the procedure of [30] with gauge fixing, as before, such that $v_1 = \varphi = \theta = 0$. We are lead to the dual vielbeins

$$\begin{aligned}e^{\hat{1}'} = \frac{e^{g+\frac{\Phi}{2}}}{8\mathcal{W}}\sqrt{\mathcal{C}}\Bigg[&e^{2k+\Phi}\Big(-\sqrt{2}\mathcal{C}e^{2g+\Phi}(\cos\psi(a\omega_2\mathcal{H} + dv_2) + \sin\psi(a\omega_1\mathcal{H} - \omega_3 v_2)) - \\ &4\mathcal{H}\sin\psi(a\omega_2\mathcal{H} + dv_2) + 4\mathcal{H}\cos\psi(a\omega_1\mathcal{H} - \omega_3 v_2)\Big) - \\ &4v_2\mathcal{C}e^{2g+\Phi}\sin\psi(a\omega_2 v_2 - dv_3) - 8\sqrt{2}v_2\cos\psi(v_2 dv_2 + \mathcal{H}dv_3) + \\ &\frac{1}{2}\mu_1\mathcal{S}e^{g+\Phi}\Big(8v_2^2\cos\psi\omega_2 + \mathcal{C}e^{2k+\Phi}\big(\cos\psi(\mathcal{C}e^{2g+\Phi}\omega_2 - 2\sqrt{2}\mathcal{H}\omega_1) + \\ &\sin\psi(\mathcal{C}e^{2g+\Phi}\omega_1 + 2\sqrt{2}\mathcal{H}\omega_2)\big)\Big)\Bigg]\end{aligned} \tag{F.11}$$

$$\begin{aligned}e^{\hat{2}'} = \frac{e^{g+\frac{\Phi}{2}}}{8\mathcal{W}}\sqrt{\mathcal{C}}\Bigg[&e^{2k+\Phi}\mathcal{C}\Big(\sqrt{2}\mathcal{C}e^{2g+\Phi}(\cos\psi(\omega_3 v_2 - a\omega_1\mathcal{H}) + a\omega_2\mathcal{H}\sin\psi + dv_2\sin\psi) - \\ &4\mathcal{H}(\cos\psi(a\omega_2\mathcal{H} + dv_2) + \sin\psi(a\omega_1\mathcal{H} - \omega_3 v_2))\Big) - \\ &4v_2\mathcal{C}e^{2g+\Phi}\cos\psi(a\omega_2 v_2 - dv_3) + 8\sqrt{2}v_2\sin\psi(v_2 dv_2 + \mathcal{H}dv_3) + \\ &\frac{1}{2}\mu_1\mathcal{S}e^{g+\Phi}\Big(-8v_2\sin\psi\omega_2 + \mathcal{C}e^{2k+\Phi}\big((\mathcal{C}e^{2g+\Phi}\omega_1 + 2\sqrt{2}\mathcal{H}\omega_2)\cos\psi - \\ &(\mathcal{C}e^{2g+\Phi}\omega_2 - 2\sqrt{2}\mathcal{H}\omega_1)\sin\psi\big)\Big)\Bigg]\end{aligned}$$

$$\begin{aligned}e^{\hat{3}'} = \frac{e^{k+\frac{\Phi}{2}}}{8\mathcal{W}}\sqrt{\mathcal{C}}\Big[&\sqrt{2}\mathcal{C}^2e^{4g+2\Phi}(a\omega_2 v_2 - dv_3) + 4v_2\mathcal{C}e^{2g+\Phi}(\omega_3 v_2 - a\omega_1\mathcal{H}) - \\ &8\sqrt{2}\mathcal{H}(v_2 dv_2 + \mathcal{H}dv_3) + \mu_1 v_2\mathcal{S}e^{g+\Phi}(4\mathcal{H}\omega_2 + \sqrt{2}\mathcal{C}e^{2g+\Phi}\omega_2)\Big]\end{aligned}$$

which upon rotating according to eq (E.12) give the vielbeins of eq (5.3.5).

