Quantum spaces arising from weighted circle actions and their non-commutative geometry.

Fairfax, Simon A

Quantum Spaces Arising from Weighted Circle Actions and their Non-Commutative Geometry

Simon A. Fairfax

Submitted to Swansea University in fulfilment of the requirements for the Degree of Doctor of Philosophy

September 2013

Declaration

This work has not previously been accepted in substance for any degree and is not being concurrently submitted in candidature for any degree.

Signed . .. (candidate)

Date ....02/08/2013.............

Statement 1

This thesis is the result of my own investigations, except where otherwise stated. Other sources are acknowledged by explicit references. A bibliography is appended.

Signed ... (candidate)

Date ....02/08/2013................

Statement 2

I hereby give consent for my thesis, if accepted, to be available for photocopying and inter-library loan, and for the title and summary to be made available to outside organisations.

Signed . .. (candidate)

Date ....02/08/2013................

Acknowledgements

I would like to express my gratitude to my supervisor Professor Tomasz Brzezinski for his help and encouragement through this project. I would also like to thank EPSRC for supporting my research and the Department of Mathematics of Swansea University for providing me the opportunity for research. I would also like to thank the examiners Ulrich Kräehmer and Jeffrey Giansiracusa for all their work during the examination process and their useful feedback.

Introduction

The purpose of this thesis is to investigate weighted actions of the circle group $U(1)$ on known quantum spaces. By introducing suitable weights we are able to construct new unexplored quantum spaces which contain the known quantum spaces in the unweighted case. Once we are able to describe the algebraic structure of these new quantum spaces we investigate their quantum geometry.

This thesis is split into two main parts with an outlook of open problems attached; the first consists of introductory material, motivation and an overview of quantum groups and their non-commutative geometry. The second part contains the results from research into quantum weighted projective spaces, in particular describes quantum weighted projective spaces, quantum weighted real projective spaces and quantum weighted Heegaard spaces; see [5], [6] and [7]. Finally, some open problems are discussed, firstly the existence of a differential calculus over the quantum weighted projective spaces. Secondly, the description of higher dimensional quantum weighted projective spaces. One possible approach for interpreting these spaces on the C^*-algebra level is by graph algebra theory; the ideas are discussed briefly in the appendices of this thesis. An outline of each chapter is given as follows.

Part I An overview of quantum groups and non-commutative geometry.

Chapter 1 Essentially quantum groups are non-commutative (Hopf) algebraic structures, hence the purpose of this chapter to describe algebraic structures with an emphasis on the theory used to developed quantum groups in later chapters. The main focus is on comodule algebras over Hopf algebras and C^*-algebras over $\mathbb{C}$ and their K-theory.

Chapter 2 Motivation for quantum groups and non-commutative geometry are described and well known examples are given, namely, both even and odd dimensional quantum spheres and Podleś quantum 2-spheres. Next our attention is turned to non-commutative geometry, motivated by classical geometry, definitions for non-commutative principal and associated bundles are set out. Non-commutative principal bundles are identified with principal comodule algebras and the geometrical importance of Hopf-Galois extensions in made. The constructions of Fredholm modules and the Chern character over an algebra are described. The differential calculi for an algebra are mentioned. Finally, the definition of a connection is given and it is shown that a strong connection is equivalent to the principality of a comodule algebra, providing a highly useful tool when performing calculations.

Part II The main results of the research into weighted circle actions on quantum algebras.

Chapter 3 Firstly, quantum weighted projective spaces $\mathcal{O}(\mathbb{WP}_q(l_0,...,l_n))$ are introduced, for coprime weights $l_0,...,l_n$, as the subalgebra of coinvariant elements of quantum spheres via a suitable $U(1)$ action, or equivalently a $\mathcal{O}(U(1))$-coaction. We concentrate on the quantum weighted projective lines, i.e. on the case $n=1$. For a pair of coprime positive integers $l_0=k, l_1=l$, we give the presentation of $\mathcal{O}(\mathbb{WP}_q(k,l))$ in terms of generators and relations and classify all irreducible representations of $\mathcal{O}(\mathbb{WP}_q(k,l))$ (up to unitary equivalence). We prove that all infinite dimensional irreducible representations are faithful. We then proceed to analyse the structure of $\mathcal{O}(\mathbb{WP}_q(k,l))$ as coinvariant subalgebras.

We prove that $\mathcal{O}(S_q^3)$ is a Hopf-Galois $\mathbb{C}[u,u^*]$-extension of $\mathcal{O}(\mathbb{WP}_q(k,l))$ or $\mathcal{O}(S_q^3)$ is a *principal* $\mathbb{C}[u,u^*]$-*comodule algebra* with coaction $\varrho_{k,l}$ if and only if $k=l=1$. This is in perfect agreement with the classical situation where it is known that the teardrop manifolds are not global quotients of the 3-sphere by a free action. On the other hand, we prove that in the case $k=1$, $\mathcal{O}(\mathbb{WP}_q(1,l))$ is a coinvariant subalgebra (or a base) of a principal $\mathbb{C}[u,u^*]$-comodule algebra that can be identified with the coordinate algebra of the quantum lens space $\mathcal{O}(L_q(l;1,l))$. We explicitly construct a suitable strong connection on $\mathcal{O}(L_q(l;1,l))$ and show that $\mathcal{O}(L_q(l;1,l))$ is not a cleft principal $\mathbb{C}[u,u^*]$-comodule algebra. Quantum weighted projectives spaces are classified as generalised Weyl algebras and it is shown that their global dimension is one when $k=1$ and infinite otherwise, reinforcing the importance of this special case.

Next construction of Fredholm modules and associated cyclic cycles or Chern characters τ_s on $\mathcal{O}(\mathbb{WP}_q(k,l))$ are presented. Using the explicit description of strong connections in $\mathcal{O}(L_q(l;1,l))$ we calculate a part of the Chern-Galois character. Finally we evaluate τ_s at the computed part of the Chern-Galois character and show that the results are different from zero. From this we conclude that the finitely generated projective $\mathcal{O}(\mathbb{WP}_q(1,l))$-module $\mathcal{L}[1]$ (associated to $\mathcal{O}(L_q(l;1,l))$) is not free, thus the principal $\mathbb{C}[u,u^*]$-comodule algebra $\mathcal{O}(L_q(l;1,l))$ is not cleft.

Finally, we construct C^*-algebras $C(\mathbb{WP}_q(k,l))$ of continuous functions on the quantum weighted projective lines and identify them as direct sums of compact operators on (separable) Hilbert spaces with adjoined identity. Through this identification we immediately deduce the K-groups of $C(\mathbb{WP}_q(k,l))$.

Chapter 4 The ideas used in the teardrop case are extended to prolonged quantum spheres $\mathcal{O}(\Sigma_q^3)$. In this case we get quantum real weighted projective spaces; the results here differ significantly from the teardrop case. Quantum real projective spaces split into two unidentical cases, depending on whether l is a even positive integer, $\mathcal{O}(\mathbb{RP}_q(l;-))$, or l an odd positive integer, $\mathcal{O}(\mathbb{RP}_q(l;+))$, each analysed in detail. The algebras $\mathcal{O}(\mathbb{RP}_q(l;\pm))$ are identified in [10] as fixed points of weighted circle actions on the coordinate algebra $\mathcal{O}(\Sigma_q^3)$ of a non-orientable quantum Seifert manifold described in [12] . As in the quantum weighted projectives case, we fully describe the algebraic structures of $\mathcal{O}(\mathbb{RP}_q(l;\pm))$ and classify all infinite dimensional representations. Furthermore, we construct quantum $U(1)$-principal bundles over the corresponding quantum spaces $\mathcal{O}(\mathbb{RP}_q(l;\pm))$ and describe associated line bundles. We show that the principal comodule algebra over $\mathcal{O}(\mathbb{RP}_q(l;-))$ is non-trivial while over $\mathcal{O}(\mathbb{RP}_q(l;+))$ turns out to be trivial (this means that all associated bundles are trivial, hence we do not mention them in the text). We also prove $\mathcal{O}(\mathbb{RP}_q(l;\pm))$ can be also understood as quotients of $\mathcal{O}(\Sigma_q^3)$ by almost free S^1-actions. Next we classify $\mathcal{O}(\mathbb{RP}_q^2(l;+))$ as a generalised Weyl algebra and comment that the negative case do not appear to fit this picture. Finally, we construct Fredholm modules and associated Chern characters.

Chapter 5 In this chapter quantum Heegaard spaces are considered from a weighting perspective. Like quantum real weighted projective spaces the quantum weighted Heegaard spaces split into two cases, however there are similiarities when considering quantum principal bundles. Firstly, we equip the coordinate algebra of the Heegaard 3-sphere $\mathcal{O}(S^3_{pq\theta})$ with $\mathbb{Z}$-gradings determined for a pair of coprime integers k,l, by setting $\deg(a) = k$, $\deg(b) = l$, where a and b are generators of the $*$-algebra $\mathcal{O}(S^3_{pq\theta})$, or, equivalently, with the weighted coaction of $\mathcal{O}(U(1))$, and study the zero degree algebras $\mathcal{O}(S_{pq}(k,l))$. These split into two cases, one in which both k and l are positive, and one in which k is positive and l is negative. We use the notation $\mathcal{O}(S_{pq}(k,l^{\pm}))$ to distinguish these cases. We list bounded irreducible $*$-representations of these algebras and identify $\mathcal{O}(S_{pq}(k,l^{\pm}))$ as the generalised Weyl algebras.

The subalgebras of $\mathcal{O}(S^3_{pq\theta})$ which admit principal $\mathcal{O}(U(1))$-coactions that fix $\mathcal{O}(S_{pq}(k,l^{\pm}))$ are identified. In the case $k = |l| = 1$ these coincide with $\mathcal{O}(S^3_{pq\theta})$. Furthermore it is shown that these principal comodule algebras are non-trivial and strong connections on them are constructed. It is shown that $\mathcal{O}(S_{pq}(k,l^{\pm}))$ are generalised Weyl algebras. Next we deal with the noncommutative geometric aspects of $\mathcal{O}(S_{pq}(k,l^{\pm}))$. More concretely, we construct Fredholm modules over $\mathcal{O}(S_{pq}(k,l^{\pm}))$ and calculate Chern numbers of line bundles associated to principal comodule algebras constructed. Finally, we study algebras of continuous functions on quantum weighted Heegaard spheres, identify them with pullbacks of Toeplitz algebras and calculate their K-groups.

Outlook

Chapter 6 An approach for describing a covariant differential calculus over the quantum teardrop space $\mathcal{O}(\mathbb{WP}_q(k,l))$ is described. It is not clear yet whether such a space exists due to the singularity in the classical case. However, we noted moving into the quantum setting the singularity is resolved in the case $k = 1$. The approach taken here involves restricting the well known covariant calculus on $\mathcal{O}(S^3_q)$ to the generators of $\mathcal{O}(\mathbb{WP}_q(k,l)) \subset \mathcal{O}(S^3_q)$.

Chapter 7 Higher dimensional quantum weighted projective spaces are briefly considered along with the inherent problem of managing these spaces given the large number of generators on the algebraic level. One possible approach for understanding these spaces on the C^*-algebra level is through graph algebras; the details are described in the next appendix.

Chapter 8 The central ideas in relation to graph C^*-algebras are introduced and basic examples described. The graph C^*-algebras of the continous functions on the quantum spheres and lens spaces are described in detail since these are the key spaces used in the analysis though this thesis. Finally, a possible connection between higher dimension quantum weighted projective spaces and graph algebras is made.

Contents

Part I

Review of algebraic and geometric concepts and moving from classical to quantum spaces

Chapter 1

Preliminaries

The aim of this chapter is to set out the relevant theory associated to algebraic structures. The content contained here will be the algebraic framework in which we work throughout. Firstly the concept of an algebra over a field is defined by introducing a concept of multiplication over a vector space satisfying certain properties. This developes to the notion of a coalgebra and next a Hopf algebra. Actions and coactions are then discussed giving rise to module and comodule algebras. Next C^*-algebras and basic properties are set-out and this leads nicely into algebraic K-theory. Basic examples are introduced to clarify these ideas. The material contained in this section can be found in; [33], [9], [17], [39].

1.1 Types of algebraic structures with basic properties

The most basic types of structures are algebras over a field and their dual coalgebras.

1.1.1 Algebra and coalgebra structures

Definition 1.1.1. (An algebra) Given a vector space A over a field k, then we say A is an *algebra* if there exists a map $m_A : A \otimes A \to A$ given by $m_A(a_1 \otimes a_2) = a_1a_2$ and an element $1_A \in A$ such that

(i) A is associative, i.e., $a_1(a_2a_3) = (a_1a_2)a_3$ for all $a_1, a_2, a_3 \in A$, and

(ii) A is unital, i.e. $1_A a = a1_A = a$ for all $a \in A$.

Since multiplication and the unit can be thought of as maps, an algebraic structure can be expressed in terms of commutative diagrams. Let $1_A : k \to A$ be the k-linear map defined by $1_A(\alpha) = \alpha 1_A$, now $1_A(1)a = a1_A(1) = a$ which implies $1_A(1)$ is the unit element in A. Now the associativity and unital conditions are equivalent to the commutativity of the following diagrams,

$$\begin{tikzcd} A\otimes A\otimes A \arrow[r, "id\otimes m_A"] \arrow[d, "m_A\otimes id"'] & A\otimes A \arrow[d, "m_A"] \\ A\otimes A \arrow[r, "m_A"'] & A, \end{tikzcd} \qquad \begin{tikzcd} A\cong k\otimes A\cong A\otimes k \arrow[r, "id_A\otimes 1_A"] \arrow[d, "1_A\otimes id_A"'] \arrow[dr, "id_A"] & A\otimes A \arrow[d, "m_A"] \\ A\otimes A \arrow[r, "m_A"'] & A, \end{tikzcd} \tag{1.1}$$

and reversing the direction of each of the arrows leads to the definition of the dual concept.

Definition 1.1.2. (Coalgebra) Given a vector space C over a field k, then we say that C is a *coalgebra* if there exists maps $\Delta_C : C \to C\otimes C$ and $\epsilon_C : C \to k$, called *comultiplication* and *counit* rendering the following commutative diagrams

$$\begin{tikzcd} C \arrow[r, "\Delta_C"] \arrow[d, "\Delta_C"'] & C\otimes C \arrow[d, "\Delta_C\otimes id_C"] \\ C\otimes C \arrow[r, "id_C\otimes\Delta_C"'] & C\otimes C\otimes C, \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "\Delta_C"] \arrow[d, "\Delta_C"'] \arrow[dr, "id_C"] & C\otimes C \arrow[d, "id_C\otimes\epsilon_C"] \\ C\otimes C \arrow[r, "\epsilon_C\otimes id_C"'] & C. \end{tikzcd} \tag{1.2}$$

The commutativity of the diagram on the left, namely $(\Delta_C\otimes id_C)\circ\Delta_C = (id_C\otimes\Delta_C)\circ\Delta_C$ is known as the coassociativity property and on the right, namely $(\epsilon_C\otimes id_C)\circ\Delta_C = (id_C\otimes\epsilon_C)\circ\Delta_C = id_C$, the counital property.

Notation In order to assist in performing calculations with the comultiplication map we need some notation. We use the Heyneman and Sweedler shorthand notation, for $c \in C$ we write

$$\Delta_C(c) = \sum_{i=1}^{n} c^i_{(1)}\otimes c^i_{(2)} = \sum c_{(1)}\otimes c_{(2)} = c_{(1)}\otimes c_{(2)},$$

hence omit summation and indices.

1.1.2 Hopf algebras

A Hopf algebra over a field k contains an algebra and a coalgebra structure which are compatible with each other, furthermore there exist a map called the antipode which behaves in a similar way to the inverse function for a group. Firstly bialgebras are defined.

Definition 1.1.3. (Bialgebra) A vector space H is called a *bialgebra* if

(i) H is an algebra with multiplication m_H and unit 1_H,

(ii) H is a coalgebra with comultiplication Δ_H and counit ϵ_H, and

(iii) Δ_H and ϵ_H and algebra maps, i.e., $\Delta_H(ab) = \Delta_H(a)\Delta_H(b) = a_{(1)}b_{(1)}\otimes a_{(2)}b_{(2)}$ where $H\otimes H$ is viewed as an algebra with multiplication given by $(h_1\otimes h_2)(g_1\otimes g_2) = h_1g_1\otimes h_2g_2$ and $\epsilon_H(ab) = \epsilon_H(a)\epsilon_H(b)$ and $\epsilon_H(1) = 1$.

Within the bialgebra set-up the compatibility of the algebra and coalgebra structure is contained in the sense that the comultiplication map Δ_H and the counit map ϵ_H are algebra maps. In fact, this property is equivalent to m_H and 1_H being coalgebra maps.

Definition 1.1.4. (Hopf algebra) A *Hopf algebra* H is a bialgebra with a map $S : H \to H$ called the *antipode*, satisfying

$$m_H \circ (id \otimes S) \circ (\Delta_H(h)) = m_H \circ (S \otimes id) \circ (\Delta_H(h)) = \epsilon_H(h)1_H.$$

The antipode can be expressed using the Sweedler notation as $h_{(1)}S(h_{(2)}) = S(h_{(1)})h_{(2)} = \epsilon_H(h)1_H$, hence we can see that the map S behaves in a similar way to an inverse function on H. The properties of H are summarised in the following proposition.

Proposition 1.1.5. *Let H be a Hopf algebra with antipode $S : H \to H$. The following properties hold:*

$$S(gh) = S(h)S(g) \text{ for each } g, h \in H \text{ and } S(1_H) = 1_H, \text{ and} \tag{1.3a}$$

$$\Delta_H(S(h)) = S(h_{(2)}) \otimes S(h_{(1)}) \text{ for each } h \in H \text{ and } \epsilon_H(S(h)) = \epsilon_H(h). \tag{1.3b}$$

Equation (1.3a) tells us the antipode is an anti-algebra map, and Equation (1.3b) tells us the antipode is an anti-coalgebra map.

Definition 1.1.6. A Hopf algebra is *commutative* if it is commutative as an algebra. It is *cocommutative* if it cocommutative as a coalgebra, i.e. if $\tau \circ \Delta = \Delta$ (the arrow-reversal version of commutativity), where τ is the map which flips the tensor product.

The most simple types of Hopf algebras are as follows.

Group Hopf Algebra

Let G be a group. The group Hopf algebra $H = kG$ (k any field) is the vector space with basis G with the following Hopf algebra structure on basis elements,

multiplication: $m_H(g_1 \otimes g_2) = g_1 g_2$ (group product), unit: $1_H = e_G$ (group identity),

comultiplication: $\Delta_H(g) = g \otimes g$, counit: $\epsilon_H(g) = 1_H$, antipode: $S_H(g) = g^{-1}$,

and extended to the whole of $H = kG$ in a linear way. Multiplication in H is derived from the group hence associativity and the unit property follows. Coassociativity follows for the whole of kG since it follows trivially for the basis elements, similarly for the counit property. The maps Δ_H and ϵ_H are multiplication maps since:

$$\Delta_H(gh) = gh \otimes gh = (g \otimes g)(h \otimes h) = \Delta_H(g)\Delta_H(h),$$

and

$$\epsilon_H(gh) = 1 = \epsilon_H(g)\epsilon_H(h).$$

Note that if G is non-commutative then so is kG and that kG is always cocommutative using this structure.

1.2 Module and comodule algebras

The next type of algebraic structures that we are interested in are module algebras and their dual comodule algebras. These play important roles in non-commutative geometry when constructing algebraic objects equivalent to topological bundles. Firstly we consider actions of algebras on vector spaces then go on to identify their dual coactions by reversing the direction of the arrows in the commutative diagrams describing the algebraic properties of actions. We then built on this concept by incorporating the algebraic structures identified in the previous section, to describe actions of Hopf algebras on algebras. Again the dual concepts within this setting are considered.

1.2.1 Actions and coactions of algebras on vector spaces

Definition 1.2.1. (Left action) Let V be a vector space and A an algebra. A *left A action* on V is a linear map $\triangleright : A \otimes V \to V$ which satisfies the following properties

(i) associativity, $(a_1 a_2) \triangleright v = a_1 \triangleright (a_2 \triangleright v)$ for all $a_1, a_2 \in A, v \in V$, and

(ii) identity, $1_A \triangleright v = v$ for all $v \in V$.

These properties can be written as commutative diagrams as,

$$\begin{tikzcd} A \otimes A \otimes V \arrow[r, "m_A \otimes id_V"] \arrow[d, "id_A \otimes \triangleright"'] & A \otimes V \arrow[d, "\triangleright"] \\ A \otimes V \arrow[r, "\triangleright"'] & V, \end{tikzcd} \qquad \begin{tikzcd} V \cong k \otimes V \arrow[r, "1_A \otimes id_V"] \arrow[dr, "id_V"'] & A \otimes V \arrow[d, "\triangleright"] \\ & V. \end{tikzcd} \tag{1.4}$$

Using the arrow-reversing notion it is clear that a coaction is defined in the following way.

Definition 1.2.2. (Right coaction) Let W be a vector space and C a coalgebra with comultiplication Δ_C and counit ϵ_C. A *right C coaction* is a linear map $\rho : W \to W \otimes C$ which satisfies the following properties

(i) coassociativity, $(id_W \otimes \Delta_C) \circ \rho = (\rho \otimes id_C) \circ \rho$, and

(ii) coidentity, $(id_W \otimes \epsilon_C) \circ \rho = id_W$.

These properties can be expressed as commutative diagrams in the following form,

$$\begin{tikzcd} W \arrow[r, "\rho"] \arrow[d, "\rho"'] & W \otimes C \arrow[d, "\rho \otimes id_C"] \\ W \otimes C \arrow[r, "id_W \otimes \Delta_C"'] & W \otimes C \otimes C, \end{tikzcd} \qquad \begin{tikzcd} W \arrow[r, "\rho"] \arrow[dr, "id_W"'] & W \otimes C \arrow[d, "id_W \otimes \epsilon_C"] \\ & W. \end{tikzcd} \tag{1.5}$$

Right actions and left coactions are defined in a similar way. Since we are using a coalgebra structure we require notation that is consistent with the Sweedler notation for the coaction to assist when performing calculations. We use the shorthand $\rho(w) = w_{(0)} \otimes w_{(1)}$, where the summation and index are implicit, this is consistent with the Sweedler notation discussed above.

1.2.2 Comodule algebras over Hopf algebras

We have seen how an algebra acts on a vector space, next we extend this idea to a Hopf algebra acting on an algebra.

Definition 1.2.3. (Left H-module algebra) Let A be an algebra and H a Hopf algebra. A is said to be a *left H-module algebra* if there is a linear map $\triangleright : H \otimes A \to A$ called the left action of H on A such that,

(i) H acts on A as a vector space (see Definition 1.2.1).

(ii) m_A commutes with the action $\triangleright$, that is, $h \triangleright (ab) = (h_{(1)} \triangleright a)(h_{(2)} \triangleright b)$.

(iii) the unit 1_A commutes with the action $\triangleright$, that is, $h \triangleright 1_A = \epsilon(h)1_A$.

Definition 1.2.4. (Right H-comodule algebra) Let B be an algebra and H a Hopf Algebra. B is said to be a *right H-comodule algebra* if there is a linear map $\rho^B : B \to B \otimes H$ called the right H-coaction on B, such that

(i) H coacts on B as a vector space (see Definition 1.2.2).

(ii) ρ^B commutes with the multiplicative structure, that is,

$$\rho^B \circ m_B = (m_B \otimes id) \circ (id \otimes id \otimes m_H) \circ (id \otimes \tau \otimes id) \circ (\rho^B \otimes \rho^B).$$

Written in Sweedler notation this comes out as $(bb')_{(0)} \otimes (bb')_{(1)} = (b_{(0)}b'_{(0)}) \otimes (b_{(1)}b'_{(1)})$.

(iii) ρ^B commutes with the unital structure, that is, $\rho^B(1_B) = 1_B \otimes 1_H$.

Definition 1.2.5. (Left H-comodule algebra) Let B be an algebra and H a Hopf Algebra. B is said to be a *left H-comodule algebra* if there is a linear map ${}^B\rho : B \to H \otimes B$ called the left H-coaction on B, such that

(i) H coacts on B as a vector space (see Definition 1.2.2).

(ii) ${}^B\rho$ commutes with the multiplicative and unital structure.

1.3 C^*-algebras and their representations

This section sets out the essentials when dealing with C^*-algebras. Basic definitions are introduced along with well known properties. Building C^*-algebras from polynomial algebras using representation spaces is discussed. This is a particularly important process since a large number of spaces we deal with arise as polynomial algebras.

1.3.1 C^*-algebras

Definition 1.3.1. ($*$-algebra) An algebra A over $\mathbb{C}$ is called a *$*$-algebra* if there exist an operation $* : A \to A$, $a \mapsto a^*$ which satisfies the following conditions, for all $a, b \in A$, $\lambda, \mu \in \mathbb{C}$,

(i) $(\lambda a + \mu b)^* = \overline{\lambda} a + \overline{\mu} b$, (ii) $(a^*)^* = a$, (iii) $(ab)^* = b^* a^*$.

The operation $*$ is known as a $*$-operation or an algebra involution.

Definition 1.3.2. (C^*-algebra) A is called a *C^*-algebra* if A is an algebra over $\mathbb{C}$ with norm $\|.\| : A \to \mathbb{C}$, $a \mapsto \|a\|$, and involution $* : A \to A$, $a \mapsto a^*$, such that A is complete with respect to the norm, with the properties that

(i) $\|ab\| \leq \|a\|\|b\|$, for all $a, b \in A$,

(ii) $\|a^* a\| = \|a\|^2$, for all $a \in A$.

Firstly note that $\|a\| = \|a^*\|$, in this case we say the involution is isometric, this can be seen by combining the properties in the definition of a C^*-algebra. Observe that $\|a^* a\| = \|a\|\|a\| \leq \|a^*\|\|a\|$ implying that $\|a\| \leq \|a^*\|$. Therefore $\|a\| \leq \|a^*\| \leq \|(a^*)^*\| = \|a\|$, showing $\|a\| = \|a^*\|$ for all $a \in A$.

Definition 1.3.3. (Sub-C^*-algebra) A non-empty subset $B \subset A$ is called a *sub-C^*-algebra* of A if it is a C^*-algebra with the operations given on A. Namely, it is norm closed and closed under the operations: addition, multiplication, involution and scalar multiplication.

Suppose A is a C^*-algebra and $F \subset A$. The sub-C^*-algebra of A generated by F, written $C^*(F)$, is the smallest sub-C^*-algebra of A that contains F. $C^*(F)$ can be expressed as follows. For each $n \in \mathbb{N}$ put

$$W_n = \{x_1 x_2 ... x_n : x_j \in F \cup F^*, \text{for } j = 1, ..., n\} \quad \text{(words of length } n\text{)}$$

where $F^* = \{x^* : x \in F\}$ and put $W = \bigcup_{n=1}^{\infty} W_n$. The set W is the set of all words in $F \cup F^*$. Using $W = W^*$ and W is closed under multiplication, we see that the linear span of W is a sub-$*$-algebra of A, hence completing the space we arrive at $C^*(F) = \overline{\text{span}(W)}$.

Example 1.3.4. Let $\mathcal{B}(\mathcal{H})$ be the set of all bounded linear operators on a Hilbert space $\mathcal{H}$. $\mathcal{B}(\mathcal{H})$ is a C^*-algebra with addition and multiplication taken in the standard way for function spaces. If $S, T \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \mathbb{C}$, then $S+T$, λS and ST are bounded operators hence contained in $\mathcal{B}(\mathcal{H})$. The norm is defined by $\|S\| = \sup\{\|S(x)\| : \|x\| = 1\}$ and S^* is defined as the operator such that $\langle x, S(y)\rangle = \langle S^*(x), y\rangle$ for all $x, y \in \mathcal{H}$.

Example 1.3.5. The Toeplitz algebra $\mathcal{T}$ is the C^*-algebra generated by the unilateral shift U acting on a separable Hilbert space $\mathcal{H}$ with orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ by $Ue_n = e_{n+1}$. The operator U^* is given by $U^*e_n = e_{n-1}$ for $n \geq 1$ and $U^*(e_0) = 0$.

From commutative C^*-algebras to non-commutative geometry. The GNS-theorem says a commutative C^*-algebra A is isomorphic to the algebra of continuous functions on a compact Hausdorff topological space, say X. Within the theory of non-commutative geometry we extend this concept to non-commutative C^*-algebras. A non-commutative C^*-algebra $\mathcal{A}$ is viewed as the algebra of continuous functions on an object say X_q, called a quantum space. We find that many of the properties of the classical space X can be carried over in a natural way to the quantum space X_q in the theory of non-commutative geometry. This is discussed in detail in chapters 2 and 3.

A corollary to the GNS-theorem is the Galfand-Naimark theorem which states: For each C^*-algebra A there exists a Hilbert space H and an isometric $*$-homomorphism $\phi : A \to \mathcal{B}(H)$. In other words, every C^*-algebra is isomorphic to a sub-C^*-algebra of $\mathcal{B}(H)$. This result is used when we classify representations of a C^*-algebra.

1.3.2 Representations of C^*-algebras

Given a $*$-algebra A we would like to extend this to a C^*-algebra where A is a dense subalgebra of the C^*-algebra. In order to perform this extension we would need to define a norm on A. This is done in the context of representation theory whereby A is represented as a subalgebra of the algebra of bounded operators on a Hilbert space; see Example 1.3.4.

Suppose V is a Hilbert space then the space $\mathrm{End}(V) = \{f : V \to V : f \text{ linear}\}$ is an algebra with multiplication given by composition and unit given by the identity map on V. Suppose that V is a left A-module with action $\triangleright : A \times V \to V$, now we can define the map

$$\pi : A \to \mathrm{End}(V), \quad \pi(a)(v) = a \triangleright v.$$

We see that π is an algebra map since $\pi(1_A)(v) = 1_A \triangleright v = v$ hence $\pi(1_A)$ is the identity map and

$$\begin{aligned}\pi(ab)(v) &= (ab) \triangleright v = a \triangleright (b \triangleright v)\\ &= a \triangleright (\pi(b)(v)) = (\pi(a)\pi(b))(v) \Longrightarrow \pi(ab) = \pi(a)\pi(b).\end{aligned}$$

Furthermore, if we have an algebra map $\pi : A \to \mathrm{End}(V)$ then we can view V as a left A-module with action given by $\bullet : A \times V \to V$, $a \bullet v = \pi(a)(v)$. V is called the representation space of A and the map π is called a representation of A in V. V being a representation space of A is equivalent to V being a left A-module.

Within non-commutative geometry $*$-algebraic structures are typically described as polynominal algebras with a collection of relations between the generators. The general strategy to extend the $*$-algebraic structure to a C^*-algebra involves classifying all representations. The process of determining the representations of an algebra is determined by the relations of the polynominal algebra. The representations allow us to view the elements of the algebra as operators over some Hilbert space. Now the C^*-algebra extension is defined as the completion of the $*$-algebra with respect to the representations.

Example 1.3.6. Consider the group algebra $A = \mathbb{C}\mathbb{Z}$ of the integers $\mathbb{Z}$ over the complex numbers. This algebra is generated by a unitary element u, meaning $u^*u = uu^* = 1$. Hence A contains polynomials in u and u^* with complex coefficients. Suppose $\pi : A \to \mathrm{End}(V)$ is a representation of A with representation space V. Since $uu^* = 1$ we require $\pi(u)\pi(u^*) = I$ where I is the identity operator on V. One possible representation π of A takes the form $\pi(u)e_n = e_{n+1}$ and $\pi(u^*)e_n = e_{n-1}$, for V a Hilbert space with orthonormal basis $\{e_n\}_{n=0}^{\infty}$. In fact, any unitary operator will give a representation for A.

1.4 K-theory

Given a C^*-algebra A then we associate to this a pair of Abelian groups which we denote by $K_0(A)$ and $K_1(A)$ called the K-groups. These K-groups contain information about C^*-algebras, hence knowing the K-groups provides an insight to the algebraic structure. The K-groups are defined in terms of the Grothendieck construction of Abelian groups.

Grothendieck constructions

Given an Abelian semigroup $(S, +)$ we can associate to this an Abelian group. Define the equivalence relation $\sim$ on $S \times S$ by $(x_1, y_1)\sim(x_2, y_2)$ if there exists $z \in S$ such that $x_1 + y_2 + z = x_2 + y_1 + z$. Now the Grothendieck group associated to S, written $G(S)$, is given by the quotient

$$G(S) = S \times S/\sim = \{[(x, y)] : (x, y) \in S \times S\}.$$

The Abelian group $(G(S), +)$ has addition $[(x_1, y_1)] + [(x_2, y_2)] = [(x_1 + x_2, y_1 + y_2)]$, neutral element $[(x, x)]$ and inverse elements given by $-[(x, y)] = [(y, x)]$. The Abelian group axoims are easily verified. For any $y \in S$ the map $\gamma_S : S \to G(S)$, $\gamma_S(x) = [(x+y, x)]$ is called the Grothendieck map. The map is additive and independent of the choice of y, furthermore can be used to describe the Grothendieck group as

$$G(S) = \{\gamma_S(x) - \gamma_S(y) : x, y \in S\}.$$

1.4.1 The K_0-group

The K_0-group is constructed by considering projections in the matrix algebra of a given C^*-algebra. Suppose A is a C^*-algebra, then any element $p \in A$ is called a *projection* if $p^2 = p = p^*$. We write $\mathcal{P}(A)$ for the set of projections in A.

Now define $\mathcal{P}_n(A) = \mathcal{P}(M_n(A))$ the set of projections in the $n \times n$ matrix algebra with entries in A, and write $\mathcal{P}_\infty(A) = \bigcup_{n=1}^{\infty} \mathcal{P}_n(A)$. We can define an equivalence relation on $\mathcal{P}_\infty$ written $\sim_0$ as follows. Suppose $p \in \mathcal{P}_n(A)$, $q \in \mathcal{P}_m(A)$ then $p\sim_0 q$ if $\exists v \in M_{m,n}(A)$ with $p = v^*v$ and $q = vv^*$. Furthermore, we have an addition on $\mathcal{P}_\infty(A)$ as follows

$$\oplus : \mathcal{P}_\infty(A) \times \mathcal{P}_\infty(A) \to \mathcal{P}_\infty(A), \qquad p \oplus q = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \in \mathcal{P}_{m+n}(A),$$

making $\mathcal{P}_\infty(A)$ a semigroup.

Given a C^*-algebra A, then we can construct an Abelian semigroup $\mathcal{D}(A) = \mathcal{P}_\infty(A)/\sim_0$ with elements $[p]_\mathcal{D}$ for each $p \in \mathcal{P}_\infty(A)$, and addition $[p]_\mathcal{D} + [q]_\mathcal{D} = [p \oplus q]_\mathcal{D}$.

Definition 1.4.1. (K_0-group for unital C^*-algebras) Let A be a unital C^*-algebra, and let $(\mathcal{D}(A), +)$ be the Abelian semigroup obtained from taking the quotient of $\mathcal{P}_\infty(A)$ by the equivalence relation $\sim_0$. The K_0-group of A is defined to be the Grothendieck group of $\mathcal{D}(A)$,

$$K_0(A) = G(\mathcal{D}(A)),$$

with Grothendieck map

$$\gamma : \mathcal{D}(A) \to K_0(A), \quad \gamma([p]_\mathcal{D}) = [p]_0.$$

Hence we can describe the group as

$$\begin{aligned} K_0(A) &= \{[p]_0 - [q]_0 : p, q \in \mathcal{P}_\infty(A)\} \\ &= \{[p]_0 - [q]_0 : p, q \in \mathcal{P}_n(A), n \in \mathbb{N}\}, \end{aligned}$$

where addition is defined by $[p]_0 + [q]_0 = [p \oplus q]_0$ with neutral element $0 = [0_A]_0$, 0_A being the zero projection in A.

Remark 1.4.2. Given a finitely generated projective module, we can associate to this a class in the K_0 group as follows. Let R be a ring and recall P is called a finitely generated projective left R-module if

1) $X = \{x_1, ..., x_n\} \subset P$ is a free generating set of P. That is, for any $p \in P$, $\exists r_1, ..., r_n \in R$ such that $p = r_1x_1 + ... + r_nx_n$ and for any linear combination $s_1x_1 + ... + s_nx_n = 0$ there is only one solution $s_1 = ... = s_n = 0$.

2) $R^n \cong P \oplus Q$ for some module Q.

Every finitely generated projective R-module arises from an idempotent element $e \in M_n(R)$, i.e. $e^2 = e \in M_n(R)$. We find the image $e(R^n)$ produces P, the kernel produces the module Q and by taking the direct sum of these modules we obtain R^n. This set-up allows us to associate to a finitely generated projective module P a class in the K_0-group using e.

Example 1.4.3. The K_0-group of the algebra $\mathbb{C}$ is given by the Abelian group $\mathbb{Z}$, that is, $K_0(\mathbb{C}) = \mathbb{Z}$.

Proof. Firstly we calculate $\mathcal{D}(\mathbb{C}) = \mathcal{P}_\infty(\mathbb{C})/\sim_0$, to do this we need to consider the trace map

$$\mathrm{Tr} : M_n(\mathbb{C}) \to \mathbb{C}, \qquad \mathrm{Tr}\begin{pmatrix} \alpha_{11} & \dots & \alpha_{1n} \\ \dots & \dots & \dots \\ \alpha_{n1} & \dots & \alpha_{nn} \end{pmatrix} = \sum_{j=1}^{n} \alpha_{jj}.$$

Now suppose $p, q \in \mathcal{P}_\infty(\mathbb{C})$, say $p \in \mathcal{P}_n$ and $q \in \mathcal{P}_m$. As described above $p \sim_0 q \Longrightarrow \exists v \in M_{m,n}(\mathbb{C})$ such that $p = v^*v$ and $q = vv^*$. Using the basic properties of the trace map, this means that $\mathrm{Tr}(p) = \mathrm{Tr}(v^*v) = \mathrm{Tr}(vv^*) = \mathrm{Tr}(q)$, i.e., p and q have the same trace, which further means the ranks of p and q have the same dimension. The matrix $p \in M_m(\mathbb{C})$ can be represented by $\left[\begin{array}{c|c} I_\alpha & 0 \\ \hline 0 & 0 \end{array}\right] \in M_m(\mathbb{C})$ where $\alpha = \operatorname{rank} p$. Similarly, $q \in M_n(\mathbb{C})$ can be represented by $\left[\begin{array}{c|c} I_\beta & 0 \\ \hline 0 & 0 \end{array}\right] \in M_n(\mathbb{C})$ where $\beta = \operatorname{rank} q$. Suppose p and q have the same rank then $\alpha = \beta$, so $m = n$ and $\mathrm{Tr}(p) = \mathrm{Tr}(q)$. In fact, p and q represent the same linear transformation under different bases and can be written as $p = A^{-1}qA$ with A being the changes of basis matrix. Since we can express $q = AA^*$ and $p = A^*A$ we find $p \sim_0 q$. Hence $\mathcal{D}(\mathbb{C})$ contains the classes of projections where each class consists of all the projections with equal dimension. So, $\mathcal{D}(\mathbb{C}) \cong \mathbb{Z}^+ = \{0, 1, 2, \ldots\}$ with the usual addition.

To calculate the K_0-group we need the property: if $(H, +)$ is an Abelian group and $S \subset H$ is a non-empty subset closed under addition then $(S, +)$ is an Abelian semigroup where $G(S) \cong \{x - y : x, y \in S\}$. We see $(\mathbb{Z}^+, +)$ is an Abelian semigroup as a subset of the Abelian group $(\mathbb{Z}, +)$. So, $K_0(\mathbb{C}) = G(\mathcal{D}(\mathbb{C})) \cong G(\mathbb{Z}^+) \cong \{x - y : x, y \in \mathbb{Z}^+\} = \mathbb{Z}$. □

Example 1.4.4. The K_0-group of the algebra $M_n(\mathbb{C})$ is given by $\mathbb{Z}$, that is, $K_0(M_n(\mathbb{C})) \cong \mathbb{Z}$.

Proof. Let $p \in P_\infty(M_n(\mathbb{C}))$, say $p \in P_m(M_n(\mathbb{C}))$, so $p \in M_{mn}(\mathbb{C})$ such that $p^2 = p = p^*$. Using a similar argument to Example 1.4.3, we find $p \sim_0 q \iff \operatorname{rank}(p) = \operatorname{rank}(q)$ in $P_\infty(M_n(\mathbb{C}))$. Hence $\mathcal{D}(M_n(\mathbb{C}))$ contains the classes of projections where each class contains projections with the same dimension, giving $\mathcal{D}(M_n(\mathbb{C})) = \mathbb{Z}^+$. Now, $K_0(M_n(\mathbb{C})) = G(\mathcal{D}(M_n(\mathbb{C})) = G(\mathbb{Z}^+) = \mathbb{Z}$. □

Example 1.4.5. Other well known examples without proof are

$$K_0(\mathcal{B}(H)) = 0, \quad K_0(\mathcal{K}(H)) = 0, \quad K_0(\mathcal{T}) = \mathbb{Z},$$

where $\mathcal{B}(H)$ are the bounded linear operators on a Hilbert space H, $\mathcal{K}(H)$ the compact operators on H and $\mathcal{T}$ the Toeplitz algebra.

1.4.2 The K_1-group

The K_1-group is constructed by considering unitary elements in the matrix algebra of a given C^*-algebra. Suppose A is a C^*-algebra, we say an element $u \in A$ is *unitary* if $uu^* = u^*u = 1$. We write $\mathcal{U}(A)$ for the set of all unitary elements in A. Similar to above we also write $\mathcal{U}_n(A) = \mathcal{U}(M_n(A))$ and $\mathcal{U}_\infty(A) = \bigcup_{n=1}^\infty \mathcal{U}_n(A)$.

We have an addition on $\mathcal{U}_\infty(A)$, for $u \in \mathcal{U}_n(A)$ $v \in \mathcal{U}_m(A)$, given by

$$\oplus : \mathcal{U}_\infty(A) \times \mathcal{U}_\infty(A) \to \mathcal{U}_\infty(A), \quad u \oplus v = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix},$$

making $\mathcal{U}_\infty(A)$ a semigroup.

We can define an equivalence relation on $\mathcal{U}_\infty(A)$ written $\sim_1$ defined as $u \sim_1 v$ if there exists a natual number $k \geq \max\{m, n\}$ such that $(u \oplus 1_{k-n}) \sim_{\mathrm{h}} (v \oplus 1_{k-m})$ where 1_r is the unit in $M_r(A)$ and $\sim_{\mathrm{h}}$ is the homotopy equivalence relation on $\mathcal{U}_k(A)$ defined as $a \sim_{\mathrm{h}} b$ if there exists a continuous path from a to b, $t \mapsto v(t)$ for $t \in [0, 1]$.

Definition 1.4.6. (K_1-group for unital C^*-algebras) Let A be a unital C^*-algebra then the K_1-group of A is defined as the quotient

$$K_1(A) = \mathcal{U}_\infty(A)/\sim_1 = \{[u]_1 : u \in \mathcal{U}_\infty(A)\},$$

with addition $[u]_1 + [v]_1 = [u \oplus v]_1$, zero element $0 = [1_n]_1$ and inverse elements $-[u]_1 = [u^*]_1$.

Example 1.4.7. $K_1(\mathbb{C}) = K_1(M_n(\mathbb{C})) = 0$, $K_1(\mathcal{B}(H)) = 0$, and $K_1(\mathcal{T}) = 0$.

