Pseudodifferential operators on compact abelian groups with applications.

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Pseudodifferential Operators on Compact Abelian Groups with Applications

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Abstract

Pseudodifferential operators on compact groups are discussed, with an emphasis on the conditions for which the theorem of Hille and Yosida holds. Some preliminary functional analysis is given including the notion of regularly dissipative operators and Pontrjagin duality. The dual group is described, especially that it is discrete. Some important inequalities, such as Young's inequality, are also stated. Generalised trigonometrical polynomials and generalised Sobolev spaces are defined on the compact group G. A finite exhaustion of the dual space is used to define pointwise convergence and to give a condition for which a generalised Sobolev space is continuously embedded in C(G) and compactly embedded into a larger Sobolev space.
The thesis defines k-ellipticity, k-smoothing operators and the k-parametrix, and proves their relation to the compactness of the embedding. It is shown that k-ellipticity is characterised by an inequality of Gårding type. Some examples of pseudodifferential operators with constant coefficients are given. Another inequality of Gårding type is proved for pseudodifferential operators with variable coefficients, and the existence of a weak solution to $(A(x, D) - \lambda)u = f$ is given under certain conditions on the adjoint $A^*(x, D)$. A variational solution of $B[\varphi, u] = (\varphi, f)$ is found, and we prove a Gårding type inequality for the sesquilinear form.

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Chapter 1

Introduction

A *differential operator*, $P(D) := \sum_\alpha a_\alpha D^\alpha$, which acts on smooth functions with compact support in $\mathbb{R}^n$ can be written in the form

$$\begin{aligned} P(D)u(x) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\xi}P(\xi)u(y)\mathrm{d}y\mathrm{d}\xi \\ &= F^{-1}\big(P(\xi)F(u(x))\big), \end{aligned}$$

where F is the Fourier transform and $P(\xi)$ is a polynomial. This is derived by applying $P(D)$ to the representation

$$u(x) = F^{-1}(F\,u(x)) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\xi}u(y)\mathrm{d}y\mathrm{d}\xi$$

and using $P(D)e^{i(x-y)\xi} = e^{i(x-y)\xi}P(\xi)$ on $\mathbb{R}^n$. We call $P(\xi)$ the *symbol* of $P(D)$.

Similarly a *pseudodifferential operator*, $P(x,D)$ on $\mathbb{R}^n$, is an operator of the form

$$P(x,D)u(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\xi}P(x,\xi)u(y)\mathrm{d}y\mathrm{d}\xi. \qquad (1.0.1)$$

In this thesis we explore pseudodifferential operators, both $P(D)$ with constant coefficients and $P(x,D)$ with variable coefficients, on a compact

Abelian group which has no inherently differentiable structure, by using a form similar to (1.0.1). Moreover, the theorem of Hille and Yosida states that an operator $A = A(x, D)$ is the generator of a *strongly continuous contraction semigroup* on $\mathrm{L}^2(\mathrm{G})$ if and only if the following three conditions hold:

1. $D(A) \subset G$ is dense.
2. A is a dissipative operator.
3. $R(\lambda - A) = X$ for some $\lambda > 0$.

The third condition is equivalent to the solving of

$$(\lambda - A)u = f$$

for all $f \in \mathrm{L}^2(\mathrm{G})$ and one $\lambda > 0$, and we concentrate on attaining this result.

Initially some functional analysis is introduced which is necessary for us to understand the behaviour of functions on compact Abelian groups. The notion of regularly dissipative operators plays an important role in some of the later theorems and is introduced in a general context.

Chapter 3 begins with a formal description of locally compact and compact Abelian groups, the latter of which we are most interested in. The reader is familiarised with the Haar measure, a translation invariant measure which exists on all locally compact Abelian groups. The Haar measure on a group G is denoted μ_G.

We give the notion of Pontrjagin duality between a group and its dual group, and see that the dual group of a compact group is discrete. This discrete nature allows us greater control, and justifies our interest in compact groups.

The convolution of two functions,

$$(f * g)(x) = \int_G f(x - y)g(y)\mathrm{d}\mu_G(y)\,,$$

is given with its most common applications, namely the convolution theorem

$$(f * g)^\wedge(\gamma) = \hat{f}(\gamma)\hat{g}(\gamma)$$

and Young's inequality

$$\|f * g\|_{L^p(G)} \leq \|f\|_{L^p(G)} \|g\|_{L^1(G)} .$$

We often assume that our symbols are *negative definite functions*, as these give rise to the most important applications, namely that a negative definite symbol $P(\gamma)$ generates a convolution semigroup of measures $(\hat{\mu}_t)_{t\geq 0}$ given by

$$\hat{\mu}_t(\gamma) = e^{-tP(\gamma)}.$$

A general introduction to negative definite functions is given in section 3.2 and more can be found in [10].

Chapter 4 begins with generalised trigonometrical polynomials on our compact group G, which have a finite Fourier series and are dense in $C(G)$ and $L^p(G)$. These generalised trigonometrical polynomials are denoted $S(G)$. We then aim to categorise the 'smoothness' of functions on $\mathrm{L}^2(\mathrm{G})$ by the rate of decay of their Fourier transforms, with a generalisation of Sobolev spaces, denoted $\mathrm{H}^r_k(\mathrm{G})$. This allows us to prove many results on the simpler space $S(G)$ and then extend to $\mathrm{H}^r_k(\mathrm{G})$.

Also a finite exhaustion is defined on the discrete dual space, as a means of defining pointwise convergence and divergence of generalised Fourier series. We use these convergence properties to give a condition for which the Sobolev spaces are continuously embedded in $C(G)$ and more importantly for which the embedding of one Sobolev space into another is *compact.* To obtain 'almost everywhere' convergence results in our case it is more involved, compare [13] and [30] for positive results in the one dimensional classical case, [18] for negative results and [1] as a survey on results in the classical multi-dimensional case.

The thesis defines k-ellipticity, k-smoothing operators and the k-parametrix, and we prove their relation to the compactness of the embedding, and see that k-ellipticity helps us to solve $A_r(D)u = f$.

In chapter 5 we show that translation invariant k-elliptic operators are characterised by an inequality of Gårding type:

$$\|A(D)\varphi\|_{r,k}^2 \geq c_0\|\varphi\|_{r+t,k}^2 - c_1\|\varphi\|_{r,k}^2 .$$

It is shown that under certain conditions, k-ellipticity, characterised by the above inequality, gives the existence of a solutiom to

$$A_r(D)u = f .$$

The d-potential and d-energy of a measure are briefly mentioned and symmetric Dirichlet forms generated by a translation invariant pseudodifferential operator are explored.
We give some examples.
Then a Gårding type inequality for pseudodifferential operators *with variable coefficients* is proved. i.e.

$$\|A(x,D)\varphi\|_{\mathrm{L}^2(\mathrm{G})}^2 \geq c_0\|\varphi\|_{r,k}^2 - c_1\|\varphi\|_{\mathrm{L}^2(\mathrm{G})}^2 .$$

In section 5.6 we prove the existence of a weak solution to $[A(x,D)-\lambda]u = f$ under some condition on the adjoint $A^*(x,D)$.
Finally, the main result of the thesis is given in section 5.7, where we give a variational solution based on conditions on the sesquilinear form

$$B[u,v] = (A(x,D)u,v)_{\mathrm{L}^2(\mathrm{G})},$$

which is then simplified to conditions on the pseudodifferential operator $A(x,D)$. In particular, we prove a Gårding type imequality for the sesquilinear form B. i.e.

$$B[u,u] \geq \lambda_0(1-\eta)\|u\|_{\mathrm{H}_k^m(\mathrm{G})}^2 - \rho\|u\|_{\mathrm{L}^2(\mathrm{G})}^2.$$

Chapter 2

Some Functional Analysis

2.1 Preliminary Functional Analysis

In this section we will introduce many notions from functional analysis which will be needed later. Theorems are frequently offered without proof, although references will be given so that the reader may understand the concepts in more detail. The thesis aims to be mainly self-contained. However, the reader is assumed to have some basic knowledge of set theory and functional analysis.

Definition 2.1.1. Let X be a complex Algebra. X is called an *Algebra with Involution* if there exists a mapping

$$\begin{aligned} X &\to X \\ x &\mapsto x^* \end{aligned}$$

such that for $\alpha, \beta \in \mathbb{C}$

$$\begin{array}{lrcl} (1) & (\alpha x + \beta y)^* & = & \overline{\alpha} x^* + \overline{\beta} y^* \\ (2) & x^{**} & = & x \\ (3) & (xy)^* & = & y^* x^* \end{array}$$

Definition 2.1.2. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a normed linear space X is said to be *weakly convergent* if a finite $\lim_{n\to\infty} f(x_n)$ exists for each continuous linear functional f on X. Then $\{x_n\}$ is said to *converge weakly* to an element $x_\infty \in X$ if $\lim_{n\to\infty} f(x_n) = f(x_\infty)$ for all continuous linear functionals f on X. Note also that x_∞ is uniquely determined, via the Hahn-Banach Theorem. More details can be found in [45], 120.

Theorem 2.1.1. *(Riesz' representation theorem)*
Let X be a Hilbert space with scalar product $(\cdot, \cdot)$ and let f be a bounded linear functional on X. Then there exists a uniquely determined vector y_f in X such that

$$f(x) = (x, y_f) \text{ for all } x \in X, \text{ and } \|f\| = \|y_f\|.$$

Conversely, any vector $y \in X$ defines a bounded linear functional f_y on X by

$$f_y(x) = (x, y) \text{ for all } x \in X, \text{ and } \|f_y\| = \|y\|.$$

2.2 Dissipative Operators in Hilbert Spaces

Dissipative operators, in particular regularly dissipative operators, are of great importance for the main results of Chapter 4. For this reason, we describe dissipative operators and their properties here, following closely the work of Tanabe, [42].

Definition 2.2.1. Let A be a linear operator in a Hilbert space X with dense domain. The operator A is called a *dissipative operator* if Re $(Au, u) \leq 0$ for all $u \in \mathrm{D}(A)$. If $-A$ is dissipative, i.e. Re $(Au, u) \geq 0$ for all $u \in \mathrm{D}(A)$, we call A an *accretive operator.*
A dissipative operator which extends a dissipative operator A, is said to be a *dissipative extension of A*. An operator A is *maximal dissipative* if the only

dissipative extension of A is the operator A itself. *Accretive extensions* and *maximal accretive operators* are defined in the same way.

Theorem 2.2.1. Let A be a linear operator with dense domain. The following statements are equivalent:

(1) A is a dissipative operator.
(2) $\|(A-\lambda)u\| \geq \mathrm{Re}\lambda\|u\|$ for all $u \in \mathrm{D}(A)$ and all λ satisfying $\mathrm{Re}\lambda > 0$.
(3) $\|(A-\lambda)u\| \geq \lambda\|u\|$ for all $u \in \mathrm{D}(A)$ and all λ satisfying $\lambda > 0$.

Theorem 2.2.2. When A is a dissipative operator, the following statements are equivalent:

(1) A is a maximal dissipative operator.
(2) $R(A-\lambda) = X$ for all λ satisfying $\mathrm{Re}\lambda > 0$.
(3) $R(A-\lambda) = X$ for some λ satisfying $\mathrm{Re}\lambda > 0$.

Theorem 2.2.3. Let A be a closed dissipative operator. A is maximal dissipative if and only if A* is a dissipative operator. In this case, A* is also maximal dissipative.

2.3 Regularly Dissipative Operators

Let X be a complex Hilbert space with inner product $(\cdot,\cdot)$ and norm $|\cdot|$. Let V be another Hilbert space with inner product $((\cdot,\cdot))$ and norm $\|\cdot\|$. We assume that V is embedded in X as a dense subspace and that the topology on V is stronger than on X. Therefore, there exists an M_0 such that $|u| \leq M_0\|u\|$ for all $u \in V$.

Definition 2.3.1. A *sesquilinear form* is a function $B(\cdot,\cdot) : V \times V \to \mathbb{C}$

which is linear in u and antilinear in v,

$$B(u_1 + u_2, v) = B(u_1, v) + B(u_2, v), \tag{2.3.1}$$

$$B(u, v_1 + v_2) = B(u, v_1) + B(u, v_2), \tag{2.3.2}$$

$$B(\lambda u, v) = \lambda B(u, v), \tag{2.3.3}$$

$$B(u, \lambda v) = \overline{\lambda} B(u, v). \tag{2.3.4}$$

Let $B(\cdot, \cdot)$ be a sesquilinear form defined on $V \times V$, which we assume is *bounded*,

$$|B(u, v)| \leq M\|u\| \, \|v\| \tag{2.3.5}$$

and satisfies *Gårding's inequality*, i.e. for some $\delta > 0$ and some $k \in \mathbb{R}$

$$\text{Re } B(u, u) \geq \delta \, \|u\|^2 - k \, |u|^2 \quad \text{for all } u \in V. \tag{2.3.6}$$

If $B(\cdot, \cdot)$ satisfies these conditions, an operator A is defined as follows:

$$\begin{cases} \text{Given } u \in V. \text{ If there exists an element } f \in X \text{ satisfying} \\ B(u, v) = (f, v) \text{ for all } v \in V, \text{ then } u \in \mathrm{D}(A) \text{ and } Au = f. \end{cases} \tag{2.3.7}$$

Denoting the set of linear functionals on V by V^*, we also note that an operator $\tilde{A}$ is defined for an element $f \in V^*$ by

$$B(u, v) = (\tilde{A}u, v) = (f, v) \tag{2.3.8}$$

and that $\tilde{A}$ is an extension of A, as shown in Tanabe [42]. In fact

$$D(A) := \{u \in V : \tilde{A}u \in X\}.$$

Frequently both A and $\tilde{A}$ are denoted simply by A, as the lack of distinction should not lead to any confusion.

Theorem 2.3.1. The sesquilinear form $B^*(\cdot, \cdot)$ defined by $B^*(u, v) = \overline{B(v, u)}$ is called an *adjoint sesquilinear form.* If $B(\cdot, \cdot)$ satisfies (2.3.5) or (2.3.6), then correspondingly $B^*(\cdot, \cdot)$ does also. Therefore $B^*(u, v) = (A^*u, v) = (u, Av)$ where A^* is the *adjoint* of A viewed as an operator in X.

Definition 2.3.2. An operator defined by (2.3.7) which satisfies the Gårding inequality for $k = 0$, is called a *regularly accretive operator.* If $-A$ is regularly accretive, A is called a *regularly dissipative operator.*
A regularly accretive operator is maximal accretive.

Definition 2.3.3. When $B^*(u,v) = B(u,v)$ holds for all $u, v \in V$ the sesquilinear form is said to be *symmetric.*

Theorem 2.3.2. If $E(\cdot,\cdot)$ is a bounded, symmetric sesquilinear form on V satisfying

$$\text{Re } E(u,u) \geq \delta \|u\|^2,$$

then A is positive definite and self-adjoint, $\mathrm{D}(A^{\frac{1}{2}}) = V$, and

$$E(u,v) = (A^{\frac{1}{2}}u, A^{\frac{1}{2}}v) \quad u, v \in V. \tag{2.3.9}$$

2.4 Semigroups and their Generators

Following chapter 4 of [34], let $(X, \|\cdot\|_X)$ be a real or complex Banach space.

Definition 2.4.1. A. A one parameter family $(T_t)_{t\geq 0}$ of bounded linear operators $T_t : X \to X$ is called a (one parameter) *semigroup of operators*, if $T_0 = \text{id}$ and $T_{s+t} = T_s \circ T_t$ hold for all $s, t \geq 0$.
B. We call $(T_t)_{t\geq 0}$ *strongly continuous* if

$$\lim_{t\to 0} \|T_t u - u\|_X = 0$$

for all $u \in X$.
C. The semigroup $(T_t)_{t\geq 0}$ is called a *contraction semigroup* if, for all $t \geq 0$

$$\|T_t\|_{X,X} \leq 1$$

holds, i.e. if each of the operators T_t is a contraction. Here $\|T_t\|_{X,X}$ denotes the operator norm.

Definition 2.4.2. Let $(T_t)_{t\geq 0}$ be a strongly continuous contraction semigroup of operators on a Banach space $(X, \|\cdot\|_X)$. The generator A of $(T_t)_{t\geq 0}$ is defined by

$$Au := \lim_{t\to 0} \frac{T_t u - u}{t} \; (\textit{strong limit}) \tag{2.4.1}$$

with domain

$$D(A) := \left\{ u \in X \,\middle|\, \lim_{t\to 0} \frac{T_t u - u}{t} \text{ exists as a strong limit} \right\}. \tag{2.4.2}$$

We now continue to follow [34] with the next theorem which encompasses the main results of the thesis.

Theorem 2.4.1. *(Hille-Yosida)* A linear operator A on a Banach space $(X, \|\cdot\|_X)$ with domain $D(A) \subset X$ is the generator of a strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ if and only if the following three conditions hold:

1. $D(A) \subset X$ is dense.
2. A is a dissipative operator. ie. $\lambda \|u\|_X \leq \|(\lambda - A)u\|_X$ for all $\lambda > 0$.
3. $R(\lambda - A) = X$ for some $\lambda > 0$.

Remark. If A^* is injective, then condition 3 of theorem 2.4.1 holds. Moreover, if for a closed and dissipative operator A, and one $\lambda > 0$ the equation

$$(\lambda - A)u = f \tag{2.4.3}$$

is uniquely solvable for all $f \in X$, then for all $\lambda > 0$ it is uniquely solvable for all $f \in X$. Therefore if (2.4.3) is uniquely solvable for one $\lambda > 0$, then condition 3 of theorem 2.4.1 holds.

Chapter 3

Fourier Analysis on Compact Abelian Groups

3.1 Compact Abelian Groups and their Dual Groups

The aim of this section is to describe compact Abelian groups and the role of the Fourier transform. Some important definitions are made, in particular dual groups and the convolution of two functions. We frequently make use of Bachman [2] and Yosida [45].

Definition 3.1.1. A non-empty set G is said to be a *topological group* if, and only if, G is a topological space and

i) G is a group.

ii) G is a Hausdorff space. i.e. given $x, y \in \mathrm{G}, x \neq y$, there is $X \subset \mathrm{G}, Y \subset \mathrm{G}$ s.t. X,Y are neighbourhoods of x,y respectively and $X \cap Y = \phi$.

iii) $f : G \to G$

$g \mapsto -g$

is continuous,

iv) $h : G \times G \to G$

$(g_1, g_2) \mapsto g_1 + g_2$

is continuous, where $+$ denotes the group operation.

Definition 3.1.2. A topological group G is said to be an *Abelian* group if $x + y = y + x$ for all $x, y \in$ G.

From now on, all groups we consider will be Abelian by assumption.

Definition 3.1.3. A topological group G is called *compact* if for each open covering of G there is a *finite* subcovering.

The following definition and theorem can be found in [44].

Definition 3.1.4. Let $\epsilon > 0$ and $N \subset X$ be a subset in a Banach space $(X, \|.\|)$. We call N an ϵ-net for the set $M \subset X$, if $M \subset \bigcup_{y \in N} B_\epsilon(y)$ holds.

Theorem 3.1.1. A set $M \subset X$ is pre-compact, i.e. its closure $\overline{M}$ is compact, if for every $\epsilon > 0$ there exists a finite ϵ-net.

Definition 3.1.5. A topological group, G, is called *locally compact* if for each $x \in G$ there is an open set $O_x \subset G$ containing x for which the closure $\overline{O_x}$ is compact.

Theorem 3.1.2. A Hausdorff compact group is locally compact.

Definition 3.1.6. Let G be a locally compact group, and S denote the Borel sets on G. For a subset $E \subset G$ and element $g \in G$, we define $E \oplus g := \{h \in G : h = e + g,\ e \in E\}$ and $g \oplus E := \{h \in G : h = g + e,\ e \in E\}$.
A *right Haar measure* is a measure on G such that

(1) the measure of any non-empty open set is positive,

(2) $\mu(E \oplus g) = \mu(E) \quad E \in S,\ g \in G.$

A *left Haar measure* satisfies (1) and $\mu(g \oplus E) = \mu(E)$. Any statement made for a right Haar measure implies a corresponding statement for the left Haar measure; Therefore we may use the right Haar measure throughout, and call it simply 'the Haar measure.'