A valid question at this point is whether there is a local diffeomorphism which maps us from the Baryonic Branch dual solution as defined in section F.1 to the solution defined as in section 5.3. The answer is yes, and it may be most easily found by comparing the dilaton as defined in eq (5.3.7) and eq (F.6) . Examining these makes it clear that one needs to transform $\mathcal{V}_3$ such that it is mapped to $\mathcal{H}$. This may be achieved with a transformation in v_3 only

$$v_3 \to v_3 + \sqrt{2}\mathcal{Z} \tag{F.12}$$

under this which

$$\mathcal{V}_3 \to \mathcal{H}, \quad \Lambda \to dv_3 \tag{F.13}$$

and so vielbeins of eq (F.4) are mapped to those of eq (5.3.5). The map on the RR sector also follows trivially whilst the NS 2-form of eq (5.3.6) is mapped to that of eq (F.5) up to an exact.

So it is clear that one may "cure" the bad asymptotics and $g_{\rho v_3}$ mixing of section F.1 either by a gauge transformation in the NS 2-from before dualisation, or by a local diffeomorphism on the dual coordinate v_3 after the duality procedure is performed.

F.3 Details of the Dual Baryonic Branch Structure

All that remains to compete the elucidation of the baryonic dual is to give supplementary details to section 5.4 on the dynamical $SU(2)$ structure. Actually, the derivation of the structure is essentially the same as that of the dual of the wrapped D5 solution in section E, so we will only focus on the differences here.

The 10-d MW Killing spinors of Baryonic Branch obey the same projection as the wrapped D5 spinors (see eq (E.5)). However, whilst the internal spinors are still parallel, they now differ by a point dependent phase $e^{i\zeta(r)} = \mathcal{C} + i\mathcal{S}$

$$\begin{aligned}\epsilon_1 &= e^A(\xi_+ \otimes (e^{i\zeta(r)/2}\eta_+) + \xi_- \otimes (e^{-i\zeta(r)/2}\eta_-)),\\ \epsilon_2 &= e^A(\xi_+ \otimes (e^{-i\zeta(r)/2}\eta_+) + \xi_- \otimes (e^{i\zeta(r)/2}\eta_-)),\end{aligned} \tag{F.14}$$

where the Minkowski warp factor is now $e^{2A} = \frac{e^{\Phi}}{\mathcal{C}}$. We now follow the steps illustrated between eqs (E.7) and (E.9) such that the $SU(3)$-structure of the Baryonic Branch takes canonical form.

The dual 10-d Killing spinors are given as in eqs (E.13),(E.14), however the

ζ^a entering into their definition are now given by

$$\zeta^1 = \frac{2\sqrt{2}e^{-g-k-\phi}v_2\cos\psi}{\sqrt{\mathcal{C}}}, \quad \zeta^2 = -\frac{2\sqrt{2}e^{-g-k-\phi}v_2\sin\psi}{\sqrt{\mathcal{C}}},$$
$$\zeta^3 = \frac{2\sqrt{2}e^{-2g-\phi}\mathcal{H}}{\sqrt{\mathcal{C}}}. \tag{F.15}$$

The new spinor $\hat{\epsilon}_2$ is:

$$\hat{\epsilon}_2 = \frac{e^{\Phi/2}}{\sqrt{\mathcal{C}}}\left(\zeta_+ \otimes (e^{-i\zeta(r)/2}\hat{\eta}_-^2) + \zeta_- \otimes (e^{i\zeta(r)/2}\hat{\eta}_+^2)\right) \tag{F.16}$$

where $\hat{\eta}_-^2$ is still given by eq (E.18).

The dynamic $SU(2)$-structure supported by the dual Baryonic Branch solution may be expressed as

$$\Phi_+ = \frac{e^A}{8}e^{-iv\wedge w}\left(k_\parallel e^{-ij} - ik_\perp\omega\right),$$
$$\Phi_- = \frac{ie^A}{8}e^{i\zeta(r)}(v+iw)\wedge\left(k_\perp e^{-ij} + ik_\parallel\omega\right). \tag{F.17}$$

The forms and functions entering into these expressions are given by

$$\begin{aligned}
e^{2A} &= \frac{e^\Phi}{\mathcal{C}}\\
e^{i\zeta(r)} &= \mathcal{C} + i\mathcal{S}\\
k_\parallel &= \frac{\sin\alpha}{\sqrt{1+\zeta.\zeta}} \qquad k_\perp = \sqrt{\frac{\cos^2\alpha+\zeta.\zeta}{1+\zeta.\zeta}}\\
z = w - i\,v &= \frac{1}{\sqrt{\cos^2\alpha+\zeta.\zeta}}\left(\sqrt{\Delta}\tilde{e}^3 + \zeta_2\sin\alpha\tilde{e}^\theta + i(\sqrt{\Delta}\tilde{e}^\rho + \zeta_2\sin\alpha\tilde{e}^\varphi)\right)\\
j &= \tilde{e}^{\rho 3} + \tilde{e}^{\varphi\theta} + \tilde{e}^{21} - v\wedge w\\
\omega &= \frac{i}{\sqrt{\cos^2\alpha+\zeta.\zeta}}\left(\sqrt{\Delta}(\tilde{e}^\varphi + i\tilde{e}^\theta) - \zeta_2\sin\alpha(\tilde{e}^\rho + i\tilde{e}^3)\right)\wedge(\tilde{e}^2 + i\tilde{e}^1),
\end{aligned} \tag{F.18}$$

