



Swansea University  
Prifysgol Abertawe



## Swansea University E-Theses

---

# On Nelson's stochastic mechanics for a semiclassical parabolic state.

Williams, Angharad

### How to cite:

---

Williams, Angharad (2012) *On Nelson's stochastic mechanics for a semiclassical parabolic state..* thesis, Swansea University.

<http://cronfa.swan.ac.uk/Record/cronfa42359>

### Use policy:

---

This item is brought to you by Swansea University. Any person downloading material is agreeing to abide by the terms of the repository licence: copies of full text items may be used or reproduced in any format or medium, without prior permission for personal research or study, educational or non-commercial purposes only. The copyright for any work remains with the original author unless otherwise specified. The full-text must not be sold in any format or medium without the formal permission of the copyright holder. Permission for multiple reproductions should be obtained from the original author.

Authors are personally responsible for adhering to copyright and publisher restrictions when uploading content to the repository.

Please link to the metadata record in the Swansea University repository, Cronfa (link given in the citation reference above.)

<http://www.swansea.ac.uk/library/researchsupport/ris-support/>

On Nelson's Stochastic Mechanics for a  
Semiclassical Parabolic State

Angharad Williams

Submitted to Swansea University in fulfilment of the requirements for the  
Degree of Doctor of Philosophy

Swansea University

2012

ProQuest Number: 10798067

All rights reserved

INFORMATION TO ALL USERS

The quality of this reproduction is dependent upon the quality of the copy submitted.

In the unlikely event that the author did not send a complete manuscript and there are missing pages, these will be noted. Also, if material had to be removed, a note will indicate the deletion.



ProQuest 10798067

Published by ProQuest LLC (2018). Copyright of the Dissertation is held by the Author.

All rights reserved.

This work is protected against unauthorized copying under Title 17, United States Code  
Microform Edition © ProQuest LLC.

ProQuest LLC.  
789 East Eisenhower Parkway  
P.O. Box 1346  
Ann Arbor, MI 48106 – 1346

# Contents

<b>Abstract</b>	<b>4</b>
<b>Acknowledgements</b>	<b>5</b>
<b>1 Preliminaries</b>	<b>6</b>
1.1 Stochastic Mechanics . . . . .	6
1.1.1 Wave Mechanics . . . . .	6
1.1.2 The Madelung Fluid . . . . .	8
1.1.3 Itô Diffusions . . . . .	10
1.1.4 Nelson's Stochastic Mechanics . . . . .	14
1.2 The Kepler Problem . . . . .	17
1.2.1 Conic Sections . . . . .	17
1.2.2 The Classical Kepler Problem . . . . .	19
1.2.3 The Levi-Civita Transformation . . . . .	23
1.2.4 The Quantum Kepler Problem . . . . .	24
<b>2 Introduction to the Two-Dimensional Parabolic Nelson Diffusion</b>	<b>27</b>
2.1 A Review of the Divine Clockwork . . . . .	27
2.1.1 The Wave Function for the Atomic Elliptic State . . . . .	27
2.1.2 The Nelson Diffusion for the Atomic Elliptic State . . . . .	30
2.1.3 The Limiting Wave Function . . . . .	30
2.1.4 The Limiting Nelson Diffusion Process . . . . .	32
2.2 The Limiting Parabolic Diffusion . . . . .	34
2.2.1 Analysis of the Parabolic Nelson Diffusion . . . . .	40
2.3 The Classical Case . . . . .	45
2.3.1 Investigating the Motion . . . . .	45
2.3.2 Calculating Explicit Time Dependence . . . . .	49
2.3.3 The Burgers Velocity Field for the Parabolic Motion . . . . .	52

<b>3</b>	<b>The First Order Correction to the Parabolic Diffusion</b>	<b>56</b>
3.1	One-Dimensional Stochastic Differential Equations . . . . .	57
3.1.1	A Combinatorial Problem . . . . .	58
3.1.2	The Integrating Factor Method . . . . .	62
3.2	Two-Dimensional Stochastic Differential Equations . . . . .	63
3.3	Time-Ordered Products . . . . .	65
3.3.1	Product Integration . . . . .	65
3.3.2	Using Time-Ordered Products . . . . .	67
3.4	The First Order Correction to the Parabolic Orbit . . . . .	70
3.4.1	The Radial Equation for the Parabolic Orbit . . . . .	71
<b>4</b>	<b>The Levi-Civita Transformation for the Parabolic Orbit</b>	<b>74</b>
4.1	The Quantum Correction to the Levi-Civita Transformation . . . . .	77
4.2	The Quantum Correction to the Cartesian Coordinates . . . . .	84
4.3	Quantum Areal Velocity . . . . .	86
<b>A</b>	<b>Simulating Diffusion Processes</b>	<b>93</b>
	<b>List of Figures</b>	<b>95</b>
	<b>Bibliography</b>	<b>96</b>

# Abstract

This thesis presents an analysis of a stochastic process characterising a parabolic motion with small random perturbations. This process arises from considerations of the Bohr correspondence limit of the atomic elliptic state. It represents the semiclassical behaviour of a particle, describing a parabolic orbit under a Coulomb potential. By first considering the analogous classical mechanical system, we investigate the difference between the classical and semiclassical systems.

Chapter 1 begins by introducing Nelson's stochastic mechanics as a reformulation of Schrödinger's wave mechanics. Comparisons are drawn between the classical and quantum Kepler problems.

In Chapter 2, we consider earlier results of Durran, Neate and Truman, together with a derivation of the parabolic state by considering the limit of the eccentricity of the semiclassical elliptic diffusion. We proceed to analyse the resulting stochastic differential equation, proving the existence of a solution in the weak sense. A complete analysis of the trajectory and time-dependence of the corresponding classical system is also provided.

Chapter 3 focuses on asymptotic series solutions to more general stochastic differential equations in both one and two dimensions. Methods considered are used to find the first order quantum correction to the parabolic orbit in terms of time-ordered products.

We conclude in Chapter 4 by applying the Levi-Civita transformation to the semiclassical orbit, yielding first order quantum corrections to both its Cartesian coordinates and areal velocity.

# Acknowledgements

Firstly I would like to thank my supervisor Professor Aubrey Truman for his immeasurable guidance and support, both prior to and during the completion of this work. I would secondly like to thank my co-supervisor Dr. Andrew Neate for his advice, motivation and technical assistance. I would also like to thank past and present staff and students at Swansea University Mathematics Department for making my time here enjoyable and worthwhile.

I most graciously thank the Engineering and Physical Sciences Research Council (EPSRC) for kindly providing the financial support which facilitated my postgraduate studies.

I wish to thank all of the friends and family who have willingly given their support and encouragement (but are too numerous for individual mention). Finally I thank my parents; Linda Williams, for every kind of support imaginable; and John Williams, for providing me with inspiration during trying times.

# Chapter 1

## Preliminaries

### 1.1 Stochastic Mechanics

Beginning with brief accounts of both quantum mechanics [18] and stochastic calculus [14, 21, 23, 27], we proceed to discuss in some detail the concept of Nelson's stochastic mechanics [24, 25], whereby the movement of a quantum particle may be modelled using a diffusion process satisfying a particular Itô stochastic differential equation.

#### 1.1.1 Wave Mechanics

According to Schrödinger's wave mechanics, the state of the quantum particle at a given time  $t$  and position  $\mathbf{x} \in \mathbb{R}^n$  is completely determined by the complex-valued wave function  $\psi(\mathbf{x}, t)$ , where  $\psi : \mathbb{R}^n \times (0, \infty) \rightarrow \mathbb{C}$ . For a particle of unit mass subject to a conservative force field given by the negative gradient of some real-valued potential function,  $-\nabla V(\mathbf{x})$ , the wave function is postulated to satisfy the time-dependent Schrödinger equation:

$$i\hbar \frac{\partial \psi(\mathbf{x}, t)}{\partial t} = -\frac{\hbar^2}{2} \Delta \psi(\mathbf{x}, t) + V(\mathbf{x})\psi(\mathbf{x}, t), \quad (1.1)$$

where  $\Delta$  denotes the Laplacian differential operator and  $\hbar = \frac{h}{2\pi}$ , the reduced Planck's constant. Throughout we consider the spin-free case.

The Schrödinger equation may be obtained by making formal operator substitutions in the classical identity for the Hamiltonian:

$$H(q, p) = \frac{p^2}{2} + V(q),$$

for generalised position and momentum coordinates  $q, p \in \mathbb{R}^n$ .