A proof for the *existence* of a left Haar measure on every locally compact (not necessarily Abelian) group is given in G.B. Folland [19], Theorem 2.10, and in H. Reiter and J.D. Stegeman [40], Theorem 3.3.2.

Definition 3.1.7. A mapping

$$\chi : G \to \mathbb{C}$$

with the properties

$$|\chi(x)| = 1 \qquad \text{for all } x \in G, \tag{3.1.1}$$

$$\chi(x+y) = \chi(x)\chi(y) \qquad \text{for all } x, y \in G, \tag{3.1.2}$$

is called a *character* of G.

Definition 3.1.8. Let G be a compact Abelian group, and let the class of all *continuous* characters on G be denoted by Γ. If we take $(\chi_1 + \chi_2)(g) = \chi_1(g)\chi_2(g)$ as the group operation, we see that Γ forms a group, and this group is called the *dual group* of G.

Following [40] we introduce on Γ the following topology. For a compact set $K \subset G$ and $\varepsilon > 0$ denote by $U(K, \varepsilon)$ the set of all $\chi \in \Gamma$ such that $|\chi(x) - 1| < \varepsilon$ for all $x \in K$. The family of these sets $U(K, \varepsilon)$ form a basis of neighbourhoods of the unit character. By translation we can now construct a topology on Γ.

Definition 3.1.9. We call a group, G, *discrete* if all subsets of G are open. Note that under this topology, even a set consisting of one discrete point is open.

Theorem 3.1.3. The dual group of a *compact* group is *discrete.* Conversely, if G is discrete, then Γ is compact.

For the proof, see [40], p. 133, which also refers to Weil(1953) and Pontrjagin(1966).

Theorem 3.1.4. *(Pontrjagin Duality Theorem)* Let Λ denote the dual group of Γ. Let y be a fixed element of G and define, for $\gamma \in \Gamma$,

$$f_y(\gamma) = \gamma(y).$$

Then the mapping

$$\begin{array}{ccc} G & \rightarrow & \Lambda \\ y & \mapsto & f_y \end{array}$$

given above is an isomorphism and a homeomorphism.

The proof of Theorem 3.1.4 is given in [2], p. 241.
Remark. If we write for a moment $G^* := \Gamma$, where Γ is the dual group of G, then we may interpret Pontrjagin's theorem as: $(G^*)^* = G$.

Definition 3.1.10. We define the *Fourier Transform* of $f \in \mathrm{L}^1(\mathrm{G})$ as

$$\hat{f}(\chi) = \int_G f(y)\overline{\chi(y)}\mathrm{d}\mu_G(y),$$

where μ_G is the Haar measure on G and $\chi \in \Gamma$.

Some immediate consequences of the definition are, for $f_1, f_2 \in \mathrm{L}^1(\mathrm{G})$,

$$(f_1 + f_2)\hat{}\ (\chi) = \hat{f}_1(\chi) + \hat{f}_2(\chi)$$

and

$$(\lambda f_1)\hat{}\ (\chi) = \lambda\hat{f}_1(\chi).$$

In the case of $\mathrm{L}^1(\mathrm{G})$, if we take

$$f^*(x) = \overline{f(-x)}$$

as the involution operation, we now have

$$(f^*)\,\hat{}\,(\gamma) = \int_G \overline{f(-x)}\;\overline{\gamma(x)}\;\mathrm{d}\mu_G(x)$$

for $\gamma \in \Gamma$. Since G is Abelian it follows that for any measurable $E \subset G$, $\mu_G(E) = \mu_G(-E)$. Therefore replacing x by -x in the above expression, we obtain

$$(f^*)\,\hat{}\,(\gamma) = \int_G \overline{f(x)}\;\overline{\gamma(-x)}\;\mathrm{d}\mu_G(x),$$

but since $\gamma \in \Gamma$ we know

$$\gamma(-x) = (-\gamma)(x) = \overline{\gamma(x)},$$

hence

$$(f^*)\,\hat{}\,(\gamma) = \int_G \overline{f(x)}\;\gamma(x)\;\mathrm{d}\mu_G(x) = \overline{\hat{f}(\gamma)}.$$

Theorem 3.1.5. *(Plancherel)*
The mapping

$$\begin{array}{rcl} \mathrm{L}^1(\mathrm{G}) \cap \mathrm{L}^2(\mathrm{G}) & \rightarrow & \mathrm{L}^2(\Gamma) \\ f & \mapsto & \hat{f} \end{array}$$

is an *isometry* onto a dense subspace of $\mathrm{L}^2(\Gamma)$, and hence may be extended to an isometry of $\mathrm{L}^2(\mathrm{G})$ onto $\mathrm{L}^2(\Gamma)$.

A detailed proof of Plancherel's Theorem on G can be found in [2], p. 235.

Remark. In Theorem 3.1.5 we used a formulation which also holds for locally compact groups. Of course, in our case of an Abelian compact group G, we always have $\mathrm{L}^2(\mathrm{G}) \subset \mathrm{L}^1(\mathrm{G})$.

Definition 3.1.11. Let $\mu \in \mathbb{M}_b^+(\mathrm{G})$ be a bounded Borel measure on G. Its *Fourier transform* $\hat{\mu} : \Gamma \rightarrow \mathbb{C}$ is given by

$$\hat{\mu}(\xi) = \int_G \overline{\xi(x)}\mu(\mathrm{d}x). \tag{3.1.3}$$

We note that

$$|\hat{\mu}(\xi)| \leq \|\mu\| = \hat{\mu}(0),$$

and that $\xi \mapsto \hat{\mu}(\xi)$ is a uniformly continuous function, see [10]. Note that on the discrete space Γ, every function is uniformly continuous.

Definition 3.1.12. We denote by $f * g$ the *convolution* of f and g, defined as

$$(f * g)(x) = \int_G f(x-y)g(y)\mathrm{d}\mu_G(y) = \int_G g(x-y)f(y)\mathrm{d}\mu_G(y) \qquad (3.1.4)$$

where $f, g \in \mathrm{L}^1(\mathrm{G})$ and μ_G is the Haar measure on G.

We now give two frequently used tools, the Convolution Theorem and Young's inequality.

Theorem 3.1.6. *(Convolution Theorem)* For $f, g \in \mathrm{L}^1(\mathrm{G})$ we have

$$(f * g)\,\hat{}\,(\gamma) = \hat{f}(\gamma)\hat{g}(\gamma). \qquad (3.1.5)$$

Proof.

$$\begin{aligned}(f * g)\,\hat{}\,(\gamma) &= \int_G \overline{\gamma(x)}(f * g)(x)\mathrm{d}\mu_G(x), \\ &= \int_G \overline{\gamma(y)}g(y)\left\{\int_G f(x-y)\overline{\gamma(x-y)}\mathrm{d}\mu_G(x)\right\}\mathrm{d}\mu_G(y), \\ &= \hat{f}(\gamma)\,\hat{g}(\gamma).\end{aligned}$$

□

Theorem 3.1.7. *(Young's Inequality)* For $f \in \mathrm{L}^p(\mathrm{G})$, $1 \leq p \leq \infty$ and $g \in \mathrm{L}^1(\mathrm{G})$ the convolution

$$(f * g)(x) = \int_G f(x-y)g(y)\mathrm{d}\mu_G(y)$$

defines an element in $\mathrm{L}^p(\mathrm{G})$ such that

$$\|f * g\|_{\mathrm{L}^p(\mathrm{G})} \leq \|f\|_{\mathrm{L}^p(\mathrm{G})}\|g\|_{\mathrm{L}^1(\mathrm{G})}. \qquad (3.1.6)$$

We will also later use the following (unnamed) inequality, which can be found in Reiter & Stegeman [40].

Theorem 3.1.8. For $k \in L^1(G)$ and $u, v \in L^2(G)$ we have

$$\left| \int_G \int_G k(x-y)u(x)v(y)\mathrm{d}\mu_G(x)\mathrm{d}\mu_G(y) \right| \leq \|k\|_{L^1(G)} \|u\|_{L^2(G)} \|v\|_{L^2(G)}. \quad (3.1.7)$$

Proof. Following the Cauchy-Schwarz Inequality with Young's Inequality yields

$$\begin{aligned} \left| \int_G \int_G k(x-y)u(x)v(y)\mathrm{d}\mu_G(x)\mathrm{d}\mu_G(y) \right| &\leq \|(k*u)v\|_{L^1(G)} \leq \|k*u\|_{L^2(G)} \|v\|_{L^2(G)} \\ &\leq \|k\|_{L^1(G)} \|u\|_{L^2(G)} \|v\|_{L^2(G)}. \end{aligned}$$

□

3.2 Negative Definite Functions and Convolution Semigroups

It is assumed for the rest of the thesis that G is a compact Abelian group. The dual group of G is therefore discrete and is denoted Γ.
In this section we introduce the notion of negative definite functions on Γ, the dual group of G, and assert a one-to-one correspondence between these negative definite functions and convolution semigroups on G.

Definition 3.2.1. A function $u : \Gamma \to \mathbb{C}$ is called *positive definite* if, for any choice of $n \in \mathbb{N}$ and vectors $\xi_1, \ldots, \xi_n \in \Gamma$, the matrix $\big(u(\xi_j - \xi_l)\big)_{j,l=1,\ldots,n}$ is positive Hermitian,

i.e.

$$\big(u(\xi_j - \xi_l)\big)_{j,l=1,\ldots,n} = \big(\overline{u(\xi_j - \xi_l)}\big)^t_{j,l=1,\ldots,n}$$

and for all $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ we have

$$\sum_{j,l=1}^{n} u(\xi_j - \xi_l)\lambda_j \overline{\lambda_l} \geq 0.$$

We now state Bochner's theorem, following Berg, Forst in [10].

Theorem 3.2.1. *(Bochner)*
A function $u : \Gamma \to \mathbb{C}$ is the Fourier transform of a measure $\mu \in \mathrm{M}_b^+(\mathrm{G})$ with total mass $\|\mu\|$, if and only if the following conditions are satisfied:

1. u is continuous
2. $u(0) = \hat{\mu}(0) = \|\mu\|$
3. u is positive definite.

Definition 3.2.2. A family of bounded Borel measures $(\mu_t)_{t\geq 0}$ on G, is called a *convolution semigroup* on G if the following conditions hold:

i) $\mu_t(G) \leq 1$ for all $t \geq 0$

ii) $\mu_s * \mu_t = \mu_{s+t}$ where $s, t \geq 0$ and $\mu_0 = \varepsilon_0$

iii) $\mu_t \to \varepsilon_0$ vaguely as $t \to 0$. i.e. for all $f \in \mathrm{C}_0(\mathrm{G}) = \mathrm{C}_b(\mathrm{G})$, it holds $\lim\limits_{t\to 0} \mu_t(f) = \lim\limits_{t\to 0} \int_G f(x)\mu_t(\mathrm{dx}) = \int_G f(x)\varepsilon_0(\mathrm{dx}) = f(0)$.

Let $(\mu_t)_{t\geq 0}$ be a convolution semigroup on G. It follows, by Bochner's Theorem 3.2.1 that the family of the Fourier transforms of μ_t, namely $(\hat{\mu}_t)_{t\geq 0}$, consists of continuous positive definite functions on Γ.

Theorem 3.2.2. Let $(\mu_t)_{t\geq 0}$ be a convolution semigroup on G. Then, there exists a function $\psi : \Gamma \to \mathbb{C}$ such that

$$\hat{\mu}_t(\gamma) = e^{-t\psi(\gamma)} \tag{3.2.1}$$

holds for all $\gamma \in \Gamma$ and $t \geq 0$.

Also following [10], we introduce negative definite functions.

Definition 3.2.3. A function $\psi : \Gamma \to \mathbb{C}$ is called *negative definite* if $\psi(0) \geq 0$ and $\gamma \mapsto e^{-t\psi(\gamma)}$ is positive definite, for $t \geq 0$. By $\mathrm{CN}(\Gamma)$ we denote the set of all (continuous) negative definite functions on Γ.

Theorem 3.2.2 therefore states that for any convolution semigroup $(\mu_t)_{t\geq 0}$ there exists a unique *negative definite* function $\psi : \Gamma \to \mathbb{C}$ such that $\hat{\mu}_t(\gamma) = e^{-t\psi(\gamma)}$ holds. A more direct definition of a negative definite function is given here:

Proposition 3.2.1. A function $\psi : \Gamma \to \mathbb{C}$ is negative definite if, and only if, for any $n \in \mathbb{N}$ and vectors $\xi_1, \ldots, \xi_n \in \Gamma$, the matrix

$$\left(\psi(\xi_j) + \overline{\psi(\xi_l)} - \psi(\xi_j - \xi_l)\right)_{j,l=1,\ldots,n}$$

is positive Hermitian.

Proposition 3.2.2. Let $\psi \in \mathrm{CN}(\Gamma)$, then we have

i) $\psi(0) \geq 0,$ in particular $\psi(0) \in \mathbb{R}$,
ii) the mapping $|\psi|^{\frac{1}{2}} : \Gamma \to \mathbb{R}$ is subadditive,
iii) for any $\gamma \in \Gamma$ it follows that $\mathrm{Re}[\psi(\gamma)] \geq \psi(0)$.

Theorem 3.2.3. *(Peetre's Inequality for negative definite functions)*
Let $\psi : \Gamma \to \mathbb{C}$ be a negative definite function, then the following inequality holds:

$$\frac{1 + |\psi(\xi)|}{1 + |\psi(\eta)|} \leq 2\big(1 + |\psi(\xi - \eta)|\big). \tag{3.2.2}$$

The proof of this inequality follows in exactly the same way as on $\mathbb{R}^n$, given in Jacob [34].

Chapter 4

Some Function Spaces

4.1 Some Function Spaces

In this section we seek to categorise the 'smoothness' of functions on G, where G is once again a compact Abelian group with countable dual group Γ. We firstly consider generalised trigonometric polynomials, S(G), and then expand to generalised Sobolev spaces, via a norm and scalar product on S(G).

Definition 4.1.1. A *generalised trigonometrical polynomial* is by definition any function $f : G \to \mathbb{C}$ having the representation

$$f(x) = \sum_{\substack{\gamma \in \Gamma \\ finite}} a_\gamma \gamma(x) \qquad , \qquad a_\gamma \in \mathbb{C}.$$

The set of all generalised trigonometrical polynomials is denoted by S(G).

A well known fact is the following:

Proposition 4.1.1. Let G and Γ be as above and $1 \leq p < \infty$. Then S(G) is dense in the spaces C(G) and L^p(G), respectively.

Since the characters in our case form a complete orthonormal system in $L^2(G)$, every $f \in L^2(G)$ can be represented as

$$f(x) = \sum_{\gamma \in \Gamma} \hat{f}(\gamma)\gamma(x)$$

where the series converges in $L^2(G)$, and this L^2-limit does not depend on the order of summation.

We want to introduce a certain family of function spaces generalising Sobolev spaces. For this reason let $k : \Gamma \to \mathbb{C}$ be a function and denote by k_* the function defined on Γ by $k_*(\gamma) = (1 + |k(\gamma)|^2)^{1/2}$. For each $r \in \mathbb{R}$, $r \geq 0$, we define on $S(G)$ the scalar product

$$(\varphi, \psi)_{r,k} = \sum_{\substack{\gamma \in \Gamma \\ finite}} k_*^{2r}(\gamma)\hat{\varphi}(\gamma)\overline{\hat{\psi}(\gamma)}. \tag{4.1.1}$$

The norm corresponding to (4.1.1) is denoted by $\|.\|_{r,k}$.
In particular, for each function $k : \Gamma \to \mathbb{C}$, we have

$$\|\varphi\|_{0,k} = \|\varphi\|_0$$

where $\|.\|_0$ denotes the norm in $L^2(G)$.

Definition 4.1.2. The completion of $S(G)$ with respect to (4.1.1) is the Hilbert space $H_k^r(G)$.

Obviously we have

Proposition 4.1.2. A. For $0 \leq s < r$ the space $H_k^r(G)$ is continuously embedded into the space $H_k^s(G)$, in particular $H_k^r(G)$ is always continuously embedded into $L^2(G)$ and

$$\|u\|_0 \leq \|u\|_{r,k} \quad , \quad u \in H_k^r(G).$$

B. Let $k_i : \Gamma \to \mathbb{C}$, $i = 1, 2$, be two functions and suppose that for all $\gamma \in \Gamma \setminus \widetilde{\Gamma}$, $\widetilde{\Gamma} \subset \Gamma$ finite, the estimate $|k_1(\gamma)| \leq c\,|k_2(\gamma)|$ holds. Then for each $r \geq 0$, $\mathrm{H}^r_{k_2}(\mathrm{G})$ is continuously embedded into the space $\mathrm{H}^r_{k_1}(\mathrm{G})$.
C. For each $u \in \mathrm{H}^r_k(\mathrm{G})$, $r \geq 0$, $\gamma \mapsto k^r_*(\gamma)\hat{u}(\gamma)$ belongs to $\mathrm{L}^2(\Gamma)$.
Conversely, whenever $\gamma \mapsto k^r_*(\gamma)v(\gamma)$ belongs to $\mathrm{L}^2(\Gamma)$ for some sequence $(v(\gamma))_{\gamma\in\Gamma}$, then

$$x \mapsto w(x) := \sum_{\gamma\in\Gamma} v(\gamma)\gamma(x)$$

belongs to $\mathrm{H}^r_k(\mathrm{G})$ and $\hat{w}(\gamma) = v(\gamma)$.
As done in [15] for the n-dimensional torus, we can characterise the dual space of $\mathrm{H}^r_k(\mathrm{G})$, for $r \geq 0$. First we define for $f \in \mathrm{L}^2(\mathrm{G})$ and $r \geq 0$ the norm

$$\|f\|_{-r,k} := \sup_{0\neq u\in \mathrm{H}^r_k(\mathrm{G})} \frac{|(f,u)_0|}{\|u\|_{r,k}} \quad . \tag{4.1.2}$$

It follows as in [15] that the set $\mathcal{L}$ of all continuous linear functionals l on $\mathrm{H}^r_k(\mathrm{G})$ given by $l(u) = (f,u)_0$ for some $f \in \mathrm{L}^2(G)$ is dense in $[\mathrm{H}^r_k(\mathrm{G})]^*$ with respect to the norm (4.1.2).

Definition 4.1.3. Let $r > 0$. The completion of $\mathrm{L}^2(\mathrm{G})$ with respect to $\|.\|_{-r,k}$ is denoted by $\mathrm{H}^{-r}_k(\mathrm{G})$.

This notation, with the considerations made above, imply immediately:

Proposition 4.1.3. For $r > 0$ we have $[\mathrm{H}^r_k(\mathrm{G})]^* = \mathrm{H}^{-r}_k(\mathrm{G})$.

Lemma 4.1.1. A. The set $\mathrm{S}(\mathrm{G})$ is dense in $\mathrm{H}^{-r}_k(\mathrm{G})$.
B. For each $\varphi \in S(\mathrm{G})$ we have

$$\|\varphi\|^2_{-r,k} = \sum_{\substack{\gamma\in\Gamma \\ finite}} k^{-2r}_*(\gamma)|\hat{\varphi}(\gamma)|^2. \tag{4.1.3}$$

Hence it follows

Proposition 4.1.4. Let $r \in \mathbb{R}$ and $u \in \mathrm{H}^r_k(\mathrm{G})$, then

$$\|u\|^2_{r,k} = \sum_{\gamma \in \Gamma} k_*^{2r}(\gamma)|\hat{u}(\gamma)|^2. \tag{4.1.4}$$

The proof of the lemma is straightforward, and we refer the reader to [15], 202-203.

Obviously Proposition 4.1.4 implies that for $r, t \in \mathbb{R}$, $r > t$, the space $\mathrm{H}^r_k(\mathrm{G})$ is continuously embedded into the space $\mathrm{H}^t_k(\mathrm{G})$.

Later in this paper, we will prove further properties of the spaces $\mathrm{H}^r_k(\mathrm{G})$, $r \in \mathbb{R}$, and related spaces. In particular we will prove some compactness results for the embedding of $\mathrm{H}^r_k(\mathrm{G})$ into $\mathrm{H}^s_k(\mathrm{G})$.

