Roth's method and the Yosida approximation for pseudodifferential operators with negative definite symbols.

Potrykus, Alexander

Roth's Method and the Yosida Approximation for Pseudodifferential Operators with Negative Definite Symbols

Alexander Potrykus

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Department of Mathematics

University of Wales Swansea

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Abstract

This thesis consists of two parts. The first one extends an idea developed by J. P. Roth. He succeeded to construct a Feller semigroup associated with a second order elliptic differential operator $L(x, D)$ by investigating the semigroups obtained by freezing the coefficients of $L(x, D)$. In Chapter 2 we show that a modification of his method works also for certain pseudodifferential operators with bounded negative definite symbols. Partly we can rely on ideas of E. Popescu. In Chapter 3 we show that if a certain pseudodifferential operator $-q(x, D)$ generates a Feller semigroup $(T_t)_{t \geqslant 0}$ then the Feller semigroups $\left(T_t^{(\nu)}\right)_{t \geqslant 0}$ generated by the pseudodifferential operators whose symbols are the Yosida approximations of $-q(x, \xi)$, i.e.

$$-q^{(\nu)}(x, \xi) = -\frac{\nu q(x, \xi)}{\nu + q(x, \xi)},$$

converge strongly to $(T_t)_{t \geqslant 0}$.

Introduction

Pseudodifferential operators with negative definite symbols and their relation to Feller semigroups are recurrent topics in this thesis. The motivation for studying this connection can be found in the field of stochastic processes. One may identify a Feller semigroup $(T_t)_{t\geqslant 0}$, which is a positivity preserving, strongly continuous contraction semigroup of linear operators defined on C_∞, with a Feller process $\big((X_t)_{t\geqslant 0}, P^x\big)_{x\in\mathbb{R}^n}$. The general idea is to use pseudodifferential operators as generators of Feller semigroups and thus to associate with reasonably "nice" pseudodifferential operators corresponding Feller processes. Once this connection is established, one may then use the fact that every pseudodifferential operator is uniquely determined by its symbol to use properties of the symbol to study properties of the Feller process, see [8], [9], [24]–[26] as well as [17].

Let us go into some more detail: A basic result from stochastic analysis is Kolmogorov's extension theorem which gives us for every Markovian semigroup of kernels $\big(p_t(x,A)\big)_{t\geqslant 0}$, $x \in \mathbb{R}^n$, $A \subset \mathbb{R}^n$ measurable, the existence of a stochastic process. The interpretation of $p_t(x,A)$ is that of the probability of being at the time t in the set A when starting at $t = 0$ in $x \in \mathbb{R}^n$. If we are now given a Feller semigroup $(T_t)_{t\geqslant 0}$ on C_∞, we can obtain such a Markovian semigroup of kernels by setting $p_t(x,A) := T_t\chi_A(x)$ where the right-hand side is well-defined if we take a monotone approximation of χ_A by continuous functions. Vice versa if we start with a Feller process a Feller semigroup $(T_t)_{t\geqslant 0}$ is given by $T_t u(x) := E\big(u(X_t)\big)$. Next we would like to identify the generator of every Feller semigroup. We mention here that there is no classification theorem that tells us that every Feller semigroup is generated by a pseudodifferential operator with negative definite symbol. But we may ask under which conditions certain pseudodifferential operators generate Feller semigroups.

The main tool for constructing operator semigroups is the Hille-Yosida Theorem as stated in Section 1.4. It turns out that a result by Ph. Courrège [6] is fundamental to make the Hille-Yosida Theorem work for pseudodifferential operators in order to construct a Feller semigroup. The result characterizes pseudodifferential operators with negative definite symbols as the only ones that satisfy the positive maximum principle, i.e. $A : D(A) \to C_\infty$, $D(A) \subset C_\infty$, satisfies the positive maximum principle on $D(A)$ whenever for $u \in D(A)$ the fact that $u(x_0) = \sup_{x\in\mathbb{R}^n} u(x) \geqslant 0$ implies $Au(x_0) \leqslant 0$. Another important theory needed in this context is the symbolic calculus for negative definite functions developed by W. Hoh in [10]. It is an extension of the classical symbolic calculus as presented in e.g. Kumano-go [19]. In this thesis we concentrate on the construction of Feller semigroups using pseudodifferential operators with negative definite symbols.

Chapter 1 contains an explanation of the notation that we use and also introduces other topics such as negative definite functions, Bernstein functions, pseudodifferential operators and some operator semigroup theory. In Chapter 2 we first explain how J. P. Roth in [23] succeeded to construct Feller semigroups associated with a second order elliptic differential operator $L(x, D)$. Then we extend his method to pseudodifferential operators with bounded negative definite symbols. In Chapter 3 we use a pseudodifferential operators $-q^{(\nu)}(x, D)$ whose symbols is the Yosida approximation of a given symbol q, i.e.

$$-q^{(\nu)}(x, D) = -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \frac{\nu q(x,\xi)}{\nu + q(x,\xi)} \hat{u}(\xi)\, d\xi.$$

and show that the sequence of corresponding semigroups $\left(T_t^{(\nu)}\right)_{t\geqslant 0}$, $\nu > 0$, that is generated by $-q^{(\nu)}(x, D)$, converges strongly to the Feller semigroup $(T_t)_{t\geqslant 0}$ for $\nu \to \infty$. We need to make the assumption that $-q(x, D)$ generates the Feller semigroup $(T_t)_{t\geqslant 0}$.

Chapter 1

Preliminaries

The main purpose of this chapter is to fix the notation used throughout this thesis and to introduce some of the concepts we need in later chapters. We begin with explaining our notation in Section 1, whereas Section 2 treats negative definite functions. Section 3 introduces pseudodifferential operators, the symbol of which are continuous negative definite functions with respect to the co-variable. For these operators the corresponding symbolic calculus is discussed. Section 4 contains some material of the theory of operator semigroups and relations to the pseudodifferential operators under consideration. In Section 2 we prove a new result about uniformly bounded continuous negative definite functions which is needed for Roth's method in Chapter 2. The proof also serves as a nice example for how negative as well as positive definite functions and convolution semigroups of measures are connected. We further would like to emphasize that the symbolic calculus as described in Section 3 is not the classical symbolic calculus. In fact to some extend one might consider the classical symbolic calculus as a special case of the symbolic calculus related to negative definite symbols. This will be very important for Chapter 3, but is not needed in Chapter 2.

1.1 Notation

Most of the notation we use is standard, hence we only point out the not so common cases. The Index of Notation that we include at the end of this thesis contains a short list of function spaces and notation related to the symbolic calculus of pseudodifferential operators.

Throughout this thesis we are going to study properties of (in general) complex-valued functions of n independent real variables and their various derivatives. In order to work conveniently with these variables, functions and derivatives we use *multiindex notation.* If we denote the variables by $x_1, \ldots, x_n$, or simply x, we may define a function u of these variables with domain $\mathbb{R}^n$ and write $u(x)$, $x \in \mathbb{R}^n$. For any multiindex $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, we define its length as the sum $|\alpha| = \alpha_1 + \ldots + \alpha_n$ and its factorial as the product $\alpha! = (\alpha_1!) \cdot \ldots \cdot (\alpha_n!)$. Furthermore we write $\alpha \leqslant \beta \in \mathbb{N}_0^n$ if one has $\alpha_j \leqslant \beta_j$ for all $j = 1, \ldots, n$. For $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}_0^n$ one defines $x^\alpha = x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}$. For the operation of taking partial derivatives, we write

$$\partial^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

We often work with smooth functions $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $(x, \xi) \mapsto q(x, \xi)$ and want to consider partial derivatives with respect to the variable $\xi \in \mathbb{R}^n$ only. In this case we write

$$\partial_\xi^\alpha q = \frac{\partial^{|\alpha|}}{\partial \xi_1^{\alpha_1} \cdots \partial \xi_n^{\alpha_n}} q.$$

We also need binomial coefficients which are defined as follows: for $a, b \in \mathbb{N}_0$,

$$\binom{a}{b} = \frac{a!}{(a-b)!b!},$$

if $0 \leqslant b \leqslant a$,

$$\binom{a}{b} = 0$$

otherwise. For $\alpha, \beta \in \mathbb{N}_0^n$ one defines the *binomial coefficients* as products

$$\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \cdot \ldots \cdot \binom{\alpha_n}{\beta_n},$$

and finds that if $\beta \leqslant \alpha$

$$\binom{\alpha}{\beta} = \frac{\alpha!}{(\alpha-\beta)!\beta!},$$

and otherwise

$$\binom{\alpha}{\beta} = 0.$$

In Chapter 3 we often use two other results, the *Binomial formula*

$$(x+y)^\alpha = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} x^\beta y^{\alpha-\beta}, \tag{1.1}$$

where x and $y \in \mathbb{R}^n$, and *Leibniz's formula*

$$\partial^\alpha(uv) = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \left(\partial^\beta u\right) \left(\partial^{\alpha-\beta} v\right), \tag{1.2}$$

for $u,\ v \in C^{|\alpha|}$. For Chapter 3 it is also useful to keep in mind that

$$\sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} = 2^{|\alpha|}. \tag{1.3}$$

which follows easily from the Binomial formula (1.1). Furthermore for functions $u : \mathbb{R} \to \mathbb{R}$ and $v : \mathbb{R}^n \to \mathbb{R}^n$ we find that

$$\partial^\alpha(u \circ v) = \sum_{j=1}^{|\alpha|} u^{(j)}(v(\cdot)) \sum \frac{\alpha!}{\delta_\beta! \delta_\gamma! \cdots \delta_\omega!} \left(\frac{\partial^\beta v}{\beta!}\right)^{\delta_\beta} \cdots \left(\frac{\partial^\omega v}{\omega!}\right)^{\delta_\omega} \tag{1.4}$$

where the second sum runs over all pairwise different multiindices

$$0 \neq \beta, \gamma, \ldots, \omega \in \mathbb{N}_0^n$$

and all $\delta_\beta, \delta_\gamma, \ldots, \delta_\omega \in \mathbb{N}$ such that $\delta_\beta \beta + \delta_\gamma \gamma + \ldots + \delta_\omega \omega = \alpha$ and $\delta_\beta + \delta_\gamma + \ldots + \delta_\omega = j$.

Another central notion we use in this thesis is that of the Fourier transformation. Let us introduce the *Schwartz space* $\mathcal{S}$ of C^∞ functions that are rapidly decreasing. We define on $\mathcal{S}$ the *Fourier transform* $\widehat{u}$ of a function u and write

$$\widehat{u}(\xi) = \mathcal{F}[u](\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} u(x)\, \mathrm{d}x. \tag{1.5}$$

We give (1.5) merely to avoid any ambiguity concerning the normalization factor $(2\pi)^{-\frac{n}{2}}$. Whenever we use a Fourier transform in this text it is meant in the sense of (1.5). We do not want to go into any details of the theory of Fourier transforms here, but will give further remarks whenever necessary. Let us only mention here that one may also extend the Fourier transform to $\mathcal{S}'$, the space of tempered distributions, hence Fourier transforms of bounded positive measures are also well defined. In Chapter 3 we need the following properties of Fourier transforms: let $u, v \in \mathcal{S}$, and denote by $u * v$ the convolution product of u and v defined as

$$(u * v)(x) = \int_{\mathbb{R}^n} u(x-y) v(y)\, \mathrm{d}y;$$

further we use the convention $\check{u}(x) = u(-x)$ and the operators $D^\alpha = (-i)^{|\alpha|}\partial^\alpha$. Then

$$\widehat{u \cdot v} = (2\pi)^{-\frac{n}{2}}(u * v), \tag{1.6}$$
$$\widehat{u * v} = (2\pi)^{\frac{n}{2}}\hat{u} \cdot \hat{v}, \tag{1.7}$$
$$\hat{\hat{u}} = \check{u}, \tag{1.8}$$
$$\widehat{D_x^\alpha u}(\xi) = \xi^\alpha \hat{u}(\xi), \tag{1.9}$$
$$\widehat{x^\alpha u}(\xi) = (-D_\xi)^\alpha \hat{u}(\xi). \tag{1.10}$$

The most important tool that is at our disposal is the *Plancherel formula*

$$\|u\|_{L^2} = \|\hat{u}\|_{L^2} \tag{1.11}$$

and it is central to obtain the results of Chapter 2 and 3. Note that one proves (1.11) first for $u \in \mathcal{S}$ and extends first the Fourier transform and then (1.11) to L^2.

There are a few other things left to mention. For an imbedding result concerning Sobolev spaces and at a few other places it is useful to have the following classical result in mind:

$$\int_{\mathbb{R}^n} \frac{1}{(1 + |x|^2)^{\frac{s}{2}}}\, \mathrm{d}x < \infty \tag{1.12}$$

if and only if $s > n$. Additionally, if $(X, \|\cdot\|_X)$ is a normed space and $x \in X$, then we denote the norm of x with respect to this normed space with $\|x\|_X$, i.e. we will explicitly write as index which space we mean. As an example take a function $u \in L^2$, then we would write $\|u\|_{L^2}$ to denote the L^2-norm of u, i.e.

$$\|u\|_{L^2} = \left(\int_{\mathbb{R}^n} |u(x)|^2\, \mathrm{d}x\right)^{\frac{1}{2}}.$$

For an operator $S : X \to Y$ we similarily write $\|S\|_{X\to Y}$ to denote the operator norm

$$\|S\|_{X\to Y} := \sup_{\|x\|_X \leqslant 1} \|Sx\|_Y.$$

1.2 Negative Definite Functions

We begin with defining what a negative definite function is. There are several ways of doing this, the first one we present is to use positive definiteness in the sense of Bochner, and the second one is the Lévy-Khinchin representation formula. References for this section are N. Jacob [15] and Chr. Berg and G. Forst [2]. We first state the *Theorem of Bochner.* The notation $\mathcal{M}_+^b(\mathbb{R}^n)$ is used to denote the space of bounded positive measures.

Theorem 1.1. *A function $\varphi : \mathbb{R}^n \to \mathbb{C}$ is the Fourier transform of a measure $\mu \in \mathcal{M}_+^b(\mathbb{R}^n)$ with finite total mass $\|\mu\|$, if and only if the following conditions are satisfied*

1. *φ is continuous*

2. *φ is positive definite.*

Then it follows

$$\varphi(0) = \widehat{\mu}(0) = (2\pi)^{-\frac{n}{2}}\|\mu\|.$$

Bochner's Theorem identifies the continuous positive definite functions with Fourier transforms of bounded measures. A more technical definition of positive definiteness is the following.

Definition 1.2. A function $\varphi : \mathbb{R}^n \to \mathbb{C}$ is called *positive definite* if for any choice of $k \in \mathbb{N}$ and vectors $\xi^1, \ldots, \xi^k \in \mathbb{R}^n$ the matrix $\left(\varphi(\xi^j - \xi^l)\right)_{j,l=1,\ldots,k}$ is positive Hermitian, i.e. for all $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ we have

$$\sum_{j,l=1}^{k} \varphi(\xi^j - \xi^l)\lambda_j\overline{\lambda_l} \geqslant 0.$$

Using the definition of positive definiteness one may now define negative definite functions:

Definition 1.3. A function $\psi : \mathbb{R}^n \to \mathbb{C}$ is called *negative definite* if

$$\psi(0) \geqslant 0$$

and

$$\xi \mapsto (2\pi)^{-\frac{n}{2}} e^{-t\psi(\xi)}$$

is positive definite for $t \geqslant 0$.

Note that the factor $(2\pi)^{-\frac{n}{2}}$ in the above definition is not mandatory. One may choose any positive value. The factor we chose is convenient when we work with Fourier transforms, see e.g. the Plancherel's formular (1.11). Another way to look at negative definite functions, and one that does not involve positive definiteness, is the *Lévy-Khinchin representation.*

Theorem 1.4. *A function $\psi : \mathbb{R}^n \to \mathbb{C}$ is* continuous negative definite *if and only if there exists a constant $c \geqslant 0$, a vector $d \in \mathbb{R}^n$, a symmetric positive semidefinite quadratic form q on $\mathbb{R}^n$ and a finite measure μ on $\mathbb{R}^n \backslash \{0\}$ such that*

$$\psi(\xi) = c + i(\langle d, \xi \rangle) + q(\xi) + \int_{\mathbb{R}^n \backslash \{0\}} \left(1 - e^{-i\langle x, \xi \rangle} - \frac{i\langle x, \xi \rangle}{1 + |x|^2}\right) \frac{1 + |x|^2}{|x|^2} \mu(\mathrm{d}x).$$

As we have now seen how positive and negative definite functions are connected, we next want to investigate the relation of negative definite functions and convolution semigroups of measures. For this purpose we start with the definition of a convolution semigroup. We denote with ε_0 the Dirac measure with unit mass in 0.

Definition 1.5. A family $(\mu_t)_{t \geqslant 0}$ of positive bounded measures on $\mathbb{R}^n$, i.e. $\mu_t \in \mathcal{M}_b^+(\mathbb{R}^n)$, with the properties

1. $\mu_t(\mathbb{R}^n) = \|\mu_t\| \leqslant 1 \quad$ for $t > 0$,
2. $\mu_t * \mu_s = \mu_{t+s} \quad$ for $t, s > 0$,
3. $\lim_{t \to 0} \mu_t = \varepsilon_0$ vaguely,

is called a *convolution semigroup* on $\mathbb{R}^n$.

It turns out that there is a one-to-one correspondence between convolution semigroups $(\mu_t)_{t \geqslant 0}$ on $\mathbb{R}^n$ and continuous negative definite functions. This identification is very useful and we frequently make use of it in Chapter 2.

Theorem 1.6. *If $(\mu_t)_{t \geqslant 0}$ is a convolution semigroup on $\mathbb{R}^n$, then there exists a uniquely determined continuous negative definite function ψ on $\mathbb{R}^n$ such that*

$$\widehat{\mu}_t(\xi) = (2\pi)^{-\frac{n}{2}} e^{-t\psi(\xi)} \tag{1.13}$$

for $t > 0$ and $\xi \in \mathbb{R}^n$. Conversely, given a continuous negative definite function ψ on $\mathbb{R}^n$, (1.13) determines a convolution semigroup $(\mu_t)_{t \geqslant 0}$ on $\mathbb{R}^n$.

We remark that in the following we only work with continuous negative definite functions. Examples of continuous negative definite functions and corresponding convolution semigroups are given at the end of this section. In the next corollary we summarize some properties and estimates for continuous negative definite functions that are needed later on.

Corollary 1.7. *Let $\psi : \mathbb{R}^n \to \mathbb{C}$ be a continuous negative definite function.*
A. Then

$$\psi(\xi) = \overline{\psi(-\xi)} \quad \text{and} \quad \operatorname{Re}\psi(\xi) \geqslant \psi(0)$$

for all $\xi \in \mathbb{R}^n$.
B. There exists a constant $C > 0$ such that

$$|\psi(\xi)| \leqslant C(1 + |\xi|^2)$$

for all $\xi \in \mathbb{R}^n$.

Next we give a generalized version of *Peetre's inequality* for negative definite functions.

Lemma 1.8. *Let $\psi : \mathbb{R}^n \to \mathbb{C}$ be a negative definite function. Then we have for $\xi, \eta \in \mathbb{R}^n$*

$$\frac{1 + |\psi(\xi)|}{1 + |\psi(\eta)|} \leqslant 2\,(1 + |\psi(\xi - \eta)|)\,. \tag{1.14}$$

Corollary 1.9 additionally summarizes some things that are mostly needed only in the proof of Theorem 1.10. We quote these results for the readers convenience but also give references.

Corollary 1.9. A.(Corollary 7.6 in [2]) *Let ψ be a negative definite function on $\mathbb{R}^n$. The function $\xi \mapsto \psi(\xi) - \psi(0)$ is negative definite.*
B.(Corollary 7.7 in [2]) *Let φ be a positive definite function on $\mathbb{R}^n$. The function $\xi \mapsto \varphi(0) - \varphi(\xi)$ is negative definite.*
C.(Proposition 7.11 in [2]) *A function $\psi : \mathbb{R}^n \to \mathbb{C}$ is negative definite if and only if there exists a sequence $(\psi_n)_{n\in\mathbb{N}}$ of functions $\psi_n : \mathbb{R}^n \to \mathbb{C}$ of the form*

$$\psi_n = a_n + \varphi_n(0) - \varphi_n,$$

where $a_n \geqslant 0$ and $\varphi_n : \mathbb{R}^n \to \mathbb{C}$ is positive definite, such that $\lim_{n\to\infty} \psi_n = \psi$ pointwise on $\mathbb{R}^n$.
D.(Lemma 2.1.1 in [15]) *For all $a \geqslant 0$ and $t \geqslant 0$ we have*

$$\left|\frac{e^{-at} - 1 + at}{t}\right| \leqslant \frac{1}{2}a^2 t;$$

furthermore for $z \in \mathbb{C}$, $\operatorname{Re} z \leqslant 0$, we have

$$|1 - e^z| \leqslant |z|.$$

Up to this point we only considered functions $\psi : \mathbb{R}^n \to \mathbb{C}$, $\xi \mapsto \psi(\xi)$, but what we are interested in are functions $q : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{C}$, $(x, \xi) \mapsto q(x, \xi)$, which are continuous negative definite in the second component, i.e. $\xi \mapsto q(x, \xi)$ is continuous negative definite for all $x \in \mathbb{R}^n$ fixed. To simplify notation and also to emphasize the connection to pseudodifferential operators, we call such a function q a *continuous negative definite symbol.*

Theorem 1.10. *Let $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $(x, \xi) \mapsto q(x, \xi)$ be a continuous negative definite symbol such that $\sup_{x,\xi\in\mathbb{R}^n} |q(x, \xi)| \leqslant C$ for $C > 0$. Then there exists a constant $m > 0$ such that for all $x \in \mathbb{R}^n$, $\xi \mapsto m - q(x, \xi)$ is positive definite.*

Proof. In the following let $x_0 \in \mathbb{R}^n$ be fixed and assume that $q(x_0, 0) = 0$. If $q(x_0, 0) \neq 0$ we may use Corollary 1.9.A to be back in the previous case again. For $t > 0$ we consider the function

$$\xi \mapsto q_t(x_0, \xi) := \frac{1}{t}\left(1 - e^{-tq(x_0,\xi)}\right)$$

which is by Corollary 1.9.B, see also the proof of Theorem 7.8 in [2], negative definite. As $q(x_0, 0) = 0$ we have that $e^{-tq(x_0,0)} = 1$, and $\xi \mapsto e^{-tq(x_0,\xi)}$ is positive definite. Then $\frac{1}{t}e^{-tq(x_0,\xi)}$ is also positive definite and $\frac{1}{t}e^{-tq(x_0,0)} = \frac{1}{t}$. By Theorem 1.6 there exists a convolution semigroup $(a_t^{x_0})_{t\geqslant 0}$ on $\mathbb{R}^n$ such that

$$(2\pi)^{-\frac{n}{2}}\frac{1}{t}e^{-tq(x_0,\xi)} = (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle}a_t^{x_0}(\mathrm{d}x),$$

i.e.

$$\frac{1}{t}e^{-tq(x_0,\xi)} = \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle}a_t^{x_0}(\mathrm{d}x).$$

Hence we get

$$a_t^{x_0}(\mathbb{R}^n) = \int_{\mathbb{R}^n} e^{-i\langle x,0\rangle}a_t^{x_0}(\mathrm{d}x) = \frac{1}{t}$$

implying $a_t(\{0\}) = 0$ as well as

$$\frac{1}{t}\left(1 - e^{-tq(x_0,\xi)}\right) = \frac{1}{t} - \frac{1}{t}e^{-tq(x_0,\xi)} = \int_{\mathbb{R}^n} \left(1 - e^{-i\langle x,\xi\rangle}\right) a_t^{x_0}(\mathrm{d}x).$$

Further we find with Corollary 1.9.D that

$$\begin{aligned}|q(x_0,\xi) - q_t(x_0,\xi)| &= \left|\frac{1}{t}\left(tq(x_0,\xi) - 1 + e^{-tq(x_0,\xi)}\right)\right| \leqslant \frac{1}{2}tq(x_0,\xi)^2 \\ &\leqslant tC',\end{aligned}$$

as q is uniformly bounded on $\mathbb{R}^{2n}$. As this estimate does neither depend on the choice of $x_0 \in \mathbb{R}^n$, nor on $\xi \in \mathbb{R}^n$, we get

$$\lim_{t\to 0} \sup_{x,\xi\in\mathbb{R}^n} |q(x,\xi) - q_t(x,\xi)| = 0. \tag{1.15}$$

An application of Corollary 1.9.C now gives us that

$$q_t(x_0,\xi) = m'_{t,x_0} + \varphi_{t,x_0}(0) - \varphi_{t,x_0}(\xi) = m_{t,x_0} - \varphi_{t,x_0}(\xi)$$

where $m_{t,x_0} = m'_{t,x_0} + \varphi_{t,x_0}(0) \geqslant 0$ and φ_{t,x_0} is continuous positive definite for $x_0 \in \mathbb{R}^n$ fixed.