In quantum mechanics, observables such as position, denoted  $Q$ , and momentum,  $P$ , become non-commuting Hermitian linear operators acting on the wave function  $\psi$ :

$$Q\psi(\mathbf{x}, t) = \mathbf{x}\psi(\mathbf{x}, t) \quad \text{and} \quad P\psi(\mathbf{x}, t) = -i\hbar\nabla\psi(\mathbf{x}, t).$$

For position and momentum operators  $Q = (Q_1, Q_2, Q_3)$  and  $P = (P_1, P_2, P_3)$ ,  $Q$  and  $P$  satisfy the canonical commutation relations for  $j, k = 1, 2, 3$ :

- $[P_j, P_k] = 0$ ,
- $[Q_j, Q_k] = 0$ ,
- $[Q_j, P_k] = i\hbar\delta_{jk}I$ ,

where  $I$  denotes the identity operator and  $[A, B]$  denotes the commutator bracket:

$$[A, B] = AB - BA.$$

The Hamiltonian operator (again for a particle of unit mass) is given by  $H$  where:

$$H\psi(\mathbf{x}, t) = -\frac{\hbar^2}{2}\Delta\psi(\mathbf{x}, t) + V(\mathbf{x})\psi(\mathbf{x}, t).$$

The eigenvalues of these operators are the possible values of the corresponding observable in the state  $\psi(\mathbf{x}, t)$ , and in particular eigenvalues of the Hamiltonian correspond to possible values of the total energy  $E$ . Letting:

$$\psi(\mathbf{x}, t) = \exp\left(-\frac{iEt}{\hbar}\right)\psi(\mathbf{x})$$

yields the time-independent Schrödinger equation:

$$H\psi(\mathbf{x}) = E\psi(\mathbf{x}),$$

for stationary states  $\psi(\mathbf{x})$ ,

Born's probabilistic interpretation of quantum mechanics states that the probability density  $\rho$  of the quantum particle in the state  $\psi$  is given by:

$$\rho(\mathbf{x}, t) = |\psi|^2 = \psi\bar{\psi}, \tag{1.2}$$

which is assumed to be normalised. Using this interpretation, the expected value of some observable  $X$ , in the state  $\psi$  may be defined:

$$\langle X \rangle = \int_{\mathbb{R}^3} \bar{\psi} X \psi \, d\mathbf{x}.$$

Consequently, the uncertainty in  $X$ , denoted  $\Delta X$  may be defined as follows:

$$(\Delta X)^2 = \int_{\mathbb{R}^3} (X - \langle X \rangle)^2 |\psi|^2 \, d\mathbf{x}.$$

For the uncertainties in position and momentum,  $\Delta Q$  and  $\Delta P$  respectively, we have Heisenberg's uncertainty principle, that is:

$$\Delta Q \Delta P \geq \frac{\hbar}{2}.$$

Since the product of the uncertainties in the operators  $Q$  and  $P$  must always be larger than some constant, it is impossible to simultaneously measure the position and momentum of a quantum particle. States where equality holds in the above are called coherent states which have minimal uncertainty. The proof of Heisenberg's uncertainty principle [18] relies on the noncommutativity of the operators  $Q$  and  $P$ , hence any pair of commutative operators correspond to two simultaneously measurable observables with common eigenvectors.

An alternative formulation of quantum mechanics is the Heisenberg picture. According to this description, it is the observables that evolve in time while the states remain constant, as opposed to the time evolution of the states and constancy of observables represented by the Schrödinger picture. Given Heisenberg's equation of motion for any observable  $X$ :

$$i\hbar \frac{d}{dt} \langle X \rangle = \langle [X, H] \rangle,$$

any observable which commutes with the Hamiltonian must be a constant of the motion.

Throughout this work, we will make much reference to the semiclassical, or correspondence limit of quantum mechanics. This is the notion that classical mechanics may be recovered from a quantum system as the reduced Planck's constant  $\hbar \rightarrow 0$ , while fixing some of the system's physical quantities.

### 1.1.2 The Madelung Fluid

As it is complex-valued, the wave function may be represented as  $\psi = \exp(R + iS)$  where  $R$  and  $S$  are real-valued functions of space and time. For this representation, first note that:

$$\nabla\psi = (\nabla R + i\nabla S)\psi,$$

and consequently:

$$\Delta\psi = |\nabla R + i\nabla S|^2\psi + (\Delta R + i\Delta S)\psi.$$

Given the above, the Schrödinger equation (1.1) becomes:

$$i\hbar \left( \frac{\partial R}{\partial t} + i \frac{\partial S}{\partial t} \right) \psi = -\frac{\hbar^2}{2} (|\nabla R + i\nabla S|^2 + \Delta R + i\Delta S) \psi + V\psi,$$

or equivalently:

$$i\hbar \frac{\partial R}{\partial t} - \hbar \frac{\partial S}{\partial t} = -\frac{\hbar^2}{2} (|\nabla R|^2 + |\nabla S|^2 + 2i\nabla R \cdot \nabla S + \Delta R + i\Delta S) + V. \quad (1.3)$$

The continuity equation for a fluid may be obtained from the above by firstly noting that given  $\psi = \exp(R + iS)$ , then the density  $\rho(\mathbf{x}, t) = \exp(2R(\mathbf{x}, t))$ . It is also necessary to introduce the velocity field  $v(\mathbf{x}, t) = \hbar\nabla S(\mathbf{x}, t)$ . Equating the imaginary parts of (1.3) yields:

$$\frac{\partial R}{\partial t} = -\hbar \left( \nabla R \cdot \nabla S + \frac{\Delta S}{2} \right).$$

Substituting the above into:

$$\frac{\partial \rho}{\partial t} = 2\rho \frac{\partial R}{\partial t},$$

gives the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0. \quad (1.4)$$

On the other hand, equating the real parts of equation (1.3) gives:

$$\hbar \frac{\partial S}{\partial t} + \frac{\hbar^2}{2} |\nabla S|^2 + V - \frac{\hbar^2}{2} (|\nabla R|^2 + \Delta R) = 0. \quad (1.5)$$

Letting  $\hat{S} = \hbar S$ , the above may be written as:

$$\frac{\partial \hat{S}}{\partial t} + \frac{|\nabla \hat{S}|^2}{2} + V - \frac{\hbar^2}{2} \frac{\Delta \exp(R)}{\exp(R)} = 0, \quad (1.6)$$

by also noting that:

$$\nabla(\exp(R)) = \nabla R \exp(R) \quad \text{and} \quad \Delta(\exp(R)) = (\Delta R + |\nabla R|^2) \exp(R).$$

The dynamical system described by the above equations (1.4) and (1.6) is the Madelung fluid model. On inspection of (1.6), it is clear that this equation is analagous to that of the the classical Hamilton-Jacobi fluid, with the additional so-called “quantum mechanical potential” given by:

$$-\frac{\hbar^2}{2} \frac{\Delta \exp(R)}{\exp(R)}.$$

Guerra [16] shows that a particle interpretation of this fluid model is possible by using stochastic processes, following Nelson’s stochastic mechanics [25].

### 1.1.3 Itô Diffusions

Following texts by Gihman and Skorohod, [14], Karatzas and Shreve, [21], McKean, [23] and Øksendal, [27] , we proceed to discuss Brownian motion and the Itô integral, leading to the theory of stochastic differential equations. This provides a precursor to the discussion of Nelson’s stochastic mechanics in Section 1.1.4.

#### Brownian Motion

Brownian motion was originally observed in 1827 by botanist Robert Brown as the seemingly random movement of pollen grains suspended in water. The history of Brownian motion is discussed more extensively in Nelson’s Dynamical Theories of Brownian Motion, [25].

The mathematical model of Brownian motion, also called the Wiener process (denoted  $B_t(\omega)$ , or equivalently  $B(t)$ ) is a temporally homogeneous Gaussian stochastic process on a probability space  $(\Omega, \mathcal{F}, P)$  with independent increments. That is, the distribution of  $B(t)$  is given by:

$$\mathbb{P}(B(t) \in (a, b)) = \frac{1}{\sqrt{2\pi t}} \int_a^b e^{-\frac{x^2}{2t}} dx,$$

where  $B(0) = 0$ ,  $(B(t+h) - B(h))$  has the same distribution as  $B(t)$  for  $t, h > 0$  and  $(B(t+h) - B(h))$  is independent of  $B(s)$  for  $0 \leq s \leq h$ .

The average value of the process is zero,  $\mathbb{E}(B(t)) = 0$ , and  $\mathbb{E}(B^2(t)) = t$ . These results may then be used to show that the covariance of the process is given by:

$$\mathbb{E}(B(t)B(s)) = \min(t, s).$$

The sample paths of Brownian motion are almost surely continuous but nowhere differentiable, thus it is impossible to define integration with respect to  $B(t)$  in the usual manner. This led to the development of Itô calculus (Itô, [20] 1944), beginning with the construction of the Itô integral.

## The Itô Integral

We follow the construction of the Itô integral as given in Øksendal, [27].

**Definition 1.1.1.** Given a probability space  $(\Omega, \mathcal{F}, P)$ , let  $\mathcal{V}$  be a class of functions  $f : [0, \infty) \times \Omega \rightarrow \mathbb{R}$  such that:

- the mapping  $(t, \omega) \mapsto f(t, \omega)$  is  $\mathcal{B}([0, \infty)) - \mathcal{F}$ -measurable.
- $f(t, \cdot)$  is  $\mathcal{F}_t$  adapted.
- $\mathbb{E}(\int_0^t f^2(s, \omega) ds) < \infty$  for all  $t \geq 0$ .

The aim of Itô calculus is to attach a reasonable mathematical interpretation to the integral:

$$\int_0^t f(s, \omega) dB_s(\omega),$$

where  $B_s(\omega)$  denotes a one-dimensional Brownian motion starting at the origin.

As with the Riemann integral, the Itô integral is defined by a limiting procedure. In the case of the Riemann integral, one first defines the integral of a step function and extends the definition to a larger class of functions by approximation. The integral of such an arbitrary function  $f \in \mathcal{V}$  is defined as the limit of the integral of step functions which converge to  $f$  in some sense. The definition of the Itô integral begins with elementary functions  $\phi \in \mathcal{V}$ . The integral is first defined for such functions, and showing that a function  $f$  from a wider class may be approximated by these elementary functions, the integral of  $f$  is defined as the limit of the integral of  $\phi$  as  $\phi \rightarrow f$ .