For later purposes we introduce the notion of a *finite exhaustion* of Γ, the discrete dual group of G. By this, we mean a sequence $(\Gamma_m)_{m \in \mathrm{N}}$ of finite subsets Γ_m of Γ with the following properties:

$$\Gamma_m \neq \phi, \qquad \Gamma_m \subsetneqq \Gamma_{m+1} \qquad \text{and} \qquad \bigcup_{m \in \mathrm{N}} \Gamma_m = \Gamma.$$

4.2 Convergence and Embedding Theorems

We want to emphasise that the results in this paper frequently depend on a fixed finite exhaustion of Γ. Although the initial choice of exhaustion $(\Gamma_m)_{m \in \mathrm{N}}$ is not important when formulating and proving these results, it must remain fixed throughout.

Definition 4.2.1. Let G be a compact Abelian group with discrete dual group Γ and let $(\Gamma_m)_{m \in \mathrm{N}}$ be a finite exhaustion of Γ. Furthermore, let $u \in \mathrm{L}^1(\mathrm{G})$. We say that the Fourier series $\sum_{\gamma \in \Gamma} \hat{u}(\gamma)\gamma(x)$ *converges μ_G almost everywhere* to $u(x)$, w.r.t. $(\Gamma_m)_{m \in \mathrm{N}}$, if the sequence $\mathrm{S}_m(x) := \sum_{\gamma \in \Gamma_m} \hat{u}(\gamma)\gamma(x)$ converges almost everywhere to u. Again μ_G denotes the Haar measure on G.

It is well known that even for the group $\mathrm{G} = \mathbb{T}^1$, the one dimensional torus, i.e. the circle, the Fourier series of an L^1-function need not converge almost everywhere to the function. This result is due to A.N. Kolmogorov [37]. But the celebrated L. Carleson theorem [13] states that for any $u \in \mathrm{L}^2(\mathbb{T}^1)$ the Fourier series of u converges $\mu_{\mathbb{T}^1}$ almost everywhere to u. This result was generalised by R. Hunt to elements in $\mathrm{L}^p(\mathbb{T}^1)$, $1 < p < \infty$, see [30].
It was C. Fefferman [18], who pointed out that Carleson's result does not extend in the natural way for elements in $\mathrm{L}^2(\mathbb{T}^2)$, where $\mathbb{T}^2$ is the two dimensional torus. He found that in this case it does depend on the way in which the summation of the series is done, i.e. on the choice of the finite exhaustion. For further discussions on this problem we refer to [1].
However, the next theorem, which is a type of Sobolev embedding theorem, also gives a sufficient condition that elements in certain subspaces of $\mathrm{L}^2(\mathrm{G})$ have a pointwise convergent Fourier series. We need the following condition.

Condition 4.2.1. Let $k : \Gamma \to \mathbb{C}$ be a fixed function and $t_0 > 0$ be a real number. We say that k fulfills Condition 4.2.1 if and only if

$$\sum_{\gamma \in \Gamma} k_*^{-2t_0}(\gamma) < \infty \tag{4.2.1}$$

holds. By this we mean that the sequence $\left(\sum_{\gamma \in \Gamma_m} k_*^{-2t_0}(\gamma) \right)_{m \in \mathbb{N}}$ converges in $\mathbb{C}$ for the finite exhaustion $(\Gamma_m)_{m \in \mathbb{N}}$. Condition 4.2.1 means exactly that the embedding $\mathrm{H}_k^{t_0}(\mathrm{G}) \to \mathrm{L}^2(\mathrm{G})$ is of Hilbert-Schmidt type.

Proposition 4.2.1. If Condition 4.2.1 holds for one finite exhaustion of Γ, then it holds for all finite exhaustions of Γ.

Proof. Given that $\left(\sum_{\gamma \in \Gamma_m} k_*^{-2t_0}(\gamma) \right)_{m \in \mathbb{N}}$ converges in $\mathbb{C}$, the partial sum satisfies $s_m = \sum_{\gamma \in \Gamma_m} k_*^{-2t_0}(\gamma) \leq K$, for a constant $K \in \mathbb{C}$ and all $m \in \mathbb{N}$.

Now let $t_m = \sum_{\gamma \in \tilde{\Gamma}_m} k_*^{-2t_0}(\gamma)$, where $(\tilde{\Gamma}_m)_{m\in\mathbb{N}}$ is a different finite exhaustion of Γ. For each $m \in \mathbb{N}$ there exists some $\alpha(m) \in \mathbb{N}$ for which $\tilde{\Gamma}_m \subset \Gamma_{\alpha(m)}$, so

$$t_m \leq s_{\alpha(m)} \leq K.$$

Thus t_m is an increasing positive sequence, bounded above, and so converges with

$$\lim_{m\to\infty} t_m \leq \lim_{m\to\infty} s_m. \tag{4.2.2}$$

Furthermore, given any $i \in \mathbb{N}$, there is a $\beta(i) \in \mathbb{N}$ such that $\Gamma_i \subset \tilde{\Gamma}_m$ for $m > \beta(i) > i$ and

$$s_i \leq t_m \leq s_{\alpha(m)} \leq K \quad \text{ for } m > \beta(i).$$

If we let $i \to \infty$, then

$$\lim_{i\to\infty} s_i \;\; \leq \lim_{m\to\infty} t_m$$

which combined with (4.2.2) implies that $(s_m)_{m\in\mathbb{N}}$ and $(t_m)_{m\in\mathbb{N}}$ converge to the same limit.

$\square$

Theorem 4.2.1. Let G, and Γ be as mentioned above, $(\Gamma_m)_{m\in\mathbb{N}}$ be any finite exhaustion of Γ and suppose that Condition 4.2.1 holds. Then the space $\mathrm{H}_k^r(\mathrm{G})$ is continuously embedded into the space $\mathrm{C}(\mathrm{G})$ provided $r \geq t_0$. More precisely, the elements of $\mathrm{H}_k^r(\mathrm{G})$ have an absolutely convergent Fourier series and for μ_G almost all $x \in \mathrm{G}$ we have

$$|u(x)| \leq c\,\|u\|_{r,k}\,. \tag{4.2.3}$$

Proof. Let $u \in \mathrm{H}_k^r(\mathrm{G})$. Then we find

$$\Big(\sum_{\gamma\in\Gamma} |\hat{u}(\gamma)|\Big)^2 = \Big(\sum_{\gamma\in\Gamma} |\hat{u}(\gamma)| k_*^{-r}(\gamma) k_*^{r}(\gamma)\Big)^2$$

$$\leq \Big(\sum_{\gamma\in\Gamma} k_*^{-2r}(\gamma)\Big)\Big(\sum_{\gamma\in\Gamma} |\hat{u}(\gamma)|^2 k_*^{2r}(\gamma)\Big)$$

$$\leq \Big(\sum_{\gamma\in\Gamma} k_*^{-2t_0}(\gamma)\Big)\Big(\sum_{\gamma\in\Gamma} |\hat{u}(\gamma)|^2 k_*^{2r}(\gamma)\Big),$$

which implies that the series $\sum\limits_{\gamma\in\Gamma} \hat{u}(\gamma)\gamma(.)$ converges absolutely and uniformly on G. Hence there exists a continuous function $\widetilde{u} : \mathrm{G} \to \mathbb{C}$ which is the uniform limit of $\Big(\sum\limits_{\gamma\in\Gamma_m} \hat{u}(\gamma)\gamma(.)\Big)_{m\in\mathbb{N}}$. But on the other hand $\Big(\sum\limits_{\gamma\in\Gamma_m} \hat{u}(\gamma)\gamma(.)\Big)_{m\in\mathbb{N}}$ converges in $\mathrm{L}^2(\mathrm{G})$ to u. Thus $u = \widetilde{u}$ μ_G-almost everywhere. Moreover, we find that

$$|\widetilde{u}(x)| = \Big|\sum_{\gamma\in\Gamma} \hat{u}(\gamma)\gamma(x)\Big| \leq \Big(\sum_{\gamma\in\Gamma} k_*^{-2t_0}(\gamma)\Big)^{1/2} \|u\|_{r,k}$$

for all $u \in \mathrm{H}_k^r(\mathrm{G})$, which implies

$$|u(x)| \leq c\,\|u\|_{r,k} \qquad \mu_G\text{-almost everywhere}$$

for all $u \in \mathrm{H}_k^r(\mathrm{G})$. We also note that $\|u\|_\infty = \sup\limits_{x\in G} |u(x)| \leq c\|u\|_{r,k}$.
i.e. in this case, $\mathrm{H}_k^r(\mathrm{G})$ is continuously embedded into C(G). Hence the theorem is proved, by Proposition 4.2.1. □

Our next aim is to give necessary and sufficient conditions for the embedding of $\mathrm{H}_k^r(\mathrm{G})$ into $\mathrm{L}^2(\mathrm{G})$, or more generally for $\mathrm{H}_k^{r+t}(\mathrm{G})$ into $\mathrm{H}_k^r(\mathrm{G})$, to be compact. For this we need the following

Definition 4.2.2. Let Γ and $(\Gamma_m)_{m\in\mathbb{N}}$ be as above and $k : \Gamma \to \mathbb{R}$ be a given function.

A. We say that a sequence $(\gamma_j)_{j\in\mathbb{N}}$, $\gamma_j \in \Gamma$, diverges to ∞, i.e. $\lim\limits_{j\to\infty} \gamma_j = \infty$, if and only if for any $m_0 \in \mathbb{N}$ there exists a number $\mathrm{N}_0 \in \mathbb{N}$ such that for all $j \geq \mathrm{N}_0$ it follows that $\gamma_j \in \Gamma \setminus \Gamma_{m_0}$ holds.
B. We write $\lim\limits_{j\to\infty} k(\gamma_j) = \infty$ if and only if for any $\mathrm{N} \in \mathbb{N}$ there exists a $m_0 \in \mathbb{N}$ such that $\gamma_j \in \Gamma \setminus \Gamma_{m_0}$ implies $k(\gamma_j) \geq \mathrm{N}$.
C. We will write $\lim\limits_{\gamma\to\infty} k(\gamma) = \infty$ if and only if for any $\mathrm{N} \in \mathbb{N}$ there exists $m_0 \in \mathbb{N}$, such that for all $\gamma \in \Gamma \setminus \Gamma_{m_0}$ it follows that $k(\gamma) \geq \mathrm{N}$ holds.

By our assumptions Γ is discrete and $(\Gamma_m)_{m\in\mathbb{N}}$ is a finite exhaustion. Suppose that $\Gamma_{m+1} \setminus \Gamma_m$ has J_m elements. Moreover, let $Z : \Gamma \to \mathbb{N}$ be a bijection such that $Z\Big|_{\Gamma_m\setminus\Gamma_{m-1}}$ is a bijection between $\Gamma_m \setminus \Gamma_{m-1}$ and the set $\mathbb{N} \cap \Big[\big(\sum\limits_{\nu=1}^{m-1} \mathrm{J}_\nu\big) + 1, \sum\limits_{\nu=1}^{m} \mathrm{J}_\nu\Big]$. Note that for $m = 0$ we set $\Gamma_0 = \phi$. Then it follows from the definition that the sequence $Z^{-1} : \mathbb{N} \to \Gamma$ diverges to ∞.
If $l(m) = \sum\limits_{\nu=1}^{m} \mathrm{J}_\nu$ and $\mathbb{N}_m = \{1, 2, \ldots, m\}$, then $Z^{-1}(\mathbb{N}_{l(m)}) = \Gamma_m$. Hence, using Z^{-1} we can consider Γ as a sequence diverging to infinity.
Whenever we now write $\gamma \to \infty$, we mean that we have fixed a mapping Z with the properties stated above, and we consider the sequence $Z^{-1} : \mathbb{N} \to \Gamma$.
Now we can prove

Theorem 4.2.2. Let $r, t \in \mathbb{R}$, $t > 0$. Moreover let G, Γ and $(\Gamma_m)_{m\in\mathbb{N}}$ be as above and consider Γ as a sequence diverging to infinity. In order that the embedding of $\mathrm{H}_k^{r+t}(\mathrm{G})$ into $\mathrm{H}_k^r(\mathrm{G})$ is compact, it is necessary and sufficient that

$$\lim_{\gamma\to\infty} k_*(\gamma) = \infty \tag{4.2.4}$$

holds.

Proof. **A.** First we prove that (4.2.4) implies the compactness of the embedding. We start by proving that the set

$$\mathrm{M} = \{u \in \mathrm{S}(\mathrm{G}), \|u\|_{r+t,k} \leq 1\}$$

is finite in $\mathrm{H}_k^r(\mathrm{G})$. For an arbitrary element $u \in \mathrm{M}$ we define the m-th segment by

$$u_{\mathrm{m}}(x) = \sum_{\gamma \in \Gamma_m} \hat{u}(\gamma)\gamma(x),$$

and we find

$$\|u - u_m\|_{r,k}^2 = \sum_{\gamma \in \Gamma \setminus \Gamma_m} k_*^{2r}(\gamma)|u(\gamma)|^2.$$

We want to prove that the set M_m of all m-th segments of elements in M is an ε-net in $\mathrm{H}_k^r(\mathrm{G})$ provided m is sufficiently large. For $\varepsilon > 0$ there exists a $m_0 \in \mathbb{N}$ such that if $\gamma \in \Gamma \setminus \Gamma_{m_0}$ it follows that

$$k_*^{-2t}(\gamma) \leq \varepsilon$$

holds. For $\gamma \in \Gamma \setminus \Gamma_{m_0}$ and $m \geq m_0$ we find

$$\begin{aligned}
\|u - u_m\|_{r,k}^2 &\leq \sum_{\gamma \in \Gamma \setminus \Gamma_{m_0}} k_*^{2r+2t}(\gamma)|\hat{u}(\gamma)|^2 k_*^{-2t}(\gamma) \\
&\leq \sum_{\gamma \in \Gamma \setminus \Gamma_{m_0}} k_*^{2r+2t}(\gamma)|\hat{u}(\gamma)|^2 \\
&\leq \|u\|_{r+t,k}^2 \\
&\leq 1,
\end{aligned}$$

since we have $u \in \mathrm{M}$, i.e. $\|u\|_{r+t,k} \leq 1$.

Hence M_{m_0} is an ε-net for M in $\mathrm{H}_k^r(\mathrm{G})$. Since Γ_m is finite, M_m is finite in $\mathrm{H}_k^r(\mathrm{G})$ for any $m \in \mathbb{N}$. Thus we know that for any $\varepsilon > 0$ there exists a finite ε-net in $\mathrm{H}_k^r(\mathrm{G})$ for the set M. By Satz 1.7 in [44], i.e. theorem 3.1.1, it follows that M is finite in $\mathrm{H}_k^r(\mathrm{G})$ which implies the compactness of the embedding of $\mathrm{H}_k^{r+t}(\mathrm{G})$ into $\mathrm{H}_k^r(\mathrm{G})$.

B. Now suppose that the embedding of $\mathrm{H}_k^{r+t}(\mathrm{G})$ into $\mathrm{H}_k^r(\mathrm{G})$ is compact, $r, t \in \mathbb{R}$, $t > 0$. We have to prove $\lim\limits_{\gamma\to\infty} k_*(\gamma) = \infty$, where $\gamma \to \infty$ is understood as stated above. Thus we consider Γ as a sequence $(\gamma_j)_{j\in\mathbb{N}}$ constructed by a certain mapping $Z : \Gamma \to \mathbb{N}$ depending on $(\Gamma_m)_{m\in\mathbb{N}}$. On G we define the function u^j by

$$u^j(x) = k_*^{-(r+t)}(\gamma_j)\gamma_j(x)$$

and we find

$$\hat{u}^j(\gamma) = \begin{cases} k_*^{-(r+t)}(\gamma_j) & , \quad \gamma = \gamma_j \\ 0 & , \quad \gamma \neq \gamma_j \end{cases}$$

which gives

$$\|u^j\|_{r+t,k}^2 = \sum_{\gamma\in\Gamma} |\hat{u}^j(\gamma)|^2\, k_*^{2(r+t)}(\gamma) = k_*^{-2(r+t)}(\gamma_j) k_*^{2(r+t)}(\gamma_j) = 1.$$

Hence the sequence $(u^j)_{j\in\mathbb{N}}$ is bounded in the Hilbert space $\mathrm{H}_k^{r+t}(\mathrm{G})$ and for this reason it has a weakly convergent subsequence denoted by $(u^{j'})$. Now let $\varphi \in \mathrm{S}(\mathrm{G})$, then it follows

$$(u^{j'}, \varphi)_{r+t,k} = \sum_{\gamma\in\Gamma} k_*^{2(r+t)}(\gamma)\hat{u}^{j'}(\gamma)\overline{\hat{\varphi}(\gamma)}$$

$$= k_*^{2(r+t)}(\gamma_{j'})\, k_*^{-(r+t)}(\gamma_{j'})\overline{\hat{\varphi}(\gamma_{j'})}$$

$$= k_*^{r+t}(\gamma_{j'})\overline{\hat{\varphi}(\gamma_{j'})}\,.$$

We claim that

$$\lim_{j'\to\infty} k_*^{r+t}(\gamma_{j'})\hat{\varphi}(\gamma_{j'}) = 0.$$

Since φ is a trigonometrical polynomial, there exists a finite set $\Gamma_\varphi \subset \Gamma$ such that

$$\varphi(x) = \sum_{\gamma\in\Gamma_\varphi} \hat{\varphi}(\gamma)\gamma(x).$$

But this implies $\hat{\varphi}(\gamma) = 0$ for all $\gamma \in \Gamma \setminus \Gamma_{m_0}$ whenever $\Gamma_\varphi \subset \Gamma_{m_0}$ and we find

$$\lim_{j' \to \infty} k_*^{2(r+t)}(\gamma_{j'})\overline{\hat{\varphi}(\gamma_{j'})} = 0.$$

Therefore, $(u^{j'})$ converges weakly in $\mathrm{H}_k^{r+t}(\mathrm{G})$ to 0.
By the compactness of the embedding of $\mathrm{H}_k^{r+t}(\mathrm{G})$ into $\mathrm{H}_k^r(\mathrm{G})$ we find $\|u^{j'}\|_{r,k} \to 0$ for $j' \to \infty$. From this we conclude by using

$$\|u^{j'}\|_{r,k}^2 = \sum_{\gamma \in \Gamma} k_*^{2r}(\gamma)|\hat{u}^{j'}(\gamma)|^2 = k_*^{-2t}(\gamma_{j'})$$

that

$$\lim_{j' \to \infty} k_*^{-2t}(\gamma_{j'}) = 0$$

or

$$\lim_{j' \to \infty} k_*^{2t}(\gamma_{j'}) = \infty \tag{4.2.5}$$

for the subsequence $(\gamma_{j'})$. But any subsequence $(\gamma_{j'})$ of $(\gamma_j)_{j \in \mathbb{N}}$ contains a subsequence $(\gamma_{j''})$ such that (4.2.5) holds and therefore (4.2.4) follows. □

Remark.
Theorem 4.2.2 can also be proved by the following known fact, pointed out by E.Lytvynov. Let $(a_i)_{i=1}^\infty$ be a fixed element of ℓ_∞. Define a bounded operator $\ell_2 \ni (x_i)_{i=1}^\infty \mapsto Ax = (\alpha_1 x_1, \alpha_2 x_2, \ldots) \in \ell_2$. Then A is compact if and only if $\lim_{k \to \infty} \alpha_k = 0$, see Chapter 9. Exercise 1.1 in [7].

4.3 Translation Invariant Pseudodifferential Operators

In the following, let G be a compact Abelian group with discrete dual group Γ on which a finite exhaustion $(\Gamma_m)_{m \in \mathbb{N}}$ is fixed. Furthermore, let $k : \Gamma \to \mathbb{C}$ be a fixed function.