Let $\mathcal{B}(0)$ be a basis for the system of neighbourhoods of 0 in $\mathbb{R}^n$, and choose for every $V \in \mathcal{B}(0)$ a continuous positive definite function f_V on $\mathbb{R}^n$ such that supp $\subset V$, $0 \leqslant f_V \leqslant 1$ and $f_V(0) = 1$. That such a function exists is shown in [2], Chapter 2. By Bochner's theorem, Theorem 1.1, there exists a positive bounded measure σ_V on $\mathbb{R}^n$ that is associated with f_V, i.e. $\widehat{\sigma_V}(\xi) = (2\pi)^{-\frac{n}{2}} f_V(\xi)$. As

$$\|\sigma_V\| = \int_{\mathbb{R}^n} 1\sigma_V(\mathrm{d}x) = \int_{\mathbb{R}^n} e^{-i\langle 0,x\rangle}\sigma_V(\mathrm{d}x) = (2\pi)^{\frac{n}{2}}\widehat{\sigma_V}(0) = f_V(0) = 1.$$

we find that σ_V has total mass 1. For $t \geqslant 0$ and $V \in \mathcal{B}(0)$ we find

$$\begin{aligned}\langle\sigma_V, q_t\rangle &= \int_{\mathbb{R}^n} q_t(x_0,\xi)\sigma_V(\mathrm{d}\xi) \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left(1 - e^{-i\langle x,\xi\rangle}\right) a_t^{x_0}(\mathrm{d}x)\sigma(\mathrm{d}\xi) \\ &= \int_{\mathbb{R}^n} \left(1 - \left(\int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle}\sigma_V(\mathrm{d}\xi)\right)\right) a_t^{x_0}(\mathrm{d}x) \\ &= \int_{\mathbb{R}^n} \left(1 - (2\pi)^{\frac{n}{2}}\widehat{\sigma_V}(x)\right) a_t^{x_0}(\mathrm{d}x) \\ &= \int_{\mathbb{R}^n} \left(1 - f_V(x)\right) a_t^{x_0}(\mathrm{d}x).\end{aligned}$$

and it follows that

$$\begin{aligned}
\lim_{V\in\mathcal{B}(0)}\langle\sigma_V, q_t(x_0,\cdot)\rangle &= \lim_{V\in\mathcal{B}(0)}\int_{\mathbb{R}^n}\big(1-f_V(x)\big)a_t^{x_0}(\mathrm{d}x)\\
&= \lim_{V\in\mathcal{B}(0)}\int_{\mathbb{R}^n\setminus\{0\}}\big(1-f_V(x)\big)a_t^{x_0}(\mathrm{d}x)\\
&= \int_{\mathbb{R}^n\setminus\{0\}} 1\, a_t^{x_0}(\mathrm{d}x)\\
&= a_t^{x_0}(\mathbb{R}^n\setminus\{0\})\\
&= a_t^{x_0}(\mathbb{R}^n).
\end{aligned}$$

By (1.15) we know there exists for every $\varepsilon > 0$ some $t_0 > 0$ such that for $x, \xi \in \mathbb{R}^n$

$$|q(x,\xi) - q_t(x,\xi)| < \varepsilon$$

for all $t \in]0, t_0[$. We want to show that $\lim_{V\in\mathcal{B}(0)}\langle\sigma_V, q(x_0,\cdot)\rangle$ exists. First note that

$$\liminf_{V\in\mathcal{B}(0)}\langle\sigma_V, q(x_0,\cdot)\rangle \leqslant \limsup_{V\in\mathcal{B}(0)}\langle\sigma_V, q(x_0,\cdot)\rangle.$$

We find for $x_0 \in \mathbb{R}^n$ fixed

$$\begin{aligned}
&\limsup_{V\in\mathcal{B}(0)}\langle\sigma_V, q(x_0,\cdot)\rangle = \limsup_{V\in\mathcal{B}(0)}\langle\sigma_V, q(x_0,\cdot) - q_t(x_0,\cdot) + q_t(x_0,\cdot)\rangle\\
&= \limsup_{V\in\mathcal{B}(0)}\langle\sigma_V, q_t(x_0,\cdot)\rangle + \limsup_{V\in\mathcal{B}(0)}\langle\sigma_V, q(x_0,\cdot) - q_t(x_0,\cdot)\rangle\\
&= a_t^{x_0}(\mathbb{R}^n) + \limsup_{V\in\mathcal{B}(0)}\int_{\mathbb{R}^n}\big(q(x_0,\xi) - q_t(x_0,\xi)\big)\sigma_V(\mathrm{d}\xi)\\
&\leqslant a_t^{x_0}(\mathbb{R}^n) + \limsup_{V\in\mathcal{B}(0)}\int_{\mathbb{R}^n}|q(x_0,\xi) - q_t(x_0,\xi)|\,\sigma_V(\mathrm{d}\xi)\\
&\leqslant a_t^{x_0}(\mathbb{R}^n) + \varepsilon,
\end{aligned}$$

recall $\sigma_V(\mathbb{R}^n) = 1$. Similarily

$$\begin{aligned}
\liminf_{V\in\mathcal{B}(0)}\langle\sigma_V, q(x_0,\cdot)\rangle &= a_t^{x_0}(\mathbb{R}^n) + \liminf_{V\in\mathcal{B}(0)}\int_{\mathbb{R}^n}\big(q(x_0,\xi) - q_t(x_0,\xi)\big)\sigma_V(\mathrm{d}\xi)\\
&\geqslant a_t^{x_0}(\mathbb{R}^n) - \liminf_{V\in\mathcal{B}(0)}\int_{\mathbb{R}^n}|q(x_0,\xi) - q_t(x_0,\xi)|\,\sigma_V(\mathrm{d}\xi)\\
&\geqslant a_t(\mathbb{R}^n) - \varepsilon.
\end{aligned}$$

Hence for all $0 < t < t_0$ and all $x_0, \xi \in \mathbb{R}^n$.

$$\begin{aligned} -\varepsilon &\leqslant \liminf_{V \in \mathcal{B}(0)} \big(\langle \sigma_V, q(x_0, \xi)\rangle - a_t^{x_0}(\mathbb{R}^n)\big) \\ &\leqslant \limsup_{V \in \mathcal{B}(0)} \big(\langle \sigma_V, q(x_0, \xi)\rangle - a_t^{x_0}(\mathbb{R}^n)\big) \\ &\leqslant \varepsilon. \end{aligned}$$

It follows that for every $x_0 \in \mathbb{R}^n$

$$m_{x_0} := \lim_{V \in \mathcal{B}(0)} \langle \sigma_V, q(x_0, \cdot)\rangle$$

exists with

$$m_{x_0} = \lim_{t \to 0} a_t^{x_0}(\mathbb{R}^n).$$

On the other hand

$$\langle \sigma_V, q(x_0, \cdot)\rangle = \int_{\mathbb{R}^n} q(x_0, \xi)\sigma_V(\mathrm{d}\xi),$$

thus

$$\left|\int_{\mathbb{R}^n} q(x_0, \xi)\sigma_V(\mathrm{d}\xi)\right| \leqslant C \int_{\mathbb{R}^n} 1\sigma_V(\mathrm{d}\xi) = C$$

which follows from the uniform boundedness of our symbol. We conclude

$$m_{x_0} = \lim_{V \in \mathcal{B}(0)} \langle \sigma_V, q(x_0, \cdot)\rangle \leqslant C$$

for all $x_0 \in \mathbb{R}^n$ proving the theorem. □

Let us now turn our attention to another important topic, Bernstein functions. These functions will be important in Chapter 3 when we discuss the Yosida approximation. Similar to the case of negative definite functions, there are several ways to look at Bernstein functions. We present two of these here.

Definition 1.11. A C^∞-function $f :]0, \infty[\to \mathbb{R}$ is called a *Bernstein function* if and only if $f \geqslant 0$ and $(-1)^p \partial^p f \leqslant 0$ for all integers $p \geqslant 1$.

Theorem 1.12. *A function $f :]0, \infty[\to \mathbb{R}$ is a Bernstein function if and only if there exist constants $a, b \geqslant 0$ and a positive measure μ on $]0, \infty[$ satisfying*

$$\int_0^\infty \frac{s}{1+s} \mu(\mathrm{d}s) < \infty,$$

such that

$$f(x) = a + bx + \int_0^\infty (1 - e^{-xs})\mu(\mathrm{d}s)$$

for $x > 0$. The triple (a, b, μ) is uniquely determined by f.

Bernstein functions are a nice tool in the theory of negative definite functions as they allow us to construct new negative definite functions out of existing ones. In order to illustrate that in more detail, let us first look at the following table that displays some continuous negative definite functions $\psi : \mathbb{R} \to \mathbb{C}$ and the corresponding convolution semigroups $(\mu_t)_{t \geqslant 0}$ on $\mathbb{R}$. We

$(\mu_t)_{t\geqslant 0}$	$\psi : \mathbb{R} \to \mathbb{C}\,,\ \xi \mapsto \psi(\xi)$
$e^{-at}\varepsilon_0\,,\ a \geqslant 0$	a
$\varepsilon_{at}\,,\ a \in \mathbb{R}$	$ia\xi$
$(4\pi t)^{-\frac{1}{2}} e^{-\frac{x^2}{4t}}\,\mathrm{d}x$	ξ^2
$\frac{t}{\pi}(t^2 + x^2)^{-1}\,\mathrm{d}x$	$\lvert\xi\rvert$
$\sum_{k=0}^{\infty} e^{-t}\frac{t^k}{k!}\varepsilon_{sk}\,,\ s \geqslant 0$	$1 - e^{-is\xi}$
$\chi_{]0,\infty[}(x)\frac{1}{\Gamma(t)}x^{t-1}e^{-x}\,\mathrm{d}x$	$\log(1 + \xi^2) + i\arctan\xi$

Table 1.1: Negative Definite Functions and Convolution Semigroups

want to calculate one example explicitly. Let us take the third entry in Table 1.1, i.e. let

$$\mu_t(\mathrm{d}x) = (4\pi t)^{-\frac{1}{2}} e^{-\frac{x^2}{4t}}\,\mathrm{d}x,$$

and denote the density with

$$g_t(x) = (4\pi t)^{-\frac{1}{2}} e^{-\frac{x^2}{4t}}$$

for $x \in \mathbb{R}$. Using Theorem 1.6 we need to calculate the Fourier transform $\widehat{\mu}_t(\xi)$ which is

$$\begin{aligned}\widehat{\mu}_t(\xi) &= (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{-ix\xi} \mu_t(\mathrm{d}x) = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{-ix\xi} g_t(x)\,\mathrm{d}x \\ &= (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{-ix\xi} (4\pi t)^{-\frac{1}{2}} e^{-\frac{x^2}{4t}}\,\mathrm{d}x \\ &= e^{-t\xi^2}.\end{aligned}$$

We find that the associated continuous negative definite function $\psi : \mathbb{R} \to \mathbb{C}$ is the function $\xi \mapsto \xi^2$. Let us now present some examples of Bernstein functions.

- $x \mapsto x^\alpha$, $\alpha \in (0,1)$
- $x \mapsto \frac{\lambda}{\lambda+x}$, $\lambda > 0$
- $x \mapsto \log(1+x)$
- $x \mapsto \sqrt{x}\arctan\frac{\lambda}{\sqrt{x}}$, $\lambda > 0$.

With regard to Chapter 3 we are particularly interested in the Bernstein function

$$x \mapsto \frac{\lambda x}{\lambda + x}, \ \lambda > 0. \tag{1.16}$$

In order to check that (1.16) is indeed a Bernstein function we use Theorem 1.12 with

$$\begin{aligned}\frac{\lambda x}{\lambda + x} &= \lambda\left(\frac{x+\lambda-\lambda}{\lambda(x+\lambda)}\right) = \lambda\left(\frac{1}{\lambda} - \frac{1}{x+\lambda}\right) \\ &= \lambda \int_0^\infty \left(e^{-\lambda s} - e^{-(x+\lambda)s}\right)\mathrm{d}s \\ &= \int_0^\infty (1-e^{-sx})\lambda e^{-\lambda s}\,\mathrm{d}s.\end{aligned}$$

For the derivatives of any Bernstein function f we have

$$\left|f^{(k)}(x)\right| \leqslant \frac{k!}{x^k} f(x), \tag{1.17}$$

where $x > 0$ and $k \in \mathbb{N}_0$. This estimate is needed in the proof of Theorem 3.8 in Chapter 3.

The following result can be found in Chr. Berg and G. Forst [2], Remark 9.20–9.22, and also in N. Jacob [15], Lemma 3.9.9. It connects Bernstein functions and continuous negative definite functions.

Theorem 1.13. *Let $f :]0,\infty[\to \mathbb{R}$ be a Bernstein function, and $\psi : \mathbb{R}^n \to \mathbb{C}$ be a continuous negative definite function. Then $f \circ \psi : \mathbb{R}^n \to \mathbb{R}$ is a continuous negative definite function.*

Let $\psi : \mathbb{R}^n \to \mathbb{C}$ be a continuous negative definite function. Then by Theorem 1.13 the function

$$\psi^\lambda : \mathbb{R}^n \to \mathbb{C}\,,\; \xi \mapsto \frac{\lambda\psi(\xi)}{\lambda + \psi(\xi)}\,,\; \lambda > 0,$$

is also continuous negative definite. The function ψ^λ is nice as it is bounded even if ψ is unbounded, and one has the pointwise convergence

$$\lim_{\lambda\to\infty} \psi^\lambda(\xi) = \psi(\xi).$$

We will pick up this topic again in Chapter 3.

1.3 The Symbolic Calculus for Pseudodifferential Operators

We already mentioned in the introduction of this chapter that the material covered in this section is not so much needed in Chapter 2, but is fundamental for Chapter 3. The symbolic calculus for negative definite symbols was developed by W. Hoh in [10], another reference is N. Jacob [16]. Let us first recall some basic things about pseudodifferential operators. Elementary properties of Fourier transformation, compare (1.9), allow us to write for $u \in \mathcal{S}$

$$\widehat{D_x^\alpha u}(\xi) = \xi^\alpha \hat{u}(\xi).$$

As $\hat{\hat{v}} = \check{v}$, compare (1.8), this leads to

$$\begin{aligned} D_x^\alpha u(x) = \widehat{\widehat{D_x^\alpha u}}(-x) &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \widehat{D_x^\alpha u}(\xi)\,\mathrm{d}\xi \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \xi^\alpha \hat{u}(\xi)\,\mathrm{d}\xi. \end{aligned}$$

Similarly we find for the differential operator $b(x,D) = \sum_{|k|\leqslant m} b_\alpha(x) D^\alpha$ the following expression:

$$b(x,D)u(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} b(x,\xi)\hat{u}(\xi)\,\mathrm{d}\xi$$

for $u \in \mathcal{S}$, where we call b the symbol of the operator $b(x, D)$. One can do pretty much the same calculations as above and finds that as $b(x, D)$ is a linear partial differential operator, the symbol b is simply the polynomial $b(x, \xi) = \sum_{|\alpha| \leqslant m} a_\alpha(x)\xi^\alpha$. We are interested in studying properties of an operator $q(x, D)$ of the form

$$q(x, D)u(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} q(x, \xi)\hat{u}(\xi)\, \mathrm{d}\xi$$

where $u \in \mathcal{S}$ and $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$, $\xi \mapsto q(x, \xi)$ is a continuous negative definite function for all $x \in \mathbb{R}^n$ fixed. We call $q(x, D)$ a *pseudodifferential operator* with negative definite symbol. One of the problems with pseudodifferential operators that have variable coefficients is that of finding inverses. Since $q(x, \xi)\hat{u}(\xi)$ is no longer the Fourier transform of $q(x, D)u$, it turns out that the operator we would get by using the symbol $\left(q(x, \xi)\right)^{-1}$ is not an exact inverse of the operator $q(x, D)$. This is one of the problems we will encounter in Section 1.4, when we discuss the Hille-Yosida theorem.

Going back to our pseudodifferential operator $q(x, D)$ with negative definite symbol, it is clear that we need to restrict the class of allowed symbols in order to develop a nice theory. In order to do so we have to impose some conditions on the negative definite functions used to define our symbol classes. Every continuous negative definite function has a Lévy-Khinchin representation, compare Theorem 1.4,

$$\psi(\xi) = c + i(\langle d, \xi\rangle) + q(\xi) + \int_{\mathbb{R}^n\setminus\{0\}} \left(1 - e^{-i\langle x,\xi\rangle} - \frac{i\langle x, \xi\rangle}{1 + |x|^2}\right) \frac{1 + |x|^2}{|x|^2}\mu(\mathrm{d}x),$$

which in case of a real-valued ψ can be written as

$$\psi(\xi) = c + q(\xi) + \int_{\mathbb{R}^n\setminus\{0\}} \left(1 - \cos(\langle x, \xi\rangle)\right)\nu(\mathrm{d}x),$$

where $\nu(\mathrm{d}x) = \frac{1+|x|^2}{|x|^2}\mu(\mathrm{d}x)$ is called the *Lévy measure* associated with ψ. We say that a continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{R}$ belongs to the class Λ if for all $l \geqslant 2$ all absolute moments of the measure ν exist, i.e.

$$M_l := \int_{\mathbb{R}^n\setminus\{0\}} |x|^l \nu(\mathrm{d}x) < \infty. \tag{1.18}$$

In the formal definition of the symbol classes we also need the function $\rho : \mathbb{N}_0 \to \mathbb{N}_0$, $k \mapsto \rho(k) = k \wedge 2$. This function allows us later on to consider asymptotic expansions of symbols. It will not be needed explicitly in any calculation later on. Now we are ready to give the definition of a symbol class.

Definition 1.14. A. Let $m \in \mathbb{R}$, $\psi \in \Lambda$ and q be a C^∞ complex-valued function defined on $\mathbb{R}^n \times \mathbb{R}^n$. Then we say that q is a *symbol of order* m, belonging to $S_\rho^{m,\psi}$, if for all α, $\beta \in \mathbb{N}_0^n$ there are constants $C_{\alpha\beta}$ such that

$$\left|\partial_x^\alpha \partial_\xi^\beta q(x,\xi)\right| \leqslant C_{\alpha\beta}\big(1+\psi(\xi)\big)^{\frac{m-\rho(|\beta|)}{2}}$$

for $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$.
B. Let $m \in \mathbb{R}$, $\psi \in \Lambda$ and suppose that q is a C^∞ complex-valued function defined on $\mathbb{R}^n \times \mathbb{R}^n$ such that

$$\left|\partial_x^\alpha \partial_\xi^\beta q(x,\xi)\right| \leqslant C_{\alpha\beta}\big(1+\psi(\xi)\big)^{\frac{m}{2}}$$

holds for all $\alpha, \beta \in \mathbb{N}_0^n$ and $x, \xi \in \mathbb{R}^n$. Then we call q a symbol of class $S_0^{m,\psi}$.

Obviously we have $S_\rho^{m,\psi} \subset S_0^{m,\psi}$. As we will soon see, the main difference between symbols of class $S_\rho^{m,\psi}$ and $S_0^{m,\psi}$ is that in the latter case we do not have any asymptotic expansion to work with. In other words, proofs that use the asymptotic expansion of symbols, e.g. the Gårding inequality, hold only for symbols of class $S_\rho^{m,\psi}$. The same will happen in Chapter 3, where some results only hold if $q \in S_\rho^{m,\psi}$. We now show that the symbolic calculus for negative definite symbols works similar to the classical one.

We begin with some further observations. The continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{R}$, $\xi \mapsto |\xi|^2$ belongs to the class Λ. This is easy to see as the Lévy measure associated with ψ through the Lévy-Khinchin formula is 0. Now by definition a classical symbol $q \in S_{1,0}^m$, $m \in \mathbb{R}$, is a C^∞ function $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ such that

$$\left|\partial_x^\alpha \partial_\xi^\beta q(x,\xi)\right| \leqslant C_{\alpha\beta}\left(1+|\xi|^2\right)^{\frac{m-|\beta|}{2}}.$$

Since for all $m \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$,

$$\left(1+|\xi|^2\right)^{\frac{m-|\beta|}{2}} \leqslant \left(1+|\xi|^2\right)^{\frac{m-\rho(|\beta|)}{2}},$$

it follows that $S_{1,0}^m \subset S_\rho^{m,|\cdot|^2} \subset S_0^{m,|\cdot|^2}$. In order to understand now how the composition of two pseudodifferential operators with symbols of class $S_\rho^{m,\psi}$ or $S_0^{m,\psi}$ works and what the symbol of the composition looks like, we introduce amplitudes, oscillatory integrals and some other related results.

Oscillatory integrals are integrals of functions which are normally not absolutely integrable, but still exist as a certain limit. The functions we want to define integrals for, are products of an oscillatory term and an amplitude with a controlled growth at infinity. For precise statements we need the notion of amplitudes. As a reference we give N. Jacob [16].

Definition 1.15. For $m \geqslant 0$ and a C^∞ complex-valued function a defined on $\mathbb{R}^n \times \mathbb{R}^n$ we say that $a \in A^m$, the space of amplitudes of order m, if

$$\left|\partial_y^\alpha \partial_\eta^\beta a(y,\eta)\right| \leqslant C_{\alpha\beta} \left(1+|y|^2\right)^{\frac{m}{2}} \left(1+|\eta|^2\right)^{\frac{m}{2}}$$

for $\alpha, \beta \in \mathbb{N}_0^n$.

Now we can give the definition of oscillatory integrals and an important estimate for Chapter 3.

Theorem 1.16. *Let $a \in A^m(\mathbb{R}^n \times \mathbb{R}^n)$ and $\chi \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ such that $\chi(0,0)=1$. Then the limit*

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-i\langle y,\eta\rangle} a(y,\eta)\chi(\varepsilon y, \varepsilon\eta)\, \mathrm{d}y\, \mathrm{d}\eta$$

exists, is independent of χ, and is denoted by $\mathrm{Os}\text{–}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle y,\eta\rangle} a(y,\eta)\, \mathrm{d}y\, \mathrm{d}\eta$. One has the estimate

$$\left|\mathrm{Os}\text{–}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle y,\eta\rangle} a(y,\eta)\, \mathrm{d}y\, \mathrm{d}\eta\right| \leqslant C_m |||a|||_{m+2n+1}, \tag{1.19}$$

where

$$|||a|||_k = \max_{|\alpha+\beta|\leqslant k} \sup_{y,\eta\in\mathbb{R}^n} \left|\left(1+|y|^2\right)^{-\frac{m}{2}} \left(1+|\eta|^2\right)^{-\frac{m}{2}} \partial_y^\alpha \partial_\eta^\beta a(y,\eta)\right|.$$

Oscillatory integrals behave in some respect like absolutely convergent integrals, we have for example that for $a \in A^m(\mathbb{R}^n \times \mathbb{R}^n)$

$$\begin{aligned} &\partial_\eta^\alpha\, \mathrm{Os}\text{–}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle y,\eta\rangle} a(y,\eta)\, \mathrm{d}y\, \mathrm{d}\eta \\ &= (-1)^{|\alpha|}\, \mathrm{Os}\text{–}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle y,\eta\rangle} \partial_\eta^\alpha a(y,\eta)\, \mathrm{d}y\, \mathrm{d}\eta. \end{aligned} \tag{1.20}$$

Another result that will be useful in Chapter 3 is

Theorem 1.17. *If $a \in A^m(\mathbb{R}^n \times \mathbb{R}^n)$, then for $c \in \mathbb{R}^n$ fixed*

$$\begin{aligned} &(2\pi)^{-n}\, \mathrm{Os}\text{–}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle y,\eta\rangle} a(y,c)\, \mathrm{d}y\, \mathrm{d}\eta \\ &= (2\pi)^{-n}\, \mathrm{Os}\text{–}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle y,\eta\rangle} a(\eta,c)\, \mathrm{d}y\, \mathrm{d}\eta \\ &= a(0,c). \end{aligned}$$

Before we proceed let us remark that if $q \in S_0^{m,\psi}$, $m \geqslant 0$, we also have that $q \in A^m$. This follows with Corollary 1.7.B from

$$\left|\partial_x^\alpha \partial_\xi^\beta q(x,\xi)\right| \leqslant C_{\alpha\beta}\left(1+\psi(\xi)\right)^{\frac{m}{2}} \leqslant C'_{\alpha\beta}\left(1+|\xi|^2\right)^{\frac{m}{2}},$$

i.e. $S_\rho^{m,\psi} \subset S_0^{m,\psi} \subset A^m$. The next Theorem enables us to define a symbolic calculus for pseudodifferential operators with negative definite symbols.

Theorem 1.18. *A. Let $\psi \in \Lambda$ and $q_1 \in S_0^{m,\psi}$, $q_2 \in S_0^{l,\psi}$. Then the oscillatory integrals*

$$q_1^*(x,\xi) = (2\pi)^{-n}\, \text{Os}-\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle y,\eta\rangle}\overline{q_1(x-y,\xi-\eta)}\, dy\, d\eta,$$

and

$$q_1\# q_2(x,\xi) = (2\pi)^{-n}\, \text{Os}-\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle y,\eta\rangle} q_1(x,\xi-\eta)q_2(x-y,\xi)\, dy\, d\eta,$$

define symbols $q_1^* \in S_0^{m,\psi}$ and $q_1\# q_2 \in S_0^{m+l,\psi}$, where q_1^* is the symbol of the operator $q_1^*(x,D)$ and $q_1\# q_2$ is the symbol of the operator $q_1(x,D)\circ q_2(x,D)$.
B. Let $\psi \in \Lambda$ and $q_1 \in S_\rho^{m,\psi}$, $q_2 \in S_\rho^{m,\psi}$. Then

$$q_1^*(x,\xi) = \overline{q_1(x,\xi)} + \sum_{j=1}^n \partial_{\xi_j} D_{x_j} q(x,\xi) + q_{r_1}(x,\xi)$$

where $q_{r_1} \in S_0^{m-2,\psi}$ and

$$q_1\# q_2(x,\xi) = q_1(x,\xi)q_2(x,\xi) + \sum_{j=1}^n \partial_{\xi_j} q_1(x,\xi) D_{x_j} q_2(x,\xi) + q_{r_2}(x,\xi)$$

where $q_{r_2} \in S_0^{m+l-2,\psi}$.

It is important to point out that we have asymptotic expansions only up to order 2, whereas in the classical calculus we have expansions modulo a term of order $-\infty$! At the end of this section we want to give some results that are well-known to hold for pseudodifferential operators with classical symbols, but can also be extended to the symbolic calculus with negative definite symbols.

Theorem 1.19. *If $q \in S_0^{\infty,\psi}$ and $u \in \mathcal{S}$, the formula*

$$q(x,D)u(x) = (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} q(x,\xi)\hat{u}(\xi)\, d\xi$$

defines a function $q(x,D)u \in \mathcal{S}$.

The anisotropic Sobolev spaces $H^{m,\psi}$ that are used in the next Theorem are more thoroughly explained in Chapter 3, but their definition can also be found in the Index of Notation.

Theorem 1.20. *Let $q \in S_0^{m,\psi}$, $m \in \mathbb{R}$, then for every $s \in \mathbb{R}$ there exists a constant C_s such that $q(x,D)u \in H^{s,\psi}$ for all $u \in H^{m+s,\psi}$ with*

$$\|q(x,D)u\|_{H^{s,\psi}} \leqslant C_s \|u\|_{H^{s+m,\psi}}.$$

1.4 Operator Semigroups

As we mentioned in the introduction, our aim is to develop a theory of pseudodifferential operators with negative definite symbols that allows us to construct stochastic processes and identify their properties. This is done using operator semigroups, and we will use this section to explain the theory in more detail. Of importance, also for Chapter 3, is the Hille-Yosida theorem, a cornerstone of the theory. As references we give A. Pazy [20] and also N. Jacob [15]. Let us begin with some simple definitions.

Definition 1.21. Let X be a Banach space. A one parameter family $(T_t)_{t\geqslant 0}$ of bounded linear operators from X to X is a *semigroup of bounded linear operators on X* if

1. $T_0 = \mathrm{id}$
2. $T_{t+s} = T_t \circ T_s$ for every $s, t \geqslant 0$.

The semigroup $(T_t)_{t\geqslant 0}$ is called a *contraction semigroup* if

$$\|T_t\| \leqslant 1$$

for all $t \geqslant 0$. It is called *uniformly continuous* if

$$\lim_{t\to 0} \|T_t - \mathrm{id}\,\| = 0, \tag{1.21}$$

and *strongly continuous* if for every $x \in X$

$$\lim_{t\to 0} \|T_t x - x\|_X = 0. \tag{1.22}$$

Instead of (1.22) we also write "$\lim_{t\to 0} T_t x = x$ strongly". A central notion in the theory of one parameter semigroups is that of the generator.

Definition 1.22. Let $(T_t)_{t\geqslant 0}$ be a one parameter semigroup of operators on a Banach space $(X, \|\cdot\|)$ as defined above. The *generator* A of $(T_t)_{t\geqslant 0}$ is defined by the strong limit

$$Ax = \lim_{t\to 0} \frac{T_t x - x}{t}$$

with domain

$$D(A) := \left\{ x \in X \; ; \; \lim_{t\to 0} \frac{T_t x - x}{t} \text{ exists as a strong limit} \right\}.$$

Let us further say that a linear bounded operator $S : X \to X$, where X is a Banach space of real-valued functions, is *positivity preserving* if $x \geqslant 0$ implies that $Sx \geqslant 0$. Now we are able to give the definition of a Feller semigroup.