An elementary function  $\phi \in \mathcal{V}$  has the form:

$$\phi(t, \omega) = \sum_{j \geq 0} e_j(\omega) \chi_{[t_j, t_{j+1})}(t),$$

where  $e_j$  is  $\mathcal{F}_{t_j}$ -measurable and  $\chi$  is the characteristic function. The Itô integral of an elementary function may be reasonably defined by:

$$\int_0^t \phi(s, \omega) dB_s(\omega) = \sum_{j \geq 0} e_j(\omega)(B_{t_{j+1}} - B_{t_j})(\omega).$$

Any function  $f \in \mathcal{V}$  may be approximated by a sequence of elementary functions  $\{\phi_n\} \subset \mathcal{V}$ :

$$\mathbb{E} \left( \int_0^t (f - \phi_n)^2 ds \right) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (1.7)$$

This then allows the Itô integral to be defined as follows:

**Definition 1.1.2.** For  $f \in \mathcal{V}$ , the Itô integral of  $f$  is defined by:

$$\int_0^t f(s, \omega) dB_s(\omega) = \lim_{n \rightarrow \infty} \int_0^t \phi_n(s, \omega) dB_s(\omega), \quad (1.8)$$

where  $\{\phi_n\} \subset \mathcal{V}$  is a sequence of elementary functions such that:

$$\mathbb{E} \left( \int_0^t (f(s, \omega) - \phi_n(s, \omega))^2 ds \right) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

In the above definition the convergence of (1.8) is in  $L^2(P)$ , and the postulated sequence of functions  $\{\phi_n\}$  exist by (1.7). By construction, the limit (1.8) is independent of the sequence of elementary functions.

The first important property to note is that the Itô integral has zero mean:

$$\mathbb{E} \left( \int_0^t f(s, \omega) dB_s(\omega) \right) = 0.$$

The fundamental Itô isometry gives that:

$$\mathbb{E} \left( \left( \int_0^t f(s, \omega) dB_s(\omega) \right)^2 \right) = \mathbb{E} \left( \int_0^t f^2(s, \omega) ds \right).$$

Furthermore it is important to note that  $\int_0^t f dB_s(\omega)$  is a martingale with respect to the canonical filtration  $\{\mathcal{F}_t\}$ .

## Stochastic Differential Equations

With a reasonable definition of the Itô integral, it is now possible to consider  $d$ -dimensional stochastic processes  $X_t$  satisfying integral equations of the form:

$$X_t = X_0 + \int_0^t b(X_s, s) ds + \int_0^t \sigma(X_s, s) dB(s),$$

where the stochastic process  $X_t$  is called an Itô diffusion and  $X_0$  is a deterministic initial condition. The above equation may be written in the conventional form:

$$dX_t = b(X_t, t) dt + \sigma(X_t, t) dB(t),$$

which is a general stochastic differential equation with drift coefficients  $b = (b_1, b_2, \dots, b_d)^T$  and diffusion coefficient matrix  $\sigma = \{\sigma_{ij}\}$ ,  $i, j = 1, 2, \dots, d$ . Here,  $B(t) = (B_1(t), B_2(t), \dots, B_d(t))$  is a  $d$ -dimensional Brownian motion, where each constituent  $B_i$  is a Brownian motion independent of  $B_j$  for  $i \neq j$ . Thus the covariance is given by:

$$\mathbb{E}(B_i(s)B_j(t)) = \delta_{ij} \min(s, t).$$

For the purposes of this work, it will be sufficient to consider stochastic differential equations (SDEs) with time-homogenous drift and constant diffusion coefficients. That is, for  $\epsilon > 0$ , equations of the form:

$$dX_t = b(X_t) dt + \epsilon dB(t). \tag{1.9}$$

It is often necessary to consider functions of the process  $X_t$  which satisfies the above SDE. Itô's formula provides a stochastic analogue of the chain rule for differentiation of deterministic functions.

**Proposition 1.1.3 (Itô's Formula).** Let  $f : \mathbb{R}^d \times [0, \infty) \rightarrow \mathbb{R}$  be a continuous function, with first order derivative in  $t$  and second order derivatives in  $x$ . Then for  $X_t$  satisfying (1.9),  $Y_t := f(X_t, t)$  is again a stochastic integral defined by:

$$dY_t = \left( \frac{\partial}{\partial t} + b \cdot \nabla + \frac{\epsilon^2}{2} \Delta \right) f(X_t, t) dt + \epsilon \nabla f(X_t, t) \cdot dB(t).$$

The forward Kolmogorov equation is a partial differential equation describing the evolution of the transition density of a stochastic process, which follows directly from an application of Itô's formula.

**Corollary 1.1.4 (Kolmogorov's Equation).** Let  $X_t$  satisfy (1.9). Assume that  $X_t$  has a smooth transition density  $\rho_t(y)$  given by:

$$\mathbb{P}(X_t \in A | X_0 = x) = \int_A \rho_t(x, y) dy$$

for a Borel set  $A \subset \mathbb{R}^d$ . Then  $\rho_t(x, y)$  satisfies the forward Kolmogorov equation:

$$\frac{\partial \rho_t(y)}{\partial t} = \frac{\epsilon^2}{2} \Delta \rho_t(y) - \nabla \cdot b \rho_t(y) \quad \text{with} \quad \lim_{t \downarrow 0} \rho_t(x, y) = \delta_x(y).$$

### Existence and Uniqueness of Solutions

**Proposition 1.1.5.** If for all  $x, y \in \mathbb{R}^d$ ,  $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$  satisfies the global Lipschitz condition  $|b(x) - b(y)| \leq K|x - y|$  for some constant  $K$  then the stochastic differential equation:

$$X_t = X_0 + \int_0^t b(X_s) ds + \epsilon \int_0^t dB(s) \quad (1.10)$$

has a unique solution.

The solution of the SDE postulated by the above Proposition 1.1.5 is meant in the strong sense, whereby an  $\mathcal{F}_t$  adapted solution may be constructed on any given probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with a given Brownian motion  $B(t)$ , where  $\mathcal{F}_t = \sigma(B(t))$ . In contrast, the existence of a weak solution means that it is possible to find a process  $X_t$  with a driving Brownian motion  $B(t)$  on some particular probability space, such that the process satisfies the SDE. A strong solution is also a weak solution, but the converse is not true in general.

The uniqueness obtained by Proposition 1.1.5 is pathwise uniqueness, which is said to hold for equation (1.10) if given two solutions  $X_t$  and  $\hat{X}_t$ , then  $\mathbb{P}[X_t = \hat{X}_t, \forall t] = 1$  on a given probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  for a given  $\mathcal{F}_t$  adapted Brownian motion  $B(t)$ . Thus given such a probability space and Brownian motion there is one and only one stochastic process satisfying the SDE (1.10). The corresponding uniqueness concept for weak solutions is uniqueness in law, which is said to hold for (1.10) if any two weak solutions  $X_t$  and  $\hat{X}_t$  (possibly on different probability spaces with different driving Brownian motions) are equal in law. Uniqueness in law is implied by pathwise uniqueness but the converse is not necessarily true. Full details may be found in Øksendal, [27].

#### 1.1.4 Nelson's Stochastic Mechanics

From the continuity equation for the Madelung fluid (1.4):



$$\frac{\partial \rho}{\partial t} = \nabla \cdot (-\hbar \exp(2R) \nabla S) = \nabla \cdot \left( \frac{\hbar}{2} \nabla \exp(2R) - \hbar \exp(2R) \nabla (R + S) \right). \quad (1.11)$$

Given that the density  $\rho = \exp(2R)$ , the above equation recapitulates the forward Kolmogorov equation (detailed in Corollary 1.1.4) for a stochastic process with drift  $b = \hbar \nabla (R + S)$  and diffusion coefficient  $\epsilon$ :

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \left( \frac{\epsilon^2}{2} \nabla \rho - \rho b \right),$$

where  $\epsilon^2 = \hbar$ .

Since it is possible to obtain the forward Kolmogorov equation directly from the Schrödinger equation, this implies that given any solution to the Schrödinger equation (the wave function in the form  $\exp(R + iS)$ ), it should be possible to construct a corresponding diffusion process:

$$dX(t) = b(X(t), t) dt + \epsilon dB(t). \quad (1.12)$$

Equivalently to wave mechanics, the movement of the quantum particle of unit mass may be modelled by the above diffusion process, with  $X(t)$  being the position of the particle at time  $t$ . This is known as the Nelson diffusion, the foundation of the theory of Nelson's stochastic mechanics [25].