Chapter 2

Quantum groups and non-commutative geometry

The aim of this chapter is to first describe the motivation behind quantum groups. The idea is to build non-commutative spaces with an inherent underlying geometry in the classical sense, which can be generalised to the whole of the non-commutative space. Naturally, the starting point is to recollect basic geometric ideas, in particular topological bundles, such as fibre bundles, vector bundles and principal bundles (we begin with a review of [2] where these concepts are discussed). By considering the space of functions from a topological space into $\mathbb{C}$ we can build commutative algebraic spaces. In order to describe truly quantum spaces, by which we mean non-commutative spaces, we need to continue building on this idea. It turns out non-commutativity is achieved by considering the algebra of polynomial functions on an affine algebraic variety, where the generators are the coordinate functions from the topological space into $\mathbb{C}$. Now the non-commutativity is attained through the relations of the generators of the algebra of coordinate functions. We refer to the non-commutative algebras as q-deformations of the classical space where q is a parameter, usual a real number in the interval $(0, 1)$ which controls the non-commutativity of the space. Typically, each value of q gives a different algebra and at the limit $q \to 1$ we recover the classical space. With the formulation of quantum groups in mind, we turn our attention to their geometry. The construction of quantum groups involves replacing classical spaces with spaces of functions, naturally their geometry is viewed in a similar way. By taking topological and geometrical concepts and replacing classical spaces by quantum spaces, or non-commutative algebras of coordinate functions, we are able to develop a very natual concept of geometry in the algebraic sense. The main focus in this thesis will be on the quantum version of principal bundles, known as quantum principal bundles. Finally, given constructions for quantum groups and non-commutative geometry we consider their differential calculi. This incorporates the concepts of connections and connection forms in the quantum setting. The application of these ideas will follow in Part II where new examples of quantum spaces are identified.

2.1 Geometric consideration: topological aspects of bundles

We begin by recalling some classical geometry, namely topological bundles; see [2].

2.1.1 Fibre bundles

Definition 2.1.1. A *fibre bundle* is a quadruple (E, π, M, F) where E, M, F are topological spaces, $\pi : E \to M$ is a continuous surjective map satisfying the local triviality condition.

The local triviality condition is satisfied if for each $x \in E$, there is an open neighbourhood $U \subset M$ of $\pi(x)$ such that $\pi^{-1}(U)$ is homeomorphic to the product space $U \times F$, in such a way that π carries over to the projection onto the first factor. That is, the following diagram commutes,

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\phi"] \arrow[d, "\pi"'] & U \times F \arrow[dl, "p_1"] \\ U. & \end{tikzcd}$$

The map p_1 is the natural projection $U \times F \to U$ and $\phi : \pi^{-1}(U) \to U \times F$ is a homeomorphism. It follows that the fibres $\pi^{-1}(m)$ are homeomorphic to F for each $m \in M$

Example 2.1.2. An example of a fibre bundle which is non-trivial is the Möbius strip. It has a circle that runs lengthwise through the centre of the strip as a base B and a line segment running vertically for the fibre F. The line segments are in fact copies of the real line, hence each $\pi^{-1}(m)$ is homeomorphic to $\mathbb{R}$ hence the Möbius strip is a fibre bundle.

2.1.2 Free actions and the principal map

Let X be a topological space which is compact and satisfies the Hausdorff property and G a compact topological group. Suppose there is a right action of G on X given by, $\triangleleft : X \times G \to X$, $x \triangleleft g = xg$.

Definition 2.1.3. An action of G on X is said to be *free* if $xg = x$ for any $x \in X$ implies that $g = e$, the group identity.

With an eye on algebraic formulations of freeness, the principal map $F^G : X \times G \to X \times X$ is defined as $(x, g) \mapsto (x, xg)$. Hence it takes an element (x, g) and produces the action of g on x in the second component and records x in the first.

Proposition 2.1.4. *G acts freely on X if and only if F^G is injective.*

Proof. "$\Longleftarrow$" Suppose the action is free, hence $xg = x$ implies that $g = e$. If $(x, xg) = (x', x'g')$ then $x = x'$ and $xg = xg'$. Applying the action of g'^{-1} to both sides of $xg = xg'$ we get $x(gg'^{-1}) = x$, which implies $gg'^{-1} = e$ so $g = g'$ by the freeness property, hence F^G is injective as required.

"$\Longrightarrow$" Suppose F^G is injective, so $F^G(x, g) = F^G(x', g')$ or $(x, xg) = (x', x'g')$ implies $x = x'$ and $g = g'$. Since $x = xe$ from the properties of the action, if $x = xg$ then $g = e$ from the injectivity property. □

Since G acts on X we can define a quotient space X/G which we label Y. This space is defined to be,

$$Y = X/G := \{[x] : x \in X\}, \qquad \text{where} \qquad [x] = xG = \{xg : g \in G\}.$$

The sets xG are called the *orbits* of the points x. They are defined as the set of elements in X to which x can be moved by the action of elements of G. The set of orbits of X under the action of G forms a partition of X, hence we can define an equivalence relation on X as,

$$x \sim y \Longleftrightarrow \exists g \in G \;\text{ such that }\; xg = y.$$

The equivalence relation is the same as saying x and y are in the same orbit, i.e., $xG = yG$. Given any quotient space, then there is a canonical surjective map defined as,

$$\pi : X \to Y = X/G, \qquad x \mapsto xG = [x],$$

which maps elements in X to their orbits. We define the pull-back along this map π to be the set,

$$X \times_Y X := \{(x, y) \in X \times X : \pi(x) = \pi(y)\}.$$

As described above, the image of the principal map F^G contains elements of X in the first leg and the action of $g \in G$ on x in the second leg. To put it another way, the image records elements of $x \in X$ in the first leg and all the elements in the same orbit as this x in the second leg. Hence we can identify the image of the canonical map as the pull back along π, namely $X \times_Y X$. This is formally proved as a part of the following proposition.

Proposition 2.1.5. *G acts freely on X if and only if the map defined by,*

$$F^G_X : X \times G \to X \times_Y X, \qquad (x, g) \mapsto (x, xg),$$

is bijective.

Proof. First note that the map is well-defined since the elements x and xg are in the same orbit hence map to the same equivalence class under π. Using Proposition 2.1.4 we can deduce that the injectivity of F^G_X is equivalent to the freeness of the action. Hence if we can show that F^G_X is surjective the proof is complete. Take $(x, y) \in X \times_Y X$, this means $\pi(x) = \pi(y)$ which implies x and y are in the same equivalence class, which in turn means they are in the same orbit. We can therefore deduce that $y = xg$ for some $g \in G$. So, $(x, y) = (x, xg) = F^G_X(x, g)$ implies $(x, y) \in \mathrm{Im}F^G_X$. Hence $\mathrm{Im}F^G_X = X \times_Y X$ completing the proof. □

2.1.3 Principal bundles

Principal bundles are bundles which arise from principal actions.

Definition 2.1.6. An action of G on X is said to be *principal* if the map F^G is both injective and proper, i.e. it is injective, continuous and such that an inverse image of a compact subset is compact.

Since the injectivity and freeness condition are equivalent we can interpret principal actions as both free and proper actions. We can also deduce that these types of actions give rise to homeomorphisms F_X^G from $X \times G$ onto the space $X \times_{X/G} X$. Principal actions lead to the concept of topological principal bundles.

Definition 2.1.7. A principal bundle is a quadruple (X, π, M, G) such that
(i) (X, π, M, G) is a fibre bundle and G is a topological group acting continuously on X with action $\triangleleft : X \times G \to X$, $x \triangleleft g = xg$;
(ii) the action $\triangleleft$ is principal;
(iii) $\pi(x) = \pi(y) \iff \exists g \in G$ such that $y = xg$; and
(iv) the induced map $X/G \to M$ is a homeomorphism.

The first two properties tell us that principal bundles are bundles in which there is a group G action on the total space X which is a principal action, i.e., principal bundles correspond to principal actions. By Definition 2.1.6, principal actions occur when the principal map is both injective and proper, or equivalently, when the action is free and proper. The third property ensures that the fibres of the bundle correspond to the orbits coming from the action and the final property implies that the quotient space can topologically be viewed as the base space on the bundle.

Example 2.1.8. Suppose X is a topoplogical space and G a topological group which acts on X from the right. The triple $(X, \pi, X/G)$ where X/G is the orbit space and π the natural projection is a bundle. A principal action of G on X makes the quadruple $(X, \pi, X/G, G)$ a principal bundle.

We describe a principal bundle (X, π, Y, G) as a G-principal bundle over (X, π, Y), or X as a G-principal bundle over Y.

2.1.4 Vector bundles

Definition 2.1.9. A vector bundle is a bundle (E, π, M) where each fibre $\pi^{-1}(m)$ is endowed with a vector space structure such that addition and scalar multiplication are continuous maps.

Any vector bundle can be understood as a bundle associated to a principal bundle in the following way. Consider a G-principal bundle (X, π, Y, G) and let V be a representation space of G, i.e. a (topological) vector space with a (continuous) left linear G-action $\triangleright :$ $G \times V \to V$, $(g, v) \mapsto g \triangleright v$. Then G acts from the right on $X \times V$ by

$$(x, v) \triangleleft g := (xg, g^{-1} \triangleright v), \quad \text{for all } x \in X,\ v \in V \text{ and } g \in G.$$

We can define $E = (X \times V)/G$, and a surjective (continuous map) $\pi_E : E \to Y$, $(x, v) \triangleleft G \mapsto \pi(x)$, and thus have a fibre bundle (E, π_E, Y, V).

Definition 2.1.10. A *section* of a bundle (E, π_E, Y) is a continuous map $s : Y \to E$ such that, for all $y \in Y$,

$$\pi_E(s(y)) = y,$$

i.e. a section is simply a section of the morphism π_E. The set of sections of E is denoted by $\Gamma(E)$.

Proposition 2.1.11. *Sections in a fibre bundle (E, π_E, Y, V) associated to a principal G-bundle X are in bijective correspondence with (continuous) maps $f : X \to V$ such that*

$$f(xg) = g^{-1} \triangleright f(x).$$

All such G-equivariant maps are denoted by $\mathrm{Hom}_G(X, V)$

Proof. Remember that $Y = X/G$. Given a map $f \in \mathrm{Hom}_G(X, V)$, define the section $s_f : Y \to E$, $xG \mapsto (x, f(x)) \triangleleft G$.

Conversely, given $s \in \Gamma(E)$, define $f_s : X \to V$ by assigning to $x \in X$ a unique $v \in V$ such that $s(xG) = (x, v) \triangleleft G$. Note that v is unique, since if $(x, w) = (x, v) \triangleleft g$, then $xg = x$ and $w = g^{-1} \triangleright v$. Freeness implies that $g = e$, hence $w = v$. The map f_s has the required equivariance property, since the element of $(X \times V)/G$ corresponding to xg is $g^{-1} \triangleright v$. □

2.2 Quantum groups

In the spirit of non-commutative geometry we would like to construct non-commutative algebras using topological spaces.

2.2.1 The algebra of functions on a topological space

The starting point of translating these geometric ideas into algebraic structures involves considering the space of functions from affine varieties into the complex plane. Let X be an affine variety and G an affine variety with a group structure which acts on X from the right. Also, write $Y = X/G$ for the quotient of X by this action. Now $\mathcal{O}(X) = \{f : X \to \mathbb{C} : f \text{ regular function}\}$ is a complex algebra with multiplication given pointwise, that is $(fg)(x) = f(x)g(x)$ for $f, g \in \mathcal{O}(X)$ and unit $1 : X \to \mathbb{C}, x \mapsto 1$; similarly for $\mathcal{O}(G)$ and $\mathcal{O}(Y)$. It is convenient to write $A = \mathcal{O}(X)$ and $H = \mathcal{O}(G)$ and note the identification $\mathcal{O}(G \times G) \cong \mathcal{O}(G) \otimes \mathcal{O}(G)$. Through this identification we have the following.

Proposition 2.2.1. *$H = \mathcal{O}(G)$ is a Hopf algebra with comultiplication $\mathcal{O}(G) \ni f \mapsto (\Delta f) \in \mathcal{O}(G) \otimes \mathcal{O}(G)$, $(\Delta f)(g, h) = f(gh)$, counit $\epsilon : \mathcal{O}(G) \to \mathbb{C}$, $\epsilon(f) = f(e)$ where $e \in G$ is the neutral element, and the antipode $S : H \to H$, $(Sf)(g) = f(g^{-1})$.*

Proof. Firstly, the associativity and the unital property of the algebraic structure; since the codomain of functions in $\mathcal{O}(G)$ is $\mathbb{C}$, associativity and unital properties follow trivially. Now the coassociativity and counital property of the coalgebra structure are confirmed. Coassociativity, let $f \in \mathcal{O}(G)$,

$$\begin{aligned}(\Delta \otimes id)(\Delta f)(x,y,z) &= (\Delta f)(xy,z) = (xy)z = x(yz) \\ &= (\Delta f)(x,yz) = ((id \otimes \Delta) \circ \Delta f)(x,y,z).\end{aligned}$$

Counital property,

$$(id \otimes \epsilon)(\Delta f)(g) = (id \otimes \epsilon)(\Delta f)(g \otimes 1_G) = (\Delta f)(g \otimes e) = f(g),$$

showing $(id \otimes \epsilon) \circ \Delta = id$; similarly for the other case. Next the antipode property,

$$\begin{aligned}(m \circ (S \otimes id) \circ \Delta)f(g) &= m(S \otimes id)(\Delta f)(g) = m((Sf_{(1)} \otimes f_{(2)}))(g) \\ &= (Sf_{(1)})(g)f_{(2)}(g) = f_{(1)}(g^{-1})f_{(2)}(g) \\ &= (\Delta f)(g^{-1},g) = f(g^{-1}g) = f(e) = \epsilon(f).\end{aligned}$$

This shows $(m \circ (S \otimes id) \circ \Delta) = \epsilon$ and again the other case is shown in a similar way. □

Using the fact that G acts on X we can construct a right coaction of H on $A = \mathcal{O}(X)$ by $\varrho^A : A \to A \otimes H$, $\varrho^A(a)(x,g) = a(xg)$. This coaction is an algebra map due to the commutativity of the coordinate functions involved.

Proposition 2.2.2. *$A = \mathcal{O}(X)$ is a right $H = \mathcal{O}(G)$-comodule with coaction given*

$$\varrho^A : A \to A \otimes H, \qquad \varrho^A(a)(x,g) = a(xg),$$

where we use the identification $\mathcal{O}(X) \otimes \mathcal{O}(G) \cong \mathcal{O}(X \times G)$.

Proof. The coassociativity and the counital properties are checked first. Let $a \in A = \mathcal{O}(G)$, $x \in X$ and $g,h \in G$, then

$$\begin{aligned}((id \otimes \Delta) \circ \varrho^A)(a)(x,g,h) &= \varrho^A(a)(x,gh) = a(x(gh)) \\ &= a((xg)h) = \varrho^A(a)(xg,h) \\ &= ((\varrho^A \otimes id) \circ \varrho^A)(a)(x,g,h),\end{aligned}$$

showing the coassociativity. Similarly,

$$\begin{aligned}((id \otimes \epsilon) \circ \varrho^A)(a)(x) &= (id \otimes \epsilon)(\varrho^A(a))(x) = (id \otimes \epsilon)(a_{(0)} \otimes a_{(1)})(x) \\ &= (a_{(0)}\epsilon(a_{(1)}))(x) = (a_{(0)}a_{(1)}(e))(x) \\ &= a_{(0)}(x)a_{(1)}(e) = \varrho^A(a)(x,e) = a(xe) = a(x),\end{aligned}$$

showing the counital property. Lastly, ϱ^A is an algebra map since,

$$\begin{aligned}\varrho^A(ab)(x,g) &= (ab)(xg) = a(xg)b(xg) \\ &= \varrho^A(a)(x,g)\varrho^A(b)(x,g) = (\varrho^A(a)\varrho^A(b))(x,g). \quad \text{QED.}\end{aligned}$$

We have considered the spaces of functions on X and G, next we consider the space of functions on Y and write $B = \mathcal{O}(Y)$ where $Y = X/G$. We can see B is a subalgebra of A by considering the map

$$\pi^* : B \to A, \qquad b \mapsto b \circ \pi,$$

where π is the canonical surjection $X \ni x \mapsto [x] \in X/G$. The map π^* is injective, since $b \neq b'$ in $\mathcal{O}(X/G)$ means there exists at least one orbit $xG = [x]$ such that $b([x]) \neq b'([x])$, but $\pi(x) = [x]$, so $b(\pi(x)) \neq b'(\pi(x))$ which implies $\pi^*(b) \neq \pi^*(b')$. Therefore, we can identify B with $\pi^*(B)$. Furthermore, $a \in \pi^*(B)$ if and only if

$$a(xg) = a(x),$$

for all $x \in X$, $g \in G$. This is the same as

$$\varrho^A(a)(x, g) = (a \otimes 1)(x, g),$$

for all $x \in X$, $g \in G$, where $1 : G \to \mathbb{C}$ is the unit function $1(g) = 1$ (the unit of H). Thus we can identify B with the coinvariants

$$B = A^{coH} := \{a \in A \mid \varrho^A(a) = a \otimes 1\}.$$

Since B is a subalgebra of A, it acts on A via the inclusion map $(ab)(x) = a(x)b(\pi(x))$, $(ba)(x) = b(\pi(x))a(x)$. We can identify $\mathcal{O}(X \times_Y X)$ with $\mathcal{O}(X) \otimes_{\mathcal{O}(Y)} \mathcal{O}(X) = A \otimes_B A$, by the map

$$\theta(a \otimes_B a')(x, y) = a(x)a'(y), \quad \text{with } \pi(x) = \pi(y).$$

Note that θ is well defined because $\pi(x) = \pi(y)$. Proposition 2.1.5 immediately yields.

Proposition 2.2.3. *The action of G on X is free if and only if $F_X^{G*} : \mathcal{O}(X \times_Y X) \to \mathcal{O}(X \times G)$, $f \mapsto f \circ F_X^G$ is bijective.*

In view of the definition of the coaction of H on A, we can identify F_X^{G*} with the *canonical map*

$$\mathsf{can} : a \otimes_B a' \mapsto [(x, g) \mapsto a(x)a'(x.g)] = a\varrho^A(a').$$

Thus the action of G on X is free if and only if this purely algebraic map is bijective.

2.2.2 The q-deformations of the classical spheres

We now put these ideas into practice and describe some well known quantum groups.

The quantum 3-sphere Take the classical 3-sphere $S^3 = \{(u, v) \in \mathbb{C}^2 : |u|^2 + |v|^2 = 1\}$. To describe the quantum version of S^3 we first note the isomorphism $S^3 \cong SU(2)$, where

$$SU(2) = \{\begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \in M_2(\mathbb{C}) : |\alpha|^2 + |\beta|^2 = 1\}.$$

We view the matrix entries of elements of $SU(2)$ as functions from $SU(2)$ into $\mathbb{C}$. Take the functions on $SU(2)$

$$\tilde{\alpha}, \tilde{\beta} : SU(2) \to \mathbb{C},$$

$$\tilde{\alpha}\begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} = \alpha, \ \tilde{\beta}\begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} = -\overline{\beta},$$

we see these functions preserve the structure of $SU(2)$ in that $\tilde{\alpha}\overline{\tilde{\alpha}} + \tilde{\beta}\overline{\tilde{\beta}} = 1$ where $1 : SU(2) \to \mathbb{C}, 1\begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} = 1$. The algebra of functions generated by $\tilde{\alpha}, \tilde{\beta}$ satisfying $\tilde{\alpha}\overline{\tilde{\alpha}} + \tilde{\beta}\overline{\tilde{\beta}} = 1$, written $\mathcal{O}(SU(2))$, is clearly commutative; as described in the introduction this type of construction will always be commutative. In general, to build non-commutative spaces we consider polynomial algebras with generators given by the functions on $SU(2)$ and the non-commutativity comes from the relations between these generators. This process is known as a q-deformation of an algebra where $q \in (0,1)$ is a parameter. The idea is that for each $q \in (0,1)$ we have a different algebraic structure, each non-commutative and by considering the limit $q \to 1$ we recover the classical space, in this case $SU(2)$. We denote this space by $\mathcal{O}(SU_q(2))$.

Example 2.2.4. The quantum group $\mathcal{O}(SU_q(2)) \cong \mathcal{O}(S^3_q)$ is described as the polynomial algebra generated by $\alpha, \beta, \gamma, \delta$ satisfying the relations

$$\alpha\beta = q\beta\alpha, \quad \alpha\gamma = q\gamma\alpha, \quad \beta\gamma = \gamma\beta, \quad \beta\delta = q\delta\beta, \quad \gamma\delta = q\delta\gamma,$$

$$\delta\alpha - q^{-1}\beta\gamma = 1, \quad \alpha\delta - q\beta\gamma = 1.$$

We have a $*$-structure given by $\alpha^* = \delta$, $\beta^* = -q\gamma$, $\gamma^* = -q^{-1}\beta$, $\delta^* = \alpha$, hence we can describe the space as the $*$-algebra generated by α and β satisfying the relations,

$$\alpha\beta = q\beta\alpha, \quad \alpha\beta^* = q\beta^*\alpha, \quad \beta\beta^* = \beta^*\beta, \tag{2.2}$$

$$\alpha\alpha^* = \alpha^*\alpha + (q^{-2} - 1)\beta\beta^*, \quad \alpha\alpha^* + \beta\beta^* = 1. \tag{2.3}$$

The coalgebra structure is given by

$$\Delta(\alpha) = \alpha \otimes \alpha + \beta \otimes \gamma, \quad \Delta(\beta) = \alpha \otimes \beta + \beta \otimes \delta, \tag{2.4}$$

$$\Delta(\gamma) = \gamma \otimes \alpha + \delta \otimes \gamma, \quad \Delta(\delta) = \delta \otimes \delta + \gamma \otimes \beta, \tag{2.5}$$

and $\epsilon(\alpha) = \epsilon(\delta) = 1, \epsilon(\beta) = \epsilon(\gamma) = 0$. These are extended to the whole of $\mathcal{O}(SU_q(2))$ as algebra maps. On the $*$-structure, this comes out as

$$\Delta(\alpha^*) = \alpha^* \otimes \alpha^* - q\beta^* \otimes \beta, \quad \Delta(\beta^*) = \alpha^* \otimes \beta^* + \beta^* \otimes \alpha, \tag{2.6}$$

and $\epsilon(\alpha^*) = 1$, $\epsilon(\beta^*) = 0$. Finally, the antipode is given by $S(\alpha) = \delta$, $S(\beta) = -q^{-1}\beta$, $S(\gamma) = -q\gamma$, $S(\delta) = \alpha$. We see that by setting $q = 1$ we get a commutative Hopf algebra which is isomorphic to the the algebra of functions on $SU(2) \cong S^3$.

The quantum 2-sphere Classically, the 2-sphere can be described as the homogeneous space of the Lie group $SU(2)$. In order to move into the quantum setting we first need to define the quantum version of homogeneous spaces. An algebra is called a *quantum homogeneous space* of a Hopf algebra A provided it is isomorphic to a subalgebra $B \subseteq A$ such that $\Delta(B) \subseteq A \otimes B$. Furthermore, A is required to be right faithfully flat as a module over B; see [36]. Now putting $A = \mathcal{O}(S_q^3)$ and using the comultiplication map described in Example 2.2.4 we have a description of the quantum 2-sphere $B = \mathcal{O}(S_q^2)$.

Proposition 2.2.5. *The algebra of polynomial functions on the quantum 2-sphere, known as the standard quantum 2-sphere denoted $\mathcal{O}(S_q^2)$, is generated by elements x and z satisfying*

$$x^* = x, \qquad zx = q^2xz, \qquad zz^* = q^2x(1-q^2x), \qquad z^*z = x(1-x), \tag{2.7}$$

where $x = \gamma\gamma^*$, $z = \alpha\gamma^*$ and $z^* = \gamma\alpha^*$ in $\mathcal{O}(S_q^3)$.

By setting $q = 1$ we notice that $\mathcal{O}(S_{q=1}^2)$ is a commutative algebra. By defining self-adjoint elements $\xi = 1-2x$, $\eta = z+z^*$ and $\zeta = i(z-z^*)$ and using relations 2.7 we arrive at $\xi^2+\eta^2+\zeta^2 = 1$. This means that $\mathcal{O}(S_{q=1}^2)$ is the polynominal algebra of functions on S^2.

Higher dimensional quantum spheres Building on these structures, odd dimensional quantum spheres are defined as follows; see [44].

Definition 2.2.6. The algebra $\mathcal{O}(S_q^{2n+1})$ of coordinate functions on the quantum sphere is the unital complex $*$-algebra with generators $z_0, z_1, \ldots, z_n$ subject to the following relations,

$$z_iz_j = qz_jz_i \quad \text{for } i<j, \qquad z_iz_j^* = qz_j^*z_i \quad \text{for } i \neq j,$$

$$z_iz_i^* = z_i^*z_i + (q^{-2}-1)\sum_{m=i+1}^{n} z_mz_m^*, \qquad \sum_{m=0}^{n} z_mz_m^* = 1,$$

where q is a real number, $q \in (0,1)$.

Even dimensional quantum spheres $\mathcal{O}(S_q^{2n})$ are obtained by taking $\mathcal{O}(S_q^{2n+1})$ and setting $z_n^* = z_n$.

2.2.3 Podleś quantum 2-spheres $\mathcal{O}(S_{q,s}^2)$

We have seen in Example 2.2.5 a quantum deformation of the standard S^2. For each $q \in (0,1)$ we get non-isomorphic structures, however there exist further non-isomorphic q-deformed spheres, parameterised by $s \in [0,1]$.

Proposition 2.2.7. *For $s \in [0,1]$, in $\mathcal{O}(S_q^3)$ let,*

$$x = (1-s^2)\gamma\gamma^* + s(\gamma\alpha + \alpha^*\gamma),$$

$$z = (1-s^2)\alpha\gamma^* + s(q\gamma^{*2} - \alpha^2),$$

$$z^* = (1-s^2)\gamma\alpha^* - s(q\gamma^2 - \alpha^{*2}).$$

The elements x, z, z^* satisfy the relations

$$x^* = x, \quad zx = q^2 xz, \quad zz^* = (s^2 + q^2 x)(1 - q^2 x), \quad z^* z = (s^2 + x)(1 - x),$$

and for each value of s, the polynominal algebra with generators x, z and the above relations is a quantum homogeneous space of $\mathcal{O}(S^3_q)$ known as the quantum 2-sphere and denoted $\mathcal{O}(S^2_{q,s})$. Furthermore $\mathcal{O}(S^2_{q,s})$ are not isomorphic for different values of s; see [32].

2.3 Non-commutative principal and associated bundles

2.3.1 Principal comodule algebras

Using Proposition 2.2.3, by replacing the topological spaces with the algebra of functions we can describe an algebraically equivalent definition of freeness. In fact, we are not restricted to commutative algebraic spaces, in full generality this leads to the following.

Definition 2.3.1. (Hopf-Galois Extensions) Let H be a Hopf algebra and A a right H-comodule algebra with coaction given by $\varrho^A : A \to A \otimes H$. Define $B := \{b \in A : \varrho^A(b) = b \otimes 1\}$. We say that $B \subseteq A$ is a *Hopf-Galois extension* if the left A-module, right H-comodule map

$$\mathsf{can} : A \otimes_B A \to A \otimes H, \qquad a \otimes_B a' \mapsto a\varrho^A(a')$$

is an isomorphism.

Proposition 2.2.3 tells us that when viewing bundles from an algebraic perspective, the freeness condition is equivalent to the Hopf-Galois extension property. Hence, the Hopf-Galois extension condition is a necessary condition to ensure a bundle is principal. Not all information about a topological space is encoded in a coordinate algebra, so to make a fuller reflection of the richness of the classical notion of a principal bundle we need to require conditions additional to the Hopf-Galois property.

Definition 2.3.2. Let H be a Hopf algebra with bijective antipode and let A be a right H-comodule algebra with coaction $\varrho^A : A \to A \otimes H$. Let B denote the coinvariant subalgebra of A. We say that A is a *principal H-comodule algebra* if:
(a) $B \subseteq A$ is a Hopf-Galois extension;
(b) the multiplication map $B \otimes A \to A$, $b \otimes a \mapsto ba$, splits as a left B-module and right H-comodule map (the equivariant projectivity).

As indicated already in [40], [13] or [24], principal comodule algebras should be understood as principal bundles in non-commutative geometry. In particular, if H is an algebra of coordinate functions on a quantum group [47], then the existence of the Haar measure together with the results of [40] mean that the freeness of the coaction implies its principality.

Example 2.3.3. (Cleft comodule algebras) Suppose B is an algebra and H a Hopf algebra. Let $A = B \otimes H$ and consider it as a right H-comodule with coaction

$$\rho^A : A \to A \otimes H, \qquad \rho^A = id_B \otimes \Delta_H, \qquad \rho^A(b \otimes h) = b \otimes h_{(1)} \otimes h_{(2)}.$$

(A, ρ^A) is a H-comodule algebra with coinvariant elements

$$A^{coH} = \{b \otimes h \in B \otimes H : \rho^A(b \otimes h) = b \otimes h \otimes 1_H\} = \{b \otimes 1_H : b \in B\} \cong B.$$

Furthermore, the canonical map

$$\mathsf{can} : (B \otimes H) \otimes_B (B \otimes H) \to B \otimes H \otimes H, \qquad \mathsf{can}(b \otimes h' \otimes h) = b \otimes h'h_{(1)} \otimes h_{(2)},$$

is an isomorphism with inverse given by

$$\mathsf{can}^{-1}(b \otimes h' \otimes h) = b \otimes h'Sh_{(1)} \otimes h_{(2)}.$$

Therefore, $A = B \otimes H$ is a Hopf-Galois extension of B. In this case we refer to A as a cleft extension or a cleft comodule algebra. Cleft comodule algebras are examples of principal comodule algebras; see Example 2.4.16 for the proof.

Let A be a right H-comodule algebra. A is cleft if there exists a right H-colinear map $j : H \to A$ that has an inverse in the convolution algebra $\mathrm{Hom}(H, A)$ and is normalised so that $j(1) = 1$. The convolution algebra is given by $\mathrm{Hom}(H, A) = \{f : H \to A : f \text{ is an algebra map}\}$ with multiplication $(f * g)(h) = f(h)g(h)$ and unit $1(h) = 1_A$. The map j is called a *cleaving map* or a *normalised total integral.* In particular, if $j : H \to A$ is an H-colinear algebra map, then it is automatically convolution invertible $j^{-1} = j \circ S$ and normalised. A comodule algebra A admitting such a map is termed a *trivial* principal comodule algebra.

2.3.2 Modules associated to principal comodule algebras

Having described non-commutative principal bundles, we can look at the associated vector bundles. First we look at the classical case and try to understand it purely algebraically. Start with a vector bundle (E, π_E, Y, V) associated to a principal G-bundle X. Since V is a vector representation space of G, also the set $\mathrm{Hom}_G(X, V)$ is a vector space. Consequently $\Gamma(E)$ is a vector space. Furthermore, $\mathrm{Hom}_G(X, V)$ is a left module of $B = \mathcal{O}(Y)$ with the action $(bf)(x) = b(\pi_E(x))f(x)$. To understand better the way in which the B-module $\Gamma(E)$ is associated to the principal comodule algebra $\mathcal{O}(X)$ we recall

Definition 2.3.4. Given a Hopf algebra H, right H-comodule A with coaction ϱ^A and left H-comodule V with coaction ${}^V\varrho$, the *cotensor product* is defined as an equaliser:

$$\begin{tikzcd} A\Box_H V \arrow[r] & A \otimes V \arrow[r, shift left, "\varrho^A\otimes\mathrm{id}"] \arrow[r, shift right, "\mathrm{id}\otimes{}^V\varrho"'] & A \otimes H \otimes V . \end{tikzcd}$$

If A is an H-comodule algebra, and $B = A^{coH}$, then $A\Box_H V$ is a left B-module with the action $b(a\Box v) = ba\Box v$. In particular, in the case of a principal G-bundle X over $Y = X/G$, for any left $\mathcal{O}(G)$-comodule V the cotensor product $\mathcal{O}(X)\Box_{\mathcal{O}(G)}V$ is a left $\mathcal{O}(Y)$-module.

Assume that V is finite dimensional. Then the dual vector space V^* is a left $\mathcal{O}(G)$-comodule with the coaction ${}^{V^*}\varrho : v \mapsto \sum v_{(-1)} \otimes v_{(0)}$ (summation implicit) determined by $\sum v_{(-1)}(g)v_{(0)} = g^{-1} \triangleright v$.

Proposition 2.3.5. *The left $\mathcal{O}(Y)$-module of sections $\Gamma(E)$ is isomorphic to the left $\mathcal{O}(Y)$-module $\mathcal{O}(X)\Box_{\mathcal{O}(G)}V$.*

Proof. First identify $\Gamma(E)$ with $\mathrm{Hom}_G(X,V)$. Let $\{v_i \in V^*,\ v^i \in V\}$ be a (finite) dual basis. Take $f \in \mathrm{Hom}_G(X,V)$, and define $\theta : \mathrm{Hom}_G(X,V) \to \mathcal{O}(X)\Box_{\mathcal{O}(G)}V$ by $\theta(f) = \sum_i v_i \circ f \otimes v^i$.

In the converse direction, define a left $\mathcal{O}(Y)$-module map

$$\theta^{-1} : \mathcal{O}(X)\Box_{\mathcal{O}(G)}V \to \mathrm{Hom}_G(X,V), \qquad a\Box v \mapsto a(-)v.$$

One easily checks that the constructed maps are mutual inverses. □

Moving away from commutative algebras of functions on topological spaces one uses Proposition 2.3.5 as the motivation for the following

Definition 2.3.6. Let A be a principal H-comodule algebra. Set $B = A^{coH}$ and let V be a left H-comodule. The left B-module $\Gamma = A\Box_H V$ is called a *module associated to the principal comodule algebra A.*

Γ is a projective left B-module, and if V is a finite dimensional vector space, then Γ is a finitely generated projective left B-module. In this case it has the meaning of a module of sections over a non-commutative vector bundle. Furthermore, its class gives an element in the K_0-group of B. If A is a cleft principal comodule algebra, then every associated module is free, since $A \cong B \otimes H$ as a left B-module and right H-comodule, so that

$$\Gamma = A\Box_H V \cong (B \otimes H)\Box_H V \cong B \otimes (H\Box_H V) \cong B \otimes V.$$

2.4 Differential calculus

The process we have followed so far is to start with geometric objects and move into a quantum setting by performing q-deformations on polynomial algebras with generators viewed as functions and relations, furthermore by setting the q-parameter to one we recover the classical geometric case. Combining the algebraic theory set out in Sections 1.1 - 1.4 and the classical geometry set out in Sections 2.1 - 2.3, we were able to discuss the geometric side of these quantum spaces. Next, it is natural to ask whether it is possible to describe a concept of calculus in the quantum setting.

2.4.1 Differential graded algebras

Definition 2.4.1. Let A be an algebra. A first order differential calculus is a pair $(\Omega^1 A, d)$ where

(i) $\Omega^1 A$ is an A-bimodule,

(ii) $d : A \to \Omega^1 A$ is a linear map which satisfies $d(ab) = (da)b + adb$ for all $a, b \in A$,

(iii) $\Omega^1 A = \text{span}\{adb : a, b \in A\}$.

The first condition says the first order differential calculus can be multiplied by elements of A, which can be thought of as functions, from both the left and right. The second condition is known as Leibniz's rule, the map d is known as the exterior derivative and the third condition gives a description of the space in terms of the exterior derivative. This definition can be extended to higher forms in the following way.

Definition 2.4.2. A differential graded algebra is an $\mathbb{N}_0$-graded algebra

$$\Omega A = \bigoplus_{n=0}^{\infty} \Omega^n A,$$

equipped with a system of operations

$$d_n : \Omega^n A \to \Omega^{n+1} A, \qquad n = 0, 1, 2, ...$$

satisfying the properties

(i) $d_{n+1} \circ d_n = 0$,

(ii) $d_{n+m}(\omega\omega') = d_{n+m}(\omega)\omega' + (-1)^n \omega d_{n+m}(\omega')$ for all $\omega \in \Omega^n A$ and $\omega' \in \Omega^m A$.

The zero-graded elements of the differential graded algebra, namely $\Omega^0 A$, is an algebra which we simply denote by A. $\Omega^n A$ are referred to as the n-forms. The first property says that d gives A the structure of a cochain complex and the second condition is known as the graded Leibniz rule as mentioned above.

Universal construction

Differential graded algebras can be constructed by taking an algebra $A = (A, m_A, 1_A)$ and defining a first order calculus as,

$$\Omega^1 A := ker(m_A) = \{\sum a \otimes a' \in A \otimes A : m_A(\sum a \otimes a') = 0\},$$

along with the operation

$$d : A \to \Omega^1 A, \qquad d(a) = 1_A \otimes a - a \otimes 1_A.$$

The left and right A-actions, making $\Omega^1 A$ into an A-bimodule, are given in the obvious way $a \triangleright (a' \otimes a'') \triangleleft a''' = aa' \otimes a''a'''$. The higher orders are obtained by defining

$$\Omega^n A := \Omega^{n-1} A \otimes_A \Omega^1 A \subset A^{\otimes(n+1)}, \qquad \text{for } n \in \mathbb{N}_0,$$

which produces $\Omega^n A = \Omega^1 A \otimes_A \Omega^1 A ... \otimes_A \Omega^1 A$ (n-times). The differential d is defined on higher forms by

$$d : \Omega^n A \to \Omega^{n+1} A,$$

$$d(a_0 \otimes a_1 \otimes ... \otimes a_n) = \sum_{i=0}^{n+1} (-1)^i a_0 \otimes a_1 \otimes ... \otimes a_{i-1} \otimes 1 \otimes a_i \otimes ... \otimes a_n.$$

The multiplication between n-forms and m-forms is given by,

$$(a_0 \otimes ... \otimes a_n)(b_0 \otimes ... \otimes b_m) = (a_0 \otimes ... \otimes a_n b_0 \otimes ... \otimes b_m).$$

This differential graded algebra $(\Omega A, d)$ is called the universal differential envelope of A. $(\Omega^1 A, d)$ is called the universal differential calculus over the algebra A.

2.4.2 Covariant differential calculi

Since we are in an algebraic setting it is natural to introduce module or comodule structures and incorporate this into the differential graded algebras setting.

Definition 2.4.3. Let (A, ρ^A) be a right H-comodule algebra where H is a bialgebra. Then we say that the first order differential calculus $(\Omega^1 A, d)$ is a (right) covariant differential calculus on A if $\Omega^1 A$ is a right H-comodule by a coaction $\varrho^{\Omega^1 A} : \Omega^1 A \to \Omega^1 A \otimes H$ such that $\varrho^{\Omega^1 A}(adb) = a_{(0)} db_{(0)} \otimes a_{(1)} b_{(1)}$ and that $d : A \to \Omega^1 A$ is a right H-comodule map.

Equivalently, the left covariant differential calculus can be defined. When A is both a left and right H-comodule algebra and the differential is both a left and right H-comodule map with commuting coactions, the covariant calculus is called bicovariant.

Proposition 2.4.4. *For (A, ρ^A) a right H-comodule algebra where H is a bialgebra, the space of universal one-forms $\Omega^1 A$ is a right H-comodule with coaction given by*

$$\rho^{\Omega^1 A} : \Omega^1 A \to \Omega^1 A \otimes H, \qquad a \otimes a' \mapsto a_{(0)} \otimes a'_{(0)} \otimes a_{(1)} a'_{(1)},$$

furthermore the differential $d : A \to \Omega^1 A$ is a right H-comodule map, thus making it a right covariant calculus.

Proof. This is a straightforward exercise using Sweedler notation. □

As set-out in Woronowicz's paper [48], within this set-up the right covariant differential calculus on a Hopf algebra A viewed as a right A-comodule via the comultiplication map can be fully described by the elements $\omega_i \in \Omega^1 A$ such that $\rho^{\Omega^1 A}(\omega_i) = \omega_i \otimes 1$ (we say that each ω_i is right-invariant under the coaction $\rho^{\Omega^1 A}$) that freely generate $\Omega^1 A$ as a right A-module. Hence all elements $x \in \Omega^1 A$ can be expressed by $x = \sum_{i=1}^{n} \omega_i a_i$, where

$\{\omega_1, ..., \omega_n\}$ are the right invariant elements and each $a_i \in A$.

The three dimensional first order covariant calculus on $\mathcal{O}(S_q^3)$. In [48] Woronowicz gives a description of a three dimensional covariant calculus on the quantum three sphere; a detailed account is given as follows. We know that $A = \mathcal{O}(S_q^3)$ is a Hopf algebra with structure fully described in Example 2.2.4.

The first order differential calculus $\Omega^1 A$ is generated by three left-invariant one-forms $\{\omega_0, \omega_+, \omega_-\}$ satisfying the A-bimodule relations,

$$\omega_0\alpha = q^{-2}\alpha\omega_0, \qquad \omega_0\beta = q^2\beta\omega_0, \qquad \omega_0\gamma = q^{-2}\gamma\omega_0, \qquad \omega_0\delta = q^2\delta\omega_0, \tag{2.8a}$$

$$\omega_+\alpha = q^{-1}\alpha\omega_+, \qquad \omega_+\beta = q\beta\omega_+, \qquad \omega_+\gamma = q^{-1}\gamma\omega_+, \qquad \omega_+\delta = q\delta\omega_+, \tag{2.8b}$$

$$\omega_-\alpha = q^{-1}\alpha\omega_-, \qquad \omega_-\beta = q\beta\omega_-, \qquad \omega_-\gamma = q^{-1}\gamma\omega_-, \qquad \omega_-\delta = q\delta\omega_-. \tag{2.8c}$$

The exterior differential is given on generators by,

$$d(\alpha) = \alpha\omega_0 - q\beta\omega_+, \quad d(\beta) = -q^2\beta\omega_0 + \alpha\omega_-, \quad d(\gamma) = \gamma\omega_0 - q\delta\omega_+, \quad d(\delta) = -q^2\delta\omega_0 + \gamma\omega_-,$$

Since $\mathcal{O}(S_q^3)$ is a Hopf algebra, we can view it as a left $\mathcal{O}(S_q^3)$-comodule with coaction given by the comultiplication map $\Delta : \mathcal{O}(S_q^3) \to \mathcal{O}(S_q^3) \otimes \mathcal{O}(S_q^3)$, $\Delta(a) = a_{(1)} \otimes a_{(2)}$ using the Sweedler notation. This extends to a coaction on the one-forms

$$\begin{aligned} \Delta_L : \Omega^1(\mathcal{O}(S_q^3)) &\to A \otimes \Omega^1(\mathcal{O}(S_q^3)) \\ \Delta_L(adb) &= a_{(1)}b_{(1)} \otimes a_{(2)}db_{(2)}. \end{aligned}$$

It is now possible to calculate the $\omega_i's$ in the form AdB to perform calculations using Δ_L. We start by considering

$$\Delta_L(d\alpha) = \alpha_{(1)} \otimes d\alpha_{(2)} = \alpha \otimes d\alpha + \beta \otimes d\gamma,$$

hence

$$\alpha_{(1)} \otimes d\alpha_{(2)} = \alpha \otimes (\alpha\omega_0 - q\beta\omega_+) + \beta \otimes (\gamma\omega_0 - q\delta\omega_+).$$

Now applying $m \circ (S \otimes id)$ to both sides gives

$$\begin{aligned} S(\alpha_{(1)})d\alpha_{(2)} &= \delta\alpha\omega_0 - q\delta\beta\omega_+ - q^{-1}\beta\gamma\omega_0 + \beta\delta\omega_+ \\ &= (\delta\alpha - q^{-1}\beta\gamma)\omega_0 \\ &= \omega_0 \end{aligned}$$

In the same way $\omega_+ = -S(\gamma_{(1)})d\gamma_{(2)}$ and $\omega_- = S(\beta_{(1)})d\beta_{(2)}$ which gives,

$$\omega_0 = \delta(d\alpha) - q^{-1}\beta(d\gamma), \qquad \omega_- = \delta(d\alpha) - q^{-1}\beta(d\gamma), \qquad \omega_+ = q\gamma(d\alpha) - \alpha(d\gamma).$$

Lastly, since we have a $*$-structure on $\mathcal{O}(S_q^3)$ we can deduce the $*$-structure on $\Omega^1(\mathcal{O}(S_q^3))$ using the fact that the exterior derivative should commute with this structure. This means

$$d(\alpha^*) = -q^2\delta\omega_0 + \gamma\omega_- = -\omega_0\delta + q\omega_-\gamma,$$

is the same as

$$(d\alpha)^* = (\alpha\omega_0 - q\beta\omega_+)^* = \omega_0^*\delta + q^2\omega_+^*\gamma,$$

giving rise to $\omega_0^* = -\omega_0$ and $\omega_+^* = q^{-1}\omega_-$.