Definition 1.23. Let $(T_t)_{t\geqslant 0}$ be a strongly continuous contraction semigroup on $(C_\infty, \|\cdot\|_\infty)$ that is positivity preserving, i.e. for $u \geqslant 0$, $T_t u \geqslant 0$. Then $(T_t)_{t\geqslant 0}$ is called a *Feller semigroup.*

Next we summarize some results for operator semigroups that are needed later on.

Lemma 1.24. *Let $(T_t)_{t\geqslant 0}$ be a strongly continuous semigroup on a Banach space $(X, \|\cdot\|)$ and denote by A its generator with domain $D(A) \subset X$.*
A. For $u \in X$ and $t \geqslant 0$ it follows that $\int_0^t T_s u \,\mathrm{d}s \in D(A)$ and

$$T_t u - u = A \int_0^t T_s u \,\mathrm{d}s.$$

B. For $u \in D(A)$ and $t \geqslant 0$ we have $T_t u \in D(A)$, i.e. $D(A)$ is invariant under T_t, and

$$\frac{\mathrm{d}}{\mathrm{d}t} T_t u = A T_t u = T_t A u.$$

C. For $u \in D(A)$ and $t \geqslant 0$ we always get

$$T_t u - u = \int_0^t A T_s u \,\mathrm{d}s = \int_0^t T_s A u \,\mathrm{d}s.$$

Definition 1.25. A linear operator $A : D(A) \to X$, $D(A) \subset X$, is called *dissipative*, more precisely X-dissipative, if

$$\|\lambda u - Au\|_X \geqslant \lambda \|u\|_X \tag{1.23}$$

holds for all $\lambda > 0$ and $u \in D(A)$.

As there is a one-to-one correspondence between Feller semigroups and Feller processes, compare e.q. [17], we may now state the aim of our theory more clearly: We want to construct Feller semigroups that are generated by pseudodifferential operators (defined on nice domains) with negative definite symbols. One then hopes to find connections between the symbol of the pseudodifferential operator and the corresponding stochastic process. For results in such a direction we refer to W. Hoh [8] and [9], and in particular to R. Schilling [24]–[27]. There are a few problems left to solve though. The most obvious one is how one actually constructs a Feller semigroup. One way is to use the Hille-Yosida theorem. Another way is to use Chernoff's theorem, refer to Chapter 2 for details on the construction, and yet another way is discussed at the end of Chapter 3. But let us stick with the classical Hille-Yosida theorem for the moment, which we give here in a slightly alterated version that is due to Lumer-Phillips.

Theorem 1.26. *A linear operator on a Banach space $(X, \|\cdot\|)$ is closable and its closure $\overline{A}$ is the generator of a strongly continuous contraction semigroup on X if and only if the following three conditions are satisfied*

A. $D(A) \subset X$ is dense

B. A is a dissipative operator

C. $R(\lambda - A)$ is dense in X for some $\lambda > 0$.

The first thing that should be noted is that this Theorem does not give us a positivity preserving semigroup. The next theorem by Ph. Courrège [5] solves this dilemma.

Theorem 1.27. *Every pseudodifferential operator*

$$-q(x, D)u(x) = -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} q(x,\xi)\hat{u}(\xi)\, d\xi,$$

with a continuous symbol $-q$ such that $\xi \mapsto q(x,\xi)$ is negative definite for any $x \in \mathbb{R}^n$ satisfies the positive maximum principle on $C_0^\infty(\mathbb{R}^n)$.

Hence we find that $-q(x, D)$ satisfies on C_0^∞ the positive maximum principle. It follows that $-q(x, D)$ is a dissipative operator on C_0^∞, i.e. it satisfies

$$\|\lambda u + q(x, D)u\|_\infty \geqslant \lambda \|u\|_\infty$$

for all $\lambda > 0$ and $u \in C_0^\infty$, compare e.g. Th. Kurtz and St. Ethier [7]. It can be shown that the resulting semigroup is then positivity preserving, compare

Theorem 4.5.3 in [15]. As $C_0^\infty \subset C_\infty$ is dense, condition A of Theorem 1.26 is fulfilled as well. Hence it remains to show Theorem 1.26.C. This condition means that for every f from a dense subset of C_∞ we need to find a $u \in C_0^\infty$ such that

$$\lambda u + q(x, D)u = f, \tag{1.24}$$

i.e. we need to solve this equation. This in turn means to find the inverse of the pseudodifferential operator $\lambda + q(x, D)$, which is even in the case of classical symbols far from easy to do. The strategy is to solve (1.24) on a larger domain than C_0^∞, on which Theorem 1.26.A and Theorem 1.26.B are still satisfied. We are not going into any more details here but simply remark that the solution is to consider pseudodifferential operators on certain scales of anisotropic Sobolev spaces $H^{s,\psi}$ and to solve (1.24) there. At the end of this first chapter we present the theorem that allows us to construct Feller semigroups as long as the symbol of the generating pseudodifferential operator satisfies certain conditions. As a reference we give W. Hoh [10].

Theorem 1.28. *Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a continuous negative definite function in the class Λ, and suppose in addition that $\psi(\xi) \geqslant c_0|\xi|^{r_0}$ for some $c_0 > 0$, $r_0 > 0$ and large $|\xi|$. If q is a negative definite symbol belonging to $S_\rho^{2,\psi}$ and satisfies $q(x,\xi) \geqslant \delta\big(1 + \psi(\xi)\big)$ for some $\delta > 0$ and all $\xi \in \mathbb{R}^n$, $|\xi|$ sufficiently large, then $-q(x, D)$ defined on C_0^∞ is closable in C_∞ and its closure is a generator of a Feller semigroup.*

Chapter 2

On Roth's Method for Pseudodifferential Operators with Bounded Negative Definite Symbols

We use the term "Roth's method" to describe a procedure used by J. P. Roth in [23] to construct Feller semigroups that are generated by second order elliptic linear partial differential operators with variable coefficients. His idea is to "freeze" the coefficients and thus obtaining a family of constant coefficient operators. Then he constructs a corresponding family of Feller semigroups, and uses certain estimates to "glue" these Feller semigroups together to obtain a Feller semigroup that is generated by the variable coefficient differential operator. This idea will be explained in detail in Section 1.

Section 2 describes how "Roth's method" works if we replace the differential operator with a pseudodifferential operator that has a negative definite symbol. We pick up some ideas suggested by E. Popescu in [21], and need to make some other modifications to Roth's original concept as outlined in Section 1 as well. The most notable one is the use of a theorem of P. R. Chernoff [4] that allows us to construct strongly continuous contraction semigroups and to identify their generators. We work with uniformly bounded symbols, i.e. $\sup_{x,\xi\in\mathbb{R}^n} |q(x,\xi)| \leqslant C$, $C > 0$, hence we want to mention that a similar result can be achieved by defining the semigroup as $(e^{-tq(x,D)})_{t\geqslant 0}$ where $-q(x,D)$ is the generator of the semigroup. As the title of this chapter suggests we emphasize here the way of construction, and not so much the final result. Roth's method is also interesting in view of Chapter 3. We already remarked at the end of Section 1.2 that the Yosida approximation of an unbounded symbol is a bounded one and that we have pointwise convergence of the Yosida approximation to the original symbol. This connection is explored further in Chapter 3.

2.1 Differential Operators

This section is based on work done by J. P. Roth [23]. Having Section 2.2 in mind, we point out which argumentations and ideas can not be carried over to the case of pseudodifferential operators. Let us begin with a definition.

Definition 2.1. Let a linear partial differential operator of second order

$$Lu(x) = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}u(x) + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i}u(x)$$

be given with coefficients $a_{ij} : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto a_{ij}(x)$ for $i, j = 1, \ldots, n$, and a vector $b(x) = \big(b_i(x)\big)_{i=1,\ldots,n}$ in $\mathbb{R}^n$. The operator L is called *uniformly elliptic* if for all $x, \xi \in \mathbb{R}^n$,

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_l \geqslant c_0|\xi|^2$$

holds with some constant $c_0 > 0$ independent of x.

The main result of this section is the following theorem.

Theorem 2.2. *Define on $C_\infty^2(\mathbb{R}^n)$ a uniformly elliptic second order linear partial differential operator*

$$Lu(x) = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}u(x) + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i}u(x). \tag{2.1}$$

Assume in addition that $a_{ij}, b_i \in C_b^2$ for all $i, j = 1, \ldots, n$ such that for a matrix P of order n we have $A(x) = P^T(x)P(x)$. Then L extends to the generator of a Feller semigroup $(T_t)_{t\geqslant 0}$ on $C_\infty(\mathbb{R}^n)$. This semigroup is given for $u \in C_\infty(\mathbb{R}^n)$ and $t \geqslant 0$ by the strong limit

$$T_t u = \lim_{m\to\infty} \left(S_{\frac{t}{m}}\right)^m (u), \tag{2.2}$$

where

$$S_t u(x) = \pi^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-|\xi|^2} u\big(x + tb(x) - 2\sqrt{t}P(x)\xi\big)\, d\xi.$$

We are going to look at the above theorem more closely and explain how J. P. Roth arrives at this result. Let us first discuss the operator S_t. Let $y \in \mathbb{R}^n$ be fixed and consider the operator L^y defined by

$$\begin{aligned} D(L^y) &= D(L) \\ L^y u(x) &= \sum_{i,j=1}^{n} a_{ij}(y) \frac{\partial^2}{\partial x_i \partial x_j} u(x) + \sum_{i=1}^{n} b_i(y) \frac{\partial}{\partial x_i} u(x), \end{aligned} \tag{2.3}$$

i.e. $L = L^x$. Note that we have constant coefficients now as $y \in \mathbb{R}^n$ fixed. It can be shown that L^y extends to the generator of a Feller semigroup $(R_t^y)_{t \geqslant 0}$ defined by

$$R_t^y u(x) = \pi^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-|\xi|^2} u\big(x + tb(y) - 2\sqrt{t} P(y)\xi\big)\, d\xi. \tag{2.4}$$

Assume S_t is the operator on $C_\infty(\mathbb{R}^n)$ defined by

$$S_t u(x) = R_t^x u(x), \tag{2.5}$$

i.e.

$$S_t u(x) = \pi^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-|\xi|^2} u\big(x + tb(x) - 2\sqrt{t} P(x)\xi\big)\, d\xi. \tag{2.6}$$

That S_t is linear and positivity preserving is obvious. The contractivity of S_t follows from the fact that $(R_t^y)_{t \geqslant 0}$ is a Feller semigroup for all $y \in \mathbb{R}^n$. It remains to show that $t \mapsto S_t u$ is strongly continuous for every $u \in C_\infty$. This follows from

$$\begin{aligned} &|S_t u(x) - S_s u(x)| \\ \leqslant &\int_{\mathbb{R}^n} e^{-|\xi|^2} \left| u\big(x + tb(x) - 2\sqrt{t} P(x)\xi\big) - u\big(x + sb(x) - 2\sqrt{s} P(x)\xi\big) \right| d\xi, \end{aligned}$$

and hence

$$\lim_{s \to t} \|S_t u - S_s u\|_\infty = 0.$$

With some effort it can further be shown that

$$\frac{\partial}{\partial t} R_t^x u(x) = R_t^x L^x u(x). \tag{2.7}$$

We obviously have $R_0^x = \mathrm{id}$, hence an application of the mean value theorem gives us for $\theta \in]0, t[$ that

$$R_t^x u(x) - u(x) = t R_{t\theta}^x L^x u(x).$$

It follows that

$$\frac{S_t u(x) - u(x)}{t} - Lu(x) = \frac{R_t^x u(x) - u(x)}{t} - Lu(x)$$
$$= R_{t\theta}^x L^x u(x) - Lu(x) = \left(R_{t\theta}^x - \mathrm{id}\right) Lu(x)$$
$$= \sum_{i,j=1}^n a_{ij}(x) \left(S_{t\theta} \frac{\partial^2}{\partial x_i x_j} u(x) - \frac{\partial^2}{\partial x_i \partial x_j} u(x) \right)$$
$$+ \sum_{i=1}^n b_i(x) \left(S_{t\theta} \frac{\partial}{\partial x_i} u(x) - \frac{\partial}{\partial x_i} u(x) \right).$$

As $t \mapsto S_t v$ is continuous for all $v \in C_\infty(\mathbb{R}^n)$ and $S_0 = \mathrm{id}$, we conclude that for $u \in D(L) = C_b^2(\mathbb{R}^n)$

$$\lim_{t \to 0} \frac{S_t u(x) - u(x)}{t} - Lu(x) = 0,$$

and hence

$$\lim_{t \to 0} \frac{S_t u(x) - u(x)}{t} = Lu(x). \tag{2.8}$$

Let us give some remarks before proceeding: It is tempting to say now that with (2.8) we have proven that L is the generator of the semigroup $(S_t)_{t \geqslant 0}$. But we have not actually shown that $(S_t)_{t \geqslant 0}$ is a semigroup, and indeed it is not. What we do know however is that $(R_t^y)_{t \geqslant 0}$ as defined in (2.4) forms a Feller semigroup with generator L^y as given in (2.3). As $S_t = R_t^x$, the family $(S_t)_{t \geqslant 0}$ is much harder to handle. We have merely shown in (2.8) a certain convergence behavior, which however falls in line nicely with Definition 1.22. Our next aim is to show that the family $(S_t)_{t \geqslant 0}$ almost defines an operator semigroup, i.e. it is a semigroup modulo some error term. Using a limiting process we then show that the error term vanishes. First we need some estimates for the operator S_t. Those are rather technical and we refer to J. P. Roth [23] for details. We mention once more that it is possible to derive these estimates explicitly because we work with a very specific operator S_t, see (2.6). For a detailed discussion and a comparison of the operators we use in Section 2.1 and Section 2.2, see the paragraph below (2.28).

From now on we use for $f \in C_b^2(\mathbb{R}^n)$ the notation $|f|$ to denote the seminorm

$$|f| := \|\nabla f\|_\infty + \|H(f)\|_\infty,$$

where

$$\nabla f(x) = \left(\frac{\partial}{\partial x_1} f(x), \ldots, \frac{\partial}{\partial x_n} f(x) \right),$$

and

$$H(f)(x) = J(\nabla f)(x),$$

the Jacobian of the gradient of f. In the following the constant K does not depend on the x variable in the matrix $A(x)$ or the vector $b(x)$.

Lemma 2.3. *There exists a constant $K > 0$ such that for all $u \in D(L)$ and $t \in [0,1]$,*

$$|S_t u| \leqslant (1 + Kt)|u|$$

For Lemma 2.3 and its proof we refer to Lemma 1 on p. 240 in J. P. Roth [23].

Definition 2.4. For all subdivisions $\Delta = (t_0 = a < t_1 < \ldots < t_m = b)$ of the interval $[a, b]$ we denote with S_Δ the operator

$$S_\Delta := S_{t_m - t_{m-1}} \circ S_{t_{m-1} - t_{m-2}} \circ \ldots \circ S_{t_1 - t_0}.$$

If Δ is a subdivision of $[a, b]$ then $\sigma(\Delta) := m$ denotes the number of divisions of Δ.

Lemma 2.5. *There exists a constant $K > 0$ such that for all $u \in D(L)$, $t \in [0,1]$ and all subdivisions Δ of $[0,t]$,*

$$|S_\Delta u| \leqslant K|u|.$$

This follows immediately from Lemma 2.3 as $[0, t] \subset [0, 1]$.

Lemma 2.6. *There exists a $K > 0$ such that for all $u \in D(L)$ and $s, t \geqslant 0$, $s + t \leqslant 1$,*

$$\|S_s \circ S_t u - S_{s+t} u\|_\infty \leqslant Kt\sqrt{s}\,|u|.$$

For details concerning this lemma and its proof we note that it corresponds to Lemma 3 on p. 242 in J. P. Roth [23].

Lemma 2.7. *For all $u \in D(L)$, $t \in [0,1]$ and all subdivisions Δ_1, Δ_2 of $[0,t]$ there exists a $K > 0$ such that*

$$\|S_{\Delta_1} u - S_{\Delta_2} u\|_\infty \leqslant K|u|\,t\sqrt{\max(\{\sigma(\Delta_1), \sigma(\Delta_2)\})}.$$

The proof can be found on p. 244 in J. P. Roth [23]. Let us now define for $t \in [0,1]$ and $n \in \mathbb{N}$,

$$T_{m,t} := \left(S_{\frac{t}{2^m}}\right)^{2^m} = S_{\frac{t}{2^m}} \circ S_{\frac{t}{2^m}} \circ \ldots \circ S_{\frac{t}{2^m}} \Big\} 2^m\text{-terms}, \qquad (2.9)$$

i.e. we have a subdivision of the interval $[0,t]$ into 2^m-pieces, each with a length of $\frac{t}{2^m}$. It follows from Lemma 2.7 that

$$\begin{aligned} \|T_{m,t}u - T_{m+1,t}u\|_\infty &\leqslant K|u|\, t\sqrt{\max\left(\left\{\frac{t}{2^m}, \frac{t}{2^{m+1}}\right\}\right)} \\ &= K|u|\, t\sqrt{\frac{t}{2^m}}. \end{aligned}$$

As a result we find that for $u \in D(L)$ as $m \to \infty$, $T_{m,t}u$ converges uniformly for $t \in [0,1]$ to an element of $C_\infty(\mathbb{R}^n)$, which we denote by $T_t u$, i.e.

$$\lim_{m\to\infty} T_{m,t}u = T_t u$$

as a strong limit. Furthermore $\|T_{m,t}\| \leqslant 1$, see (2.9), and as $D(L) = C^2_\infty \subset C_\infty$ dense, this uniform convergence holds for all $u \in C_\infty(\mathbb{R}^n)$.

We obtain a family $(T_t)_{t\in[0,1]}$ of positivity preserving operators on $C_\infty(\mathbb{R}^n)$ with $\|T_t\| \leqslant 1$. As $t \mapsto S_t u$ is strongly continuous for all $u \in C_\infty$ we also have that $t \mapsto T_{m,t}u$ is strongly continuous for all $m \in \mathbb{N}$ and all $u \in C_\infty$. Hence $t \mapsto T_t f$ is strongly continuous on $[0,1]$ for all $u \in C_\infty$ as a uniform limit of continuous functions. Moreover, $T_0 = \mathrm{id}$ as $S_0 = \mathrm{id}$.

We now show that $(T_t)_{t\in[0,1]}$ has the semigroup property. Let $s, t \geqslant 0$ such that $s + t \leqslant 1$. Then it follows from Lemma 2.7 that

$$\left\|\left(T_{m,s} \circ T_{m,t}\right) u - T_{m,s+t}u\right\|_\infty \leqslant K|u|\,(s+t)\sqrt{\frac{s+t}{2^m}}.$$

By taking the limit $m \to \infty$ we get for $u \in D(L)$,

$$(T_s \circ T_t)u = T_{t+s}u.$$

Hence $T_s \circ T_t = T_{s+t}$ and one may extend the family $(T_t)_{t\in[0,1]}$ to a Feller semigroup $(T_t)_{t\geqslant 0}$.

We are left with identifying the generator $\big(A, D(A)\big)$ of $(T_t)_{t\geqslant 0}$. This is worked out in detail by J. P. Roth [23], p. 245–248, and in order to be selfcontained we outline the basic ideas. The following Lemmas will be helpful.

Lemma 2.8. *Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence such that $\alpha_n \in \mathcal{D}(\mathbb{R}^n)$, $\alpha_n \geqslant 0$, $\int_{\mathbb{R}^n} \alpha_n(x)\,\mathrm{d}x = 1$ and $\operatorname{supp}\alpha_n \subset B\left(0, \frac{1}{n}\right)$, where $B(0,r)$ is the open ball with center 0 and radius r. Then there exists a $K > 0$ such that for all $t \in [0,1]$ and for all $u \in D(B)$,*

$$\left\|S_t\left(u * \alpha_n\right) - S_t(u) * \alpha_n\right\| \leqslant K|u|\frac{t}{n}.$$

Lemma 2.9. *There exists $K > 0$ such that for all $t \in [0,1]$ and all subdivision $\Delta \in [0,t]$ and all $u \in D(B)$,*

$$\|S_t u - S_\Delta u\| \leqslant Kt\sqrt{t}|u|.$$

Denote by $\left(\widetilde{A}, D(\widetilde{A})\right)$ the operator

$$D(\widetilde{A}) = \left\{u \in C^2(\mathbb{R}^n)\ ;\ \partial^\alpha f \in C_\infty(\mathbb{R}^n) \text{ for } |\alpha| \leqslant 2\right\}$$

and

$$\widetilde{A}u(x) = Lu(x)$$

with L as in (2.1). We may use $D(\widetilde{A})$ as domain of each of the operators L^y, compare (2.3), by our ellipticity assumption. In addition for $f \in D(\widetilde{A})$ it follows that

$$\lim_{t\to 0} \frac{S_t f - f}{t} = \widetilde{A}f, \tag{2.10}$$

compare (2.8). For the following we need a "nice" subset of $C_\infty(\mathbb{R}^n)$. For $\lambda > 0$ let E_λ denote the set of all $f \in C_\infty(\mathbb{R}^n)$ such that there exists a sequence $(f_n)_{n\in\mathbb{N}}$, $f_n \in D(\widetilde{A})$, $|f| \leqslant \lambda$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} \|f - f_n\|_\infty = 0$. Now set

$$E := \bigcup_{\lambda>0} E_\lambda \subset C_\infty(\mathbb{R}^n). \tag{2.11}$$

Using Lemma 2.9 and the definition of E_λ we find that for all $\lambda > 0$, all $f \in E_\lambda$ and all $t \in [0,1]$ it holds

$$\|T_t f - S_t f\|_\infty \leqslant K\lambda t\sqrt{t}$$

implying that for $f \in E$ the two limits

$$\lim_{t\to 0} \frac{1}{t}(T_t f - f)$$

and

$$\lim_{t\to 0}\frac{1}{t}(S_t f - f)$$

exist either both and they are equal, or both do not exist. But for $f \in D(\widetilde{A})$ we know $\lim_{t\to 0}\frac{1}{t}S_t f - f = \widetilde{A}f$ implying that the generator $(A, D(A))$ of $(T_t)_{t\geqslant 0}$ is an extension of $(\widetilde{A}, D(\widetilde{A}))$.

Denote by $(\widetilde{A}^-, D(\widetilde{A}^-))$ the closure of $(\widetilde{A}, D(\widetilde{A}))$ in $C_\infty(\mathbb{R}^n)$. We want to show that $(A, D(A)) = (\widetilde{A}^-, D(\widetilde{A}^-))$ and we know already $\widetilde{A}^- \subset A$. Hence it remains to prove that $A \subset \widetilde{A}^-$. Since $D(A)$ is invariant under T_t, i.e. $T_t(D(A)) \subset D(A)$ and since by Lemma 2.5 we have for all $\lambda > 0$ and all $t \in [0,1]$ that $T_t(E_\lambda) \subset E_{K\lambda}$, it follows that E is invariant under T_t too, i.e. $T_t(E) \subset E$. Hence $E \cap D(A)$ is invariant under $(T_t)_{t\geqslant 0}$ and it is also a dense subset of $C_\infty(\mathbb{R}^n)$. Hence, by Th. Kurtz and St. Ethier [7] Proposition I.3.3 the space $E \cap D(A)$ is a core for $(A, D(A))$ and it is sufficient to show that $\widetilde{A}^-$ is an extension of $A|_{E\cap D(A)}$. Using the mollifier results from 2.9 we find for $f \in E_\lambda \cap D(A)$, $\lambda > 0$, and $(\alpha_n)_{n\in\mathbb{N}}$ as in Lemma 2.9 that for all $t \in [0,1]$ it holds

$$\|S_t(f * \alpha_n) - (S_t f) * \alpha_n\|_\infty \leqslant \frac{K\lambda}{n}t$$

implying for all $t \in [0,1]$ and $\lambda > 0$

$$\left\|\frac{1}{t}\big(S_t(f * \alpha_n) - f * \alpha_n\big) - \frac{1}{t}(S_t f - f) * \alpha_n\right\|_\infty \leqslant \frac{K\lambda}{n}.$$

For $n \to \infty$ we find $f * \alpha_n \to f$ and $\widetilde{A}(f * \alpha_n) \to Af$, hence $f \in D(\widetilde{A}^-)$ and $\widetilde{A}^- f = Af$, i.e. $\widetilde{A}^-$ is an extension of A proving that $\widetilde{A}^- = A$ as closed operators.

It should be mentioned that our result in the next section reads basically the same as Theorem 2.2, but the construction of the Feller semigroup through a strong limit is completely different. The estimates from Lemma 2.4 to Lemma 2.7 are only possible because we work with very specific operators S_t — still the general ideas of this section can also be applied in Section 2.2.

2.2 Pseudodifferential Operators

We point out that most of this section can also be found in our joint article with N. Jacob [18]. Let us begin with considering a pseudodifferential operator

$$q(x,D)u(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} q(x,\xi)\hat{u}(\xi)\,\mathrm{d}\xi$$

for $u \in \mathcal{S}(\mathbb{R}^n)$. In order to make Roth's method work for pseudodifferential operators we need to assume that $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $(x,\xi) \mapsto q(x,\xi)$, is a continuous function satisfying

$$1.\ \xi \mapsto q(x,\xi) \text{ is negative definite for all } x \in \mathbb{R}^n \tag{2.12}$$

$$2.\ \sup_{x,\xi\in\mathbb{R}^n} |q(x,\xi)| \leqslant C. \tag{2.13}$$

Now we are in the position to use Theorem 1.7 to find that there exists a constant $m > 0$ that does not depend on $x, \xi \in \mathbb{R}^n$ such that

$$\xi \mapsto m - q(x,\xi) \tag{2.14}$$

is positive definite for all $x \in \mathbb{R}^n$. Then the Theorem of Bochner, Theorem 1.1, gives us then the existence of a measure $\nu^x \in \mathcal{M}_+^b(\mathbb{R}^n)$ such that

$$\widehat{\nu^x}(\xi) = m - q(x,\xi) \tag{2.15}$$

for all $x \in \mathbb{R}^n$. Using Theorem 1.1 again and Definition 1.3 we find

$$\begin{aligned} \|\nu^x\| &= (2\pi)^{\frac{n}{2}}\widehat{\nu^x}(0) \\ &= (2\pi)^{\frac{n}{2}}\big(m - q(x,0)\big) \\ &\leqslant (2\pi)^{\frac{n}{2}} m. \end{aligned} \tag{2.16}$$

Let us point out that the estimate in (2.16) holds uniformly for all $x \in \mathbb{R}^n$. This fact in turn follows from the uniform boundedness of the symbol q, (2.13).