with ζ^a defined by eq (F.15). Specifically the vielbeins $\tilde{e}$ that the structure is expressed in terms of a rotation of those in eq (F.11). First one preforms a rotation by α

$$\begin{aligned}
\hat{e}^\varphi &= \cos\alpha e^\varphi + \sin\alpha e^{2'}\\
\hat{e}^2 &= -\sin\alpha e^\varphi + \cos\alpha e^{2'}\\
\hat{e}^a &= e^a \text{ for } a \neq \varphi, 2',
\end{aligned} \tag{F.19}$$

and then rotates these vielbeins to get $\tilde{e} = R\hat{e}$, where the matrix R is given by eq (E.19) with ζ^a by eq (F.15).

G On the Numerics for Chapter 6

Our goal is to numerically find some particular solutions of the equations

$$\begin{aligned} \dot{a} &= -\frac{c}{2a} + \frac{a^5 f^2}{8b^4c^3}, & \dot{b} &= -\frac{c}{2b} - \frac{a^2(a^2-3c^2)f^2}{8b^3c^3}, \\ \dot{c} &= -1 + \frac{c^2}{2a^2} + \frac{c^2}{2b^2} - \frac{3a^2f^2}{8b^4}, & \dot{f} &= -\frac{a^4f^3}{4b^4c^3}. \end{aligned} \tag{G.1}$$

In general this system will have four integration constants. We can find series solutions of these equations as $r \to 0$ and choose the the zeroth order term in each expansion to be the independent parameter. Thus, generally the IR expansions will have the form,

$$a(r) \sim a_0 + a_1(a_0, b_0, c_0, f_0)r + a_2(a_0, b_0, c_0, f_0)r^2 + \cdots \tag{G.2}$$

and similar expressions for all the other functions. However, we are interested in solutions dual to a 4 dimensional field theory, thus we want the 3-cycle that the D6 brane wraps to shrink to zero as $r \to 0$. From the IIA metric (6.2.10) we see that this requirement fixes $a_0 = 0$, $c_0 = 0$ and we are left with only two independent parameters in the IR, b_0 and f_0 that we label R_0 and q_0R_0 respectively. Similarly, in the UV generically we have 4 independent parameters but since we want solutions with a stabilized dilaton, we set the coefficient of the linear term in the dilaton expansion to zero and are left with three independent parameters R_1, q_1, h_1 in terms of which a UV solution to arbitrary order can be found.

To find numerical solutions we have the choice of starting in the IR and integrate forward or start in the UV and integrate backwards. We choose to solve the equations of motion starting from the IR, using the IR expansions as boundary conditions. Our motivations for doing so are two-fold. First, the parameter space in the IR is smaller, $\{R_0, q_0\}$, than the one in the UV, $\{R_1, q_1, h_1\}$, this facilitates the search of a solution with the required behaviour. Second, the expansion of the equations of motion around $r = 0$ is less computationally-intensive than the one around $r \to \infty$ allowing us to use very high order expansions as boundary conditions. More precisely, in our code we use IR expansions of the functions $a(r)$, $b(r)$, $c(r)$, $f(r)$ up to order $\mathcal{O}(r^{27})$ as boundary conditions. By way of illustration, we present here the IR expansions up to order $\mathcal{O}(r^{13})$,