2.4.3 Strong connections

Horizontal forms and connections in Hopf-Galois extensions. Classically, given a principal G-bundle X over M, a connection is a map $\Pi : T^*X \to T^*X$ which projects elements of the cotangent space T^*X onto the horizontal one-forms. The horizontal one-forms being the one-forms excluding the tangent vectors to the fibres of X. With this is mind we can turn our attention to the quantum setting.

Definition 2.4.5. Let (A, ρ^A) be a right H-comodule algebra, where H is a bialgebra and set $B = A^{coH}$. The A-subbimodule $\Omega^1_{hor}A$ of $\Omega^1 A$ generated by all $d(b)$, $b \in B$, is called a module of horizontal one-forms. This is written explicitly as

$$\Omega^1_{hor}A = A(\Omega^1 B)A = \{a_i \otimes b_i a'_i - a_i b_i \otimes a'_i : a_i, a'_i \in A, b_i \in B\},$$

i.e., the horizontal one-forms in a Hopf-Galois extension is the A-bimodule generated by the one-forms restricted to the coinvariant elements.

An equivalent definition can be made using the following short exact sequence.

$$0 \longrightarrow \Omega^1_{hor}A \xrightarrow{i} A \otimes A \xrightarrow{\pi} A \otimes_B A \longrightarrow 0, \tag{2.9}$$

where $\pi : A \otimes A \to A \otimes_B A$ is the canonical projection defining $A \otimes_B A$ and $i : \Omega^1_{hor}A \to A \otimes A$ is the natural embedding of the subset into the bigger set.

We have seen that for a right H-comodule algebra A, where H is a Hopf algebra, the canonical map is defined by $\mathsf{can} : A \otimes_B A :\to A \otimes H$, $a \otimes a' \mapsto aa'_{(0)} \otimes a'_{(1)}$, where $B = A^{coH}$.

Definition 2.4.6. We call the map $\overline{\mathsf{can}}$ the *lifted canonical map*, defined as,

$$\overline{\mathsf{can}} : A \otimes A \to A \otimes H, \qquad a \otimes a' \mapsto aa'_{(0)} \otimes a'_{(1)}.$$

The restriction of the lifted canonical map to the first order differential calculus $\Omega^1 A \subset A \otimes A$ is called the *vertical map*, which is written explicitly as,

$$\mathsf{ver} : \Omega^1 A \to A \otimes H^+, \qquad a \otimes a' \mapsto aa'_{(0)} \otimes a'_{(1)},$$

where $H^+ = \ker \epsilon_H$. The map is well-defined since applying $(id \otimes \epsilon_H)$ to the image gives $aa'_{(0)}\epsilon(a'_{(1)}) \otimes 1 = aa' \otimes 1 = 0$.

Proposition 2.4.7. *$A \otimes H^+$ is a right H-comodule with coaction given by*

$$\rho^{A\otimes H^+} : A \otimes H^+ \to A \otimes H^+ \otimes H, \qquad a \otimes h \mapsto a_{(0)} \otimes h_{(2)} \otimes a_{(1)}(Sh_{(1)})h_{(3)},$$

where H^+ is viewed as a right H-comodule via the adjoint coaction,

$$Ad : H^+ \to H^+ \otimes H, \qquad h \mapsto h_{(2)} \otimes (Sh_{(1)})h_{(3)}.$$

Furthermore, $\mathsf{ver} : \Omega^1 A \to A \otimes H^+$ is a right H-comodule map.

Proof. $A \otimes H^+$ is a right H-comodule. Firstly check coassociativity,

$$\begin{aligned}(\rho^{A\otimes H^+} \otimes id) \circ \rho^{A\otimes H^+}(a \otimes h) &= a_{(0)(0)} \otimes h_{(2)(2)} \otimes a_{(0)(1)}(Sh_{(2)(1)})h_{(2)(3)} \otimes a_{(1)}(Sh_{(1)})h_{(3)} \\ &= a_{(0)} \otimes h_{(3)} \otimes a_{(1)}(Sh_{(2)})h_{(4)} \otimes a_{(2)}(Sh_{(1)})h_{(5)},\end{aligned}$$

the first equality follows from the definition of the coaction and the second a re-labelling of indices.

$$\begin{aligned}(id \otimes \Delta_H) \circ \rho^{A\otimes H^+}(a \otimes h) &= a_{(0)} \otimes h_{(2)} \otimes a_{(1)(1)}(Sh_{(1)})_{(1)}h_{(3)(1)} \otimes a_{(1)(2)}(Sh_{(1)})_{(2)}h_{(3)(2)} \\ &= a_{(0)} \otimes h_{(2)} \otimes a_{(1)(1)}(Sh_{(1)(2)})h_{(3)(1)} \otimes a_{(1)(2)}(Sh_{(1)(1)})h_{(3)(2)} \\ &= a_{(0)} \otimes h_{(3)} \otimes a_{(1)}(Sh_{(2)})h_{(4)} \otimes a_{(2)}(Sh_{(1)})h_{(5)},\end{aligned}$$

the second equality uses the antihomomorphism property of the antipode and the third is a relabelling. Next the counit property is checked,

$$\begin{aligned}(id \otimes \epsilon) \circ \rho^{A\otimes H^+}(a \otimes h) &= a_{(0)} \otimes h_{(2)} \otimes \epsilon(a_{(1)})\epsilon(Sh_{(1)})\epsilon(h_{(3)}) \\ &= a_{(0)}\epsilon(a_{(1)}) \otimes h_{(2)}\epsilon(S(h_{(1)})\epsilon(h_{(3)}) \otimes 1 \\ &= a \otimes h_{(1)}\epsilon(h_{(2)}) \otimes 1 \\ &= a \otimes h \otimes 1.\end{aligned}$$

the second equality uses the fact that the counit is an algebra map and the third uses the counital property associated to H. Finally, we show *ver* is a right H-comodule map. $\Omega^1 A$ is viewed as a right H-comodule via the coaction $\rho^{\Omega^1 A} : \Omega^1 A \to \Omega^1 A \otimes H$, $a \otimes a' \mapsto a_{(0)} \otimes a'_{(0)} \otimes a_{(1)}a'_{(1)}$.

$$\begin{aligned}(\rho^{A\otimes H^+} \circ ver)(a \otimes a') &= \rho^{A\otimes H^+}(aa'_{(0)} \otimes a'_{(1)}) \\ &= a_{(0)(0)}a'_{(0)(0)} \otimes a'_{(1)(2)} \otimes a_{(0)(1)}a'_{(0)(1)}(Sa'_{(1)(1)})a'_{(1)(3)}, \\ &= a_{(0)}a'_{(0)} \otimes a'_{(2)} \otimes a_{(1)}\epsilon(a'_{(1)})a'_{(3)}, \\ &= a_{(0)}a'_{(0)} \otimes a'_{(1)} \otimes a_{(1)}a'_{(2)}.\end{aligned}$$

and

$$\begin{aligned}(ver \otimes id) \circ \rho^{\Omega^1 A}(a \otimes a') &= (ver \otimes id)(a_{(0)} \otimes a'_{(0)} \otimes a_{(1)}a'_{(1)}) \\ &= a_{(0)}a'_{(0)} \otimes a'_{(1)} \otimes a_{(1)}a'_{(2)}.\end{aligned}$$

□

Proposition 2.4.8. *The following statements are equivalent:*

(a) $B \subset A$ is a Hopf-Galois extension;

(b) the following sequence is short exact

$$0 \longrightarrow \Omega^1_{hor} A \xrightarrow{i} \Omega^1 A \xrightarrow{ver} A \otimes H^+ \longrightarrow 0, \tag{2.10}$$

where i is the inclusion.

Proof. Firstly, we re-state condition (b) in an equivalent way. The sequence

$$0 \longrightarrow \Omega^1_{hor}A \xrightarrow{i} \Omega^1 A \xrightarrow{\mathsf{ver}} A \otimes H^+ \longrightarrow 0, \tag{2.11}$$

is exact, if, and only if,

$$0 \longrightarrow \Omega^1_{hor}A \xrightarrow{i} A \otimes A \xrightarrow{\overline{\mathsf{can}}} A \otimes H \longrightarrow 0. \tag{2.12}$$

" $\Leftarrow$ " We can read straight from (2.12) that $ker\ \overline{\mathsf{can}} = \Omega^1_{hor}A$, and since the vertical map is a restriction of $\overline{\mathsf{can}}$ from $A \otimes A$ to $\Omega^1 A$, its kernel is $\Omega^1_{hor}A \subset \Omega^1 A$. " $\Rightarrow$ " Using the identities $A \otimes A \cong \Omega^1 A \oplus A$ and $A \otimes H \cong (A \otimes H^+) \oplus A$ as left A-modules, we see that $ker\ \overline{\mathsf{can}} = ker\ \overline{\mathsf{can}}|_{\Omega^1 A} = ker\ \mathsf{ver} = \Omega^1_{hor}A$.

Now suppose $B \subset A$ is a Hopf-Galois extension, so $\mathsf{can} : A \otimes_B A \to A \otimes H$ is bijective. Consider the sequence

$$0 \longrightarrow \Omega^1_{hor}A \xrightarrow{i} A \otimes A \xrightarrow{\overline{\mathsf{can}}} A \otimes H \longrightarrow 0. \tag{2.13}$$

recalling $\overline{\mathsf{can}} = \mathsf{can} \circ \pi : A \otimes A \to A \otimes H$; see Definition 2.4.6. Since can is bijective, it is surjective and π is a projetion hence surjective, we can deduce $\overline{\mathsf{can}}$ is sujective. Also,

$$ker\ \overline{\mathsf{can}} = ker\ (\mathsf{can} \circ \pi) = ker\ \pi = \Omega^1_{hor}A,$$

where the penultimate equality follows since can is injective and the final equality by Definition 2.4.5, hence the sequence 2.12 is exact.

Now suppose 2.12 is an exact sequence, hence $\overline{\mathsf{can}}$ is surjective. Let $(a \otimes h) \in A \otimes H$, so $\overline{\mathsf{can}}(a' \otimes a'') = a \otimes h$ for some $(a' \otimes a'') \in A \otimes A$ since $\overline{\mathsf{can}}$ is surjective. Now,

$$\overline{\mathsf{can}}(a' \otimes a'') = (\mathsf{can} \circ \pi)(a' \otimes a'') = a \otimes h \Longrightarrow a \otimes h = \mathsf{can}(\pi(a' \otimes a'')),$$

so can is also surjective. Suppose $(a \otimes_B a') \in A \otimes_B A$ such that $\mathsf{can}(a \otimes_B a') = 0$. Since $\overline{\mathsf{can}} = \mathsf{can} \circ \pi$, there is an element $(c \otimes c') \in A \otimes A$ such that $\overline{\mathsf{can}}(c \otimes c') = 0$ and $\pi(c \otimes c') = a \otimes a'$. But from the exactness of the sequence $ker\ \overline{\mathsf{can}} = \Omega^1_{hor}A$, so $c \otimes c' \in \Omega^1_{hor}A$ and by definition $ker\ \pi = \Omega^1_{hor}A$, so $a \otimes a' = 0$ and can is injective, showing that can is bijective. □

Definition 2.4.9. A *connection* in a Hopf-Galois extension is a k-linear map $\Pi : \Omega^1 A \to \Omega^1 A$ such that

(i) $\Pi \circ \Pi = \Pi$ (the connection is an idempotent map),

(ii) $ker\ \Pi = \Omega^1_{hor}A$ (the kernel corresponds to the horizontal one-forms),

(iii) $\Pi(adb) = a\Pi(db)$, for all $a, b \in A$ (Π is a left A-module map),

(iv) $(\Pi \otimes id) \circ \rho^{\Omega^1 A} = \rho^{\Omega^1 A} \circ \Pi$ (the connection is a right H-comodule map).

The first property shows that the connection produces a splitting into horizontal and vertical parts. This can be seen as follows, for $\omega \in \Omega^1 A$,

$$\omega = id(\omega) = (id(\omega) - \Pi(\omega)) + \Pi(\omega) = \omega_H + \omega_V,$$

where $\omega_H = (id(\omega) - \Pi(\omega))$ and $\omega_V = \Pi(\omega)$, and

$$\Pi(\omega_H) = \Pi(id(\omega) - \Pi(\omega)) = \Pi(\omega) - \Pi^2(\omega) = 0 \Longrightarrow \omega_H \in \Omega^1_{hor} A.$$

Hence the connection provides a splitting of $\Omega^1 A$ into horizontal and vertical parts.

In the classical case connections in a principal bundle are in one-to-one correspondence with connections forms. Given a G-principal bundle X over M, then the connection forms are differential forms on X with values in the Lie algebra of G that are covariant under the underlying action of the bundle. In the quantum setting we have the following.

Definition 2.4.10. A *connection form* in a Hopf–Galois extension $B \subset A$ is a k-linear map $\omega : H^+ \to \Omega^1 A$ such that

(i) $(\rho^{\Omega^1 A} \circ \omega) = (\omega \otimes H) \circ \mathrm{Ad}$,

(ii) $\mathsf{ver} \circ \omega = 1_A \otimes id_{H^+}$.

The first property informs us that a connection form should be a right H-comodule map and the second property that the ver map applied to the connection form gives the identity.

Theorem 2.4.11. *In a Hopf-Galois extension $B \subset A$ by a Hopf algebra H, connections and connection forms are in one-to-one correspondence.*

Proof. Since we are working in a Hopf-Galois extension we know that the sequence is short exact of left A-module and right H-comodules,

$$0 \longrightarrow \Omega^1_{hor} A \xrightarrow{i} \Omega^1 A \xrightarrow{\mathsf{ver}} A \otimes H^+ \longrightarrow 0. \tag{2.14}$$

Suppose $\Pi : \Omega^1 A \to \Omega^1 A$ is a connection, hence there is a splitting of $\Omega^1 A$ into horizontal part and vertical part which means $\Omega^1 A = \Omega^1_{hor} A \oplus A \otimes H^+$ and there is a left A-module and right H-comodule map $\bar{\omega} : A \otimes H^+ \to \Omega^1 A$ such that $\mathsf{ver} \circ \bar{\omega} = id$. The map defined by $\omega : H^+ \to \Omega^1 A$, $\omega(h) = \bar{\omega}(1 \otimes h)$ is a connection form.

Now suppose that $\omega : H^+ \to \Omega^1 A$ is a connection form, and define $\bar{\omega} : A \otimes H^+ \to \Omega^1 A$ by $\bar{\omega}(a \otimes h) = a\omega(h)$. The map $\bar{\omega}$ is a left A-module and right H-comodule map and it a splitting of the above exact sequence since,

$$(\mathsf{ver} \circ \bar{\omega})(a \otimes h) = \mathsf{ver}(a\omega(h)) = a(\mathsf{ver} \circ \omega)(h) = a(1_A \otimes h) = a \otimes h.$$

Now the map defined by $\Pi = \bar{\omega} \circ \mathsf{ver} : \Omega^1 A \to \Omega^1 A$ is a connection. □

Strong connections. We have seen connections defined in a Hopf-Galois setting, next we consider connections in a module.

Definition 2.4.12. Let B be an algebra and Γ a left B-module. A *connection* in Γ is a k-linear map

$$\nabla : \Gamma \to \Omega^1 B \otimes_B \Gamma,$$

such that

$$\nabla(bx) = d(b) \otimes_B x + b\nabla(x),$$

for all $b \in B, x \in \Gamma$.

Definition 2.4.13. Let $\Pi : \Omega^1 A \to \Omega^1 A$ be a connection in a Hopf-Galois extension $B \subset A$. The right H-comodule map,

$$D : A \to \Omega^1_{hor} A, \qquad D = d - \Pi \circ d,$$

is called a *covariant derivative* corresponding to Π. Furthermore, the connection Π is called a *strong connection* if $D(A) \subset (\Omega^1 B)A$.

Now we can state a strong connection Π in a Hopf-Galois extension $B \subset A$ induces a connection in the left B-module A. The connection is given by the covariant derivative D, as stated in above definition. The existence of a strong connection in a Hopf-Galois extension is summarised in the following theorem.

Theorem 2.4.14. *A strong connection in a Hopf-Galois extension $B \subset A$ by a Hopf algebra H exists, if, and only if, A is H-equivariantly projective as a left B-module, i.e., there exists a left B-module, right H-comodule splitting of the multiplication map $m_A : B \otimes A \to A$.*

Normally it is not practical to find a splitting of the multiplication map in order to prove the existence of a strong connection. Instead we need additional tools which make calculations more manageable. This is given in the following theorem.

Proposition 2.4.15. *A right H-comodule algebra A with coaction $\varrho^A : A \to A \otimes H$ is principal if and only if it admits a* strong connection form, *that is if there exists a map $\omega : H \longrightarrow A \otimes A$, such that*

$$\omega(1) = 1 \otimes 1, \tag{2.15a}$$

$$m_A \circ \omega = 1_A \circ \varepsilon, \tag{2.15b}$$

$$(\omega \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \varrho) \circ \omega, \tag{2.15c}$$

$$(S \otimes \omega) \circ \Delta = (\sigma \otimes \mathrm{id}) \circ (\varrho \otimes \mathrm{id}) \circ \omega. \tag{2.15d}$$

Here $m_A : A \otimes A \to A$ denotes the multiplication map, $1_A : \mathbb{C} \to A$ is the unit map, $\Delta : H \to H \otimes H$ is the comultiplication, $\varepsilon : H \to \mathbb{C}$ counit and $S : H \to H$ the (bijective) antipode of the Hopf algebra H, and $\sigma : A \otimes H \to H \otimes A$ is the flip.

Proof. If a strong connection form ω exists, then the inverse of the canonical map can (see Definition 2.3.2 (a)) is the composite

$$A \otimes H \xrightarrow{\mathrm{id} \otimes \omega} A \otimes A \otimes A \xrightarrow{m_A \otimes \mathrm{id}} A \otimes A \longrightarrow A \otimes_B A\,,$$

while the splitting of the multiplication map (see Definition 2.3.2 (b)) is given by

$$A \xrightarrow{\varrho^A} A \otimes H \xrightarrow{\mathrm{id} \otimes \omega} A \otimes A \otimes A \xrightarrow{m_A \otimes \mathrm{id}} B \otimes A\,.$$

Conversely, if $B \subseteq A$ is a principal comodule algebra, then ω is the composite

$$H \xrightarrow{1_A \otimes \mathrm{id}} A \otimes H \xrightarrow{\mathsf{can}^{-1}} A \otimes_B A \xrightarrow{\mathrm{id} \otimes s} A \otimes_B B \otimes A \xrightarrow{\cong} A \otimes A\,,$$

where s is the left B-linear right H-colinear splitting of the multiplication $B \otimes A \to A$. □

Example 2.4.16. Let A be cleft comodule algebra over Hopf algebra H. Hence there exists a right H-colinear map $j : H \to A$ that has an inverse in the convolution algebra $\mathrm{Hom}(H, A)$ and is normalised so that $j(1) = 1$; see Example 2.3.3. Writing j^{-1} for the convolution inverse of j, one easily observes that

$$\omega : H \to A \otimes A, \qquad h \mapsto (j^{-1} \otimes j)(\Delta(h)),$$

is a strong connection form. Hence a cleft comodule algebra is an example of a principal comodule algebra.

Example 2.4.17. Let H be a Hopf algebra of a compact quantum group. By the Woronowicz theorem [47], H admits an invariant Haar measure, i.e. a linear map $\Lambda : H \to \mathbb{C}$ such that, for all $h \in H$,

$$h_{(1)}\Lambda(h_{(2)}) = \Lambda(h), \qquad \Lambda(1) = 1.$$

where $\Delta(h) = h_{(1)} \otimes h_{(2)}$ is the Sweedler notation for the comultiplication. Next, assume that the lifted canonical map:

$$\overline{\mathsf{can}} : A \otimes A \to A \otimes H, \qquad a \otimes a' \mapsto a\varrho(a'), \tag{2.16}$$

is surjective, and write

$$\ell : H \to A \otimes A, \qquad \ell(h) = \sum \ell(h)^{[1]} \otimes \ell(h)^{[2]},$$

for the $\mathbb{C}$-linear map such that $\overline{\mathsf{can}}(\ell(h)) = 1 \otimes h$, for all $h \in H$. Then, by a theorem of Schneider [40], A is a principal H-comodule algebra. Explicitly, a strong connection form is

$$\omega(h) = \Lambda\left(h_{(1)}\ell\left(h_{(2)}\right)^{[1]}{}_{(1)}\right)\Lambda\left(\ell\left(h_{(2)}\right)^{[2]}{}_{(1)}S\left(h_{(3)}\right)\right)\ell(h_{(2)})^{[1]}{}_{(0)} \otimes \ell(h_{(2)})^{[2]}{}_{(0)},$$

where the coaction is denoted by the Sweedler notation $\varrho^A(a) = a_{(0)} \otimes a_{(1)}$; see [4].

2.5 Fredholm modules and the Chern character

In this section we give the details relating to the construction of Fredholm modules over a $*$-algebra.

2.5.1 Constructing even Fredholm modules

Definition 2.5.1. An *even Fredholm module* over a $*$-algebra $\mathcal{A}$ is a quadruple $(\mathfrak{V}, \pi, F, \gamma)$, where $\mathfrak{V}$ is a Hilbert space of a representation π of $\mathcal{A}$ and F and γ are operators on $\mathfrak{V}$ such that $F^* = F$, $F^2 = I$, $\gamma^2 = I$, $\gamma F = -\gamma F$, and, for all $a \in \mathcal{A}$, the commutator $[F, \pi(a)]$ is a compact operator.

In general, if A is a bounded operator on a separable Hilbert space $\mathcal{H}$ with orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$, then we say A is trace class if

$$\mathrm{Tr}\ |A| = \sum_{k\in\mathbb{N}} \langle (A^*A)^{\frac{1}{2}} e_k, e_k\rangle < \infty.$$

In this case we can define the trace of A by the convergent series

$$\mathrm{Tr}\ A := \sum_{k\in\mathbb{N}} \langle A e_k, e_k\rangle.$$

Definition 2.5.2. A Fredholm module $(\mathfrak{V}, \pi, F, \gamma)$ over a $*$-algebra $\mathcal{A}$, is said to be *1-summable* if the commutator $[F, \pi(a)]$ is a trace class operator for all $a \in \mathcal{A}$.

2.5.2 The cyclic homology of an algebra and the Chern character

A chain complex $(A_\bullet, d_\bullet)$ is defined as a sequence of vector spaces $\ldots, A_{-2}, A_{-1}, A_0, A_1, \ldots$ connected by homomorphisms $d_n : A_n \to A_{n-1}$ such that $d_n \circ d_{n+1} = 0$ for all $n \in \mathbb{Z}$. We write this as

$$\cdots \xleftarrow{d_{n-1}} A_{n-1} \xleftarrow{d_n} A_n \xleftarrow{d_{n+1}} A_{n+1} \ \cdots .$$

The elements $x \in A_n$ are called the chains and we say x has degree (or dimension) n. The maps $d_\bullet$ are called the boundary maps (or differential maps). The elements of the image of d_n are called the boundaries and the elements of the kernel are called the cycles. The family of groups $H_\bullet(A)$ is called the homology of A, where the n^{th}-homology group is defined as

$$H_n(A) = \mathrm{Ker}\ d_n / \mathrm{Im}\ d_{n+1}, \qquad n \in \mathbb{Z}.$$

There are many different types of homologies in the context of non-commutative geometry, we are interested in the cyclic homology of an algebra. Firstly we need to define a bicomplex.

Definition 2.5.3. A bicomplex $A_{\bullet,\bullet}$ is a family of vector spaces $A_{p,q}$ with two families of homomorphisms

$$\partial' : A_{p,q} \to A_{p-1,q}, \qquad \partial'' : A_{p,q} \to A_{p,q-1},$$

for any $p, q \in \mathbb{Z}$, satisfying

$$\partial'\partial' = 0, \qquad \partial'\partial'' + \partial''\partial' = 0, \qquad \partial''\partial'' = 0.$$

Definition 2.5.4. For an algebra A consider the bicomplex $CC_\bullet(A)$ given by

$$\begin{tikzcd}
{} \arrow[d, "\partial_3"] & {} \arrow[d, "-\partial'_3"] & {} \arrow[d, "\partial_3"] & {} \arrow[d, "-\partial'_3"] & & \\
A\otimes A\otimes A \arrow[d, "\partial_2"] & A\otimes A\otimes A \arrow[l, "\tilde{\tau}_2"'] \arrow[d, "-\partial'_2"] & A\otimes A\otimes A \arrow[l, "N_2"'] \arrow[d, "\partial_2"] & A\otimes A\otimes A \arrow[l, "\tilde{\tau}_2"'] \arrow[d, "-\partial'_2"] & {} \arrow[l, "N_2"'] & \dots \\
A\otimes A \arrow[d, "\partial_1"] & A\otimes A \arrow[l, "\tilde{\tau}_1"'] \arrow[d, "-\partial'_1"] & A\otimes A \arrow[l, "N_1"'] \arrow[d, "\partial_1"] & A\otimes A \arrow[l, "\tilde{\tau}_1"'] \arrow[d, "-\partial'_1"] & {} \arrow[l, "N_1"'] & \dots \\
A & A \arrow[l, "\tilde{\tau}_0"'] & A \arrow[l, "N_0"'] & A \arrow[l, "\tilde{\tau}_0"'] & {} \arrow[l, "N_0"'] & \dots ,
\end{tikzcd}$$

with boundary maps

$$\partial'_n(a_0 \otimes a_1 \otimes \dots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_n,$$

$$\partial_n(a_0 \otimes a_1 \otimes \dots \otimes a_n) = \partial'_n(a_0 \otimes a_1 \otimes \dots \otimes a_n) + (-1)^n a_n a_0 \otimes a_1 \otimes \dots \otimes a_{n-1},$$
$$\tau_n(a_0 \otimes a_1 \otimes \dots \otimes a_n) = (-1)^n a_n \otimes a_0 \otimes a_1 \otimes \dots \otimes a_{n-1},$$

and

$$\tilde{\tau}_n = Id_{A^{\otimes(n+1)}} - \tau_n, \qquad N_n = \sum_{i=1}^{n} (\tau_n)^i.$$

We refer to $CC_\bullet(A)$ as a *cyclic bicomplex*. In order to define a homology of the cyclic bicomplex, we need to interpret this as a chain complex. This can be done by viewing the terms of degree n of a chain complex by

$$\bigoplus_{i+j=n, i,j>0} CC_{i,j}(A) = \bigoplus_{j=0}^{n} A^{\otimes j+1}.$$

The homology of this chain complex is known as the cyclic homology of A and is denoted $HC_\bullet(A)$. The first two groups come out as

$$HC_0(A) = A/\{\text{Im } \partial_1 + \text{Im } \tilde{\tau}_0\}, \qquad HC_1(A) = \text{Ker } \partial_1/\{\text{Im } \partial_2 + \text{Im } \tilde{\tau}_1\}.$$

We now turn our attention to the Chern character. The Chern character is a homomorphism from $K_0(A)$ into the homology group $HC_{2n}(A)$ for some $n \in \mathbb{N}$ constructed as follows. Take a class $[P] \in K_0(A)$ where P is a finitely generated projective A-module. Suppose P has dual basis $\{x_1, \dots, x_n, \pi_1, \dots, \pi_n\}$ where $x_i \in P$ and $\pi_i \in {}_A\text{Hom}(P, A)$, so any $p \in P$ can be written $p = \sum_{i=1}^{n} \pi_i(p) x_i$. The matrix $E = (E_{ij})_{i,j=1}^{n} := (\pi_j(x_i))_{i,j=1}^{n}$ is an idempotent in $M(A)$. With the idempotent E we can associate a $2n$-cycle in the cyclic bicomplex $CC_\bullet(A)$. Firstly, define

$$\widetilde{ch}_n := \sum_{i_1, \dots, i_n} E_{i_1 i_2} \otimes E_{i_2 i_3} \otimes \dots \otimes E_{i_n i_1}, \tag{2.17}$$

and then the $2n$-cycle

$$\bigoplus_{l=0}^{2n}(-1)^{\frac{l}{2}}\frac{l!}{\lfloor\frac{l}{2}\rfloor!}\widetilde{ch}_l(E). \tag{2.18}$$

The class of this $2n$-cycle does not depend on the choice of P or E in $[P]$, hence it defines an Abelian group map

$$\mathrm{ch}_n : K_0(A) \to HC_{2n}(A). \tag{2.19}$$

known as the Chern map.

2.5.3 The Chern-Connes pairing

Let $\mathcal{A}$ be a principal comodule algebra with coaction ρ and coinvariant subalgebra $\mathcal{B}$. In this subsection first we follow [34] (see also [29] and [18]), and associate even Fredholm modules to the algebra $\mathcal{B}$ and use them to construct traces or cycles in the cyclic bicomplex $CC_\bullet(\mathcal{B})$. The latter are then used to calculate the Chern number of a non-commutative line bundle associated to the quantum principal bundle $\mathcal{A}$ over the subalgebra $\mathcal{B}$.

Line bundles associated to a principal comodule algebra

As explained in [19] any strong connection in $\mathcal{A}$ can be used to construct finitely generated projective modules over the coinvariant subalgebra $\mathcal{B}$. To achieve this one needs to take any finite dimensional left comodule W over H (or a finite dimensional corepresentation of H) and then the cotensor product $A\Box_H W$ is a finitely generated projective left B-module. In the case $H = \mathcal{O}(U(1))$, these projective modules, or projective modules of sections of line bundles associated to a principal comodule algebra $\mathcal{A}$, are defined as

$$\mathcal{L}[n] := \{x \in \mathcal{A} : \rho(x) = x \otimes u^n\}, \qquad n \in \mathbb{Z}.$$

In other words, $\mathcal{L}[n]$ is the degree n component of $\mathcal{A}$ when the latter is viewed as a $\mathbb{Z}$-graded algebra. An idempotent $E[n]$ for $\mathcal{L}[n]$ is given in terms of a strong connection ω,

$$E[n]_{ij} = \omega(u^n)^{[2]}{}_i\omega(u^n)^{[1]}{}_j \in \mathcal{B}, \tag{2.20}$$

where $\omega(u^n) = \sum_i \omega(u^n)^{[1]}{}_i \otimes \omega(u^n)^{[2]}{}_i$.

The Chern-Connes pairing

Suppose $(\mathfrak{V}, \pi, F, \gamma)$ is an even Fredholm module over the algebra $\mathcal{B}$. We associate a cyclic cycle of $\mathcal{B}$ or a *Chern character* τ by

$$\tau(x) = \mathrm{Tr}\ (\gamma\pi(x)), \quad x \in \mathcal{B}$$

Now, the traces of powers of each of the $E[n]$ make up a cycle in the cyclic complex of $\mathcal{B}$, whose corresponding class in homology $HC_\bullet(\mathcal{B})$ is known as the Chern character of $\mathcal{L}[n]$; see Section 2.5.2. In particular Tr $E[n]$ can be paired with the Chern character associated to a Fredholm module over $\mathcal{B}$ to give an integer, which identifies isomorphism classes of the $E[n]$.

In general, the result of the combined process that to an isomorphism class of a corepresentation of a Hopf algebra H assigns the Chern character of the B-module associated to the H-principal comodule algebra A is known as the *Chern-Galois character.*

Using the Chern-Galois character to show non-cleftness

The construction of traces τ provides one with an alternative way of proving that the principal comodule algebra $\mathcal{A}$ is not cleft. This involves evaluating τ at the zero-component of the *Chern-Galois* character of the principal $\mathcal{O}(U(1))$-comodule algebra $\mathcal{A}$.

The traces of powers of each of the $E[n]$ make up a cycle in the cyclic complex of $\mathcal{B}$. In particular, the traces of $E[n]$ form the zero-component of the Chern-Galois character of $\mathcal{A}$. Should $\mathcal{A}$ be a cleft principal comodule algebra, then every module $\mathcal{L}[n]$ would be free. Thus an alternative way of showing that $\mathcal{A}$ is not cleft is to prove that, say, $\mathcal{L}[1]$ is not a free left $\mathcal{B}$-module. For this it suffices to show that Tr $E[1]$ is a non-trivial element of $HC_0(\mathcal{B})$ by proving that $\tau(\text{Tr } E[1]) \neq 0$.

Part II

Examples of weighted $U(1)$-actions on non-commutative algebras

Chapter 3

Quantum teardrops

3.1 Quantum weighted projective spaces

The motivation for quantum teardrop spaces starts with the well-known Hopf fibration whereby classically the 3-sphere S^3 and the circle $U(1)$ are used to describe the 2-sphere S^2. In this example, we have an action $\triangleright : U(1) \times S^3 \to S^3$ given by $u \triangleright (z_1, z_2) = (uz_1, uz_2)$ where $u \in U(1)$ and $(z_1, z_2) \in S^3 \subset \mathbb{C}^2$, so $|z_1|^2 + |z_2|^2 = 1$. Now taking the quotient of S^3 by $U(1)$ using the usual equivalence relation $(z_1, z_2) \sim (w_1, w_2) \iff$ there exists $u \in U(1)$ such that $u \triangleright (z_1, z_2) = (w_1, w_2)$, we deduce that $S^2 = S^3/U(1)$. Moving to the quantum setting, the quantum homogeneous space $\mathcal{O}(S^2_q)$ can be described by the fixed point space of the action of $U(1)$ on $\mathcal{O}(S^3_q)$, see Definition 2.2.4, defined on generators as $e^{i\theta} \triangleright \alpha = e^{i\theta}\alpha$ and $e^{i\theta} \triangleright \beta = e^{i\theta}\beta$. This idea can be extended by introducing weights to the action $\triangleright : U(1) \times S^3 \to S^3$ described in the Hopf fibration, the resulting spaces are called weighted projective spaces. This is formally defined as follows.

Definition 3.1.1. (Weighted projective spaces) Given $n + 1$ pairwise coprime numbers $l_0, \ldots l_n$, one can define the action of the group $U(1)$ on S^{2n+1} by $u \cdot (z_0, z_1, \ldots, z_n) = (u^{l_0} z_0, u^{l_1} z_1, \ldots, u^{l_n} z_n)$, where $(z_0, ..., z_n) \in S^{2n+1}$, i.e. $\sum_{i=0}^{n} |z_i|^2 = 1$, now weighted projective spaces are defined as the quotients of S^{2n+1} by this action,

$$\mathbb{WP}(l_0, l_1, \ldots, l_n) = S^{2n+1}/U(1).$$

By introducing the weighting to the action the resulting quotients are not necesarily manifolds. In fact, weighted projective spaces are examples of orbifolds, that are not global quotients of manifolds by finite groups. (Orbifolds are locally modelled as quotients of open subsets of $\mathbb{C}^n$ by finite groups). For $n = 1$, $\mathbb{WP}(1, 1)$ is the two-sphere $S^2 = \mathbb{CP}^1$, while $\mathbb{WP}(1, l)$ is the teardrop orbifold studied by Thurston [43].

This formulation of weighted projective spaces makes the definition of *quantum weighted projective spaces* particularly straightforward as it allows one to follow the general strategy in which a classical space is replaced by the coordinate algebra of the quantum space. Also, the action of the classical group on a space is replaced by the coaction of the coordinate algebra of the corresponding quantum group on the coordinate algebra of the quantum space; the coordinate algebra of the quantum quotient space then arises as the fixed points or coinvariants of this coaction. Recalling the algebra $\mathcal{O}(S_q^{2n+1})$ of coordinate functions on the quantum sphere is the unital complex $*$-algebra with generators $z_0, z_1, \ldots, z_n$ subject to the following relations:

$$z_i z_j = q z_j z_i \quad \text{for } i < j, \qquad z_i z_j^* = q z_j^* z_i \quad \text{for } i \neq j, \tag{3.1}$$

$$z_i z_i^* = z_i^* z_i + (q^{-2} - 1) \sum_{m=i+1}^{n} z_m z_m^*, \qquad \sum_{m=0}^{n} z_m z_m^* = 1, \tag{3.2}$$

where q is a real number, $q \in (0,1)$; see 2.2.6. For any $n+1$ pairwise coprime numbers $l_0, \ldots, l_n$, one can define the coaction of the Hopf algebra $\mathcal{O}(U(1)) = \mathbb{C}[u, u^*]$, where u is a unitary and grouplike generator, as

$$\varrho_{l_0,\ldots,l_n} : \mathcal{O}(S_q^{2n+1}) \to \mathcal{O}(S_q^{2n+1}) \otimes \mathbb{C}[u, u^*], \qquad z_i \mapsto z_i \otimes u^{l_i}, \qquad i = 0, 1, \ldots, n. \tag{3.3}$$

This coaction is then extended to the whole of $\mathcal{O}(S_q^{2n+1})$ so that $\mathcal{O}(S_q^{2n+1})$ is a right $\mathbb{C}[u, u^*]$-comodule algebra.

Definition 3.1.2. (Quantum weighted projective spaces) The algebra of coordinate functions on the quantum weighted projective space is defined as the subalgebra of $\mathcal{O}(S_q^{2n+1})$ containing all elements invariant under the coaction $\varrho_{l_0,\ldots,l_n}$, i.e.

$$\mathcal{O}(\mathbb{WP}_q(l_0, l_1, \ldots, l_n)) = \mathcal{O}(S_q^{2n+1})^{co\mathbb{C}[u,u^*]} := \{x \in \mathcal{O}(S_q^{2n+1}) \mid \varrho_{l_0,\ldots,l_n}(x) = x \otimes 1\}.$$

In the case $l_0 = l_1 = \cdots = 1$ one obtains the algebra of functions on the quantum complex projective space $\mathcal{O}(\mathbb{CP}_q^n)$ [45] (see also [35]).

In the above definition of quantum weighted projective spaces we followed the general strategy which requires one to replace actions of groups by coactions of Hopf algebras. However, due to the very simple nature of $U(1)$, the Hopf algebra $\mathcal{O}(U(1))$ is isomorphic to the group algebra $\mathbb{C}\mathbb{Z}$ of the Pontrjagin dual $\mathbb{Z}$ of $U(1)$. Given a group G, a comodule algebra of the group algebra $\mathbb{C}G$ is the same as a G-graded algebra. Thus to define the $\mathcal{O}(U(1))$-coaction $\varrho_{l_0,\ldots,l_n}$ (3.3) is the same as to give a suitable $\mathbb{Z}$-grading to $\mathcal{O}(S_q^{2n+1})$, compatible with the algebra and $*$- structure of $\mathcal{O}(S_q^{2n+1})$. The generators z_i of $\mathcal{O}(S_q^{2n+1})$ are homogeneous elements of degree l_i (the z_i^* have degree $-l_i$), and $\mathcal{O}(\mathbb{WP}_q(l_0, l_1, \ldots, l_n))$ is simply the degree 0 part of such $\mathbb{Z}$-graded $\mathcal{O}(S_q^{2n+1})$.

3.2 The coordinate algebra of the quantum teardrop and its representations

We concentrate on the quantum weighted projective lines, i.e. on the case $n = 1$. For a pair of coprime positive integers k, l, we give the presentation of $\mathcal{O}(\mathbb{WP}_q(k, l))$ in terms of generators and relations and classify all irreducible representations of $\mathcal{O}(\mathbb{WP}_q(k, l))$ (up to unitary equivalence)

The algebra of coordinate functions on the quantum three-sphere $\mathcal{O}(S_q^3)$ is the same as the algebra of coordinate functions on the quantum group $\mathcal{O}(SU_q(2))$, i.e. $\mathcal{O}(S_q^3) = \mathcal{O}(SU_q(2))$; see [47]. The generators of the latter are traditionally denoted by $\alpha = z_0$ and $\beta = z_1^*$. In terms of α and β relations (4.2) come out as

$$\alpha\beta = q\beta\alpha, \qquad \alpha\beta^* = q\beta^*\alpha, \qquad \beta\beta^* = \beta^*\beta, \alpha\alpha^* = \alpha^*\alpha + (q^{-2}-1)\beta\beta^*, \qquad \alpha\alpha^* + \beta\beta^* = 1,$$

where $q \in (0,1)$. Setting $k = l_0$ and $l = l_1$, the coaction $\varrho_{k,l}$ of $\mathbb{C}[u,u^*]$ on $\mathcal{O}(S_q^3)$ takes the form

$$\alpha \mapsto \alpha \otimes u^k, \qquad \beta \mapsto \beta \otimes u^{*l} = \beta \otimes u^{-l}, \tag{3.2}$$

and $\mathcal{O}(\mathbb{WP}_q(k,l))$ is the coinvariant subalgebra of $\mathcal{O}(S_q^3)$. Equivalently, we can view $\mathcal{O}(S_q^3)$ as a $\mathbb{Z}$-graded $*$-algebra generated by α of degree k and β of degree $-l$; $\mathcal{O}(\mathbb{WP}_q(k,l))$ is the degree zero part of $\mathcal{O}(S_q^3)$.

Theorem 3.2.1. *(i) The algebra $\mathcal{O}(\mathbb{WP}_q(k,l))$ is the $*$-subalgebra of $\mathcal{O}(S_q^3)$ generated by $a = \beta\beta^*$ and $b = \alpha^l\beta^k$.*

(ii) The elements a and b satisfy the following relations

$$a^* = a, \quad ab = q^{-2l}ba, \quad bb^* = q^{2kl}a^k\prod_{m=0}^{l-1}(1-q^{2m}a), \quad b^*b = a^k\prod_{m=1}^{l}(1-q^{-2m}a). \tag{3.3}$$

(iii) $\mathcal{O}(\mathbb{WP}_q(k,l))$ is isomorphic to the $*$-algebra generated by generators a, b and relations (3.3).

Proof. (i) A basis for the space $\mathcal{O}(S_q^3)$ consists of all elements of the form $\alpha^p\beta^r\beta^{*s}$ and $\alpha^{*p}\beta^r\beta^{*s}$, $p,r,s \in \mathbb{N}$. Since the coaction is an algebra map,

$$\varrho_{k,l}(\alpha^p\beta^r\beta^{*s}) = \alpha^p\beta^r\beta^{*s} \otimes u^{kp-lr+ls}, \qquad \varrho_{k,l}(\alpha^{*p}\beta^r\beta^{*s}) = \alpha^{*p}\beta^r\beta^{*s} \otimes u^{-kp-lr+ls}.$$

The first of these elements is coinvariant provided $kp - lr + ls = 0$, i.e. $kp = l(r-s)$. Since k and l are coprime numbers, p must be divisible by l, i.e. $p = ml$ for some $m \in \mathbb{N}$. Therefore, $r = mk + s$ and the only coinvariant elements among the $\alpha^p\beta^r\beta^{*s}$ are those of the form

$$\alpha^{ml}\beta^{mk}\beta^s\beta^{*s} \sim (\alpha^l\beta^k)^m(\beta\beta^*)^s.$$

By similar arguments, the only coinvariant elements among terms of the form $\alpha^{*p}\beta^r\beta^{*s}$ are scalar multiples of $(\alpha^l\beta^k)^{*m}(\beta\beta^*)^r$. We conclude that $\mathcal{O}(\mathbb{WP}_q(k,l))$ is a subalgebra of $\mathcal{O}(S_q^3)$ generated by a self-adjoint element $a = \beta\beta^*$ and by $b = \alpha^l\beta^k$.