Next we use Theorem 1.5.C and find that for $x_0 \in \mathbb{R}^n$ fixed there exists a convolution semigroup $(\mu_t^{x_0})_{t\geqslant 0}$ on $\mathbb{R}^n$ such that

$$\widehat{\mu_t^{x_0}}(\xi) = (2\pi)^{-\frac{n}{2}} e^{-tq(x_0,\xi)}. \tag{2.17}$$

Furthermore, a Feller semigroup $(V_t^{x_0})_{t\geqslant 0}$ can be associated with the convolution semigroup $(\mu_t^{x_0})_{t\geqslant 0}$ using the identity

$$V_t^{x_0}u(x) = \int_{\mathbb{R}^n} u(x-y)\mu_t^{x_0}(\mathrm{d}y) \tag{2.18}$$

for $u \in C_\infty$. For more details see Example 4.1.3 in [15]. On $\mathcal{S}$ we find for the generator A of $(V_t^{x_0})_{t\geqslant 0}$, compare Example 4.1.12 in [15],

$$Au(x) = -q(x_0, D)u(x) = -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} q(x_0,\xi)\widehat{u}(\xi)\,\mathrm{d}\xi. \tag{2.19}$$

Using the equality (2.15) we find

$$\begin{aligned}
&-q(x_0, D)u(x) \\
&= -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} q(x_0,\xi)\widehat{u}(\xi)\,\mathrm{d}\xi \\
&= -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \big(q(x_0,\xi) + m - m\big)\widehat{u}(\xi)\,\mathrm{d}\xi \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \big(m - q(x_0,\xi)\big)\widehat{u}(\xi)\,\mathrm{d}\xi - (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} m\widehat{u}(\xi)\,\mathrm{d}\xi \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \widehat{\nu^x}(\xi)\widehat{u}(\xi)\,\mathrm{d}\xi - mu(x) \\
&= \mathcal{F}^{-1}[\widehat{\nu^x}\widehat{u}](x) - mu(x) \\
&= (2\pi)^{-\frac{n}{2}}(\nu^x * u)(x) - mu(x) \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} u(x-y)\nu^x(\mathrm{d}y) - mu(x).
\end{aligned} \tag{2.20}$$

In particular it follows as $u \in \mathcal{S}$ is bounded and by (2.16) that

$$\begin{aligned}
|-q(x_0,D)u(x)| &\leqslant (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} |u(x-y)|\nu^x(\mathrm{d}y) + m|u(x)| \\
&\leqslant (2\pi)^{-\frac{n}{2}} \|u\|_\infty \|\nu^x\| + m\|u\|_\infty \\
&\leqslant (2\pi)^{-\frac{n}{2}} \|u\|_\infty (2\pi)^{\frac{n}{2}} m + m\|u\|_\infty \\
&= 2m\|u\|_\infty,
\end{aligned}$$

i.e.

$$\|q(x_0,D)u\|_\infty \leqslant 2m\|u\|_\infty. \tag{2.21}$$

Our calculation in (2.20) gives us that we may write the operator $q(x_0, D)$ in the following form

$$-q(x_0,D)u(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} u(x-y)\nu^x(\mathrm{d}y) - mu(x). \tag{2.22}$$

Using this representation, one obtains from (2.21) that the operator $q(x_0, D)$ maps C_∞ into C_∞.

Let us now define on $\mathcal{S}$ the pseudodifferential operator

$$W_t u(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} e^{-tq(x,\xi)} \widehat{u}(\xi)\, \mathrm{d}\xi. \tag{2.23}$$

First we note that

$$\begin{aligned} V_t^{x_0} u(x) &= \int_{\mathbb{R}^n} u(x-y)\mu_t^{x_0}(\mathrm{d}y) \\ &= \int_{\mathbb{R}^n} (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} \widehat{u}(\xi)\, \mathrm{d}\xi\, \mu_t^{x_0}(\mathrm{d}y) \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} \mu_t^{x_0}(\mathrm{d}y)\, \widehat{u}(\xi)\, \mathrm{d}\xi \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \int_{\mathbb{R}^n} e^{-i\langle y,\xi\rangle} \mu_t^{x_0}(\mathrm{d}y)\, \widehat{u}(\xi)\, \mathrm{d}\xi \\ &= \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \widehat{\mu_t^{x_0}}(\xi) \widehat{u}(\xi)\, \mathrm{d}\xi \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} e^{-tq(x_0,\xi)} \widehat{u}(\xi)\, \mathrm{d}\xi, \end{aligned} \tag{2.24}$$

and observe for $u \in \mathcal{S}$,

$$W_t u(x) = V_t^x u(x). \tag{2.25}$$

When we described Roth's method in Section 1 we made a similar observation in (2.5), though compared to our situation now it was far more obvious. The fact that our operators involve Fourier transforms makes them harder to handle. Next let $x_0 \in \mathbb{R}^n$ be fixed and we find

$$|V_t^{x_0} u(x)| = \left| \int_{\mathbb{R}^n} u(x-y)\mu_t^{x_0}(\mathrm{d}y) \right| \leqslant \|u\|_\infty, \tag{2.26}$$

and thus

$$\sup_{x\in\mathbb{R}^n} |V_t^x u(x)| \leqslant \|u\|_\infty. \tag{2.27}$$

We use identity (2.25) to conclude for $u \in \mathcal{S}$

$$\|W_t u\|_\infty = \sup_{x\in\mathbb{R}^n} |W_t u(x)| = \sup_{x\in\mathbb{R}^n} |V_t^x u(x)| \leqslant \|u\|_\infty. \tag{2.28}$$

Equations (2.26), (2.27) and (2.28) outline our strategy from now on: First we freeze the coefficients at $x_0 \in \mathbb{R}^n$ and show an estimate for the operator $V_t^{x_0}$ that is independent of x_0. Then we use (2.25) to arrive at the same estimate also for the operator W_t. With regard to Section 2.1 one could say that the operator $V_t^{x_0}$ here is the operator $R_t^{x_0}$ there, and the same holds also for the operators W_t and S_t respectively. Let us explore this connection in more detail. Recall that the operator L^y was defined as

$$L^y u(x) = \sum_{i,j=1}^{n} a_{ij}(y) \frac{\partial^2}{\partial x_i x_j} u(x) + \sum_{i=1}^{n} b_i(y) \frac{\partial}{\partial x_i} u(x),$$

compare (2.4). Then

$$L^y u(x) = \widehat{\widehat{L^y u}}(-x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \widehat{L^y u}(\xi)\, \mathrm{d}\xi.$$

As

$$\begin{aligned}
\widehat{L^y u}(\xi) &= \sum_{i,j=1}^{n} a_{ij}(y) \mathcal{F}\left[\frac{\partial^2}{\partial x_i \partial x_j} u\right](\xi) + \sum_{i=1}^{n} b_i(y) \mathcal{F}\left[\frac{\partial}{\partial x_i} u\right](\xi) \\
&= \sum_{i,j=1}^{n} a_{ij}(y) \xi_i \xi_j \widehat{u}(\xi) - i \sum_{i=1}^{n} b_i(y) \xi_i \widehat{u}(\xi) \\
&= \left(\langle \xi, A(y)\xi\rangle - i\langle b(y), \xi\rangle\right) \widehat{u}(\xi),
\end{aligned}$$

it follows that

$$L^y u(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} l(y,\xi) \widehat{u}(\xi)\, \mathrm{d}\xi,$$

where

$$l(y,\xi) = \langle \xi, A(y)\xi\rangle - i\langle b(y), \xi\rangle.$$

Let $(\rho_t^y)_{t\geqslant 0}$ be the corresponding convolution semigroup for $y \in \mathbb{R}^n$ fixed:

$$\widehat{\rho_t^y}(\xi) = (2\pi)^{-\frac{n}{2}} e^{-tl(y,\xi)}$$

and define the Feller semigroup $(R_t^y)_{t\geqslant 0}$ by

$$R_t^y u(x) = \int_{\mathbb{R}^n} u(x-z) \rho_t^y(\mathrm{d}z) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} e^{-tl(y,\xi)} \widehat{u}(\xi)\, \mathrm{d}\xi,$$

compare calculation (2.24). Set $\tau_y u(x) := u(x+y)$, and note

$$\begin{aligned}R_t^y u(x) &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} e^{-t\langle \xi, A(y)\xi\rangle} e^{i\langle tb(y),\xi\rangle} \widehat{u}(\xi)\, \mathrm{d}\xi \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} e^{-t\langle \xi, A(y)\xi\rangle} \widehat{\tau_{tb(y)} u}(\xi)\, \mathrm{d}\xi.\end{aligned}$$

The associated convolution semigroup $(\eta_t)_{t\geqslant 0}$ with the negative definite function $\xi \mapsto |\xi|^2$ is the Brownian semigroup. As by assumption $A(y) = P(y)^T P(y)$, we then get

$$\begin{aligned}R_t^y u(x) &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} e^{-t\langle \xi, A(y)\xi\rangle} e^{i\langle tb(y),\xi\rangle} \widehat{u}(\xi)\, \mathrm{d}\xi, \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} e^{-t\langle \xi, P(y)^T P(y)\xi\rangle} \widehat{\tau_{tb(y)} u}(\xi)\, \mathrm{d}\xi, \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} e^{-t|P(y)\xi|^2} \widehat{\tau_{tb(y)} u}(\xi)\, \mathrm{d}\xi \\ &= \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \widehat{\eta_t^{P(y)}}(\xi) \widehat{\tau_{tb(y)} u}(\xi)\, \mathrm{d}\xi \\ &= \int_{\mathbb{R}^n} \big(\tau_{tb(y)} u\big)\big(x - P(y)z\big) \eta_t^{P(y)}(\mathrm{d}z) \\ &= \int_{\mathbb{R}^n} u\big(x + tb(y) - P(y)z\big) (4\pi t)^{-\frac{n}{2}} e^{-\frac{|P(y)z|^2}{4t}}\, \mathrm{d}z \\ &= (4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} u\big(x + tb(y) - 2\sqrt{t}P(y)\xi\big) e^{-|\xi|^2} (2\sqrt{t})^n\, \mathrm{d}\xi \\ &= \pi^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-|\xi|^2} u\big(x + tb(y) - 2\sqrt{t}P(y)\xi\big).\end{aligned}$$

Now we can make our above statement precise. Given the specific symbol l we arrive at exactly the same operator R_t^y as in Section 2.1. For this case the result at the end of this section reads the same as the result in Section 2.1, we just use a different technique. Of course our result extends this result, as we admit different symbols than l.

We proceed with showing that the operator W_t maps C_∞ into C_∞. We will give two different proofs, the second one was contributed by R. L. Schilling [28].

For the first proof we need to make one additional assumption on the symbol q:

$$|\partial_\xi^\alpha q(x,\xi)| \leqslant C_\alpha \tag{2.29}$$

for all $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leqslant 2$ and $x, \xi \in \mathbb{R}^n$, i.e. for Lemma 2.10 condition (2.29) replaces (2.13).

Lemma 2.10. *Let q satisfy* (2.12) *and* (2.29). *Then for $u \in \mathcal{S}$ the function $W_t u$ belongs to C_∞.*

Proof. Since $\widehat{u} \in \mathcal{S}$ the properties of $x \mapsto q(x,\xi)$ imply that $x \mapsto W_t u(x)$ is continuous. In order to prove that $W_t u(x) \to 0$ as $|x| \to \infty$ we observe

$$\begin{aligned}
|x|^2 W_t u(x) &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} |x|^2 e^{i\langle x,\xi\rangle} e^{-tq(x,\xi)} \widehat{u}(\xi)\,d\xi \\
&= -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \Delta_\xi \left(e^{i\langle x,\xi\rangle}\right) e^{-tq(x,\xi)} \widehat{u}(\xi)\,d\xi \\
&= -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \Delta_\xi \left(e^{-tq(x,\xi)} \widehat{u}(\xi)\right) d\xi.
\end{aligned}$$

From (2.29) and as the Schwartz space is closed under multiplication by polynomials, we deduce that $\Delta_\xi\big(e^{-tq(x,\xi)}\widehat{u}(\xi)\big)$ is bounded by an L^1-function which is independent of x. It follows that

$$\big||x|^2 W_t u(x)\big| \leqslant C'$$

for $C' > 0$ and all $x \in \mathbb{R}^n$ and thus

$$\lim_{|x|\to\infty} \big||x| W_t u(x)\big| = 0.$$

□

Together with (2.28) we now know that $W_t : \mathcal{S} \to C_\infty$ is a contraction , i.e. $\|W_t u\|_\infty \leqslant \|u\|_\infty$ for $u \in \mathcal{S}$. As $\mathcal{S} \subset C_\infty$ dense we may extend the operator W_t and denote its extension again by $W_t : C_\infty \to C_\infty$, $\|W_t\|_{C_\infty \to C_\infty} \leqslant 1$.

As mentioned before, the proof we now present was contributed by R. L. Schilling [28]. Additional to assumptions (2.12) and (2.13) we need here further

$$\lim_{\eta\to 0} \sup_{x\in\mathbb{R}^n} \big|q(x,0) - q(x,\eta)\big| = 0. \tag{2.30}$$

Let us also introduce another definition

Definition 2.11. A family $(\mu^x)_{x\in\mathbb{R}^n}$ of measures $\mu^x \in \mathcal{M}_+^1(\mathbb{R}^n)$ is called *tight* if

$$\lim_{R\to\infty} \sup_{x\in\mathbb{R}^n} \mu^x\big(B_R^c(0)\big) = 0.$$

Lemma 2.12. *Let q satisfy (2.12), (2.13) and (2.30). If $x_0 \in \mathbb{R}^n$, $t \geqslant 0$ and $\mu_t^{x_0} \in \mathcal{M}_+^1(\mathbb{R}^n)$ such that $\widehat{\mu_t^{x_0}}(\xi) = (2\pi)^{-\frac{n}{2}} e^{-tq(x_0,\xi)}$, then the family of measures $(\mu_t^{x_0})_{x_0 \in \mathbb{R}^n}$ is tight.*

Proof. Let $f \in C_0^\infty(\mathbb{R}^n)$ such that

$$f(x) = \begin{cases} 1 & , |x| \leqslant 1 \\ z & , \text{where } z \in [0,1] \text{ for } 1 < |x| \leqslant 2 \\ 0 & , |x| > 2, \end{cases}$$

then $f_k(x) := f(\frac{x}{k})$ satisfies

$$f_k(x) = \begin{cases} 1 & , |x| \leqslant k \\ z & , \text{where } z \in [0,1] \text{ for } k < |x| \leqslant 2k \\ 0 & , |x| > 2k. \end{cases}$$

Further we get, compare Lemma 3.1.9.B in [15], that $\widehat{f_k}(\xi) = k^n \widehat{f}(k\xi)$. We calculate that

$$\begin{aligned} \int_{\mathbb{R}^n} f_R(y) \mu_t^{x_0}(\mathrm{d}y) &= \int_{\mathbb{R}^n} (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \widehat{f_R}(\xi)\, \mathrm{d}\xi\, \mu_t^{x_0}(\mathrm{d}y) \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \mu_t^{x_0}(\mathrm{d}y) \widehat{f_R}(\xi)\, \mathrm{d}\xi \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-i<x,-\xi>} \mu_t^{x_0}(\mathrm{d}y) \widehat{f_R}(\xi)\, \mathrm{d}\xi \\ &= \int_{\mathbb{R}^n} \widehat{\mu_t^{x_0}}(-\xi) \widehat{f_R}(\xi)\, \mathrm{d}\xi. \end{aligned}$$

It follows that

$$\begin{aligned} \mu_t^{x_0}(B_{2R}(0)) &\geqslant \int_{\mathbb{R}^n} f_R(y) \mu_t^{x_0}(\mathrm{d}y) \\ &= \int_{\mathbb{R}^n} \widehat{\mu_t^{x_0}}(-\xi) \widehat{f_R}(\xi)\, \mathrm{d}\xi \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} R^n \widehat{f}(R\xi) e^{-tq(x_0,-\xi)}\, \mathrm{d}\xi \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \widehat{f}(\eta) e^{-tq\left(x_0,-\frac{\eta}{R}\right)}\, \mathrm{d}\eta, \end{aligned}$$

where we used the change-of-variable theorem in the last step. Further we have

$$\begin{aligned}\mu_t^{x_0}(\mathbb{R}^n) &= \int_{\mathbb{R}^n} 1\,\mu_t^{x_0}(\mathrm{d}x) = \int_{\mathbb{R}^n} e^{-i\langle x,0\rangle}\mu_t^{x_0}(\mathrm{d}x) \\ &= (2\pi)^{\frac{n}{2}}\widehat{\mu}_t^{x_0}(0) = e^{-tq(x_0,0)},\end{aligned} \tag{2.31}$$

as well as

$$\begin{aligned}1 = f(0) &= \mathcal{F}^{-1}[\widehat{f}](0) \\ &= (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{i\langle 0,\eta\rangle}\widehat{f}(\eta)\,\mathrm{d}\eta \\ &= (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\widehat{f}(\eta)\,\mathrm{d}\eta.\end{aligned} \tag{2.32}$$

Using the equalities (2.31) and (2.32), and Corollary 1.6.D we find

$$\begin{aligned}\mu_t^{x_0}\left(B_{2R}^c(0)\right) &= \mu_t^{x_0}(\mathbb{R}^n) - \mu_t^{x_0}\left(B_{2R}(0)\right) \\ &\leqslant e^{-tq(x_0,0)} - (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\widehat{f}(\eta)e^{-tq\left(x_0,-\frac{\eta}{R}\right)}\,\mathrm{d}\eta \\ &= (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\widehat{f}(\eta)e^{-tq(x_0,0)}\,\mathrm{d}\eta - (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\widehat{f}(\eta)e^{-tq\left(x_0,-\frac{\eta}{R}\right)}\,\mathrm{d}\eta \\ &= (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\widehat{f}(\eta)\left(e^{-tq(x_0,0)} - e^{-tq\left(x_0,-\frac{\eta}{R}\right)}\right)\mathrm{d}\eta \\ &\leqslant (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\left|\widehat{f}(\eta)\right|\left|e^{-tq(x_0,0)}\left(1 - e^{-t\left(q\left(x_0,-\frac{\eta}{R}\right)-q(x_0,0)\right)}\right)\right|\mathrm{d}\eta \\ &\leqslant (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\left|\widehat{f}(\eta)\right|\left|1 - e^{-t\left(q\left(x_0,-\frac{\eta}{R}\right)-q(x_0,0)\right)}\right|\mathrm{d}\eta \\ &\leqslant (2\pi)^{-\frac{n}{2}}t\int_{\mathbb{R}^n}\left|\widehat{f}(\eta)\right|\left|q\left(x_0,-\frac{\eta}{R}\right) - q(x_0,0)\right|\mathrm{d}\eta.\end{aligned}$$

As Lemma 1.5.B gives us that $|q(x,\xi)| \leqslant C(1+|\xi|^2)$ we may apply Lebesgue's theorem on dominated convergence to find using (2.30)

$$\begin{aligned}\lim_{R\to\infty}\sup_{x\in\mathbb{R}^n}\mu_t^x\left(B_{2R}^c(0)\right) &\leqslant (2\pi)^{-\frac{n}{2}}t\lim_{R\to\infty}\sup_{x\in\mathbb{R}^n}\int_{\mathbb{R}^n}\left|\widehat{f}(\eta)\right|\left|q\left(x,-\frac{\eta}{R}\right) - q(x,0)\right|\mathrm{d}\eta \\ &\leqslant (2\pi)^{-\frac{n}{2}}t\int_{\mathbb{R}^n}\left|\widehat{f}(\eta)\right|\lim_{R\to\infty}\sup_{x\in\mathbb{R}^n}\left|q\left(x,-\frac{\eta}{R}\right) - q(x,0)\right|\mathrm{d}\eta \\ &= 0.\end{aligned}$$

□

Lemma 2.13. *Let W_t be the pseudodifferential operator as defined in* (2.23), *where q satisfies* (2.12), (2.13) *and* (2.30). *Then W_t maps $C_\infty(\mathbb{R}^n)$ into itself, i.e. $W_t : C_\infty(\mathbb{R}^n) \to C_\infty(\mathbb{R}^n)$.*

Proof. From Theorem 4.5.7.B in [15] we know that

$$W_t : C_0^\infty(\mathbb{R}^n) \to C(\mathbb{R}^n)$$

is well defined. As $C_0^\infty \subset C_\infty$ dense and (2.28) holds we may extend this operator to

$$W_t : C_\infty(\mathbb{R}^n) \to C(\mathbb{R}^n).$$

It remains to show that

$$\lim_{|x|\to\infty} W_t u(x) = 0$$

for $u \in C_\infty$. Let $\varepsilon > 0, u \in C_\infty(\mathbb{R}^n)$ and $K > 0$ such that $|u(x)| \leqslant \varepsilon$ for all $|x| > K$. Then we have for all $|x| > 2K$,

$$\begin{aligned}
|W_t u(x)| &= \left| \int_{\mathbb{R}^n} u(x-y)\mu_t^x(\mathrm{d}y) \right| \\
&= \left| \int_{B_K(x)} u(x-y)\mu_t^x(\mathrm{d}y) + \int_{B_K^c(x)} u(x-y)\mu_t^x(\mathrm{d}y) \right| \\
&\leqslant \|u\|_\infty \, \mu_t^x\big(B_K(x)\big) + \varepsilon \, \mu_t^x\big(B_K^c(x)\big) \\
&\leqslant \|u\|_\infty \, \mu_t^x\big(B_K^c(0)\big) + \varepsilon.
\end{aligned}$$

The last inequality follows because for $|x| > 2K$ we have $B_K(x) \subset B_K^c(0)$. Moreover, μ_t^x is a sub-probability measure and hence $\mu_t^x\big(B_K^C(c)\big) \leqslant 1$. We get

$$|W_t u(x)| \leqslant \|u\|_\infty \sup_{x\in\mathbb{R}^n} \mu_t^x\big(B_K^c(0)\big) + \varepsilon.$$

Applying Lemma 2.12 we finally conclude that

$$\lim_{|x|\to\infty} |W_t u(x)| \leqslant \varepsilon$$

for all $\varepsilon > 0$. □

Let us now continue to construct the Feller semigroup, either assuming Lemma 2.10 or Lemma 2.13. The additional restriction imposed on the symbol by both Lemma is not needed in the following proofs. As far as similarities to Section 1 are concerned, this is the point where we have to go a different way. We simply do not have the estimates from Lemma 2.5 to Lemma 2.7 at our disposal. A different way was proposed by E. Popescu in [21] who used a theorem of P. R. Chernoff to construct the Feller semigroup.

Theorem 2.14. *Let $(S_t)_{t\geqslant 0}$ be a family of strongly continuous linear contractions on a Banach space $(X, \|\cdot\|)$ with $S_0 = id$. Assume that the strong derivative S_0' is densely defined and suppose that*

$$\lim_{m\to\infty} \left\| \left(S_{\frac{t}{m}}\right)^m u - T_t u \right\| = 0 \tag{2.33}$$

holds for all $u \in X$. Then $(T_t)_{t\geqslant 0}$ is a strongly continuous contraction semigroup on X and its generator A extends S_0'. Moreover the strong convergence of $\left(S_{\frac{t}{m}}\right)^m$ to T_t is uniform for t in compact intervals.

This theorem basically gives us the existence of the limit similar to (2.2), where S_t should again be replaced by W_t. Chernoff's theorem also allows us to identify the generator. Hence it is left to check the conditions of Theorem 2.14. In the following we need Lemma 2.15 and Lemma 2.16, as reference compare A. Pazy [20].

Lemma 2.15. *Let $\left(S, D(S)\right)$ and $\left(T, D(T)\right)$ be linear and bounded generators of strongly continuous contraction semigroups $\left(e^{tS}\right)_{t\geqslant 0}$ and $\left(e^{tT}\right)_{t\geqslant 0}$ on the Banach space $(X, \|\cdot\|)$ such that $D(S) = D(T) = X$. Then*

$$\left\|e^{tS} - e^{tT}\right\| \leqslant t\,\|S - T\|$$

for all $t \geqslant 0$.

Lemma 2.16. *Let T be a linear and bounded operator on the Banach space $(X, \|\cdot\|)$ such that $\|T\| \leqslant 1$. Then*

$$\left\|e^{n(T-\mathrm{id})}x - T^n x\right\| \leqslant \sqrt{n}\,\|Tx - x\|$$

for all $n \geqslant 1$ and for all $x \in X$.

In order to use Chernoff's theorem we need to show that

$$\left(\left(W_{\frac{t}{m}}\right)^m u\right)_{m\in\mathbb{N}}$$

is a Cauchy-sequence for every $u \in C_\infty$. This is done in the following Proposition where we use some ideas of E. Popescu [21].

Proposition 2.17. *Let $q(x,\xi)$ satisfy conditions* (2.12), (2.13), (2.29) *and consider the operators $\left(W_t, C_\infty(\mathbb{R}^n)\right)$ and $\left(V_t^{x_0}, C_\infty(\mathbb{R}^n)\right)$ as introduced above. Then*

$$\lim_{\substack{m\to\infty\\ l\to\infty}} \left\| \left(W_{\frac{t}{m}}\right)^m u - \left(W_{\frac{t}{l}}\right)^l u \right\|_\infty = 0 \tag{2.34}$$

holds for $u \in C_\infty(\mathbb{R}^n)$ uniformly for t in compact intervals.