Since the sample paths of the Nelson diffusion are nowhere differentiable, it is necessary to define the mean forward and backward derivatives of a function of  $X(t)$  as follows:

$$D_{\pm} f(X(t), t) := \lim_{h \downarrow 0} \mathbb{E} \left( \frac{f(X(t \pm h), t \pm h) - f(X(t), t)}{\pm h} \middle| X(t) \right).$$

An application of Itô's formula gives the mean forward derivative as:

$$D_+ f(X(t), t) = \left( \frac{\partial}{\partial t} + b \cdot \nabla + \frac{\epsilon^2}{2} \Delta \right) f(X(t), t).$$

The mean forward velocity is defined as the mean forward derivative of the process  $X(t)$  itself, which is just given by the drift of the Nelson diffusion:

$$b_+(X(t), t) := D_+ X(t) = b(X(t), t).$$

Partial integration then implies the mean backward derivative is given by:

$$D_- f(X(t), t) = \left( \frac{\partial}{\partial t} + b_+ - \epsilon^2 \ln \rho \cdot \nabla + \frac{\epsilon^2}{2} \Delta \right) f(x(t), t),$$

which allows the mean backward velocity to be defined as follows:

$$b_-(X(t), t) := D_- X(t) = b_+(X(t), t) - \epsilon^2 \nabla \ln \rho(X(t), t).$$

Since it is necessary to have a notion of second derivative for processes such as (1.12), Nelson conventionally defines the stochastic acceleration to be the following combination of operators acting on  $X(t)$ :

$$a(X(t), t) = \frac{1}{2}(D_+ D_- + D_- D_+) X(t).$$

It follows that:

$$a(X(t), t) = \left( \frac{\partial v}{\partial t} + (v \cdot \nabla)v - (u \cdot \nabla)u - \frac{\epsilon^2}{2} \Delta u \right) (X(t), t), \quad (1.13)$$

where  $u$  is called the osmotic velocity and  $v$  the current velocity, defined as follows:

$$u := \frac{1}{2}(b_+ - b_-) \quad \text{and} \quad v := \frac{1}{2}(b_+ + b_-).$$

From (1.11), the drift, and hence the mean forward derivative of the Nelson diffusion may be written explicitly in terms of the real-valued functions  $R$  and  $S$ , as  $\epsilon^2 \nabla(R + S)$ . It follows that the mean backward derivative is given by  $b_- = \epsilon^2 \nabla(R - S)$ , with  $u = \epsilon^2 \nabla R$  and  $v = \epsilon^2 \nabla S$ ; the respective osmotic and current velocities. Note that the current velocity of the Nelson diffusion is the same as the velocity field of the Madelung fluid as given in the previous section, with  $\epsilon^2 = \hbar$ . Substituting the osmotic and current velocities into the above expression for the stochastic acceleration (1.13) yields:

$$a(X(t), t) = -\nabla \left( -\epsilon^2 \frac{\partial S}{\partial t} + \frac{\epsilon^4}{2} (|\nabla R|^2 + \Delta R - |\nabla S|^2) \right) (X(t), t). \quad (1.14)$$

Recall, that after writing the wave function in the form  $\exp(R + iS)$  and equating real parts of the Schrödinger equation (1.5), we have:

$$V = -\epsilon^2 \frac{\partial S}{\partial t} + \frac{\epsilon^2}{4} (|\nabla R|^2 + \Delta R - |\nabla S|^2), \quad (1.15)$$

with  $V$  being the potential function from the Schrödinger equation. Substituting (1.15) into (1.14) finally gives:

$$a(X(t), t) = -\nabla V(X(t)).$$

The above equation is the Nelson-Newton law for a particle of unit mass. Since the stochastic acceleration is equal to the negative gradient of some potential function, this provides a stochastic analogue of Newton's second law for classical mechanics.

## 1.2 The Kepler Problem

We begin by considering the two-dimensional classical Kepler problem by first discussing the conic sections [31] which constitute the basic trajectories arising from it. The classical problem [15, 19] outlines the mechanics that should be regained by taking the correspondence limit of the quantum Kepler problem [18], which is discussed thereafter.

### 1.2.1 Conic Sections

In polar coordinates, relative to the focus as the origin, any conic section may be represented as the locus of points  $(r, \theta)$  for which:

$$r = \frac{k}{1 - e \cos \theta}, \quad (1.16)$$

where  $k > 0$  is a simple scale factor governing the size of the conic, and  $e$  is a parameter known as the eccentricity, which determines the shape of the curve. In the case that:

- $e = 0$ , equation (1.16) defines a circle (since the radius  $r = k$  and is therefore constant),
- $0 < e < 1$  the equation defines an ellipse,
- $e = 1$  the equation defines a parabola,
- $e > 1$  the equation defines a hyperbola.

The eccentricity may be thought of as a measure of how deviant a conic is from the circular case.

The conic sections have particular importance in astronomy, since Kepler first noted that the orbits of all planets about the sun are elliptic. Newton later generalised this, modelling trajectories of other celestial bodies as generalised conic sections.

Equivalently to the polar form (1.16), an ellipse may be defined relative to the centre as the origin in Cartesian coordinates by:

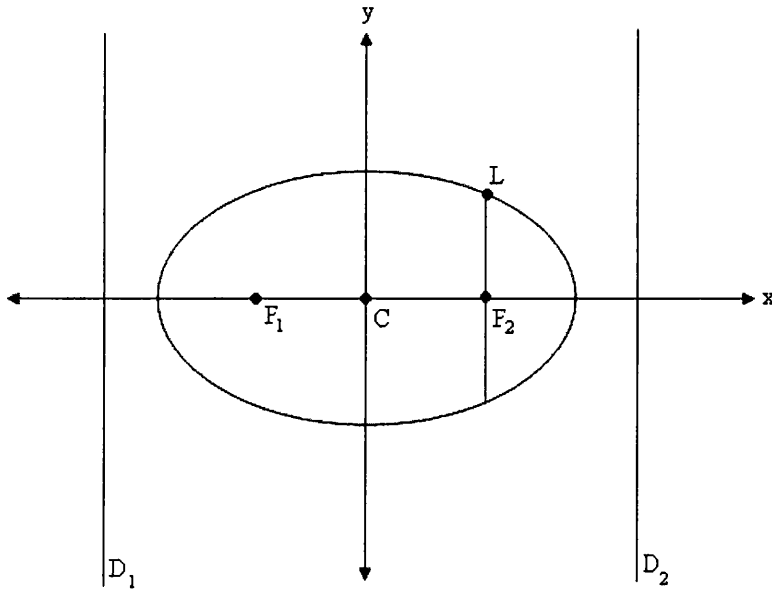


Figure 1.1: The foci and directrices of an ellipse.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad (1.17)$$

where  $a$  and  $b$  are the lengths of the semimajor and semiminor axes of the ellipse respectively, as shown by Figure 1.1.

The two foci of the ellipse are labelled  $F_1$  and  $F_2$ , which are equidistant from the centre  $C$ , which we take as the origin of the coordinate system, with the axes parallel to the axes of the ellipse. The lines  $D_1$  at  $x = -a/e$  and  $D_2$  at  $x = a/e$  are known as the directrices of the ellipse. For each focus-directrix pair,  $F_i$  and  $D_i$ , and any point  $P$  on the ellipse, the distance between  $F_i$  and  $P$  is  $e$  times the perpendicular distance between  $P$  and the directrix  $D_i$ . That is,  $|PF_i| = e|PD_i|$  for  $i = 1, 2$ . The chord perpendicular to the major axis through either of the foci is known as the latus rectum, with its length denoted by  $2l$ . In this representation of the ellipse, the eccentricity is defined to be the ratio of the semimajor axis to the distance between the centre and one of the foci, hence this distance is given by  $ae$ . The coordinates of the point  $L$  are thus  $(ae, l)$ . Substituting this point into the Cartesian representation of an ellipse, (1.17) gives:

$$e^2 + \frac{l^2}{b^2} = 1,$$

so we may represent the length of the semiminor axis by:

$$b = \frac{l}{\sqrt{1 - e^2}}. \quad (1.18)$$

The distance property states that for a given point  $P$  on the ellipse:

$$|PF_1| + |PF_2| = 2a.$$

Since the point  $B$  is equidistant from each of the foci, the distance between  $B$  and either of the foci must be  $a$ . Pythagoras' Theorem then gives:

$$b^2 + (ae)^2 = a^2.$$

Substituting (1.18) into the above yields the following expression for the length of the semimajor axis:

$$a = \frac{l}{1 - e^2}. \quad (1.19)$$

As with the ellipse, the parabola has an equivalent Cartesian representation:

$$y^2 = 4ax. \quad (1.20)$$

The parabola has just one focus-directrix pair, the point  $(a, 0)$  and the line at  $x = -a$  when defined by the equation above with the origin of the coordinate system is at the apex of the parabola. (Figure 1.2.)

The parabola is the locus of all points for which their distance from the focus and perpendicular distance from the directrix are equal; a special case of the above property for the ellipse with  $e = 1$ . The parabola may be thought of as an ellipse with one of its foci at infinity, since the above expression for the length of the ellipse's semimajor axis (1.19) tends to infinity as the eccentricity tends to 1.

## 1.2.2 The Classical Kepler Problem

The classical Kepler problem is a special case of the two-body problem in classical mechanics which arises from applying the universal law of gravitation and Newton's second law to the motion of a planet around the sun.

According to the universal law of gravitation, any two particles attract each other with a force that is directly proportional to the product of their

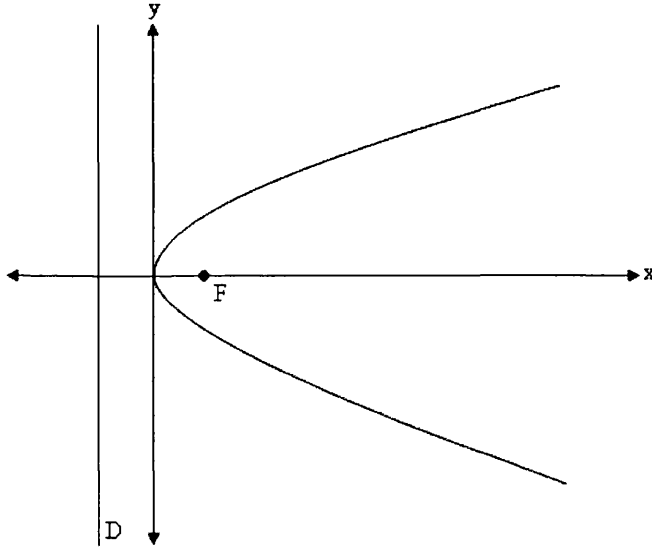


Figure 1.2: The focus and directrix of a parabola.

masses and inversely proportional to the square of the distance between them. Given two particles with masses  $m_A$  and  $m_B$  and position vectors  $\mathbf{r}_A$  and  $\mathbf{r}_B$  respectively, the force exerted on particle  $A$  due to particle  $B$  is given by:

$$\mathbf{F}_{AB} = -\frac{\gamma m_A m_B (\mathbf{r}_A - \mathbf{r}_B)}{|\mathbf{r}_A - \mathbf{r}_B|^3},$$

where  $\gamma$  is the universal constant of gravitation. The term  $|\mathbf{r}_A - \mathbf{r}_B|$  appears cubed in the denominator since the numerator contains  $(\mathbf{r}_B - \mathbf{r}_A)$ .