(ii) An elementary calculation that uses equations (3.1) confirms that (3.3) are indeed relations that a and b satisfy.

(iii) Denote by $\mathcal{O}(\widetilde{\mathbb{WP}_q(k,l)})$ the unital $*$-algebra generated by generators a and b and relations (3.3). Parts (i) and (ii) establish the existence of the following surjective $*$-algebra map

$$\theta : \mathcal{O}(\widetilde{\mathbb{WP}_q(k,l)}) \to \mathcal{O}(\mathbb{WP}_q(k,l)), \qquad a \mapsto \beta\beta^*, \quad b \mapsto \alpha^l\beta^k. \tag{3.4}$$

Note that the Diamond Lemma immediately implies that the elements

$$a^m b^n,\ a^m b^{*n'}, \qquad m,n \in \mathbb{N},\ n' \in \mathbb{N}\setminus\{0\}, \tag{3.5}$$

form a basis for $\mathcal{O}(\widetilde{\mathbb{WP}_q(k,l)})$. The map θ sends these elements to

$$(\beta\beta^*)^m(\alpha^l\beta^k)^n \sim \alpha^{ln}\beta^{k(m+n)}\beta^{*m}, \qquad (\beta\beta^*)^m(\beta^{*k}\alpha^{*l})^{n'} \sim \alpha^{*ln'}\beta^m\beta^{*k(m+n')},$$

respectively. As these are linearly independent elements of $\mathcal{O}(S^3_q)$, the map θ is injective, hence an isomorphism of $*$-algebras as required. $\square$

In the remainder of this section we study representations of $\mathcal{O}(\mathbb{WP}_q(k,l))$ identified with the $*$-algebra $\mathcal{O}(\widetilde{\mathbb{WP}_q(k,l)})$ generated by a and b subject to relations (3.3). This analysis leads to an alternative proof that the map θ (3.4) is an isomorphism.

Proposition 3.2.2. *Up to a unitary equivalence, the following is the list of all bounded irreducible $*$-representations of $\mathcal{O}(\mathbb{WP}_q(k,l))$.*

(i) One dimensional representation

$$\pi_0 : a \mapsto 0, \qquad b \mapsto 0. \tag{3.6}$$

(ii) Infinite dimensional representations $\pi_s : \mathcal{O}(\mathbb{WP}_q(k,l)) \to \mathrm{End}(V_s)$, labelled by $s = 1,2,\dots,l$. For each s, the separable Hilbert space $V_s \simeq l^2(\mathbb{N})$ has orthonormal basis e^s_p, $p \in \mathbb{N}$, and

$$\pi_s(a)e^s_p = q^{2(lp+s)}e^s_p, \quad \pi_s(b)e^s_p = q^{k(lp+s)}\prod_{r=1}^{l}\left(1-q^{2(lp+s-r)}\right)^{1/2}e^s_{p-1}, \quad \pi_s(b)e^s_0 = 0. \tag{3.7}$$

Proof. First consider the case when $\pi(a) = 0$. The relation $b^*b = a^k\prod_{m=1}^{l}(1-q^{-2m}a)$ implies that $\pi(b) = 0$, and this is the one dimensional representation.

Let V denote the irreducible representation space in which $\pi(a) \neq 0$. It is immediate from the relation $ab = q^{-2l}ba$ that $\ker(\pi(a)) = V$ or $\ker(\pi(a)) = 0$. Since the first case is excluded by assumption $\pi(a) \neq 0$, we conclude $\ker(\pi(a)) = 0$. Suppose that the spectrum of $\pi(a)$ is discrete and consists only of $0, q^2, q^4, ..., q^{2l}$. If v is an eigenvector of $\pi(a)$ with eigenvalue q^{2i}, then the relation $ab = q^{-2l}ba$ implies that $w = \pi(b)v$ is an eigenvector with eigenvalue q^{2i-2l}, or $w = 0$. Consider $\widetilde{w} = \pi(b^*)v$. Applying $\pi(b)$ to both sides gives

$$\pi(b)\widetilde{w} = \pi(bb^*)v = q^{2k(l+i)}\prod_{m=0}^{l-1}(1-q^{2(m+i)})v.$$

So, if $\widetilde{w} = 0$ then $q^{2k(l+i)}\prod_{m=0}^{l-1}(1-q^{2(m+i)}) = 0$, which cannot be true. Hence $\widetilde{w} \neq 0$ which implies $w \neq 0$. Therefore the spectrum also contains q^{2i-2l}, which contradicts the assumption that $0, q^2, ..., q^{2l}$ are the only elements of the spectrum of $\pi(a)$. Thus there must exist $\lambda \in \mathrm{sp}(\pi(a))$ such that $\lambda \neq q^{2i}$ for $i \in \{1, 2, ..., l\}$. This means that there exists a sequence $(\xi_n)_{n \in \mathbb{N}}$ of unit vectors in the representation space V such that

$$\lim_{n \to \infty} \|\pi(a)\xi_n - \lambda\xi_n\| = 0. \tag{3.8}$$

We show that there exists $N \in \mathbb{N}$ and $C > 0$ such that $\|\pi(b^*)\xi_n\| \geq C$, for all $n \geq \mathbb{N}$, using the relation $b^*b = a^k \prod_{m=1}^{l}(1 - q^{-2m}a)$. By the remainder theorem, this relation can be expressed in the following format:

$$b^*b = a^k \prod_{m=1}^{l}(1 - q^{-2m}a) = (a - \lambda)p(a) + \lambda^k \prod_{m=1}^{l}(1 - q^{-2m}\lambda),$$

where $p(a)$ is a polynomial in the variable a of degree less than $k + l$. The triangle inequality and the norm property $\|x\|\|y\| \geq \|xy\|$ imply that:

$$\begin{aligned}
\|\pi(a^k)\prod_{m=1}^{l}(1 - q^{-2m}\pi(a))\xi_n\| &\geq \|\lambda^k \prod_{m=1}^{l}(1 - q^{-2m}\lambda)\| - \|p(a)(\pi(a) - \lambda I)\xi_n\| \\
&\geq \|\lambda^k \prod_{m=1}^{l}(1 - q^{-2m}\lambda)\| - \|p(a)\|\|(\pi(a) - \lambda I)\xi_n\|.
\end{aligned}$$

Therefore,

$$\|\pi(b^*)\|\|\pi(b)\xi_n\| \geq \|\lambda^k \prod_{m=1}^{l}(1 - q^{-2m}\lambda)\| - \|p(a)\|\|(\pi(a) - \lambda I)\xi_n\|,$$

so that

$$\|\pi(b)\xi_n\| \geq \frac{\|\lambda^k \prod_{m=1}^{l}(1 - q^{-2m}\lambda)\|}{\|\pi(b^*)\|} - \frac{\|p(a)\|}{\|\pi(b^*)\|}\|(\pi(a) - \lambda I)\xi_n\|. \tag{3.9}$$

Since $\|\lambda^k \prod_{m=1}^{l}(1 - q^{-2m}\lambda)\|/\|\pi(b^*)\|$ is positive, the existence of the desired N and C follows from (3.8) above. Hence we conclude that

$$\eta_n := \frac{\pi(b)\xi_n}{\|\pi(b)\xi_n\|},$$

are unit vectors for $n \geq N$. Our goal now is to show that

$$\lim_{n \to \infty} \|\pi(b)\eta_n - q^{-2l}\lambda\eta_n\| = 0,$$

which is the same as asserting $q^{-2l}\lambda \in \mathrm{sp}(\pi(b))$. Assuming $n \geq N$, hence $\|\pi(b)\xi_n\| \geq C$, and using the relation $ab = q^{-2l}ba$ we obtain

$$\begin{aligned}
\|\pi(a)\eta_n - q^{-2l}\lambda\eta_n\| &= \frac{\|\pi(a)\pi(b)\xi_n - q^{-2l}\lambda\pi(b)\xi_n\|}{\|\pi(b)\xi_n\|} \\
&\leq \frac{\|\pi(b)\|}{q^{2l}C}\|\pi(a)\xi_n - \lambda\xi_n\| \xrightarrow[n \to \infty]{} 0.
\end{aligned}$$

Hence we have shown that if $\lambda \in \mathrm{sp}(\pi(a))$ then $q^{-2l}\lambda \in \mathrm{sp}(\pi(a))$. So the spectrum contains $\lambda, q^{-2l}\lambda, q^{-4l}\lambda, \ldots$. Since we require this sequence to be bounded there must exist an $n \in \mathbb{N}$ such that $q^{-2nl}\lambda = \lambda_0$, the largest possible eigenvalue, i.e. $\lambda = q^{2nl}\lambda_0$ for some $n \in \mathbb{N}$. Hence we have shown that $\mathrm{sp}(\pi(a)) \subset \{q^{2nl}\lambda_0 : n \in \mathbb{N}\} \cup \{0\}$. The implication of this calculation is that there exists a unit vector ξ such that $\pi(a)\xi = \lambda_0\xi$. We now use this fact to calculate directly the representations.

It follows from the relation $ab^{*^p} = q^{2lp}b^{*^p}a$ that

$$\pi(a)(\pi(b^{*^p})\xi) = q^{2lp}\lambda_0\pi(b^{*^p})\xi,$$

which shows that $\pi(b^{*^p})\xi$ are eigenvectors of $\pi(a)$ which have distinct eigenvalues $q^{2lp}\lambda_0$. Thus the vectors:

$$e_p = \frac{\pi(b^{*^p})\xi}{\|\pi(b^{*^p})\xi\|}, \quad p \in \mathbb{N},$$

form an orthonormal system. We now show that the span of the e_p is closed under the action of the algebra:

$$\pi(a)e_p = \frac{\pi(a)\pi(b^{*^p})\xi}{\|\pi(b^{*^p})\xi\|} = \frac{\pi(q^{2lp}b^{*^p}a)\xi}{\|\pi(b^{*^p})\xi\|} = \frac{q^{2lp}\lambda_0\pi(b^{*^p})\xi}{\|\pi(b^{*^p})\|},$$

therefore,

$$\pi(a)e_p = q^{2lp}\lambda_0 e_p.$$

On the other hand,

$$\pi(b)e_p = \frac{\pi(b)\pi(b^{*^p})\xi}{\|\pi(b^{*^p})\xi\|} = \frac{\pi(bb^*)\pi(b^{*^{p-1}})\xi}{\|\pi(b^{*^p})\xi\|}.$$

Now,

$$\begin{aligned}
\|\pi(b^{*^p})\xi\| &= \langle \pi(b^{*^p})\xi, \pi(b^{*^p})\xi\rangle^{\frac{1}{2}} \\
&= \langle \pi(b)\pi(b^{*^p})\xi, \pi(b^{*^{p-1}})\xi\rangle^{\frac{1}{2}} \\
&= \langle q^{2klp}\lambda_0^{2k}\prod_{r=1}^{l}(1-\lambda_0 q^{2(l(p-1)+r)})\pi(b^{*^{p-1}})\xi, \pi(b^{*^{p-1}})\xi\rangle^{\frac{1}{2}} \\
&= q^{klp}\lambda_0^{k}\prod_{r=1}^{l}(1-\lambda_0 q^{2(l(p-1)+r)})^{\frac{1}{2}}\|\pi(b^{*^{p-1}})\xi\|.
\end{aligned}$$

Hence

$$\pi(b)e_p = \frac{\pi(b)\pi(b^{*^p})\xi}{\|\pi(b^{*^p})\xi\|} = \frac{q^{2lpk}\lambda_0^{2k}\prod_{r=1}^{l}(1-\lambda_0 q^{2(l(p-1)+r)})\pi(b^{*^{p-1}})}{q^{lpk}\lambda_0^{k}\prod_{r=1}^{l}(1-\lambda_0 q^{2(l(p-1)+r)})^{\frac{1}{2}}\|\pi(b^{*^{p-1}})\|},$$

which simplifies to

$$\pi(b)e_p = q^{lpk}\lambda_0^{k}\prod_{r=1}^{l}(1-\lambda_0 q^{2(l(p-1)+r)})^{\frac{1}{2}}e_{p-1}.$$

Reversing the order of multiplication by using the substition $r' = l - r$ we arrive at the following result:

$$\pi(b)(e_p) = q^{lpk}\lambda_0^k \prod_{r'=1}^{l} (1 - \lambda_0 q^{2(lp-r')})^{\frac{1}{2}} e_{p-1}.$$

Considering the case when $p = 0$ we see that $\pi(b)e_0 = \pi(b)\xi = 0$ since $\pi(b)$ acts as a stepping down operator on the basis elements e_p. This implies that

$$\lambda_0^k \prod_{r'=1}^{l} (1 - \lambda_0 q^{-2r'}) = 0,$$

therefore $\lambda_0 = 0$, which corresponds to the one dimensional case, or $\lambda_0 = q^{2s}$ for some $s \in \{1, 2, ..., l\}$, which relates to the infinite dimensional case. □

Proposition 3.2.3. *Each of the representations π_s is a faithful representation.*

Proof. We use reasoning similar to that in the proof of [21, Proposition 1]. Consider an arbitrary element of $\mathcal{O}(\mathbb{WP}_q(k,l))$ expressed as a linear combination of the basis (3.5),

$$x = \sum_{m,n} \mu_{m,n} a^m b^n + \sum_{m,n'} \nu_{m,n'} a^m b^{*n'},$$

$\mu_{m,n}, \nu_{m,n'} \in \mathbb{C}$, and suppose that $\pi_s(x) = 0$, i.e. $\pi_s(x)e_p^s = 0$, for all $p \in \mathbb{N}$. Since the application of $\pi_s(a^m b^n)$ to e_p^s does not increase the index p, while the application of $\pi_s(a^m b^{*n'})$ increases p, the vanishing condition splits into two cases, which can be dealt with separately, and we deal only with the first of them. The condition

$$\pi_s\left(\sum_{m,n} \mu_{m,n} a^m b^n\right) e_p^s = 0, \qquad \text{for all } p \in \mathbb{N},$$

is equivalent to the system of equations

$$\sum_{m,n} \mu_{m,n} q^{nk[l(p-(n-1)/2)+s]} q^{2m[l(p-n)+s]} \prod_{r=1}^{ln} \left(1 - q^{2(lp+s-r)}\right)^{1/2} e_{p-n}^s = 0, \quad \text{for all } p \in \mathbb{N},$$

(use (3.3) repetitively). Since this must be true for all $n \le p$, and the vectors e_{p-n}^s are linearly independent for different n, we obtain a system of equations, one for each n,

$$\sum_{m} \mu_{m,n} q^{2m[l(p-n)+s]} = 0, \qquad \text{for all } p \in \mathbb{N}. \tag{3.10}$$

There are only finitely many non-zero coefficients $\mu_{m,n}$. Let N be the smallest natural number such that $\mu_{m,n} = 0$, for all $m > N$. Define

$$\lambda_{p,n} = q^{2[l(p-n)+s]}.$$

Then equations (3.10) for $\mu_{m,n}$ take the form

$$\sum_{m=0}^{N} \mu_{m,n}\lambda_{p,n}^{m} = 0, \qquad \text{for all } p \in \mathbb{N}.$$

The matrix of coeffcients of the first $N+1$ equations (for $p = 0, 1, \ldots, N$) has the Vandermonde form, and is invertible since $\lambda_{p,n} \neq \lambda_{p',n}$ if $p \neq p'$ (remember that $q \in (0,1)$). Therefore, $\mu_{m,n} = 0$ is the only solution to (3.10). Similarly one proves that necessarily $\nu_{m,n'} = 0$ and concludes that π_s is an injective map. □

Finally, we look at the way representations of $\mathcal{O}(\mathbb{WP}_q(k,l))$ are related to representations of the quantum sphere algebra $\mathcal{O}(S_q^3)$.

Proposition 3.2.4. *Let $\pi : \mathcal{O}(S_q^3) \to \mathrm{End}(V)$ denote the representation of $\mathcal{O}(S_q^3)$ given on an orthonormal basis e_n, $n \in \mathbb{N}$ of V by [44]*

$$\pi(\alpha)e_n = \left(1 - q^{2n}\right)^{1/2} e_{n-1}, \qquad \pi(\beta)e_n = q^{n+1}e_n, \tag{3.11}$$

(see also [49]). Then there exists an algebra isomorphism $\phi : \mathrm{End}(\bigoplus_{s=1}^{l} V_s) \to \mathrm{End}(V)$ rendering commutative the following diagram,

$$\begin{tikzcd} \mathcal{O}(\mathbb{WP}_q(k,l)) \arrow[r, "\theta"] \arrow[d, "\oplus_s \pi_s"'] & \mathcal{O}(S_q^3) \arrow[d, "\pi"] \\ \mathrm{End}(\bigoplus_{s=1}^{l} V_s) \arrow[r, "\phi"] & \mathrm{End}(V), \end{tikzcd} \tag{3.12}$$

where θ is the $*$-algebra map given by formulae (3.4).

Proof. Consider the vector space isomorphism,

$$\hat{\phi} : \bigoplus_{s=1}^{l} V_s \to V, \qquad e_p^s \mapsto e_{lp+s-1},$$

and let $\phi : \mathrm{End}(\bigoplus_{s=1}^{l} V_s) \to \mathrm{End}(V)$, $f \mapsto \hat{\phi}\circ f \circ \hat{\phi}^{-1}$, be the induced algebra isomorphism. Using (3.11) one easily finds that

$$\pi(\alpha^l\beta^k)e_n = q^{k(n+1)} \prod_{r=1}^{l} \left(1 - q^{2(n-r+1)}\right)^{1/2} e_{n-1}, \qquad \pi(\beta\beta^*)e_n = q^{2(n+1)}e_n.$$

This immediately implies that, for all $x \in \mathcal{O}(\mathbb{WP}_q(k,l))$,

$$\hat{\phi}\left(\pi_s(x)e_p^s\right) = \pi\left(\theta(x)\right)\hat{\phi}(e_p^s).$$

Therefore, the induced map ϕ makes the diagram (3.12) commute as required.
□

3.3 Quantum teardrops and quantum principal bundles

At the limit $q \to 1$, $\mathcal{O}(\mathbb{WP}_q(k,l))$ becomes a commutative algebra (of polynomials on the weighted projective space) but the polynomials on the right hand side of the third (equivalently, fourth) relation in (3.3) have multiple roots at $a = 0$ and $a = 1$. The multiplicity of roots indicates that weighted projective lines have singularities, in particular they are not smooth manifolds unless $k = l = 1$. On the other hand, in the case $k = 1$ (which corresponds to the teardrop) and $q \neq 1$ the defining relations (3.3) of $\mathcal{O}(\mathbb{WP}_q(1,l))$ have no repeated roots. This suggests that by moving from the commutative singular manifold $\mathbb{WP}(1,l)$ to the noncommutative quantum space $\mathcal{O}(\mathbb{WP}_q(1,l))$ one is able to resolve singularities and obtain a smooth quantum manifold, very much in the spirit of noncommutative crepant resolutions [46]. In this section we show that in the special case where $k = 1$ we can in fact construct non-commutative principal bundle over $\mathcal{O}(\mathbb{WP}_q(1,l))$, the quantum teardrop.

In the first part of this section we prove that only in the case of the quantum 2-sphere (i.e. $k = l = 1$), $\mathcal{O}(S_q^3)$ is a principal $\mathcal{O}(U(1))$-comodule algebra over $\mathcal{O}(\mathbb{WP}_q(k,l))$. In that case we are dealing with the well-known quantum version of the Hopf fibration.

Theorem 3.3.1. *The algebra of $\mathcal{O}(S_q^3)$ is a principal $\mathcal{O}(U(1))$-comodule algebra over $\mathcal{O}(\mathbb{WP}_q(k,l))$ by the coaction $\varrho_{k,l}$ if and only if $k = l = 1$.*

Proof. If $k = l = 1$, then $\mathcal{O}(\mathbb{WP}_q(1,1)) = \mathcal{O}(S_q^2)$, and it is known that $\mathcal{O}(S_q^3)$ is a principal comodule algebra that describes the quantum Hopf fibration (over the standard Podleś sphere); see [13] or [22]. We assume, therefore, that $k \neq l$ (i.e. $(k,l) \neq (1,1)$), and show that $1 \otimes u$ is not in the image of the canonical map in that case. We proceed by identifying a basis for $\mathcal{O}(SU_q(2)) \otimes \mathcal{O}(SU_q(2))$ and applying the canonical map to observe the form of the image. The ultimate aim is to show that the canonical map is not surjective by proving that element $1 \otimes u$, which is in the codomain of the canonical map, is not in the image.

A basis for $\mathcal{O}(SU_q(2)) \otimes \mathcal{O}(SU_q(2))$ consists of

$$\begin{array}{ll} \alpha^h \beta^m \beta^{*n} \otimes \alpha^{\bar{h}} \beta^{\bar{m}} \beta^{*\bar{n}}, & \alpha^h \beta^m \beta^{*n} \otimes \beta^{\bar{m}} \beta^{*\bar{n}} \alpha^{*\bar{p}}, \\ \beta^m \beta^{*n} \alpha^{*p} \otimes \alpha^{\bar{h}} \beta^{\bar{m}} \beta^{*\bar{n}}, & \beta^m \beta^{*n} \alpha^{*p} \otimes \beta^{\bar{m}} \beta^{*\bar{n}} \alpha^{*\bar{p}}, \end{array}$$

where $h, m, n, \bar{h}, \bar{m}, \bar{n}, \bar{p} \in \mathbb{N}$. Hence, applying the canonical map we conclude that every element in the image of can is a linear combination of:

$$\begin{array}{ll} \alpha^{h+\bar{h}} \beta^{m+\bar{m}} \beta^{*n+\bar{n}} \otimes u^{k\bar{h}-l\bar{m}+l\bar{n}}, & \beta^{m+\bar{m}} \beta^{*n+\bar{n}} \alpha^h \alpha^{*\bar{p}} \otimes u^{-l\bar{m}+l\bar{n}-k\bar{p}}, \\ \beta^{m+\bar{m}} \beta^{*n+\bar{n}} \alpha^{*p} \alpha^{\bar{h}} \otimes u^{k\bar{h}-l\bar{m}+l\bar{n}}, & \beta^{m+\bar{m}} \beta^{*n+\bar{n}} \alpha^{*p+\bar{p}} \otimes u^{-l\bar{m}+l\bar{n}-k\bar{p}}, \end{array} \tag{3.13}$$

where $h, m, n, \bar{h}, \bar{m}, \bar{n}, \bar{p} \in \mathbb{N}$.

To obtain identity in the first leg we must use one of the following relations (3.1) or equations which include terms of the form $\alpha^{*n}\alpha^m$ or $\alpha^n\alpha^{*m}$. A straightforward calculation gives the following:

$$\alpha^m\alpha^{*n} = \begin{cases} \prod_{p=1}^{m}(1-q^{2p-2}\beta\beta^*) & \text{when } m=n, \\ \alpha^{m-n}\prod_{p=1}^{n}(1-q^{2p-2}\beta\beta^*) & \text{when } m>n, \\ \prod_{p=1}^{m}(1-q^{2p-2}\beta\beta^*)\alpha^{*n-m} & \text{when } n>m, \end{cases}$$

and

$$\alpha^{*n}\alpha^m = \begin{cases} \prod_{p=1}^{m}(1-q^{-2p}\beta\beta^*) & \text{when } m=n, \\ \alpha^{*n-m}\prod_{p=1}^{m}(1-q^{-2p}\beta\beta^*) & \text{when } n>m, \\ \prod_{p=1}^{n}(1-q^{-2p}\beta\beta^*)\alpha^{m-n} & \text{when } n<m. \end{cases}$$

We see that to obtain identity in the first leg we require the powers of α and α^* to be equal. We now construct all possible elements of the domain which map to $1 \otimes u$ after applying the canonical map.

Case 1: Use the second term in (3.13) to obtain $\alpha^N\alpha^{*N}$. In this case: $h = \bar{p} = N$, $n+\bar{n} = m+\bar{m} = 0$. Since $n, \bar{n}, m, \bar{m} \in \mathbb{N}$ we must have $n = \bar{n} = m = \bar{m} = 0$. Also $-l\bar{m}+l\bar{n}-k\bar{p} = 1$, which implies that $-k\bar{p} = 1$, hence there are no possible terms.

Case 2: Use the third term in (3.13) to obtain $\alpha^{*N}\alpha^N$. In this case $\bar{h} = p = N$, $m = \bar{m} = n = \bar{n} = 0$. Also $k\bar{h}-l\bar{m}+l\bar{n} = 1$, which implies that $k\bar{h} = 1$, hence $k = 1$ and $\bar{h} = 1$. Therefore, the only terms of the form $\alpha^{*N}\alpha^N$ are when $N = 1$ and in this case $k = 1$. We now look at the other terms which are of the form $\beta\beta^*$ so that we can use the relation $\alpha^*\alpha+q^{-2}\beta\beta^* = 1$. Four possibilities need be considered, one for each of the terms in (3.13). In the case of the first of these terms $h = \bar{h} = 0$, $m+\bar{m} = 1$, $n+\bar{n} = 1$ and $k\bar{h}-l\bar{m}+l\bar{n} = 1$, which implies that $l(\bar{n}-\bar{m}) = 1$, hence $l = 1$ and $\bar{n}-\bar{m} = 1$. The only solution is: $l = 1$, $n = \bar{m} = 0$, $m = \bar{n} = 1$, $h = \bar{h} = 0$. A similar approach can be used when considering the remaining terms in (3.13) to conclude that in all four cases one is forced to require $l = 1$. Therefore, it is impossible to obtain a term of the form $1\otimes u$ when both k and l are not simultaneously equal to one. This shows that the canonical map is not sujective, hence not an isomorphism, implying that $\mathcal{O}(\mathbb{WP}_q(k,l)) \subseteq \mathcal{O}(SU_q(2))$ is not a Hopf-Galois extension when k and l are not both one. $\square$

Theorem 3.3.1 asserts that the defining action (3.2) of $U(1)$ on the quantum group $\mathcal{O}(SU_q(2))$ does not make it a total space of a quantum $U(1)$-principal bundle over the quantum weighted projective space $\mathcal{O}(\mathbb{WP}_q(k,l))$, unless $k = l = 1$ (the case of the quantum Hopf fibration). The remainder of this section is devoted to construction of a quantum $U(1)$-principal bundle over the quantum teardrop $\mathcal{O}(\mathbb{WP}_q(1,l))$ with the total space provided by the *quantum lens space* $\mathcal{O}(L_q(l;1,l))$. The coordinate algebra of the *quantum lens space* $\mathcal{O}(L_q(l;1,l))$ is defined as follows [30].

Definition 3.3.2. The coordinate (or group) algebra of the cyclic group $\mathbb{Z}_l$, $\mathcal{O}(\mathbb{Z}_l)$, is a Hopf $*$-algebra generated by a unitary grouplike element w satisfying $w^l = 1$. $\mathcal{O}(SU_q(2))$ is a right $\mathcal{O}(\mathbb{Z}_l)$-comodule $*$-algebra with the coaction

$$\varrho : \mathcal{O}(SU_q(2)) \to \mathcal{O}(SU_q(2)) \otimes \mathcal{O}(\mathbb{Z}_l), \qquad \alpha \mapsto \alpha \otimes w, \quad \beta \mapsto \beta \otimes 1.$$

$\mathcal{O}(L_q(l;1,l))$ is defined as the coinvariant subalgebra of $\mathcal{O}(SU_q(2))$ under the coaction ϱ.

To view $\mathcal{O}(SU_q(2))$ as a comodule algebra of a group algebra of the finite cyclic group $\mathbb{Z}_l$ is the same as equipping $\mathcal{O}(SU_q(2))$ with a $\mathbb{Z}_l$-grading compatible with multiplication and $*$-involution. From the graded algebra point of view, α has degree 1 and β has degree 0. $\mathcal{O}(L_q(l;1,l))$ is the degree zero part of $\mathcal{O}(SU_q(2))$ (all calculations of degrees modulo l).

Proposition 3.3.3. *(i) The algebra $\mathcal{O}(L_q(l;1,l))$ is the $*$-subalgebra of $\mathcal{O}(S_q^3)$ generated by $c := \alpha^l$ and $d := \beta$*

(ii) The elements c and d satisfy the following relations

$$cd = q^l dc, \qquad cd^* = q^l d^* c, \qquad dd^* = d^* d, \tag{3.14}$$

$$cc^* = \prod_{m=0}^{l-1}(1 - q^{2m} dd^*), \qquad c^* c = \prod_{m=1}^{l}(1 - q^{-2m} dd^*). \tag{3.15}$$

(iii) Universally, $\mathcal{O}(L_q(l;1,l))$ can be defined as a $*$-algebra generated by c and d subject to relations (3.14) *and* (3.15).

Proof. Follow similar arguments to those in Section 3.2 . □

Again, following the same techniques as in Section 3.2 one can classify – up to unitary equivalence – all irreducible $*$-representations of $\mathcal{O}(L_q(l;1,l))$.

Proposition 3.3.4. *There is a family of one-dimensional representations π_0^λ defined as*

$$\pi_0^\lambda(c) = \lambda, \qquad \pi_0^\lambda(d) = 0, \qquad \lambda \in \mathbb{C},\ |\lambda| = 1.$$

For every $s = 1, 2, \ldots, l$, there is a family of infinite-dimensional representations π_s^λ, $\lambda \in \mathbb{C}$, $|\lambda| = 1$. The action of π_s^λ on an orthonormal basis $e_p^{\lambda,s}$, $p \in \mathbb{N}$, for its representation space $V_s^\lambda \simeq l^2(\mathbb{N})$ is given by

$$\pi_s^\lambda(c) e_p^{\lambda,s} = \prod_{m=1}^{l}(1 - q^{2(pl+s-m)})^{1/2} e_{p-1}^{\lambda,s}, \qquad \pi_s^\lambda(d) e_p^{\lambda,s} = \lambda q^{pl+s} e_p^{\lambda,s}.$$

Proof. Use the same arguments as in Section 3.2. □

As for quantum teardrops, there is a vector space isomorphism

$$\phi^\lambda : \bigoplus_{s=1}^{l} V_s^\lambda \to V^\lambda, \qquad e_p^{\lambda,s} \mapsto e_{lp+s-1}^\lambda,$$

which embeds the direct sum of representations π_s^λ in the representation π^λ of $\mathcal{O}(SU_q(2))$,

$$\pi^\lambda(\alpha) e_n^\lambda = \left(1 - q^{2n}\right)^{1/2} e_{n-1}^\lambda, \qquad \pi^\lambda(\beta) e_n^\lambda = \lambda q^{n+1} e_n^\lambda.$$

Here e_n^λ, $n \in \mathbb{N}$, is an orthonormal basis for the representation space V^λ.

$\mathcal{O}(L_q(l;1,l))$ is a right comodule algebra over the Hopf algebra $\mathcal{O}(U(1)) = \mathbb{C}[u,u^*]$, $u^{-1} = u^*$. The coaction $\varrho_l : \mathcal{O}(L_q(l;1,l)) \to \mathcal{O}(L_q(l;1,l)) \otimes \mathcal{O}(U(1))$ is given on generators c and d by

$$\varrho_l : c \mapsto c \otimes u, \qquad d \mapsto d \otimes u^*.$$

It is an easy to check that

$$\mathcal{O}(L_q(l;1,l))^{co\mathcal{O}(U(1))} \simeq \mathcal{O}(\mathbb{WP}_q(1,l)),$$

through the identification $a = cd$, $b = dd^*$. Equivalently, $\mathcal{O}(L_q(l;1,l))$ can be viewed as a $\mathbb{Z}$-graded $*$-algebra generated by c of degree 1 and d of degree -1, and $\mathcal{O}(\mathbb{WP}_q(1,l))$ is the degree zero part of $\mathcal{O}(L_q(l;1,l))$ graded in that way.

Theorem 3.3.5. *The coordinate algebra of the quantum lens space $A = \mathcal{O}(L_q(l;1,l))$ is a principal $H = \mathcal{O}(U(1))$-comodule algebra over $B = \mathcal{O}(\mathbb{WP}_q(1,l))$.*

Proof. Using Proposition 2.4.15 we show that the right $\mathcal{O}(U(1))$-comodule $\mathcal{O}(L_q(l;1,l))$ is principal by constructing a strong connection. A strong connection for $\mathcal{O}(L_q(l;1,l))$ is defined recursively as follows. Firstly $\omega(1) = 1 \otimes 1$ and then for all $n > 0$,

$$\omega(u^n) = c^*\omega(u^{n-1})c - \sum_{m=1}^{l} (-1)^m q^{-m(m+1)} \binom{l}{m}_{q^{-2}} d^m d^{*m-1} \omega(u^{n-1}) d^*, \tag{3.16}$$

$$\omega(u^{-n}) = c\omega(u^{-n+1})c^* - \sum_{m=1}^{l} (-1)^m q^{m(m-1)} \binom{l}{m}_{q^2} d^{m-1} d^{*m} \omega(u^{-n+1}) d, \tag{3.17}$$

where, for all $x \in \mathbb{R}$, the *deformed* or *q-binomial* coefficients $\binom{l}{m}_x$ are defined by the following polynomial equality in an indeterminate t

$$\prod_{m=1}^{l} (1 + x^{m-1}t) = \sum_{m=0}^{l} x^{m(m-1)/2} \binom{l}{m}_x t^m. \tag{3.18}$$

Before we check that ω has the properties of a strong connection; see Proposition 2.4.15, we observe that since we are dealing with $\mathcal{O}(U(1))$-comodules, conditions (2.15a) and (2.15d) have straightforward meaning in terms of $\mathbb{Z}$-graded algebras. $\mathcal{O}(U(1))$ is a $\mathbb{Z}$-graded $*$-algebra generated by u of degree 1. $\mathcal{O}(L_q(l;1,l)) \otimes \mathcal{O}(L_q(l;1,l))$ is a $\mathbb{Z}$-bigraded space with $x \otimes y$ of degree (r,s) for all $x \in \mathcal{O}(L_q(l;1,l))$ of degree $-r$ and $y \in \mathcal{O}(L_q(l;1,l))$ of degree s (note the change of sign). Conditions (2.15c) and (2.15d) mean that ω is a left and right degree preserving map. By construction, ω has property (2.15a). The remaining properties are proven by induction on n. That $m_A(\omega(u)) = \varepsilon(u) = 1$ follows by the second of equations (3.15) combined with (3.18). Applying $\mathrm{id} \otimes \varrho_l$ to the right hand side of (3.16) (with $n = 1$), one immediately obtains that $(\mathrm{id} \otimes \varrho_l)(\omega(u)) = \omega(u) \otimes u$, as required for (2.15c). Similarly one checks (2.15d).

Now, assume that $\omega(u^{n-1})$ satisfies conditions (2.15b)–(2.15d). Then, multiplying the right hand side of (3.16) and using $m_A(\omega(u^{n-1})) = \varepsilon(u^{n-1}) = 1$, we obtain

$$m_A(\omega(u^n)) = c^*c - \sum_{m=1}^{l} (-1)^m q^{-m(m+1)} \binom{l}{m}_{q^{-2}} d^m d^{*m} = 1,$$

by (3.15) and (3.18). Since ϱ_l is an algebra map and $(\mathrm{id}\otimes\varrho_l)(\omega(u^{n-1})) = \omega(u^{n-1})\otimes u^{n-1}$ by inductive assumption, we can compute

$$\begin{aligned}(\mathrm{id}\otimes\varrho_l)(\omega(u^n)) =& c^*(\mathrm{id}\otimes\varrho_l)(\omega(u^{n-1})c) \\ & - \sum_{m=1}^{l}(-1)^m q^{-m(m+1)}\binom{l}{m}_{q^{-2}} d^m d^{*m-1}(\mathrm{id}\otimes\varrho_l)(\omega(u^{n-1})d^*) \\ =& c^*\omega(u^{n-1})c\otimes u^n - \sum_{m=1}^{l}(-1)^m q^{-m(m+1)}\binom{l}{m}_{q^{-2}} d^m d^{*m-1}\omega(u^{n-1})d^*\otimes u^n \\ =& \omega(u^n)\otimes u^n,\end{aligned}$$

as required. The left colinearity of ω (2.15d) is proven in a similar way. The case of $\omega(u^{-n})$ is treated in the same manner. □

Recall that a principal H-comodule algebra A is said to be *cleft* if there exists a convolution invertible, right H-colinear map $j : H \to A$ such that $j(1) = 1$. The special case of this is when j is an algebra map (then its convolution inverse is $j\circ S$) and this corresponds to the trivial quantum principal bundle. In the case of comodule algebras over $\mathcal{O}(U(1)) = \mathbb{C}[u, u^{-1}]$ the necessary condition for a map $j : \mathcal{O}(U(1)) \to A$ to be convolution invertible is that $j(u)$ is an invertible element (unit) of A. Arguing as in [22, Appendix] we obtain

Lemma 3.3.6. *The principal $\mathcal{O}(U(1))$-comodule algebra $\mathcal{O}(L_q(l;1,l))$ is not cleft.*

Proof. Multiples of 1 are the only invertible elements of $\mathcal{O}(SU_q(2))$; see [22, Appendix]. Since $\mathcal{O}(L_q(l;1,l))$ is a subalgebra of $\mathcal{O}(SU_q(2))$ the same can be said about $\mathcal{O}(L_q(l;1,l))$. Thus any convolution invertible map $j : \mathcal{O}(U(1)) \to \mathcal{O}(L_q(l;1,l))$ must have the form $j(u) = \lambda 1$, for some $\lambda\in\mathbb{C}^*$. This, however, violates the right $\mathcal{O}(U(1))$-colinearity of j or, equivalently, that j is a $\mathbb{Z}$-degree preserving map. □

The surjectivity of the canonical map in Definition 2.3.2 corresponds to the freeness of the coaction ϱ of H on A. By Theorem 3.3.1 we know that if $(k,l)\neq(1,1)$, then the coaction $\varrho_{k,l}$ of $\mathcal{O}(U(1))$ on $\mathcal{O}(SU_q(2))$ is not free. However, Theorem 3.3.5 implies that $\varrho_{1,l}$ is *almost free* in the following sense.

Definition 3.3.7. Let H be a Hopf algebra and let A be a right H-comodule algebra with coaction $\varrho : A \to A\otimes H$. We say that the coaction is *almost free* if the cokernel of the (lifted) canonical map

$$\overline{\mathsf{can}} : A\otimes A \to A\otimes H, \qquad a\otimes a' \mapsto a\varrho(a'),$$

is finitely generated as a left A-module.

Corollary 3.3.8. *The coaction $\varrho_{1,l}$ is almost free.*

Proof. Note that the $*$-algebra inclusion

$$\iota : \mathcal{O}(L_q(l;1,l)) \hookrightarrow \mathcal{O}(SU_q(2)), \qquad c\mapsto\alpha^l, \quad d\mapsto\beta,$$

makes the following diagram commute

$$\begin{tikzcd}
\mathcal{O}(L_q(l;1,l)) \arrow[r, "\iota"] \arrow[d, "\varrho_l"'] & \mathcal{O}(SU_q(2)) \arrow[d, "\varrho_{1,l}"] \\
\mathcal{O}(L_q(l;1,l)) \otimes \mathcal{O}(U(1)) \arrow[r, "\iota\otimes(-)^l"] & \mathcal{O}(SU_q(2)) \otimes \mathcal{O}(U(1)),
\end{tikzcd}$$

where $(-)^l : u \to u^l$. The surjectivity of the canonical map $\mathcal{O}(L_q(l;1,l)) \otimes \mathcal{O}(L_q(l;1,l)) \to \mathcal{O}(L_q(l;1,l)) \otimes \mathcal{O}(U(1))$ (proven in Theorem 3.3.5) implies that $1 \otimes u^{ml} \in \mathrm{Im}(\overline{\mathsf{can}})$, $m \in \mathbb{Z}$, where $\overline{\mathsf{can}}$ is the (lifted) canonical map corresponding to the coaction $\varrho_{1,l}$. This means that $\mathcal{O}(SU_q(2)) \otimes \mathbb{C}[u^l, u^{-l}] \subseteq \mathrm{Im}(\overline{\mathsf{can}})$. Therefore, there is a short exact sequence of left $\mathcal{O}(SU_q(2))$-modules

$$(\mathcal{O}(SU_q(2)) \otimes \mathbb{C}[u, u^{-1}])/(\mathcal{O}(SU_q(2)) \otimes \mathbb{C}[u^l, u^{-l}]) \longrightarrow \mathrm{coker}(\overline{\mathsf{can}}) \longrightarrow 0.$$

The left $\mathcal{O}(SU_q(2))$-module $(\mathcal{O}(SU_q(2)) \otimes \mathbb{C}[u, u^{-1}])/(\mathcal{O}(SU_q(2)) \otimes \mathbb{C}[u^l, u^{-l}])$ is finitely generated, hence so is $\mathrm{coker}(\overline{\mathsf{can}})$. □

3.4 Quantum weighted projective spaces as generalised Weyl algebras

Generalised Weyl algebras are defined in [3] as follows.

Definition 3.4.1. Let $\mathcal{D}$ be a ring, $\sigma = (\sigma_1, ..., \sigma_n)$ a set of commuting automorphisms of $\mathcal{D}$ and $\tilde{a} = (\tilde{a_1}, ..., \tilde{a_n})$ a set of (non-zero) elements of the centre $Z(\mathcal{D})$ of $\mathcal{D}$ such that $\sigma_i(\tilde{a_j}) = \tilde{a_j}$ for all $i \neq j$. The associated *generalised Weyl algebra* $\mathcal{D}(\sigma, \tilde{a})$ *of degree* n is a ring generated by $\mathcal{D}$ and the $2n$ indeterminates $X_1^+, ..., X_n^+, X_1^-, ..., X_n^-$ subject to the following relations, for all $\alpha \in \mathcal{D}$:

$$X_i^- X_i^+ = \tilde{a_i}, \qquad X_i^+ X_i^- = \sigma_i(\tilde{a_i}), \qquad X_i^{\pm}\alpha = \sigma_i^{\pm}(\alpha) X_i^{\pm}, \tag{3.19}$$

$$[X_i^-, X_j^-] = [X_i^+, X_j^+] = [X_i^+, X_j^-] = 0, \quad \forall i \neq j, \tag{3.20}$$

where $[x, y] = xy - yx$.

We call $\tilde{a}$ the *defining elements* and σ the *defining automorphisms* of $\mathcal{D}(\sigma, \tilde{a})$. Note that in the degree one case the relations (3.20) are null.

Proposition 3.4.2. *The algebras of coordinate functions $\mathcal{O}(\mathbb{WP}_q(k,l))$ are degree one generalised Weyl algebras.*

Proof. Set $\mathcal{D} = \mathbb{C}[a]$. In this case $X^+ = b$, $X^- = b$ and automorphism σ of $\mathcal{D}$ and defining elements $\widetilde{a}$ are

$$\sigma(a) = q^{2l} a, \qquad \widetilde{a} = a^k \prod_{m=1}^{l} (1 - q^{-2m} a).$$

Now

$$\sigma(\widetilde{a}) = q^{2kl} a \prod_{m=1}^{l} (1 - q^{-2m+2l} a) \Longrightarrow \sigma(\widetilde{a}) = q^{2kl} a \prod_{m'=1}^{l} (1 - q^{2m'} a) = bb^* = X^+ X^-,$$

by making the substitution $m' = l - m$. The remaining relations can be verified in a similar way. □

One of the key theorems associated to generalised Weyl algebras [3, Theorem 1.6] provides an insight to the global dimension of such algebras. Since we have shown that the coordinate algebra $\mathcal{O}(\mathbb{WP}_q(k,l))$ of the quantum weighted projective line is a degree one generalised Weyl algebra, we can use [3, Theorem 1.6] to conclude that the global dimension of $\mathcal{O}(\mathbb{WP}_q(k,l))$ is equal to 2 if $k = 1$ and is infinite otherwise.[1] This can be used as an indication that, for the quantum teardrop case $k = 1$, the classical singularity has been removed (although it is not clear yet, whether $\mathcal{O}(\mathbb{WP}_q(1,l))$ is a Calabi-Yau algebra).