Proof. Let $u \in C_\infty(\mathbb{R}^n)$ and $t \geqslant 0$. We note that

$$\begin{aligned}
&\left\| \left(W_{\frac{t}{m}}\right)^m u - \left(W_{\frac{t}{l}}\right)^l u \right\|_\infty \\
&\leqslant \left\| \left(W_{\frac{t}{m}}\right)^m u - e^{m(W_{\frac{t}{m}} - \mathrm{id})} u \right\|_\infty + \left\| e^{m(W_{\frac{t}{m}} - \mathrm{id})} u - e^{l(W_{\frac{t}{l}} - \mathrm{id})} u \right\|_\infty \\
&\quad + \left\| \left(W_{\frac{t}{l}}\right)^l u - e^{l(W_{\frac{t}{l}} - \mathrm{id})} u \right\|_\infty .
\end{aligned}$$

By definition W_t is obviously linear and as shown in (2.28) we have $\|W_t u\|_\infty \leqslant \|u\|_\infty$. Applying Lemma 2.16 to $A = W_{\frac{t}{m}}, n = m$, and to $A = W_{\frac{t}{l}}, n = l$, we arrive at

$$\left\| \left(W_{\frac{t}{m}}\right)^m u - e^{m(W_{\frac{t}{m}} - \mathrm{id})} u \right\|_\infty \leqslant \sqrt{m} \left\| W_{\frac{t}{m}} u - u \right\|_\infty , \tag{2.35}$$

and

$$\left\| \left(W_{\frac{t}{l}}\right)^l u - e^{l(W_{\frac{t}{l}} - \mathrm{id})} u \right\|_\infty \leqslant \sqrt{l} \left\| W_{\frac{t}{l}} u - u \right\|_\infty , \tag{2.36}$$

respectively. For $x_0 \in \mathbb{R}^n$ fixed we further get using (2.21) and Lemma 1.24.A that

$$\begin{aligned}
\left| V^{x_0}_{\frac{t}{m}} u(x) - u(x) \right| &= \left| \int_0^{\frac{t}{m}} V^{x_0}_s q(x_0, D) u(x) \, \mathrm{d}s \right| \\
&\leqslant \frac{t}{m} \left\| V^{x_0}_s \right\| \left\| q(x_0, D) u \right\|_\infty \\
&\leqslant \frac{t}{m} C_0 \|u\|_\infty ,
\end{aligned} \tag{2.37}$$

and

$$\left| V^{x_0}_{\frac{t}{l}} u(x) - u(x) \right| \leqslant \frac{t}{l} C_0 \|u\|_\infty ,$$

respectively. It follows that

$$\left| W_{\frac{t}{m}} u(x) - u(x) \right| = \left| V^{x}_{\frac{t}{m}} u(x) - u(x) \right| \leqslant \frac{t}{m} C \|u\|_\infty ,$$

and

$$\left| W_{\frac{t}{l}} u(x) - u(x) \right| \leqslant \frac{t}{l} C \|u\|_\infty$$

with some $C > 0$. Taking the supremum over $x \in \mathbb{R}^n$ we find

$$\left\| W_{\frac{t}{m}} u - u \right\|_\infty \leqslant \frac{t}{m} C \|u\|_\infty \quad \text{and} \quad \left\| W_{\frac{t}{l}} u - u \right\|_\infty \leqslant \frac{t}{l} C \|u\|_\infty .$$

Using the estimates in (2.35) and (2.36) we conclude

$$\lim_{m\to\infty}\left\|\left(W_{\frac{t}{m}}\right)^m u - e^{m(W_{\frac{t}{m}}-\mathrm{id})}u\right\|_\infty \leqslant \lim_{m\to\infty}\frac{t}{\sqrt{m}}\,C\,\|u\|_\infty = 0,$$

as well as

$$\lim_{l\to\infty}\left\|\left(W_{\frac{t}{l}}\right)^l u - e^{l(W_{\frac{t}{l}}-\mathrm{id})}u\right\|_\infty \leqslant \lim_{l\to\infty}\frac{t}{\sqrt{l}}\,C\,\|u\|_\infty = 0$$

uniformly for t in compact intervals. It remains to show that

$$\lim_{\substack{m\to\infty\\ l\to\infty}}\left\|e^{m(W_{\frac{t}{m}}-\mathrm{id})}u - e^{l(W_{\frac{t}{l}}-\mathrm{id})}u\right\|_\infty = 0$$

holds uniformly for t in compact intervals. Now we temporarily set $r := \frac{t}{m}, s := \frac{t}{l}$ and apply Lemma 2.15 to get

$$\begin{aligned}
&\left\|e^{t\frac{W_{\frac{t}{m}}-\mathrm{id}}{\frac{t}{m}}}u - e^{t\frac{W_{\frac{t}{l}}-\mathrm{id}}{\frac{t}{l}}}u\right\|_\infty \\
&= \left\|e^{t\frac{W_r-\mathrm{id}}{r}}u - e^{t\frac{W_s-\mathrm{id}}{s}}u\right\|_\infty \\
&\leqslant t\sup_{\|v\|_\infty\leqslant 1}\left\|\frac{W_r-\mathrm{id}}{r}v - \frac{W_s-\mathrm{id}}{s}v\right\|_\infty \|u\|_\infty \\
&= t\sup_{\|v\|_\infty\leqslant 1}\left\|\frac{W_{\frac{t}{m}}-\mathrm{id}}{\frac{t}{m}}v - \frac{W_{\frac{t}{l}}-\mathrm{id}}{\frac{t}{l}}v\right\|_\infty \|u\|_\infty .
\end{aligned} \tag{2.38}$$

In order to handle

$$\left\|\frac{W_{\frac{t}{m}}-\mathrm{id}}{\frac{t}{m}}v - \frac{W_{\frac{t}{l}}-\mathrm{id}}{\frac{t}{l}}v\right\|_\infty$$

we want to use the fact that $(V_t^{x_0})_{t\geqslant 0}, x_0 \in \mathbb{R}^n$, is a uniformly continuous semigroup on $C_\infty(\mathbb{R}^n)$. This follows from Theorem 1.2 in A. Pazy [20]. Writing for a moment $V_t^{x_0} = e^{-tq(x_0,D)}$ we find following A. Pazy [20] that

$$\left\|V_t^{x_0} - \mathrm{id}\right\| = \left\|e^{-tq(x_0,D)} - \mathrm{id}\right\| \leqslant t\left\|q(x_0,D)\right\| e^{t\|q(x_0,D)\|} \tag{2.39}$$

and from this derive for fixed $x_0 \in \mathbb{R}^n$

$$\begin{aligned}
&\left| \frac{V^{x_0}_{\frac{t}{m}} - \mathrm{id}}{\frac{t}{m}} v(x) - \frac{V^{x_0}_{\frac{t}{l}} - \mathrm{id}}{\frac{t}{l}} v(x) \right| \\
&= \left| \frac{m}{t} \int_0^{\frac{t}{m}} V^{x_0}_s q(x_0, D) v(x) \,\mathrm{d}s - \frac{l}{t} \int_0^{\frac{t}{l}} V^{x_0}_s q(x_0, D) v(x) \,\mathrm{d}s \right| \\
&\leqslant \left| \frac{m}{t} \int_0^{\frac{t}{m}} \left(V^{x_0}_s q(x_0, D) v(x) - q(x_0, D) v(x) \right) \mathrm{d}s \right| \\
&\qquad + \left| \frac{l}{t} \int_0^{\frac{t}{l}} \left(V^{x_0}_s q(x_0, D) v(x) - q(x_0, D) v(x) \right) \mathrm{d}s \right|
\end{aligned}$$

$$\begin{aligned}
&= \left| \frac{m}{t} \int_0^{\frac{t}{m}} (V^{x_0}_s - \mathrm{id}\,) q(x_0, D) v(x) \,\mathrm{d}s \right| + \left| \frac{l}{t} \int_0^{\frac{t}{l}} (V^{x_0}_s - \mathrm{id}\,) q(x_0, D) v(x) \,\mathrm{d}s \right| \\
&\leqslant \frac{m}{t} \frac{t}{m} \sup_{\substack{x_0 \in \mathbb{R}^n \\ 0 \leqslant s \leqslant \frac{t}{m}}} \left(\| V^{x_0}_s - \mathrm{id} \| \, \| q(x_0, D) v \|_\infty \right) \\
&\qquad + \frac{l}{t} \frac{t}{l} \sup_{\substack{x_0 \in \mathbb{R}^n \\ 0 \leqslant s \leqslant \frac{t}{l}}} \left(\| V^{x_0}_s - \mathrm{id} \| \, \| q(x_0, D) v \|_\infty \right) \\
&\leqslant \sup_{\substack{x_0 \in \mathbb{R}^n \\ 0 \leqslant s \leqslant \frac{t}{m}}} \left(s \, \| q(x_0, D) \| \, e^{s \| q(x_0, D) \|} \, \| q(x_0, D) \| \right) \| v \|_\infty \\
&\qquad + \sup_{\substack{x_0 \in \mathbb{R}^n \\ 0 \leqslant s \leqslant \frac{t}{l}}} \left(s \, \| q(x_0, D) \| \, e^{s \| q(x_0, D) \|} \, \| q(x_0, D) \| \right) \| v \|_\infty \\
&\leqslant \frac{t}{m} \, C_0^2 \, e^{\frac{t}{m} C_0} \, \| v \|_\infty + \frac{t}{l} \, C_0^2 \, e^{\frac{t}{l} C_0} \, \| v \|_\infty .
\end{aligned}$$

It follows that

$$\lim_{\substack{m \to \infty \\ l \to \infty}} \left(\frac{t}{m} \, C_0^2 \, e^{\frac{t}{m} C_0} \, \| v \|_\infty + \frac{t}{l} \, C_0^2 \, e^{\frac{t}{l} C_0} \, \| v \|_\infty \right) = 0.$$

Now

$$\begin{aligned}
\left\| \frac{V^{x_0}_{\frac{t}{m}} - \mathrm{id}}{\frac{t}{m}} v - \frac{V^{x_0}_{\frac{t}{l}} - \mathrm{id}}{\frac{t}{l}} v \right\|_\infty &= \sup_{x \in \mathbb{R}^n} \left| \frac{V^{x_0}_{\frac{t}{m}} - \mathrm{id}}{\frac{t}{m}} v(x) - \frac{V^{x_0}_{\frac{t}{l}} - \mathrm{id}}{\frac{t}{l}} v(x) \right| \\
&\leqslant \frac{t}{m} \, C_0^2 \, e^{\frac{t}{m} C_0} \, \| v \|_\infty + \frac{t}{l} \, C_0^2 \, e^{\frac{t}{l} C_0} \, \| v \|_\infty
\end{aligned}$$

and thus

$$\lim_{\substack{m\to\infty\\ l\to\infty}} \sup_{x_0\in\mathbb{R}^n} \left\| \frac{V^{x_0}_{\frac{t}{m}} - \mathrm{id}}{\frac{t}{m}} v - \frac{V^{x_0}_{\frac{t}{l}} - \mathrm{id}}{\frac{t}{l}} v \right\|_\infty = 0$$

uniformly for t in compact intervals. Since

$$\left| \frac{W_{\frac{t}{m}} - \mathrm{id}}{\frac{t}{m}} v(x) - \frac{W_{\frac{t}{l}} - \mathrm{id}}{\frac{t}{l}} v(x) \right| = \left| \frac{V^x_{\frac{t}{m}} - \mathrm{id}}{\frac{t}{m}} v(x) - \frac{V^x_{\frac{t}{l}} - \mathrm{id}}{\frac{t}{l}} v(x) \right|$$

we find

$$\begin{aligned}
&\lim_{\substack{m\to\infty\\ l\to\infty}} \left\| \frac{W_{\frac{t}{m}} - \mathrm{id}}{\frac{t}{m}} v - \frac{W_{\frac{t}{l}} - \mathrm{id}}{\frac{t}{l}} v \right\|_\infty \\
&\leqslant \lim_{\substack{m\to\infty\\ l\to\infty}} \sup_{x_0\in\mathbb{R}^n} \left\| \frac{V^x_{\frac{t}{m}} - \mathrm{id}}{\frac{t}{m}} v - \frac{V^x_{\frac{t}{l}} - \mathrm{id}}{\frac{t}{l}} v \right\|_\infty \\
&= 0.
\end{aligned}$$

Remembering (2.38) we finally get that

$$\begin{aligned}
&\lim_{\substack{m\to\infty\\ l\to\infty}} \left\| e^{m(W_{\frac{t}{m}} - \mathrm{id})} u - e^{l(W_{\frac{t}{l}} - \mathrm{id})} u \right\|_\infty \\
&= \lim_{\substack{m\to\infty\\ l\to\infty}} \left\| e^{t\frac{W_{\frac{t}{m}} - \mathrm{id}}{\frac{t}{m}}} u - e^{t\frac{W_{\frac{t}{l}}\,\mathrm{id}}{\frac{t}{l}}} u \right\|_\infty \\
&\leqslant t \lim_{\substack{m\to\infty\\ l\to\infty}} \sup_{\|v\|_\infty\leqslant 1} \left\| \frac{W_{\frac{t}{m}} - \mathrm{id}}{\frac{t}{m}} v - \frac{W_{\frac{t}{l}} - \mathrm{id}}{\frac{t}{l}} v \right\|_\infty \|u\|_\infty \\
&\leqslant t \sup_{\|v\|_\infty\leqslant 1} \lim_{\substack{m\to\infty\\ l\to\infty}} \left\| \frac{W_{\frac{t}{m}} - \mathrm{id}}{\frac{t}{m}} v - \frac{W_{\frac{t}{l}} - \mathrm{id}}{\frac{t}{l}} v \right\|_\infty \|u\|_\infty \\
&= 0
\end{aligned}$$

uniformly for t in compact intervals. □

Observe that in Proposition 2.17 the most difficult part is the estimate for the middle term

$$\left\| e^{m\left(W_{\frac{t}{m}} - \mathrm{id}\right)} u - e^{l\left(W_{\frac{t}{l}} - \mathrm{id}\right)} u \right\|_\infty .$$

Theorem 2.18. *Let $q(x,\xi)$ satisfy conditions* (2.12), (2.13), (2.29) *and define on $S(\mathbb{R}^n)$ the operator $-q(x,D)$ by*

$$-q(x,D)u(x) = -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} q(x,\xi)\hat{u}(\xi)\,\mathrm{d}\xi. \tag{2.40}$$

Then $-q(x,D)$ extends to the generator $\big(A, C_\infty(\mathbb{R}^n)\big)$ of a strongly continuous contraction semigroup $(T_t)_{t\geqslant 0}$ on $C_\infty(\mathbb{R}^n)$.

Proof. Let $\big(W_t, C_\infty(\mathbb{R}^n)\big)$ and $\big(V_t^{x_0}, C_\infty(\mathbb{R}^n)\big)$ denote the pseudodifferential operators as defined in (2.23) and (2.18). We want to apply Theorem 2.14 to the family $(W_t)_{t\geqslant 0}$ and hence need to show that $(W_t)_{t\geqslant 0}$ is a family of strongly continuous linear contractions on $C_\infty(\mathbb{R}^n)$ with $W_0 = \mathrm{id}$. Further it is needed that $\left(\left(W_{\frac{t}{m}}\right)^m u\right)_{m\in\mathbb{N}}$ converges to some limit $T_t u$. Since

$$W_0 u(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} e^{-0q(x,\xi)}\hat{u}(\xi)\,\mathrm{d}\xi = \mathcal{F}^{-1}[\hat{u}](x) = u(x),$$

it follows that $W_0 = \mathrm{id}$. The contraction property of $(W_t)_{t\geqslant 0}$ was proved in (2.28) and Proposition 2.17, in particular (2.34), gives us the existence of a strong limit. Hence for $t > 0$ we may define on $C_\infty(\mathbb{R}^n)$ the operator

$$T_t u := \lim_{m\to\infty} \left(W_{\frac{t}{m}}\right)^m u.$$

It remains to show that the family $(W_t)_{t\geqslant 0}$ is strongly continuous, i.e. we need to check

$$\lim_{s\uparrow t} \|W_t u - W_s u\|_\infty = 0 \tag{2.41}$$

and

$$\lim_{s\downarrow t} \|W_t u - W_s u\|_\infty = 0. \tag{2.42}$$

For $s < t$ and fixed $x_0 \in \mathbb{R}^n$ we have

$$\begin{aligned} \left\|V_t^{x_0} u - V_s^{x_0} u\right\|_\infty &= \left\|V_s^{x_0} V_{t-s}^{x_0} u - V_s^{x_0} u\right\|_\infty \\ &= \left\|V_s^{x_0}(V_{t-s}^{x_0} u - u)\right\|_\infty \leqslant \left\|V_s^{x_0}\right\| \left\|V_{t-s}^{x_0} u - u\right\|_\infty \\ &\leqslant \left\|V_{t-s}^{x_0} u - u\right\|_\infty. \end{aligned}$$

As calculated in (2.37) we have that

$$|V_{t-s}^{x_0} u(x) - u(x)| \leqslant (t-s)\, C_0 \|u\|_\infty$$

which yields

$$\lim_{s\uparrow t} \sup_{x_0\in\mathbb{R}^n} \sup_{x\in\mathbb{R}^n} |V_{t-s}^{x_0}u(x) - u(x)| = \lim_{s\uparrow t} \sup_{x_0\in\mathbb{R}^n} \|V_{t-s}^{x_0}u - u\|_\infty = 0$$

implying

$$\lim_{s\uparrow t} \sup_{x_0\in\mathbb{R}^n} \|V_t^{x_0}u - V_s^{x_0}u\|_\infty = 0.$$

Observing that

$$|W_tu(x) - W_su(x)| = |V_t^xu(x) - V_s^xu(x)|$$

we may deduce

$$\lim_{s\uparrow t} \|W_tu - W_su\|_\infty = 0$$

which proves (2.41). In order to prove (2.42) note that for $s > t$ we have

$$\|V_t^{x_0}u - V_s^{x_0}u\|_\infty = \|V_t^{x_0}V_{s-t} - V_t^{x_0}u\|_\infty \leqslant \|V_t^{x_0}\| \; \|V_{s-t}^{x_0}u - u\|_\infty$$

and we may argue as above. Finally, before we can apply Theorem 2.14 we have to show that the strong derivative W_0' is densely defined. We show that $A = W_0' = -q(x, D)$, i.e. that

$$\lim_{t\to 0} \left\| \frac{W_tu - u}{t} + q(x, D)u \right\|_\infty = 0.$$

Since $(V_t^{x_0})_{t\geqslant 0}$ is a uniformly continuous semigroup we find using (2.39)

$$\begin{aligned}
&\left| \frac{V_t^{x_0}u(x) - u(x)}{t} + q(x_0, D)u(x) \right| \\
=&\left| \frac{1}{t} \int_0^t V_s^{x_0}q(x_0, D)u(x)\,\mathrm{d}s - q(x_0, D)u(x) \right| \\
=&\left| \frac{1}{t} \int_0^t (V_s^{x_0} - \mathrm{id})q(x_0, D)u(x)\,\mathrm{d}s \right| \\
\leqslant&\frac{1}{t}\, t\Big(\sup_{\substack{x_0\in\mathbb{R}^n \\ 0\leqslant s\leqslant t}} \|V_s^{x_0} - \mathrm{id}\| \Big) \|q(x_0, D)u\|_\infty \\
=&\Big(\sup_{\substack{x_0\in\mathbb{R}^n \\ 0\leqslant s\leqslant t}} s\, \|q(x_0, D)\|\; e^{s\|q(x_0,D)\|} \Big) \|q(x_0, D)u\|_\infty \\
\leqslant& t\, C_0^2\, e^{tC_0} \|u\|_\infty.
\end{aligned}$$

This implies

$$\lim_{t\to 0} \sup_{x_0\in\mathbb{R}^n} \sup_{x\in\mathbb{R}^n} \left| \frac{V_t^{x_0}u(x) - u(x)}{t} + q(x_0, D)u(x) \right| = 0.$$

Since

$$\left| \frac{W_t u(x) - u(x)}{t} + q(x, D)u(x) \right| = \left| \frac{V_t^x - u(x)}{u(x)} + q(x, D)u(x) \right| \leqslant t\, C_0^2\, e^{tC_0}\, \|u\|_\infty$$

we get

$$\lim_{t\to 0} \left\| \frac{W_t u - u}{t} + q(x, D)u \right\|_\infty = 0$$

proving the theorem. □

It is still left to show that the semigroup is positivity preserving. This follows easily by the same argumentation that was given in Section 1.4, for details we refer to Theorem 4.5.3 in [15]. Hence we may finally note

Corollary 2.19. *In the situation of Theorem 2.18 the semigroup $(T_t)_{t\geqslant 0}$ is a Feller semigroup.*

Proof. Note that $A = -q(x, D)$ satisfies on $\mathcal{S}(\mathbb{R}^n) \subset C_\infty(\mathbb{R}^n)$ the positive maximum principle, compare Ph. Courrège [5]. Furthermore we have shown that $(A, \mathcal{S}(\mathbb{R}^n))$ extends the generator of a strongly continuous contraction semigroup $(T_t)_{t\geqslant 0}$. Thus we may use Theorem 3.1 and get that $R(\lambda - A) = C_\infty(\mathbb{R}^n)$ for some $\lambda > 0$. As $\mathcal{S}(\mathbb{R}^n) \subset C_\infty(\mathbb{R}^n)$ is dense this allows us to apply Theorem 1.26 and we conclude that $-q(x, D)$ extends to the generator of a Feller semigroup $(T_t)_{t\geqslant 0}$ on $C_\infty(\mathbb{R}^n)$. □

Chapter 3

The Yosida Approximation of Pseudodifferential Operators

We explained in Section 1.4 how to construct Feller semigroups that have as generator a pseudodifferential operator with negative definite symbol. The main tool we used was the Hille-Yosida theorem, Theorem 1.26, which gives us the existence of a Feller semigroup under certain conditions. If one is interested in details of the construction, one needs to examine the proof of Theorem 1.26 more closely. This is discussed in greater detail in Section 3.1. In Section 3.2 we then give an approximation procedure for Feller semigroups by using the Yosida approximation of negative definite symbols. An important role in this chapter plays the Hille-Yosida theorem, which we use here in a different version than in Section 1.4. In Section 3.2 we heavily rely upon the symbolic calculus we introduced in Section 1.3.

3.1 The Hille-Yosida construction

In this section we will talk about the Hille-Yosida construction of a Feller semigroup. Note that Theorem 1.26 gives us first of all an existence result of a strongly continuous contraction semigroup if the operator A satisfies certain conditions. For further properties of the semigroup we will now look at the proof of Theorem 1.26. As a reference for this section we give N. Jacob [15]. Let us begin with stating the Hille-Yosida theorem in two different version, of which the second one is identical to Theorem 1.26. Both versions are due to Lumer-Philips. The connection between both versions is explained afterwards.

Theorem 3.1. *A linear operator $(A, D(A))$ on a Banach space $(X, \|\cdot\|)$ is the generator of a strongly continuous contraction semigroup $(T_t)_{t\geqslant 0}$ on X if* *and only if the following three conditions hold*

A. *$D(A) \subset X$ is dense,*

B. *A is a dissipative operator,*

C. *$R(\nu - A) = X$ for some $\nu > 0$.*

Theorem 3.2. *A linear operator $\big(A, D(A)\big)$ on a Banach space $\big(X, \|\cdot\|_X\big)$ is closable and its closure $\overline{A}$ is the generator of a strongly continuous contraction semigroup $(T_t)_{t\geqslant 0}$ on X if and only if the following three conditions hold*

A. *$D(A) \subset X$ is dense,*

B. *A is a dissipative operator,*

C. *$R(\nu - A)$ is dense in X for some $\nu > 0$.*

Using the next Lemma it is easy to show that Theorem 3.2 follows from Theorem 3.1.

Lemma 3.3. *Let A be a densely defined linear dissipative operator on a Banach space $\big(X, \|\cdot\|_X\big)$. Then A is closable and $\overline{R(\nu - A)} = R(\nu - \overline{A})$ for all $\nu > 0$.*

Assume that the linear operator A is closable and satisfies the conditions of Theorem 3.2. It follows that as $D(A) \subset X$ is dense we also have that $D(\overline{A}) \subset X$ is dense. Further, as A is dissipative, also $\overline{A}$ is dissipative. In order to see this, recall the definition of dissipativity, i.e. we have to show

$$\left\|\nu u - \overline{A}u\right\|_X \geqslant \nu\|u\|_X \tag{3.1}$$

for all $\nu \geqslant 0$ and $u \in D(\overline{A})$. Now by definition we have $\overline{A}u = Au$ for $u \in D(A)$. As $D(A) \subset D(\overline{A})$ dense we find for every $u \in D(\overline{A})$ and approximating sequence $(u_n)_{n\in\mathbb{N}} \subset D(A)$ such that $\lim_{n\to\infty}\|u_n - u\|_X = 0$ and $Au_n \to \overline{A}u$. But this proves (3.1). And with Lemma 3.3 we find that $R(\nu - \overline{A}) = \overline{R(\nu - A)} = X$, i.e. $\overline{A}$ satisfies all conditions from Theorem 3.1. With a similar argument one can also prove the other direction, i.e. if $\overline{A}$ is the generator of a strongly continuous contraction semigroup, then A satisfies the conditions of Theorem 3.2.

We are going to work with Theorem 3.1 from now on, but note that for the actual construction of a Feller semigroup with pseudodifferential operators we prefer Theorem 3.2, compare also Section 1.4. The reason is that our operators $q(x, D)$ are not closed on their domain, but closable. In order to understand how the proof of Theorem 3.1 works, we need the notion of the Yosida approximation of A. Let us denote by $\rho(A)$ the resolvent set of an operator A which consists of all $\nu \in \mathbb{C}$ such that $\nu - A$ is surjective and has a continuous inverse $(\nu - A)^{-1}$ defined on $R(\nu - A) = X$.

Theorem 3.4. *Let A be a closed and dissipative operator which is densely defined on a Banach space $(X, \|\cdot\|_X)$. We assume that $(0, \infty) \subset \rho(A)$. The* Yosida approximation *of A is defined for $\nu > 0$ by*

$$A_\nu = \nu A(\nu - A)^{-1}. \tag{3.2}$$

It has the following properties

A. *For all $\nu > 0$ the operator A_ν is bounded on X and the semigroup $(e^{tA_\nu})_{t\geqslant 0}$ is a strongly continuous contraction semigroup.*

B. *For all $\nu, \mu > 0$ we have*

$$A_\nu A_\mu = A_\mu A_\nu. \tag{3.3}$$

C. *For $u \in D(A)$ it follows that*

$$\lim_{\nu\to\infty} \|A_\nu u - Au\|_X = 0. \tag{3.4}$$

Let us make a few remarks. The first is that although the operator A need not to be bounded, the operator A_ν as defined in (3.2) always is. Secondly, it is nice to have commutativity as given in (3.3). Note that if $q_1, q_2 \in S_0^{\infty,\psi}$ are two pseudodifferential operators with variable coefficients then $q_1(x, D) \circ q_2(x, D) \neq q_2(x, D) \circ q_1(x, D)$, i.e. q_1 and q_2 in general do not commutate. This makes life in Theorem 3.13 much harder. Thirdly, and this is most important, we have a strong convergence of $A_\nu u$ to Au for all $u \in D(A)$, i.e. we may approximate the unbounded operator A by the bounded one A_ν.

Now we are ready to give some of the details of the proof of Theorem 3.1. Assume that the conditions of Theorem 3.1 hold. It follows from Lemma 4.1.26 and Lemma 4.1.27 in [15] that A satisfies the assumption of Theorem 3.4. Hence we may define the Yosida approximation

$$A_\nu = \nu A(\nu - A)^{-1} \tag{3.5}$$

of A, and use Theorem 3.4.A to write

$$T_t^\nu = e^{tA_\nu}, \tag{3.6}$$

where $(T_t^\nu)_{t\geqslant 0}$ denotes the strongly continuous contraction semigroup on X generated by A_ν. Note that it is a simple calculation to check that for every bounded linear operator $A : X \to X$, a uniformly continuous semigroup is defined by $\left(e^{tA}\right)_{t\geqslant 0}$ and its generator is A. We may apply Lemma 2.15 to find

$$\|T_t^\nu u - T_t^\mu u\|_X \leqslant t\,\|A_\nu u - A_\mu u\|_X \tag{3.7}$$

for all $u \in X$, $t \geqslant 0$ and $\nu, \mu > 0$. As Theorem 3.4.C tells us that

$$\lim_{\substack{\nu\to\infty\\ \mu\to\infty}} \|A_\nu u - A_\mu u\| = 0, \tag{3.8}$$

we find that the strong limit

$$\lim_{\nu\to\infty} T_t^\nu =: T_t u \tag{3.9}$$

exists for all $u \in D(A)$. That $(T_t)_{t\geqslant 0}$ is a strongly continuous contraction semigroup with generator $\overline{A}$ can then be shown with some effort. We stop here because only this first part of the proof is of interest to us.