$$
\begin{aligned}
a(r) = {} & \frac{r}{2} - \frac{(2+q_0^2)r^3}{(288R_0^2)} + \frac{(74+29q_0^2-31q_0^4)r^5}{(69120R_0^4)} + \\
& \frac{(-7274+546q_0^2+5043q_0^4-2473q_0^6)r^7}{(34836480R_0^6)} + \\
& \frac{(-2767396+2066644q_0^2+1326639q_0^4-2267840q_0^6+761969q_0^8)r^9}{(60197437440R_0^8)} + \\
& \frac{P_{10}(q_0)r^{11}}{(158921234841600R_0^{10})} - \frac{P_{12}(q_0)r^{13}}{(2975005516234752 00R_0^{12})} \\
P_{10}(q_0) = {} & -1732820552+2661492292q_0^2-714674162q_0^4-1616450167q_0^6+ \\
& 1494468524q_0^8-388078387q_0^{10} \\
P_{12}(q_0) = {} & 809180302184-1936619471316q_0^2+ \\
& 1686929485098q_0^4+13678188077q_0^6 \\
& -1046636256642q_0^8+694139577405q_0^{10}-148147907158q_0^{12}
\end{aligned}
\tag{G.3}
$$

$$
\begin{aligned}
b(r) = {} & R_0 - \frac{(-2+q_0^2)r^2}{(R_0 16)} - \frac{(13-21q_0^2+11q_0^4)r^4}{(1152R_0^3)} + \\
& \frac{(3268-8866q_0^2+9149q_0^4-3209q_0^6)r^6}{(1658880R_0^5)} \\
& + \frac{Pb_8(q_0)r^8}{(557383680R_0^7)} + \frac{Pb_{10}(q_0)r^{10}}{(1203948748800R_0^9)} + \frac{Pb_{12}(q_0)r^{12}}{(3814109636198400R_0^{11})} \\
Pb_{12}(q_0) = {} & -96075595496+555977381336q_0^2- \\
& 1393711678048q_0^4+1890154422552q_0^6 \\
& -1451154850145q_0^8+596013842074q_0^{10}-102144488257q_0^{12} \\
Pb_{10}(q_0) = {} & 120346756-576435426q_0^2+1165086146q_0^4-1196194108q_0^6+ \\
& 617593365q_0^8-127804976q_0^{10} \\
Pb_8(q_0) = {} & -235082+885868q_0^2-1355526q_0^4+938210q_0^6-244621q_0^8
\end{aligned}
\tag{G.4}
$$

$$
\begin{aligned}
c(r) = & -\frac{r}{2} + \frac{(8-5q_0^2)r^3}{(288R_0^2)} - \frac{(232-353q_0^2+157q_0^4)r^5}{(34560R_0^4)} + \\
& \frac{(31168-76440q_0^2+68637q_0^4-21286q_0^6)r^7}{(17418240R_0^6)} \\
& + \frac{Pc_{10}(q_0)r^{11}}{(39730308710400R_0^{10})} + \frac{Pc_8(q_0)r^9}{(15049359360R_0^8)} + \frac{Pc_{12}(q_0)r^{13}}{(74375137905868800R_0^{12})} \\
Pc_{10}(q_0) = & \, 5716032512 - 24717750400q_0^2 + 44863517744q_0^4 - \\
& 41761366916q_0^6 + 19753037956q_0^8 - 3779455283q_0^{10} \\
Pc_8(q_0) = & -7527424 + \quad 25507072q_0^2 - 34570320q_0^4 + 21451291q_0^6 - 5080615q_0^8 \\
Pc_{12}(q_0) = & -3137711476736 + 16500424668672q_0^2 - \\
& 37556084710560q_0^4 + 46609546892530q_0^6 - \\
& 33023463748437q_0^8 + 12612429685326q_0^{10} - 2023272290207q_0^{12}
\end{aligned}
\tag{G.5}
$$

$$
\begin{aligned}
f(r) = & \, q_0R_0 + \frac{q_0^3r^2}{R_016} + \frac{q_0^3(-14+11q_0^2)r^4}{(1152R_0^3)} + \frac{q_0^3(2152-3473q_0^2+1492q_0^4)r^6}{(829440R_0^5)} \\
& + \frac{Pf_8(q_0)r^{10}}{(1203948748800R_0^9)} + \\
& \frac{Pf_6(q_0)r^8}{(557383680R_0^7)} + \frac{Pf_{10}(q_0)r^{12}}{(238381852262400R_0^{11})} \\
Pf_8(q_0) = & \, q_0^3(170283008 - 568700672q_0^2 + 744979116q_0^4 - \\
& 446064434q_0^6 + 102094739q_0^8) \\
Pf_6(q_0) = & \, q_0^3(-329536 + 813288q_0^2 - 705252q_0^4 + 210349q_0^6) \\
Pf_{10}(q_0) = & \, q_0^3(-8376443008 + 35383047296q_0^2 - 62151718900q_0^4 + \\
& 55981055275q_0^6 - 25662630839q_0^8 + 4767879802q_0^{10})
\end{aligned}
\tag{G.6}
$$