3.5 Fredholm modules and the Chern-Connes pairing for quantum teardrops

In this section first we follow associate even Fredholm modules to algebras $\mathcal{O}(\mathbb{WP}_q(k,l))$ and use them to construct traces or cycles in the cyclic bicomplex $CC_\bullet(\mathcal{O}(\mathbb{WP}_q(k,l)))$. The latter are then used to calculate the Chern number of a non-commutative line bundle associated to the quantum principal bundle $\mathcal{O}(L_q(l;1,l))$ over the quantum teardrop $\mathcal{O}(\mathbb{WP}_q(1,l))$; see subsection 2.5.3.

Proposition 3.5.1. *For every $s = 1, 2, \ldots, l$, let $\mathfrak{V}_s = V_s \oplus V_0$, where V_s is the Hilbert space $l^2(\mathbb{N})$ of representation π_s and $V_0 = \oplus\mathbb{C} \simeq l^2(\mathbb{N})$, which we take to be representation space of $\pi = \oplus\pi_0$; see Proposition 3.2.2. Define $\overline{\pi}_s := \pi_s \oplus \pi$,*

$$F = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad \gamma = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

Then $(\mathfrak{V}_s, \overline{\pi}_s, F, \gamma)$ are 1-summable Fredholm modules over $\mathcal{O}(\mathbb{WP}_q(k,l))$. The corresponding Chern characters are

$$\tau_s(a^m b^n) = \begin{cases} \frac{q^{2ms}}{1-q^{2ml}} & \text{if } n = 0,\ m \neq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.21}$$

Here $n \in \mathbb{Z}$ and, for a positive n, b^{-n} means b^{*n}.

Proof. It is obvious that $F^* = F$, $F^2 = \gamma^2 = I$ and $F\gamma + \gamma F = 0$. Next, by a straightforward calculation, for all $x \in \mathcal{O}(\mathbb{WP}_q(k,l))$,

$$[F, \pi(x)] = \begin{pmatrix} 0 & \pi(x) - \pi_s(x) \\ \pi_s(x) - \pi(x) & 0 \end{pmatrix}.$$

[1] We are grateful to Ulrich Krähmer for pointing this out to us.

Using the formulae in Proposition 3.2.2 one easily finds, for all $m, n, p \in \mathbb{N}$,

$$\pi_s(a^m b^n)e_p^s = q^{nk[lp-(n-1)/2+s]+2m[l(p-n)+s]} \prod_{r=1}^{ln} (1 - q^{2(lp+s-r)})^{1/2} e_{p-n}^s;$$

compare the proof of Proposition 3.2.3. This implies that, for positive m, $\pi_s(a^m b^n)$ are trace class operators, as $\mathrm{Tr}\ (\pi_s(a^m b^n)) = 0$ if $n \neq 0$ and

$$\mathrm{Tr}\ (\pi_s(a^m)) = \sum_p q^{2m(lp+s)} = \frac{q^{2ms}}{1 - q^{2ml}}. \tag{3.22}$$

Since $\pi_0(a^m b^n) = 0$, if $(m, n) \neq (0, 0)$, and $\pi_0(1) = 1$, we conclude that, for all $x \in \mathcal{O}(\mathbb{WP}_q(k, l))$, $\pi(x) - \pi_s(x)$ is a trace class operator. Therefore, $(\mathfrak{V}_s, \overline{\pi}_s, F, \gamma)$ is a 1-summable Fredholm module.

Finally,

$$\tau_s(x) = \mathrm{Tr}\ (\gamma \overline{\pi}_s(x)) = \mathrm{Tr}\ (\pi_s(x) - \pi(x)),$$

and the formula (3.21) follows by equation (3.22). □

Note that the form of the Chern character of $(\mathfrak{V}_s, \overline{\pi}_s, F, \gamma)$ is independent of k. In the case $l = 1$, necessarily $s = 1$ and τ_1 coincides with the trace calculated for the quantum 2-sphere in [34]. Similarly to the case studied in [34], the characters τ_s on $\mathcal{O}(\mathbb{WP}_q(k, l))$ factor through the algebra map

$$\mathcal{O}(\mathbb{WP}_q(k, l)) \to \mathbb{C}[a], \qquad a^m b^n \mapsto \delta_{n,0} a^m.$$

On the polynomial algebra $\mathbb{C}[a]$ the characters are given by Jackson's integrals. More precisely, define $\hat{\tau}_s$ by the commutative diagram

$$\begin{tikzcd} \mathcal{O}(\mathbb{WP}_q(k,l)) \arrow[rr, "\tau_s"] \arrow[dr] & & \mathbb{C} \\ & \mathbb{C}[a]. \arrow[ur, "\hat{\tau}_s"'] & \end{tikzcd}$$

Then

$$\hat{\tau}_s(f) = \frac{1}{1 - q^{2l}} \int_0^{q^{2s}} \frac{f(a)}{a} d_{q^{2l}} a,$$

where the Jackson integral is defined by the formula

$$\int_0^x f(a) d_q a = \lim_{y \to 0} (1 - q) \sum_{r \in \mathbb{N}} (xq^r f(xq^r) - yq^r f(yq^r)),$$

for all $x \in \mathbb{R}$ and all f in $\mathbb{C}[a, a^{-1}]$.

Next, we show that $\mathcal{O}(L_q(l; 1, l))$ is not a cleft comodule algebra by calculating the Chern character of the zero-component of the line bundle $\mathcal{L}[1]$; see subsection 2.5.3.

Lemma 3.5.2. *The zero-component of the Chern character of $\mathcal{L}[1]$ is the class of*

$$\operatorname{Tr} E[1] = 1 + \sum_{m=1}^{l} (-1)^m q^{m(m-1)} \left(1 - q^{-2ml}\right) \binom{l}{m}_{q^2} a^m. \tag{3.23}$$

Proof. First, observe that the formula (3.18) yields the following identity for q-binomial coefficients:

$$\binom{l}{m}_{x^{-1}} = x^{m(m-l)} \binom{l}{m}_{x}. \tag{3.24}$$

Next, remember that $dd^* = a \in \mathcal{O}(\mathbb{WP}_q(1,l))$. Having these observations at hand the rest is a straightforward calculation:

$$\begin{aligned}
\operatorname{Tr} E[1] &= cc^* - \sum_{m=1}^{l} (-1)^m q^{-m(m+1)} \binom{l}{m}_{q^{-2}} d^m d^{*m} \\
&= \prod_{m=0}^{l-1} (1 - q^{2m} a) - \sum_{m=1}^{l} (-1)^m q^{2m(m-l)-m(m+1)} \binom{l}{m}_{q^2} a^m \\
&= 1 + \sum_{m=1}^{l} (-1)^m \left(q^{m(m-1)} - q^{m(m-1)-2ml}\right) \binom{l}{m}_{q^2} a^m \\
&= 1 + \sum_{m=1}^{l} (-1)^m q^{m(m-1)} \left(1 - q^{-2ml}\right) \binom{l}{m}_{q^2} a^m,
\end{aligned}$$

where the first equality follows by (3.16), the second by (3.15) and (3.24), while the third equality is a consequence of (3.18). □

Proposition 3.5.3. *For all $s = 1, 2, \ldots, l$, let τ_s be the cyclic cycle on $\mathcal{O}(\mathbb{WP}_q(1,l))$ constructed in Proposition 3.5.1. Then $\tau_s(\operatorname{Tr} E[1]) = 1$. Consequently, the left $\mathcal{O}(\mathbb{WP}_q(1,l))$-module $\mathcal{L}[1]$ is not free.*

Proof. Use (3.21) and (3.23) to calculate

$$\begin{aligned}
\tau_s(\operatorname{Tr} E[1]) &= \sum_{m=1}^{l} (-1)^m q^{m(m-1)} \left(1 - q^{-2ml}\right) \binom{l}{m}_{q^2} \frac{q^{2ms}}{1 - q^{2ml}} \\
&= -\sum_{m=1}^{l} (-1)^m q^{m(m-1)} \binom{l}{m}_{q^2} q^{2m(s-l)} \\
&= 1 - \sum_{m=0}^{l} (-1)^m q^{m(m-1)} \binom{l}{m}_{q^2} q^{2m(s-l)} \\
&= 1 - \prod_{m=1}^{l} \left(1 - q^{2(s-l+m-1)}\right) = 1 - \prod_{m=1}^{l} \left(1 - q^{2(s-m)}\right) = 1.
\end{aligned}$$

The last equality follows from the observation that since $s = 1, 2, \ldots, l$, one of the factors in the product must vanish. The fourth equality is a consequence of the definition of q-binomial coefficients (3.18). □

3.6 Algebras of continuous functions on quantum teardrops and their K-theory

The C^*-algebra $C(\mathbb{WP}_q(k,l))$ of continuous functions on the quantum weighted projective space $\mathbb{WP}_q(k,l)$ is defined as the subalgebra of bounded operators on the Hilbert space $\bigoplus_{s=1}^{l} V_s$ obtained as the completion of $\bigoplus_{s=1}^{l} \pi_s\left(\mathcal{O}(\mathbb{WP}_q(k,l))\right)$; see Proposition 3.2.2. In this section we show that this C^*-algebra is isomorphic to the direct sum of compact operators with adjoined identity.

Proposition 3.6.1. *Let $\mathcal{K}_s$ denote the algebra of all compact operators on the Hilbert space V_s. There is a split-exact sequence of C^*-algebra maps*

$$0 \longrightarrow \bigoplus_{s=1}^{l} \mathcal{K}_s \longrightarrow C(\mathbb{WP}_q(k,l)) \longrightarrow \mathbb{C} \longrightarrow 0. \tag{3.25}$$

Proof. We use a method of proof similar to that of [42][Proposition 1.2]. Write $\pi^{\oplus}$ for $\bigoplus_{s=1}^{l} \pi_s$. A basis for $\bigoplus_{s=1}^{l} V_s$ consists of eigenvectors e_p^s of $\pi^{\oplus}(a)$ with distinct eigenvalues $q^{2(lp+s+1)}$. Since, for all s, $q^{2(lp+s+1)} \to 0$ as $p \to \infty$, $\pi^{\oplus}(a)$ is a compact operator. Similarly, matrix coefficients $q^{lp+s+1}\prod_{r=0}^{l-1}\left(1-q^{2(lp+s-r)}\right)^{1/2}$ of $\pi_s(b)$ tend to 0 as p tends to infinity, hence also $\pi^{\oplus}(b)$ is a compact operator; compare the proof of Proposition 3.5.1. This proves that the kernel of the projection of $C(\mathbb{WP}_q(k,l))$ on the identity component $\mathbb{C}$ contains only compact operators.

The spectrum of $\pi^{\oplus}(a)$ consists of distinct numbers

$$\mathrm{sp}\left(\pi^{\oplus}(a)\right) = \{0\} \cup \{q^{2(lp+s)} \mid s = 1,2,\dots l,\ p \in \mathbb{N}\};$$

see the proof of Proposition 3.2.2. By functional calculus, for any s and p there are operators $f_{p,s}(\pi^{\oplus}(a))$ in $C(\mathbb{WP}_q(k,l))$ with spectrum given by

$$f_{p,s} : \mathrm{sp}\left(\pi^{\oplus}(a)\right) \to \mathbb{C}, \qquad 0 \mapsto 0, \qquad q^{2(ln+t)} \mapsto \delta_{s,t}\delta_{p,n}.$$

Hence $C(\mathbb{WP}_q(k,l))$ contains all orthogonal projections P_p^s to one-dimensional spaces spanned by the e_p^s. More explicitly, these are obtained as limits:

$$q^{-2ns} \prod_{r=0,\, r\neq p}^{n} \frac{\pi_s(a) - q^{2(lr+s)}}{q^{2lp} - q^{2lr}} \xrightarrow[n\to\infty]{} P_p^s. \tag{3.26}$$

Next, noting that $\pi_s(b)$ and $\pi_s(b^*)$ are shift-by-one operators with non-zero coefficients all the remaining generators of $\mathcal{K}_s$ (and hence of $\bigoplus_{s=1}^{l} \mathcal{K}_s$) can be obtained as products of the rescaled $\pi_s(b)$ and $\pi_s(b^*)$. Finally, the first map in the sequence (3.25) is injective since all the π_s are faithful representations. □

The following corollaries are straightforward consequences of Proposition 3.6.1.

Corollary 3.6.2. *The C^*-algebra $C(\mathbb{WP}_q(k,l))$ is isomorphic to the direct sum of l-copies of algebras of compact operators with the adjoined identity.*

Classically, in the unweighted case $\mathbb{WP}(1,1)$ becomes the 2-sphere S^2, and we know $C^*(S^2)$ corresponds to the compact operators with unit adjoined: this is consistent with our findings. In the case $l > 1$ we have a repeated root at the point a with order l, and when we consider the algebra of continuous functions on $\mathbb{WP}(k,l)$, Corollary 3.6.2 suggests each root corresponds to a copy of the compact operators. On the quantum level, Corollary 3.6.2 implies in particular that $C(\mathbb{WP}_q(k,l)) \ncong C(\mathbb{WP}_q(k,l'))$ if $l \neq l'$ and that $C(\mathbb{WP}_q(k,l)) \cong C(\mathbb{WP}_q(k',l))$, for all k, k'. This is compatible with the interpretation of equations (3.3) as partially resolving singularities of the classical weighted projective lines. The persistence of the multiple root at $a = 0$ indicates the singularity at 0 (of the classical weighted projective line) is not resolved, hence on the topological level different values of k correspond to the same quantum manifold. The separating of roots at $a = 1$ indicates the resolution of singularity (of multiplicity l) at 1; resolutions of singularities of different multiplicities might produce non-isomorphic manifolds.

Corollary 3.6.3. *The K-groups of $C(\mathbb{WP}_q(k,l))$ are:*

$$K_0\left(C(\mathbb{WP}_q(k,l))\right) = \mathbb{Z}^{l+1}, \qquad K_1\left(C(\mathbb{WP}_q(k,l))\right) = 0.$$

Proof. This follows immediately from Proposition 3.6.1 by recalling that $K_0(\mathcal{K}) = K_0(\mathbb{C}) = \mathbb{Z}$ and $K_1(\mathcal{K}) = K_1(\mathbb{C}) = 0$, where $\mathcal{K}$ is the C^*-algebra of compact operators on a separable Hilbert space; see Examples 1.4.3, 1.4.5 and 1.4.7. □

The first of the $\mathbb{Z}$ in $K_0\left(C(\mathbb{WP}_q(k,l))\right)$ corresponds to the rank of free modules, the remaining ones are generated by the classes of projections P_0^s; see (3.21). The cyclic cycles τ_s constructed in Proposition 3.5.1 (see (3.21)) extend to cycles on $C(\mathbb{WP}_q(k,l)$. Since, for any x that is not a multiple of identity, $\tau_s(x) = \mathrm{Tr}\,(\pi_s(x))$ we immediately conclude that $\tau_s(P_0^s) = 1$, and the index pairing between the K-theory and cyclic homology of $C(\mathbb{WP}_q(k,l)$ is given by

$$\langle[\tau_s],[P_0^t]\rangle = \delta_{s,t}\tau_s(P_0^t) = \delta_{s,t}.$$

Chapter 4

Quantum real projective spaces

In the previous chapter we took odd dimensional quantum spheres $\mathcal{O}(S_q^{2n+1})$ and viewed them as $\mathcal{O}(U(1))$-comodule algebras; by introducing weights to the coaction we described complex weighted projective spaces. We now consider a similar process over prolonged quantum spheres $\mathcal{O}(\Sigma_q^{2n+1})$. These spaces were studied in [10] and are constructed by taking the cotensor product of even dimensional quantum spheres $\mathcal{O}(S_q^{2n})$, viewed as right $\mathcal{O}(\mathbb{Z}_2)$-comodules, with the algebra of Laurent polynomials $\mathbb{C}[u, u^*]$ viewed as left $\mathcal{O}(\mathbb{Z}_2)$-comodules. By introducing a suitable weighted $\mathcal{O}(U(1))$-coaction on $\mathcal{O}(\Sigma_q^{2n+1})$ we are able to describe quantum weighted real projective spaces $\mathcal{O}(\mathbb{RP}_q(l_0, ..., l_n))$.

4.1 Weighted circle actions on prolonged spheres

In this section we recall the definitions of algebras we are interested in.

4.1.1 The $\mathcal{O}(\Sigma_q^{2n+1})$ and $\mathcal{O}(\mathbb{RP}_q(l_0, ..., l_n))$ coordinate algebras

Recalling (see Definition 2.2.6 and the following remark), for q be a real number $0 < q < 1$, the coordinate algebra $\mathcal{O}(S_q^{2n})$ of the even-dimensional quantum sphere is the unital complex $*$-algebra with generators $z_0, z_1, \ldots, z_n$, subject to the following relations:

$$z_i z_j = q z_j z_i \quad \text{for } i < j, \qquad z_i z_j^* = q z_j^* z_i \quad \text{for } i \neq j, \tag{4.1a}$$

$$z_i z_i^* = z_i^* z_i + (q^{-2} - 1) \sum_{m=i+1}^{n} z_m z_m^*, \qquad \sum_{m=0}^{n} z_m z_m^* = 1, \qquad z_n^* = z_n. \tag{4.1b}$$

$\mathcal{O}(S_q^{2n})$ is a $\mathbb{Z}_2$-graded algebra with $\deg(z_i) = 1$ and so is $\mathbb{C}[u, u^*]$ (with $\deg(u) = 1$). In other words, $\mathcal{O}(S_q^{2n})$ is a right $\mathbb{C}\mathbb{Z}_2$-comodule algebra and $\mathbb{C}[u, u^*]$ is a left $\mathbb{C}\mathbb{Z}_2$-comodule algebra, hence one can consider the cotensor product algebra, see Definition 2.3.4, $\mathcal{O}(\Sigma_q^{2n+1}) := \mathcal{O}(S_q^{2n})\Box_{\mathbb{C}\mathbb{Z}_2}\mathbb{C}[u, u^*]$. It was shown in [10] that, as a unital $*$-algebra $\mathcal{O}(\Sigma_q^{2n+1})$ has generators $\zeta_0, ..., \zeta_n$ and a central unitary ξ which are related in the following way.

$$\zeta_i \zeta_j = q \zeta_j \zeta_i \quad \text{for } i < j, \qquad \zeta_i \zeta_j^* = q \zeta_j^* \zeta_i \quad \text{for } i \neq j, \tag{4.2a}$$

$$\zeta_i\zeta_i^* = \zeta_i^*\zeta_i + (q^{-2}-1)\sum_{m=i+1}^{n}\zeta_m\zeta_m^*, \qquad \sum_{m=0}^{n}\zeta_m\zeta_m^* = 1, \qquad \zeta_n^* = \zeta_n\xi. \tag{4.2b}$$

For any choice of $n+1$ pairwise coprime numbers $l_0, ..., l_n$ one can define the coaction of the Hopf algebra $\mathcal{O}(U(1)) = \mathbb{C}[u, u^*]$ on $\mathcal{O}(\Sigma_q^{2n+1})$ as

$$\rho_{l_0,...,l_n} : \mathcal{O}(\Sigma_q^{2n+1}) \to \mathcal{O}(\Sigma_q^{2n+1}) \otimes \mathbb{C}[u, u^*], \qquad \zeta_i \mapsto \zeta_i \otimes u^{l_i}, \qquad \xi \mapsto \xi \otimes u^{-2l_n}, \tag{4.3}$$

for $i = 0, 1, ..., n$. This coaction is then extended to the whole of $\mathcal{O}(\Sigma_q^{2n+1})$ so that $\mathcal{O}(\Sigma_q^{2n+1})$ is a right $\mathbb{C}[u, u^*]$-comodule algebra.

The algebra of coordinate functions on the quantum real weighted projective space is now defined as the subalgebra of $\mathcal{O}(\Sigma_q^{2n+1})$ containing all coinvariant elements, i.e.,

$$\mathcal{O}(\mathbb{RP}_q(l_0, ..., l_n)) = \mathcal{O}(\Sigma_q^{2n+1})^{\mathcal{O}(U(1))} := \{x \in \mathcal{O}(\Sigma_q^{2n+1}) : \rho_{l_0,...,l_n}(x) = x \otimes 1\}.$$

4.1.2 The 2D quantum real projective space $\mathcal{O}(\mathbb{RP}_q(k, l)) \subset \mathcal{O}(\Sigma_q^3(k, l))$.

To gain a clear understanding of this space the $n = 1$ case is described in detail. Within this set-up k, l are coprime numbers, $\mathcal{O}(\Sigma_q^3)$ is generated by ζ_0, ζ_1 and a central unitary ξ such that

$$\zeta_0\zeta_1 = q\zeta_1\zeta_0, \quad \zeta_0\zeta_1^* = q\zeta_1^*\zeta_0, \tag{4.4a}$$

$$\zeta_0\zeta_0^* = \zeta_0^*\zeta_0 + (q^{-2}-1)\zeta_1^2\xi, \qquad \zeta_0\zeta_0^* + \zeta_1^2\xi = 1, \qquad \zeta_1^* = \zeta_1\xi. \tag{4.4b}$$

The linear basis of $\mathcal{O}(\Sigma_q^3)$ is

$$\{\zeta_0^r\zeta_1^s\xi^t, \ \zeta_0^{*r}\zeta_1^s\xi^t \mid r, s, \in \mathbb{N}, \ t \in \mathbb{Z}\}. \tag{4.5}$$

On generators the coaction $\varrho_{k,l}$ is given by

$$\zeta_0 \mapsto \zeta_0 \otimes u^k, \qquad \zeta_1 \mapsto \zeta_1 \otimes u^l, \qquad \xi \mapsto \xi \otimes u^{-2l}, \tag{4.6}$$

and extended to the whole of $\mathcal{O}(\Sigma_q^3)$ so that the coaction is an algebra map. We denote this comodule algebra by $\mathcal{O}(\Sigma_q^3(k, l))$.

It turns out that the two-dimensional quantum real projective spaces split into two cases depending not wholly on the parameter k, but instead whether k is either even or odd, and hence only the cases $k = 1$ and $k = 2$ need be considered [12].

The odd case.

For $k = 1$, $\mathcal{O}(\mathbb{RP}_q^2(l; -))$ is a polynomial $*$-algebra generated by a, b, c_- which satisfy the relations:

$$a = a^*, \qquad ab = q^{-2l}ba, \qquad ac_- = q^{-4l}c_-a, \qquad b^2 = q^{3l}ac_-, \qquad bc_- = q^{-2l}c_-b, \tag{4.7a}$$

$$bb^* = q^{2l}a\prod_{m=0}^{l-1}(1 - q^{2m}a), \qquad b^*b = a\prod_{m=1}^{l}(1 - q^{-2m}a), \tag{4.7b}$$

$$b^*c_- = q^{-l}\prod_{m=1}^{l}(1-q^{-2m}a)b, \qquad c_-b^* = q^l b\prod_{m=0}^{l-1}(1-q^{2m}a), \tag{4.7c}$$

$$c_-c_-^* = \prod_{m=0}^{2l-1}(1-q^{2m}a), \qquad c_-^*c_- = \prod_{m=1}^{2l}(1-q^{-2m}a). \tag{4.7d}$$

The embedding of generators of $\mathcal{O}(\mathbb{RP}_q^2(l;-))$ into $\mathcal{O}(\Sigma_q^3)$ or the isomorphism of $\mathcal{O}(\mathbb{RP}_q^2(l;-))$ with the coinvariants of $\mathcal{O}(\Sigma_q^3(k,l))$ is provided by

$$a \mapsto \zeta_1^2\xi, \qquad b \mapsto \zeta_0^l\zeta_1\xi, \qquad c_- \mapsto \zeta_0^{2l}\xi. \tag{4.8}$$

There is a family of one-dimensional representations of $\mathcal{O}(\mathbb{RP}_q^2(l;-))$ labelled by $\theta \in [0,1)$ and given by

$$\pi_\theta(a) = 0, \qquad \pi_\theta(b) = 0, \qquad \pi_\theta(c_-) = e^{2\pi i\theta}. \tag{4.9}$$

All other irreducible representations are infinite dimensional, labelled by $r = 1,\ldots,l$, and given by

$$\pi_r(a)e_n^r = q^{2(ln+r)}e_n^r, \quad \pi_r(b)e_n^r = q^{ln+r}\prod_{m=1}^{l}\left(1-q^{2(ln+r-m)}\right)^{1/2}e_{n-1}^r, \quad \pi_r(b)e_0^r = 0, \tag{4.10a}$$

$$\pi_r(c_-)e_n^r = \prod_{m=1}^{2l}\left(1-q^{2(ln+r-m)}\right)^{1/2}e_{n-2}^r, \qquad \pi_r(c_-)e_0^r = \pi_r(c_-)e_1^r = 0, \tag{4.10b}$$

where e_n^r, $n \in \mathbb{N}$, is an orthonormal basis for the representation space $\mathcal{H}_r \cong l^2(\mathbb{N})$.

The even case.

For $k = 2$ and l odd, $\mathcal{O}(\mathbb{RP}_q^2(l;+))$ is a polynomial $*$-algebra generated by a, c_+ which satisfy the relations:

$$a^* = a, \qquad ac_+ = q^{-2l}c_+a, \tag{4.11a}$$

$$c_+c_+^* = \prod_{m=0}^{l-1}(1-q^{2m}a), \qquad c_+^*c_+ = \prod_{m=1}^{l}(1-q^{-2m}a). \tag{4.11b}$$

The embedding of generators of $\mathcal{O}(\mathbb{RP}_q^2(l;+))$ into $\mathcal{O}(\Sigma_q^3)$ or the isomorphism of $\mathcal{O}(\mathbb{RP}_q^2(l;+))$ with the coinvariants of $\mathcal{O}(\Sigma_q^3(k,l))$ is provided by

$$a \mapsto \zeta_1^2\xi, \qquad c_+ \mapsto \zeta_0^l\xi. \tag{4.12}$$

There is a family of one-dimensional representations of $\mathcal{O}(\mathbb{RP}_q^2(l;+))$ labelled by $\theta \in [0,1)$ and given by

$$\pi_\theta(a) = 0, \qquad \pi_\theta(c_+) = e^{2\pi i\theta}. \tag{4.13}$$

All other irreducible representations are infinite dimensional, labelled by $r = 1,\ldots,l$, and given by

$$\pi_r(a)e_n^r = q^{2(ln+r)}e_n^r, \quad \pi_r(c_+)e_n^r = \prod_{m=1}^{l}\left(1-q^{2(ln+r-m)}\right)^{1/2}e_{n-1}^r, \quad \pi_r(c_+)e_0^r = 0, \tag{4.14}$$

where e_n^r, $n \in \mathbb{N}$ is an orthonormal basis for the representation space $\mathcal{H}_r \cong l^2(\mathbb{N})$.

4.2 Quantum weighted real projective spaces and quantum principal bundles

Our next aim is to construct quantum principal bundles with base spaces given by $\mathcal{O}(\mathbb{RP}_q^2(l;\pm))$ and fibre structures given by the circle Hopf algebra $\mathcal{O}(U(1)) \cong \mathbb{C}[u,u^*]$. The question arises which quantum space (i.e. a $\mathbb{C}[u,u^*]$-comodule algebra with coinvariants isomorphic to $\mathcal{O}(\mathbb{RP}_q^2(l;\pm))$) we should consider as the total space within this construction. We look first at the coactions of $\mathbb{C}[u,u^*]$ on $\mathcal{O}(\Sigma_q^3)$ that define $\mathcal{O}(\mathbb{RP}_q(k,l))$, i.e. at the comodule algebras $\mathcal{O}(\Sigma_q^3(k,l))$.

4.2.1 The (non-)principality of $\mathcal{O}(\Sigma_q^3(k,l))$.

Theorem 4.2.1. *Let $A = \mathcal{O}(\Sigma_q^3(k,l))$ be the right $H = \mathcal{O}(U(1))$ comodule algebra with coaction $\rho_{k,l} = \rho$. The subalgebra of coinvariant elements $B = \mathcal{O}(\mathbb{RP}_q(k,l)) \subset \mathcal{O}(\Sigma_q^3(k,l))$ is not a Hopf-Galois extension when $(k,l) \neq (1,1)$.*

Proof. We aim to show that the canonical map is not an isomorphism by showing that the image does not contain $1\otimes u$, i.e. it cannot be surjective since we know $1\otimes u$ is in the codomain. We begin by identifying a basis for the algebra $\mathcal{O}(\Sigma_q^3(k,l)) \otimes \mathcal{O}(\Sigma_q^3(k,l))$; observing the relations (4.4a) and (4.4b) it is clear that a basis for $\mathcal{O}(\Sigma_q^3(k,l))$ is given by linear combinations of elements of the form,

$$b_1 = b_1(p_1,p_2,p_3) = \zeta_0^{p_1}\zeta_1^{p_2}\xi^{p_3}, \qquad b_2 = b_2(\bar{p_1},\bar{p_2},\bar{p_3}) = \zeta_0^{\bar{p_1}}\zeta_1^{\bar{p_2}}\xi^{*\bar{p_3}},$$

$$b_3 = b_3(q_1,q_2,q_3) = \zeta_0^{*q_1}\zeta_1^{q_2}\xi^{q_3}, \qquad b_4 = b_4(\bar{q_1},\bar{q_2},\bar{q_3}) = \zeta_0^{*\bar{q_1}}\zeta_1^{\bar{q_2}}\xi^{*\bar{q_3}},$$

noting that all powers are non-negative. Hence a basis for $\mathcal{O}(\Sigma_q^3(k,l)) \otimes \mathcal{O}(\Sigma_q^3(k,l))$ is given by linear combinations of elements of the form,

$$b_i \otimes b_j, \qquad \text{where} \quad i,j \in \{1,2,3,4\}, \tag{4.15}$$

applying the canonial map gives elements of the form,

$$can(b_i \otimes b_j) = b_i\rho(b_j) = b_ib_j \otimes u^{deg(b_j)}, \qquad \text{where} \quad i,j \in \{1,2,3,4\}. \tag{4.16}$$

The next stage is to construct all possible elements in $A \otimes A$ which map to $1 \otimes u$. To obtain the identity in the first leg we must use one of the following relations:

$$\zeta_0^m\zeta^{*n}_0 = \begin{cases} \prod_{p=0}^{m-1}(1-q^{2p}\zeta_1^2\xi) & \text{when } m=n, \\ \zeta_0^{m-n}\prod_{p=0}^{n-1}(1-q^{2p}\zeta_1^2\xi) & \text{when } m>n, \\ \prod_{p=0}^{m-1}(1-q^{2p}\zeta_1^2\xi)\zeta_0^{*n-m} & \text{when } n>m, \end{cases} \tag{4.17a}$$

$$\zeta_0^{*n}\zeta_0^m = \begin{cases} \prod_{p=1}^{m}(1-q^{-2p}\zeta_1^2\xi) & \text{when } m=n, \\ \zeta_0^{*n-m}\prod_{p=1}^{m}(1-q^{-2p}\zeta_1^2\xi) & \text{when } n>m, \\ \prod_{p=1}^{n}(1-q^{-2p}\zeta_1^2\xi)\zeta^{m-n} & \text{when } n<m. \end{cases} \tag{4.17b}$$

or

$$\xi\xi^* = \xi^*\xi = 1$$

We see that to obtain an element which has one term with the identity in the first leg we require the powers of ζ_0 and ζ_0^* to be equal. We now construct all possible elements of the domain which map to $1 \otimes u$ after applying the canonical map.

Case 1: use the first relation to obtain $\zeta_0^m \zeta_0^{*m}$ $(m > 0)$; this can be done in fours ways; using $b_1\rho(b_3)$, $b_1\rho(b_4)$, $b_2\rho(b_3)$ and $b_2\rho(b_4)$. Now,

$$b_1\rho(b_3) \sim \zeta_0^{p_1}\zeta_0^{*q_1}\zeta_1^{p_2+q_2}\xi^{p_3+q_3} \otimes u^{-kq_1+lq_2-2lq_3} \Longrightarrow p_1 = q_1 = m, p_2 = q_2 = 0, p_3 = q_3 = 0,$$

where we use $\sim$ when the elements differ by a complex number. This implies

$$-kq_1 + lq_2 - 2lq_3 = 1 \Longrightarrow -mk = 1,$$

hence no possible terms. A similar calculation for the three other cases shows that $1 \otimes u$ cannot be obtained as an element of the image of the canonical map in this case.

Case 2: use the second relation to obtain $\zeta_0^{*n}\zeta_0^n$ $(n > 0)$; this can be done in four ways $b_3\rho(b_1)$, $b_3\rho(b_2)$, $b_3\rho(b_2)$ and $b_4\rho(b_2)$. Now,

$$b_3\rho(b_1) \sim \zeta_0^{*q_1}\zeta_0^{p_1}\zeta_1^{p_2+q_2}\xi^{p_3+q_3} \otimes u^{kp_1+lp_2-2lp_3} \Longrightarrow p_1 = q_1 = n, p_2 = q_2 = 0, p_3 = q_3 = 0$$

and

$$nk = 1 \Longrightarrow n = 1 \text{ and } k = 1.$$

Note that $k = 1$ is not a problem provided l is not equal to 1. This is reviewed at the next stage of the proof. The same conclusion is reached in all four cases.

In all possibilities $\zeta_0^{*n}\zeta_0^n$ appears only when $n = 1$, in which case the relation simplifies to $\zeta_0^*\zeta_0 = 1 - q^{-2}\zeta_1^2\xi$, so the next stage involves constructing elements in the domain which map to $\zeta_1^2\xi$. There are eight possibilities altogether to be checked: $b_1\rho(b_1)$, $b_1\rho(b_2)$, $b_1\rho(b_3)$, $b_1\phi(b_4)$, $b_3\rho(b_1)$, $b_3\rho(b_2)$, $b_3\rho(b_3)$ and $b_3\rho(b_4)$. The first case gives:

$$b_1\rho(b_1) \sim \zeta_0^{2p_1}\zeta_1^{2p_2}\xi^{2p_3} \otimes u^{kp_1+lp_2-2lp_3} \Longrightarrow 2p_1 = 0,\ 2p_2 = 2,\ 2p_3 = 1,$$

and

$$kp_1 + lp_2 - 2lp_3 = 1 \Longrightarrow p_1 = 0, p_2 = 1, p_3 \text{ has no possible values and } l = 1.$$

Hence $1\otimes u$ cannot be obtained as an element in the image in this case. Similar calculations for the remaining possibilities show that either $1 \otimes u$ is not in the image of the canonical map, or that if $1 \otimes u$ is in the image then $k = l = 1$.

Case 3: finally, it seems possible that $1 \otimes u$, using the third relation, could be in the image of the canonical map. All possible elements in the domain which could potentially map to this element are constructed and investigated. There are eight possibilities: $b_1\phi(b_2)$, $b_1\phi(b_4)$, $b_2\phi(b_1)$, $b_2\phi(b_3)$, $b_3\phi(b_2)$, $b_3\phi(b_4)$, $b_4\phi(b_1)$ and $b_4\phi(b_3)$. The first possibility comes out as

$$b_1\phi(b_2) \sim \zeta_0^{p_1+\bar{p}_1}\zeta_1^{p_2+\bar{p}_2}\xi^{p_3}\xi^{\bar{p}_3} \otimes u^{k\bar{p}_1+l\bar{p}_2+2l\bar{p}_3} \Longrightarrow p_1 = \bar{p}_1 = 0, p_2 = \bar{p}_2 = 0, p_3 = \bar{p}_3 = 1.$$

Also

$$k\bar{p_1} + l\bar{p}_2 + 2l\bar{p}_3 = 1 \Longrightarrow 2l = 1,$$

which implies there are no terms. The same conclusion can be reached for the remaining relations.

This concludes that $1 \otimes u$, which is contained in $A \otimes H$, is not in the image of the canonical map, proving that this map is not surjective and ultimately not an isomorphism when k and l are both not simultaneously equal to 1, completing the proof that $B \subset A$ is not a Hopf-Galois extension in this case. □

Theorem 4.2.1 tells us that if we use $\mathcal{O}(\Sigma_q^3(k,l))$ as our total space, then we are forced to put $(k,l) = (1,1)$ to ensure that the required Hopf-Galois condition does not fail. A consequence of this would be the generators ζ_0 and ζ_1 would have grade 1. This suggests that the space $\mathcal{O}(\Sigma_q^3(k,l))$ is too restrictive as there is no freedom with the weights k or l, and that we should in fact consider a subalgebra of this space which would offer some choice. Theorem 4.2.1 indicates that the desired subalgebra should have generators with grades 1 to ensure the Hopf-Galois condition is satisfied. This process is similar to those followed in [5], where the bundles over the quantum teardrops $\mathbb{WP}_q(1,l)$ have the total spaces provided by the quantum lens spaces and structure groups provided by the circle group $U(1)$. We follow a similar approach in the sense that we view $\mathcal{O}(\Sigma_q^3(k,l))$ as a right H-comodule algebra, where H is the Hopf algebra of a suitable cyclic group.

4.2.2 The k odd case $\mathcal{O}(\mathbb{RP}_q^2(l;-))$.

The principal $\mathcal{O}(U(1))$-comodule algebra over $\mathcal{O}(\mathbb{RP}_q^2(l;-))$.

Take the group Hopf $*$-algebra $H = \mathcal{O}(\mathbb{Z}_l)$ which is generated by a unitary group-like element w and satisfies the relation $w^l = 1$. The algebra $\mathcal{O}(\Sigma_q^3)$ is a right $\mathcal{O}(\mathbb{Z}_l)$-comodule $*$-algebra with coaction

$$\mathcal{O}(\Sigma_q^3) \to \mathcal{O}(\Sigma_q^3) \otimes \mathcal{O}(\mathbb{Z}_l), \qquad \zeta_0 \mapsto \zeta_0 \otimes w, \quad \zeta_1 \mapsto \zeta_1 \otimes 1, \quad \xi \mapsto \xi \otimes 1. \tag{4.18}$$

Note that the $\mathbb{Z}_l$-degree of the generator ξ is determined by the degree of ζ_1: The relation $\zeta_1^* = \zeta_1\xi$ and that the coaction must to compatible with all relations imply that $\deg(\zeta_1^*) = \deg(\zeta_1) + \deg(\xi)$. Since ζ_1 has degree zero, ξ must also have degree zero.

The next stage of the process is to find the coinvariant elements of $\mathcal{O}(\Sigma_q^3)$ given the coaction defined above.

Proposition 4.2.2. *The fixed point subalgebra of the above coaction is isomorphic to the algebra $\mathcal{O}(\Sigma_q^3(l;-))$, generated by x, y and z subject to the following relations*

$$y^* = yz, \qquad xy = q^l yx, \qquad xx^* = \prod_{p=0}^{l-1}(1 - q^{2p}y^2z), \qquad x^*x = \prod_{p=1}^{l}(1 - q^{-2p}y^2z), \tag{4.19}$$

and z is a central unitary element. The embedding of $\mathcal{O}(\Sigma_q^3(l;-))$ into $\mathcal{O}(\Sigma_q^3)$ is given by $x \mapsto \zeta_0^l$, $y \mapsto \zeta_1$ and $z \mapsto \xi$.

Proof. Clearly ζ_1, ξ, ζ_0^l and ζ_0^{*l} are coinvariant elements of $\mathcal{O}(\Sigma_q^3)$. Apply the coaction to the basis (4.5) to obtain

$$\zeta_0^r\zeta_1^s\xi^t \mapsto \zeta_0^r\zeta_1^s\xi^t \otimes w^r, \qquad \zeta_0^{*r}\zeta_1^s\xi^t \mapsto \zeta_0^{*r}\zeta_1^s\xi^t \otimes w^{-r}.$$

These elements are coinvariant, provided $r = r'l$. Hence every coinvariant element is a polynomial in ζ_1, ξ, ζ_0^l and ζ_0^{*l}. Relations (4.19) are now easily derived from (4.4) and (4.17). □

The algebra $\mathcal{O}(\Sigma_q^3(l;-))$ is a right $\mathcal{O}(U(1))$-comodule with coaction defined as,

$$\phi : A \to A \otimes \mathcal{O}(U(1)), \qquad x \mapsto x \otimes u, \qquad y \mapsto y \otimes u, \qquad z \mapsto z \otimes u^{-2}. \tag{4.20}$$

The second and third relation in (4.19) tell us that the grade of z must be double the grade of y^* since xx^* and x^*x have degree zero, and so

$$\deg(y^2z) = \deg(y^2) + \deg(z) = 2\deg(y) + \deg(z) = 0 \Longrightarrow \deg(z) = -2\deg(y) = 2\deg(y^*).$$

The fixed points of the algebra $\mathcal{O}(\Sigma_q^3(l;-))$ under the coaction ϕ is found by identifying the basis elements of the algebra $\mathcal{O}(\Sigma_q^3(l;-))$ and applying these to the coaction. The image of the basis elements under this coaction will then provide the coinvariance condition.

Proposition 4.2.3. *$\mathcal{O}(\Sigma_q^3(l;-))$ is a right $\mathcal{O}(U(1))$-comodule algebra with $\mathcal{O}(\Sigma_q^3(l;-))^{co\mathcal{O}(U(1))}$ being isomorphic to $\mathcal{O}((\mathbb{RP}_q(l;-))$, i.e., the subalgebra of invariant elements under the coaction ϕ corresponds to the real weighted projective spaces (the negative case).*

Proof. We aim to show that the $*$-subalgebra of $\mathcal{O}(\Sigma_q^3(l;-))$ of elements which are invariant under the coaction is generated by x^2z, xyz and y^2z. The isomorphism of $\mathcal{O}(\Sigma_q^3(l;-))^{co\mathcal{O}(U(1))}$ with $\mathcal{O}((\mathbb{RP}_q(l;-))$ is obtained by using the embedding of $\mathcal{O}(\Sigma_q^3(l;-))$ in $\mathcal{O}(\Sigma_q^3)$ described in Proposition 4.2.2, i.e. $y^2z \mapsto \zeta_1\xi \mapsto a$, $xyz \mapsto \zeta_0^l\zeta_1\xi \mapsto b$ and $x^2z \mapsto \zeta_0^{2l}\xi \mapsto c_-$.

The algebra $\mathcal{O}(\Sigma_q^3(l;-))$ is spanned by elements of the type $x^ry^sz^t$, $x^{*r}y^sz^t$, where $r, s \in \mathbb{N}$ and $t \in \mathbb{Z}$. Applying the coaction ϕ to these basis elements gives $x^ry^sz^t \mapsto x^ry^sz^t \otimes u^{r+s-2t}$. Hence $x^ry^sz^t$ is ϕ-invariant if and only if $2t = r+s$. If r is even, then s is even and

$$x^ry^sz^t = x^ry^sz^{(r+s)/2} = (x^2z)^{r/2}(y^2z)^{s/2}.$$

If r is odd, then so is s and

$$x^ry^sz^t = x^ry^sz^{(r+s)/2} \sim (x^2z)^{(r-1)/2}(y^2z)^{(s-1)/2}(xyz).$$

The case of $x^{*r}y^sz^t$ is dealt with similarly, thus proving that all coinvariants of ϕ are polynomials in x^2z, xyz, y^2z and their $*$-conjugates. □

The main result of this section is contained in the following

Theorem 4.2.4. *$\mathcal{O}(\Sigma_q^3(l;-))$ is a non-cleft principal $\mathcal{O}(U(1))$-comodule algebra over $\mathcal{O}(\mathbb{RP}_q(l;-))$ via the coaction ϕ.*

Proof. To prove that $\mathcal{O}(\Sigma_q^3(l;-))$ is a principal $\mathcal{O}(U(1))$-comodule algebra over $\mathcal{O}(\mathbb{RP}_q(l;+))$ we employ Proposition 2.4.15 and construct a strong connection form as follows.