Let us now look at the situation we are facing. We want to construct a strongly continuous contraction semigroup using a pseudodifferential operator

$$q(x,D)u(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} q(x,\xi)\hat{u}(\xi)\,\mathrm{d}\xi,$$

that is defined on a suitable domain, and where $q \in S_0^{\infty,\psi}$, compare Definition 1.15. Right now it is less important to remember what difficulties one encounters when trying to show that the operator $q(x,D)$ satisfies the conditions of Theorem 3.2, but to understand how the Yosida approximation is used in this case. Recalling our previous calculations, (3.2)–(3.9), we may first of all define

$$-q_\nu(x,D) = -\nu q(x,D)\big(\nu + q(x,D)\big)^{-1}, \tag{3.10}$$

the Yosida approximation of the operator $-q(x,D)$. Then one may write

$$T_t^\nu = e^{-tq_\nu(x,D)},$$

and as in (3.9) we find that

$$T_t u := \lim_{\nu\to\infty} T_t^\nu u = \lim_{\nu\to\infty} e^{-tq_\nu(x,D)} u$$

exists. One could remark now that we have a nice approximation of the semigroup $(T_t)_{t\geqslant 0}$ by the semigroup $(T_t^{\nu})_{t\geqslant 0}$ which in turn uses the Yosida approximation $q_\nu(x,D)$ of $-q(x,D)$. But the problem here is to actually calculate $q_\nu(x,D)$! As (3.10) shows this involves finding the inverse of $\nu + q(x,D)$ which is hard to do in our case as we have variable coefficients. One important fact to notice is

$$q_\nu(x,D) \neq q^{(\nu)}(x,D)$$

where

$$q^{(\nu)}(x,D)u(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \nu q(x,\xi)\big(\nu + q(x,\xi)\big)^{-1}\hat{u}(\xi)\,\mathrm{d}\xi, \qquad (3.11)$$

i.e. $q_\nu(x,D)$ is not equal to the pseudodifferential operator with symbol

$$\nu q(x,\xi)\big(\nu + q(x,\xi)\big)^{-1} = \frac{\nu q(x,\xi)}{\nu + q(x,\xi)}.$$

The idea that we pursue in Section 2 is to use the operator $q^{(\nu)}(x,D)$ as a replacement of the operator $q_\nu(x,D)$ and to see whether we still have

$$T_t u = \lim_{\nu\to\infty} e^{-tq^{(\nu)}(x,D)}u, \qquad (3.12)$$

for suitable u. If this works out it would confirm our intuition, as

$$q^{(\nu)}(x,\xi) = \frac{\nu q(x,\xi)}{\nu + q(x,\xi)} \longrightarrow q(x,\xi)$$

for $\nu \to \infty$.

3.2 Approximation of Feller Semigroups

Before we proceed with explaining how the operator $q^{(\nu)}(x,D)$ can be used to approximate the Feller semigroup $(T_t)_{t\geqslant 0}$, compare (3.10)–(3.12) for a short outline of the idea, we introduce some new function spaces. We will mainly work on scales of anisotropic Sobolev spaces and then use an imbedding theorem to extend our results to C_∞. If we set

$$\lambda^{s,\psi}(\xi) := \big(1 + \psi(\xi)\big)^{\frac{s}{2}} \qquad (3.13)$$

where $s \in \mathbb{R}$ and $\psi : \mathbb{R}^n \to \mathbb{R}$ is a continuous negative definite function, then we call

$$H^{s,\psi} := H^{s,\psi}(\mathbb{R}^n) = \big\{u \in \mathcal{S}'(\mathbb{R}^n)\,;\, \big\|\lambda^{s,\psi}\hat{u}\big\|_{L^2} < \infty\big\} \qquad (3.14)$$

the *anisotropic Sobolev space of order* $s \in \mathbb{R}$ *associated with* ψ. In Theorem 1.20 we already gave one example that makes use of these spaces, i. e. Theorem 1.20 tells us that the operator $q(x,D) : H^{s+m,\psi} \to H^{s,\psi}$ is continuous for all $s \in \mathbb{R}$, if $q \in S_0^{m,\psi}$. We will prove a result similar to that one later on. It is also important to have estimate (1.19) in mind. Let us now explain how the imbedding $H^{s,\psi} \hookrightarrow C_\infty$ works. For convenience we quote Theorem 3.10.12 in [15]. In Theorem 3.5 the space $B_p^{s,\psi}$ is used which is defined as

$$B_p^{s,\psi} := B_p^{s,\psi}(\mathbb{R}^n) = \left\{ u \in \mathcal{S}'(\mathbb{R}^n) \, ; \, \left\| \lambda^{s,\psi} \widehat{u} \right\|_{L^p} < \infty \right\}. \tag{3.15}$$

Theorem 3.5. *Let* $\psi : \mathbb{R}^n \to \mathbb{C}$ *be a continuous negative definite function and suppose that for some* $m \in \mathbb{N}_0$ *we have*

$$\frac{(1+|\cdot|^2)^{\frac{m}{2}}}{(1+|\psi(\cdot)|)^{\frac{s}{2}}} \in L^q$$

such that $1 \leqslant p < \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Then it follows that* $B_p^{s,\psi} \subset C_\infty^m$ *and the estimate*

$$\sup_{x \in \mathbb{R}^n} |\partial^\alpha u(x)| \leqslant c \|u\|_{B_p^{s,\psi}}$$

holds for all $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leqslant m$, *and* $u \in B_p^{s,\psi}$.

It is obvious that $H^{s,\psi} = B_2^{s,\psi}$. Hence for our purposes set $p = 2$ and $m = 0$ in Theorem 3.5. Then $q = 2$ and one needs to check that $(1 + |\psi(\cdot)|)^{-\frac{s}{2}} \in L^2$. To make this work we need that $\psi(\xi) \geqslant c|\xi|^r$ for some $c > 0$, $r > 0$ and sufficiently large $|\xi|$. It follows that for $rs > n$ and constants $\tilde{c}$, c_0', $c_0 > 0$ we find

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left(\frac{1}{(1+|\psi(\xi)|)^{\frac{s}{2}}} \right)^2 \mathrm{d}\xi = \int_{\mathbb{R}^n} \frac{1}{(1+|\psi(\xi)|)^s} \mathrm{d}\xi \\
&\leqslant \tilde{c} \int_{\mathbb{R}^n} \frac{1}{(1+c|\xi|^r)^s} \mathrm{d}\xi \leqslant c_0' \int_{\mathbb{R}^n} \frac{1}{(1+c|\xi|)^{rs}} \mathrm{d}\xi \\
&\leqslant c_0,
\end{aligned}$$

and thus $(1 + |\psi(\cdot)|)^{-\frac{s}{2}} \in L^2$. This allows us to apply Theorem 3.5 to find that $H^{s,\psi} \subset C_\infty$ with $\|u\|_\infty \leqslant c\|u\|_{H^{s,\psi}}$ for $u \in H^{s,\psi}$. But this gives us the imbedding $H^{s,\psi} \hookrightarrow C_\infty$. Let us also remark that $H^{s,\psi}$ is dense in C_∞ as $\mathcal{S}$ is dense in $H^{s,\psi}$ as well as in C_∞.

As we have just seen, the function ψ has to satisfy some additional growth condition for the imbedding result $H^{s,\psi} \hookrightarrow C_\infty$. On the other hand, ψ has to be of class Λ, compare Section 1.3, to be an admissible function for the definition of a symbol class. It turns out that we need for our results a few other conditions as well. Hence from now on let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a continuous negative definite function such that

A. ψ is of class Λ, (3.16)

B. $\psi(\xi) \geqslant c|\xi|^r$ for all $c > 0$, $r > 0$ and sufficiently large $|\xi|$. (3.17)

Let $q \in S_0^{m,\psi}$ for $m \in \mathbb{R}$, where ψ satisfies conditions A and B above, such that

C. $q(x,\xi) \geqslant K\lambda^{m,\psi}(\xi)$ (3.18)

for $K > 0$ and all $x, \xi \in \mathbb{R}^n$. Let us also observe

$$q^{(\nu)} := \frac{\nu q}{\nu + q}, \tag{3.19}$$

and note that

$$q - q^{(\nu)} = \frac{q^2}{\nu + q}. \tag{3.20}$$

If we denote by $\mathbf{1}$ the function $(x,\xi) \mapsto 1$, then as $\mathbf{1} \in S_0^{0,\psi}$ and $\mathbf{1} = (\nu+q)\frac{1}{\nu+q}$ it follows that $\frac{1}{\nu+q} \in S_0^{-m,\psi}$. Hence we conclude that $q^{(\nu)} \in S_0^{0,\psi}$ as well as $q - q^{(\nu)} \in S_0^{m,\psi}$.

Now we are in a position to state our aim more clearly. We consider the operators $q(x,D)$ and $q^{(\nu)}(x,D)$ on the Banach spaces $H^{s,\psi}$, $s \in \mathbb{R}$, and want to prove a convergence result similar to (3.4), i.e. for $u \in H^{m,\psi}$

$$\lim_{\nu\to\infty} \left\| q^{(\nu)}(x,D)u - q(x,D)u \right\|_{H^{s,\psi}} = 0.$$

This is done in Theorem 3.9 which relies heavily on Theorem 3.8. For Theorem 3.8 it is vital to have an L^2-estimate for pseudodifferential operators which is given in Theorem 3.7. There are several L^2-estimates for pseudodifferential operators in the literature, see e.g. [12], [3], but the one we choose is taken from [13] by I. L. Hwang. We made this choice because the estimate for the pseudodifferential operator involves the estimate of its symbol, see Definition 1.14. We also present the proof here, as we use a different normalization of the Fourier transform and it is the constant in the estimate that we are interested in. But we want to point out that Theorem 3.7 is not our result, and can be found in [13]. For the proof of Theorem 3.7 we need Lemma 3.6

Lemma 3.6. *If $v \mapsto U(v)$ is a semilinear form on $\mathcal{S}$ satisfying $|U(v)| \leqslant C\|v\|_{L^2}$, then there exists a unique $u \in L^2$ such that $U(v) = (u,v)_{L^2}$ for $v \in \mathcal{S}$, and one has $\|u\|_{L^2} \leqslant C$.*

This Lemma is a variant of the Riesz representation theorem.

Theorem 3.7. *Let $q \in C^{2n}(\mathbb{R}^n)$ be such that*

$$|\partial_x^\alpha \partial_\xi^\beta q(x,\xi)| \leqslant C_{\alpha\beta}$$

for all $\alpha, \beta \in \{0,1\}^n$. Then the pseudodifferential operator $q(x,D)$ satisfies

$$\|q(x,D)u\|_{L^2} \leqslant 2^{-\frac{3}{4}n}\,\pi^{\frac{3}{4}n} \sum_{\alpha,\beta,\gamma\in\{0,1\}^n} \binom{\gamma}{\alpha} C_{\alpha\beta}\|u\|_{L^2}. \tag{3.21}$$

Proof. We set

$$\chi_\xi(x) := e^{i\langle x,\xi\rangle}\prod_{j=1}^n (1+ix_j)^{-1}$$

and for $u \in \mathcal{S}$ we define

$$\Psi(x,\xi) := \int_{\mathbb{R}^n} \chi_\xi(x-y)u(y)\,\mathrm{d}y.$$

Further denote by $\tau_y u$ the function $\tau_y u(x) = u(y-x)$. We want to show that $\Psi \in L^2(\mathbb{R}^{2n})$ and hence calculate

$$\begin{aligned}
&\|\Psi\|_{L^2}^2 = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\Psi(x,\xi)|^2\,\mathrm{d}x\,\mathrm{d}\xi \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left| e^{i\langle x,\xi\rangle}\int_{\mathbb{R}^n} e^{-i\langle y,\xi\rangle}\chi_0(x-y)u(y)\,\mathrm{d}y\right|^2 \mathrm{d}x\,\mathrm{d}\xi \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} e^{-i\langle x-z,\xi\rangle}\chi_0(z)u(x-z)\,\mathrm{d}z\right|^2 \mathrm{d}x\,\mathrm{d}\xi \\
&= (2\pi)^{\frac{n}{2}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left|(2\pi)^{-\frac{n}{2}} e^{-i\langle x,\xi\rangle}\int_{\mathbb{R}^n} e^{-i\langle z,-\xi\rangle}\chi_0(z)\tau_x u(z)\,\mathrm{d}z\right|^2 \mathrm{d}x\,\mathrm{d}\xi \\
&= (2\pi)^{\frac{n}{2}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\mathcal{F}[\chi_0\cdot\tau_x u](-\xi)|^2\,\mathrm{d}x\,\mathrm{d}\xi = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left|\left(\widehat{\chi_0} * \widehat{\tau_x u}\right)(-\xi)\right|^2 \mathrm{d}x\,\mathrm{d}\xi \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\mathcal{F}[\widehat{\chi_0} * \widehat{\tau_x u}](\xi)|^2\,\mathrm{d}\xi\,\mathrm{d}x = (2\pi)^{\frac{n}{2}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left|\widehat{\widehat{\chi_0}}(\xi)\widehat{\widehat{\tau_x u}}(\xi)\right|^2 \mathrm{d}\xi\,\mathrm{d}x \\
&= (2\pi)^{\frac{n}{2}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\widetilde{\chi_0}(\xi)\widetilde{\tau_x u}(\xi)|^2\,\mathrm{d}\xi\,\mathrm{d}x = (2\pi)^{\frac{n}{2}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\chi_0(\xi)\tau_x u(\xi)|^2\,\mathrm{d}\xi\,\mathrm{d}x
\end{aligned}$$

$$= (2\pi)^{\frac{n}{2}} \int_{\mathbb{R}^n} |\chi_0(\xi)|^2 \left(\int_{\mathbb{R}^n} |u(x-\xi)|^2 \, \mathrm{d}x \right) \mathrm{d}\xi$$

$$= (2\pi)^{\frac{n}{2}} \|u\|_{L^2}^2 \prod_{j=1}^{n} \int_{\mathbb{R}^n} \left| \frac{1}{1+i\xi_j} \right|^2 \mathrm{d}\xi_j$$

$$= (2\pi)^{\frac{n}{2}} \pi^n \|u\|_{L^2}^2,$$

as

$$\int_{\mathbb{R}} \left| \frac{1}{1+i\xi_j} \right|^2 \mathrm{d}\xi_j = \int_{\mathbb{R}} \frac{1}{1+\xi_j^2} \, \mathrm{d}\xi_j = \pi.$$

Hence $\Psi \in L^2(\mathbb{R}^{2n})$ with a norm equal to $2^{\frac{n}{4}} \pi^{\frac{3}{4}n} \|u\|_{L^2}$. For the moment define

$$g(x,\xi) := \partial_x^\alpha \left(e^{-i\langle x,\xi\rangle} \Psi(x,\xi) \right),$$

and calculate

$$\|g\|_{L^2}^2 = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |g(x,\xi)|^2 \, \mathrm{d}x \, \mathrm{d}\xi = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| \partial_x^\alpha \left(e^{-i\langle x,\xi\rangle} \Psi(x,\xi) \right) \right|^2 \mathrm{d}x \, \mathrm{d}\xi$$

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| \partial_x^\alpha \left(e^{-i\langle x,\xi\rangle} \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} \chi_0(x-y) u(y) \, \mathrm{d}y \right) \right|^2 \mathrm{d}x \, \mathrm{d}\xi$$

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} e^{-i\langle y,\xi\rangle} (\partial_x^\alpha \chi_0(x-y)) u(y) \, \mathrm{d}y \right|^2 \mathrm{d}x \, \mathrm{d}\xi$$

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} e^{-i\langle x-z,\xi\rangle} (\partial_z^\alpha \chi_0(z)) \tau_x u(z) \, \mathrm{d}z \right|^2 \mathrm{d}x \, \mathrm{d}\xi$$

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| (2\pi)^{\frac{n}{2}} \mathcal{F} \left[(\partial_\xi^\alpha \chi_0) \cdot \tau_x u \right] (-\xi) \right|^2 \mathrm{d}x \, \mathrm{d}\xi$$

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| \left(\widehat{\partial_\xi^\alpha \chi_0} * \widehat{\tau_x u} \right) (-\xi) \right|^2 \mathrm{d}x \, \mathrm{d}\xi = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| \mathcal{F} \left[\widehat{\partial_\xi^\alpha \chi_0} * \widehat{\tau_x u} \right] (\xi) \right|^2 \mathrm{d}x \, \mathrm{d}\xi$$

$$= (2\pi)^{\frac{n}{2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| \widehat{\widehat{\partial_\xi^\alpha \chi_0}}(\xi) \widehat{\widehat{\tau_x u}}(\xi) \right|^2 \mathrm{d}x \, \mathrm{d}\xi = (2\pi)^{\frac{n}{2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| \widetilde{\partial_\xi^\alpha \chi_0}(\xi) \widetilde{\tau_x u}(\xi) \right|^2 \mathrm{d}x \, \mathrm{d}\xi$$

$$= (2\pi)^{\frac{n}{2}} \int_{\mathbb{R}^n} \left| \partial_\xi^\alpha \chi_0(\xi) \right|^2 \left(\int_{\mathbb{R}^n} |u(x-\xi)|^2 \, \mathrm{d}x \right) \mathrm{d}\xi = (2\pi)^{\frac{n}{2}} \|u\|_{L^2}^2 \int_{\mathbb{R}^n} \left| \partial_\xi^\alpha \chi_0(\xi) \right|^2 \mathrm{d}\xi$$

$$= (2\pi)^{\frac{n}{2}} \|u\|_{L^2}^2 \prod_{j=1}^{n} \int_{\mathbb{R}} \left| \partial_{\xi_j}^{\alpha_j} \frac{1}{1+i\xi_j} \right|^2 \mathrm{d}\xi_j$$

$$= (2\pi)^{\frac{n}{2}} \pi^n 2^{-|\alpha|} \|u\|_{L^2}^2,$$

as for $h(x) = \frac{1}{1+ix}$ we have $h'(x) = -\frac{i}{(1+ix)^2}$, hence $|h'(x)| = \frac{1}{|(1+ix)^2|}$, and thus

$$\int_{\mathbb{R}} |h'(x)|^2 \, \mathrm{d}x = \int_{\mathbb{R}} \frac{1}{|(1+ix)^2|^2} \, \mathrm{d}x = \frac{\pi}{2}.$$

Therefore $\partial_x^\alpha \left(e^{-ix\cdot\xi}\Psi(x,\xi)\right) \in L^2(\mathbb{R}^{2n})$ with a norm equal to $2^{\frac{n-2|\alpha|}{4}}\pi^{\frac{3}{4}n}\|u\|_{L^2}$. Further we note that

$$\begin{aligned}
&\left(\prod_{j=1}^{n}(1+\partial_{\xi_j})\right)\Psi(x,\xi) = \left(\prod_{j=1}^{n}(1+\partial_{\xi_j})\right)\int_{\mathbb{R}^n}\chi_\xi(x-y)u(y)\,\mathrm{d}y \\
&= \left(\prod_{j=1}^{n}(1+\partial_{\xi_j})\right)\int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle}\chi_0(x-y)u(y)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^n}\prod_{j=1}^{n}(1+\partial_{\xi_j})\left(e^{i\langle x-y,\xi\rangle}\right)\chi_0(x-y)u(y)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^n}\prod_{j=2}^{n}(1+\partial_{\xi_j}\left((1+i(x_1-y_1))e^{i\langle x-y,\xi\rangle}\right)\chi_0(x-y)u(y)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^n}\prod_{j=1}^{n}\left(1+i(x_j-y_j)\right)e^{i\langle x-y,\xi\rangle}\prod_{j=1}^{n}\left(1+i(x_j-y_j)\right)^{-1}u(y)\,\mathrm{d}y \\
&= (2\pi)^{\frac{n}{2}}e^{i\langle x,\xi\rangle}\widehat{u}(\xi).
\end{aligned}$$

Next we define for $v \in \mathcal{S}$ the function

$$\Phi(x,\xi) := (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\chi_{-x}(\xi-\eta)\widehat{v}(\eta)\,\mathrm{d}\eta.$$

In a similar way as before we get

$$\begin{aligned}
&\|\Phi\|_{L^2}^2 = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|\Phi(x,\xi)|^2\,\mathrm{d}x\,\mathrm{d}\xi \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left|(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}e^{i\langle\xi-\eta,x\rangle}\chi_0(\xi-\eta)\widehat{v}(\eta)\,\mathrm{d}\eta\right|^2\mathrm{d}x\,\mathrm{d}\xi \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left|(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}e^{-i\langle\zeta,x\rangle}\chi_0(\zeta)\tau_\xi\widehat{v}(\zeta)\,\mathrm{d}\zeta\right|^2\mathrm{d}x\,\mathrm{d}\xi \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left|\mathcal{F}\left[\chi_0\cdot\tau_\xi\widehat{v}\right](x)\right|^2\mathrm{d}x\,\mathrm{d}\xi \\
&= (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left|\left(\widehat{\chi_0}*\widehat{\tau_\xi\widehat{v}}\right)(x)\right|^2\mathrm{d}x\,\mathrm{d}\xi
\end{aligned}$$

$$
\begin{aligned}
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left|\mathcal{F}\left[\widehat{\chi_0} * \tau_\xi \widehat{v}\right](x)\right|^2 \mathrm{d}x\,\mathrm{d}\xi \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left|\widehat{\widehat{\chi_0}}(x)\widehat{\tau_\xi \widehat{v}}(x)\right|^2 \mathrm{d}x\,\mathrm{d}\xi = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left|\widecheck{\chi_0}(x)\widecheck{\tau_\xi \widehat{v}}(x)\right|^2 \mathrm{d}x\,\mathrm{d}\xi \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\chi_0(x)\tau_\xi \widehat{v}(x)|^2 \,\mathrm{d}x\,\mathrm{d}\xi = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\chi_0(x)\widehat{v}(\xi - x)|^2 \,\mathrm{d}x\,\mathrm{d}\xi \\
&= \int_{\mathbb{R}^n} |\chi_0(x)|^2 \left(\int_{\mathbb{R}^n} |\widehat{v}(\xi - x)|^2 \,\mathrm{d}\xi\right) \mathrm{d}x = \|\widehat{v}\|_{L^2}^2 \int_{\mathbb{R}^n} |\chi_0(x)|^2 \,\mathrm{d}x \\
&= \pi^n \|v\|_{L^2}^2,
\end{aligned}
$$

which means that $\Phi \in L^2(\mathbb{R}^{2n})$ with a norm equal to $\pi^{\frac{n}{2}}\|v\|_{L^2}$.
Furthermore,

$$
\left(\prod_{j=1}^{n}\left(1 - \partial_{x_j}\right)\right)\Phi(x,\xi) = \left(\prod_{j=1}^{n}(1 - \partial_{x_j})\right)(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} \chi_{-x}(\xi - \eta)\widehat{v}(\eta)\,\mathrm{d}\eta
$$

$$
\begin{aligned}
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \prod_{j=1}^{n}\left(1 - \partial_{x_j}\right)\left(e^{-i\langle \xi-\eta, x\rangle}\right)\chi_0(\xi - \eta)\widehat{v}(\eta)\,\mathrm{d}\eta \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \prod_{j=2}^{n}\left(1 - \partial_{x_j}\right)\left(e^{-i\langle \xi-\eta, x\rangle} - (-i)(\xi_1 - \eta_1)e^{-i\langle \xi-\eta, x\rangle}\right) \\
&\qquad \cdot \chi_0(\xi - \eta)\widehat{v}(\eta)\,\mathrm{d}\eta \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \prod_{j=1}^{n}\left(1 + i(\xi_j - \eta_j)\right)e^{-i\langle \xi-\eta, x\rangle}\prod_{j=1}^{n}\left(1 + i(\xi_j - \eta_j)\right)^{-1}\widehat{v}(\eta)\,\mathrm{d}\eta \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-i\langle \xi-\eta, x\rangle}\widehat{v}(\eta)\,\mathrm{d}\eta \\
&= e^{-i\langle x,\xi\rangle}v(x).
\end{aligned}
$$

Now we define

$$
\left(q(x,D)u, v\right) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} q(x,\xi)\widehat{u}(\xi)\overline{v}(x)\,\mathrm{d}x\,\mathrm{d}\xi,
$$

and get by using the previous calculations and estimates

$$
\begin{aligned}
\big(q(x,D)u,v\big) &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} q(x,\xi)\widehat{u}(\xi)\overline{v}(x)\,\mathrm{d}x\,\mathrm{d}\xi \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} q(x,\xi) e^{-i\langle x,\xi\rangle}\left(\prod_{j=1}^{n}(1+\partial_{\xi_j})\right)\big(\Psi(x,\xi)\big) \\
&\qquad \cdot \overline{e^{i\langle x,\xi\rangle}\left(\prod_{j=1}^{n}(1-\partial_{x_j})\right)\big(\Phi(x,\xi)\big)}\,\mathrm{d}x\,\mathrm{d}\xi \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} q(x,\xi) e^{-i\langle x,\xi\rangle}\left(\prod_{j=1}^{n}(1+\partial_{\xi_j})\right)\big(\Psi(x,\xi)\big) \\
&\qquad \cdot \left(\prod_{j=1}^{n}(1-\partial_{x_j})\right)\big(\overline{\Phi}(x,\xi)\big)\,\mathrm{d}x\,\mathrm{d}\xi \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left(\prod_{j=1}^{n}(1+\partial_{\xi_j})\right)\big(q(x,\xi)\big) e^{-i\langle x,\xi\rangle}\Psi(x,\xi) \\
&\qquad \cdot \left(\prod_{j=1}^{n}(1-\partial_{x_j})\right)\big(\overline{\Phi}(x,\xi)\big)\,\mathrm{d}x\,\mathrm{d}\xi \\
&= (2\pi)^{-\frac{n}{2}} \sum_{\beta\in\{0,1\}^n} (-1)^{|\beta|} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \partial_\xi^\beta q(x,\xi) e^{-i\langle x,\xi\rangle}\Psi(x,\xi)\left(\prod_{j=1}^{n}(1-\partial_{x_j})\right) \\
&\qquad \cdot \big(\overline{\Phi}(x,\xi)\big)\,\mathrm{d}x\,\mathrm{d}\xi \\
&= (2\pi)^{-\frac{n}{2}} \sum_{\beta,\gamma\in\{0,1\}^n} (-1)^{|\beta|} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \partial_x^\gamma\left(\partial_\xi^\beta q(x,\xi) e^{-i\langle x,\xi\rangle}\Psi(x,\xi)\right)\overline{\Phi}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi \\
&= (2\pi)^{-\frac{n}{2}} \sum_{\alpha,\beta,\gamma\in\{0,1\}^n} (-1)^{|\beta|}\binom{\gamma}{\alpha} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \partial_x^\alpha\partial_\xi^\beta q(x,\xi)\partial_x^{\gamma-\alpha}\big(e^{-i\langle x,\xi\rangle}\Psi(x,\xi)\big) \\
&\qquad \cdot \overline{\Phi}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi \\
&= (2\pi)^{-\frac{n}{2}} \sum_{\alpha,\beta,\gamma\in\{0,1\}^n} (-1)^{|\beta|}\binom{\gamma}{\alpha} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \overline{\Phi}(x,\xi)\big(\partial_x^\alpha\partial_\xi^\beta q(x,\xi)\big) \\
&\qquad \cdot \partial_x^{\gamma-\alpha}\big(e^{-i\langle x,\xi\rangle}\Psi(x,\xi)\big)\,\mathrm{d}x\,\mathrm{d}\xi.
\end{aligned}
$$

We want to apply Lemma 3.6, hence we need to show that

$$\big|\big(q(x,D)u,v\big)\big| \leqslant C_q\|u\|_{L^2}\|v\|_{L^2}.$$

We calculate

$$
\begin{aligned}
&\left|\left(q(x,D)u,v\right)\right| \\
&\leqslant (2\pi)^{-\frac{n}{2}} \sum_{\alpha,\beta,\gamma\in\{0,1\}^n} \left|(-1)^{|\beta|}\right| \binom{\gamma}{\alpha} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left|\overline{\Phi}(x,\xi)\right| \left|\partial_x^\alpha \partial_\xi^\beta q(x,\xi)\right| \\
&\qquad \cdot \left|\partial_x^{\gamma-\alpha}\left(e^{-i\langle x,\xi\rangle}\Psi(x,\xi)\right)\right| \,\mathrm{d}x\,\mathrm{d}\xi \\
&\leqslant (2\pi)^{-\frac{n}{2}} \sum_{\alpha,\beta,\gamma\in\{0,1\}^n} \binom{\gamma}{\alpha} C_{\alpha\beta} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left|\overline{\Phi}(x,\xi)\right| \left|\partial_x^{\gamma-\alpha}\left(e^{-i\langle x,\xi\rangle}\Psi(x,\xi)\right)\right| \,\mathrm{d}x\,\mathrm{d}\xi \\
&\leqslant (2\pi)^{-\frac{n}{2}} \sum_{\alpha,\beta,\gamma\in\{0,1\}^n} \binom{\gamma}{\alpha} C_{\alpha\beta} \|\Phi\|_{L^2} \left\|\partial_x^{\gamma-\alpha}\left(e^{-i\langle x,\xi\rangle}\Psi\right)\right\|_{L^2} \\
&\leqslant (2\pi)^{-\frac{n}{2}} \sum_{\alpha,\beta,\gamma\in\{0,1\}^n} \binom{\gamma}{\alpha} C_{\alpha\beta} \pi^{\frac{n}{2}} \|v\|_{L^2} 2^{\frac{n-2|\gamma-\alpha|}{4}} \pi^{\frac{3}{4}n} \|u\|_{L^2} \\
&= 2^{-\frac{3}{4}n}\,\pi^{\frac{3}{4}n} \sum_{\alpha,\beta,\gamma\in\{0,1\}^n} \binom{\gamma}{\alpha} C_{\alpha\beta} \|u\|_{L^2}\|v\|_{L^2}.
\end{aligned}
$$

Using Theorem 3.6 we conclude that

$$
\|q(x,D)u\|_{L^2} \leqslant 2^{-\frac{3}{4}n}\,\pi^{\frac{3}{4}n} \sum_{\alpha,\beta,\gamma\in\{0,1\}^n} \binom{\gamma}{\alpha} C_{\alpha\beta}\, \|u\|_{L^2}.
$$

□

From now on we will only be interested in symbols $q \in S_0^{m,\psi}$ where $m > 0$, as for $m \leqslant 0$ the symbol q and its derivatives are bounded, and there are several ways to construct a corresponding Feller semigroup, compare e.g. Chapter 2.