Using 40-digit **WorkingPrecision** in **NDSolve**, Mathematica 8, we generate, using the IR expansions as boundary conditions, solutions that extend in the UV. We observe that not for all values of $\{R_0, q_0\}$ we get solutions with stabilized dilaton. Thus, the behavior of the dilaton serves as a first indication of a potential solution with the required UV behavior. We use UV expansions up to order $\mathcal{O}(1/r^9)$ for all the functions. We show here, as an example, the UV expansion for $a(r)$.

$$
\begin{aligned}
a(r) =& \frac{r}{\sqrt{6}} - \frac{\sqrt{3}q_1R_1}{\sqrt{2}} + \frac{21\sqrt{3}R_1{}^2}{\sqrt{2}\,16r} + \frac{63\sqrt{3}q_1R_1{}^3}{\sqrt{2}\,16r^2} + \\
& \frac{9\sqrt{3}\left(672q_1{}^2+221\right)R_1{}^4}{\sqrt{2}\,512r^3} + \frac{81\sqrt{3}q_1\left(224q_1{}^2+221\right)R_1{}^5}{\sqrt{2}512r^4} + \\
& \frac{\sqrt{3}\left(2048h_1 + 1377\left(768q_1{}^4+1632q_1{}^2+137\right)R_1{}^6\right)}{\sqrt{2}\,8192r^5} + \\
& + \frac{3\sqrt{\frac{3}{2}}q_1R_1(10240h_1 + 81(11645 + 68000q_1^2 + 22272q_1^4)R_1^6)}{8192r^6} \\
& + \frac{1}{3670016r^7}(27\sqrt{\frac{3}{2}}R_1^2(8192h_1(27+560q_1^2) + 27(583399 + 17765952q_1^2 + \\
& 68376576q_1^4 + 20299776q_1^6)R_1^6)) \\
& + \frac{1}{524288r^8}(27\sqrt{\frac{3}{2}}q_1R_1^3(8192h_1(81+560q_1^2) + \\
& 81(583399 + 7332032q_1^2 + 20121600q_1^4 + 5849088q_1^6)R_1^6)) \\
& \frac{9\sqrt{\frac{3}{2}}R_1^4}{8388608\,r^9}Pa_9^{UV}(q_1,R_1,h_1) + \cdots
\end{aligned}
\tag{G.7}
$$

where,

$$
\begin{aligned}
Pa_9^{UV}(q_1,R_1,h_1) =& \left(4096h_1(3941 + 93312q_1^2 + 322560q_1^4)\right. \\
& + 243(3297681 + 129163840q_1^2 \qquad\qquad (G.8)\\
& \left.+ 912975360q_1^4 + 1851260928q_1^6 + 528482304q_1^8)R_1^6\right)
\end{aligned}
\tag{G.9}
$$

We then have to analyse if this candidate solution obtained by forward integration has indeed a UV where the functions are given by eq (6.2.33) or not. To this end, we define a mismatch function,

$$
m = \sum_i \left(\log\left(|f_i^{numerical}(r_{match})|\right) - \log\left(|f_i^{expansion}(r)|\right) \right)^2, \tag{G.10}
$$

where $f_i \in \{a,\ b,\ c,\ f\}$, $f_i^{numerical}$ refers to the solution obtained by forward integration and $f_i^{expansion}$ refers to the UV expansion. We then minimize m using **NMinimize** and **AccuracyGoal** $= 20$. If the minimization procedure yields a small value ($m \leq 10^{-4}$) this setup determines the UV parameters R_1, q_1, h_1 for which our numerical solution has the required UV behavior. Some sample solutions obtained with this procedure are presented in Figure 6.2. Note that we choose to normalize the dilaton such that

$$(g_s N)^{3/4} e^{2\phi_0/3} = 1 \tag{G.11}$$

where $\phi_0 \equiv \phi(r=0)$.