Definition 4.3.1. For any $t \in \mathbb{R}$ we denote by $\sum^t(k)$ the set of all functions $A : \Gamma \to \mathbb{C}$ such that with some positive constant $c_{t,A}$

$$|A(\gamma)| \leq c_{t,A} k_*^t(\gamma) \tag{4.3.1}$$

holds for all $\gamma \in \Gamma$. Moreover let

$$\sum^\infty(k) = \bigcup_{t \in \mathbb{R}} \sum^t(k) \, . \tag{4.3.2}$$

The *k-degree* of an element $A \in \sum^\infty(k)$ is defined by

$$deg_k(A) := \inf\{t \in \mathbb{R}; |A(\gamma)| \leq c_{t,A} k_*^t(\gamma), \, \gamma \in \Gamma\} \, .$$

Now, any $A \in \sum^\infty(k)$ defines an operator A(D) on S(G) into itself by

$$[A(\mathrm{D})\varphi]\hat{}\,(\gamma) := A(\gamma)\hat{\varphi}(\gamma) \qquad , \qquad \varphi \in \mathrm{S(G)} \, . \tag{4.3.3}$$

Since

$$\varphi(x) = \sum_{\substack{\gamma \in \Gamma \\ finite}} \hat{\varphi}(\gamma)\gamma(x) \, ,$$

A(D) is well defined on S(G) and

$$A(\mathrm{D})\varphi(x) = \sum_{\substack{\gamma \in \Gamma \\ finite}} A(\gamma)\hat{\varphi}(\gamma)\gamma(x) \, .$$

We call A the *symbol* of the operator A(D).

Definition 4.3.2. By $\mathcal{A}^t(k)$ we denote the set of all operators generated by a symbol $A \in \sum^\infty(k)$ with k-degree less than or equal to t.

Moreover we set $\mathcal{A}^\infty(k) = \bigcup_{t \in \mathbb{R}} \mathcal{A}^t(k)$.

For any $A \in \sum^\infty(k)$ the operator A(D) belongs to $\mathcal{A}^\infty(k)$. Conversely, given $\mathrm{A(D)} \in \mathcal{A}^\infty(k)$, then there exists a number $t \in \mathbb{R}$ such that $\mathrm{A(D)} \in \mathcal{A}^t(k)$, hence for any $\epsilon > 0$ we find $A \in \sum^{t+\epsilon}(k)$ and A is the symbol of A(D).

Let $\mathrm{A(D)} \in \mathcal{A}^t(k)$, $\mathrm{B(D)} \in \mathcal{A}^r(k)$ and $\alpha, \beta \in \mathbb{C}$. Since $\alpha\mathrm{A(D)} \in \mathcal{A}^t(k)$ and $\beta\mathrm{B(D)} \in \mathcal{A}^r(k)$, we find $\alpha\mathrm{A(D)} + \beta\mathrm{B(D)} \in \mathcal{A}^{t\vee r}(k)$. Hence for any $t \in \mathbb{R}$ $\mathcal{A}^t(k)$ is a complex linear space, and for $t < r = t \vee r$ it follows that $\mathcal{A}^t(k)$ is a subspace of $\mathcal{A}^r(k)$. For the product of two operators it follows by a straightforward calculation (see [15], Lemma 1.1) that we have:

Lemma 4.3.1. Let $\mathrm{A(D)} \in \mathcal{A}^t(k)$, $\mathrm{B(D)} \in \mathcal{A}^r(k)$ and $T(D) := B(D) \circ A(D)$. Then T(D) has the function $\gamma \mapsto T(\gamma) := B(\gamma)A(\gamma)$ as its symbol, in particular we have $T(D) \in \mathcal{A}^{t+r}(k)$.

Thus we have found that $\mathcal{A}^\infty(k)$ is an algebra, moreover,

$$i) \qquad \mathcal{A}^\infty(k) = \bigcup_{t\in\mathbb{R}} \mathcal{A}^t(k)$$

$$ii) \qquad \alpha\mathcal{A}^t(k) + \beta\mathcal{A}^t(k) \subset \mathcal{A}^t(k)$$

$$iii) \qquad \mathcal{A}^r(k) \circ \mathcal{A}^t(k) \subset \mathcal{A}^{t+r}(k).$$

Definition 4.3.3. Let G,Γ and $(\Gamma_m)_{m\in\mathbb{N}}$ be as above.

A. An operator $\mathrm{A(D)} \in \mathcal{A}^t(k)$, $t \in \mathbb{R}$, is said to be *k-elliptic*, if there exists an $m_0 \in \mathbb{N}$ and a constant $c > 0$ such that

$$|\mathrm{A}(\gamma)| \geq c k_*^t(\gamma) \tag{4.3.4}$$

holds for all $\gamma \in \Gamma \setminus \Gamma_{m_0}$.

B. An operator $\mathrm{R(D)} \in \mathcal{A}^\infty(k)$ is said to be a *k-smoothing* operator, if $deg_k\mathrm{R} = -\infty$.

C. An operator $\mathrm{R(D)} \in \mathcal{A}^\infty(k)$ is said to be *finite*, if R(D)(S(G)) is finite dimensional.

D. We define

$$\mathcal{A}^{-\infty}(k) = \bigcap_{t\in\mathbb{R}} \mathcal{A}^t(k)\ .$$

Remarks.

1. An operator R(D) is finite if and only if its symbol R equals zero in the complement of a finite subset of Γ .

2. Any finite operator is k-smoothing for any k. Indeed, let $\Gamma' = \{\gamma \in \Gamma; \mathrm{R}(\gamma) \neq 0\}$ and for $t \in \mathbb{R}$ let

$$\mathrm{C}_{t,R,k} := \frac{\max_{\gamma \in \Gamma'} |\mathrm{R}(\gamma)|}{\min_{\gamma \in \Gamma'} k_*^t(\gamma)} .$$

Then we have for all $\gamma \in \Gamma$

$$|\mathrm{R}(\gamma)| \leq \mathrm{C}_{t,R,k}\, k_*^t(\gamma)$$

thus $\mathrm{R(D)} \in \mathcal{A}^t(k)$ for any $t \in \mathbb{R}$.

3. It is not true that any k-smoothing operator is finite. A counterexample is given by the operator $R : \mathbb{Z}^n \to \mathbb{C}$ where $R(a) := \exp(-|k_*(a)|)$, in [15], p.199.

Definition 4.3.4. Let $\mathrm{A(D)} \in \mathcal{A}^t(k)$, $t \in \mathbb{R}$. An operator $\mathrm{B(D)} \in \mathcal{A}^{-t}(k)$ with symbol $\mathrm{B} \in \sum^{-t}(k)$ is said to be a *k-parametrix* for A(D) if there exists a k-smoothing operator R(D) such that

$$B(D) \circ A(D) = \mathrm{id} - \mathrm{R(D)}. \tag{4.3.5}$$

The following theorem is fundamental for all that follows,

Theorem 4.3.1. An operator $\mathrm{A(D)} \in \mathcal{A}^t(k), t \in \mathbb{R}$, is k-elliptic if and only if there exists a k-parametrix $\mathrm{B(D)} \in \mathcal{A}^{-t}(k)$ such that the operator R(D) in (4.3.5) is finite.

Proof. **A.** Let A(D) be k-elliptic. Then there exists by definition $m_0 \in \mathbb{N}$ such that for all $\gamma \in \Gamma \setminus \Gamma_{m_0}$

$$|A(\gamma)|^{-1} \leq c\, k_*^{-t}(\gamma)$$

holds with some constant $c > 0$. We define on Γ the function $B : \Gamma \to \mathbb{C}$ by

$$B(\gamma) := \begin{cases} 0 & \gamma \in \Gamma_{m_0} \\ A(\gamma)^{-1} & \gamma \in \Gamma \setminus \Gamma_{m_0} \end{cases}$$

which satisfies the estimate

$$|B(\gamma)| \leq c\, k_*^{-t}(\gamma) \text{ for all } \gamma \in \Gamma.$$

Hence $B \in \sum^{-t}(k)$ and $B(D) : S(G) \to S(G)$ belongs to $\mathcal{A}^{-t}(k)$.
On the other hand, let

$$R(\gamma) := \begin{cases} 1 & \gamma \in \Gamma_{m_0} \\ 0 & \gamma \in \Gamma \setminus \Gamma_{m_0}\,. \end{cases}$$

Then $R \in \sum^{-\infty}(k) := \bigcap\limits_{t \in \mathbb{R}} \sum^t(k)$ and the operator R(D) is finite. Thus we find

$$(B(D) \circ A(D)\varphi)\hat{}\,(\gamma) = B(\gamma)A(\gamma)\hat{\varphi}(\gamma) = \begin{cases} 0 & \gamma \in \Gamma_{m_0} \\ \hat{\varphi}(\gamma) & \gamma \in \Gamma \setminus \Gamma_{m_0} \end{cases}$$

which implies

$$B(D) \circ A(D) = \mathrm{id} - R(D)$$

and B(D) is a k-parametrix of A(D).
B. Now let $B(D) \in \mathcal{A}^{-t}(k)$ be a k-parametrix of $A(D) \in \mathcal{A}^t(k)$, in particular we know that $B(D)$ has a symbol $B \in \sum^{-t}(k)$ and

$$B(D) \circ A(D) = \mathrm{id} - R(D)$$

holds with a finite operator $R(D)$. It follows that there exists a function $R : \Gamma \to \mathbb{C}$ and $m_0 \in \mathbb{N}$ such that $R(\gamma) = 0$ on $\Gamma \setminus \Gamma_{m_0}$. Hence we find $B(\gamma)A(\gamma) = 1$ for all $\gamma \in \Gamma \setminus \Gamma_{m_0}$.
Since $B \in \sum^{-t}(k)$ we get

$$|A(\gamma)| = |B(\gamma)|^{-1} \geq c\, k_*^t(\gamma)$$

for all $\gamma \in \Gamma \setminus \Gamma_{m_0}$ which implies the second part of the theorem. $\square$

Our next aim is to extend elements of $\mathcal{A}^\infty(k)$ continuously to some spaces $\mathrm{H}_k^r(\mathrm{G})$.

Proposition 4.3.1. Let $r \in \mathbb{R}$ and $k : \Gamma \to \mathbb{C}$ be a fixed function. Moreover let $A \in \sum^t(k)$. Then we have for all $\varphi \in \mathrm{S}(\mathrm{G})$

$$\|A(D)\varphi\|_{r,k} \leq c\,\|\varphi\|_{r+t,k}. \tag{4.3.6}$$

Proof. For $\varphi \in \mathrm{S}(\mathrm{G})$ we find

$$\begin{aligned} \|A(D)\varphi\|_{r,k}^2 &= \sum_{\gamma\in\Gamma} \left|[A(D)\varphi]\hat{}\,(\gamma)\right|^2 k_*^{2r}(\gamma) \\ &\leq c \sum_{\gamma\in\Gamma} k_*^{2t}(\gamma)|\hat{\varphi}(\gamma)|^2 k_*^{2r}(\gamma) \\ &= c\,\|\varphi\|_{r+t,k}^2 \,. \end{aligned}$$

Note that all sums are finite since $\varphi \in \mathrm{S}(\mathrm{G})$. □

Using Proposition 4.1.1 and Proposition 4.3.1 we can extend, for each $r \in \mathbb{R}$ any operator $A(D) : \mathrm{S}(\mathrm{G}) \to \mathrm{S}(\mathrm{G})$ with symbol $A \in \sum^t(k)$ continuously to an operator $A_r(D) : \mathrm{H}_k^r(\mathrm{G}) \to \mathrm{H}_k^{r-t}(\mathrm{G})$.

Moreover, combining Lemma 4.3.1 and Proposition 4.3.1 we find for two operators $A(D)$ and $B(D)$ with symbols $A \in \sum^t(k)$ and $B \in \sum^s(k)$, respectively, and their product $T(D) = A(D) \circ B(D)$, that

$$\|T(D)\varphi\|_{t+s+r,k} \leq c\,\|\varphi\|_{r,k} \quad , \quad r \in \mathbb{R}. \tag{4.3.7}$$

Thus, using the notation introduced above for the extended operators we have, for any $r \in \mathbb{R}$,

$$T_r(D) = A_{r-s}(D) \circ B_r(D) \tag{4.3.8}$$

where $A_{r-s}(D) : \mathrm{H}_k^{r-s}(\mathrm{G}) \to \mathrm{H}_k^{r-s-t}(\mathrm{G})$ and $B_r(D) : \mathrm{H}_k^r(\mathrm{G}) \to \mathrm{H}_k^{r-s}(\mathrm{G})$, hence $T_r(D) := \mathrm{H}_k^r(\mathrm{G}) \to \mathrm{H}_k^{r-s-t}(\mathrm{G})$.

Given $A \in \sum^t(k)$ and $f \in \mathrm{H}_k^{r-t}(\mathrm{G}), r \in \mathbb{R}$, we want to solve the equation

$$A_r(D)u = f. \tag{4.3.9}$$

Suppose that $A(D)$ is k-elliptic. Then $\Gamma' = \{\gamma \in \Gamma; A(\gamma) = 0\}$ is finite. Let $\mathcal{L}(\Gamma') = \{\varphi \in \mathrm{S(G)}; \hat{\varphi}(\gamma) = 0 \text{ for } \gamma \in \Gamma'\}$. The closure of $\mathcal{L}(\Gamma')$ in $\mathrm{H}_k^r(\mathrm{G})$ is denoted by $\mathcal{N}_r$ and $\mathcal{N}'$ is the finite dimensional subspace of $\mathrm{H}_k^r(\mathrm{G})$ defined by $\mathcal{N}' = \{\varphi \in \mathrm{S(G)}; \hat{\varphi}(\gamma) = 0 \text{ for } \gamma \in \Gamma \setminus \Gamma'\}$. Obviously $\mathcal{N}' \subset \mathrm{H}_k^r(\mathrm{G})$ for any $r \in \mathbb{R}$.

Theorem 4.3.2. Let $A \in \sum^t(k)$ and $A(D)$ be k-elliptic. Then for equation (4.3.9) Fredholm's alternative holds. ie. (4.3.9) only has a solution $u \in \mathrm{H}_k^r(\mathrm{G})$ for $f \in \mathcal{N}_{r-t}$. This solution is unique up to an arbitrary element of $\mathcal{N}'$.

Proof. A. Let $\mathcal{N}' = \{0\}$, i.e. $A(\gamma) \neq 0$ for all $\gamma \in \Gamma$. Then we consider the function u defined by

$$\hat{u}(\gamma) = \frac{\hat{f}(\gamma)}{A(\gamma)} \quad , \quad \gamma \in \Gamma .$$

We have to prove that u belongs to $\mathrm{H}_k^r(\mathrm{G})$. By the k-ellipticity of $A(D)$ it follows that $|A(\gamma)|^{-1} \leq c'\, k_*^{-t}(\gamma)$ holds for all $\gamma \in \Gamma$. Therefore we find

$$\begin{aligned} \|u\|_{r,k}^2 &= \sum_{\gamma \in \Gamma} |\hat{u}(\gamma)|^2 k_*^{2r}(\gamma) = \sum_{\gamma \in \Gamma} |\hat{f}(\gamma)|^2 |A(\gamma)|^{-2} k_*^{2r}(\gamma) \\ &\leq c'^2 \sum_{\gamma \in \Gamma} |\hat{f}(\gamma)|^2 k_*^{2(r-t)}(\gamma) = c'^2 \|f\|_{r-t,k}^2 . \end{aligned}$$

Obviously u is a solution of (4.3.9) and since now $\mathcal{N}_{r-t} = \mathrm{H}_k^{r-t}(\mathrm{G})$, this case is proved.

B. Suppose $\mathcal{N}' \neq \{0\}$, i.e. $\Gamma' \neq \phi$. In order that (4.3.9) has a solution $u \in \mathrm{H}_k^r(\mathrm{G})$, $A(\gamma)\hat{u}(\gamma) = \hat{f}(\gamma)$ must hold for all $\gamma \in \Gamma$, which implies $\hat{f}(\gamma) = 0$ for all $\gamma \in \Gamma'$, i.e. $f \in \mathcal{N}_{r-t}$.

On the other hand, for $f \in \mathcal{N}_{r-t}$, if we define

$$\hat{u}(\gamma) = \begin{cases} \frac{\hat{f}(\gamma)}{A(\gamma)} & , \gamma \in \Gamma \setminus \Gamma' \\ 0 & , \gamma \in \Gamma' \end{cases}$$

then $u \in \mathrm{H}_k^r(\mathrm{G})$ and it is a solution of (4.3.9), the first statement is proved as before. Now let $g \in \mathcal{N}'$, i.e. $\hat{g}(\gamma) = 0$ for $\gamma \in \Gamma \setminus \Gamma'$.
We find

$$[A_r(D)(u+g)]\hat{\ }(\gamma) = A(\gamma)(\hat{u}(\gamma) + \hat{g}(\gamma)) = \hat{f}(\gamma)$$

for all $\gamma \in \Gamma$.
Finally we let $A_r(D)u = f$ and $A_r(D)v = f$ for some $u, v \in \mathrm{H}_k^r(\mathrm{G})$. This implies

$$0 = [A_r(D)(u-v)]\hat{\ }(\gamma) = A(\gamma)(\hat{u}(\gamma) - \hat{v}(\gamma))$$

for all $\gamma \in \Gamma$. That means $\hat{u}(\gamma) = \hat{v}(\gamma)$ for all $\gamma \in \Gamma \setminus \Gamma'$, hence $u - v \in \mathcal{N}'$ and the second part of the theorem is proved. □

Using the notion of a k-parametrix, Theorem 4.3.2 can be formulated another way,

Theorem 4.3.3. Let $A_r(D) \in \mathcal{A}^t(k)$, $t \in \mathbb{R}$, be a k-elliptic operator and let $r \in \mathbb{R}$. Then there exists an operator $B_{r-t}(D) : \mathrm{H}_k^{r-t}(\mathrm{G}) \to \mathrm{H}_k^r(\mathrm{G})$ with symbol $B \in \sum^{-t}(k)$, such that with a finite operator $R_r(D)$,

$$B_{r-t}(D) \circ A_r(D) = \mathrm{id}_r - R_r(D) \tag{4.3.10}$$

holds, i.e. $B_{r-t}(D)$ is a k-parametrix for $A_r(D)$. Here id_r denotes the identity on $\mathrm{H}_k^r(\mathrm{G})$.

Proof. Define $B \in \sum^{-t}(k)$ by

$$B(\gamma) = \begin{cases} A(\gamma)^{-1} \quad , & \gamma \in \Gamma \setminus \Gamma' \\ 0 \quad , & \gamma \in \Gamma' \end{cases}$$

and recall that a k-elliptic operator $A(D) \in \mathcal{A}^t(k)$ always has a symbol $A \in \sum^t(k)$. □

4.4 Ellipticity and Compactness of Embeddings

The purpose of this section is to make clear the connection between k-smoothing operators, k-ellipticity and compactness of the embedding of $\mathrm{H}_k^{r+t}(\mathrm{G})$ into $\mathrm{H}_k^r(\mathrm{G})$. First we prove,

Proposition 4.4.1. Let $k : \Gamma \to \mathbb{C}$ and $r \in \mathbb{R}$. For any k-smoothing operator $R_r(D)$ we have

$$R_r(D)\, u \in \bigcap_{s\in\mathbb{R}} \mathrm{H}_k^s(\mathrm{G}) \;=: \mathrm{H}_k^\infty(\mathrm{G}) \tag{4.4.1}$$

for all $u \in \mathrm{H}_k^r(\mathrm{G})$.

Proof. We have to prove for all $s \in \mathbb{R}$ and all $u \in \mathrm{H}_k^r(\mathrm{G})$ that $\|R_r(D)u\|_{s,k}$ is finite. Now, since $R(D)$ is a k-smoothing operator it follows that

$$|R(\gamma)| \leq \; c_s\, k_*^s(\gamma)$$

for all $s \in \mathbb{R}$ and $\gamma \in \Gamma$.
This implies

$$\|R_r(D)u\|_{s,k}^2 = \sum_{\gamma\in\Gamma} |[R_r(D)u]\hat{}(\gamma)|^2 k_*^{2s}(\gamma)$$

$$= \sum_{\gamma\in\Gamma} |R(\gamma)|^2\, |\hat{u}(\gamma)|^2\, k_*^{2s}(\gamma)$$

$$\leq c_{r,s} \sum_{\gamma\in\Gamma} k_*^{2(r-s)}(\gamma)\, k_*^{2s}(\gamma)\, |\hat{u}(\gamma)|^2$$

$$= c_{r,s}\|u\|_{r,k}^2.$$

□

The following results are dependent on the finite exhaustion $(\Gamma_m)_{m\in\mathbb{N}}$ of Γ, so it is important to keep the same finite exhaustion throughout.