Theorem 3.8. *Let ψ be a continuous negative definite function satisfying* (3.16) *and* (3.17). *Moreover assume $q \in S_0^{m,\psi}$, $m > 0$, satisfying* (3.18) *is given.*

A. For every $s \in \mathbb{R}$ there exists a constant C_s independent of ν such that $\left(q - q^{(\nu)}\right)(x,D)u \in H^{s,\psi}$ for all $u \in H^{2m+s,\psi}$ with

$$
\left\|\left(q - q^{(\nu)}\right)(x,D)u\right\|_{H^{s,\psi}} \leqslant \frac{1}{\nu} C_s \|u\|_{H^{2m+s,\psi}},
$$

i.e. the operator $\left(q - q^{(\nu)}\right)(x,D)$ maps the space $H^{2m+s,\psi}$ continuously into the space $H^{s,\psi}$.

B. For every $s \in \mathbb{R}$ there exists a constant C_s' independent of ν such that $\big(q - q^{(\nu)}\big)(x, D)u \in H^{s,\psi}$ for all $u \in H^{m+s,\psi}$ with

$$\big\|\big(q - q^{(\nu)}\big)(x, D)u\big\|_{H^{s,\psi}} \leqslant C_s' \|u\|_{H^{m+s,\psi}},$$

i.e. the operator $\big(q - q^{(\nu)}\big)(x, D)$ maps the space $H^{m+s,\psi}$ continuously into the space $H^{s,\psi}$.

Proof. Let $\kappa \in \{m, 2m\}$ and define

$$r^{(\nu)} := \lambda^{s,\psi} \# \left(q - q^{(\nu)}\right) \# \lambda^{-\kappa-s,\psi}.$$

As $r^{(\nu)} \in S_0^{0,\psi}$ we may use Theorem 3.7 to find

$$\begin{aligned}
&\left\|\left(q - q^{(\nu)}\right)(x, D)u\right\|_{H^{s,\psi}} = \left\|\lambda^{s,\psi}\mathcal{F}\left[\left(q - q^{(\nu)}\right)(x, D)u\right]\right\|_{L^2} \\
&= \left\|\mathcal{F}\left[\lambda^{s,\psi}(D)\left(q - q^{(\nu)}\right)(x, D)u\right]\right\|_{L^2} = \left\|\lambda^{s,\psi}(D)\left(q - q^{(\nu)}\right)(x, D)u\right\|_{L^2} \\
&= \left\|\lambda^{s,\psi}(D)\left(q - q^{(\nu)}\right)(x, D)\lambda^{-\kappa-s,\psi}(D)\lambda^{\kappa+s,\psi}(D)u\right\|_{L^2} \\
&\leqslant \left\|\lambda^{s,\psi}(D)\left(q - q^{(\nu)}\right)(x, D)\lambda^{-\kappa-s,\psi}(D)\right\|_{L^2\to L^2}\left\|\lambda^{\kappa+s,\psi}(D)u\right\|_{L^2} \\
&= \left\|r^{(\nu)}(x, D)\right\|_{L^2\to L^2}\left\|\mathcal{F}\left[\lambda^{\kappa+s,\psi}(D)u\right]\right\|_{L^2} = \left\|r^{(\nu)}(x, D)\right\|_{L^2\to L^2}\left\|\lambda^{\kappa+s,\psi}\widehat{u}\right\|_{L^2} \\
&= \left\|r^{(\nu)}(x, D)\right\|_{L^2\to L^2}\|u\|_{H^{\kappa+s,\psi}}.
\end{aligned}$$

Note that for part A of the theorem we take $\kappa = 2m$ and for part B we take $\kappa = m$. Now it remains to show that

$$\left\|r^{(\nu)}(x, D)u\right\|_{L^2} \leqslant \frac{1}{\nu}C_s\|u\|_{L^2}.$$

In principle this is just an application of Theorem 3.7, but we are particularly interested in the $\frac{1}{\nu}$-decay of the constant

$$C_q = 2^{-\frac{3}{4}n}\pi^{\frac{3}{4}n}\sum_{\alpha,\beta,\gamma\in\{0,1\}^n}\binom{\gamma}{\alpha}C_{\alpha\beta}$$

compare (3.21). In the above expression we want to explicitly calculate the constant $C_{\alpha\beta}$. Going back to Theorem 3.7, we find that this constant is derived by estimating $\partial_x^\alpha\partial_\xi^\beta r^{(\nu)}(x, \xi)$, hence this is the expression we want to investigate more thoroughly!

Let us begin with writing

$$\begin{aligned}
r^{(\nu)} &= \lambda^{s,\psi} \# \left(q - q^{(\nu)}\right) \# \lambda^{-\kappa-s,\psi} \\
&= \lambda^{s,\psi} \# \Big(\left(q - q^{(\nu)}\right) \# \lambda^{-\kappa-s,\psi}\Big).
\end{aligned}$$

Note that with Theorem 1.17

$$\begin{aligned}
&\left(\left(q-q^{(\nu)}\right)\#\lambda^{-\kappa-s,\psi}\right)(x,\xi)\\
&=(2\pi)^{-n}\,\mathrm{Os-}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}e^{-i\langle y,\eta\rangle}\left(q-q^{(\nu)}\right)(x,\xi-\eta)\lambda^{-\kappa-s,\psi}(\xi)\,\mathrm{d}y\,\mathrm{d}\eta\\
&=(2\pi)^{-n}\lambda^{-\kappa-s,\psi}(\xi)\,\mathrm{Os-}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}e^{-i\langle y,\eta\rangle}\left(q-q^{(\nu)}\right)(x,\xi-\eta)\,\mathrm{d}y\,\mathrm{d}\eta\\
&=\lambda^{-\kappa-s,\psi}(\xi)\left(q-q^{(\nu)}\right)(x,\xi-0)\\
&=\left(q-q^{(\nu)}\right)(x,\xi)\lambda^{-\kappa-s,\psi}(\xi),
\end{aligned}$$

as well as for $q_1,q_2\in S_0^{\infty,\psi}$

$$\begin{aligned}
&\partial_x^\alpha\partial_\xi^\beta\left(q_1\#q_2\right)(x,\xi)\\
&=(2\pi)^{-n}\,\mathrm{Os-}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}e^{-i\langle y,\eta\rangle}\left(\partial_x^\alpha\partial_\xi^\beta q_1(x,\xi-\eta)q_2(x-y,\xi)\right)\,\mathrm{d}y\,\mathrm{d}\eta\\
&=(2\pi)^{-n}\,\mathrm{Os-}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}e^{-i\langle y,\eta\rangle}\sum_{\beta_1\leqslant\beta}\binom{\beta}{\beta_1}\sum_{\alpha_1\leqslant\alpha}\binom{\alpha}{\alpha_1}\partial_x^{\alpha_1}\partial_\xi^{\beta_1}q_1(x,\xi-\eta)\\
&\qquad\cdot\partial_x^{\alpha-\alpha_1}\partial_\xi^{\beta-\beta_1}q_2(x-y,\xi)\,\mathrm{d}y\,\mathrm{d}\eta\\
&=\sum_{\beta_1\leqslant\beta}\sum_{\alpha_1\leqslant\alpha}\binom{\beta}{\beta_1}\binom{\alpha}{\alpha_1}\left(\left(\partial_x^{\alpha_1}\partial_\xi^{\beta_1}q_1\right)\#\left(\partial_x^{\alpha-\alpha_1}\partial_\xi^{\beta-\beta_1}q_2\right)\right)(x,\xi).
\end{aligned}$$

Hence we get

$$\begin{aligned}
&\left|\partial_x^\alpha\partial_\xi^\beta r^{(\nu)}(x,\xi)\right|\\
&=\Bigg|\sum_{\beta_1\leqslant\beta}\sum_{\alpha_1\leqslant\alpha}\binom{\beta}{\beta_1}\binom{\alpha}{\alpha_1}\\
&\qquad\cdot\left(\left(\partial_x^{\alpha_1}\partial_\xi^{\beta_1}\lambda^{s,\psi}\right)\#\left(\partial_x^{\alpha-\alpha_1}\partial_\xi^{\beta-\beta_1}\left(q-q^{(\nu)}\right)\#\lambda^{-\kappa-s,\psi}\right)\right)(x,\xi)\Bigg|\\
&\leqslant\sum_{\beta_1\leqslant\beta}\sum_{\alpha_1\leqslant\alpha}\binom{\beta}{\beta_1}\binom{\alpha}{\alpha_1}\Bigg|\mathrm{Os-}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}e^{-i\langle y,\eta\rangle}\left(\partial_x^{\alpha_1}\partial_\xi^{\beta_1}\lambda^{s,\psi}(\xi-\eta)\right)\\
&\qquad\cdot\left(\partial_x^{\alpha-\alpha_1}\partial_\xi^{\beta-\beta_1}\left(q-q^{(\nu)}\right)(x-y,\xi)\lambda^{-\kappa-s,\psi}(\xi)\right)\,\mathrm{d}y\,\mathrm{d}\eta\Bigg|.
\end{aligned}$$

Set

$$c_{x,\xi}(y,\eta):=\left(\partial_x^{\alpha_1}\partial_\xi^{\beta_1}\lambda^{s,\psi}(\xi-\eta)\right)\left(\partial_x^{\alpha-\alpha_1}\partial_\xi^{\beta-\beta_1}\left(q-q^{(\nu)}\right)(x-y,\xi)\lambda^{-\kappa-s,\psi}(\xi)\right);$$

we want to check that $c_{x,\xi}(y,\eta)$ is an amplitude and use the estimate (1.19) with the oscillatory integral above. We need the estimate

$$\begin{aligned}
&\left|\partial_y^\gamma \partial_\eta^\delta c_{x,\xi}(y,\eta)\right| \\
&= \Bigg|\partial_y^\gamma \partial_\eta^\delta \Bigg(\left(\partial_x^{\alpha_1}\partial_\xi^{\beta_1}\lambda^{s,\psi}(\xi-\eta)\right) \\
&\qquad \cdot \left(\partial_x^{\alpha-\alpha_1}\partial_\xi^{\beta-\beta_1}\left(q-q^{(\nu)}\right)(x-y,\xi)\lambda^{-\kappa-s,\psi}(\xi)\right)\Bigg)\Bigg| \\
&= \Bigg|\partial_y^\gamma \Bigg(\sum_{\delta_1\leqslant\delta}\binom{\delta}{\delta_1}\left(\partial_\eta^{\delta_1}\partial_x^{\alpha_1}\partial_\xi^{\beta_1}\lambda_\psi^{s}(\xi-\eta)\right) \\
&\qquad \cdot \left(\partial_\eta^{\delta-\delta_1}\partial_x^{\alpha-\alpha_1}\partial_\xi^{\beta-\beta_1}\left(q-q^{(\nu)}\right)(x-y,\xi)\lambda^{-\kappa-s,\psi}(\xi)\right)\Bigg)\Bigg| \\
&\leqslant \sum_{\delta_1\leqslant\delta}\sum_{\gamma_1\leqslant\gamma}\binom{\delta}{\delta_1}\binom{\gamma}{\gamma_1}\left|\partial_y^{\gamma_1}\partial_\eta^{\delta_1}\partial_x^{\alpha_1}\partial_\xi^{\beta_1}\lambda^{s,\psi}(\xi-\eta)\right| \\
&\qquad \cdot \left|\partial_y^{\gamma-\gamma_1}\partial_\eta^{\delta-\delta_1}\partial_x^{\alpha-\alpha_1}\partial_\xi^{\beta-\beta_1}\left(q-q^{(\nu)}\right)(x-y,\xi)\lambda^{-\kappa-s,\psi}(\xi)\right|.
\end{aligned}$$

It is easy to see that

$$\left|\partial_y^{\gamma_1}\partial_\eta^{\delta_1}\partial_x^{\alpha_1}\partial_\xi^{\beta_1}\lambda^{s,\psi}(\xi-\eta)\right| \leqslant C_1\lambda^{s-|\beta_1|-|\delta_1|,\psi}(\xi-\eta) \leqslant C_1\lambda^{s,\psi}(\xi-\eta). \qquad (3.22)$$

For the second factor we get

$$\begin{aligned}
&\left|\partial_y^{\gamma-\gamma_1}\partial_\eta^{\delta-\delta_1}\partial_x^{\alpha-\alpha_1}\delta_\xi^{\beta-\beta_1}\left(q-q^{(\nu)}\right)(x-y,\xi)\lambda^{-\kappa-s,\psi}(\xi)\right| \\
&= \left|\partial_\xi^{\beta-\beta_1}\left(\left(\partial_y^{\gamma-\gamma_1}\partial_\eta^{\delta-\delta_1}\partial_x^{\alpha-\alpha_1}\left(q-q^{(\nu)}\right)(x-y,\xi)\right)\lambda^{-\kappa-s,\psi}(\xi)\right)\right| \\
&\leqslant \sum_{\beta_2\leqslant\beta-\beta_1}\binom{\beta-\beta_1}{\beta_2}\left|\partial_\xi^{\beta_2}\lambda^{-\kappa-s,\psi}(\xi)\right| \\
&\qquad \cdot \left|\partial_y^{\gamma-\gamma_1}\partial_\eta^{\delta-\delta_1}\partial_x^{\alpha-\alpha_1}\partial_\xi^{\beta-\beta_1-\beta_2}\left(q-q^{(\nu)}\right)(x-y,\xi)\right|.
\end{aligned}$$

Again we look at both factors and get that

$$\left|\partial_\xi^{\beta_2}\lambda^{-\kappa-s,\psi}(\xi)\right| \leqslant C_2\lambda^{-\kappa-s-|\beta_2|,\psi}(\xi) \leqslant C_2\lambda^{-\kappa-s,\psi}(\xi), \qquad (3.23)$$

as well as

$$
\begin{aligned}
&\left|\partial_y^{\gamma-\gamma_1}\partial_\eta^{\delta-\delta_1}\partial_x^{\alpha-\alpha_1}\partial_\xi^{\beta-\beta_1-\beta_2}\left(q-q^{(\nu)}\right)(x-y,\xi)\right| \\
&= \left|\partial_y^{\gamma-\gamma_1}\partial_\eta^{\delta-\delta_1}\partial_x^{\alpha-\alpha_1}\partial_\xi^{\beta-\beta_1-\beta_2} q(x-y,\xi)^2 \frac{1}{\nu+q(x-y,\xi)}\right| \\
&= \left|\partial_y^{\gamma-\gamma_1}\partial_\eta^{\delta-\delta_1}\partial_x^{\alpha-\alpha_1}\sum_{\beta_3\leqslant\beta-\beta_1-\beta_2}\binom{\beta-\beta_1-\beta_2}{\beta_3}\left(\partial_\xi^{\beta_3} q(x-y,\xi)^2\right)\right. \\
&\qquad \left.\cdot\left(\partial_\xi^{\beta-\beta_1-\beta_2-\beta_3}\frac{1}{\nu+q(x-y,\xi)}\right)\right| \\
&= \left|\sum_{\beta_3\leqslant\beta-\beta_1-\beta_2}\sum_{\alpha_2\leqslant\alpha-\alpha_1}\binom{\beta-\beta_1-\beta_2}{\beta_3}\binom{\alpha-\alpha_1}{\alpha_2}\partial_y^{\gamma-\gamma_1}\partial_\eta^{\delta-\delta_1}\right. \\
&\qquad \left.\cdot\left(\left(\partial_x^{\alpha_2}\partial_\xi^{\beta_3} q(x-y,\xi)^2\right)\left(\partial_x^{\alpha-\alpha_1-\alpha_2}\partial_\xi^{\beta-\beta_1-\beta_2-\beta_3}\frac{1}{\nu+q(x-y,\xi)}\right)\right)\right| \\
&\leqslant \sum_{\beta_3\leqslant\beta-\beta_1-\beta_2}\sum_{\alpha_2\leqslant\alpha-\alpha_1}\sum_{\delta_2\leqslant\delta-\delta_1}\sum_{\gamma_2\leqslant\gamma-\gamma_1}\binom{\beta-\beta_1-\beta_2}{\beta_3}\binom{\alpha-\alpha_1}{\alpha_2}\binom{\delta-\delta_1}{\delta_2} \\
&\qquad \cdot\binom{\gamma-\gamma_1}{\gamma_2}\left|\partial_y^{\gamma_2}\partial_\eta^{\delta_2}\partial_x^{\alpha_2}\partial_\xi^{\beta_3} q(x-y,\xi)^2\right| \\
&\qquad \cdot\left|\partial_y^{\gamma-\gamma_1-\gamma_2}\partial_\eta^{\delta-\delta_1-\delta_2}\partial_x^{\alpha-\alpha_1-\alpha_2}\partial_\xi^{\beta-\beta_1-\beta_2-\beta_3}\frac{1}{\nu+q(x-y,\xi)}\right|.
\end{aligned}
$$

We get

$$
\left|\partial_y^{\gamma_2}\partial_\eta^{\delta_2}\partial_x^{\alpha_2}\partial_\xi^{\beta_3} q(x-y,\xi)^2\right| \leqslant C_3\lambda^{2m,\psi}(\xi) \tag{3.24}
$$

as $q^2 \in S_0^{2m,\psi}$ for $q \in S_0^{m,\psi}$. Now define a function $f^{(\nu)} :]0,\infty[\to \mathbb{R}$ as

$$
f^{(\nu)}(x) = \frac{1}{\nu+x},
$$

and note that $f^{(\nu)}$ is a Bernstein function, compare Section 1.2. Thus we may use the estimate (1.17) to find

$$
\left|\left(f^{(\nu)}\right)^{(j)}(x)\right| \leqslant \frac{j!}{x^j} f(x). \tag{3.25}
$$

Then for $\vartheta = (\gamma - \gamma_1 - \gamma_2, \delta - \delta_1 - \delta_2, \alpha - \alpha_1 - \alpha_2, \beta - \beta_1 - \beta_2) \in \mathbb{N}_0^{4n}$ and $\kappa = 2m$ we get, using (1.4), (3.18) and (3.25),

$$\left|\partial_y^{\gamma-\gamma_1-\gamma_2}\partial_\eta^{\delta-\delta_1-\delta_2}\partial_x^{\alpha-\alpha_1-\alpha_2}\partial_\xi^{\beta-\beta_1-\beta_2}\frac{1}{\nu+q(x-y,\xi)}\right|$$

$$\leqslant C_4\sum_{j=1}^{|\vartheta|}\left|\left(f^{(\nu)}\right)^{(j)}\left(q(x-y,\xi)\right)\right|$$

$$\cdot \sum_{\substack{A_1+...+A_j=\gamma-\gamma_1-\gamma_2\\ B_1+...+B_j=\delta-\delta_1-\delta_2\\ C_1+...+C_j=\alpha-\alpha_1-\alpha_2\\ D_1+...D_j=\beta-\beta_1-\beta_2-\beta_3}}\prod_{l=1}^{j}\left|\partial_y^{A_l}\partial_\eta^{B_l}\partial_x^{C_l}\partial_\xi^{D_l}q(x-y,\xi)\right|$$

$$\leqslant C_4\sum_{j=1}^{|\vartheta|}\frac{j!}{q(x-y,\xi)^j}\cdot\frac{1}{\nu+q(x-y,\xi)}$$

$$\cdot \sum_{\substack{A_1+...+A_j=\gamma-\gamma_1-\gamma_2\\ B_1+...+B_j=\delta-\delta_1-\delta_2\\ C_1+...+C_j=\alpha-\alpha_1-\alpha_2\\ D_1+...D_j=\beta-\beta_1-\beta_2-\beta_3}}\prod_{l=1}^{j}\left|\partial_y^{A_l}\partial_\eta^{B_l}\partial_x^{C_l}\partial_\xi^{D_l}q(x-y,\xi)\right| \tag{3.26}$$

$$\leqslant C_4\sum_{j=1}^{|\vartheta|}j!\frac{1}{\nu+q(x-y,\xi)}\sum_{\substack{A_1+...+A_j=\gamma-\gamma_1-\gamma_2\\ B_1+...+B_j=\delta-\delta_1-\delta_2\\ C_1+...+C_j=\alpha-\alpha_1-\alpha_2\\ D_1+...D_j=\beta-\beta_1-\beta_2-\beta_3}}\prod_{l=1}^{j}\frac{\left|\partial_y^{A_l}\partial_\eta^{B_l}\partial_x^{C_l}\partial_\xi^{D_l}q(x-y,\xi)\right|}{q(x-y,\xi)} \tag{3.27}$$

$$\leqslant C_4\sum_{j=1}^{|\vartheta|}j!\frac{1}{\nu}\sum_{\substack{A_1+...+A_j=\gamma-\gamma_1-\gamma_2\\ B_1+...+B_j=\delta-\delta_1-\delta_2\\ C_1+...+C_j=\alpha-\alpha_1-\alpha_2\\ D_1+...D_j=\beta-\beta_1-\beta_2-\beta_3}}\prod_{l=1}^{j}\frac{C_{B_lD_l}\lambda^{m,\psi}(\xi)}{K\lambda^{m,\psi}(\xi)} \tag{3.28}$$

$$\leqslant \frac{1}{\nu}C_5. \tag{3.29}$$

For $\kappa = m$ we need to estimate the above differently. The estimate from

(3.27) to (3.28) changes to

$$
\begin{aligned}
&C_4 \sum_{j=1}^{|\vartheta|} j! \frac{1}{\nu + q(x-y,\xi)} \sum_{\substack{A_1+\ldots+A_j=\gamma-\gamma_1-\gamma_2\\ B_1+\ldots+B_j=\delta-\delta_1-\delta_2\\ C_1+\ldots+C_j=\alpha-\alpha_1-\alpha_2\\ D_1+\ldots D_j=\beta-\beta_1-\beta_2-\beta_3}} \prod_{l=1}^{j} \frac{\left|\partial_y^{A_l}\partial_\eta^{B_l}\partial_x^{C_l}\partial_\xi^{D_l} q(x-y,\xi)\right|}{q(x-y,\xi)}\\
&\leqslant C_4 \sum_{j=1}^{|\vartheta|} j! \frac{1}{q(x-y,\xi)} \sum_{\substack{A_1+\ldots+A_j=\gamma-\gamma_1-\gamma_2\\ B_1+\ldots+B_j=\delta-\delta_1-\delta_2\\ C_1+\ldots+C_j=\alpha-\alpha_1-\alpha_2\\ D_1+\ldots D_j=\beta-\beta_1-\beta_2-\beta_3}} \prod_{l=1}^{j} \frac{C_{B_l D_l}\lambda^{m,\psi}(\xi)}{K\lambda^{m,\psi}(\xi)}\\
&\leqslant C_5'\lambda^{-m,\psi}(\xi).
\end{aligned}
\tag{3.30}
$$

For $\kappa = 2m$ we collect the estimates (3.22), (3.23), (3.24), (3.26), and using Peetre's inequality (1.14), we finally end up with

$$
\begin{aligned}
\left|\partial_y^\gamma\partial_\eta^\delta c_{x,\xi}(y,\eta)\right| &\leqslant \frac{1}{\nu}C_6\, \lambda^{s,\psi}(\xi-\eta)\lambda^{-2m-s,\psi}(\xi)\lambda^{2m,\psi}(\xi)\\
&= \frac{1}{\nu}C_6\, \lambda^{s,\psi}(\xi-\eta)\lambda{-s,\psi(\xi)}\\
&\leqslant \frac{1}{\nu}C_6\, 2^{|s|}\lambda^{|s|,\psi}(\eta)\lambda^{s,\psi}(\xi)\lambda^{-s,\psi}(\xi)\\
&= \frac{1}{\nu}C_6\, 2^{|s|}\lambda^{|s|,\psi}(\eta)\\
&\leqslant \frac{1}{\nu}C_6\, 2^{|s|}\left(1+|\eta|^2\right)^{\frac{|s|}{2}}\left(1+|y|^2\right)^{\frac{|s|}{2}}.
\end{aligned}
$$