We work in the sidereal reference frame, the inertial frame where the sun is fixed at the origin and axes are determined by distant gravitating masses. Due to spherical symmetry and radial mass dependence, we may treat a planet as a gravitating point mass subject to a force according to the universal law of gravitation. Since the mass of the planet is so small compared to the mass of the sun, the force exerted on the sun by the planet may be considered negligible. Letting  $\mathbf{r}$  be the position vector of the planet, applying Newton's second law gives:

$$m_p \ddot{\mathbf{r}} = -\frac{\gamma m_s m_p \mathbf{r}}{r^3},$$

for  $m_p$  and  $m_s$  being the respective masses of the planet and the sun and  $r = |\mathbf{r}|$ . Dividing by  $m_p$  yields:

$$\ddot{\mathbf{r}} = -\frac{\mu\mathbf{r}}{r^3}, \quad (1.21)$$

the equation of motion for the Kepler problem where  $\mu = \gamma m_s$  is the constant known as the gravitational mass of the sun. Note that the scalar potential for the Kepler problem is the Coulomb potential:

$$V(r) = -\frac{\mu}{r}.$$

Kepler's Laws of planetary motion may then be derived by integrating the above equation of motion. Firstly note that for  $r^2 = \mathbf{r} \cdot \mathbf{r}$ , differentiation with respect to time gives:

$$\mathbf{r} \cdot \dot{\mathbf{r}} = r\dot{r}. \quad (1.22)$$

Taking the scalar product of (1.21) with  $\dot{\mathbf{r}}$ , the equation of motion becomes:

$$\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} = -\frac{\mu\mathbf{r} \cdot \dot{\mathbf{r}}}{r^3} = -\frac{\mu r\dot{r}}{r^3} = -\frac{\mu\dot{r}}{r^2}$$

by using the identity (1.22). Integrating the above, we see that:

$$\frac{d}{dt} \left( \frac{\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}}{2} - \frac{\mu}{r} \right) = 0,$$

hence:

$$\frac{\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}}{2} - \frac{\mu}{r} = E, \quad (1.23)$$

a constant which gives the energy per unit mass of the planet. Next, taking the vector product of  $\mathbf{r}$  with (1.21) automatically implies the constancy of the angular momentum vector. Note that  $\mathbf{r} \times \ddot{\mathbf{r}} = 0$ , which integrates to give:

$$\frac{d}{dt}(\mathbf{r} \times \dot{\mathbf{r}}) = 0.$$

Hence  $\mathbf{L} = \mathbf{r} \times \dot{\mathbf{r}}$ , the angular momentum per unit mass of the planet is a constant vector. This in turn implies that the areal velocity:

$$\frac{dA}{dt} = \frac{|\mathbf{L}|}{2},$$

is constant.

Taking the vector product of the equation of motion (1.21) with the angular momentum  $\mathbf{L}$  then gives:

$$\mathbf{L} \times \ddot{\mathbf{r}} = \frac{\mu \dot{\mathbf{r}}}{r^2} - \frac{\mu \dot{\mathbf{r}}}{r},$$

which integrates to give:

$$\frac{d}{dt} \left( \mathbf{L} \times \dot{\mathbf{r}} + \frac{\mu \mathbf{r}}{r} \right) = 0.$$

From the above equation we see that

$$\mathbf{Z} := \mathbf{L} \times \dot{\mathbf{r}} + \frac{\mu \mathbf{r}}{r}$$

is also a constant vector. This is the Lenz-Runge vector, which points in the direction of the orbit's major axis. Taking the scalar product of  $\mathbf{Z}$  with the position vector  $\mathbf{r}$  leads to the polar equation for the planet's orbit since:

$$\mathbf{r} \cdot \mathbf{Z} = \mathbf{r} \cdot \left( \mathbf{L} \times \dot{\mathbf{r}} + \frac{\mu \mathbf{r}}{r} \right) = -L^2 + \mu r,$$

for  $L = |\mathbf{L}|$ . On the other hand  $\mathbf{r} \cdot \mathbf{Z} = rZ \cos \theta$ , where  $Z = |\mathbf{Z}|$ , and  $\theta$  is the angle between the vectors  $\mathbf{r}$  and  $\mathbf{Z}$ , which is equal to the polar angle of the position of the planet since the Lenz-Runge vector is parallel to the orbit's major axis. Hence  $rZ \cos \theta = -L^2 + \mu r$ , and the polar equation of the orbit may be written in the form:

$$\frac{l}{r} = 1 - e \cos \theta. \quad (1.24)$$

This is the polar equation of a conic section, where we have set the eccentricity  $e = \frac{Z}{\mu}$  and the length of the semilatus rectum  $l = \frac{L^2}{\mu}$ . From the previous section, we know that we may classify the shape of conics according to the value of the parameter  $e$ . We may now relate this parameter to the conservation laws for the orbits as follows:

$$Z^2 = \mu^2 + L^2 \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} - \frac{2L^2 \mu}{r},$$

therefore:

$$e^2 = \frac{Z^2}{\mu^2} = 1 + \frac{2L^2}{\mu^2} \left( \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} - \frac{\mu}{r} \right) = 1 + \frac{2L^2 E}{\mu^2}.$$

Hence in terms of the angular momentum and energy constants  $L$  and  $E$  and the gravitational mass of the sun, an orbit's eccentricity is given by:

$$e = \sqrt{1 + \frac{2L^2 E}{\mu^2}}. \quad (1.25)$$



The above equation shows that for bound periodic orbits  $E < 0$  is required, since this value would allow  $0 < e < 1$ . The zero energy state gives  $e = 1$ , and thus a parabolic orbit, and values of  $E$  greater than 1 will define hyperbolic orbits with  $e > 1$ .

### 1.2.3 The Levi-Civita Transformation

The Levi-Civita transformation provides a connection between the two most fundamental problems in classical mechanics by simplifying the two-dimensional Kepler problem into a simple harmonic oscillator.

The transformation requires the position vector in the Kepler problem to be represented as a time-dependent point in the complex plane  $z(t) = x(t) + iy(t)$ . The new coordinate system used is a complex number  $w$ , defined by  $w^2 = z$ . A full account of the technicalities of the transformation may be found in either Arnold, Kozlov and Neishtadt, [1] or Steifel and Schiefele, [32].

Throughout,  $t$  will denote the physical time variable, with  $\dot{z} = \frac{dz}{dt}$ . The new time variable required is defined by

$$ds = \frac{dt}{|z|}.$$

This is the so-called fictitious time variable and  $\frac{dz}{ds}$  will be denoted by  $z'$ .

In terms of the complex variable  $z$ , the Kepler problem is:

$$\ddot{z} + \frac{\mu z}{|z|^3} = 0. \quad (1.26)$$

Letting  $|z| = r$ , the derivatives with respect to physical time may be computed in terms of the fictitious time derivatives as follows:

$$\dot{z} = \frac{dz}{dt} = \frac{dz}{ds} \frac{ds}{dt} = \frac{z'}{r},$$

and:

$$\ddot{z} = \frac{d}{dt} \left( \frac{z'}{r} \right) = \frac{z''}{r^2} - \frac{r' z'}{r^3}.$$

Substituting in the derivatives, the above Kepler problem (1.26) becomes:

$$r z'' - r' z' + \mu z = 0. \quad (1.27)$$

Now defining  $w^2 = z$ , differentiation with respect to time variable  $s$  gives:

$$z' = 2ww' \quad \text{and} \quad z'' = 2(w')^2 + 2ww''.$$

In terms of  $w$ , (1.27) becomes:

$$r(2w' + 2ww'') - 2r'ww' + \mu w^2 = 0. \quad (1.28)$$

Since  $r = |z| = |w|^2$ , then  $r = w\bar{w}$ , where  $\bar{w}$  denotes the complex conjugate of  $w$ . The product rule for differentiation then yields  $r' = w'\bar{w} + w\bar{w}'$ . These expressions for  $r$  and  $r'$  may then be substituted into the above expression (1.28), finally yielding:

$$w'' + w \left( \frac{\frac{\mu}{2} - |w'|^2}{|w|^2} \right) = 0. \quad (1.29)$$

Recall the energy conservation for the Kepler problem, (1.23). In terms of  $z$ :

$$E = \frac{|z|^2}{2} - \frac{\mu}{|z|} \quad \implies \quad E = \frac{|z'|^2}{2|z|^2} - \frac{\mu}{|z|}.$$

Replacing  $z'$  by  $2ww'$  gives energy conservation in terms of  $w$  as follows:

$$E = \frac{2|w'|^2}{|w|^2} - \frac{\mu}{|w|^2}. \quad (1.30)$$

Given that the above expression (1.30) is a constant, note that the coefficient

$$w'' + \omega^2 w = 0,$$

where  $\omega^2 = -\frac{E}{2}$

Therefore under the well-defined time change and change of variable from

where  $\omega^2 = -\frac{E}{2}$

Therefore under the well-defined time change and change of variable from  $z$  to  $w$ , the equation of motion for the Kepler problem has been reduced to a simple harmonic oscillator in the above form, provided  $E < 0$ . The effect is to transform bounded motion on an ellipse into simple harmonic motion. The result of the transformation is unbounded motion for both  $E = 0$  and  $E > 0$ .