A natural question to ask is to what extent integrating back with the parameters found through the minimization procedure will reproduce the integrated forward solution. Since the IR expansions are of very high order ($\mathcal{O}(r^{27})$) while the UV expansions are only of order $\mathcal{O}(1/r^9)$ we expect that the UV solution will not be very accurate in the IR. We present plots comparing the backward and forward integrated solutions in Figure G.1. In order to verify that the small discrepancies in the IR are due to accumulated numerical error we evaluate the residual. Namely, we define a function res_k that evaluates the equation of motion for $k(r)$ using the numerical solution. If the solution were exact res_k should be identically zero. Since it is a numerical solution there will always be certain deviation form zero.

$$\begin{aligned}
res_a(r) &= |\dot{a}_{num} + \frac{c_{num}}{2a_{num}} - \frac{a_{num}^5 f_{num}^2}{8b_{num}^4 c_{num}^3}|, \\
res_b(r) &= |\dot{b}_{num} + \frac{c_{num}}{2b_{num}} + \frac{a_{num}^2(a_{num}^2 - 3c_{num}^2) f_{num}^2}{8b_{num}^3 c_{num}^3}|, \\
res_c(r) &= |\dot{c}_{num} + 1 - \frac{c_{num}^2}{2a_{num}^2} - \frac{c_{num}^2}{2b_{num}^2} + \frac{3a_{num}^2 f_{num}^2}{8b_{num}^4}|, \\
res_f(r) &= |\dot{f}_{num} + \frac{a_{num}^4 f_{num}^3}{4b_{num}^4 c_{num}^3}|.
\end{aligned} \tag{G.12}$$

In Figure G.2 we see that the integrated forward solution is more accurate for all values of r. Also note, (Figure G.2 a, b and d) that the integrated back solution fails considerably close to the IR ($resa(r_{IR}) \sim 10^{-2}$) and this explains the differences in figure G.1.