Define $\omega : \mathcal{O}(U(1)) \to \mathcal{O}(\Sigma_q^3(l;-)) \otimes \mathcal{O}(\Sigma_q^3(l;-))$ to be,

$$\omega(1) = 1 \otimes 1 \tag{4.21a}$$

$$\omega(u^n) = x^*\omega(u^{n-1})x - \sum_{m=1}^{l}(-1)^m q^{-m(m+1)}\binom{l}{m}_{q^{-2}} y^{2m-1}z^m\omega(u^{n-1})y \tag{4.21b}$$

$$\omega(u^{*n}) = x\omega(u^{-n+1})x^* - \sum_{m=1}^{l}(-1)^m q^{m(m-1)}\binom{l}{m}_{q^2} y^{2m-1}z^{m-1}\omega(u^{-n+1})yz, \tag{4.21c}$$

where, for all $s \in \mathbb{R}$, the *deformed* or *q-binomial* coefficients $\binom{l}{m}_s$ are defined by the polynomial equality given in 3.18 (indeterminate t). The map ω has been designed such that normalisation property (2.15a) is automatically satisfied. Considering property (2.15b), multiplying both legs on the right hand side of equation (4.21a) clearly gives 1. To check property (2.15b) from equations (4.21b) and (4.21c) take a bit more work. We use proof by induction, but first have to derive an identity to assist with the calculation. Set $s = q^{-2}$, $t = -q^{-2}y^*y$ in (3.18) to arrive at,

$$\sum_{m=1}^{l}(-1)^m q^{m(m-1)}\binom{l}{m}_{q^{-2}} y^{*m}y^m = \prod_{m=1}^{l}(1 + q^{-2(m-1)}(-q^{-2}y^*y)) - 1,$$

which simplifies to,

$$\sum_{m=1}^{l}(-1)^m q^{m(m+1)}\binom{l}{m}_{q^{-2}} y^{2m}z^m = \prod_{m=1}^{l}(1 - q^{-2m}y^2z) - 1. \tag{4.22}$$

Now to start the induction process we consider the case $n = 0$; clearly $(m \circ \omega)(1) = 1$ providing the basis. Next, we assume that the relation holds for $n = N$, that is $(m \circ \omega)(u^N) = 1$, and consider the case $n = N + 1$.

$$\omega(u^{N+1}) = x^*\omega(u^N)x - \sum_{m=1}^{l}(-1)^m q^{-m(m+1)}\binom{l}{m}_{q^{-2}} y^{2m-1}z^m\omega(u^N)y,$$

applying the multiplication map to both sides and using the induction hypothesis,

$$\begin{aligned}(m \circ \omega)(u^{N+1}) &= x^*x - \sum_{m=1}^{l}(-1)^m q^{-m(m+1)}\binom{l}{m}_{q^{-2}} y^{2m-1}z^m \\ &= x^*x - (x^*x - 1) = 1,\end{aligned}$$

showing property (2.15b) holds for all $u^n \in \mathcal{O}(U(1))$, where $n \in \mathbb{N}$. To show this property holds for each u^{*n} we adopt the same strategy; this is omitted from the proof as it does not hold further insight, instead repetition of similar arguments.

Property (2.15c); again we adopt a strategy of proof by induction. Applying $(\mathrm{id}\otimes\phi)$ to (4.21a) then we obviously get $(\omega\otimes\mathrm{id})\circ\Delta$. Next applying $\omega(u)$ to (4.21b) gives

$$\begin{aligned}
x^*\otimes x\otimes u &- \sum_{m=1}^{l}(-1)^m q^{-m(m-1)}\binom{l}{m}_{q^2} y^{2m-1}z^m\otimes y\otimes u \\
&= \left(x^*\otimes x-\sum_{m=1}^{l}(-1)^m q^{-m(m-1)}\binom{l}{m}_{q^2} y^{2m-1}z^m\otimes y\right)\otimes u \\
&= \omega(u)\otimes u=(\omega\otimes\mathrm{id})\circ\Delta(u).
\end{aligned}$$

This shows that property (2.15c) holds for equation (4.21b) when $n=1$. We now assume the property holds for $n=N-1$, hence $(id\otimes\phi)\circ w(u^{N-1})=(\omega\otimes id)\circ\Delta(u^{N-1})=\omega(u^{N-1})\otimes u^{N-1}$, and consider the case $n=N$.

$$\begin{aligned}
(\mathrm{id}\otimes\phi)(w(u^N)) &= (\mathrm{id}\otimes\phi)\left(x^*\omega(u^{N-1})x-\sum_{m=1}^{l}(-1)^m q^{-m(m-1)}\binom{l}{m}_{q^{-2}} y^{2m-1}z^m\omega(u^{N-1})y\right) \\
&= x^*((id\otimes\phi)(\omega(u^{N-1})x))-\sum_{m=1}^{l}(-1)^m q^{-m(m-1)}\binom{l}{m}_{q^{-2}} y^{2m-1}z^m((id\otimes\phi)(\omega(u^{N-1}) \\
&= x^*\omega(u^{N-1})x\otimes u^{N-1}-\sum_{m=1}^{l}(-1)^m q^{-m(m-1)}\binom{l}{m}_{q^{-2}} y^{2m-1}z^m\omega(u^{N-1})y\otimes u^{N-1} \\
&= \omega(u^N)\otimes u^N \\
&= (\omega\otimes\mathrm{id})\circ\Delta(u^N),
\end{aligned}$$

hence property (2.15c) is satisfied for all $u^n\in\mathcal{O}(U(1))$ where $n\in\mathbb{N}$. The case for u^{*n} is proved in a similar manner, as is property (2.15d). Again, the details are omitted as the process is identical. This completes the proof that ω is a strong connection form, hence $\mathcal{O}(\Sigma_q^3(l,-))$ is a principal comodule algebra.

To determine whether the constructed comodule algebra is cleft we need to identify invertible elements in $\mathcal{O}(\Sigma_q^3(l,-))$. We observe,

$$\mathcal{O}(\Sigma_q^3(l,-))\subset\mathcal{O}(\Sigma_q^3)\cong\mathcal{O}(S_q^2)\square_{\mathcal{O}(\mathbb{Z}_2)}\mathcal{O}(U(1))\subset\mathcal{O}(S_q^2)\otimes\mathcal{O}(U(1)).$$

Since only the non-zero scalar multiples of 1 are invertible elements of the quantum sphere, an invertible element of $\mathcal{O}(S_q^2)\otimes\mathcal{O}(U(1))$ must be of the form $1\otimes v$, where v is an invertible element of $\mathcal{O}(U(1))$. On the other hand only the non-zero multiples of u^n are invertible in $\mathcal{O}(U(1))$. Hence the only invertible elements in the algebraic tensor product $\mathcal{O}(S_q^2)\otimes\mathcal{O}(U(1))$ are scalar multiples of $1\otimes u^n$ for $n\in\mathbb{N}$. Now we can conclude that the only invertible elements in $\mathcal{O}(S_q^2)\square_{\mathcal{O}(\mathbb{Z}_2)}\mathcal{O}(U(1))$ are the elements of the form $1\otimes u^n$. These elements correspond to the elements ξ^n in $\mathcal{O}(\Sigma_q^3)$, which in turn correspond to z^n in $\mathcal{O}(\Sigma_q^3(l,-))$.

Suppose $j:H\to A$ is the cleaving map; to ensure the map is convolution invertible we are forced to put $u\mapsto z^n$, since the only invertible elements in A are powers of z. Since u has degree 1 in $H=\mathcal{O}(U(1))$ and z has degree -2 in $\mathcal{O}(\Sigma_q^3(l,-))$, the map j fails to preserve the degrees hence it is not colinear. Hence $\mathcal{O}(\Sigma_q^3(l,-))$ is a non-cleft principal comodule algebra. □

Almost freeness of the coaction $\varrho_{1,l}$.

At the classical limit, $q \to 1$, the algebras $\mathcal{O}(\mathbb{RP}_q(l;-))$ represent singular manifolds or orbifolds. It is known that every orbifold can be obtained as a quotient of a manifold by an *almost free* action. The latter means that the action has finite (rather than trivial as in the free case) stabiliser groups. In the algebraic level freeness is encoded in the bijectivity of the canonical map can, or, more precisely, in the surjectivity of the lifted canonical maps $\overline{\mathsf{can}}$ (2.16). The surjectivity of $\overline{\mathsf{can}}$ means the triviality of the cokernel of $\overline{\mathsf{can}}$, thus the size of the cokernel of $\overline{\mathsf{can}}$ can be treated as a measure of the size of the stabiliser groups. This leads to the following notion proposed in [5]

Definition 4.2.5. Let H be a Hopf algebra and let A be a right H-comodule algebra with coaction $\varrho^A : A \to A \otimes H$. We say that the coaction is *almost free* if the cokernel of the (lifted) canonical map

$$\overline{\mathsf{can}} : A \otimes A \to A \otimes H, \qquad a \otimes a' \mapsto a\varrho^A(a'),$$

is finitely generated as a left A-module.

Although the coaction ϕ defined in the preceding section is free, at the classical limit $q \to 1$, $\mathcal{O}(\Sigma_q^3(l,-))$ represents a singular manifold or an orbifold. On the other hand, at the same limit, $\mathcal{O}(\Sigma_q^3)$ corresponds to a genuine manifold, one of the Seifert three-dimensional non-orientable manifolds; see [41]. It is therefore natural to ask, whether the coaction $\varrho_{1,l}$ of $\mathcal{O}(U(1))$ on $\mathcal{O}(\Sigma_q^3)$ which has $\mathcal{O}(\mathbb{RP}_q(l;-))$ as fixed points is almost free in the sense of Definition 4.2.5.

Proposition 4.2.6. *The coaction $\varrho_{1,l}$ is almost free.*

Proof. Denote by $\iota_- : \mathcal{O}(\Sigma_q^3(l,-)) \hookrightarrow \mathcal{O}(\Sigma_q^3)$, the $*$-algebra embedding described in Proposition 4.2.2. One easily checks that the following diagram

$$\begin{tikzcd}
\mathcal{O}(\Sigma_q^3(l,-)) \arrow[r, "\iota_-"] \arrow[d, "\phi"'] & \mathcal{O}(\Sigma_q^3) \arrow[d, "\varrho_{1,l}"] \\
\mathcal{O}(\Sigma_q^3(l,-)) \otimes \mathcal{O}(U(1)) \arrow[r, "\iota_- \otimes (-)^l"] & \mathcal{O}(\Sigma_q^3) \otimes \mathcal{O}(U(1)),
\end{tikzcd}$$

where $(-)^l : u \to u^l$ is commutative. The principality or freeness of ϕ proven in Theorem 4.2.4 implies that $1 \otimes u^{ml} \in \operatorname{Im}(\overline{\mathsf{can}})$, $m \in \mathbb{Z}$, where $\overline{\mathsf{can}}$ is the (lifted) canonical map corresponding to coaction $\varrho_{1,l}$. This means that $\mathcal{O}(\Sigma_q^3) \otimes \mathbb{C}[u^l, u^{-l}] \subseteq \operatorname{Im}(\overline{\mathsf{can}})$. Therefore, there is a short exact sequence of left $\mathcal{O}(\Sigma_q^3)$-modules

$$(\mathcal{O}(\Sigma_q^3) \otimes \mathbb{C}[u,u^{-1}])/(\mathcal{O}(\Sigma_q^3) \otimes \mathbb{C}[u^l,u^{-l}]) \longrightarrow \operatorname{coker}(\overline{\mathsf{can}}) \longrightarrow 0.$$

The left $\mathcal{O}(\Sigma_q^3)$-module $(\mathcal{O}(\Sigma_q^3)) \otimes \mathbb{C}[u,u^{-1}])/(\mathcal{O}(\Sigma_q^3) \otimes \mathbb{C}[u^l,u^{-l}])$ is finitely generated, hence so is $\operatorname{coker}(\overline{\mathsf{can}})$. □

Associated modules or sections of line bundles.

One can construct modules associated to the principal comodule algebra $\mathcal{O}(\Sigma_q^3(l,-))$ following the procedure outlined at the end of Section 2.3; see Definition 2.3.6.

Every one-dimensional comodule of $\mathcal{O}(U(1)) = \mathbb{C}[u,u^*]$ is determined by the grading of the basis element of $\mathbb{C}$, say 1. More precisely, for any integer n, $\mathbb{C}$ is a left $\mathcal{O}(U(1))$-comodule with the coaction

$$\varrho_n : \mathbb{C} \to \mathbb{C}[u,u^*] \otimes \mathbb{C}, \qquad 1 \mapsto u^n \otimes 1.$$

Identifying $\mathcal{O}(\Sigma_q^3(l,-)) \otimes \mathbb{C}$ with $\mathcal{O}(\Sigma_q^3(l,-))$ we thus obtain, for each coaction ϱ_n

$$\Gamma[n] := \mathcal{O}(\Sigma_q^3(l,-))\Box_{\mathcal{O}(U(1))}\mathbb{C} \cong \{f \in \Sigma_q^3(l,-) \mid \phi(f) = f \otimes u^n\} \subset \mathcal{O}(\Sigma_q^3(l,-)).$$

In other words, $\Gamma[n]$ consists of all elements of $\mathcal{O}(\Sigma_q^3(l,-))$ of $\mathbb{Z}$-degree n. In particular $\Gamma[0] = \mathcal{O}(\mathbb{RP}_q(l;-))$. Each of the $\Gamma[n]$ is a finitely generated projective left $\mathcal{O}(\mathbb{RP}_q(l;-))$-module, i.e. it represents the module of sections of the non-commutative line bundle over $\mathbb{RP}_q(l;-)$. The idempotent matrix $E[n]$ defining $\Gamma[n]$ can be computed explicitly from a strong connection form ω (see equations (4.21)) in the proof of Theorem 4.2.4) following the procedure described in [11]. Write $\omega(u^n) = \sum_i \omega(u^n)^{[1]}{}_i \otimes \omega(u^n)^{[2]}{}_i$. Then

$$E[n]_{ij} = \omega(u^n)^{[2]}{}_i \omega(u^n)^{[1]}{}_j \in \mathcal{O}(\mathbb{RP}_q(l;-)). \tag{4.23}$$

For example, for $l = 2$ and $n = 1$, using (4.21b) and (4.21a) as well as redistributing numerical coefficients we obtain

$$E[1] = \begin{pmatrix} (1-a)(1-q^2a) & q^{-1}\sqrt{1+q^{-2}}\, b & iq^{-3}ba \\ q^{-1}\sqrt{1+q^{-2}}\, b^* & q^{-2}(1+q^{-2})\, a & iq^{-4}\sqrt{1+q^{-2}}\, a^2 \\ iq^{-3}b^* & iq^{-4}\sqrt{1+q^{-2}}\, a & -q^{-6}a^2 \end{pmatrix} \tag{4.24}$$

Although the matrix $E[1]$ is not hermitian, the left-hand upper 2×2 block is hermitian. On the other hand, once $\mathcal{O}(\mathbb{RP}_q(2;-))$ is completed to the C^*-algebra $C(\mathbb{RP}_q(2;-))$ of continuous functions on $\mathbb{RP}_q(2;-)$ (and then identified with the suitable pullback of two algebras of continuous functions over the quantum real projective space; see [12]), then a hermitian projector can be produced out of $E[1]$ by using the Kaplansky formula; see [20, page 88].

The traces of tensor powers of each of the $E[n]$ make up a cycle in the cyclic complex of $\mathcal{O}(\mathbb{RP}_q(l;-))$, whose corresponding class in the cyclic homology $HC_\bullet(\mathcal{O}(\mathbb{RP}_q(l;-)))$ is known as the Chern character of $\Gamma[n]$. Again, as an illustration we compute the trace of $E[1]$ for general l.

Lemma 4.2.7. *The zero-component of the Chern character of* $\Gamma[1]$ *is the class of*

$$\mathrm{Tr}\ E[1] = 1 + \sum_{m=1}^{l} (-1)^m q^{m(m-1)} \left(1 - q^{-2ml}\right) \binom{l}{m}_{q^2} a^m. \tag{4.25}$$

Proof. Setting $s \to s^{-1}$ and comparing two sides of the formula (3.18) we obtain:

$$\binom{l}{m}_{s^{-1}} = s^{m(m-l)} \binom{l}{m}_s . \tag{4.26}$$

Compute

$$\begin{aligned}
\operatorname{Tr} E[1] &= xx^* - \sum_{m=1}^{l} (-1)^m q^{-m(m+1)} \binom{l}{m}_{q^{-2}} y^{2m} z^m \\
&= \prod_{m=0}^{l-1} (1 - q^{2m} a) - \sum_{m=1}^{l} (-1)^m q^{2m(m-l)-m(m+1)} \binom{l}{m}_{q^2} a^m \\
&= 1 + \sum_{m=1}^{l} (-1)^m \left(q^{m(m-1)} - q^{m(m-1)-2ml} \right) \binom{l}{m}_{q^2} a^m \\
&= 1 + \sum_{m=1}^{l} (-1)^m q^{m(m-1)} \left(1 - q^{-2ml} \right) \binom{l}{m}_{q^2} a^m ,
\end{aligned}$$

where the first equality follows by (4.21b) and (4.21a), the second by (4.19) and (4.26) combined with the identification of a as $y^2 z$, while the third equality is a consequence of (3.18). □

4.2.3 The k even case $\mathcal{O}(\mathbb{RP}_q(l;+))$.

The principal $\mathcal{O}(U(1))$-comodule algebra over $\mathcal{O}(\mathbb{RP}_q^2(l;+))$.

In the same light as the negative case we aim to construct quantum principal bundles with base spaces $\mathcal{O}(\mathbb{RP}_q(l;+))$, and proceed by viewing $\mathcal{O}(\Sigma_q^3)$ as a right H'-comodule algebra, where H' is a Hopf-algebra of functions on a finite cyclic group. The aim is to construct the total space $\mathcal{O}(\Sigma_q^3(l,+))$ of the bundle over $\mathcal{O}(\mathbb{RP}_q(l;+))$ as the coinvariant subalgebra of $\mathcal{O}(\Sigma_q^3)$. $\mathcal{O}(\Sigma_q^3(l,+))$ must contain generators $\zeta_1^2\xi$ and $\zeta_0^l\xi$ of $\mathcal{O}(\mathbb{RP}_q(l;+))$. Suppose $H' = \mathcal{O}(\mathbb{Z}_m)$ and $\Phi : \mathcal{O}(\Sigma_q^3) \to \mathcal{O}(\Sigma_q^3) \otimes H'$ is a coaction. We require Φ to be compatible with the algebraic relations and to give zero $\mathbb{Z}_m$-degree to $\zeta_1^2\xi$ and $\zeta_0^l\xi$. These requirements yield

$$2\deg(\zeta_1) + \deg(\xi) = 0 \bmod m, \qquad l\deg(\zeta_0) + \deg(\xi) = 0 \bmod m.$$

Bearing in mind that l is odd, the simplest solution to these requirements is provided by $m = 2l$, $\deg(\xi) = 0$, $\deg(\zeta_0) = 2$, $\deg(\zeta_1) = l$. This yields the coaction

$$\Phi : \mathcal{O}(\Sigma_q^3) \to \mathcal{O}(\Sigma_q^3) \otimes \mathcal{O}(\mathbb{Z}_{2l}), \qquad \zeta_0 \mapsto \zeta_0 \otimes v^2, \quad \zeta_1 \mapsto \zeta_1 \otimes v^l, \quad \xi \mapsto \xi \otimes 1,$$

where v, $v^{2l} = 1$ is the unitary generator of $\mathcal{O}(\mathbb{Z}_{2l})$. Φ is extended to the whole of $\mathcal{O}(\Sigma_q^3)$ so that Φ is an algebra map, making $\mathcal{O}(\Sigma_q^3)$ a right $\mathcal{O}(\mathbb{Z}_{2l})$-comodule algebra.

Proposition 4.2.8. *The fixed point subalgebra of the coaction Φ is isomorphic to the $*$-algebra $\mathcal{O}(\Sigma^3_q(l,+))$ generated by x', y' an central unitary z' subject to the following relations:*

$$x'y' = q^{2l}y'x', \qquad x'y'^* = q^{2l}y'^*x', \qquad y'y'^* = q^4y'^*y', \qquad y'^* = y'z'^2, \tag{4.27a}$$

$$x'x'^* = \prod_{p=0}^{l-1}(1-q^{2p}y'z'), \qquad x'^*x' = \prod_{p=1}^{l}(1-q^{-2p}y'z'). \tag{4.27b}$$

The isomorphism between $\mathcal{O}(\Sigma^3_q(l,+))$ and the coinvariant subalgebra of $\mathcal{O}(\Sigma^3_q)$ is given by $x' \mapsto \zeta_0^l$, $y' \mapsto \zeta_1^2$ and $z' \mapsto \xi$.

Proof. Clearly ζ_1^2, ξ, ζ_0^l and ζ_0^{*l} are coinvariant elements of $\mathcal{O}(\Sigma^3_q)$. Apply the coaction Φ to the basis (4.5) to obtain

$$\zeta_0^r\zeta_1^s\xi^t \mapsto \zeta_0^r\zeta_1^s\xi^t \otimes v^{2r+ls}, \qquad \zeta_0^{*r}\zeta_1^s\xi^t \mapsto \zeta_0^{*r}\zeta_1^s\xi^t \otimes v^{-2r+ls}.$$

These elements are coinvariant, provided $2r+ls = 2ml$ in the first case or $-2r+ls = 2ml$ in the second. Since l is odd, s must be even and then $r = r'l$, hence the invariant elements must be of the form

$$(\zeta_0^l)^{r'}(\zeta_1^2)^{s/2}\xi^t, \qquad (\zeta_0^{*l})^{r'}(\zeta_1^2)^{s/2}\xi^t$$

as required. Relations (4.27) are now easily derived from (4.4) and (4.17).we seek the fixed points with respect to the coaction Φ. $\square$

The algebra $\mathcal{O}(\Sigma^3_q(l,+))$ is a right $\mathcal{O}(U(1))$-comodule with coaction defined as,

$$\Omega : \mathcal{O}(\Sigma^3_q(l,+)) \to \mathcal{O}(\Sigma^3_q(l,+)) \otimes \mathcal{O}(U(1)), \qquad x' \mapsto x' \otimes u,\ y' \mapsto y' \otimes u,\ z' \mapsto z' \otimes u^{-1}. \tag{4.28}$$

The first three relations (4.27a) bear no information on the possible gradings of the generators of $\mathcal{O}(\Sigma^3_q(l,+))$, however the final relation of (4.27a) tells us that the grade of y'^* must have the same grade of z' since,

$$\deg(y'^*) = -\deg(y') = \deg(y') + 2\deg(z'),$$

hence,

$$2\deg(y'^*) = 2\deg(z') \text{ or } \deg(y'^*) = \deg(z').$$

This is consistent with relations (4.27b) since the left hand sides, $x'x'^*$ and x'^*x', have degree zero, as does the right hand sides since,

$$\deg(y'z') = \deg(y') + \deg(y'^*) = \deg(y') + (-\deg(y')) = 0. \tag{4.29}$$

The coaction Ω in 4.28 is defined by giving x' and y' grade 1, and setting the grade of z' as -1 to ensure it's compatible with the relations of the algebra $\mathcal{O}(\Sigma^3_q(l,+))$.

Proposition 4.2.9. *The right $\mathcal{O}(U(1))$-comodule algebra $\mathcal{O}(\Sigma^3_q(l,+))$ has $\mathcal{O}(\mathbb{RP}_q(l;+))$ as its subalgebra of coinvariant elements under the coaction Ω.*

Proof. The fixed points of the algebra $\mathcal{O}(\Sigma_q^3(l,+))$ under the coaction Ω are found using the same method as in the odd k case. A basis for the algebra $\mathcal{O}(\Sigma_q^3(l,+))$ is given by $x'^r y'^s z'^t$, $x'^{*r} y'^s z'^t$, where $r, s \in \mathbb{N}$ and $t \in \mathbb{Z}$. Since the conjugate of the third and fourth type of basis element is proportional to either the first or second type, the analysis can be focused on first and second type of basis elements.

Applying the coaction Ω to these basis elements gives,

$$x'^r y'^s z'^t \mapsto x'^r y'^s z'^t \otimes u^{a+b-c}.$$

Hence the invariance of $x'^r y'^s z'^t$ is equivalent to $t = r + s$. Simple substitution and re-arranging gives,

$$x'^r y'^s z'^t = x'^r y'^s z'^{r+s} = (x'z')^r (y'z')^s.$$

Hence providing generators $x'z'$ and $y'z'$. Repeating the process for the second type of basis element gives the $*$-conjugates of $x'z'$ and $y'z'$. Using Proposition 4.2.8 we can see that $a = \zeta_1^2 \xi = y'z'$ and $c_+ = \zeta_0^l \xi = x'z'$. □

In contrast to the odd k case, although $\mathcal{O}(\Sigma_q^3(l,+))$ is a principal comodule algebra it yields trivial principal bundle over $\mathcal{O}(\mathbb{RP}_q(l;+))$.

Proposition 4.2.10. *The right $\mathcal{O}(U(1))$-comodule algebra $\mathcal{O}(\Sigma_q^3(l,+))$ is cleft.*

Proof. The cleaving map is given by,

$$j : \mathcal{O}(U(1)) \to \mathcal{O}(\Sigma_q^3(l,+)), \qquad j(u) = z',$$

which is an algebra map since $z^{*'}$ is a central unitary in $\mathcal{O}(\Sigma_q^3(l,+))$, hence must be convolution invertible. Also, j is a right $\mathcal{O}(U(1))$-comodule map since,

$$(\Omega \circ j)(u) = \Omega(z^{*'}) = z^{*'} \otimes u = j(u) \otimes u = (j \otimes id) \circ \Delta(u),$$

completing the proof. □

Whether a different nontrivial principal $\mathcal{O}(U(1))$-comodule algebra over $\mathcal{O}(\mathbb{RP}_q^2(l;+))$ can be constructed or whether such a possibility is ruled out by deeper geometric reasons remains to be seen.

Almost freeness of the coaction $\varrho_{2,l}$.

As was the case for $\mathcal{O}(\Sigma_q^3(l,-))$, the principality of $\mathcal{O}(\Sigma_q^3(l,+))$ can be used to determine that the $\mathcal{O}(U(1))$-coaction $\varrho_{2,l}$ on $\mathcal{O}(\Sigma_q^3)$ that defines $\mathcal{O}(\mathbb{RP}_q^2(l;+))$ is almost free.

Proposition 4.2.11. *The coaction $\varrho_{2,l}$ is almost free.*

Proof. Denote by $\iota_+ : \mathcal{O}(\Sigma_q^3(l,+)) \hookrightarrow \mathcal{O}(\Sigma_q^3)$, the $*$-algebra embedding described in Proposition 4.2.8. One easily checks that the following diagram

$$\begin{tikzcd}
\mathcal{O}(\Sigma_q^3(l,+)) \arrow[r, "\iota_+"] \arrow[d, "\Omega"'] & \mathcal{O}(\Sigma_q^3) \arrow[d, "\varrho_{2,l}"] \\
\mathcal{O}(\Sigma_q^3(l,+)) \otimes \mathcal{O}(U(1)) \arrow[r, "\iota_+ \otimes (-)^{2l}"] & \mathcal{O}(\Sigma_q^3) \otimes \mathcal{O}(U(1)),
\end{tikzcd}$$

where $(-)^{2l} : u \to u^{2l}$ is commutative. By the arguments analogous to those in the proof of Proposition 4.2.6 one concludes that there is a short exact sequence of left $\mathcal{O}(\Sigma_q^3)$-modules

$$(\mathcal{O}(\Sigma_q^3) \otimes \mathbb{C}[u, u^{-1}])/(\mathcal{O}(\Sigma_q^3) \otimes \mathbb{C}[u^{2l}, u^{-2l}]) \longrightarrow \mathrm{coker}(\overline{\mathsf{can}}) \longrightarrow 0,$$

where $\overline{\mathsf{can}}$ is the lifted canonical map corresponding to coaction $\varrho_{2,l}$. The left $\mathcal{O}(\Sigma_q^3)$-module $(\mathcal{O}(\Sigma_q^3)) \otimes \mathbb{C}[u, u^{-1}])/(\mathcal{O}(\Sigma_q^3) \otimes \mathbb{C}[u^{2l}, u^{-2l}])$ is finitely generated, hence so is $\mathrm{coker}(\overline{\mathsf{can}})$. $\square$

4.3 Quantum real projective spaces as Generalised Weyl algebras

In this section we consider quantum real projective spaces from the point of view of generalised Weyl algebras (see Definition 3.4.1).

Proposition 4.3.1. *The algebra of coordinate functions $\mathcal{O}(\mathbb{RP}_q^2(l;+))$ is a degree one generalised Weyl algebra.*

Proof. Set $\mathcal{D} = \mathbb{C}[a]$. In this case $X^+ = c_+$, $X^- = c_+^*$ and automorphism σ of $\mathcal{D}$ and the defining element $\widetilde{a}$ are

$$\sigma(a) = q^{2l}a, \qquad \widetilde{a} = \prod_{m=1}^{l}(1 - q^{-2m}a).$$

Now

$$\sigma(\widetilde{a}) = \prod_{m=1}^{l}(1 - q^{-2m+2l}a) \Longrightarrow \sigma(\widetilde{a}) = \prod_{m'=1}^{l}(1 - q^{2m'}a) = bb^* = X^+X^-,$$

by making the substitution $m' = l - m$. The remaining relations can be verified in a similar way. $\square$

$\mathcal{O}(\mathbb{RP}_q^2(l;-))$ does not appear to be a generalised Weyl algebra. However, $\mathcal{O}(\Sigma_q^3(l;-))$ the total space of the quantum principal bundle is a generalised Weyl algebra, as is $\mathcal{O}(\Sigma_q^3(l;+))$.

Proposition 4.3.2. *The algebras of coordinate functions $\mathcal{O}(\Sigma_q^3(l;\pm))$ are degree one generalised Weyl algebras.*

Proof. Consider $\mathcal{O}(\Sigma_q^3(l;-))$. Set $\mathcal{D} = \mathbb{C}[y,z]$. In this case $X^+ = x^*$, $X^- = x$ and automorphism σ of $\mathcal{D}$ and the defining element $\widetilde{a}$ are

$$\sigma(y) = q^{-l}a, \qquad \sigma(z) = z, \qquad \widetilde{a} = \prod_{p=0}^{l-1}(1 - q^{2p}y^2z).$$

Now

$$\sigma(\widetilde{a}) = \prod_{p=0}^{l-1}(1 - q^{2p-2l}y^2z) \Longrightarrow \sigma(\widetilde{a}) = \prod_{p'=0}^{l-1}(1 - q^{-2p'}y^2z) = x^*x = X^+X^-,$$

by making the substitution $p' = l - p$. The remaining relations can be verified in a similar way. Similarly for the case $\mathcal{O}(\Sigma_q^3(l;+))$ by defining the ground ring $\mathcal{D} = \mathbb{C}[y', z']$, $X^+ = x'^*, X^- = x'$, defining automorphisms $\sigma(y') = q^{-2l}y'$ and $\sigma(y') = q^{-2l}y'$ and defining element $\widetilde{a} = \prod_{p=0}^{l-1}(1 - q^{2p}y'z')$. □

4.4 Algebras of continuous functions on real weighted projective spaces and their K-theory

The C^*-algebras $C(\mathbb{RP}_q(l;\pm))$ of continuous functions on the quantum real weighted projective spaces are defined as the completions of $\oplus_{r=1}^{l}\pi_r(\mathcal{O}(\mathbb{RP}_q(l;\pm)))$, for representations π_r given in equations 4.10a, 4.10b and 4.14.

These spaces (plus their K-groups) were calculated in [12] and turn out to be

$$C(\mathbb{RP}_q(l;+)) \cong \mathcal{T} \oplus_\sigma \mathcal{T} \oplus_\sigma \dots \oplus_\sigma \mathcal{T} \qquad (l \text{ copies}),$$

where $\mathcal{T}$ is the Toeplitz algebra generated by unilateral shift operator U (see Example 1.3.5) and $\sigma : \mathcal{T} \to C(S^1)$ the symbol map $\sigma(U) = u$, where u is the unitary generator of $C(S^1)$. And

$$C(\mathbb{RP}_q(l;-)) \cong C(\mathbb{RP}_q^2) \oplus_{\overline{\sigma}} C(\mathbb{RP}_q^2) \oplus_{\overline{\sigma}} \dots \oplus_{\overline{\sigma}} C(\mathbb{RP}_q^2) \qquad (l \text{ copies}),$$

where $C(\mathbb{RP}_q^2)$ is defined in [26] (Theorem 4.8) and can be identified with the C^*-algebra generated by the shift-by-two operator V. The map $\overline{\sigma} : C(\mathbb{RP}_q^2) \to C(S^1)$ is given by $V \mapsto u$.

Finally, the K-groups come out as

$$K_0(C(\mathbb{RP}_q(l;+))) \cong \mathbb{Z}^l, \qquad K_1(C(\mathbb{RP}_q(l;+))) \cong 0,$$

and

$$K_0(C(\mathbb{RP}_q(l;-))) \cong \mathbb{Z}_2 \oplus \mathbb{Z}^l, \qquad K_1(C(\mathbb{RP}_q(l;-))) \cong 0.$$

Chapter 5

Quantum Heegaard spaces

Quantum Heegaard spaces may be considered as a complement to quantum teardrops in so much as, on the operator algebraic level, they include the generic Podleś two-spheres [37] for parameter $s \neq 0$ in Proposition 2.2.7. On the algebraic level, however, they are defined through integer gradings on an algebra significantly different from that of the coordinate algebra of the quantum $SU(2)$-group.

The coordinate algebra of the Heegaard quantum 3-sphere $\mathcal{O}(S^3_{pq\theta})$ [14], [1] is defined for parametres $0 \leq p, q, \theta < 1$, for θ irrational when non-zero, as the complex $*$-algebra generated by a and b satisfying the relations,

$$ab = e^{2\pi i\theta} ba, \qquad ab^* = e^{-2\pi i\theta} b^* a, \tag{5.1a}$$

$$a^*a - paa^* = 1 - p, \qquad b^*b - qbb^* = 1 - q, \qquad (1 - aa^*)(1 - bb^*) = 0. \tag{5.1b}$$

$\mathcal{O}(S^3_{pq\theta})$ contains two copies of the quantum disc with parametres p and q as $*$-subalgebras and can be interpreted as obtained by glueing of two quantum solid tori [1]. To describe the algebraic structure of $\mathcal{O}(S^3_{pq\theta})$ it is convenient to define (self-adjoint) $A := 1 - aa^*$, $B := 1 - bb^*$. In terms of these elements the relations (5.1b) can be recast as:

$$AB = BA = 0, \quad Aa = paA, \quad Ba = aB, \quad Ab = bA, \quad Bb = qbB. \tag{5.2}$$

A linear basis for $\mathcal{O}(S^3_{pq\theta})$ consists of all $A^k a^l b^m$, $A^k a^{*l} b^m$, $B^k a^l b^m$, $B^k a^{*l} b^m$ and their $*$-conjugates, where $k, l, m \in \mathbb{N}$; see [1].

Standardly, $\mathcal{O}(S^3_{pq\theta})$ is considered as a $\mathbb{Z}$-graded algebra (compatible with the $*$-structure in the sense that the $*$-operation changes the grade to its negative) in two different ways. First a and b are given an equal grade, say, 1. The degree zero subalgebra is generated by polynomials A, B and ab^* and is known as a coordinate algebra of the *mirror quantum sphere* [25]. Second, a and b are given opposite grades, say 1 for a and -1 for b. The degree zero subalgebra, generated by A, B and ab, was introduced in [15], where it was shown that its C^*-completion is isomorphic to the algebra of continuous functions on the generic (or non-standard) Podleś quantum sphere; Proposition 2.2.7, $s \neq 0$.

We prefer to interpret $\mathbb{Z}$-gradings geometrically as algebraic coactions of the coordinate Hopf algebra $\mathcal{O}(U(1))$ of the circle group. $\mathcal{O}(U(1))$ can be identified with the $*$-algebra

$\mathbb{C}[u, u^*]$ of polynomials in variables u and u^* satisying $uu^* = u^*u = 1$. The Hopf algebra structure is given by

$$\Delta(u) = u \otimes u, \qquad \epsilon(u) = 1, \qquad S(u) = u^*.$$

The two $\mathbb{Z}$-gradings described above correspond to coactions $a \mapsto a \otimes u$, $b \mapsto b \otimes u$, and $a \mapsto a \otimes u$, $b \mapsto b \otimes u^*$, respectively. It has been shown in [26], [28] that these coactions are principal, i.e. they make $\mathcal{O}(S^3_{pq\theta})$ into a principal $\mathcal{O}(U(1))$-comodule algebra [11].

5.1 The coordinate algebras of quantum weighted Heegaard spheres

In this section we gather algebraic properties of quantum weighted Heegaard spheres.

5.1.1 The definition of quantum weighted Heegaard spheres

A weighted circle coaction on $\mathcal{O}(S^3_{pq\theta})$ consistent with the algebraic relations (5.1a) and (5.1b) is defined for k, l coprime integers, by

$$\begin{aligned} \phi_{k,l} : \mathcal{O}(S^3_{pq\theta}) &\to \mathcal{O}(S^3_{pq\theta}) \otimes \mathcal{O}(U(1)), \\ a \mapsto a \otimes u^k, &\qquad b \mapsto b \otimes u^l, \end{aligned} \tag{5.3}$$

and extended to the whole of $\mathcal{O}(S^3_{pq\theta})$ so that $\mathcal{O}(S^3_{pq\theta})$ is a right $\mathcal{O}(U(1))$-comodule algebra. The fixed point subalgebras of $\mathcal{O}(S^3_{pq\theta})$ with respect to coactions $\phi_{k,l}$ are called the coordinate algebras of *quantum weighted Heegaard spheres*. Equipping $\mathcal{O}(S^3_{pq\theta})$ with coaction $\phi_{k,l}$ is equivalent to making it into a $\mathbb{Z}$-graded algebra with grading determined by $\deg(a) = k$, $\deg(b) = l$. At this stage we need to consider the possible signs for weights k and l. It turns out that the fixed point subalgebra splits into two cases depending on the signs of the weights k and l. Each case can be described by firstly putting $k > 0$ and $l > 0$ and secondly $k > 0$ and $l < 0$. We write these spaces as,

$$\mathcal{O}(S^2_{pq}(k, l^\pm)) := \mathcal{O}(S^3_{pq\theta})^{co\mathcal{O}(U(1))} = \{x \in \mathcal{O}(S^3_{pq\theta}) : \phi_{k,l}(x) = x \otimes 1\},$$

where the $+$ sign indicates positive values for l and the negative sign corresponds to negative values of l. Before we describe these algebras in detail, we first need some tools for calculation purposes. The following lemma can be proven by induction.

Lemma 5.1.1. *For all $m, n \in \mathbb{N}$,*

$$a^{*m}a^n = \begin{cases} a^{*m-n}\prod_{i=1}^n(1 - p^iA), & m \geq n \\ \prod_{i=1}^m(1 - p^iA)a^{n-m}, & m \leq n. \end{cases} \tag{5.4a}$$

$$a^m a^{*n} = \begin{cases} a^{m-n}\prod_{i=1}^n(1 - p^{-i+1}A), & m \geq n \\ \prod_{i=1}^m(1 - p^{-i+1}A)a^{*n-m}, & m \leq n, \end{cases} \tag{5.4b}$$

and similarly with a replaced by b (and hence A by B) and p by q.

With these at hand we can describe the coordinate algebras of quantum weighted Heegaard spheres by generators and relations.

Theorem 5.1.2. *The algebra $\mathcal{O}(S^2_{pq}(k,l^+))$ is the subalgebra of $\mathcal{O}(S^3_{pq\theta})$ generated by $A = 1 - aa^*$, $B = 1 - bb^*$ and $C_+ = a^l b^{*k}$ satifying the relations*

$$A^* = A, \quad B^* = B, \quad AB = BA = 0, \quad AC_+ = p^l C_+ A, \quad BC_+ = q^{-k} C_+ B, \tag{5.5}$$

$$C_+^* C_+ = \prod_{i=1}^{l}\prod_{j=1}^{k}(1 - p^i A)(1 - q^{j-k} B), \qquad C_+ C_+^* = \prod_{i=1}^{l}\prod_{j=1}^{k}(1 - p^{i-l} A)(1 - q^j B), \tag{5.6a}$$

alternatively, since $AB = 0$, we can express (5.6a) *as*

$$C_+^* C_+ = \prod_{i=1}^{l}(1 - p^i A) + \prod_{j=1}^{k}(1 - q^{j-k} B) - 1, \quad C_+ C_+^* = \prod_{i=1}^{l}(1 - p^{i-l} A) + \prod_{j=1}^{k}(1 - q^j B) - 1. \tag{5.6b}$$

$\mathcal{O}(S^2_{pq}(k,l^-))$ is the subalgebra of $\mathcal{O}(S^3_{pq\theta})$ generated by $A = 1 - aa^*$, $B = 1 - bb^*$ and $C_- = a^{*|l|} b^{*k}$, satifying the relations

$$A^* = A, \quad B^* = B, \quad AB = BA = 0, \quad AC_- = p^l C_- A, \quad BC_- = q^{-k} C_- B, \tag{5.7}$$

$$C_-^* C_- = \prod_{i=1}^{|l|}\prod_{j=1}^{k}(1 - p^{i+l} A)(1 - q^{j-k} B), \qquad C_- C_-^* = \prod_{i=1}^{|l|}\prod_{j=1}^{k}(1 - p^i A)(1 - q^j B), \tag{5.8a}$$

alternatively, since $AB = 0$, we can express (5.8a) *as*

$$C_-^* C_- = \prod_{i=1}^{|l|}(1 - p^{i+l} A) + \prod_{j=1}^{k}(1 - q^{j-k} B) - 1, \quad C_- C_-^* = \prod_{i=1}^{|l|}(1 - p^i A) + \prod_{j=1}^{k}(1 - q^j B) - 1. \tag{5.8b}$$

Proof. The first stage is to identify the fixed point subalgebra with respect to the given coaction. A basis for $\mathcal{O}(S^3_{pq\theta})$ is given by $A^k a^\mu b^\nu$ for $k \geq 0, \mu, \nu \in \mathbb{Z}$ and $B^k a^\mu b^\nu$ for $k > 0, \mu, \nu \in \mathbb{Z}$, where, for $\mu, \nu < 0$, $a^\mu = a^{*|\mu|}$ and $b^\nu = b^{*|\nu|}$ is a convenient notation; see [1]. First note that powers of A and B are automatically fixed by the weighted coaction $\phi_{k,l}$. Next,

$$\phi_{k,l}(a^\mu b^\nu) = a^\mu b^\nu \otimes u^{k\mu + l\nu} = a^\mu b^\nu \otimes 1 \Longrightarrow k\mu + l\nu = 0 \quad \text{(the coinvariance condition)}.$$

This means that basis elements of the form $a^\mu b^\nu$ are fixed under the coaction provided $k\mu = -l\nu$, which in turn means that $k|(-l\nu)$. On the other hand k and l are coprime so in fact $k|(-\nu)$ or $\alpha k = -\nu$ for some $\alpha \in \mathbb{Z}$. Substituting this back into the coinvariance condition gives $k\mu = l(\alpha k)$, i.e. $\mu = l\alpha$ and $\nu = -k\alpha$. So,

$$a^{\mu}b^{\nu} = a^{l\alpha}b^{-k\alpha} = (a^l)^{\alpha}(b^{*k})^{\alpha} \sim (a^l b^{*k})^{\alpha},$$

concluding that $a^l b^{*k}$ is a generator from the set of coinvariant elements. This gives the full description of $\mathcal{O}(S^2_{pq}(k,l^+))$ as the subalgebra of $\mathcal{O}(S^3_{pq\theta})$ generated by A, B and $C_+ = a^l b^{*k}$. Similarly, when l is negative $\mathcal{O}(S^2_{pq}(k,l^-))$ is generated by A, B and $C_- = a^{*-l}b^{*k}$.