## 1.2.4 The Quantum Kepler Problem

The quantum Kepler problem is the quantum mechanical problem describing the interaction between the nucleus and single electron in the hydrogen atom due to the Coulomb potential  $V(r) = -\frac{\mu}{r}$ , for some force constant  $\mu$  and

$r = |\mathbf{x}|$  for a three-dimensional vector  $\mathbf{x}$ . The stationary (unit mass) Schrödinger equation for this problem is:

$$-\frac{\hbar^2}{2}\Delta\psi(\mathbf{x}) - \frac{\mu}{r}\psi(\mathbf{x}) = E\psi(\mathbf{x}). \quad (1.31)$$

Finding the eigenvalues of the Hamiltonian:

$$H = -\frac{\hbar^2}{2}\Delta\psi(\mathbf{x}) - \frac{\mu}{r}\psi(\mathbf{x}),$$

gives the permissible values of  $E$  in the above equation (1.31). These values are the so-called spectrum of hydrogen given by the Balmer formula:

$$E_n = -\frac{1}{2\hbar^2 n^2}$$

where multiplicative constants may be assumed equal to 1 and  $n$  is a parameter known as the principal quantum number, to be defined below.

Quantum mechanical angular momentum may then be defined as  $L = Q \times P$  where in explicit component form:

$$L = (Q_2 P_3 - Q_3 P_2, Q_3 P_1 - Q_1 P_3, Q_1 P_2 - Q_2 P_1).$$

The commutation relations above for the operators  $Q$  and  $P$  then imply the commutation relations for the angular momentum ( $j, k = 1, 2, 3$ ):

- $[L_j, Q_k] = i\hbar\epsilon_{jkl}Q_l,$
- $[L_j, P_k] = i\hbar\epsilon_{jkl}P_l,$
- $[L_j, L_k] = i\hbar\epsilon_{jkl}L_l,$

by using the permutation tensor of three indices  $\epsilon_{jkl}$ . In more generality, any three operators which satisfy the third relation in the above may be called angular momentum operators.

It is also necessary to define the total angular momentum  $L^2$  by  $L_1^2 + L_2^2 + L_3^2$ . Each component of the angular momentum  $L_j$  for  $j = 1, 2, 3$  commutes with the total angular momentum operator, and therefore it is possible to simultaneously measure total angular momentum and one of its components.

As for the classical system, the total angular momentum is a constant of the motion since both  $L^2$  and  $L_3$  commute with the Hamiltonian  $H$ . Hence given the operators  $L^2$ ,  $L_3$  and  $H$ , it is possible to find a common eigenstate of all three, the state in which they may be simultaneously measured.

Analogously to the concept introduced for the classical problem in Section 1.2.2, another constant of the motion, the Hamilton-Lenz-Runge vector, is defined by:

$$Z = (Z_1, Z_2, Z_3) = \frac{1}{\sqrt{-2E}} \left( \frac{P \times L - L \times P}{2} - \frac{Q}{|Q|} \right),$$

where  $E$  is the total energy, the eigenvalue of the Hamiltonian. Moreover:

$$[Z_1, Z_2] = i\hbar L_3, \quad [Z_2, L_3] = i\hbar Z_1, \quad [L_3, Z_1] = i\hbar Z_2.$$

Therefore the constants of the motion  $Z_1$ ,  $Z_2$  and  $L_3$  form a set of angular momentum operators.

Due to the spherical symmetry of the Coulomb potential, it is possible to separate variables into spherical polar coordinates  $(r, \theta, \phi)$  so that the wave function becomes  $\psi = R(r)\Theta(\theta)\Phi(\phi)$ . After rewriting the above Schrödinger equation in terms of spherical polars, it is possible to find its solution, a wave function written in the form:

$$\psi_{n,l,m} = R(r)Y_l^m(\theta, \phi)$$

where the function  $Y_l^m$  is a spherical harmonic of degree  $l$ . The numbers  $n$ ,  $l$  and  $m$  denote the principal quantum number, orbital angular momentum and magnetic quantum number respectively (with  $n \in \mathbb{N}$ ,  $l = 0, 1, \dots, n-1$  and  $m = -l, \dots, l$ ). These numbers are defined by the following eigenvalue relations:

$$L^2 \Psi_{n,l,m} = \hbar^2 l(l+1) \Psi_{n,l,m}, \quad L_3 \Psi_{n,l,m} = \hbar m \Psi_{n,l,m}, \quad H \Psi_{n,l,m} = -\frac{1}{2\hbar^2 n^2} \Psi_{n,l,m}.$$

for the operators  $L^2$ ,  $L_3$  and  $H$ , the commutativity of which assures the attainability of their simultaneous eigenvalues.

These concepts will be required in our subsequent consideration of specific Schrödinger wave functions, namely the atomic circular and atomic elliptic states.

# Chapter 2

## Introduction to the Two-Dimensional Parabolic Nelson Diffusion

### 2.1 A Review of the Divine Clockwork

In the publication *The Divine Clockwork* [9] by Durran, Neate and Truman, Nelson's stochastic mechanics (as detailed in Section 1.1.4) is used as a context in which to analyse the Bohr correspondence limit of the atomic elliptic state for the Coulomb potential. This representation of the semiclassical limit allows for investigation of particle trajectories which are then shown to converge to Keplerian motion on an ellipse, thus obtaining Kepler's laws of planetary motion in a quantum mechanical setting.

#### 2.1.1 The Wave Function for the Atomic Elliptic State

Following the results of an earlier work by Lena, Delande and Gay [5], Durran, Neate and Truman use the Schrödinger wave function for the atomic circular state in  $\mathbb{R}^3$  to obtain the wave function for the atomic elliptic state. Recall that  $\Psi_{n,l,m}$  is the Schrödinger wave function for the Hamiltonian with the Coulomb potential with force constant  $\mu$ :

$$H(P, Q) = \frac{P^2}{2} - \frac{\mu}{|Q|},$$

with  $Q$  and  $P$  being the respective position and momentum operators for the orbiting quantum particle, and  $n$ ,  $l$  and  $m$  the principal, orbital angular momentum and magnetic quantum numbers. Consistent with this notation,

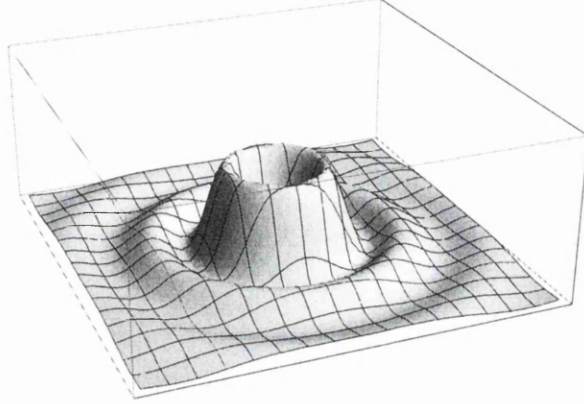


Figure 2.1: The probability density  $|\Psi_{n,n-1,n-1}|^2$  of the atomic circular state in the plane  $z = 0$ .

the atomic circular state is denoted  $\Psi_{n,n-1,n-1}$ . For this state it follows that:

$$\langle L_1 \rangle = \langle L_2 \rangle = 0 \quad \text{and} \quad \langle L_3 \rangle = \hbar(n-1),$$

for the angular momentum operator  $L = (L_1, L_2, L_3)$ . Recalling the quantum Hamilton-Lenz-Runge vector  $Z = (Z_1, Z_2, Z_3)$  (see Section 1.2.4), for the state  $\Psi_{n,n-1,n-1}$ :

$$\langle Z_1 \rangle = \langle Z_2 \rangle = \langle Z_3 \rangle = 0.$$

Since only one component of angular momentum has a nonzero expected value, the associated classical motion occurs in a plane. Given that the classical Hamilton-Lenz-Runge vector is proportional to the eccentricity of the Kepler orbit,  $\Psi_{n,n-1,n-1}(\mathbf{x})$  corresponds to the orbit with eccentricity 0, a circle. The probability density  $|\Psi_{n,n-1,n-1}|^2$  is shown to be concentrated on concentric circles in Figure 2.1.

The state considered in The Divine Clockwork is the atomic elliptic state:

$$\Psi_{\epsilon,n} = \exp\left(-\frac{i\theta Z_2}{\epsilon^2}\right) \Psi_{n,n-1,n-1},$$

where  $\exp\left(-\frac{i\theta Z_2}{\epsilon^2}\right)$  is a unitary operator acting on the wave function with some parameter  $\theta$ . (Here  $\epsilon^2 = \hbar$ .) For the state  $\Psi_{\epsilon,n}$ :

$$\langle L_3 \rangle = \epsilon^2(n-1) \cos \theta \quad \text{and} \quad \langle Z_1 \rangle = \epsilon^2(n-1) \sin \theta,$$

with:

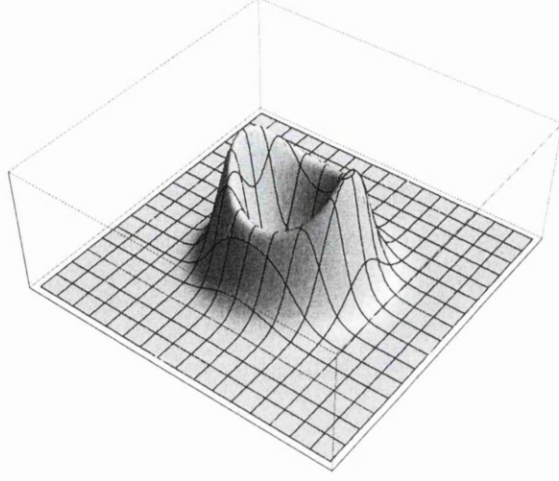


Figure 2.2: The probability density  $|\Psi_{\epsilon,n}|^2$  for the atomic elliptic state in the plane  $z = 0$ .