In the proof of Theorem 4.2.2 we already used the notion $\lim_{\gamma\to\infty} k(\gamma) = 0$ for some function $k : \Gamma \to \mathbb{C}$. By this we mean that for any $\varepsilon > 0$ there exists $m_0 \in \mathbb{N}$ such that $\gamma \in \Gamma \setminus \Gamma_{m_0}$ implies $|k(\gamma)| < \varepsilon$. Obviously $\lim_{\gamma\to\infty} k(\gamma) = \infty$ yields $\lim_{\gamma\to\infty} k^{-1}(\gamma) = 0$ analogous to Definition 4.2.2.C.

Proposition 4.4.2. Let $k : \Gamma \to \mathbb{C}$ and suppose that the embedding of $\mathrm{H}^1_k(\mathrm{G})$ into $\mathrm{L}^2(\mathrm{G})$ is compact. Then it follows for any k-smoothing operator $R(D)$ that

$$\lim_{\gamma\to\infty} |R(\gamma)| = 0. \tag{4.4.2}$$

Proof. By Theorem 4.2.2, the compactness of the embedding yields

$$\lim_{\gamma\to\infty} k_*^{-1}(\gamma) = 0.$$

Moreover $R(\gamma) \in \sum^{-1}(k)$, and the theorem follows. □

Our main result in this section is the following theorem.

Theorem 4.4.1. Let $k : \Gamma \to \mathbb{C}$ be an arbitrary function. In order that the existence of a k-parametrix $B(D) \in \mathcal{A}^{-t}(k)$ for an operator $A(D) \in \mathcal{A}^t(k)$, $t \in \mathbb{R}$, always implies that $A(D)$ is k-elliptic it is necessary and sufficient that the embedding of $\mathrm{H}^1_k(\mathrm{G})$ into $\mathrm{L}^2(\mathrm{G})$ is compact.

Proof. A. Suppose that the embedding is compact and that $A(D) \in \mathcal{A}^t(k)$ has a k-parametrix $B(D) \in \mathcal{A}^{-t}(k)$ with symbol $B \in \sum^{-t}(k)$. Then there exists an operator $R(D) \in \mathcal{A}^{-\infty}(k)$ such that

$$B(D) \circ A(D) = \mathrm{id} - R(\gamma)$$

or

$$B(\gamma)A(\gamma) = 1 - R(\gamma) \qquad \text{for all } \gamma \in \Gamma.$$

By Proposition 4.4.2 it follows using the compactness of the embedding that

$$\lim_{\gamma \to \infty} |R(\gamma)| = 0,$$

hence there exists $m_0 \in \mathbb{N}$ such that $1 - |R(\gamma)| \geq \frac{1}{2}$ for all $\gamma \in \Gamma \setminus \Gamma_{m_0}$. For these γ we find

$$|B(\gamma)|\,|A(\gamma)| = |1 - R(\gamma)| \geq \frac{1}{2}$$

and therefore

$$|A(\gamma)| \geq \frac{1}{2}\,|B(\gamma)|^{-1}\,.$$

Since $B \in \sum^{-t}(k)$ it follows that

$$|B(\gamma)| \leq c\,k_*^{-t}(\gamma) \text{ for all } \gamma \in \Gamma.$$

But for $\gamma \in \Gamma \setminus \Gamma_{m_0}$ this gives

$$|A(\gamma)| \geq \frac{1}{2c} k_*^t(\gamma)\,,$$

i.e. the k-ellipticity of $A(D)$.

B. Now suppose that the embedding is not compact. By Theorem 4.2.2 we know that there exists a sequence $(\gamma_j)_{j \in \mathbb{N}}$ with $\gamma_j \in \Gamma$, such that

$$\lim_{j \to \infty} \gamma_j = \infty \qquad \text{and} \qquad k_*(\gamma_j) \leq c.$$

The sequence $(k_*(\gamma_j))_{j \in \mathbb{N}}$ has a subsequence $(k_*(\gamma_{j'}))$ converging in $\mathbb{R}$ to some limit $c' \geq 1$; remembering that we always have $k_*(\gamma) \geq 1$. Thus for any $\varepsilon > 0$ there exists $m_0 \in \mathbb{N}$ such that for $\gamma_{j'} \in \Gamma \setminus \Gamma_{m_0}$ it follows that $|k_*(\gamma_{j'}) - c'| < \varepsilon$. Let Γ' be the set of all elements of the sequence $(\gamma_{j'})$. We define the mapping $R : \Gamma \to \mathbb{C}$ by

$$R(\gamma) = \begin{cases} 1 \;, & \gamma \in \Gamma' \\ 0 \;, & \gamma \in \Gamma \setminus \Gamma'. \end{cases}$$

Since $\lim_{\gamma_{j'} \to \infty} k(\gamma_{j'}) = c'$ it follows that there exists $m_1 \in \mathbb{N}$ such that

$$\frac{c'}{2} \leq k_*(\gamma') \leq 2c'$$

holds for all $\gamma' \in \Gamma' \setminus \Gamma_{m_1}$. For these γ' we find for $t > 0$ and $t < 0$

$$\left(\frac{2}{c'}\right)^t k_*^t(\gamma') \geq 1$$

and

$$\left(\frac{1}{2c'}\right)^t k_*^t(\gamma') \geq 1,$$

respectively. Thus for any $t \in \mathbb{R}$ and all $\gamma' \in \Gamma'$, with a suitable constant, we have

$$|R(\gamma')| \leq c_t \, k_*^t(\gamma'). \tag{4.4.3}$$

Trivially $|R(\gamma)| \leq c_t \, k_*^t(\gamma)$ holds for all $\gamma \in \Gamma \setminus \Gamma'$. Thus $R(D)$ belongs to $\mathcal{A}^{-\infty}(k)$. Let $s \in \mathbb{R}$ be fixed and consider

$$A(\gamma) := \begin{cases} 0 & , \quad \gamma \in \Gamma' \\ k_*^s(\gamma) & , \quad \gamma \in \Gamma \setminus \Gamma' . \end{cases}$$

Obviously $A(D) \in \mathcal{A}^s(k)$. Moreover, let

$$B(\gamma) := \begin{cases} 0 & , \quad \gamma \in \Gamma' \\ k_*^{-s}(\gamma) & , \quad \gamma \in \Gamma \setminus \Gamma' , \end{cases}$$

which yields $B \in \mathcal{A}^{-s}(k)$. Thus for all $\gamma \in \Gamma$ we find

$$B(\gamma)A(\gamma) = 1 - R(\gamma)$$

or

$$B(D) \circ A(D) = \mathrm{id} - R(D).$$

Hence $A \in \mathcal{A}^s(k)$ has a k-parametrix $B(D) \in \mathcal{A}^{-s}(k)$ but by construction $A(D)$ is not k-elliptic. This proves the theorem. $\square$

4.5 Operators with Variable Coefficients

We start with

Definition 4.5.1. For any $t \in \mathbb{R}$, we denote by $\sum_x^t(k)$ the set of all continuous functions $A : G \times \Gamma \to \mathbb{C}$ such that with some positive constant $c_{t,A}$,

$$|A(x,\gamma)| \leq c_{t,A}\, k_*^t(\gamma) \tag{4.5.1}$$

holds for all $\gamma \in \Gamma$ and all $x \in$ G. As before, $\sum_x^\infty(k) := \bigcup_{t\in\mathbb{R}} \sum_x^t(k)$, and symbols in $\sum_x^t(k)$ have k-degree less than or equal to t.

Elements in $\sum_x^t(k)$ are called *symbols*. If $A \in \sum_x^t(k)$ such that for every $x \in G$ the mapping $A(x,\cdot) : \Gamma \to \mathbb{C}$, $\gamma \mapsto A(x,\gamma)$, is negative definite we call A a *negative definite symbol.*

Definition 4.5.2. Each $A \in \sum_x^\infty(k)$ defines a *pseudodifferential operator with variable coefficients,* $A(\cdot, D) : S(G) \to C(G)$, by

$$A(x,D)u(x) = \sum_{\gamma\in\Gamma} A(x,\gamma)\hat{u}(\gamma)\gamma(x). \tag{4.5.2}$$

Definition 4.5.3. By $\mathcal{A}_x^t(k)$ we denote the set of all pseudodifferential operators with variable coefficients which are generated by a symbol $A \in \sum_x^t(k)$.

Definition 4.5.4. We denote the *Fourier Transform of a symbol*

$$\begin{aligned} A \ :\ & \mathrm{G} \times \Gamma \to \mathbb{C} \\ & (x,\gamma) \ \mapsto A(x,\gamma) \end{aligned}$$

with respect to x by

$$\hat{A}(\eta,\gamma) = \int_G A(x,\gamma)\overline{\eta(x)}\mathrm{d}\mu_G(x) \tag{4.5.3}$$

where $\eta, \gamma \in \Gamma$ and μ_G is the Haar measure on G. Note that $x \mapsto A(x,\gamma)$ is a continuous, hence integrable function and for $\gamma \in \Gamma$ fixed, the integral in (4.5.3) is well defined.

For $\gamma \in \Gamma$ fixed, as with the translation invariant case, we have in $L^2(G)$ the representation

$$A(x,\gamma) = \sum_{\eta\in\Gamma} \hat{A}(\eta,\gamma)\eta(x) \ . \tag{4.5.4}$$

However one must take care when applying the Fourier transform as unlike the translation invariant case where we had equation (4.3.3), we now have the following result:

Lemma 4.5.1. Let $A(\cdot\,, D) : S(G) \to C(G)$ with continuous symbol $A : G \times \Gamma \to \mathbb{C}$. Then for $u \in S(G)$ it follows that

$$\big[A(\cdot\,, D)u\big]\,\hat{}\,(\gamma) = \sum_{\xi\in\Gamma} \hat{A}(\gamma-\xi,\xi)\hat{u}(\xi). \tag{4.5.5}$$

Proof. By definition,

$$A(x,D)u(x) = \sum_{\xi\in\Gamma} A(x,\xi)\hat{u}(\xi)\xi(x) \ .$$

Therefore

$$\begin{aligned} \big[A(\cdot\,, D)u(\cdot)\big]\,\hat{}\,(\gamma) &= \int_G \overline{\gamma(x)} \left(\sum_{\xi\in\Gamma} A(x,\xi)\hat{u}(\xi)\xi(x) \right) \mathrm{d}\mu_G(x) \\ &= \sum_{\xi\in\Gamma} \int_G [\xi-\gamma](x) A(x,\xi)\hat{u}(\xi)\mathrm{d}\mu_G(x) \\ &= \sum_{\xi\in\Gamma} \int_G \overline{[\gamma-\xi]}(x) A(x,\xi)\hat{u}(\xi)\mathrm{d}\mu_G(x) \\ &= \sum_{\xi\in\Gamma} \hat{A}(\gamma-\xi,\xi)\hat{u}(\xi), \end{aligned}$$

where we used that for $u \in S(G)$ all sums are finite. $\square$

Theorem 4.5.1. Let A(x,D) have a continuous symbol $A : G \times \Gamma \to \mathbb{C}$, and let $k : \Gamma \to \mathbb{R}$.

If

$$|\hat{A}(\zeta,\gamma)| \le c_1 g(\zeta) k_*^s(\gamma)\,, \tag{4.5.6}$$

where $g \in \mathrm{L}^1(\Gamma)$, i.e. $\sum\limits_{\gamma\in\Gamma} g(\gamma) < \infty$, and $\hat{A}$ is the Fourier transform of the symbol A with respect to the first variable, then the pseudodifferential operator A(x,D) maps $\mathrm{H}_k^s(\mathrm{G})$ into $\mathrm{L}^2(\mathrm{G})$, and furthermore we have

$$\|A(x,D)u\|_{\mathrm{L}^2(\mathrm{G})} \le \tilde{c}\,\|u\|_{\mathrm{H}_k^s(\mathrm{G})}. \tag{4.5.7}$$

Proof. Consider a trigonometric polynomial $u \in \mathrm{S}(\mathrm{G})$

$$\begin{aligned} A(x,D)u(x) &= \sum_{\gamma\in\Gamma} A(x,\gamma)\hat{u}(\gamma)\gamma(x) \\ &= \sum_{\gamma\in\Gamma}\left(\sum_{\xi\in\Gamma}\hat{A}(\xi,\gamma)\xi(x)\right)\hat{u}(\gamma)\gamma(x) \\ &= \sum_{\gamma\in\Gamma}\sum_{\xi\in\Gamma}\hat{A}(\xi,\gamma)\hat{u}(\gamma)[\xi+\gamma](x)\,. \end{aligned}$$

Using the substitution $\xi+\gamma=\eta$ we find

$$\begin{aligned} A(x,D)u(x) &= \sum_{\gamma\in\Gamma}\sum_{\eta\in\Gamma}\hat{A}(\eta-\gamma,\gamma)\hat{u}(\gamma)\eta(x) \\ &= \sum_{\eta\in\Gamma}\sum_{\gamma\in\Gamma}\hat{A}(\eta-\gamma,\gamma)\hat{u}(\gamma)\eta(x)\,. \end{aligned}$$

Therefore

$$\begin{aligned} \|A(x,D)u\|^2_{\mathrm{L}^2(\mathrm{G})} &= \int_G \left|\sum_{\eta\in\Gamma}\sum_{\gamma\in\Gamma}\hat{A}(\eta-\gamma,\gamma)\hat{u}(\gamma)\eta(x)\right|^2 \mathrm{d}\mu_G(x) \\ &= \sum_{\eta\in\Gamma}\left|\sum_{\gamma\in\Gamma}\hat{A}(\eta-\gamma,\gamma)\hat{u}(\gamma)\right|^2 \\ &\le c_1^2\sum_{\eta\in\Gamma}\left(\sum_{\gamma\in\Gamma} g(\eta-\gamma)\,k_*^s(\gamma)\,|\hat{u}(\gamma)|\right)^2 \\ &= c_1^2\sum_{\eta\in\Gamma}\left(g*\{k_*^s\,|\hat{u}|\}\right)^2(\eta)\,. \end{aligned}$$

Then by Young's Inequality:

$$\begin{aligned}
\|A(x,D)u\|^2_{\mathrm{L}^2(\mathrm{G})} &\leq c_1^2\left(\sum_{\eta\in\Gamma} g(\eta)\right)^2\left(\sum_{\gamma\in\Gamma} k_*^{2s}(\gamma)|\hat{u}(\gamma)|^2\right)^2 \\
&= c_1^2 \quad \|g\|^2_{\mathrm{L}^1(\Gamma)} \quad \|u\|^2_{\mathrm{H}^s_k(\mathrm{G})} \\
&= \tilde{c}\,\|u\|^2_{\mathrm{H}^s_k(\mathrm{G})}.
\end{aligned}$$

Since $\mathrm{H}^r_k(\mathrm{G})$ is the completion of $\mathrm{S}(\mathrm{G})$ and $A(x,D)$ is linear, we can continuously extend $A(x,D)$ to $\mathrm{H}^r_k(\mathrm{G})$ and the theorem is proved. □

Chapter 5

Estimates for Some Operators

5.1 Ellipticity and Lower Bounds for Translation Invariant Pseudodifferential Operators

Let G, Γ and $(\Gamma_m)_{m\in\mathbb{N}}$ be as before. We want to characterise elliptic operators by a certain inequality of Gårding type.

Theorem 5.1.1. Let $r \in \mathbb{R}$. For any k-elliptic operator $A(D) \in \mathcal{A}^t(k)$ there exists two constants $c_0 > 0$ and $c_1 \geq 0$ such that

$$\|A(D)\varphi\|_{r,k}^2 \geq c_0 \|\varphi\|_{r+t,k}^2 - c_1 \|\varphi\|_{r,k}^2 \tag{5.1.1}$$

holds for all $\varphi \in \mathrm{S(G)}$.

Proof. Let $\varphi \in \mathrm{S(G)}$ and $r \in \mathbb{R}$. Using the k-ellipticity of $A(D)$ we find that there exists $m_0 \in \mathbb{N}$ such that

$$\begin{aligned}
\sum_{\gamma\in\Gamma\setminus\Gamma_{m_0}} |A(\gamma)|^2\,|\hat{\varphi}(\gamma)|^2\,k_*^{2r}(\gamma) &\geq c \sum_{\gamma\in\Gamma\setminus\Gamma_{m_0}} |\hat{\varphi}(\gamma)|^2\,k_*^{2r+2t}(\gamma) \\
&= c\sum_{\gamma\in\Gamma}|\hat{\varphi}(\gamma)|^2\,k_*^{2r+2t}(\gamma) - c\sum_{\gamma\in\Gamma_{m_0}}|\hat{\varphi}(\gamma)|^2\,k_*^{2r+2t}(\gamma) \\
&= c\,\|\varphi\|^2_{r+t,k} - c\sum_{\gamma\in\Gamma_{m_0}} |\hat{\varphi}(\gamma)|^2\,k_*^{2r+2t}(\gamma)\;,
\end{aligned}$$

thus

$$\|A(D)\varphi\|^2_{r,k} \geq c\|\varphi\|^2_{r+t,k} + \sum_{\gamma\in\Gamma_{m_0}} \left(|A(\gamma)|^2 - c\,k_*^{2t}(\gamma)\right) k_*^{2r}(\gamma)\,|\hat{\varphi}(\gamma)|^2\,. \quad (5.1.2)$$

Taking $c_0 = c$ and $c_1 = \max_{\gamma\in\Gamma_{m_0}} \left||A(\gamma)|^2 - c\,k_*^{2t}(\gamma)\right|$, the theorem follows. □

Corollary 5.1.1. Let $r, s \in \mathbb{R}$ and $A(D) \in \mathcal{A}^t(k)$ be a k-elliptic operator. Then there exist constants $c_0 > 0$ and $c_{s,t} \geq 0$ such that

$$\|A(D)\varphi\|^2_{r,k} \geq c_0\,\|\varphi\|^2_{r+t,k} - c_{s,t}\,\|\varphi\|^2_{s,k} \quad (5.1.3)$$

holds for all $\varphi \in \mathrm{S(G)}$.

Proof. By (5.1.2) we find

$$\begin{aligned}
\|A(D)\varphi\|^2_{r,k} &\geq c\,\|\varphi\|^2_{r+t,k} + \sum_{\gamma\in\Gamma_{m_0}} \left(|A(\gamma)|^2 - c\,k_*^{2t}(\gamma)\right) |\hat{\varphi}(\gamma)|^2\,k_*^{2r}(\gamma) \\
&\geq c\,\|\varphi\|^2_{r+t,k} + \sum_{\gamma\in\Gamma_{m_0}} \frac{|A(\gamma)|^2 - c\,k_*^{2t}(\gamma)}{k_*^{2s-2t}(\gamma)}\,|\hat{\varphi}(\gamma)|^2\,k_*^{2s}(\gamma).
\end{aligned}$$

Taking $c_0 = c$ and $c_{s,t} = \max_{\gamma\in\Gamma_{m_0}} \frac{\left||A(\gamma)|^2 - c_0\,k_*^{2t}(\gamma)\right|}{k_*^{2s-2t}(\gamma)}$, we get the desired result. □

For Corollary 5.1.1 we can prove the following converse:

Theorem 5.1.2. Suppose that $\mathrm{H}^1_k(\mathrm{G})$ is compactly embedded into $\mathrm{L}^2(\mathrm{G})$ and that $A(D) \in \mathcal{A}^t(k)$ fulfills (5.1.3). Then the operator $A(D)$ is k-elliptic.