Next we determine the relations between the generators for both algebras $\mathcal{O}(S^2_{pq}(k,l^{\pm}))$, considering the positive case first. Equations (5.2) immediately imply that $A^* = A$, $B^* = B$, $AB = 0$, $AC_+ = p^l C_+ A$, and $BC_+ = q^{-k}C_+B$. By Lemma 5.1.1,

$$C_+C_+^* = (a^l b^{*k})(b^k a^{*l}) = (a^l a^{*l})(b^{*k}b^k) = \prod_{i=1}^{l}\prod_{j=1}^{k}(1-p^{i-l}A)(1-q^jB),$$

and

$$C_+^*C_+ = (a^l b^{*k})^*(a^l b^{*k}) = (a^{*l}a^l)(b^k b^{*k}) = \prod_{i=1}^{l}\prod_{j=1}^{k}(1-p^iA)(1-q^{j-k}B).$$

The relations for the negative case are proven by similar arguments. □

5.1.2 Representations of $\mathcal{O}(S^2_{pq}(k,l^{\pm}))$

Bounded irreducible $*$-representations of coordinate algebras $\mathcal{O}(S^2_{pq}(k,l^{\pm})$ are derived and classified by standard methods applicable to all algebras of this kind; see for example the proofs of [26, Theorem 2.1] or [5, Proposition 2.2].

Proposition 5.1.3. *Up to unitary equivalence, the following is the list of all bounded irreducible $*$-representations of $\mathcal{O}(S^2_{pq}(k,l^{\pm}))$. For all $m \in \mathbb{N}$, let $\mathfrak{V}_m \cong l^2(\mathbb{N})$ be a separable Hilbert space with orthonormal basis e^m_p for $p \in \mathbb{N}$. For $s = 0,1,...,|l|-1$, $t = 0,...,k-1$, the representations $\pi^1_s : \mathcal{O}(S^2_{pq}(k,l^+)) \to \mathrm{End}(\mathfrak{V}_s)$, $\pi^2_t : \mathcal{O}(S^2_{pq}(k,l^+)) \to \mathrm{End}(\mathfrak{V}_t)$ and $\pi^{-1}_s : \mathcal{O}(S^2_{pq}(k,l^-)) \to \mathrm{End}(\mathfrak{V}_s)$ and $\pi^{-2}_t : \mathcal{O}(S^2_{pq}(k,l^-)) \to \mathrm{End}(\mathfrak{V}_t)$ are given by*

$$\pi^{\pm1}_s(A)e^s_n = p^{n|l|+s}e^s_n, \qquad \pi^{\pm1}_s(B)e^s_n = 0, \qquad \pi^1_s(C_+)e^s_n = \prod_{i=1}^{l}(1-p^{i+nl+s})^{1/2}e^s_{n+1}, \quad (5.9a)$$

$$\pi^{-1}_s(C_-)e^s_n = \prod_{i=1}^{|l|}(1-p^{i+(n-1)|l|+s})^{1/2}e^s_{n-1}, \quad (5.9b)$$

$$\pi^{\pm2}_t(A)e^t_n = 0, \quad \pi^{\pm2}_t(B)e^t_n = q^{nk+t}e^t_n, \quad \pi^{\pm2}_t(C_{\pm})e^t_n = \prod_{j=1}^{k}(1-q^{j+(n-1)k+t})^{1/2}e^t_{n-1}. \quad (5.9c)$$

Furthermore, there are one-dimensional representations in each case given by $A, B \mapsto 0, C_{\pm} \mapsto \lambda$ where $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$, which we denote by $\pi^{\pm}_{\lambda}$.

5.1.3 Quantum Heegaard spaces as generalised Weyl algebras

We now turn our attention to generalised Weyl algebras; see Definition 3.4.1, in relation to quantum Heegaard spaces.

Proposition 5.1.4. *The algebras of coordinate functions $\mathcal{O}(S^2_{pq}(k, l^\pm))$ are degree one generalised Weyl algebras.*

Proof. Set $\mathcal{D} = \mathbb{C}[A, B]/\langle AB, BA\rangle$. In the positive case $\mathcal{O}(S^2_{pq}(k, l^+))$, $X^+ = C_+^*$ and $X^- = C_+$, the automorphism σ_+ of $\mathcal{D}$ and the defining element $\tilde{a}$ are

$$\sigma_+(A) = p^l A, \qquad \sigma_+(B) = q^{-k}B, \qquad \tilde{a} = \prod_{i=1}^{l}\prod_{j=1}^{k}(1 - p^{i-l}A)(1 - q^j B).$$

In the negative case $\mathcal{O}(S^2_{pq}(k, l^-))$, $X^+ = C_-$ and $X^- = C_-^*$, the automorphism σ_- of $\mathcal{D}$ and the defining element $\tilde{a}$ are

$$\sigma_-(A) = p^{-l}A, \qquad \sigma_-(B) = q^k B, \qquad \tilde{a} = \prod_{i=1}^{|l|}\prod_{j=1}^{k}(1 - p^{i+l}A)(1 - q^{j-k}B).$$

That these satisfy all the axioms of a degree one generalised Weyl algebra can be checked by routine calculations. □

In Section 3.4 we have shown that the coordinate algebra $\mathcal{O}(\mathbb{WP}_q(k, l))$ of quantum weighted projective spaces is a degree one generalised Weyl algebra, and then used [3, Theorem 1.6] to show the global dimension of $\mathcal{O}(\mathbb{WP}_q(k, l))$ is finite if $k = 1$ and is infinite otherwise. Unfortunately, the hypothesis of [3, Theorem 1.6] fails in the quantum Heegaard case since the basic ring $\mathcal{D}$ contains zero divisors. On the other hand, one should not expect the global dimension of $\mathcal{O}(S^2_{pq}(k, l^\pm))$ to be finite: on the classical level the relation $AB = 0$ implies that there is a singularity at the origin, which persists in the quantum case.

5.2 Quantum weighted Heegaard spheres and quantum principal bundles

We aim to construct $\mathcal{O}(U(1))$-principal comodule algebras with coinvariant subalgebra $\mathcal{O}(S^2_{pq}(k, l^\pm))$. These can be understood geometrically as coordinate algebras of principal circle bundles over the quantum weighted Heegaard sphere. We follow the general strategy (employed previously in [5] and [6]) of defining a cyclic group algebra coaction on $\mathcal{O}(S^3_{pq\theta})$ where the space of coinvariant elements with respect to this coaction forms the total space.

5.2.1 Circle bundles over $\mathcal{O}(S^2_{pq}(k, l^+))$

Since the fixed point algebra of $(\mathcal{O}(S^3_{pq\theta}), \phi_{k,l})$, for positive l, is generated by $C_+ = a^l b^{*k}$, A and B, we need to define a comodule structure on $\mathcal{O}(S^3_{pq\theta})$ over the cyclic group algebra

$\mathcal{O}(\mathbb{Z}_m)$ that keeps these generators in the invariant part. In terms of the $\mathbb{Z}_m$-grading this means

$$\deg(a^l b^{*k}) = l\ \deg(a) - k\ \deg(b) = 0 \bmod m.$$

This equation is satisfied by setting $\deg(a) = k$, $\deg(b) = l$ and $m = kl$. The grading is equivalent to a coaction $\Lambda_{k,l} : \mathcal{O}(S^3_{pq\theta}) \to \mathcal{O}(S^3_{pq\theta}) \otimes \mathcal{O}(\mathbb{Z}_{kl})$ given by $a \mapsto a \otimes u^k$, $b \mapsto b \otimes u^l$, which is extended in the usual way to make $\Lambda_{k,l}$ an algebra map and hence $(\mathcal{O}(S^3_{pq\theta}), \Lambda_{k,l})$ a comodule algebra. The fixed point subalgebra is generated by $x = a^l, y = b^k, z = A, w = B$. This can be seen by taking a basis element and applying $\Lambda_{k,l}$: $A^\lambda a^\mu b^\nu$ is coinvariant if and only if $k\mu + l\nu = 0 \bmod kl$, hence $\mu = l\phi$ and $\nu = k\delta$, for some $\phi, \delta \in \mathbb{Z}$. This means that $A^\lambda a^\mu b^\nu = A^\lambda (a^l)^\phi (b^k)^\delta$ so a^l, b^k and A are generators. By considering the other basis element the final generator B is identified. The resulting algebra $\mathcal{A}_{k,l} = \mathcal{O}(S^3_{pq\theta})^{co\mathcal{O}(\mathbb{Z}_{kl})}$ is the quotient of $\mathbb{C}[w, x, y, z]$ by the relations

$$xy = e^{2\pi i\theta kl} yx, \qquad x^* y = e^{-2\pi i \theta kl} y x^*, \tag{5.10a}$$

$$xx^* = \prod_{i=1}^{l}(1 - p^{i-l}z), \quad x^*x = \prod_{i=1}^{l}(1 - p^i z), \quad yy^* = \prod_{i=1}^{k}(1 - q^{i-k}w), \quad y^*y = \prod_{i=1}^{k}(1 - q^i w), \tag{5.10b}$$

$$z^* = z, \quad w^* = w, \quad wz = zw = 0, \quad xw = wx, \quad yz = zy, \quad yw = q^{-k} wy, \quad xz = p^{-l} zx. \tag{5.10c}$$

The circle group algebra coacts on $\mathcal{A}_{k,l}$ by

$$\rho_{k,l} : \mathcal{A}_{k,l} \to \mathcal{A}_{k,l} \otimes \mathcal{O}(U(1)), \qquad w \mapsto w \otimes 1,\ x \mapsto x \otimes u,\ y \mapsto y \otimes u,\ z \mapsto z \otimes 1,$$

making $(\mathcal{A}_{k,l}, \rho_{k,l})$ a $\mathcal{O}(U(1))$-comodule algebra. The fixed points of this comodule algebra are generated by $\alpha = w$, $\beta = z$, $\gamma = xy^*$, and thus are isomorphic to $\mathcal{O}(S^2_{pq}(k, l^+))$ via the map $\alpha \mapsto B, \beta \mapsto A, \gamma \mapsto C_+$.

Theorem 5.2.1. *$(\mathcal{A}_{k,l}, \rho_{k,l})$ is a principal $\mathcal{O}(U(1))$-comodule algebra over $\mathcal{O}(S^2_{pq}(k, l^+))$.*

Proof. To prove principality we construct a *strong connection* for $\mathcal{A}_{k,l}$; see Proposition 2.4.15. A strong connection is defined by setting $\omega(1) = 1 \otimes 1$ and then recursively for $n \in \mathbb{N}$,

$$\omega(u^n) = x^* \omega(u^{n-1}) x + f(z) y^* \omega(u^{n-1}) y, \tag{5.11a}$$

$$\omega(u^{-n}) = x \omega(u^{-n+1}) x^* + f(p^{-l} z) y \omega(u^{-n+1}) y^*, \tag{5.11b}$$

where $f(z) = 1 - \prod_{i=1}^{l}(1 - p^i z)$. That this defined ω satisfies the conditions of a strong connection is proven by induction.

Condition (2.15b): Note that, since y^*y is a polynomial in w with constant term 1, $zw = 0$ and $f(z)$ has the zero constant term, $f(z)y^*y = f(z)$. Hence, in the case $n = 1$,

$$(\mu \circ \omega)(u) = x^*x + f(z)y^*y = x^*x + 1 - \prod_{i=1}^{l}(1 - p^i z) = 1,$$

by (5.10b). Now assume that $(\mu \circ \omega)(u^n) = 1$ and consider

$$(\mu \circ \omega)(u^{n+1}) = x^*(\mu \circ \omega(u^n))x + f(z)y^*(\mu \circ \omega(u^n))y = x^*x + f(z)y^*y = 1,$$

where the second equality is the inductive assumption. The proof for negative powers of u is essentially the same, so the details are omitted.

Condition (2.15c): Consider ω for positive powers of u; putting $n = 1$ gives,

$$\begin{aligned}((\mathrm{id} \otimes \rho_{k,l}) \circ \omega)(u) &= (\mathrm{id} \otimes \rho_{k,l})(x^* \otimes x + f(z)y^* \otimes y) \\ &= x^* \otimes x \otimes u + f(z)y^* \otimes y \otimes u = \omega(u) \otimes u.\end{aligned}$$

This is the basis for the inductive proof. Now assume that

$$((\mathrm{id} \otimes \rho_{k,l}) \circ \omega)(u^n) = ((\omega \otimes \mathrm{id}) \circ \Delta)(u^n) = \omega(u^n) \otimes u^n,$$

and consider

$$\begin{aligned}((\mathrm{id} \otimes \rho_{k,l}) \circ \omega)(u^{n+1}) &= (\mathrm{id} \otimes \rho_{k,l})(x^*\omega(u^n)x + f(z)y^*\omega(u^n)y) \\ &= x^*\omega(u^n)x \otimes u^{n+1} + f(z)y^*\omega(u^n)y \otimes u^{n+1} = \omega(u^{n+1}) \otimes u^{n+1},\end{aligned}$$

where the second equality follows from the induction hypothesis and $\rho_{k,l}$ being an algebra map. The proof for ω taking negative powers of u follows a similar argument.

Condition (2.15d): As in the previous conditions, the proofs for positive and negative powers of u are similar, hence only the positive case is displayed. Note that here we need to show that

$$((\widetilde{w} \otimes \mathrm{id}) \circ (\rho_{k,l} \otimes id) \circ \omega)(u^n) = ((S \otimes \omega) \circ \Delta)(u^n) = u^{-n} \otimes \omega(u^n), \tag{5.12}$$

for all n. The case $n = 1$ follows by the same argument as in the preceding proof. Assume that equation (5.12) is true for an $n \in \mathbb{N}$, and consider

$$\begin{aligned}((\widetilde{w} \otimes \mathrm{id}) \circ (\rho_{k,l} \otimes \mathrm{id}) \circ \omega)(u^{n+1}) &= ((\widetilde{w} \otimes \mathrm{id}) \circ (\rho_{k,l} \otimes \mathrm{id}))(x^*\omega(u^N)x + f(z)y^*\omega(u^n)y) \\ = u^{-n-1} \otimes x^*\omega(u^n)x + u^{-n-1} \otimes f(z)y^*\omega(u^n)y &= u^{-(n+1)} \otimes \omega(u^{n+1}),\end{aligned}$$

where the second equality follows from the induction hypothesis and $\rho_{k,l}$ being an algebra map. This completes the proof of (2.15d) for positive powers of u.

Since $(\mathcal{A}_{k,l}, \rho_{k,l})$ is a comodule algebra admitting a strong connection it is principal; see [19], [11]. □

Proposition 5.2.2. *The principal $\mathcal{O}(U(1))$-comodule algebra $\mathcal{A}_{k,l}$ is not cleft.*

Proof. Since $\mathcal{A}_{k,l} \subseteq \mathcal{O}(S^3_{pq\theta})$ and, by [23, Theorem 1.10], the only invertible elements in $\mathcal{O}(S^3_{pq\theta})$ are multiples of 1, the only invertible elements in $\mathcal{A}_{k,l}$ are also the multiples of 1. The convolution invertible map $j : \mathcal{O}(U(1)) \to \mathcal{A}_{k,l}$ must take the form $j(u) = \alpha 1$ for some $\alpha \in \mathbb{C}^*$. However this violates the right $\mathcal{O}(U(1))$-colinearity of j. □

5.2.2 Circle bundles over $\mathcal{O}(S^2_{pq}(k, l^-))$

Following the process in the case l positive, a suitable coaction $\Phi_{k,l} : \mathcal{O}(S^3_{pq\theta}) \to \mathcal{O}(S^3_{pq\theta}) \otimes \mathcal{O}(\mathbb{Z}_{k|l|})$ is arrived at as given by $a \mapsto a \otimes u^k, b \mapsto b \otimes u^l$. The fixed point subalgebra is generated by $x = a^{|l|}, y = b^k, z = A, w = B$. This coincides with the algebra $\mathcal{A}_{k,-l}$, hence positive and negative values of l give rise to the same total space of the quantum principal bundle. The principal coaction of $\mathcal{O}(U(1))$ on $\mathcal{A}_{k,-l}$ that fixes $\mathcal{O}(S^2_{pq}(k, l^-))$ is defined by $x \mapsto x \otimes u$, $y \mapsto y \otimes u^*$, $z \mapsto z \otimes 1$ and $w \mapsto w \otimes 1$.

5.3 Fredholm modules and the Chern-Connes pairing for $\mathcal{O}(S^2_{pq}(k,l^\pm))$

In this section first we associate even Fredholm modules to the algebras $\mathcal{O}(S^3_{pq\theta})$ and use them to construct traces or cyclic cycles on $\mathcal{O}(S^3_{pq\theta})$. The latter are then used to calculate the Chern number of a non-commutative line bundle associated to the quantum principal bundle $\mathcal{A}_{k,|l|}$ over the quantum weighted Heegaard spaces $\mathcal{O}(S^2_{pq}(k,l^\pm))$.

5.3.1 Fredholm modules

Representations for Fredholm modules are constructed from irreducible representations of $\mathcal{O}(S^2_{pq}(k,l^\pm))$ listed in Proposition 5.1.3. In the positive l case we take a cue from [25] and, for every $s=0,1,2,\ldots,l-1$, $t=0,1,2,\ldots,k-1$, consider the representation $\pi_{s,t}$ obtained as a direct sum of representations π^1_s and π^2_t. The Hilbert space of this representation is denoted by $\mathfrak{V}_{s,t}$ and we choose its orthonormal basis $f^{s,t}_m$, $m\in\mathbb{Z}$ as follows. For m positive the $f^{s,t}_m$ correspond to the basis elements e^s_m of the representation space of π^1_s, and for negative m, the $f^{s,t}_m$ correspond to the e^t_{-m-1} of the Hilbert space of π^2_t. In addition to $\pi_{s,t}$ we also consider the integral of one-dimensional representations, $\pi_c=\int_{\lambda\in S^1}\pi^+_\lambda \mathrm{d}\lambda$. The representation space of π_c can be identified with $\mathfrak{V}_{s,t}$, so that

$$\pi_c(A)=\pi_c(B)=0,\quad \pi_c(C^\mu_+)f^{s,t}_m=f^{s,t}_{m+\mu},\quad \pi_c(C^{*\mu}_+)f^{s,t}_m=f^{s,t}_{m-\mu}, \tag{5.13}$$

for all $m\in\mathbb{Z}$, $\mu\in\mathbb{N}$.

Proposition 5.3.1. *For all $s=0,1,2,\ldots,|l|-1$, $t=0,1,2,\ldots,k-1$, $(\mathfrak{V}_{s,t}\oplus\mathfrak{V}_{s,t},\bar{\pi}_{s,t}:=\pi_{s,t}\oplus\pi_c,F,\gamma)$, where*

$$F=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix},\qquad \gamma=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix},$$

is a 1-summable Fredholm module over $\mathcal{O}(S^2_{pq}(k,l^+))$, while $(\mathfrak{V}_s\oplus\mathfrak{V}_t,\bar{\pi}^-_{s,t}:=\pi^{-1}_s\oplus\pi^{-2}_t,F,\gamma)$ is a 1-summable Fredholm module over $\mathcal{O}(S^2_{pq}(k,l^+))$.

The corresponding Chern characters are

$$\tau^\pm_{s,t}(A^\lambda C^\mu_\pm)=\begin{cases}\frac{p^{\lambda s}}{1-p^{\lambda|l|}} & \text{if } \mu=0,\ \lambda\neq 0,\\ 0 & \text{otherwise},\end{cases}\qquad \tau^\pm_{s,t}(B^\lambda C^\mu_\pm)=\begin{cases}\frac{q^{\lambda t}}{1-q^{\lambda k}} & \text{if } \mu=0,\ \lambda\neq 0,\\ 0 & \text{otherwise}.\end{cases} \tag{5.14}$$

Here $\mu\in\mathbb{Z}$ and, for a positive μ, $C^{-\mu}_\pm$ means $C^{*\mu}_\pm$.

Proof. It is obvious that $F^*=F$, $F^2=\gamma^2=I$ and $F\gamma+\gamma F=0$. We first deal with the positive l case. By a straightforward calculation, for all $x\in\mathcal{O}(S^2_{pq}(k,l^+))$,

$$[F,\pi(x)]=\begin{pmatrix}0 & \pi_c(x)-\pi_{s,t}(x)\\ \pi_{s,t}(x)-\pi_c(x) & 0\end{pmatrix}.$$

We show that $\pi_{s,t}(x)-\pi_c(x)$ is trace class for x in the basis $\{A^\lambda C^\mu_+ : \lambda\in\mathbb{N},\mu\in\mathbb{Z}\}\cup\{B^\lambda C^\mu_+ : \lambda\in\mathbb{N},\mu\in\mathbb{Z}\}$. Using the formulae in Proposition 5.1.3 one easily finds, for all

$\lambda, \mu \in \mathbb{N}$,

$$\pi_{s,t}(A^\lambda C_+^\mu) f_m^{s,t} = \begin{cases} p^{\lambda((m+\mu)l+s)} \prod_{i=1}^{l\mu}(1-p^{i+s+ml})^{1/2} f_{m+\mu}^{s,t} & m \geq 0, \\ 0 & m = -1, \ldots, -\mu, \\ \delta_{\lambda,0} \prod_{i=1}^{k\mu}(1-q^{i-(m+1+\mu)k+t})^{1/2} f_{m+\mu}^{s,t} & m < -\mu. \end{cases} \tag{5.15a}$$

and

$$\pi_{s,t}(C_+^{*\mu} A^\lambda) f_m^{s,t} = \begin{cases} 0 & m = 0, 1, \ldots, \mu-1, \\ p^{\lambda(ml+s)} \prod_{i=1}^{l\mu}(1-p^{i+s+(m-\mu)l})^{1/2} f_{m-\mu}^{s,t} & m \geq \mu, \\ \delta_{\lambda,0} \prod_{i=1}^{k\mu}(1-q^{i-(m+1)k+t})^{1/2} f_{m-\mu}^{s,t} & m < 0. \end{cases} \tag{5.15b}$$

Set $X_{\lambda,\mu}^+ := \pi_{s,t}(A^\lambda C_+^\mu) - \pi_c(A^\lambda C_+^\mu)$. Using (5.13) we find

$$({X^+}^*_{0,\mu} X^+_{0,\mu})^{\frac{1}{2}} f_m^{s,t} = \begin{cases} 1 - \prod_{i=1}^{l\mu}(1-p^{i+s+ml})^{1/2} f_m^{s,t} & m \geq 0, \\ 0 & m = -\mu, \ldots, -1 \\ 1 - \prod_{i=1}^{k\mu}(1-q^{i+t-(m+\mu)l})^{1/2} f_m^{s,t} & m < -\mu. \end{cases}$$

We need to estimate the two infinite series. For the first series, note that all factors in the product are less than one, hence

$$\begin{aligned} 1 - \prod_{i=1}^{l\mu}(1-p^{i+s+ml})^{\frac{1}{2}} &\leq 1 - \prod_{i=1}^{l\mu}(1-p^{i+s+ml}) \\ &= p^{s+ml} P_{l\mu}(p) + p^{2(s+ml)} P_{2l\mu-1}(p) + \ldots + p^{l(s+ml)} P_{\frac{1}{2}l\mu(l\mu+1)}(p), \end{aligned}$$

where each P_i is a fixed polynomial of degree i independent of m. Now summing over $m \geq 0$, each term is a geometric series, hence the series converges. Similarly for $m < 0$, using an analogous identity, we find that the sums are convergemt for all μ, so $X_{0,\mu}^+$ are trace-class operators.

Suppose $\lambda \neq 0$, then since $\pi_c(A) = 0$ we arrive at,

$$({X_{\lambda,\mu}^+}^* X_{\lambda,\mu}^+)^{\frac{1}{2}} f_m^{s,t} = \begin{cases} p^{\lambda((m+\mu)l+s)} \prod_{i=1}^{l\mu}(1-p^{i+s+ml})^{1/2} f_m^{s,t} & m \geq \mu, \\ 0 & m < \mu. \end{cases} \tag{5.16}$$

Hence,

$$\begin{aligned} \sum_{m=0}^{\infty} \langle ({X_{\lambda,\mu}^+}^* X_{\lambda,\mu}^+)^{\frac{1}{2}} f_m^{s,t}, f_m^{s,t} \rangle &= \sum_{m=0}^{\infty} p^{\lambda((m+\mu)l+s)} \prod_{i=1}^{l\mu}(1-p^{i+s+ml})^{1/2} \\ &\leq \sum_{m=0}^{\infty} p^{\lambda((m+\mu)l+s)} = \frac{p^{\lambda(s+\mu l)}}{1-p^{\lambda l}} < \infty, \end{aligned}$$

the inequality following since the product is less than one. This implies that $(\pi_{s,t} - \pi_c)(A^\lambda C_+^\mu)$ are trace-class operators, for all values of λ and μ. By defining $\tilde{X}_{\lambda,\mu}^+ = (\pi_{s,t} - \pi_c)(B^\lambda C_+^\mu)$, in an analogous way it can be shown $(\pi_{s,t} - \pi_c)(B^\lambda C_+^\mu)$ are also trace class operators; hence we have shown that $(\mathfrak{V}_{s,t} \oplus \mathfrak{V}_{s,t}, \bar{\pi}_{s,t}, F, \gamma)$ is a 1-summable Fredholm module over $\mathcal{O}(S^2_{pq}(k, l^+))$.

For the negative l case, using Proposition 5.1.3 we arrive at

$$\pi_s^{-1}(A^\lambda)e^s_m = p^{\lambda(m|l|+s)}e^s_m, \quad \pi_s^{-1}(B)e^s_m = 0, \quad \pi_s^{-1}(C^\mu_-)e^s_m = \prod_{i=1}^{\mu|l|}(1 - p^{i+s+(m-\mu)|l|})^{1/2}e^s_{m-\mu},$$

$$\pi_t^{-2}(A)e^t_m = 0, \quad \pi_t^{-2}(B^\lambda)e^t_m = q^{\lambda(mk+s)}e^t_m, \quad \pi_t^2(C^\mu_-)e^t_m = \prod_{i=1}^{k\mu}(1 - q^{i+t+(m-\mu)k})^{1/2}e^t_{m-\mu}.$$

With these at hand it is straightforward to check that the operators $X^-_{\lambda,\mu} := \pi_s^{-1}(A^\lambda C^\mu_-) - \pi_t^{-2}(A^\lambda C^\mu_-)$, $\tilde{X}^-_{\lambda,\mu} := \pi_s^{-1}(B^\lambda C^\mu_-) - \pi_t^{-2}(B^\lambda C^\mu_-)$, the only non-zero entries of the commutator of F with $\bar{\pi}^-_{s,t}$ evaluated at the basis elements of $\mathcal{O}(S^2_{pq}(k, l^-))$, are trace class.

Finally, we can calculate the Chern characters on the basis elements of $\mathcal{O}(S^2_{pq}(k, l^\pm))$ using $\tau^\pm_{s,t}(x) = \mathrm{Tr}\ (\gamma\bar{\pi}^\pm_{s,t}(x))$. First, for $\lambda \neq 0$,

$$\tau^\pm_{s,t}(A^\lambda C^\mu_\pm) = \mathrm{Tr}\ (\gamma\bar{\pi}^\pm_{s,t}(A^\lambda C^\mu_\pm)) = \mathrm{Tr}\ (X^\pm_{\lambda,\mu}) = \delta_{\mu,0}\sum_{m=0}^{\infty} p^{\lambda(ml+s)} = \delta_{\mu,0}\frac{p^{\lambda s}}{1-p^{\lambda|l|}},$$

where we noted that if $\mu \neq 0$, then all diagonal entries of $X^\pm_{\lambda,\mu}$ are zero. Similarly, by considering the traces of $\tilde{X}^\pm_{\lambda,\mu}$ we find that $\tau^\pm_{s,t}(B^\lambda C^\mu_+) = \delta_{\mu,0}\frac{q^{\lambda t}}{1-q^{\lambda k}}$, for $\lambda \neq 0$. □

5.3.2 The Chern-Connes pairing

The Chern characters associated to line bundles over the comodule $\mathcal{A}_{k,l}$ with coaction $\varrho_{k,l}$ are calculated.

Theorem 5.3.2. *For all $s = 0, 1, \ldots, l-1$, $t = 0, 1, \ldots, k-1$, let $\tau^+_{s,t}$ be the cyclic cocycle on $\mathcal{O}(S^2_{pq}(k, l^+))$ constructed in Proposition 5.3.1. Let $E[n]$ be the idempotent determined by $\omega(u^n)$ in (5.11). Then $\tau^+_{s,t}(\mathrm{Tr}\ E[n]) = -n$. Consequently, for $n \neq 0$, the left $\mathcal{O}(S^2_{pq}(k, l^+))$-modules $\mathcal{L}[n]$ corresponding to $E[n]$ are not free.*

Proof. We prove the theorem for the positive values of n. The negative n case is proven in a similar way. Define $f(z) = 1 - \prod_{i=1}^{l}(1 - p^i z)$ and note that, for all $s = 0, 1, \ldots, l-1$, $f(p^{s-l}) = 1$.

Lemma 5.3.3. *For positive n,*

$$\omega(u^n) = \sum_i \omega(u^{n-1})^{[1]}{}_i x^* \otimes x\omega(u^{n-1})^{[2]}{}_i + \sum_i \omega(u^{n-1})^{[1]}{}_i f(z)y^* \otimes y\omega(u^{n-1})^{[2]}{}_i \quad (5.18)$$

Proof. This is proven by induction. For $n = 1$, this is simply equation (5.11a) with $n = 1$. Assume that equation (5.18) is true for any $r \leq n$. Then, using (5.11a) and the inductive assumption, we can compute

$$\begin{aligned}
\omega(u^{n+1}) &= x^*\omega(u^n)x + f(z)y^*\omega(u^n)y \\
&= \sum_i x^*\omega(u^{n-1})^{[1]}{}_i \underbrace{x^* \otimes x}\, \omega(u^{n-1})^{[2]}{}_i x \\
&\quad + \sum_i x^*\omega(u^{n-1})^{[1]}{}_i \underline{f(z)y^* \otimes y}\omega(u^{n-1})^{[2]}{}_i x \\
&\quad + \sum_i f(z)y^*\omega(u^{n-1})^{[1]}{}_i \underbrace{x^* \otimes x}\, \omega(u^{n-1})^{[2]}{}_i y \\
&\quad + \sum_i f(z)y^*\omega(u^{n-1})^{[1]}{}_i \underline{f(z)y^* \otimes y}\omega(u^{n-1})^{[2]}{}_i y \\
&= \sum_i \omega(u^n)^{[1]}{}_i x^* \otimes x\omega(u^n)^{[2]}{}_i + \sum_i \omega(u^n)^{[1]}{}_i f(z)y^* \otimes y\omega(u^n)^{[2]}{}_i,
\end{aligned}$$

where we indicated grouping of terms over which the definition (5.11a) of the strong connection ω is applied. □

Lemma 5.3.4. *For all positive n, $\mathrm{Tr}\ E[n] = g_n(z)$ is a polynomial in z independent of w such that*

$$g_{n+1}(z) = \left(1 - f\left(p^{-l}z\right)\right) g_n\left(p^{-l}z\right) + f(z)g_n(z). \tag{5.19}$$

Proof. By (5.18) and the definition of the idempotents $E[n]$,

$$\begin{aligned}
\mathrm{Tr}\ E[n] &= \sum_i x\omega(u^{n-1})^{[2]}{}_i\omega(u^{n-1})^{[1]}{}_i x^* + \sum_i y\omega(u^{n-1})^{[2]}{}_i\omega(u^{n-1})^{[1]}{}_i f(z)y^* \\
&= x\mathrm{Tr}\ E[n-1]x^* + y\mathrm{Tr}\ E[n-1]f(z)y^*.
\end{aligned}$$

In particular, since $\mathrm{Tr}\ E[0] = 1$,

$$\mathrm{Tr}\ E[1] = xx^* + yf(z)y^* = xx^* + yy^*f(z) = 1 - f\left(p^{-l}z\right) + f(z), \tag{5.20}$$

where we used (5.10) (expressed in terms of the polynomial f), in particular the fact that z commutes with y, that yy^* is a polynomial in w with constant term 1, $wz = 0$ and f has the zero constant term. Therefore, $\mathrm{Tr}\ E[1]$ is a polynomial in z only (not in w), and it satisfies (5.19) with $g_0 = 1$.

Assume, inductively, that $\mathrm{Tr}\ E[n] = g_n(z)$. Then, again extracting the same information from (5.10) as before and, additionally, using the commutation rule between z and x^* we obtain,

$$\begin{aligned}
g_{n+1}(z) &= xg_n(z)x^* + yg_n(z)y^*f(z) = xx^*g_n\left(p^{-l}z\right) + yg_n(z)y^*f(z) \\
&= \left(1 - f\left(p^{-l}z\right)\right) g_n\left(p^{-l}z\right) + f(z)g_n(z),
\end{aligned}$$

as required. □

With Lemma 5.3.3 and Lemma 5.3.4 at hand we can complete the proof of Theorem 5.3.2. Since $z = A$, the Chern number

$$\mathrm{ch}_n := \tau^+_{s,t}(\mathrm{Tr}\ E[n]) = \tau^+_{s,t}(g_n(z)),$$

is obtained by evaluating the powers of z in polynomial expansion of $g_n(z)$ using formulae (5.14). We will proceed by induction on n, but first write

$$f(z) = \sum_{m=1}^{l} c_m^l z^m, \qquad g_n(z) = \sum_{r=0}^{N} d_r^n z^r.$$

Note that $f(0) = 0$, hence in view of (5.20), $g_0(0) = 1$, and, consequently $g_n(0) = 1$, by (5.19). Therefore, $d_0^n = 1$.

Apply (5.20) and (5.14) to calculate

$$\begin{aligned} \mathrm{ch}_1 &= \tau_{s,t}^+(g_1(z)) = \tau_{s,t}^+\left(1 - f\left(p^{-l}z\right) + f(z)\right) \\ &= \tau_{s,t}^+\left(1 - \sum_{m=1}^{l} c_m^l (p^{-lm} - 1) z^m\right) = -\sum_{m=1}^{l} c_m^l p^{-lm+sm} = -f(p^{s-l}) = -1. \end{aligned}$$

The last equality follows from the observation that since $s = 0, 1, \ldots, l-1$, one of the factors in the product must vanish. Next, assume that $\mathrm{ch}_n = -n$, that is

$$\sum_{r=1}^{N} d_r^n \frac{p^{sr}}{1 - p^{lr}} = -n. \tag{5.21}$$

Then, using (5.19)

$$\begin{aligned} \mathrm{ch}_{n+1} &= \sum_{r=0}^{N} d_r^n p^{-lr} \tau_{s,t}^+(z^r) - \sum_{m=1}^{l}\sum_{r=0}^{N} c_m^l d_r^n p^{-l(m+r)} \tau_{s,t}^+(z^{m+r}) + \sum_{m=1}^{l}\sum_{r=0}^{N} c_m^l d_r^n \tau_{s,t}^+(z^{m+r}) \\ &= \sum_{r=1}^{N} d_r^n \frac{p^{(s-l)r}}{1 - p^{lr}} - \sum_{m=1}^{l}\sum_{r=0}^{N} c_m^l d_r^n \frac{p^{-l(m+r)} - 1}{1 - p^{l(m+r)}} p^{(m+r)s} \\ &= \sum_{r=1}^{N} d_r^n \frac{p^{(s-l)r}}{1 - p^{lr}} - f(p^{s-l}) \sum_{r=0}^{N} d_r^n p^{(r-l)s} \\ &= \sum_{r=1}^{N} d_r^n \left(\frac{p^{(s-l)r}}{1 - p^{lr}} - p^{(r-l)s}\right) - 1 = \sum_{r=1}^{N} d_r^n \frac{p^{sr}}{1 - p^{lr}} - 1 = -n - 1, \end{aligned}$$

by inductive assumption (5.21). This completes the proof of the theorem. □

5.4 Continuous functions on the quantum weighted Heegaard spheres

The C^*-algebras $C(S^2_{pq}(k, l^\pm))$ of continuous functions on the quantum weighted Heegaard spheres are defined as completions of the direct sum of representations classified in Proposition 5.1.3. In this section we identify these algebras as pullbacks of the Toeplitz algebra and calculate their K-groups, closely following the approach of [42] and[27] (see also [12]).

We think about the Toeplitz algebra $\mathcal{T}$ concretely as the C^*-algebra generated by an unilateral shift U acting on a separable Hilbert space $\mathcal{V}$ with orthonormal basis e_n by $Ue_n = e_{n+1}$ ($\mathcal{V}$ could be any of the spaces $\mathcal{V}_s$ in Proposition 5.1.3). $\mathcal{T}$ can be thought of as the algebra of continuous functions on the quantum unit disc, and the restriction of these functions to the boundary circle S^1 yields the *symbol map*, $\sigma : \mathcal{T} \to C(S^1)$, $U \to u$, where u is the unitary generator of $C(S^1)$. Given $k \in \mathbb{N}$, $l \in \mathbb{Z}$, define

$$\mathcal{T}^{k,l} = \{(x_1, \ldots, x_{k+l}) \in \mathcal{T}^{\oplus k+l} \mid \sigma(x_1) = \ldots = \sigma(x_l) = \sigma^{-1}(x_{l+1}) = \ldots = \sigma^{-1}(x_{k+l})\}$$

for positive l and

$$\mathcal{T}^{k,l} = \{(x_1, \ldots, x_{k-l}) \in \mathcal{T}^{\oplus k-l} \mid \sigma(x_1) = \ldots = \sigma(x_{k-l})\},$$

for negative l. In this section we prove

Theorem 5.4.1. *For all $k \in \mathbb{N}$, $l \in \mathbb{Z}$,*

$$C(S^2_{pq}(k, l^{\pm})) \cong \mathcal{T}^{k,l}. \tag{5.22}$$

Consequently,

$$K_1(C(S^2_{pq}(k, l^{\pm}))) = 0, \qquad K_0(C(S^2_{pq}(k, l^{\pm}))) = \mathbb{Z}^{k+|l|}. \tag{5.23}$$

Proof. To see that (5.23) follows from (5.22), we observe the exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \xrightarrow{\sigma} C(S^1) \longrightarrow 0, \tag{5.24}$$

that characterises the Toeplitz algebra in terms of compact operators on $\mathcal{V}$, yields the exact sequence

$$0 \longrightarrow \mathcal{K}^{\oplus k+|l|} \longrightarrow \mathcal{T}^{k,l} \longrightarrow C(S^1) \longrightarrow 0. \tag{5.25}$$

The sequence (5.25) gives rise to a six-term exact sequence of K-groups, which can be studied precisely as in [12, Section 4.2] to aid the derivation of the K-groups as stated.

Let $J_A^{\pm}$, $J_B^{\pm}$ denote the closed $*$-ideals of $C(S^2_{pq}(k, l^{\pm}))$ obtained by completing of ideals $\mathcal{J}_A^{\pm}$, $\mathcal{J}_B^{\pm}$ of $\mathcal{O}(S^2_{pq}(k, l^{\pm}))$ generated by A and B, and let $\psi : C(S^2_{pq}(k, l^{\pm})) \to C(S^2_{pq}(k, l^{\pm}))/(J_A^{\pm} \oplus J_B^{\pm})$ be the canonical surjection. The image of ψ is generated by $\psi(C_{\pm})$. In view of the relations (5.6a) and (5.8a), $\psi(C_{\pm})$ is a unitary operator, hence $C(S^2_{pq}(k, l^{\pm}))/(J_A^{\pm} \oplus J_B^{\pm}) \cong C(S^1)$. Applying $\pi_s^{\pm 1}$, $\pi_t^{\pm 2}$ to $\mathcal{J}_A^{\pm} \oplus \mathcal{J}_B^{\pm}$, one finds that the images contain only compact operators on the corresponding representation spaces $\mathfrak{V}_s$, $\mathfrak{V}_t$. On the other hand, these images contain all orthogonal projections onto one-dimensional subspaces and all step-by-one operators with non-zero weights, hence their completions contain all compact operators. In this way $J_A^{\pm} \oplus J_B^{\pm}$ can be identified with the direct sum $\mathcal{K}^{k+|l|}$. Since the direct sum of all irreducible representations of a C^*-algebra is faithful,

$$\bigoplus_{s=0}^{l-1} \pi_s^{\pm 1} \oplus \bigoplus_{t=0}^{k-1} \pi_t^{\pm 2} \oplus \bigoplus_{\lambda \in S^1} \pi_\lambda^{\pm},$$

are faithful representations of $C(S^2_{pq}(k, l^{\pm}))$. In the same way as in [42], $\pi^{\pm}_{\lambda}$ factor through $\pi^{\pm 1}_s$, $\pi^{\pm 2}_t$, hence also

$$\pi^{\pm} := \bigoplus_{s=0}^{l-1} \pi^{\pm 1}_s \oplus \bigoplus_{t=0}^{k-1} \pi^{\pm 2}_t$$

are faithful. It is clear that the images of $\pi^{\pm}$ are contained in $\mathcal{T}^{k+|l|}$. On the other hand, by inspecting formulae in Proposition 5.1.3 one easily finds that $\pi^1_s(C_+) - U$, $\pi^{-1}(C_-) - U^*$, $\pi^{\pm 2}(C_{\pm}) - U^*$ are step-by-one operators with coefficients tending to zero. Therefore, they are compact operators and thus are in the kernel of the symbol map. This implies that the image of $\pi^{\pm}$ is contained in $\mathcal{T}^{k,l}$. Summarizing the above discussion we obtain commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K}^{\oplus k+|l|} \arrow[r] \arrow[d, equal] & C(S^2_{pq}(k, l^{\pm})) \arrow[r] \arrow[d, hook, "\pi^{\pm}"'] & C(S^1) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{K}^{\oplus k+|l|} \arrow[r] & \mathcal{T}^{k,l} \arrow[r] & C(S^1) \arrow[r] & 0.
\end{tikzcd}$$

This implies the isomorphism (5.22). □

Part III

Outlook

Chapter 6

Covariant differential calculus on the quantum teardrop space

It would be interesting to see whether there exists a covariant differential calculus, and if so, whether there is a distinction between the k parameter. We have seen that moving from the classical to the quantum space we have effectively smoothed out the singularity in the special case $k = 1$. It may be the case that only when $k = 1$ the elements invariant under the coaction freely generate the module of one-forms; see Section 2.4.2. We describe here the first steps in this direction.

The quantum teardrop spaces were constructed by taking the quantum three sphere $\mathcal{O}(S_q^3)$ and viewing this as a $\mathcal{O}(U(1))$-comodule algebra using a weighted circle coaction denoted $\rho_{k,l}$. Since we have Woronowicz's description of a first order calculus on $\mathcal{O}(S_q^3)$ there is scope to build a first order calculus on the teardrop spaces since $\mathcal{O}(\mathbb{WP}_q(k,l))$ is a subalgebra of $\mathcal{O}(S_q^3)$. The process would involve restricting Woronowicz's description to the generators of $\mathcal{O}(\mathbb{WP}_q(k,l))$. It remains to be seen whether this process produces a two-dimensional first order differential calculus on $\mathcal{O}(\mathbb{WP}_q(k,l))$.