$$\langle L_1 \rangle = \langle L_2 \rangle = \langle Z_2 \rangle = \langle Z_3 \rangle = 0.$$

Given the above relations,  $\Psi_{\epsilon,n}$  corresponds to a classical elliptic trajectory, with eccentricity given by  $e = \sin \theta$ . The probability density  $|\Psi_{\epsilon,n}|^2$  is concentrated on the ellipse, as shown in Figure 2.2. A clever argument from [5] using coherent state representation and the Kustaanheimo-Stiefel transformation yields:

$$\Psi_{\epsilon,n}(\mathbf{x}) = \exp\left(-\frac{n\mu}{\lambda^2}|\mathbf{x}|\right)\mathcal{L}_{n-1}(n\nu), \quad (2.1)$$

where:

$$\nu = \frac{\mu}{\lambda^2} \left( |\mathbf{x}| - \frac{x}{e} - \frac{iy\sqrt{1-e^2}}{e} \right), \quad (2.2)$$

and  $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$ ,  $\epsilon^2 = \hbar$ ,  $\lambda = n\epsilon^2$  and  $\mathcal{L}_{n-1}$  denotes a Laguerre polynomial, defined consistently with Buchholz [3]:

$$\mathcal{L}_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n} (e^{-x} x^n)$$

with the exponential function denoted  $e^x$ .

## 2.1.2 The Nelson Diffusion for the Atomic Elliptic State

By writing  $\Psi_{\epsilon,n} = \exp(R_{\epsilon,n} + iS_{\epsilon,n})$  for real-valued  $R_{\epsilon,n}$  and  $S_{\epsilon,n}$ , as in Section 1.1.4, it is possible to construct a Nelson diffusion associated with the atomic elliptic state. The diffusion  $X_{\epsilon,n}$  satisfies:

$$dX_{\epsilon,n}(t) = b_{\epsilon,n}(X_{\epsilon,n}(t)) dt + \epsilon dB(t)$$

with:

$$b_{\epsilon,n} = \epsilon^2 \nabla(R_{\epsilon,n} + S_{\epsilon,n}).$$

The wave function  $\Psi_{\epsilon,n}$  satisfies the time-independent Schrödinger equation in the form:

$$-\frac{\epsilon^4}{2} \Delta \Psi_{\epsilon,n}(\mathbf{x}) - \frac{\mu}{|\mathbf{x}|} \Psi_{\epsilon,n}(\mathbf{x}) = E_n \Psi_{\epsilon,n}(\mathbf{x}),$$

where  $E_n = -\frac{\mu}{2\lambda^2}$ , the eigenvalue of the Hamiltonian. We may then define:

$$Z_{\epsilon,n} := -i\epsilon^2 \frac{\nabla \Psi_{\epsilon,n}(\mathbf{x})}{\Psi_{\epsilon,n}} = \epsilon^2 \nabla(S_{\epsilon,n} - iR_{\epsilon,n}), \quad (2.3)$$

Note that the drift of the Nelson diffusion may be given by:

$$b_{\epsilon,n} = \text{Re}(Z_{\epsilon,n}(\mathbf{x})) - \text{Im}(Z_{\epsilon,n}(\mathbf{x})).$$

Using equations (2.1) and (2.2) from the previous section, in Cartesian coordinates:

$$Z_{\epsilon,n} = \frac{i\mu}{\lambda} \left( 1 - \frac{\mathcal{L}'_{n-1}(n\nu)}{\mathcal{L}_{n-1}(n\nu)} \right) \frac{\mathbf{x}}{|\mathbf{x}|} + \frac{\mu}{\lambda e} \frac{\mathcal{L}'_{n-1}(n\nu)}{\mathcal{L}_{n-1}(n\nu)} (i, -\sqrt{1-e^2}, 0). \quad (2.4)$$

The time evolution of the Nelson diffusion defined above is simulated in Figure 2.3, its trajectory converging rapidly to an ellipse.

## 2.1.3 The Limiting Wave Function

The Bohr correspondence limit of the wave function  $\Psi_{\epsilon,n}$  is its limit as  $n \rightarrow \infty$  and  $\epsilon \rightarrow 0$ , while fixing  $\lambda = n\epsilon^2$  as a real number to ensure the energy level and angular momentum remain fixed. The limiting wave function is found by first defining the Bohr correspondence limit of the function  $Z_{\epsilon,n}$  (equation (2.3)) as follows:



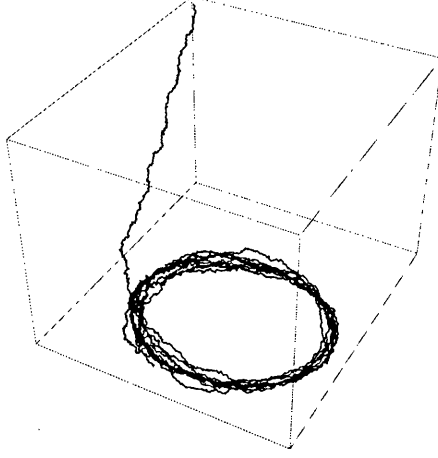


Figure 2.3: Simulation of the Nelson diffusion  $X_{\epsilon,n}$  for the atomic elliptic state in three dimensions.

$$Z_{0,\infty}(\mathbf{x}) := \lim_{\substack{n \uparrow \infty, \epsilon \downarrow 0 \\ \lambda = n\epsilon^2}} Z_{\epsilon,n}(\mathbf{x}).$$

Due to the dependence of  $Z_{\epsilon,n}$  on Laguerre polynomials, we require the following lemma from [9]:

**Lemma 2.1.1.** Let  $\mathcal{L}_n(x)$  denote the  $n$ th Laguerre polynomial and  $\lambda$  be a fixed real number. Then:

$$\lim_{\substack{n \uparrow \infty, \epsilon \downarrow 0 \\ \lambda = n\epsilon^2}} \frac{\mathcal{L}'_{n-1}(n\nu)}{\mathcal{L}_{n-1}(n\nu)} = \frac{1}{2} \left( 1 - \sqrt{1 - \frac{4}{\nu}} \right).$$

*Proof.* For Laguerre polynomials:

$$\frac{\mathcal{L}'_{n-1}(v)}{\mathcal{L}_{n-1}(v)} = \frac{n-1}{v} - \frac{n-1}{v} \frac{\mathcal{L}_{n-2}(v)}{\mathcal{L}_{n-1}(v)}.$$

Setting  $v = n\nu$ :

$$\lim_{\substack{n \uparrow \infty, \epsilon \downarrow 0 \\ \lambda = n\epsilon^2}} \frac{\mathcal{L}'_{n-1}(n\nu)}{\mathcal{L}_{n-1}(n\nu)} = \frac{1}{\nu} - \frac{1}{\nu} \lim_{\substack{n \uparrow \infty, \epsilon \downarrow 0 \\ \lambda = n\epsilon^2}} \frac{\mathcal{L}_{n-2}(n\nu)}{\mathcal{L}_{n-1}(n\nu)}.$$

If the limit:

$$p = \lim_{\substack{n \uparrow \infty, \epsilon \downarrow 0 \\ \lambda = n\epsilon^2}} \frac{\mathcal{L}_{n-2}(n\nu)}{\mathcal{L}_{n-1}(n\nu)}$$

exists and is non-zero, then  $p$  satisfies:

$$\frac{1}{p} - (2 - \nu) + p = 0, \quad (2.5)$$

due to the recurrence relation for Laguerre polynomials:

$$n\mathcal{L}_n(v) - (2n - 1 - v)\mathcal{L}_{n-1}(v) + (n - 1)\mathcal{L}_{n-2}(v) = 0.$$

Solving the quadratic (2.5) for  $p$  yields the result. □

Applying the above result to the expression for  $Z_{\epsilon,n}$  in Cartesians, (2.4) gives:

$$Z_{0,\infty}(\mathbf{x}) = \frac{i\mu}{2\lambda} \left( 1 + \sqrt{1 - \frac{4}{\nu}} \right) \frac{\mathbf{x}}{|\mathbf{x}|} + \frac{\mu}{2\lambda e} \left( 1 - \sqrt{1 - \frac{4}{\nu}} \right) (i, -\sqrt{1 - e^2}, 0).$$

The limiting wave function  $\Psi_\epsilon$  is the formal wave function satisfying:

$$Z_{0,\infty}(\mathbf{x}) = i\epsilon^2 \frac{\nabla \Psi_\epsilon(\mathbf{x})}{\Psi_\epsilon(\mathbf{x})},$$

which is given by:

$$\Psi_\epsilon = \nu^{\lambda/\epsilon^2} \left( 1 + \sqrt{1 - \frac{4}{\nu}} \right)^{2\lambda/\epsilon^2} \exp \left( -\frac{\mu}{\lambda\epsilon^2} |\mathbf{x}| + \frac{\lambda\nu}{2\epsilon^2} \left( 1 - \sqrt{1 - \frac{4}{\nu}} \right) \right).$$

It is important to note that as this wave function corresponds to Bohr's limit, it is only an approximate solution of the Schrödinger equation. Comparing Figures 2.2 and 2.4, note that the density  $|\psi_\epsilon|^2$  is highly concentrated on the ellipse.