Proof. Taking $\varphi = \gamma$, equation (5.1.3) yields

$$|A(\gamma)|^2 \, k_*^{2r}(\gamma) \geq c_0 \, k_*^{2r+2t}(\gamma) - c_1 \, k_*^{2s}(\gamma)$$

and therefore

$$|A(\gamma)|^2 \geq c_0 \, k_*^{2t}(\gamma) - c_1 \, k_*^{2s-2r}(\gamma) \tag{5.1.4}$$

for all $\gamma \in \Gamma$. Since the embedding is compact, by Theorem 4.2.2 we find that $\lim_{\gamma \to \infty} k_*(\gamma) = \infty$. Thus there exists $m_0 \in \mathbb{N}$ such that $\gamma \in \Gamma \setminus \Gamma_{m_0}$ implies

$$c_1 \, k_*^{2s-2r}(\gamma) \leq \frac{c_0}{2} k_*^{2t}(\gamma) \; ,$$

provided that $s - r < 0$. Hence we find for $\gamma \in \Gamma \setminus \Gamma_{m_0}$

$$|A(\gamma)|^2 \geq \frac{c_0}{2} k_*^{2t}(\gamma) \; ,$$

which implies the k-ellipticity of $A(D)$. □

5.2 Fundamental Solutions and Potentials

Let G be a compact Abelian group with discrete dual group and let $(\Gamma_m)_{m \in \mathbb{N}}$ be a finite exhaustion of Γ. In the following we suppose that Condition 4.2.1 holds, i.e. for a fixed function, $k : \Gamma \to \mathbb{C}$, there exists a real number $t_0 > 0$ such that

$$\sum_{\gamma \in \Gamma} k_*^{-2t_0}(\gamma) < \infty \tag{5.2.1}$$

holds. By this we mean that the sequence $\left(\sum_{\gamma \in \Gamma_m} k_*^{-2t_0}(\gamma) \right)_{m \in \mathbb{N}}$ converges in $\mathbb{R}$.

In Section 4.2 we have proved that (5.2.1) implies the compactness of the embedding of $\mathrm{H}_k^r(\mathrm{G})$ into $\mathrm{H}_k^s(\mathrm{G})$ for $r > s$. We denote by $\mathcal{M}(\mathrm{G})$ the set of all (complex valued) regular Borel, i.e. Baire, measures on G.

Proposition 5.2.1. Let k be as above and $\mu \in \mathcal{M}(\mathrm{G})$. Then it follows that $\mu \in \mathrm{H}_k^{-t_0}(\mathrm{G})$, where t_0 is the real number in (5.2.1).

Proof. We have to show that

$$\sum_{\gamma \in \Gamma} |\hat{\mu}(\gamma)|^2 k_*^{-2t_0}(\gamma)$$

is finite. By (5.2.1) it is sufficient to prove $|\hat{\mu}(\gamma)| \leq$ constant for all $\gamma \in \Gamma$. But for any positive Borel measure we find

$$|\hat{\mu}(\gamma)| = \left| \int_G \gamma(-x) \mathrm{d}\mu(x) \right| \leq \int_G \mathrm{d}\mu(x) = \mu(\mathrm{G}) < \infty \, ,$$

since G is a compact group. Here we follow the work of H. Bauer [3]. Given an arbitrary complex-valued measure μ it follows that

$$|\mu| \leq |\mu_+^r| + |\mu_-^r| + |\mu_+^i| + |\mu_-^i| \, ,$$

where μ_+^r, μ_-^r, μ_+^i, and μ_-^i denote the positive and negative parts of the real and imaginary part of μ, respectively. Thus, taking
$\widetilde{\mu} = |\mu_+^r| + |\mu_-^r| + |\mu_+^i| + |\mu_-^i|$ we get

$$\sum_{\gamma \in \Gamma} |\hat{\mu}(\gamma)|^2 k_*^{-2t_0}(\gamma) \leq \widetilde{\mu}(\mathrm{G})^2 \sum_{\gamma \in \Gamma} k_*^{-2t_0}(\gamma) < \infty \, .$$

□

We know that $\mathrm{L}^2(\mathrm{G}) \subset \mathrm{L}^1(\mathrm{G})$, hence elements of $\mathrm{L}^2(\mathrm{G})$ may be interpreted as complex valued measures.

Remark. Let $\mathrm{T} := \left\{ t_0 \in \mathbb{R}; \sum_{\gamma \in \Gamma} k_*^{-2t_0}(\gamma) < \infty \right\}$ and set $t' := \inf\{t_0; t_0 \in \mathrm{T}\}$. It follows that $\mathcal{M}(\mathrm{G}) \subset \mathrm{H}_k^{-t'-\epsilon}(\mathrm{G})$ for any $\epsilon > 0$. But it is more convenient for us to consider a fixed space $\mathrm{H}_k^{-t_0}(\mathrm{G})$ for some $t_0 \in \mathrm{T}$.

Definition 5.2.1. By $\delta \in \mathrm{H}_k^{-t_0}(\mathrm{G})$, with t_0 as in equation (5.2.1), we denote the measure given by $\hat{\delta}(\gamma) = 1$ for all $\gamma \in \Gamma$.

Proposition 5.2.2. For any $u \in \mathrm{H}_k^{t_0}(\mathrm{G})$ we have

$$\int_G u(x)\delta(\mathrm{d}x) = (u,\delta)_0 = u(0)\ .$$

Proof. Note that by Proposition 4.1.2, $\mathrm{H}_k^{-t_0}(\mathrm{G})$ and $\mathrm{H}_k^{t_0}(\mathrm{G})$ are in duality with respect to the scalar product in $\mathrm{L}^2(\mathrm{G})$. Moreover as we have proved in Theorem 4.2.1, $\mathrm{H}_k^{t_0}(\mathrm{G})$ is continuously embedded into $\mathrm{C}(\mathrm{G})$, the space of all continuous functions on G.
Since $\gamma(0) = 1$ for any $\gamma \in \Gamma$, we find

$$\begin{aligned} \textstyle\int_G u(x)\delta(\mathrm{d}x) = (u,\delta)_0 &= \sum_{\gamma\in\Gamma} \hat{u}(\gamma)\hat{\delta}(\gamma) \\ &= \sum_{\gamma\in\Gamma} \hat{u}(\gamma)\gamma(0) = u(0)\ . \end{aligned}$$

□

Definition 5.2.2. Suppose that k satisfies (5.2.1) and let $A \in \sum^r(k)$, $r \in \mathbb{R}$. We call $g_A \in \mathrm{H}_k^{r-t_0}(\mathrm{G})$ a *fundamental solution* of the operator $A(D)$ if

$$A_{r-t_0}(D)g_A = \delta \tag{5.2.2}$$

holds in $\mathrm{H}_k^{-t_0}(\mathrm{G})$.

Proposition 5.2.3. Let $A \in \sum^r(k)$, k satisfying (5.2.1), be a k-elliptic operator such that $A(\gamma) \neq 0$ for all $\gamma \in \Gamma$. Then there exists a fundamental solution of $A(D)$.

Proof. Equation (5.2.2) is equivalent to

$$A(\gamma)\hat{g}_A(\gamma) = 1 \text{ for all } \gamma \in \Gamma,$$

hence for a solution to exist we must have $A(\gamma) \neq 0$ for all $\gamma \in \Gamma$.
Under this condition we find

$$\hat{g}_A(\gamma) = A(\gamma)^{-1}$$

and the k-ellipticity of $A(D)$ yields

$$\sum_{\gamma\in\Gamma} |\hat{g}_A(\gamma)|^2 k_*^{2r-2t_0}(\gamma) \le c \sum_{\gamma\in\Gamma} k_*^{-2r}(\gamma) k_*^{2r-2t_0}(\gamma)$$

$$= c \sum_{\gamma\in\Gamma} k_*^{-2t_0}(\gamma) < \infty\,,$$

which implies $g_A \in \mathrm{H}_k^{r-t_0}(\mathrm{G})$ and moreover $\hat{g}_A \in \sum^{-r}(k)$. $\square$

Suppose now that $A(D)$ has a real symbol $A \in \sum^r(k)$ and that

$$A(\gamma) \ge c\, k_*^r(\gamma) \tag{5.2.3}$$

holds for all $\gamma \in \Gamma \setminus \Gamma_{m_0}$ for some $m_0 \in \mathbb{N}$, i.e. $A(D)$ is k-elliptic.
Then there exists a constant $d \ge 0$ such that $A(\gamma) + d \neq 0$ holds for all $\gamma \in \Gamma$. In that case, Proposition 5.2.3 implies the existence of a fundamental solution g_{A+d} of the operator $A(D) + d$.
Using the notion of a fundamental solution and the remark made above, we can give another formulation of Theorem 4.3.1,

Theorem 5.2.1. Let k be as in (5.2.1), $A(D)$ a k-elliptic operator with symbol $A \in \sum^r(k)$ then there exists a complex number d such that $A(\gamma)+d \neq 0$ holds for all $\gamma \in \Gamma$.
Moreover let $f \in \mathrm{L}^2(\mathrm{G})$. For any solution of the equation

$$A_r(D)u = f$$

we have the representation

$$u(x) - Q(D)u(x) = g_{A+d}(D)f(x)\,, \tag{5.2.4}$$

where $Q(D)$ is defined by the function

$$Q(.) = d(A(.) + d)^{-1} \in \sum^{-r}(k). \tag{5.2.5}$$

Proof. Since $A(D)$ is k-elliptic there is a bound $c > 0$, s.t. $|A(\gamma)| > c$ up to a finite number of γ's. Therefore there exists a complex number d such that $A(\gamma) + d \neq 0$ holds for all $\gamma \in \Gamma$. Since $A(\gamma) + d \neq 0$ we have

$$(A(\gamma) + d)\, \hat{u}(\gamma) = \hat{f}(\gamma) + d\, \hat{u}(\gamma),$$

hence

$$\hat{u}(\gamma) = \big(A(\gamma) + d\big)^{-1} \hat{f}(\gamma) + d \big(A(\gamma) + d\big)^{-1} \hat{u}(\gamma)$$

or

$$\hat{u}(\gamma) - d\big(A(\gamma) + d\big)^{-1} \hat{u}(\gamma) \;=\; \hat{g}_{A+d}(\gamma)\hat{f}(\gamma).$$

It remains to prove that Q is an element of $\sum^{-r}(k)$.
But as shown in Section 4.2, $\lim\limits_{\gamma\to\infty} k_*(\gamma) = \infty$, hence $\lim\limits_{\gamma\to\infty} d\, k_*^{-1}(\gamma) = 0$, and the k-ellipticity of A yields $|A(\gamma)| \geq \; c\, k_*^r(\gamma)$ for all $\gamma \in \Gamma \setminus \Gamma_{m_0}$ for some $m_0 \in \mathbb{N}$. This implies $|Q(\gamma)| \leq \; c\, k_*^{-r}(\gamma)$ for all $\gamma \in \Gamma$. □

Remark. In Section 4.2 we have given a reasonable definition of $\lim\limits_{\gamma\to\infty} k_*(\gamma) = \infty$ using the finite exhaustion $(\Gamma_m)_{m\in\mathbb{N}}$ of Γ.
Specifically that $\lim\limits_{\gamma\to\infty} k_*(\gamma) = \infty$ if and only if, for any $\mathrm{N} \in \mathbb{N}$, there exists an $m_0 \in \mathbb{N}$ such that $\gamma \in \Gamma \setminus \Gamma_{m_0}$ implies $k_*(\gamma) \geq \mathrm{N}$.

Now let $A(D)$ be a k-elliptic operator with symbol $A \in \sum^r(k)$ and suppose that for some $d \in \mathbb{C}$, the fundamental solution $g_{A+d} \in \mathrm{H}_k^{r-t_0}(\mathrm{G})$ exists. Here t_0 is the real number appearing in (5.2.1) which is assumed to be satisfied by k.

Definition 5.2.3. Let $\mu \in \mathcal{M}(\mathrm{G}) \subset \mathrm{H}_k^{-t_0}(\mathrm{G})$ and $g_{A+d} \in \mathrm{H}_k^{r-t_0}(\mathrm{G})$ be defined as above and let $r - t_0 \geq 0$. Then the *d-potential*, U^{μ}_{A+d} of μ with respect to $A(D)$, is defined as

$$\mathrm{U}^{\mu}_{A+d}(x) = \int_G g_{A+d}(x-y)\mathrm{d}\mu(y) \tag{5.2.6}$$

Under the assumptions of Definition 5.2.3, we have by a formal calculation

$$\begin{aligned}
\mathrm{U}^{\mu}_{A+d}(x) &= \sum_{\gamma\in\Gamma} \left[\mathrm{U}^{\mu}_{A+d}\right]^{\frown}(\gamma)\gamma(x) \\
&= \sum_{\gamma\in\Gamma} \hat{g}_{A+d}(\gamma)\hat{\mu}(\gamma)\gamma(x) \\
&= \sum_{\gamma\in\Gamma} \tfrac{1}{A(\gamma)+d}\hat{\mu}(\gamma)\gamma(x)\ .
\end{aligned}$$

Using Proposition 5.2.1 we get further

$$\begin{aligned}
\left\|\mathrm{U}^{\mu}_{A+d}\right\|^2_{r-t_0,k} &= \sum_{\gamma\in\Gamma} \left|\hat{g}_{A+d}(\gamma)\right|^2 \left|\hat{\mu}(\gamma)\right|^2 k_*^{2r-2t_0}(\gamma) \\
&\leq \left(|\mu|(\mathrm{G})\right)^2 \sum_{\gamma\in\Gamma} \left|\hat{g}_{A+d}(\gamma)\right|^2 k_*^{2r-2t_0}(\gamma) \\
&\leq c\left\|\hat{g}_{A+d}(\gamma)\right\|^2_{r-t_0,k}
\end{aligned}$$

and thus we have proved

Corollary 5.2.1. Let $A(D)$ and d be as in Definition 5.2.3.
Then U^{μ}_{A+d} belongs to $\mathrm{H}^{r-t_0}_k(\mathrm{G})$ for any $\mu \in \mathcal{M}(\mathrm{G})$.

By an easy calculation we find the following global regularity result:

Corollary 5.2.2. Let $A(D)$ and d be as in Definition 5.2.3. Furthermore let $\mu \in \mathcal{M}(\mathrm{G}) \cap \mathrm{H}^s_k(\mathrm{G})$, $s \geq t_0$. Then we have $\mathrm{U}^{\mu}_{A+d} \in \mathrm{H}^{s+r-t_0}_k(\mathrm{G})$.

Finally in this section, we introduce the notion of the energy of a measure.

Definition 5.2.4. Let $A(D)$ and d be as above and $\mu \in \mathcal{M}(\mathrm{G})$. The *d-energy* $\varepsilon_{A+d}(\mu)$ of μ with respect to $A(D)$ is defined by

$$\varepsilon_{A+d}(\mu) := \int_G \mathrm{U}^{\mu}_{A+d}(x)\,\mathrm{d}\mu(x), \tag{5.2.7}$$

provided the integral is finite.

Lemma 5.2.1. Let $\mu \in \mathcal{M}(\mathrm{G})$ have a density $v_\mu \in \mathrm{H}_k^{r-t_0}(\mathrm{G})$ with respect to the Haar measure μ_G. Then $\varepsilon_{A+d}(\mu)$ is finite.

Proof. By (5.2.7) we have

$$\varepsilon_{A+d}(\mu) = \int_G \mathrm{U}^\mu_{A+d}(x)\, v_\mu(x)\, \mathrm{d}\mu_G(x)$$

and Proposition 4.1.4 implies

$$\int_G \left|\mathrm{U}^\mu_{A+d}(x)\, v_\mu(x)\right|\, \mathrm{d}\mu_G(x) \leq \left\|\mathrm{U}^\mu_{A+d}\right\|_{t-t_0,k} \|v_\mu\|_{t_0-r,k}\,.$$

$\square$

Now we can prove

Theorem 5.2.2. Suppose that k satisfies (5.2.1) and let $A(D)$ be a k-elliptic operator with real symbol $A \in \sum^r(k)$. Furthermore choose $d \in \mathbb{R}$ such that $A(\gamma)+d \geq 0$ holds for all $\gamma \in \Gamma$ and let $\mu \in \mathcal{M}(\mathrm{G})$ have a real-valued density $v_\mu \in \mathrm{H}_k^{r-t_0}(\mathrm{G})$. Then we have

$$\varepsilon_{A+d}(\mu) = \int_G \left|(A(D)+d)^{\frac{1}{2}}\, \mathrm{U}^\mu_{A+d}(x)\right|^2 \mathrm{d}\mu_G(x)\,. \tag{5.2.8}$$

Proof. First observe that

$$\mathrm{U}^\mu_{A+d}(x) = (g_{A+d} * \mu)(x) = (g_{A+d} * v_\mu)(x).$$

Then it follows that

$$\begin{aligned}
\int_G \mathrm{U}^\mu_{A+d}(x)\, v_\mu(x)\, \mathrm{d}\mu_G(x) &= \sum_{\gamma\in\Gamma} \left[\mathrm{U}^\mu_{A+d}\right]^\frown(\gamma)\ \hat{\mu}(\gamma) \\
&= \sum_{\gamma\in\Gamma} \left(A(\gamma)+d\right)^{-1} \hat{\mu}(\gamma)\, \hat{\mu}(\gamma) \\
&= \sum_{\gamma\in\Gamma} \left(A(\gamma)+d\right) \frac{\hat{\mu}(\gamma)}{A(\gamma)+d}\, \frac{\hat{\mu}(\gamma)}{A(\gamma)+d}
\end{aligned}$$

$$= \sum_{\gamma\in\Gamma} \left| (A(\gamma)+d)^{\frac{1}{2}} \frac{\hat{\mu}(\gamma)}{A(\gamma)+d} \right|^2$$

$$= \int_G \left| (A(\gamma)+d)^{\frac{1}{2}} \ \mathrm{U}^{\mu}_{A+d}(x) \right|^2 \mathrm{d}\mu_G(x) \,.$$

□

Example 5.2.1. Let $\mathrm{G} = \mathbb{T}^n$ be the n-dimensional torus and $A(k) = |k|^2$, $k \in \mathbb{Z}^n$. In this case (5.2.8) gives, after some calculations, the well known result, $(d > 0)$

$$\begin{aligned} \varepsilon_{-\Delta+d}(\mu) &= c \int_0^{2\pi} \cdots \int_0^{2\pi} \left| \sum_{j=1}^{n} \frac{\partial}{\partial\theta_j} \mathrm{U}^{\mu}_{-\Delta+d}(\theta_1,\ldots,\theta_n) \right| \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_n \\ &\quad + c\, d \int_0^{2\pi} \cdots \int_0^{2\pi} \left| \sum_{j=1}^{n} \mathrm{U}^{\mu}_{-\Delta+d}(\theta_1,\ldots,\theta_n) \right|^2 \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_n \end{aligned} \tag{5.2.9}$$

5.3 Translation Invariant Dirichlet Forms

Let G be a compact Abelian group with discrete dual group and let $(\Gamma_m)_{m\in\mathrm{N}}$ be a fixed finite exhaustion. By $\mathrm{CN}(\Gamma)$ we denote the set of all (continuous) negative definite functions $\psi : \Gamma \to \mathbb{C}$ as in definition 3.2.3.
In the following we often need

Condition 5.3.1. The negative definite functions under consideration are assumed to be real valued. Moreover, we assume $\{\gamma \in \Gamma,\, \psi(\gamma) = \psi(0)\} = \{0\}$. Finally, it is assumed that there exists $\mathrm{N}_0 \in \mathbb{N}$ and a constant $c_0 > 0$ such that $\gamma \in \Gamma \setminus \Gamma_{\mathrm{N}_0}$ implies $\psi(\gamma) \geq c_0$.

Lemma 5.3.1. Suppose that $\psi \in \mathrm{CN}(\Gamma)$ and that Condition 5.3.1 holds. Then it follows with some constant $c > 0$ that

$$\psi(\gamma) \geq c\,\psi_*(\gamma) \tag{5.3.1}$$

holds for all $\gamma \in \Gamma \setminus \Gamma_{N_0}$.

Proof. We have to prove

$$\psi(\gamma) \geq c\,(1+\psi(\gamma)^2)^{\frac{1}{2}}$$

for all $\gamma \in \Gamma \setminus \Gamma_{N_0}$.
By our assumptions we have $\psi(\gamma)^2 \geq c_0^2$ for all $\gamma \in \Gamma \setminus \Gamma_{N_0}$ and taking $c = c_0(1+c_0^2)^{-\frac{1}{2}}$, i.e. that $c < 1$ and $c_0 = c\,(1-c^2)^{-\frac{1}{2}}$, we find

$$\psi(\gamma)^2(1-c^2) \geq c^2$$

or

$$\psi(\gamma)^2 \geq c^2(1+\psi(\gamma)^2).$$

□

By Lemma 5.3.1 it seems to be reasonable to consider operators with symbols $\psi \in \sum^1(\psi)$, where ψ is a negative definite function satisfying Condition 5.3.1. But it turns out that it is more convenient to take $\psi \in \sum^2(\psi^{\frac{1}{2}})$, where by Proposition 3.2.2 it follows that for a real valued function ψ the function $\psi^{\frac{1}{2}}$ is well defined.