Then following Theorem 1.16 we get

$$
\begin{aligned}
|||c_{x,\xi}|||_{|s|+2n+1} &= \max_{|y+\delta|\leqslant|s|+2n+1}\ \sup_{y,\eta\in\mathbb{R}^n} \left|\left(1+|\eta|^2\right)^{-\frac{|s|}{2}}\left(1+|y|^2\right)^{-\frac{|s|}{2}}\partial_y^\gamma\partial_\eta^\delta c_{x,\xi}(y,\eta)\right|\\
&\leqslant \max_{|y+\delta|\leqslant|s|+2n+1}\ \sup_{y,\eta\in\mathbb{R}^n}\left|\frac{1}{\nu}\,C_6\,2^{|s|}\right|\\
&\leqslant \frac{1}{\nu}C_7.
\end{aligned}
$$

By the estimate (1.19) this yields

$$\begin{aligned}
&\left|\partial_x^\alpha \partial_\xi^\beta r^{(\nu)}(x,\xi)\right| \\
&\leqslant \sum_{\beta_1\leqslant\beta}\sum_{\alpha_1\leqslant\alpha}\binom{\beta}{\beta_1}\binom{\alpha}{\alpha_1}(2\pi)^{-n}\left|\mathrm{Os}-\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle y,\eta\rangle} c_{x,\xi}(y,\eta)\,\mathrm{d}y\,\mathrm{d}\eta\right| \\
&\leqslant \sum_{\beta_1\leqslant\beta}\sum_{\alpha_1\leqslant\alpha}\binom{\beta}{\beta_1}\binom{\alpha}{\alpha_1} C_8 \,|||c_{x,\xi}|||_{|s|+2n+1} \\
&\leqslant \frac{1}{\nu}\, C'_{\alpha\beta}.
\end{aligned}$$

Now we use this result together with (3.21), and conclude for the operator $r^{(\nu)}(x,D)$ that

$$\begin{aligned}
\left\|r^{(\nu)}(x,D)u\right\|_{L^2} &\leqslant 2^{-\frac{3}{4}n}\,\pi^{\frac{3}{4}n}\sum_{\alpha,\beta,\gamma\in\{0,1\}^n}\binom{\gamma}{\alpha}\frac{1}{\nu}\,C'_{\alpha\beta}\,\|u\|_{L^2} \\
&\leqslant \frac{1}{\nu}\,C_s\,\|u\|_{L^2}.
\end{aligned}$$

This proves part A of the Theorem. For part B, take $\kappa = m$ and collect the estimates (3.22), (3.23), (3.24), (3.30), and use Peetre's inequality (1.14) to find

$$\begin{aligned}
\left|\partial_y^\gamma \partial_\eta^\delta c_{x,\xi}(y,\eta)\right| &\leqslant C'_6 \lambda^{s,\psi}(\xi-\eta)\lambda^{-m-s,\psi}(\xi)\lambda^{2m,\psi}(\xi)\lambda^{-m,\psi}(\xi) \\
&= C'_6 \lambda^{s,\psi}(\xi-\eta)\lambda^{-s,\psi}(\xi) \\
&\leqslant C'_6 2^{|s|}\lambda^{|s|,\psi}(\eta)\lambda^{s,\psi}(\xi)\lambda^{-s,\psi}(\xi) \\
&= C'_6 2^{|s|}\lambda^{|s|,\psi}(\eta) \\
&\leqslant C'_6 2^{|s|}\left(1+|\eta|^2\right)^{\frac{|s|}{2}}\left(1+|y|^2\right)^{\frac{|s|}{2}}.
\end{aligned}$$

Following Theorem 1.16 we get

$$\begin{aligned}
|||c_{x,\xi}|||_{|s|+2n+1} &= \max_{|y+\delta|\leqslant|s|+2n+1}\ \sup_{y,\eta\in\mathbb{R}^n}\left|\left(1+|\eta|^2\right)^{-\frac{|s|}{2}}\left(1+|y|^2\right)^{-\frac{|s|}{2}}\partial_y^\gamma\partial_\eta^\delta c_{x,\xi}(y,\eta)\right| \\
&\leqslant \max_{|y+\delta|\leqslant|s|+2n+1}\ \sup_{y,\eta\in\mathbb{R}^n}\left|C'_6\,2^{|s|}\right| \\
&\leqslant C'_7.
\end{aligned}$$

Estimate (1.19) then yields

$$\begin{aligned}
&\left|\partial_x^\alpha \partial_\xi^\beta r^{(\nu)}(x,\xi)\right| \\
&\leqslant \sum_{\beta_1\leqslant\beta}\sum_{\alpha_1\leqslant\alpha}\binom{\beta}{\beta_1}\binom{\alpha}{\alpha_1}(2\pi)^{-n}\left|\mathrm{Os-}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle y,\eta\rangle}c_{x,\xi}(y,\eta)\,\mathrm{d}y\,\mathrm{d}\eta\right| \\
&\leqslant \sum_{\beta_1\leqslant\beta}\sum_{\alpha_1\leqslant\alpha}\binom{\beta}{\beta_1}\binom{\alpha}{\alpha_1}C_8' \,|||c_{x,\xi}|||_{|s|+2n+1} \\
&\leqslant C''_{\alpha\beta}.
\end{aligned}$$

Now we use this result together with (3.21), and conclude for the operator $r^{(\nu)}(x,D)$ that

$$\begin{aligned}
\left\|r^{(\nu)}(x,D)u\right\|_{L^2} &\leqslant 2^{-\frac{3}{4}n}\,\pi^{\frac{3}{4}n}\sum_{\alpha,\beta,\gamma\in\{0,1\}^n}\binom{\gamma}{\alpha}C''_{\alpha\beta}\,\|u\|_{L^2} \\
&\leqslant C_s'\,\|u\|_{L^2}.
\end{aligned}$$

This concludes part B of the theorem. □

Theorem 3.9. *Let ψ be a continuous negative definite function satisfying (3.16) and (3.17). Moreover assume $q \in S_0^{m,\psi}$, $m > 0$, satisfying (3.18) is given. Then for every $s \in \mathbb{R}$ and $u \in H^{m+s,\psi}$ we have*

$$\lim_{\nu\to\infty}\left\|q(x,D)u - q^{(\nu)}(x,D)u\right\|_{H^{s,\psi}} = 0.$$

Proof. First note that since C_0^∞ is dense in $H^{2m+s,\psi}$ as well as in $H^{m+s,\psi}$, and $H^{2m+s,\psi} \subset H^{m+s,\psi}$, we find for every $u \in H^{m+s,\psi}$ a sequence $(u_\mu)_{\mu\in\mathbb{N}}$, $u_\mu \in H^{2m+s,\psi}$, such that $u_\mu \to u$ in $H^{m+s,\psi}$ as $\mu \to \infty$. Then by Theorem 3.8.A we get for $u \in H^{m+s,\psi}$ and $s \in \mathbb{R}$

$$\begin{aligned}
&\left\|q(x,D)u - q^{(\nu)}(x,D)u\right\|_{H^{s,\psi}} = \left\|\left(q-q^{(\nu)}\right)(x,D)u\right\|_{H^{s,\psi}} \\
&= \left\|\left(q-q^{(\nu)}\right)(x,D)(u-u_\mu+u_\mu)\right\|_{H^{s,\psi}} \\
&\leqslant \left\|\left(q-q^{(\nu)}\right)(x,D)u_\mu\right\|_{H^{s,\psi}} + \left\|\left(q-q^{(\nu)}\right)(x,D)(u-u_\mu)\right\|_{H^{s,\psi}} \\
&\leqslant \frac{1}{\nu}C_s\|u_\mu\|_{H^{2m+s,\psi}} + \left\|\left(q-q^{(\nu)}\right)(x,D)(u-u_\mu)\right\|_{H^{s,\psi}}.
\end{aligned}$$

As $u \in H^{m+s,\psi}$ and $u_\mu \in H^{2m+s,\psi}$ we find that $u - u_\mu \in H^{m+s,\psi}$. Thus it follows from Thereom 3.8.B that

$$\left\|q(x,D)u - q^{(\nu)}(x,D)u\right\|_{H^{s,\psi}} \leqslant \frac{1}{\nu}C_s\|u_\mu\|_{H^{2m+s,\psi}} + C'\|u-u_\mu\|_{H^{m+s,\psi}},$$

where C' is independent of ν. Taking first the limit $\nu \to \infty$ we find

$$\limsup_{\nu\to\infty} \left\|q(x,D)u - q^{(\nu)}(x,D)u\right\|_{H^{s,\psi}} \leqslant C'\|u-u_\mu\|_{H^{m+s,\psi}}.$$

For $\mu \to \infty$ we conclude now

$$\limsup_{\nu\to\infty} \left\|q(x,D)u - q^{(\nu)}(x,D)u\right\|_{H^{s,\psi}} \leqslant \lim_{\mu\to\infty} C'\|u-u_\mu\|_{H^{m+s,\psi}} = 0,$$

implying

$$\lim_{\nu\to\infty} \left\|q(x,D)u - q^{(\nu)}(x,D)u\right\|_{H^{s,\psi}} = 0.$$

□

Using Theorem 3.8 we may now prove how to approximate an existing Feller semigroup that is generated by a pseudodifferential operator $-q(x,D)$. We emphasize that we do not show the existence of the Feller semigroup. This has to be shown first, see Section 1.4. We give further remarks concerning this topic at the end of this section. The following theorem can be found in N. Jacob [16].

Theorem 3.10. *Let ψ be a continuous negative definite function satisfying (3.16), (3.17) as well as $\lim_{|\xi|\to\infty}\psi(\xi) = \infty$. Moreover assume $q \in S_\rho^{m,\psi}$, $m > 0$, satisfying (3.18) is given and real-valued. Then for $s > -m$ we have*

$$\frac{K^2}{4}\|u\|^2_{H^{m+s,\psi}} \leqslant \|q(x,D)u\|^2_{H^{s,\psi}} + C\|u\|^2_{H^{m+s-\frac{1}{2},\psi}}$$

for all $u \in H^{m+s,\psi}$.

Lemma 3.11. *Let ψ be a continuous negative definite function satisfying (3.16) and (3.17). Moreover assume $q \in S_\rho^{m,\psi}$, $m > 0$, satisfying (3.18) is given. Then for all $l > \frac{1}{2}$ and $k \in \mathbb{N}$ with $l - km \leqslant 0$ we have for $u \in H^{l,\psi}$*

$$\|u\|_{H^{l,\psi}} \leqslant C_l \left\|q(x,D)^k u\right\|_{L^2} + C_l'\|u\|_{L^2}.$$

Proof. As $l - km > -km$ we may use Theorem 3.10 to find

$$\begin{aligned}\frac{K^2}{4}\|u\|^2_{H^{l,\psi}} &\leqslant \left\|q(x,D)^k u\right\|^2_{H^{l-km,\psi}} + C_1\|u\|^2_{H^{l-\frac{1}{2},\psi}} \\ &\leqslant \left\|q(x,D)^k u\right\|^2_{L^2} + C_1\|u\|^2_{H^{l-\frac{1}{2},\psi}}.\end{aligned} \tag{3.31}$$

As $2l - 1 > 0$ and $\lim_{|\xi|\to\infty}\psi(\xi) = \infty$, we may find for $\varepsilon > 0$ and large $|\xi|$ that

$$\lambda^{2l-1,\psi}(\xi) \leqslant \varepsilon\lambda^{2l,\psi}(\xi) + C(\varepsilon),$$

which leads to

$$\|u\|^2_{H^{l-\frac{1}{2},\psi}} = \int_{\mathbb{R}^n} \lambda^{2l-1,\psi}(\xi)\,|\hat{u}(\xi)|^2\,\mathrm{d}\xi \leqslant \int_{\mathbb{R}^n} \left(\varepsilon\lambda^{2l,\psi}(\xi) + C(\varepsilon)\right)|\hat{u}(\xi)|^2\,\mathrm{d}\xi$$
$$= \varepsilon\|u\|^2_{H^{l,\psi}} + C(\varepsilon)\|u\|^2_{L^2}.$$

Then it follows from (3.31) that

$$\frac{K^2}{4}\|u\|^2_{H^{l,\psi}} \leqslant \left\|q(x,D)^k u\right\|^2_{L^2} + C_1\varepsilon\|u\|^2_{H^{l,\psi}} + C_1C(\varepsilon)\|u\|^2_{L^2},$$

and thus, as $\frac{K^2}{4} - C_1\varepsilon > 0$,

$$\|u\|^2_{H^{l,\psi}} \leqslant \left(\frac{K^2}{4} - C_1\varepsilon\right)^{-1}\left\|q(x,D)^k u\right\|^2_{L^2} + \left(\frac{K^2}{4} - C_1\varepsilon\right)^{-1} C_1C(\varepsilon)\|u\|^2_{L^2}.$$

Hence we have

$$\|u\|_{H^{l,\psi}} \leqslant C_l\left\|q(x,D)^k u\right\|_{L^2} + C_l'\|u\|_{L^2},$$

where $C_l = \left(\frac{K^2}{4} - C_1\varepsilon\right)^{-\frac{1}{2}}$ and $C_l' = \left(\frac{K^2}{4}C_1\varepsilon\right)^{-\frac{1}{2}}\left(C_1C(\varepsilon)\right)^{\frac{1}{2}}$. □

The next Theorem is the main result of this chapter.

Theorem 3.12. *Let ψ be a continuous negative definite function satisfying (3.16) and (3.17). Moreover assume $q \in S^{m,\psi}_{\rho}$, $m > 0$, satisfying (3.18) is given. Assume also that the pseudodifferential operator $-q(x,D)$ extends to the generator of a Feller semigroup. Then if $s > \max\{\frac{n}{r}, \frac{1}{2}\}$, $k \in \mathbb{N}$ such that $2m + s - km \leqslant 0$ we have for $u \in H^{km,\psi}$*

$$\left\|e^{-tq^{(\nu)}(x,D)}u - e^{-tq(x,D)}u\right\|_\infty \leqslant \frac{1}{\nu}C\|u\|_{H^{km,\psi}}.$$

Proof. Let us note that by our assumptions it follows from Theorem 3.5 that $H^{s,\psi} \hookrightarrow C_\infty$ and further we have $H^{km,\psi} \subset H^{s,\psi}$. For $u \in H^{km,\psi}$ we may use Lemma 1.24

$$\left\|e^{-tq^{(\nu)}(x,D)}u - e^{-tq(x,D)}u\right\|_\infty$$
$$\leqslant \int_0^t \left\|e^{-tq^{(\nu)}(x,D)}\right\|_{L^\infty\to L^\infty}\left\|\left(q^{(\nu)}(x,D) - q(x,D)\right)e^{-(t-r)q(x,D)}u\right\|_\infty \mathrm{d}r$$
$$\leqslant \int_0^t C_1\left\|\left(q^{(\nu)}(x,D) - q(x,D)\right)e^{-(t-r)q(x,D)}u\right\|_\infty \mathrm{d}r.$$

Then, using Theorem 3.9 and Lemma 3.12, we find

$$\begin{aligned}
&\left\|\left(q^{(\nu)}(x,D)-q(x,D)\right)e^{(t-r)q(x,D)}u\right\|_{\infty}\\
&\leqslant C_2\left\|\left(q^{\nu}(x,D)-q(x,D)\right)e^{-(t-r)q(x,D)}u\right\|_{H^{s,\psi}}\\
&\leqslant C_2\frac{1}{\nu}C_3\left\|e^{-(t-r)q(x,D)}u\right\|_{H^{2m+s,\psi}}\\
&\leqslant \frac{1}{\nu}C_2C_3\left(\left\|q(x,D)^k e^{-(t-r)q(x,D)}u\right\|_{L^2}+\left\|e^{-(t-r)q(x,D)}u\right\|_{L^2}\right)\\
&=\frac{1}{\nu}C_2C_3\left(\left\|e^{-(t-r)q(x,D)}q(x,D)^k u\right\|_{L^2}+\left\|e^{-(t-r)q(x,D)}u\right\|_{L^2}\right)
\end{aligned}$$

As $H^{km,\psi}\subset L^2$ it follows

$$\begin{aligned}
&\frac{1}{\nu}C_2C_3\left(\left\|e^{-(t-r)q(x,D)}q(x,D)^k u\right\|_{L^2}+\left\|e^{-(t-r)q(x,D)}u\right\|_{L^2}\right)\\
&\leqslant \frac{1}{\nu}C_2C_3\left\|e^{-(t-r)q(x,D)}\right\|_{L^2\to L^2}\left(\|q(x,D)^k u\|_{L^2}+\|u\|_{L^2}\right)\\
&\leqslant \frac{1}{\nu}C_2C_3C_4\left(\left\|q(x,D)^k u\right\|_{L^2}+\|u\|_{L^2}\right)\\
&\leqslant \frac{1}{\nu}C\|u\|_{H^{km,\psi}}
\end{aligned}$$

□

Finally, we conclude

Corollary 3.13. *In the situation of Theorem 3.12 we get for $u\in C_\infty$*

$$\lim_{\nu\to\infty}\left\|e^{-tq^{(\nu)}(x,D)}u-e^{-tq(x,D)}u\right\|_{\infty}=0.$$

Proof. As $H^{km,\psi}\subset C_\infty$ dense, we find for every $u\in C_\infty$ a sequence $(u_\mu)_{\mu\in\mathbb{N}}\subset H^{km,\psi}$ such that $\lim_{\mu\to\infty}\|u-u_\mu\|_\infty=0$. Then it follows from Theorem 3.12

$$\begin{aligned}
&\left\|e^{-tq^{(\nu)}(x,D)}u-e^{-tq(x,D)}u\right\|_{\infty}\\
&=\Big\|e^{-tq^{(\nu)}(x,D)}u-e^{-tq^{(\nu)}(x,D)}u_\mu+e^{-tq^{(\nu)}(x,D)}u_\mu-e^{-tq(x,D)}u_\mu\\
&\qquad+e^{-tq(x,D)}u_\mu-e^{-tq(x,D)}u\Big\|_{\infty}\\
&\leqslant\left\|e^{-tq^{(\nu)}(x,D)}u-e^{-tq^{(\nu)}(x,D)}u_\mu\right\|_{\infty}+\left\|e^{-tq^{(\nu)}(x,D)}u_\mu-e^{-tq(x,D)}u_\mu\right\|_{\infty}\\
&\qquad+\left\|e^{-tq(x,D)}u_\mu-e^{-tq(x,D)}u\right\|_{\infty}\\
&\leqslant\left\|e^{-tq^{(\nu)}(x,D)}\right\|_{L^\infty\to L^\infty}\|u-u_\mu\|_\infty+\frac{1}{\nu}C\|u_\mu\|_{H^{km,\psi}}\\
&\qquad+\left\|e^{-tq(x,D)}\right\|_{L^\infty\to L^\infty}\|u-u_\mu\|_\infty\\
&\leqslant\|u-u_\mu\|_\infty+\frac{1}{\nu}C\|u_\mu\|_{H^{km,\psi}}+\|u-u_\mu\|_\infty.
\end{aligned}$$

When taking the limit we get for $\mu \in \mathbb{N}$ fixed

$$\limsup_{\nu\to\infty} \left\| e^{-tq^{(\nu)}(x,D)}u - e^{-tq(x,D)}u \right\|_\infty \leqslant 2\|u - u_\mu\|_\infty.$$

We conclude

$$\lim_{\nu\to\infty} \left\| e^{-tq^{(\nu)}(x,D)}u - e^{-tq(x,D)}u \right\|_\infty \leqslant \lim_{\mu\to\infty} 2\|u - u_\mu\|_\infty = 0.$$

□

Going back to Section 3.1, in particular (3.12), we have now suceeded in showing that a Feller semigroup $(T_t)_{t\geqslant 0}$ that has as generator an extension of $-q(x,D)$, is given by

$$T_t u = \lim_{\nu\to\infty} T_t^{(\nu)} u = \lim_{\nu\to\infty} e^{-tq^{(\nu)}(x,D)}u,$$

for $u \in C_\infty$. In contrary to (3.10) it is easy to write down the operator $q^{(\nu)}(x,D)$, compare (3.11). One last remark we want to make is that our approximation only works if we already know that the Feller semigroup exists. In order to use our approximation to prove also the existence of the semigroup $(T_t)_{t\geqslant 0}$ with generator $-q(x,D)$ one would need to show that

$$\lim_{\nu\to\infty} \left\| e^{-tq^{(\nu)}(x,D)}u - e^{-tq^{(\eta)}(x,D)}u \right\|_\infty = 0$$

for $u \in C_\infty$, but the fact that $q^{(\nu)}(x,D)$ and $q^{(\eta)}(x,D)$ do not commutate makes things difficult.

Index of Notation

Geometry of $\mathbb{R}^n$ and Multiindices

For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $|x| = (x_1^2 + \dots + x_n^2)^{\frac{1}{2}}$, $B_R = \{x \in \mathbb{R}^n\,;\ |x| \leqslant R\}$, and for $\xi = (\xi_1, \dots, x_n) \in \mathbb{R}^n$, $\langle x, \xi \rangle = x_1\xi_1 + \dots + x_n\xi_n$; for $a, b \in \mathbb{R}$, (a, b) open interval in $\mathbb{R}$, $[a, b]$ closed interval in $\mathbb{R}$;
For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, $\alpha! = (\alpha_1!) \cdots (\alpha_n!)$, $\alpha \leqslant \beta \Leftrightarrow \alpha_j \leqslant \beta_j$ for all j, $\binom{\alpha}{\beta} = \frac{\alpha!}{(\alpha-\beta)!\beta!}$ if $\beta \leqslant \alpha$, $\binom{\alpha}{\beta} = 0$ otherwise.

Spaces of functions defined on $\mathbb{R}^n$

$\mathcal{S} := \left\{u \in C^\infty\,;\ |u|_k := \max_{|\alpha+\beta| \leqslant k} \sup_{x \in \mathbb{R}^n} \left|x^\alpha \partial^\beta u(x)\right| < \infty \text{ for all } k \in \mathbb{N}_0\right\}$

$C_0^\infty := \{u \in C^\infty\,;\ \operatorname{supp}(u) \text{ is compact}\}$

$C_\infty^m := \left\{u \in C^m\,;\ \text{for all } \varepsilon \text{ there exists a compact set } K \subset \mathbb{R}^n \text{ such that } |\partial^\alpha u(x)| < \varepsilon \text{ if } x \in K^c \text{ for } |\alpha| \leqslant m\right\}$

$C_b^m := \{u \in C^m\,;\ \partial^\alpha u \text{ is continuous and bounded for all } |\alpha| \leqslant m\}$

$L^p :=$ space of measurable functions u such that the norm

$$\|u\|_{L^p} := \left(\int_{\mathbb{R}^n} |u(x)|^p\right)^{\frac{1}{p}}$$

if $1 \leqslant p < \infty$ or

$$\|u\|_{L^\infty} = \|u\|_\infty = \sup_{x \in \mathbb{R}^n} |u(x)|$$

is finite. Furthermore we use the notation

$$(u, v) = \int_{\mathbb{R}^n} u(x)\overline{v(x)}\,\mathrm{d}x$$

if $u\overline{v} \in L^1$.

Spaces of distributions defined on $\mathbb{R}^n$

$\mathcal{S}'$: space of tempered distributions, i.e., space of semi-linear forms on $\mathcal{S} \ni \varphi \mapsto (u, \varphi) \in \mathbb{C}$ such that

$$|(u,\varphi)| \leqslant C|\varphi|_N$$

for $C \geqslant 0$ and $N \in \mathbb{N}_0$.

$H^{s,\psi} := \left\{ u \in \mathcal{S}' \,;\, \|u\|_{H^{s,\psi}} := \left\| \left(1+\psi(\cdot)\right)^{\frac{s}{2}} \widehat{u} \right\|_{L^2} < \infty \right\}$

Amplitudes, oscillatory integrals and pseudo-differential operators

$A^m := \Big\{ a \in C^\infty(\mathbb{R}^{2n})\,;\, |||a|||_k = \max_{|\alpha+\beta|\leqslant k} \sup_{y,\eta\in\mathbb{R}^n} \Big| \left(1+|y|^2\right)^{-\frac{m}{2}} \left(1+|\eta|^2\right)^{-\frac{m}{2}}$
$\partial_y^\alpha \partial_\eta^\beta a(y,\eta) \Big| < \infty \text{ for all } k \in \mathbb{N}_0 \Big\}$

$\mathrm{Os}-\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\langle y,\eta\rangle} a(y,\eta)\,\mathrm{d}y\,\mathrm{d}\eta := \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\langle y,\eta\rangle} a(y,\eta)\chi(\varepsilon y, \varepsilon\eta)\,\mathrm{d}y\,\mathrm{d}\eta$
where $\chi \in \mathcal{S}(\mathbb{R}^{2n})$ such that $\chi(0,0) = 1$.

$\Lambda := \{\psi : \mathbb{R}^n \to \mathbb{R}^n \,;\, \psi \text{ continuous negative definite such that (1.18) holds}\}$

$S_0^{m,\psi} := \Big\{ q \in C^\infty(\mathbb{R}^{2n})\,;\, \left|\partial_x^\alpha \partial_\xi^\beta q(x,\xi)\right| \leqslant C_{\alpha\beta}\left(1+\psi(\xi)\right)^{\frac{m}{2}} \text{ for } \psi \in \Lambda,$
$\alpha,\beta \in \mathbb{N}_0^n \text{ and } x,\xi \in \mathbb{R}^n \Big\}$

$S_\rho^{m,\psi} := \Big\{ q \in S_0^{m,\psi}\,;\, \left|\partial_x^\alpha \partial_\xi^\beta q(x,\xi)\right| \leqslant C_{\alpha\beta}\left(1+\psi(\xi)\right)^{\frac{m-\rho(|\beta|)}{2}} \text{ for } \psi \in \Lambda,$
$\alpha,\beta \in \mathbb{N}_0^n \text{ and } x,\xi \in \mathbb{R}^n \Big\}$

$S_0^{\infty,\psi} := \bigcup_{m\in\mathbb{R}} S_0^{m,\psi}\,;\, S_0^{-\infty,\psi} := \bigcap_{m\in\mathbb{R}} S_0^{m,\psi}$

$q_1^*(x,\xi) = (2\pi)^{-n}\,\mathrm{Os}-\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle y,\eta\rangle} \overline{q_1(x-y,\xi-\eta)}\,\mathrm{d}y\,\mathrm{d}\eta$

$(q_1\#q_2)(x,\xi) = (2\pi)^{-n}\,\mathrm{Os}-\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle y,\eta\rangle} q_1(x,\xi-\eta) q_2(x-y,\xi)\,\mathrm{d}y\,\mathrm{d}\eta$

Miscellaneous

$\mathcal{M}_+^b :=$ space of positive bounded measures on $\mathbb{R}^n$

$\mathcal{M}_+^1 :=$ space of positive measures on $\mathbb{R}^n$ with total mass 1

$\|\mu\| := \mu(\mathbb{R}^n)$ for $\mu \in \mathcal{M}_+^b$

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