6.1 The first steps to finding a first order differential calculus on $\mathcal{O}(\mathbb{WP}_q(k,l))$

Consider $\mathcal{O}(S_q^3)$ as a right $\mathcal{O}(U(1))$-comodule with weighted coaction $\rho_{k,l} : \alpha \mapsto \alpha \otimes u^k, \beta \mapsto \beta \otimes u^{-l}$ see Equation (3.2). We look to extend this coaction to the one-forms $\Omega^1(\mathcal{O}(S_q^3))$ in such as way that the exterior derivative is a right $\mathcal{O}(U(1))$-comodule map, meaning that the following diagram should commute,

$$
\begin{tikzcd}
\mathcal{O}(S_q^3) \arrow[r, "\rho_{k,l}"] \arrow[d, "d"'] & \mathcal{O}(S_q^3) \otimes \mathcal{O}(U(1)) \arrow[d, "d\otimes id"] \\
\Omega^1(\mathcal{O}(S_q^3)) \arrow[r, "\rho^{\Omega^1(\mathcal{O}(S_q^3))}"'] & \Omega^1(\mathcal{O}(S_q^3)) \otimes \mathcal{O}(U(1)).
\end{tikzcd}
$$

Since moving clockwise we get

$$(d \otimes id)\rho_{k,l}(\alpha) = (\alpha\omega_0 - q\beta\omega_+) \otimes u^k,$$

so it must be the case that

$$\rho^{\Omega^1(\mathcal{O}(S_q^3))} d(\alpha) = \rho^{\Omega^1(\mathcal{O}(S_q^3))}(\alpha\omega_0 - q\beta\omega_+) = (\alpha\omega_0 - q\beta\omega_+) \otimes u^k.$$

Hence $\alpha\omega_0$ and $\beta\omega_+$ have degree k, but since α has degree k, ω_0 must have degree zero. Similarly, since β has degree $-l$, ω_+ must have degree $k+l$. In a same manner we require ω_- to have degree $-k-l$. This gives the following description of the coaction on the one-forms,

$$\rho^{\Omega^1(\mathcal{O}(S_q^3))} : \Omega^1(\mathcal{O}(S_q^3)) \to \Omega^1(\mathcal{O}(S_q^3)) \otimes \mathcal{O}(U(1))$$

given by

$$\omega_0 \mapsto \omega_0 \otimes 1, \qquad \omega_+ \mapsto \omega_+ \otimes u^{k+l}, \qquad \omega_- \mapsto \omega_- \otimes u^{-k-l}.$$

Proposition 6.1.1. *Let $B = \mathcal{O}(\mathbb{WP}(k,l))$. $\Omega^1(B)$ is a B-bimodule generated by*

$$\Gamma_1^- = \alpha\gamma\omega_-, \qquad \Gamma_2^- = \alpha^{l+1}\beta^{k-1}\omega_-, \qquad \Gamma_3^- = \delta^{l-1}\gamma^{k+1}\omega_-,$$
$$\Gamma_1^+ = \beta\delta\omega_+, \qquad \Gamma_2^+ = \alpha^{l-1}\beta^{k+1}\omega_+, \qquad \Gamma_3^+ = \gamma^{k-1}\delta^{l+1}\omega_+,$$
$$\Gamma_0^0 = \alpha^l\beta^k\omega_0, \qquad \Gamma_1^0 = \gamma^k\delta^l\omega_0,$$

with $*$-structure

$$\Gamma_1^{-*} = -q^2\Gamma_1^+, \qquad \Gamma_2^{-*} = (-1)^{k-1}q^{l+2}\Gamma_3^+, \qquad \Gamma_3^{-*} = q^{k(l-1)+1}\Gamma_2^+, \qquad \Gamma_0^{0*} = (-1)^{k+1}q^{2l-k}\Gamma_1^0.$$

The relations are given are as follows:
<u>On Γ^0:</u>

$$a\Gamma_0^0 = q^{-2l}\Gamma_0^0 a, \qquad a\Gamma_1^0 = q^{2l}\Gamma_1^0 a, \qquad b\Gamma_0^0 = q^{l-k}\Gamma_0^0 b, \qquad b\Gamma_1^0 = q^{l-2k}\Gamma_1^0 b.$$

<u>On Γ^+:</u>

$$a\Gamma_1^+ = q^2\Gamma_1^+ a, \qquad a\Gamma_2^+ = q^{-2(l-1)}\Gamma_2^+ a, \qquad a\Gamma_3^+ = q^{2(l+1)}\Gamma_3^+ a,$$

$$b\Gamma_1^+ = q^{k+1}\Gamma_1^+(1-a), \qquad b\Gamma_2^+ = q^{2l}\Gamma_2^+ b, \qquad b\Gamma_3^+ = (-q)^{-(k-1)}q^{-l}\Gamma_3^+(q^{2kl}a^{k-1}\prod_{p=1}^{l}(1-q^{2p-2}a)).$$

<u>On Γ^-:</u>

$$a\Gamma_1^- = q^{-4}\Gamma_1^- a, \qquad a\Gamma_2^- = q^{-(l+2)}\Gamma_2^- a, \qquad a\Gamma_3^- = q^{2l-1}\Gamma_3^- a,$$
$$b\Gamma_1^- = q^{2(l-k)}\Gamma_1^- b, \qquad b\Gamma_2^- = q^{-2k}\Gamma_2^- b, \qquad b\Gamma_3^- = (-1)^k q^{l-2k}\Gamma_3^- b.$$

Proof. $\Omega^1(B)$ must contain elements with zero grading since B contains all elements in $\mathcal{O}(S_q^3)$ which are invariant under the coaction $\rho_{k,l}$. To determine which zero-degree elements generate $\Omega^1(B)$ we use the exterior derviative map on the generators of B.

$$\begin{aligned}
d(a) = d(\beta\beta^*) &= (d\beta)\beta^* + \beta(d\beta^*) = -q(d\beta)\gamma - q\beta d\gamma \\
&= -q((-q^2\beta\omega_0 + \alpha\omega_-)\gamma + \beta(\gamma\omega_0 - q\delta\omega_+)) \\
&= q\beta\gamma\omega_0 - \alpha\gamma\omega_- - q\beta\gamma\omega_0 + q^2\beta\delta\omega_+ \\
&= -\alpha\gamma\omega_- + q^2\beta\delta\omega_+,
\end{aligned}$$

this gives generators $\alpha\gamma\omega_-$ and $\beta\delta\omega_+$. Next $d(b) = d(\alpha^l\beta^k) = (d\alpha^l)\beta^k + \alpha^l(d\beta^k)$, hence we need simplified versions of $(d\alpha^l)$ and $(d\beta^k)$.

$$\begin{aligned}
(d\alpha^l) &= (d\alpha)\alpha^{l-1} + \alpha(d\alpha^{l-1}) \\
&= (d\alpha)\alpha^{l-1} + \alpha(d\alpha)\alpha^{l-2} + \alpha^2(d\alpha^{l-2}) \\
&= (d\alpha)\alpha^{l-1} + \alpha(d\alpha)\alpha^{l-2} + \alpha^2(d\alpha)\alpha^{l-3} + \dots + \alpha^{l-1}d(\alpha),
\end{aligned}$$

using the relation $(d\alpha)\alpha^k = q^{-2k}(\alpha^{k+1}\omega_0 - q\alpha^k\beta\omega_+)$ for each non-negative integer k,

$$\begin{aligned}
(d\alpha^l) =& q^{-2(l-1)}(\alpha^l\omega_0 - q\alpha^{l-1}\beta\omega_+) + \alpha(q^{-2(l-2)}(\alpha^{l-1}\omega_0 - q\alpha^{l-2}\beta\omega_+)) + \\
&\dots + \alpha^{l-2}q^{-2}\alpha(\alpha\omega_0 - q\beta\omega_+) + \alpha^{l-1}(\alpha\omega_0 - q\beta\omega_+) = \sum_{n=1}^{l} q^{-2(l-n)}(\alpha^l\omega_0 - q\alpha^{l-1}\beta\omega_+) \quad (6.3) \\
=& c_l(\alpha^l\omega_0 - q\alpha^{l-1}\beta\omega_+)
\end{aligned}$$

where $c_l \in \mathbb{C}$. Similarly, using the relations

$$(d\beta^k) = (d\beta)\beta^{k-1} + \beta(d\beta)\beta^{k-2} + \dots + \beta^{k-1}(d\beta), \qquad (d\beta)\beta^n = q^n(\alpha\beta^n\omega_- - q^{n+2}\beta^{n+1}\omega_0)$$

where n is a non-negative integer, we can deduce

$$(d\beta^k) = d_k(\alpha\beta^{k-1}\omega_- - q^{k+1}\beta^k\omega_0), \qquad d_k \in \mathbb{C}. \tag{6.4}$$

Combining equations (6.3) and (6.4),

$$\begin{aligned}
d(b) = d(\alpha^l\beta^k) &= (d\alpha^l)\beta^k + \alpha^l(d\beta^k) \\
&= c_l(\alpha^l\omega_0 - q\alpha^{l-1}\beta\omega_+)\beta^k + d_k\alpha^l(\alpha\beta^{k-1}\omega_- - q^{k+1}\beta^k\omega_0) \qquad (6.5) \\
&= c_l(q^{2k}\alpha^l\beta^k\omega_0 - q^{k+1}\alpha^{l-1}\beta^{k+1}\omega_+) + d_k(q^{2k}\alpha^{l+1}\beta^{k-1}\omega_- - q^{k+1}\alpha^l\beta^k\omega_0),
\end{aligned}$$

this means $\alpha^{l-1}\beta^{k+1}\omega_+$, $\alpha^{l+1}\beta^{k-1}\omega_-$ and $\alpha^l\beta^k\omega_0$ are generators. By considering $d(b^*)$, or alternatively by taking the $*$-conjugate on these generators, we arrive at the final generators. The B-module structure is obtained using $a = \beta\beta^*$ and $b = \alpha^l\beta^k$ and the relations 2.2, 2.3 and 2.8. □

Chapter 7

Higher dimensional weighted projective spaces

The aim of this section is to introduce higher dimensional quantum weighted projective spaces and identify any areas of difficulty.

7.1 The three-dimensional quantum weighted projective space $\mathcal{O}(\mathbb{WP}_q(k,l,m))$

In order to deal with the three dimensional quantum weighted projective space we are required to work within $\mathcal{O}(S_q^5)$. Using Definition 2.2.6 and relabelling the generators to keep the notation consistent with the two dimensional case, we see $\mathcal{O}(S_q^5)$ is generated by,

$$z_0 = \alpha, \qquad z_1^* = \beta, \qquad z_2^* = \gamma. \tag{7.1}$$

These generators satisfy the following relations,

$$\alpha\beta = q\beta\alpha, \qquad \alpha\gamma = q\gamma\alpha, \qquad \beta\gamma = q\gamma\beta, \qquad \alpha\beta^* = q\beta^*\alpha, \qquad \alpha\gamma^* = q\gamma^*\alpha, \tag{7.2a}$$

$$\beta\gamma^* = q\gamma^*\beta, \qquad \gamma\alpha^* = q\alpha^*\gamma, \qquad \gamma\beta^* = q\beta^*\gamma, \qquad \beta\alpha^* = q\alpha^*\beta, \tag{7.2b}$$

$$\alpha\alpha^* = \alpha^*\alpha + (q^{-2}-1)(\beta\beta^* + \gamma\gamma^*)\ ,\ \beta\beta^* = \beta^*\beta + (q^{-2}-1)\gamma\gamma^*, \tag{7.2c}$$

$$\gamma\gamma^* = \gamma^*\gamma\ ,\ \alpha\alpha^* + \beta\beta^* + \gamma\gamma^* = 1. \tag{7.2d}$$

Now by setting $l_0 = k$, $l_1 = l$ and $l_2 = m$ to be positive integers, the coaction $\rho_{k,l,m}$ of $\mathbb{C}[u, u^*]$ on $\mathcal{O}(S_q^5)$ takes the form,

$$\alpha \mapsto \alpha \otimes u^k\ ,\ \beta \mapsto \beta \otimes u^{-l}\ ,\ \gamma \mapsto \gamma \otimes u^{-m}, \tag{7.3}$$

and $\mathcal{O}(\mathbb{WP}_q(k,l,m))$ is defined as the incovariant subalgebra of $\mathcal{O}(S_q^5)$.

Proposition 7.1.1. *The algebra $\mathcal{O}(\mathbb{WP}_q(k,l,m))$ is the $*$-subalgebra of $\mathcal{O}(S_q^5)$ generated by*

$$\beta\beta^*, \quad \gamma\gamma^*, \quad \alpha^l\beta^k, \quad \beta^{*m}\gamma^l, \quad \alpha^m\gamma^k, \quad \alpha^{\lambda_1}\beta^{\lambda_2}\gamma, \quad \alpha\beta^{\lambda_3}\gamma^{*\lambda_4}, \quad \alpha^{\lambda_5}\beta\gamma^{\lambda_6}, \tag{7.4}$$

for some $\lambda_i \in \mathbb{N}_0$.

Proof. $\mathcal{O}(S_q^5)$ has a basis given by

$$\alpha^{i_1}\beta^{i_2}\beta^{*i_3}\gamma^{i_4}\gamma^{*i_5}, \qquad \alpha^{*j_1}\beta^{j_2}\beta^{*j_3}\gamma^{i_4}\gamma^{*j_5}, \qquad \text{each } i_n, j_n \in \mathbb{N}_0.$$

Applying the coaction $\rho_{k,l,m}$ to these we get the coinvariance conditions,

$$k(i_1) + l(-i_2 + i_3) + m(-i_4 + i_5) = 0, \tag{7.5a}$$

$$k(-j_1) + l(-j_2 + j_3) + m(-j_4 + j_5) = 0. \tag{7.5b}$$

Case 1 Using $\gcd(k,l) = 1$. This implies that $m = \lambda k + \mu l$ for some $\lambda, \mu \in \mathbb{Z}$. Since $m, l, k > 0$, λ, μ cannot both be negative. Re-scaling λ, μ using $m = (\lambda + \nu l)k + (\mu - \nu k)l$, $\nu \in \mathbb{Z}$, we can find $\tilde{\lambda} > 0$ and $\tilde{\mu} < 0$ such that $m = \tilde{\lambda}k + \tilde{\mu}l$. Substituting this into 7.5a gives

$$k(i_1 - \tilde{\lambda}i_4 + \tilde{\lambda}i_5) = l(i_2 - i_3 + \tilde{\mu}i_4 - \tilde{\mu}i_5). \tag{7.6}$$

This implies k divides the right hand side hence it divides $i_2 - i_3 + \tilde{\mu}i_4 - \tilde{\mu}i_5$ since k and l and coprime, i.e.

$$k\theta = i_2 - i_3 + \tilde{\mu}i_4 - \tilde{\mu}i_5 \quad \Longrightarrow i_2 = k\theta + i_3 - \tilde{\mu}i_4 - \tilde{\mu}i_5, \quad \theta \in \mathbb{Z}. \tag{7.7}$$

Similarly, l divides the left hand side of Equation 7.6, hence

$$l\theta = i_1 - \tilde{\lambda}i_4 + \tilde{\lambda}i_5 \Longrightarrow i_1 = l\theta - \tilde{\lambda}i_4 - \tilde{\lambda}i_5. \tag{7.8}$$

Substituting Equations 7.7 and 7.8 into basis element

$$\alpha^{i_1}\beta^{i_2}\beta^{*i_3}\gamma^{i_4}\gamma^{*i_5} \sim (\alpha^l\beta^k)^\theta(\beta\beta^*)^{i_3}(\alpha^{\tilde{\lambda}}\beta^{-\tilde{\mu}}\gamma)^{i_4}(\alpha^{-\tilde{\lambda}}\beta^{\tilde{\mu}}\gamma^*)^{i_5}, \tag{7.9}$$

giving rise to generators

$$\alpha^l\beta^k, \qquad \beta\beta^*, \qquad \alpha^{|\tilde{\lambda}|}\beta^{|\tilde{\mu}|}\gamma,$$

and their $*$-conjugates.

Case 2 Using $\gcd(l,m) = 1$. Following the same process we get the following list of generators and their $*$-conjugates

$$\beta\beta^*, \qquad \gamma\gamma^*, \qquad \beta^{*m}\gamma^l, \qquad \alpha\beta^\sigma\gamma^{*\Sigma},$$

for $\sigma, \Sigma \in \mathbb{N}_0$. Similarly in the third case; by considering $\gcd(k,m) = 1$ we get the generators (and their $*$-conjugates)

$$\alpha^m\gamma^k, \qquad \alpha^\xi\beta\gamma^\omega, \qquad \gamma\gamma^*, \qquad \beta\beta^*,$$

for $\xi, \omega \in \mathbb{N}_0$. □

As we can see moving from two-dimensional to three-dimensional quantum weighted projective spaces we are required to deal with many more generators. This makes the process for calculating the relations between generators quite long and not easy. In particular it becomes difficult to classify the representations of this space. One alternative approach to understanding quantum weighted projective spaces in higher dimensions is to consider them from a graph algebra point of view.

Chapter 8

Graph algebras

8.1 Graph C^*-algebra approach

The overall aim associated to graph algebras is to represent a C^*-algebra in terms of a directed graph. A directed graph consists of collections of vertices and edges along with two maps, called the range and source, which determine the starting and ending vertices of each edge, hence describe the shape of the directed graph. Using the Gelfand-Naimark theorem we see that every C^*-algebra can be viewd as a collection of operators on some Hilbert space. In terms of directed graphs, to each edge, say e, we associate a partial isometry written S_e and to each vertex v a projection written P_v. Now the composition of operators is dependent on the form of the directed graph. C^*-algebras constructed in this way are called graph algebras and have numerous benefits including giving a description of ideal structures ([30] Section 1.3) and determining their K-theory ([30] Section 1.2).

8.1.1 The set-up and basic concepts

Definition 8.1.1. (Directed Graphs) A *directed graph* $E = (E^0, E^1, r, s)$ consists of two countable sets E^0, E^1 and two maps $r, s : E^1 \to E^0$. The set E^0 consists of vertices and E^1 consists of edges between these vertices. Each edge e has starting and finishing point given by the maps r and s, $s(e)$ being the source of e and $r(e)$ being the range of e.

Definition 8.1.2. Given a directed graph $E = (E^0, E^1, r, s)$ then $\mu = \mu_1 ... \mu_n$ is called a *path* in E provided $r(\mu_i) = s(\mu_{i+1})$ for $i = 1, ..., n-1$. We say μ is a path of length n and write $|\mu| = n$. We also write E^n of the paths of length n and $E^* = \cup_{n=0}^{\infty} E^n$.

It is convenient to write r_E and s_E when dealing with more than one directed graph to avoid confusion. We call vertices which do not receive an edge a *source* and vertices which do not emit an edge a *sink*. A *row-finite* graph is a directed graph in which all vertices receive finitely many edges.

Convention: the convention used when describing graph algebras is of paramount importance. There are two main approaches to describing graph algebras, each with there own advantages and disadvantages, each approach varying between authors; see [38], [30]. We use the convention that partial isometries S_e move in the opposite direction to the edge (as in [30]), hence S_e^* moves along the directed edge. Consequently, we compose operators in the graph algebra in the usual way from right to left.

Definition 8.1.3. (Cuntz-Krieger E-family) Let E be a row-finite graph and $\mathcal{H}$ a Hilbert Space. A *Cuntz-Krieger E-family* $\{S, P\}$ on $\mathcal{H}$ consists of a set $\{P_v : v \in E^0\}$ of mutually orthogonal projections and a set $\{S_e : e \in E^1\}$ of partial isometries on $\mathcal{H}$, such that

$$S_e^* S_e = P_{r(e)} \text{ for all } e \in E^1; \text{ and} \tag{8.1}$$

$$P_v = \sum_{e \in E^1 : s(e) = v} S_e S_e^* \text{ whenever } v \text{ is not a sink.} \tag{8.2}$$

Given a directed graph E then the corresponding algebra, i.e. the Cuntz-Krieger E-family, is known as the graph algebra associated to E. Graph algebra are C^*-algebra and are written as $C^*(E)$ or $C^*(\{S_e, P_v\})$. The next proposition tells us how to multiply operators in a graph algebra.

Proposition 8.1.4. *Suppose $E = (E^0, E^1, r, s)$ is a row finite graph and $C^*(\{S_e, P_v\})$ a graph algebra associated to E, then for $e, f \in E^1$*

$$\{S_e S_e^* : e \in E^1\} \text{ are mutually orthogonal projections;} \tag{8.3a}$$

$$S_e^* S_f \neq 0 \Longrightarrow e \neq f; \tag{8.3b}$$

$$S_e S_f \neq 0 \Longrightarrow r(e) = s(f), \text{ so } ef \text{ is a path; and} \tag{8.3c}$$

$$S_e S_f^* \neq 0 \Longrightarrow r(e) = r(f). \tag{8.3d}$$

Proof. Part (8.3*a*), operators of the form $S_e S_e^*$ are projections since

$$(S_e S_e^*)^2 = S_e(S_e^* S_e)S_e^* = S_e P_{r(e)} S_e^* = S_e S_e^* = (S_e S_e^*)^*,$$

using (8.1) and the $*$ property for operators. And mutually orthogonal since,

$$(S_e S_e^*)(S_f S_f^*) = (S_e S_e^* P_{s(e)})(P_{s(f)} S_f S_f^*) = 0, \qquad \text{if } s(e) \neq s(f),$$

since projections $P_{s(e)}$ and $P_{s(f)}$ are mutually orthogonal in $C^*(E)$. If $s(e) = s(f) = v \in E^0$, then using Equation 8.2, P_v is the sum of $S_e S_e^*$, $S_f S_f^*$ and other projections. It follows $S_e S_e^*$ and $S_f S_f^*$ are mutually orthogonal since a projection written as a sum of projections must contain mutually orthogonal terms. Part (8.3*b*), using the partial isometry relation $S = SS^*S$ and part (8.3*a*), for $e \neq f$ we see that $S_e^* S_f = S_e^*(S_e S_e^*)(S_f S_f^*)S_f = 0$. Part (8.3*c*) follows since $S_e S_f = (S_e P_{r(e)})(P_{s(f)} S_f) = 0$ whenever $r(e) \neq s(f)$. And similarly for part (8.3*d*), $S_e S_f^* = (S_e P_{r(e)})(P_{r(f)} S_f^*) = 0$ whenever $r(e) \neq r(f)$. □

To a path $\mu = \mu_1 ... \mu_n \in E^n \subset E^*$ we associate an operator in $C^*(E)$ as $S_\mu := S_{\mu_1} S_{\mu_2} ... S_{\mu_n}$ and $S_v := P_v$ for $v \in E^0$. Now S_μ is a partial isometry. Calculating the initial projection:

$$S_\mu^* S_\mu = (S_{\mu_n}^* ... S_{\mu_2}^*)(S_{\mu_1}^* S_{\mu_1})(S_{\mu_2} ... S_{\mu_n}) = (S_{\mu_n}^* ... S_{\mu_2}^*)(S_{\mu_2} ... S_{\mu_n}),$$

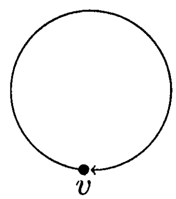
since $S_{\mu_1}^* S_{\mu_1} = P_{r(\mu_1)} = P_{s(\mu_2)}$ and $P_{s(\mu_2)} S_{\mu_2} = S_{\mu_2}$. Repeating this process we get $S_\mu^* S_\mu = P_{r(\mu)}$. Furthermore, $P_{s(\mu)} S_\mu S_\mu^* = S_\mu S_\mu^*$, hence the final projection is a subspace of $P_{s(\mu)}\mathcal{H}$. Given this description we can describe a graph algebra of a row-finite graph E as

$$C^*(E) = \overline{\text{span}}\{S_\mu S_\nu^* : \mu, \nu \in E^*, r(\mu) = r(\nu)\}. \tag{8.4}$$

8.1.2 Examples of graph algebras

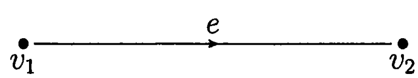
Next we calculate graph algebras of basic directed graphs to illistrate the concepts discussed above.

Example 8.1.5. Take the graph with one vertex v, and one edge e which is a loop at v.



The graph algebra is generated by partial isometry S_e and projection P_v. The Cuntz-Krieger relations state $S_e^*S_e = P_v$ and $S_eS_e^* = P_v$, and since there is only one vertex $P_v = I$ (the identity), hence $S_eS_e^* = S_e^*S_e = I$. The graph algebra is generated by a unitary operator S_e hence $C^*(E) \cong C(U(1))$.

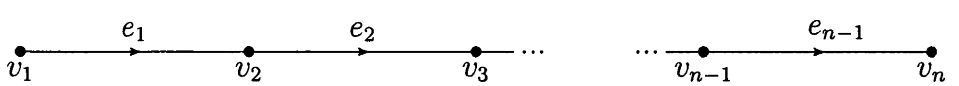
Example 8.1.6. Consider the directed graph with two vertices v_1 and v_2, adjointed by one edge e.



The graph algebra is generated by partial isometry S_e, and mutually orthogonal projections P_{v_1} and P_{v_2}. The Cuntz-Kriger relations state that $S_e^*S_e = P_{v_2}$ and $S_eS_e^* = P_{v_2}$. Since P_{v_1} and P_{v_2} are mutually orthogonal projections we also require $P_{v_1} + P_{v_2} = I$.

We can try putting $P_{v_1} = \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$ and $P_{v_2} = \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)$ in $M_2(\mathbb{C})$ then the mutually orthogonality condition is obviously satisfied. Now putting $S_e = \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$, hence $S_e^* = \left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)$, the Cuntz-Krieger relations are easily verified; $S_e^*S_e = \left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right) = P_{v_2}$ and $S_eS_e^* = \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right) = P_{v_1}$. Hence the graph algebra is generated by the two-by-two unit matrices and hence corresponds to the algebra $M_2(\mathbb{C})$.

Example 8.1.7. Consider the directed graph E with vertices $E^0 = \{v_1, ..., v_n\}$, edges $E^1 = \{e_1, ..., e_{n-1}\}$ and range and source functions given by $s(e_i) = v_i$ and $r(e_i) = v_{i+1}$ for $i = 1, ..., n-1$,



Extending the principle used in Example 8.1.6 we find the graph algebra is given by $C^*(E) \cong M_n(\mathbb{C})$.

Example 8.1.8. Consider the directed graph E given by,

$$\begin{tikzcd} & v_2 \arrow[dr, "f"] & \\ v_1 \arrow[ur, "e"] \arrow[rr, "g"'] & & v_3 \end{tikzcd}$$

Suppose $\mu, \nu \in E^*$, then using the algebraic properties of partial isometries $S_\mu S_\nu^* = S_\mu P_{r(\mu)} S_\nu^*$. Now unless $r(\mu) = v_3$ we can apply the Cuntz-Krieger at $r(\mu)$, and repeat until the path ends at a sink, in this case at v_3. Hence,

$$\begin{aligned} P_{v_1} &= S_e S_e^* + S_g S_g^* = S_e P_{v_2} S_e^* + S_g S_g^* \\ &= S_e (S_f S_f^*) S_e^* + S_g S_g^* \\ &= S_{ef} S_{ef}^* + S_g S_g^*, \end{aligned} \tag{8.5}$$

the other second Cuntz-Krieger relation is $P_{v_2} = S_f S_f^*$, note that the edge f ends at v_3, and the first Cuntz-Krieger relations are $P_{v_2} = S_e^* S_e$, $P_{v_3} = S_f^* S_f$ and $P_{v_3} = S_g^* S_g$. Now using Equation (8.4) the graph algebra can be expressed as

$$\begin{aligned} C^*(E) &= \overline{\operatorname{span}}\{S_\mu S_\nu^* : \mu, \nu \in E^*, r(\mu) = r(\nu) = v_3\} \\ &= \operatorname{span}\{S_\mu S_\nu^* : \mu, \nu \in \{v_3, f, g, ef\}\}. \end{aligned} \tag{8.6}$$

Since v_3 is a sink, two paths $\mu, \nu \in E^*$ such that $r(\mu) = r(\nu) = v_3$ cannot satisfy $\nu = \mu\nu'$ unless $\mu = \nu$. Hence,

$$(S_\mu S_\nu^*)(S_\alpha S_\beta^*) = \begin{cases} S_\mu S_\beta^* & \alpha = \nu \\ 0 & \text{otherwise.} \end{cases} \tag{8.7}$$

Thus $\{S_\mu S_\nu^* : \mu, \nu \in \{v_3, f, g, ef\}\}$ is a set of matrix units which spans $C^*(E)$ and hence is isomorphic to $M_4(\mathbb{C})$.

Example 8.1.9. Suppose E is a finite directed graph with no cycles, and $w_1, ..., w_k$ are sinks in E. Then for every Cuntz-Krieger E-family $\{S, P\}$ in which P_v is non-zero we have

$$C^*(S, P) \cong \bigoplus_{i=1}^{k} M_{|r^{-1}(w_i)|}(\mathbb{C}), \tag{8.8}$$

where $r^{-1}(w_i) = \{\mu \in E^* : r(\mu) = w_i\}$.

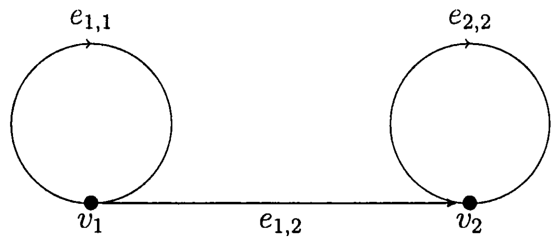
8.2 Quantum spheres as graph algebras

Recall from Definition 2.2.6 the definition of odd dimensional quantum spheres on the algebraic level. As discussed in [31], on the C^*-algebra level the space $C(S_q^3)$ of continuous functions on the quantum 3-sphere is generated by α, β such that

$$\alpha^*\alpha + \beta^*\beta = I, \qquad \alpha\alpha^* + q^2\beta^*\beta = I, \tag{8.9a}$$

$$\alpha\beta = q\beta\alpha, \qquad \alpha\beta^* = q\beta^*\alpha, \qquad \beta^*\beta = \beta\beta^*. \tag{8.9b}$$

Furthermore, $C(S_q^3)$ are isomorphic as C^*-algebras for $q \in [0, 1)$. This C^*-algebra can be described as a graph algebra where the underlying graph is typically denoted L_3 given by



The graph algebra $C^*(L_3)$ is the universal C^*-algebra generated by projections P_{v_1}, P_{v_2} and partial isometries $S_{e_{1,1}}$, $S_{e_{1,2}}$ and $S_{e_{2,2}}$, subject to the following relations

$$P_{v_1} = S^*_{e_{1,1}}S_{e_{1,1}} = S_{e_{1,1}}S^*_{e_{1,1}} + S_{e_{1,2}}S^*_{e_{1,2}} \tag{8.10a}$$

$$P_{v_2} = S^*_{e_{1,2}}S_{e_{1,2}} = S^*_{e_{2,2}}S_{e_{2,2}} = S_{e_{2,2}}S^*_{e_{2,2}}. \tag{8.10b}$$

Proposition 8.2.1. *The C^*-algebra isomorphism $\phi : C(S_q^3) \to C^*(L_3)$, for $q \in (0, 1)$, is given by*

$$\alpha \mapsto \sum_{n=0}^{\infty}(\sqrt{1-q^{2(n+1)}} - \sqrt{1-q^{2n}})(S_{e_{1,1}} + S_{e_{1,2}})^{n+1}(S^*_{e_{1,1}} + S^*_{e_{1,2}})^n, \tag{8.11a}$$

$$\beta \mapsto \sum_{n=0}^{\infty} q^n(S_{e_{1,1}} + S_{e_{1,2}})^n S_{e_{2,2}}(S^*_{e_{1,1}} + S^*_{e_{1,2}})^n. \tag{8.11b}$$

and for $q = 0$,

$$\alpha \mapsto S_{e_{1,1}} + S_{e_{1,2}}, \qquad \beta \mapsto S_{e_{2,2}}. \tag{8.12}$$

In fact for any $n = 1, 2, \ldots$ the C^*-algebra $C(S_q^{2n-1})$ is isomorphic with $C^*(L_{2n-1})$. The graph L_{2n-1} has n vertices labelled $v_1, \ldots, v_n$ and $n(n+1)/2$ edges labelled $\bigcup_{i=1}^n\{e_{i,j} : j = i, \ldots, n\}$ with $s(e_{i,j}) = v_i$ and $r(e_{i,j}) = v_j$. Also, even dimensional quantum spheres are graph algebras too; see [31].

8.3 Quantum lens spaces as graph algebras

Recall from Definition 3.3.2 we defined quantum lens spaces on the algebraic level $\mathcal{O}(L_q(l; 1, l))$ as the fixed point space of the quantum 3-sphere $\mathcal{O}(S_q^3)$ under the $\mathcal{O}(\mathbb{Z}_l)$-coaction ϱ. On the C^*-algebra level, quantum lens spaces are defined as follows.

Definition 8.3.1. Fix an integer $p \geq 2$, let $m_1, ..., m_n$ be n integers relatively prime to p and set $\theta = e^{2\pi i/p}$. The map

$$\tilde{\Lambda} : C(S_q^{2n-1}) \to C(S_q^{2n-1}), \qquad z_i \mapsto \theta^{m_i} z_i, \tag{8.13}$$

for $i = 1, ..., n$ is an automorphism of order p. For $q \in (0, 1)$ we define the C^*- algebra $C(L_q(p; m_1, ..., m_n))$ of continuous functions on the quantum lens space as the fixed point algebra corresponding to $\tilde{\Lambda}$,

$$C(L_q(p; m_1, ..., m_n)) = C(S_q^{2n-1})^{\tilde{\Lambda}}. \tag{8.14}$$

Since we have an interpretation of $C(S_q^{2n-1})$ as a graph algebra, it is possible to translate our understanding of $C(L_q(p; m_1, ..., m_n))$ in terms on graph algebra theory. Let $f : C(S_q^{2n-1}) \to C^*(L_{2n-1})$ be the C^*-algebra isomorphism and consider the map

$$\Lambda = f\tilde{\Lambda}f^{-1} : C^*(L_{2n-1}) \to C^*(L_{2n-1}), \qquad \Lambda(P_{v_i}) = P_{v_i}, \quad \Lambda(S_{e_{ij}}) = \theta^{m_i} S_{e_{ij}},$$

for $i, j \in \{1, ..., n\}$, this implies that

$$C(L_q(p; m_1, ..., m_n)) \cong C^*(L_{2n-1})^{\Lambda}. \tag{8.15}$$

In [30], Hong and Szymański showed that $C(L_q(p; m_1, ..., m_n))$ is a graph algebra and when on to determine the underlying directed graph.

8.4 Quantum weighted projective spaces as possible graph algebras

The ideas here do not seem too different to those in the previous section. On the C^*-algebra level we can view $C(\mathbb{WP}_q(l_0, ..., l_n))$ as the fixed point subspace of $C(S_q^{2n+1})$ under the action

$$\tilde{\xi} : U(1) \times C(S_q^{2n+1}) \to C(S_q^{2n+1}), \qquad z_i \mapsto u^{l_i} z_i,$$

for $u \in U(1)$ and $i = 0, 1, ..., n$, which means

$$C(\mathbb{WP}_q(l_0, ..., l_n)) = C(S_q^{2n+1})^{\tilde{\xi}}. \tag{8.16}$$

Now by considering the map

$$\xi = \phi\tilde{\xi}\phi^{-1} : C^*(L_{2n+1}) \to C^*(L_{2n+1}), \qquad \Lambda(P_{v_i}) = P_{v_i}, \quad \Lambda(S_{e_{ij}}) = u^{l_i} S_{e_{ij}},$$

for $i, j \in \{0, 1, ..., n\}$, we find that

$$C(\mathbb{WP}_q(l_0, ..., l_n)) \cong C^*(L_{2n+1})^{\xi}, \tag{8.17}$$

giving us a description of higher dimensional quantum weighted projective spaces.

Bibliography

[1] P. Baum, P.M. Hajac, R. Matthes & W. Szymański, *The K-theory of Heegaard-type quantum 3-spheres*, K-Theory 35 (2005), 159-186.

[2] P. Baum, P.M. Hajac, R. Matthes & W. Szymański, *Noncommutative geometry approach to principal and associated bundles*, arXiv:math/0701033.

[3] V.V. Bavula *Tensor homological minimal algebras, global dimension of the tensor product of algebras and of generalized Weyl algebras,* Bull. Sci. Math. 120 (1996), 293-335.

[4] E.J. Beggs & T. Brzeziński, *An explicit formula for a strong connection.* Appl. Categor. Str. 2008, 378 16, 57-63.

[5] T. Brzeziński & S.A. Fairfax, *Quantum teardrops*, Commun. Math. Phys. 316 (2012), 151-170.

[6] T. Brzeziński & S.A. Fairfax, *Quantum principal bundles over quantum real projective spaces*, Axioms 1 (2012), 201-225.

[7] T. Brzeziński & S.A. Fairfax, *Weighted circle actions on the Heegaard quantum sphere*, arXiv:1305.5942 (2013).

[8] T. Brzeziński, & S. Majid, *Quantum group gauge theory on quantum spaces.* Commun. Math. Phys. 157, 591-638 (1993) Erratum 167, 235 (1995).

[9] T. Brzeziński & R. Wisbauer, *Corings and Comodules* London Mathematical Society Lecture Notes Series 309 (2003).

[10] T. Brzeziński & B. Zieliński, *Quantum principal bundles over quantum real projective spaces.* J. Geom. Phys. 62 (2012), no. 5, 1097-1107.

[11] T. Brzeziński & P.M. Hajac, *The Chern-Galois character.* C. R. Math. Acad. Sci. Paris 338 (2004), no. 2, 113-116.

[12] T. Brzeziński, *Circle actions on a quantum Seifert manifold.* Proceedings of Science (CORFU2011) (2012), 055.

[13] T. Brzeziński & S. Majid, *Quantum group gauge theory on quantum spaces.* Comm. Math. Phys.157 (1993), 591-638. Erratum 167 (1995), 235.

[14] D. Calow & R. Matthes, *Connections on locally trivial quantum principal fibre bundles,* J. Geom. Phys. 41 (2002), 114-165.

[15] D. Calow & R. Matthes, *Covering and gluing of algebras and differential algebras, J. Geom. Phys.* 32 (2000), 364-396.

[16] A. Connes, *Noncommutative Geometry*, Academic Press, New York 1994.

[17] K. R. Davidson, *C^*-Algebras by Example.* American Mathematical Society (1996).

[18] F. D'Andrea & G. Landi, *Bounded and unbounded Fredholm modules for quantum projective spaces*, J. K-Theory 6 (2010), 231-240.

[19] L. Dąbrowski, H. Grosse & P.M. Hajac, *Strong connections and Chern-Connes pairing in the Hopf-Galois theory*, Comm. Math. Phys. 220 (2001), 301-331.

[20] J.M. Gracia-Bondia, J.C. Vårilly & H. Figueroa, *Elements of Noncommutative Geometry.* 393 Birkhauser, Boston, 2001.

[21] P.M. Hajac, R. Matthes & W. Szymański, *Graph C^*-algebras and $\mathbb{Z}_2$-quotients of quantum spheres.* Rep. Math. Phys. 51, 215224 (2003).

[22] P.M. Hajac & S. Majid, *Projective module description of the q-monopole.* Commun. Math. Phys. 206, 247-264 (1999).

[23] P.M. Hajac, A. Rennie & B. Zieliński, *The K-theory of Heegaard quantum lens spaces*, Preprint arXiv:1110.5897 (2011).

[24] P.M. Hajac, *Strong connections on quantum principal bundles.* Commun. Math. Phys. 182 (1996,), 579-617.

[25] P.M. Hajac, R. Matthes & W. Szymański, *Noncommutative index theory of mirror quantum spheres,* Comptes Rendus Math. (Acad. Sci. Paris Ser. I) 343 (2006), 731-736.

[26] P.M. Hajac, R. Matthes & W. Szymański *A locally trivial quantum Hopf fibration,* Alg. Rep. Theory 9 (2006), 121-146.

[27] P.M. Hajac, R. Matthes & W. Szymański, *Quantum real projective space, disc and spheres.* Algebr. Represent. Theory **6** (2003), 169–192.

[28] P.M. Hajac, U. Krähmer, R. Matthes & B. Zieliński *Piecewise principal comodule algebras,* J. Noncommut. Geom. 5 (2011), 591-614.

[29] E. Hawkins & G. Landi, *Fredholm modules for quantum Euclidean spheres*, J. Geom. Phys. 49 (2004), 272-293.

[30] J.H. Hong, W. Szymański *Quantum lens spaces and graph algebras.* Pacific J. Math. 211 (2003), no. 2, 249-263.

[31] J. H. Hong & W. Szymański *Quantum spheres and projective spaces as graph algebras.* Comm. Math. Phys. 232 (2002), no. 1, 157-188.

[32] U. Kräehmer *On the non-standard Podles spheres*, C^*-algebras and elliptic theory II (2008), 145-147.

[33] S. Majid, *A Quantum Group Primer.* London Mathematical Society Lecture Notes Series 292 (2002).

[34] T. Masuda, Y. Nakagami & J. Watanabe, *Noncommutative differential geometry on the quantum two sphere of Podleś.* I. An algebraic viewpoint, K - Theory 5 (1991), 151-175.

[35] U. Meyer, *Projective quantum spaces.* Lett. Math. Phys. 35, 91-97 (1995).

[36] E.F. Müller, H.-J. Schneider, *Quantum homogeneous spaces with faithfully flat module structures*, Israel Journal of Mathematics 111 (1999) 157-190.

[37] P. Podleś, *Quantum spheres*, Lett. Math. Phys., 14 (1987),193-202.

[38] I. Raeburn, *Graph algebras.* CBMS Regional Conference Series in Mathematics, 103. American Mathematical Society. ISBN: 0-8218-3660-9.

[39] M. Rordam, F. Larsen & N.J. Lausten *An introduction to K-theory for C^*-algebras* London Mathematical Society Student Text 49 (2000).

[40] H.-J. Schneider, *Principal homogeneous spaces for arbitrary Hopf algebras.* Israel J. Math. 72 (1990),167-195.

[41] P. Scott, *The geometries of 3-manifolds.* Bull. London Math. Soc. 1983, 15, 401-487.

[42] A.J.-L. Sheu *Quantization of the Poisson SU(2) and its Poisson homogeneous space – the 2-sphere*, Comm. Math. Phys. **135** (1991), 217–232.

[43] W.P. Thurston, *Three-dimensional geometry and topology.* Edited by S. Levy Princeton Mathematical Series, 35, Princeton, NJ: Princeton University Press 1997.

[44] L.L. Vaksman & Ya.S. Soibel'man, *An algebra of functions on the quantum group SU(2).* (Russian) Funkt. Anal. i Pril. 22, 1-14 (1988); translation in Funct. Anal. Appl. 22, 170-181 (1988).

[45] L.L. Vaksman & Ya.S. Soibel'man, *Algebra of functions on the quantum group $SU(n+1)$, and odd-dimensional quantum spheres (Russian).* Alg. i Anal. 2, 101-120 (1990); translation in Leningrad Math. J. 2, 1023-1042 (1991).

[46] M. Van den Bergh, *Non-commutative crepant resolutions (with some corrections).* http://arxiv.org/abs/math/0211064v2 (2002).Also in: The Legacy ofNiels Henrik Abel, O.A. Laudal, R. Piene (eds.), Springer-Verlag, Berlin, 2004, pp. 749-770.