Proposition 5.3.1. Suppose that ψ satisfies Condition 5.3.1. Then $\psi \in \sum^2(\psi^{\frac{1}{2}})$ and $\psi(D)$ is $\psi^{\frac{1}{2}}$-elliptic.

Proof. Obviously we have

$$|\psi(\gamma)| \leq \left(\psi_*^{\frac{1}{2}}(\gamma)\right)^2.$$

On the other hand, since $\psi(\gamma) \geq c_0$ for $\gamma \in \Gamma \setminus \Gamma_{N_0}$, we find for these γ's

$$\psi(\gamma) \geq c\left((1+\psi^{\frac{1}{2}}(\gamma)^2)^{\frac{1}{2}}\right)^2$$

$$= c\left(1+\psi(\gamma)\right),$$

where

$$c = c_0(1+c_0)^{-1}.$$

$\square$

In order to construct a fundamental solution for the operator $\psi(D)$, $\psi \in \sum^2(\psi^{\frac{1}{2}})$, we assume (5.2.1), i.e. it is supposed with some $t_0 > 0$ that

$$\sum_{\gamma\in\Gamma} \left(\psi_*^{\frac{1}{2}}(\gamma)\right)^{-2t_0} < \infty \tag{5.3.2}$$

holds. From this, Condition 5.3.1 and Proposition 3.2.2 we find that for all $d \in \mathbb{R}$, $d > 0$, it follows that $\psi(\gamma) + d \neq 0$.
Thus, by Proposition 5.2.3 we get

Theorem 5.3.1. Let $\psi \in \mathrm{CN}(\Gamma)$ satisfy Condition 5.3.1 and (5.3.2). Then for any $d \in \mathbb{R}$, $d > 0$, there exists a fundamental solution $g_{\psi+d} \in \mathrm{H}^{2-t_0}_{\psi^{\frac{1}{2}}}(\mathrm{G})$ for the operator $\psi(D) + d$. Moreover, if $\psi(0) \neq 0$, and so $\psi(\gamma) > 0$ for all $\gamma \in \Gamma$, then there also exists a fundamental solution $g_\psi \in \mathrm{H}^{2-t_0}_{\psi^{\frac{1}{2}}}(\mathrm{G})$ for the operator $\psi(D)$.

By Proposition 5.2.1 we have $\mathcal{M}(\mathrm{G}) \subset \mathrm{H}^{-t_0}_{\psi^{\frac{1}{2}}}(\mathrm{G})$ and for this reason d-potentials may be defined, thus we get

Corollary 5.3.1. Suppose $\psi \in \mathrm{CN}(\Gamma)$ satisfies the assumptions of Theorem 5.3.1. Moreover, let $\mu \in \mathcal{M}(\mathrm{G})$ and U^μ_{A+d} be the d-potential of μ with respect to $A = \psi(D)$, $d \in \mathbb{R}$, $d > 0$. Then it follows that $\mathrm{U}^\mu_{A+d} \in \mathrm{H}^{2-t_0}_{\psi^{\frac{1}{2}}}(\mathrm{G})$ by Corollary 5.2.1.

Our next aim is to prove that d-potentials of measures with densities $v_\mu \in \mathrm{H}^{2-t_0}_{\psi^{\frac{1}{2}}}(\mathrm{G})$ form a Dirichlet space in the sense of Beurling and Deny [12]. For the following considerations we refer to [23] as a general reference.

Definition 5.3.1. Let E be a closed symmetric sesquilinear form with domain $\mathrm{D}(E)$, a dense subspace of $\mathrm{L}^2(\mathrm{G})$, and suppose that E is non-negative, $E(u,u) \geq 0$, for all $u \in \mathrm{D}(E)$, and E is closed.
The form E is called a *symmetric Dirichlet form* if, and only if, $u \in \mathrm{D}(E)$ implies that $v := (0 \wedge u) \vee 1 \in \mathrm{D}(E)$ and

$$E(v,v) \leq E(u,u).$$

The pair $(E, \mathrm{D}(E))$ is called a *symmetric Dirichlet space.*

A Dirichlet form E is said to be regular if $\mathrm{D}(E) \cap \mathrm{C}(\mathrm{G})$ is dense in $\mathrm{D}(E)$, equipped with the topology generated by $E_1(u,v) := E(u,v) + (u,v)_0$, and also in $\mathrm{C}(\mathrm{G})$ with respect to the topology generated by the sup-norm. Moreover, let us call a Dirichlet form *local*, if having $u, v \in \mathrm{D}(E)$,
$\operatorname{supp} u \cap \operatorname{supp} v = \phi$ implies $E(u,v) = 0$.
The following theorem is due to Beurling and Deny [12], see also [14] p.190. It characterises all translation invariant Dirichlet forms on G. By definition E is translation invariant if for any $x \in \mathrm{G}$ and all $u \in \mathrm{D}(E)$ it follows that $\tau_x u \in \mathrm{D}(E)$ and $E(\tau_x u, \tau_x u) = E(u,u)$, where $\tau_x u(y) = u(y-x)$.

Theorem 5.3.2. Let G be a compact Abelian group with discrete dual group Γ. Then there exists an element $\psi \in \mathrm{CN}(\Gamma)$ such that a translation invariant Dirichlet form E satisfies

$$E(u,v) = \sum_{\gamma \in \Gamma} \psi(\gamma) \hat{u}(\gamma) \overline{\hat{v}(\gamma)} \tag{5.3.3}$$

with domain $\mathrm{D}(E) \subset \mathrm{L}^2(\mathrm{G})$, given by

$$\mathrm{D}(E) = \left\{ u \in \mathrm{L}^2(\mathrm{G}), \sum_{\gamma \in \Gamma} \psi(\gamma) |\hat{u}(\gamma)|^2 < \infty \right\}. \tag{5.3.4}$$

Now we find

Theorem 5.3.3. Let $\psi : \Gamma \to \mathbb{R}$ be a negative definite function such that Condition 5.3.1 holds. Then the Dirichlet form (5.3.3) is generated by the operator $\psi(D)$ and $\mathrm{D}(E) = \mathrm{H}^1_{\psi^{\frac{1}{2}}}(\mathrm{G})$, i.e.

$$E(u,v) = \left(\psi(D)^{\frac{1}{2}}u, \psi(D)^{\frac{1}{2}}v\right)_0 \tag{5.3.5}$$

for all $u, v \in \mathrm{H}^1_{\psi^{\frac{1}{2}}}(\mathrm{G})$.

Proof. Since $\mathrm{S}(\mathrm{G}) \subset \mathrm{C}(\mathrm{G})$ the domain of the minimal extension of (5.3.5) is just $\mathrm{H}^1_{\psi^{\frac{1}{2}}}(\mathrm{G})$. Now Plancherel's theorem 3.1.5 gives for $\varphi, \psi \in \mathrm{S}(\mathrm{G})$

$$\left(\psi(D)^{\frac{1}{2}}\varphi, \psi(D)^{\frac{1}{2}}\psi\right)_0 = \sum_{\gamma\in\Gamma} \psi(\gamma)^{\frac{1}{2}}\hat{\varphi}(\gamma)\psi(\gamma)^{\frac{1}{2}}\overline{\hat{\psi}(\gamma)}$$

$$= \sum_{\gamma\in\Gamma} \psi(\gamma)\hat{\varphi}(\gamma)\overline{\hat{\psi}(\gamma)} = E(\varphi, \psi).$$

$\square$

Using Theorem 1.3.1 in [23], (5.3.5) can be characterised as follows:

$$\mathrm{H}^2_{\psi^{\frac{1}{2}}}(\mathrm{G}) = \mathrm{H}^1_{\psi}(\mathrm{G}) = \mathrm{D}(\psi(D)) \subset \mathrm{D}(E) = \mathrm{H}^1_{\psi^{\frac{1}{2}}}(\mathrm{G})$$

and

$$E(u,v) = (\psi(D)u, v)_0$$

for $u \in \mathrm{D}(\psi(D))$ and $v \in \mathrm{D}(E)$.

Now, if $\psi(D)$ is a *local* operator, i.e. $\operatorname{supp}\psi(D)u \subset \operatorname{supp} u$ for all $u \in \mathrm{D}(\psi(D))$, it follows that E is a local form provided that for any $u \in \mathrm{H}^2_{\psi^{\frac{1}{2}}}(\mathrm{G})$ there exists a sequence $(u_n)_{n\in\mathbb{N}}$, $u_n \in \mathrm{H}^2_{\psi^{\frac{1}{2}}}(\mathrm{G})$ such that u_n converges to u in $\mathrm{H}^2_{\psi^{\frac{1}{2}}}(\mathrm{G})$ and $\operatorname{supp} u_n \subset \operatorname{supp} u$ for all n.

Indeed, in this case it follows for $u, v \in \mathrm{H}^2_{\psi^{\frac{1}{2}}}(\mathrm{G})$ with $\operatorname{supp} u \cap \operatorname{supp} v = \phi$ that

$$\begin{aligned} E(u,v) &= \lim_{n\to\infty} E(u_n, v) \\ &= \lim_{n\to\infty} \int_G \psi(D)\, u_n\, v \,\mathrm{d}\mu_G = 0\,, \end{aligned}$$

since $\operatorname{supp}\psi(D)u_n \subset \operatorname{supp} u_n \subset \operatorname{supp} u$ and by our assumptions it follows that $\operatorname{supp}\psi(D)u_n \cap \operatorname{supp} v = \phi$.

Note that by Theorem 18.27 in [10] an operator $A(D)$ with symbol $\psi \in \mathrm{CN}(\Gamma)$ is local if and only if $\psi(\gamma) = c + q(\gamma)$, where $c \geq 0$ is a constant and q is a non-negative quadratic form on Γ.

5.4 Examples

We give two translation invariant examples:

Example 5.4.1. In [17] J. Douglas used the sesquilinear form

$$D(\varphi,\psi) = \frac{1}{16\pi}\int_0^{2\pi}\int_0^{2\pi} \frac{(\varphi(\theta)-\varphi(\theta'))(\psi(\theta)-\psi(\theta'))}{\sin^2(\theta-\theta')/2}\mathrm{d}\theta\mathrm{d}\theta'$$

in order to solve the Plateau problem. But a short calculation shows that with $\lambda(k) = |k|$

$$D(\varphi,\psi) = \sum_{k\in\mathbb{Z}} \lambda(k)\hat{\varphi}(k)\overline{\hat{\psi}(k)}, \qquad \varphi,\psi \in \mathrm{S}(\mathbb{T}^1).$$

Thus D is defined on $\mathrm{H}^1_{\lambda^{1/2}}(\mathbb{T}^1) = \mathrm{H}^{1/2}(\mathbb{T}^1)$, the classical Sobolev space on the one dimensional torus $\mathbb{T}^1$.

Since $\lambda^{1/2} : \mathbb{Z} \to \mathbb{R}$ is a negative definite function, which also satisfies Condition 5.3.1, when taking $\Gamma_m = \{z, |z| \leq m\}$, we can apply the results of the previous section to the Dirichlet form D. The operator generating D is given by $\lambda(D)$, a non-local pseudodifferential operator.

The next example is due to A. Bendikov [4], see also Chr. Berg [8] and [9].

Example 5.4.2. Let $\mathbb{T}^\infty = \mathbb{T}^{\mathbb{N}}$ be the infinite dimensional torus. Its dual group may be identified with $\mathbb{Z}^{(\infty)}$, the set of all sequences of integers with the property that all of its elements but a finite number are equal to zero. Thus any $\gamma \in \mathbb{Z}^{(\infty)}$ is represented by a sequence $(\gamma^j)_{j\in\mathbb{N}}$, $\gamma^j \in \mathbb{Z}$, and only finitely many γ^j's are different from zero.
Let $a = (a_j)_{j\in\mathbb{N}}$ be a sequence of positive integers and $c > 0$ a real number. Consider the mapping $A_c^a : \mathbb{Z}^{(\infty)} \to \mathbb{R}$,

$$\gamma \mapsto A_c^a(\gamma) = \sum_{j=1}^{\infty} a_j \, (\gamma^j)^2 + c\,.$$

Note that this sum is always finite! Moreover, A_c^a is a negative definite function on $\mathbb{Z}^{(\infty)}$. Setting $k(\gamma) = [A_c^a(\gamma)]^{\frac{1}{2}}$ we find that A_c^a is defined on $\mathrm{H}_k^2(\mathbb{T}^\infty)$ generates a Dirichlet form on $\mathrm{H}_k^1(\mathbb{T}^\infty)$.

In addition, we mention

Example 5.4.3. Consider the set of rational numbers

$$\mathbb{Q} = \left\{ \frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{N} \right\}.$$

with addition as the group operation. Taking the discrete topology on $\mathbb{Q}$, we know that $\mathbb{Q}$ is a discrete group, and so it is the dual group of an Abelian compact group. We denote this group G so that $G^* = \mathbb{Q}$ and $\mathbb{Q}^*$ is isomorphic to G by theorem 3.1.4.
In this case we cannot choose a finite exhaustion which is compatible with the natural group and order structure on $\mathbb{Q}$. Therefore the convergence properties of Fourier series become more complicated. We refer only to the papers [26] and [27] for a taste of the subject.

5.5 An Inequality of Gårding Type for Pseudodifferential Operators with Variable Coefficients

One of the conditions for a variational solution of the equation $B[\phi, u] = (\phi, f)$, explored in section 5.7 later, is an inequality of Gårding type. This section aims to provide the conditions on the pseudodifferential operator which give rise to such an inequality. We begin with

Lemma 5.5.1. Let $A_1 \in \sum_x^\infty(k)$, be a symbol such that

$$|\widehat{A_1}(\gamma - \xi, \xi)| \leq h(\gamma - \xi)\, k_*^t(\xi) \tag{5.5.1}$$

where $h \in \mathrm{L}^1(\Gamma)$. Then for the corresponding pseudodifferential operator $A_1(x, D)$ it holds that

$$\|A_1(x, D)u\|_{\mathrm{L}^2(\mathrm{G})} \leq \|h\|_{\mathrm{L}^1(\Gamma)}\, \|u\|_{\mathrm{H}_k^t(\mathrm{G})}\,. \tag{5.5.2}$$

Proof. By Theorems 3.1.5, 3.1.8 and Lemma 4.5.1, we find

$$\begin{aligned}
\left|(A_1(x, D)u, v)_{\mathrm{L}^2(\mathrm{G})}\right| &= \left|([A_1(x, D)u]\,\hat{}\,, \hat{v})_{\mathrm{L}^2(\Gamma)}\right| \\
&= \sum_{\gamma\in\Gamma}\sum_{\xi\in\Gamma} \left|\hat{A}_1(\gamma - \xi, \xi)\right|\, |\hat{u}(\xi)|\, |\overline{\hat{v}(\gamma)}| \\
&\leq \sum_{\gamma\in\Gamma}\sum_{\xi\in\Gamma} h(\gamma - \xi)|k_*^t(\xi)|\, |\hat{u}(\xi)|\, |\overline{\hat{v}(\gamma)}| \\
&\leq \|h\|_{\mathrm{L}^1(\Gamma)}\, \|u\|_{\mathrm{H}_k^t(\mathrm{G})}\, \|v\|_{\mathrm{H}_k^0(\mathrm{G})}\,.
\end{aligned}$$

Therefore,

$$\|A_1(x, D)u\|_{\mathrm{L}^2(\mathrm{G})} = \frac{\left|(A_1(x, D)u, v)_{\mathrm{L}^2(\mathrm{G})}\right|}{\|v\|_{\mathrm{H}_k^0(\mathrm{G})}} \leq \|h\|_{\mathrm{L}^1(\Gamma)}\, \|u\|_{\mathrm{H}_k^t(\mathrm{G})}\,.$$

□

We have already proved that for a translation invariant pseudodifferential operator of order t, for $\gamma \in \Gamma \setminus \Gamma_{m_0}$, with Γ_{m_0} finite,

$$A(\gamma) \geq c\, k_*^t(\gamma)$$

implies

$$\|A(D)u\|_{r,k} \geq c_0 \|u\|_{r+t,k} - c_1 \|u\|_0 .$$

Now consider a pseudodifferential operator A(x,D) of order t, such that for fixed $x_0 \in$ G,

$$A(x_0, \gamma) \geq c\, k_*^t(\gamma) \qquad \text{for all } \gamma \in \Gamma \setminus \Gamma_{m_0},$$

and Condition (5.5.1) of Lemma 5.5.1 holds for

$$A_1(x, \gamma) := A(x, \gamma) - A(x_0, \gamma).$$

Then

$$A(x, D) = A(x_0, D) + \big(A(x, D) - A(x_0, D)\big)$$

where

$$\|A(x_0, D)u\|_0 \geq c_0 \|u\|_{t,k} - c_1 \|u\|_0 . \tag{5.5.3}$$

Therefore by Lemma 5.5.1,

$$\begin{aligned} \|A(x, D)u\|_0 &\geq \|A(x_0, D)u\|_0 - \big\|\big(A(x, D) - A(x_0, D)\big)u\big\|_0 \\ &\geq c_0 \|u\|_{t,k} - c_1 \|u\|_0 - \|h\|_{\mathrm{L}^1(\Gamma)} \|u\|_{t,k} \\ &\geq \big(c_0 - \|h\|_{\mathrm{L}^1(\Gamma)}\big) \|u\|_{t,k} - c_1 \|u\|_0 . \end{aligned}$$

For a pseudodifferential operator A(x,D) of order t, satisfying Condition (5.5.1) with

$$A_1(x, D) := A(x, D) - A(x_0, D),$$

we have proved the following theorem,

Theorem 5.5.1. Let $A(x, D)$ be as above, and in addition assume that for h in (5.5.2) it follows that $\|h\|_{\mathrm{L}^1(\Gamma)} < c_0$, where c_0 is as in (5.5.3). Then there exist constants $c_1 \geq 0$ and $c_2 > 0$ such that the Gårding-type inequality

$$\|A(x, D)u\|_0 \geq c_2 \|u\|_{t,k} - c_1 \|u\|_0 \tag{5.5.4}$$

holds.

5.6 On Weak Solutions to A(x,D)u = f

We start with

Definition 5.6.1. Let T be a pseudodifferential operator with adjoint T^*. Let $f \in \mathrm{L}^2(\mathrm{G})$. We call $u \in \mathrm{L}^2(\mathrm{G})$ a *weak L^2-solution* of the equation $Tu = f$ if we have

$$\int_G u T^* v \, \mathrm{d}\mu_G(x) = \int_G f \, v \, \mathrm{d}\mu_G(x),$$

for all $v \in D(T^*)$.

We want to prove

Theorem 5.6.1. Let $A(x, D) \in \mathcal{A}_x^2(k)$ be a pseudodifferential operator with variable coefficients, and let $A^*(x, D)$ be the adjoint of A(x,D) defined by

$$\int_G \big(A(x, D)u\big) \, v \, \mathrm{d}\mu_G(x) = \int_G u \, \big(A^*(x, D)v\big) \mathrm{d}\mu_G(x)$$

for all $u, v \in \mathrm{H}_k^2(\mathrm{G})$. Also assume that for some $f \in \mathrm{L}^2(\mathrm{G})$ and all $u \in \mathrm{L}^2(\mathrm{G})$

$$\left| \int_G f \, u \, \mathrm{d}\mu_G(x) \right| \leq c \left\| \big[A^*(x, D) - \lambda\big] \, u \right\|_{\mathrm{L}^2(\mathrm{G})} \tag{5.6.1}$$

holds. Then the equation $\big[A(x, D) - \lambda\big] \, u = f$ admits a weak L^2-solution.