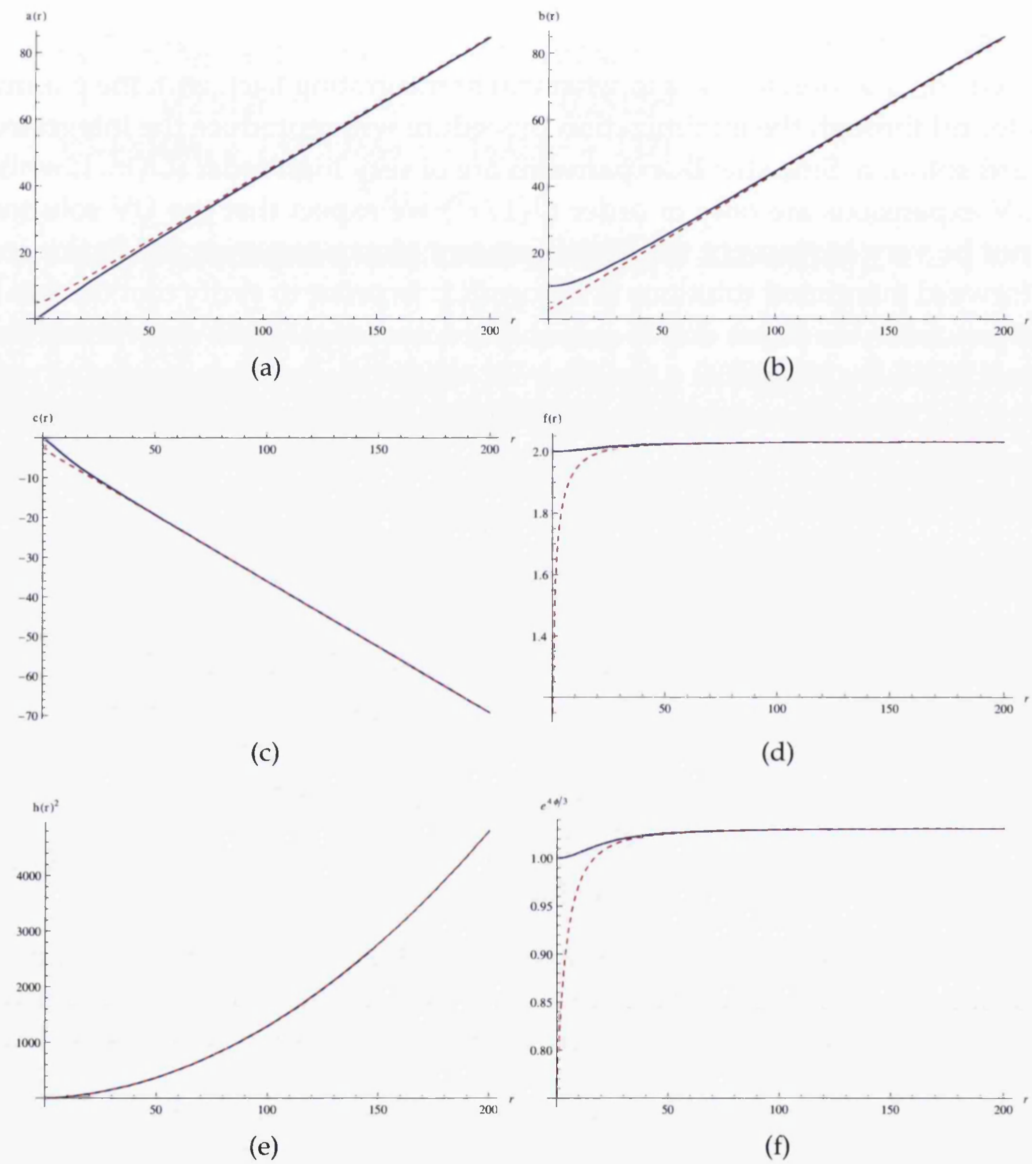

Figure G.1: The blue curves are the result of forward integration with $R_0 = 10$, $q_0 = 1/5$. After the minimization procedure we obtain the UV parameters $q_1 = 1.31946$, $R_1 = -2.03087$, $h_1 = -1.9733$ and plot (dashed red lines) the result of integrating back with these parameters to show that it coincides with the forward integration. The small discrepancies in the IR are due to accumulated numerical error. The mismatch function for this solution is $m < 10^{-4}$. We also plot $h(r)^2$ and $e^{4\phi/3}$ defined in eq (6.2.10)

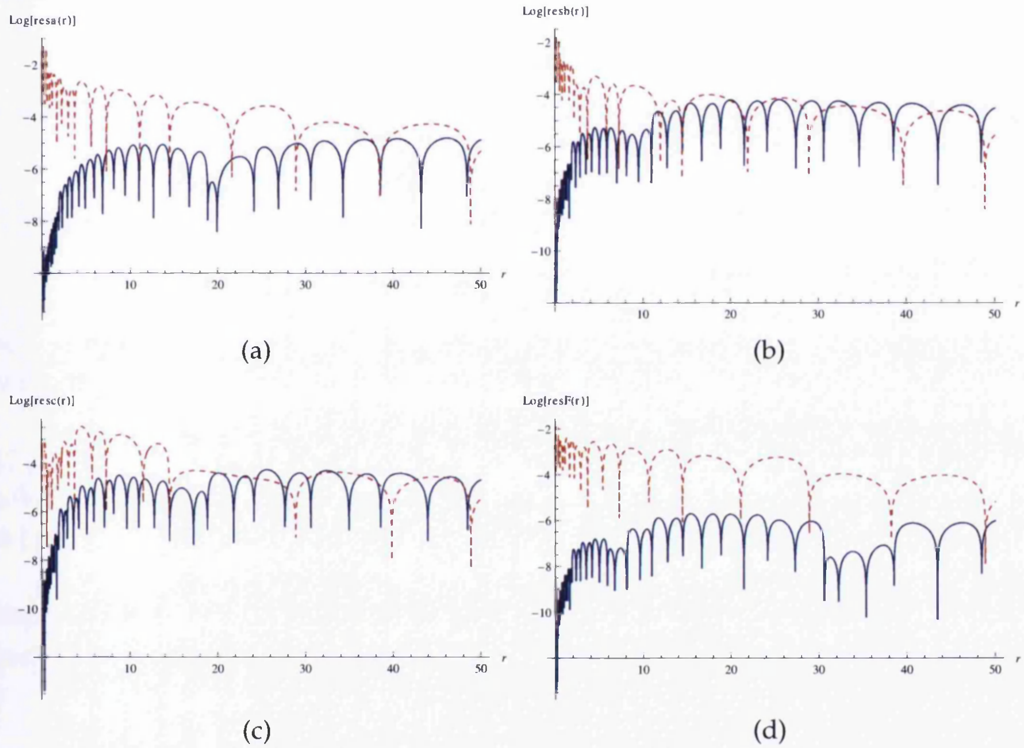

Figure G.2: $\log_{10}$ plot of the residuals defined in eq (G.12). The solid blue line is for the solution obtained by integrating forward (IR to UV) , dashed line is for the solution obtained by integrating from the UV back to the IR.

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