Proof. Consider the set

$$W_\lambda := \left\{ w \in \mathrm{L}^2(\mathrm{G}) : \text{s.t. } (A^*(x,D) - \lambda)v = w \text{ for some } v \in \mathrm{H}^2_k(\mathrm{G}) \right\}.$$

W_λ is a linear subset of $\mathrm{L}^2(\mathrm{G})$ and since $A^*(x,D)$ maps $\mathrm{H}^2_k(\mathrm{G})$ continuously into $\mathrm{L}^2(\mathrm{G})$, W_λ is a non-empty set. On W_λ we define the linear functional

$$\begin{aligned} F_\lambda \; &: \; W_\lambda \to \mathbb{C} \\ w \; &\mapsto F_\lambda w := \int_G v\, f \,\mathrm{d}\mu_G(x) \end{aligned}$$

where $v \in \mathrm{H}^2_k(\mathrm{G})$ is a solution of the equation

$$(A^*(x,D) - \lambda)v = w.$$

Note that F_λ is independent of v. If $\tilde{v}$ is another solution of the equation, $(A^*(x,D) - \lambda)\tilde{v} = w$, then by the assumption (5.6.1)

$$\begin{aligned} \left| \int_G f(v - \tilde{v}) \mathrm{d}\mu_G(x) \right| &\leq c \left\| \left(A^*(x,D) - \lambda\right)(v - \tilde{v}) \right\|_{\mathrm{L}^2(\mathrm{G})} \\ &= c\, \|w - w\|_{\mathrm{L}^2(\mathrm{G})} = 0\,. \end{aligned}$$

Also, by the assumption,

$$\begin{aligned} |F_\lambda w| = \left| \int_G v\, f \,\mathrm{d}\mu_G(x) \right| &\leq c \left\| \left(A^*(x,D) - \lambda\right) v \right\|_{\mathrm{L}^2(\mathrm{G})} \\ &= c\, \|w\|_{\mathrm{L}^2(\mathrm{G})}\,. \end{aligned}$$

Therefore F_λ is a continuous linear functional on $W_\lambda \subset \mathrm{L}^2(\mathrm{G})$. So by the Hahn-Banach theorem, we may extend F_λ to a continuous linear functional on $\mathrm{L}^2(\mathrm{G})$, which we also denote F_λ.

For this reason, we have that, for some $u \in \mathrm{L}^2(\mathrm{G})$,

$$F_\lambda w = \int_G u\, w \,\mathrm{d}\mu_G(x)$$

holds for all $w \in \mathrm{L}^2(\mathrm{G})$.

For $v \in \mathrm{H}_k^2(\mathrm{G})$, we know that $\big(A^*(x,D)v - \lambda v\big)$ belongs to W_λ, therefore

$$\int_G u\big(A^*(x,D)v - \lambda v\big)\mathrm{d}\mu_G(x) = \int_G f\ v\ \mathrm{d}\mu_G(x)$$

which shows that $u \in \mathrm{L}^2(\mathrm{G})$ is a weak L^2-solution of the equation $A(x,D)u - \lambda u = f$. $\square$

5.7 A Variational Solution Using Conditions on the Sesquilinear Form

In this section we will investigate variational solutions of certain pseudodifferential operators.

Theorem 5.7.1. Let $k : \Gamma \to \mathbb{C}$ be a negative definite function (this condition may be stronger than necessary). Assume that the sesquilinear form defined by a pseudodifferential operator with variable coefficients $A(x,D) : \mathrm{H}_k^{2m}(\mathrm{G}) \to \mathrm{L}^2(\mathrm{G})$, as $B[u,v] := (A(x,D)u, v)$ satisfies

$$|B[u,v]| \leq \tilde{c}\ \|u\|_{\mathrm{H}_k^m(\mathrm{G})}\ \|v\|_{\mathrm{H}_k^m(\mathrm{G})} \tag{5.7.1}$$

$$\mathrm{Re}\ B[u,u] \geq c\ \|u\|^2_{\mathrm{H}_k^m(\mathrm{G})} - \lambda_0\ \|u\|^2_{\mathrm{L}^2(\mathrm{G})} \tag{5.7.2}$$

and assume that the embedding $\mathrm{H}_k^m(\mathrm{G}) \to \mathrm{L}^2(\mathrm{G})$ is compact.

Then the Fredholm Alternative holds for the problem:

$$B[\varphi, u] = (\varphi, f) \qquad \text{for all } \varphi \in \mathrm{H}_k^m(\mathrm{G}). \tag{5.7.3}$$

More precisely,

either for any $f \in \mathrm{L}^2(\mathrm{G})$, there is a unique solution $u \in \mathrm{H}_k^m(\mathrm{G})$, of

$$B[\varphi, u] = (\varphi, f) \qquad \text{for all } \varphi \in \mathrm{H}_k^m(\mathrm{G}),$$

or there are a finite number of linearly independent solutions
$v_j\ (j = 1, 2, ..., n),\ v_j \in \mathrm{H}_k^m(\mathrm{G})$, of

there are a finite number of linearly independent solutions
$v_j\ (j = 1, 2, ..., n),\ v_j \in \mathrm{H}_k^m(\mathrm{G})$, of

In the second case, for the existence of a solution to (5.7.3) it is necessary and sufficient that $(f, v) = 0$ for all $v \in \mathrm{span}\{v_1, ..., v_n\}$. If a solution exists, it is only unique up to an additive element in $\mathrm{span}\{v_1, ..., v_n\}$.

Proof. We refer the reader to the formulation of Fredholm's alternative given by I.S. Louhivaara and C. Simader in [38].
Take a fixed $\lambda \geq \lambda_0,\ \lambda \neq 0$, where λ_0 is the constant appearing in the Gårding inequality (5.7.2).
Let $A_\lambda(x, D) = A(x, D) + \lambda$. The sesquilinear form associated with $A_\lambda(x, D)$ is

$$B_\lambda[u, v] = B[u, v] + \lambda(u, v).$$

By Riesz representation theorem and the Gårding inequality, for any $g \in \mathrm{L}^2(\mathrm{G})$ there exists a unique solution $w \in \mathrm{H}_k^m(\mathrm{G})$ of

$$B_\lambda[\varphi, w] = (\varphi, g) \qquad \text{for all } \varphi \in \mathrm{H}_k^m(\mathrm{G}). \tag{5.7.4}$$

Formally we set $w = A_\lambda^{-1}(x, D)g$. Note that this is not the inverse operator, rather it is the unique $w \in \mathrm{H}_k^m(\mathrm{G})$ for each $g \in \mathrm{L}^2(\mathrm{G})$ in (5.7.4).
Then since $B[\varphi, u] = (\varphi, f)$ is equivalent to $B_\lambda[\varphi, u] = (\varphi, \lambda u + f)$ for all $\varphi \in \mathrm{H}_k^m(\mathrm{G})$ and $u \in \mathrm{H}_k^m(\mathrm{G})$; u satisfies $B[\varphi, u] = (\varphi, f)$ if and only if

$$u = A_\lambda^{-1}(x, D)(\lambda u + f).$$

That is to say, u satisfies $B[\varphi, u] = (\varphi, f)$ if and only if

$$u - Tu = f_1 \qquad (\text{where } T = \lambda A_\lambda^{-1} \text{ and } f_1 = A_\lambda^{-1} f)\ .$$

From $B_\lambda[\varphi, w] = (\varphi, g)$, it follows that

$$c\, \|w\|^2_{\mathrm{H}^m_k(\mathrm{G})} \le \left|B_\lambda[w, w]\right| \le |(w, g)| \le \|w\|_{\mathrm{H}^m_k(\mathrm{G})} \|g\|_{\mathrm{L}^2(\mathrm{G})} ,$$

i.e. we know

$$\begin{aligned} c\, \|w\|_{\mathrm{H}^m_k(\mathrm{G})} &\le \|g\|_{\mathrm{L}^2(\mathrm{G})} , \\ c\, \|A_\lambda^{-1} g\|_{\mathrm{H}^m_k(\mathrm{G})} &\le \|g\|_{\mathrm{L}^2(\mathrm{G})} , \\ c\, \|\lambda A_\lambda^{-1} g\|_{\mathrm{H}^m_k(\mathrm{G})} &\le \|\lambda g\|_{\mathrm{L}^2(\mathrm{G})} , \\ c\, \|T g\|_{\mathrm{H}^m_k(\mathrm{G})} &\le \lambda \|g\|_{\mathrm{L}^2(\mathrm{G})} . \end{aligned}$$

Hence T maps bounded sets of $\mathrm{L}^2(\mathrm{G})$ into bounded sets in $\mathrm{H}^m_k(\mathrm{G})$. Applying the compactness of the embedding $\mathrm{H}^m_k(\mathrm{G})$ into $\mathrm{L}^2(\mathrm{G})$, which we denote i, we conclude that $i \circ T$ maps bounded sets in $\mathrm{L}^2(\mathrm{G})$ into compact sets in $\mathrm{L}^2(\mathrm{G})$. Thus the Fredholm-Riesz-Schauder theory can be applied, and it follows

either (a) for every $f_1 \in \mathrm{L}^2(\mathrm{G})$ there exists a unique solution of

$$u - Tu = f_1 ,$$

or (b) there are nontrivial solutions of $u - Tu = 0$ and the equation $u - Tu = f_1$ has a solution if and only if $(f_1, v_j) = 0$ for a finite number of functions v_j $(1 \le j \le n)$, which forms a basis of the space of solutions of $v - T^* v = 0$.

If case (a) holds, there exists a solution of

$$B[\varphi, u] = (\varphi, f)$$

for any $f \in \mathrm{L}^2(\mathrm{G})$. It is unique because $f = 0$ implies $f_1 = 0$ and consequently $u = 0$.

To consider case (b), note that the Gårding inequality and conditions which lead to equation (5.7.4) are also true for $A^*(x, D)$ by Theorem 2.3.1, with a possibly different λ_0 constant. We will denote the new constant λ_0^* ie.

$$\mathrm{Re}\ B^*[u, u] \ge c\, \|u\|^2_{\mathrm{H}^m_k(\mathrm{G})} - \lambda_0^*\, \|u\|^2_{\mathrm{L}^2(\mathrm{G})} . \tag{5.7.5}$$

Taking λ larger than both λ_0 and λ_0^*, we shall prove that

$$T^* = \lambda\big(A^*(x,D) + \lambda\big)^{-1} . \tag{5.7.6}$$

Indeed, by the definition of T^*,

$$(T^*v, g) = (v, Tg).$$

Setting

$$Tg = h \qquad T^*v = w$$

we have

$$\begin{aligned} B_\lambda[\varphi, h] &= \big(\varphi, A_\lambda(x,D)h\big) \\ &= \big(\varphi, A_\lambda(x,D)Tg\big) \\ &= \big(\varphi, A_\lambda(x,D)\lambda\, A_\lambda^{-1}(x,D)g\big) \\ &= \big(\varphi, \lambda g\big) \\ &= \lambda\big(\varphi, g\big) \end{aligned}$$

and in particular,

$$B_\lambda[w, h] = \lambda(w, g).$$

We set $\lambda(A^* + \lambda)^{-1}v = \tilde{w}$. Then $\tilde{w} \in \mathrm{H}_k^m(\mathrm{G})$ satisfies

$$B_\lambda[\tilde{w}, \psi] = \lambda(v, \psi) \qquad \text{for all } \psi \in \mathrm{H}_k^m(\mathrm{G}) ,$$

because

$$\begin{aligned} B_\lambda[\tilde{w}, \psi] &= \big(\tilde{w}, A_\lambda(x,D)\psi\big) \\ &= \big(\lambda(A^* + \lambda)^{-1}v, (A + \lambda)\psi\big) \\ &= \big(\lambda(A + \lambda)^*(A^* + \lambda)^{-1}v, \psi\big) \\ &= \lambda\big(v, \psi\big) . \end{aligned}$$

Hence

$$B_\lambda[\tilde{w}, h] = \lambda(v, h) = \lambda(v, Tg) = \lambda(T^*v, g) = \lambda(w, g).$$

If we combine

$$\begin{aligned} B_\lambda[w,h] &= \lambda(w,g) \\ B_\lambda[\tilde{w},h] &= \lambda(w,g) \\ \textit{and} \quad B_\lambda[\varphi,h] &= \lambda(\varphi,g)\ , \end{aligned}$$

we get

$$0 = B_\lambda[w-\tilde{w},h] = \lambda(w-\tilde{w},g)\ .$$

Since g is arbitrary, we must have $w-\tilde{w}=0$. i.e. $T^*v = \lambda(A^*+\lambda)^{-1}v$. This completes the proof of equation (5.7.6).

From equation (5.7.6) we conclude that $T^*v - v = 0$ for $v \in \mathrm{H}_k^m(\mathrm{G})$, if and only if

$$B[v,\varphi] = 0 \qquad \text{for all } \varphi \in \mathrm{H}_k^m(\mathrm{G})\ ,$$

because

$$\begin{aligned} T^*v - v &= 0 \\ \lambda(A^*+\lambda)^{-1}v &= v \\ \lambda v &= (A^*+\lambda)v \\ A^*v &= 0 \\ B[v,\varphi] &= (A^*v,\varphi) = 0 \qquad \forall \varphi \in \mathrm{H}_k^m(\mathrm{G})\ . \end{aligned}$$

To complete the proof, we must show that the conditions$(f_1,v_j) = 0$ are equivalent to the conditions $(f,v_j)=0$.

This follows from the equalities:

$$(f_1,v_j) = (A_\lambda^{-1}f,v_j) = (f,(A_\lambda^{-1})^*v_j) = \frac{1}{\lambda}(f,T^*v_j) = \frac{1}{\lambda}(f,v_j).$$

□

We will now give conditions implying (5.7.1) and (5.7.2), respectively. As in Section 5.5, for a symbol $A \in \sum_x^{2m}(k)$ we have the decomposition

$$\begin{aligned} A(x,\gamma) &= A(x_0,\gamma) + (A(x,\gamma) - A(x_0,\gamma)) \\ &= A(x_0,\gamma) + A_1(x,\gamma) \end{aligned}$$

with

$$\begin{aligned} A(x_0,\gamma) &\leq c_0\, k_*^{2m}(\gamma), \\ A(x_0,\gamma) &\geq \lambda_0\, k_*^{2m}(\gamma), \end{aligned}$$

and

$$|\hat{A}_1(\gamma-\xi,\xi)| \;\leq\; \tilde{h}(\gamma-\xi)\, k_*^{2m}(\xi).$$

Then for $u, v \in S(G)$, we may decompose the associated sesquilinear form,

$$\begin{aligned} B[u,v] &= (A(x_0,D)u,v) - (A_1(x,D)u,v) \\ &= B_1[u,v] - B_2[u,v]. \end{aligned}$$

The first part satisfies

$$|B_1[u,v]| \leq c_0\, \|u\|_{\mathrm{H}_k^m(\mathrm{G})}\, \|v\|_{\mathrm{H}_k^m(\mathrm{G})}$$

and

$$B_1[u,u] = \sum_{\gamma\in\Gamma} A(x_0,\gamma)|\hat{u}(\gamma)|^2 \geq \lambda_0\, \|u\|^2_{\mathrm{H}_k^m(\mathrm{G})},$$

and the second part satisfies, by Peetre's inequality,

$$\begin{aligned} |B_2[u,v]| &\leq \sum_{\gamma\in\Gamma}\sum_{\xi\in\Gamma} \tilde{h}(\gamma-\xi) k_*^{2m}(\xi)\, |\hat{u}(\xi)|\, |\hat{v}(\gamma)| \\ &= \sum_{\gamma\in\Gamma}\sum_{\xi\in\Gamma} \tilde{h}(\gamma-\xi) \frac{k_*^m(\xi)}{k_*^m(\gamma)}\, k_*^m(\xi)\, |\hat{u}(\xi)|\, k_*^m(\gamma)\, |\hat{v}(\gamma)| \\ &\leq 2^{|m|} \sum_{\gamma\in\Gamma}\sum_{\xi\in\Gamma} \tilde{h}(\gamma-\xi)\, k_*^{|m|}(\gamma-\xi)\, k_*^m(\xi)\, |\hat{u}(\xi)|\, k_*^m(\gamma)\, |\hat{v}(\gamma)| \\ &\leq 2^{|m|}\, \|\tilde{h}(\cdot)\, k_*^{|m|}(\cdot)\|_{L^1(\Gamma)}\, \|u\|_{\mathrm{H}_k^m(\mathrm{G})}\, \|v\|_{\mathrm{H}_k^m(\mathrm{G})}, \end{aligned}$$

provided $\tilde{h}(\cdot)\, k_*^{|m|}(\cdot) \in L^1(\Gamma)$.

Theorem 5.7.2. Let $A \in \sum_x^{2m}(k)$ have the decomposition

$$A(x,\gamma) = A(x_0,\gamma) + (A(x,\gamma) - A(x_0,\gamma)) = A(x_0,\gamma) + A_1(x,\gamma) \qquad (5.7.7)$$

for some fixed $x_0 \in G$ and assume

$$A(x_0,\gamma) \leq c_0\, k_*^{2m}(\gamma), \qquad (5.7.8)$$

$$A(x_0,\gamma) \geq \lambda_0\, k_*^{2m}(\gamma), \qquad (5.7.9)$$

and

$$|\hat{A}_1(\zeta,\gamma)| \leq \tilde{h}(\zeta)\, k_*^{2m}(\gamma). \qquad (5.7.10)$$

A. If $\tilde{h}(\cdot)k_*^m(\cdot) \in L^1(\Gamma)$ then the sesquilinear form $B[\cdot,\cdot]$ defined on $\mathrm{S(G)}$ by

$$B[u,v] = (A(x,D)u,v) \qquad (5.7.11)$$

has a continuous extension to $\mathrm{H}_k^m(\mathrm{G})$ and it holds

$$|B[u,v]| \leq c\, \|u\|_{\mathrm{H}_k^m(\mathrm{G})}\, \|v\|_{\mathrm{H}_k^m(\mathrm{G})}. \qquad (5.7.12)$$

B. If $\tilde{h}(\cdot)k_*^m(\cdot) \in L^1(\Gamma)$ and for some $\eta \in (0,1)$ it holds

$$2^{|m|}\|\tilde{h}(\cdot)\, k_*^m(\cdot)\|_{L^1(\Gamma)} \leq \eta\, \lambda_0 \qquad (5.7.13)$$

then on $\mathrm{H}_k^m(\mathrm{G})$, $B[\cdot,\cdot]$ satisfies the Gårding inequality

$$B[u,u] \geq \lambda_0(1-\eta)\, \|u\|^2_{\mathrm{H}_k^m(\mathrm{G})}. \qquad (5.7.14)$$

Remark. If we relax condition (5.7.9) to

$$A(x_0,\gamma) \geq \lambda_0\, k_*^{2m}(\gamma) \qquad (5.7.15)$$

for all $\gamma \in \Gamma \setminus \Gamma'$, where $\Gamma' \subset \Gamma$ is a finite subset, then estimate (5.7.14) becomes

$$B[u,u] \geq \lambda_0(1-\eta)\|u\|^2_{\mathrm{H}_k^m(\mathrm{G})} - \rho\, \|u\|^2_{\mathrm{L}^2(\mathrm{G})}. \qquad (5.7.16)$$

For this we need only rewrite

$$\begin{aligned} B_1[u,u] &= \sum_{\gamma\in\Gamma} A(x_0,\gamma)|\hat{u}(\gamma)|^2 \\ &= \sum_{\gamma\in\Gamma\backslash\Gamma'} A(x_0,\gamma)|\hat{u}(\gamma)|^2 + \sum_{\gamma\in\Gamma'} A(x_0,\gamma)|\hat{u}(\gamma)|^2 \\ &\geq \lambda_0 \sum_{\gamma\in\Gamma} k_*^{2m}(\gamma)|\hat{u}(\gamma)|^2 + \sum_{\gamma\in\Gamma'} (A(x_0,\gamma) - k_*^{2m}(\gamma))|\hat{u}(\gamma)|^2 \\ &\geq \|u\|^2_{\mathrm{H}_k^m(\mathrm{G})} - \rho\|u\|^2_{\mathrm{L}^2(\mathrm{G})}, \end{aligned}$$

where $\rho = \max_{\gamma\in\Gamma'} |A(x_0,\gamma) - k_*^{2m}(\gamma)|$.

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