### 2.1.4 The Limiting Nelson Diffusion Process

It is now possible to construct the limiting Nelson diffusion process  $X_\epsilon$  corresponding to the wave function  $\Psi_\epsilon$ . As given by equation (2.3), for  $\Psi_\epsilon = \exp(R_\epsilon + iS_\epsilon)$ :

$$Z_{0,\infty}(\mathbf{x}) = \epsilon^2 \nabla(S_\epsilon - iR_\epsilon).$$

The drift term of the diffusion process  $X_\epsilon$  is given by:

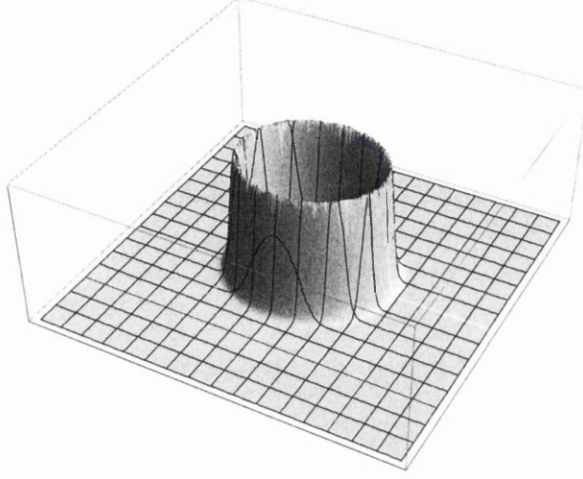


Figure 2.4: The probability density  $|\Psi_\epsilon|^2$  of the limiting elliptic diffusion in the plane  $z = 0$ .

$$b^e(\mathbf{x}) = \epsilon^2 \nabla(R_\epsilon + S_\epsilon) = \operatorname{Re}(Z_{0,\infty}(\mathbf{x})) - \operatorname{Im}(Z_{0,\infty}(\mathbf{x})).$$

Finally, the limiting Nelson diffusion process corresponding the limiting wave function  $\Psi_\epsilon$  satisfies:

$$dX_\epsilon(t) = b^e(X_\epsilon(t)) dt + \epsilon dB(t),$$

where in Cartesian coordinates,  $\mathbf{x} = (x, y, z)$  and  $b = (b_x^e, b_y^e, b_z^e)$  with:

$$\begin{aligned} b_x^e(\mathbf{x}) &= \frac{\mu}{2\lambda} \left( (\alpha + \beta - 1) \frac{1}{e} - (\alpha + \beta + 1) \frac{x}{|\mathbf{x}|} \right), \\ b_y^e(\mathbf{x}) &= \frac{\mu}{2\lambda} \left( (\alpha - \beta - 1) \frac{\sqrt{1-e^2}}{e} - (\alpha + \beta + 1) \frac{y}{|\mathbf{x}|} \right), \\ b_z^e(\mathbf{x}) &= -\frac{\mu}{2\lambda} (\alpha + \beta + 1) \frac{z}{|\mathbf{x}|}, \end{aligned}$$

and:

$$\begin{aligned} \alpha &= \left( \frac{1}{2} \sqrt{\frac{(e|\mathbf{x}| - x - \frac{4\lambda^2 e}{\mu})^2 + (1-e^2)y^2}{(e|\mathbf{x}| - x)^2 + (1-e^2)y^2}} \right. \\ &\quad \left. + \frac{1}{2} \frac{(e|\mathbf{x}| - x - \frac{2\lambda^2 e}{\mu})^2 + (1-e^2)y^2 - \frac{4\lambda^4 e^2}{\mu^2}}{(e|\mathbf{x}| - x)^2 + (1-e^2)y^2} \right)^{\frac{1}{2}}, \quad (2.6) \end{aligned}$$

$$\beta = \frac{-2\lambda^2 e \sqrt{1-e^2} y}{\mu(|\mathbf{x}| - x)^2 + (1-e^2)y^2} \alpha. \quad (2.7)$$

Note that for  $\nu$ , as defined by equation (2.2):

$$\sqrt{1 - \frac{4}{\nu}} = \alpha + i\beta.$$

By first defining:

$$\tilde{\alpha} = \frac{\mu}{\lambda^2} \left( |\mathbf{x}| - \frac{x}{e} \right) \quad \text{and} \quad \tilde{\beta} = -\frac{\mu y \sqrt{1-e^2}}{\lambda^2 e},$$

the functions  $R_\epsilon$  and  $S_\epsilon$  may be given explicitly:

$$R_\epsilon = \frac{\lambda}{2\epsilon^2} (\ln(\tilde{\alpha}^2 + \tilde{\beta}^2) + 2 \ln((1+\alpha)^2 + \beta^2) + (1-\alpha)\tilde{\alpha} + \beta\tilde{\beta}) - \frac{\mu|\mathbf{x}|}{\lambda\epsilon^2},$$

$$S_\epsilon = \frac{\lambda}{\epsilon^2} \left( \arg(\tilde{\alpha} + i\tilde{\beta}) + 2\arg(1 + \alpha + i\beta) + \frac{1}{2}\tilde{\beta}(1-\alpha) - \frac{1}{2}\beta\tilde{\alpha} \right).$$

Moreover:

$$\nabla R_\epsilon = -\frac{\mu}{2e\lambda\epsilon^2} \left( (1+\alpha) \frac{e\mathbf{x}}{|\mathbf{x}|} + ((1-\alpha), \beta\sqrt{1-e^2}, 0) \right), \quad (2.8)$$

and:

$$\nabla S_\epsilon = -\frac{\mu}{2e\lambda\epsilon^2} \left( \frac{\beta\mathbf{x}}{|\mathbf{x}|} + (-\beta, (1-\alpha)\sqrt{1-e^2}, 0) \right). \quad (2.9)$$

## 2.2 The Limiting Parabolic Diffusion

As demonstrated in The Divine Clockwork [9], the limiting Nelson diffusion process corresponding to the limiting wave function for the atomic elliptic state satisfies the stochastic differential equation:

$$dX(t) = b^\epsilon(X(t)) dt + \epsilon dB(t), \quad (2.10)$$

where  $\epsilon$  is a small quantum mechanical parameter, and  $B(t)$  is a multi-dimensional Brownian motion.

Henceforth we will be working in two dimensions, and so we begin by considering the restriction of the limiting Nelson diffusion to the putative plane

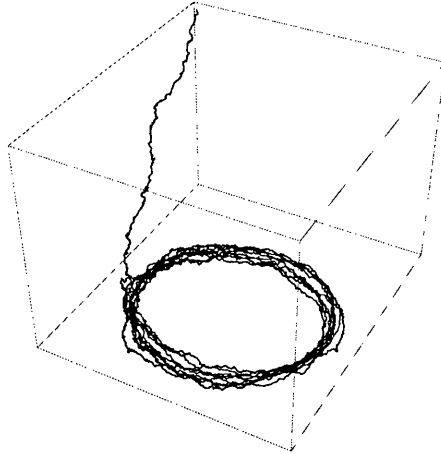


Figure 2.5: Simulation of the limiting elliptic diffusion process  $X_\epsilon$  in three-dimensions.

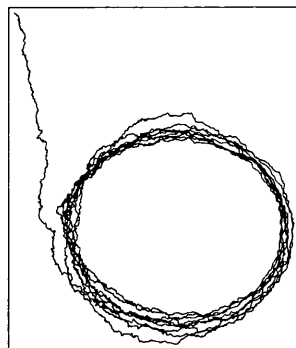


Figure 2.6: Simulation of the limiting elliptic diffusion process in two dimensions.

of motion  $z = 0$ , as discussed by Durran, Neate, Truman and Wang, [10]. A simulation of this process is shown by Figure 2.6. We consider the general stochastic differential equation above, (2.10), with the two-dimensional drift  $b^e(X(t)) = (b_x^e(X(t)), b_y^e(X(t)))$  fully defined by the following:

$$b_x^e(X(t)) = \frac{\mu}{2\lambda} \left( (\alpha + \beta - 1) \frac{1}{e} - (\alpha + \beta + 1) \frac{x}{r} \right), \quad (2.11)$$

$$b_y^e(X(t)) = \frac{\mu}{2\lambda} \left( (\alpha - \beta - 1) \frac{\sqrt{1 - e^2}}{e} - (\alpha + \beta + 1) \frac{y}{r} \right), \quad (2.12)$$

where  $X(t) = (x(t), y(t))$  in Cartesian coordinates with  $r$  as the length of the vector  $(x(t), y(t))$ , the functions  $\alpha$  and  $\beta$  are defined by equations (2.6) and (2.7) (with  $|\mathbf{x}| = r$ ),  $e$  is the eccentricity of the elliptical orbit,  $\mu$  is the force constant and  $\lambda$  is a positive constant defined by the relation:

$$a = \frac{\lambda^2}{\mu}, \quad (2.13)$$

where  $a$  denotes the length of the semimajor axis of the elliptic orbit.

It has been shown in [9] that the limiting Nelson diffusion defined according to the above equations (2.10) to (2.12) characterises Keplerian motion on the Kepler ellipse. We now investigate the behaviour of this motion in two-dimensions as the eccentricity  $e$  tends to 1 from below, the value which characterises a parabolic orbit in a classical situation. We fix  $l$ , the semi-latus rectum of the elliptical orbit, and for clarity, we may choose suitable units which allow  $\mu = 1$ . Using (2.13) and the relation from Chapter 1:

$$a = \frac{l}{1 - e^2},$$

we have:

$$\lambda^2 = \frac{l}{1 - e^2}. \quad (2.14)$$

As in Section 2.1, by defining:

$$\nu = \frac{\mu}{\lambda^2} \left( r - \frac{x}{e} - \frac{iy\sqrt{1 - e^2}}{e} \right), \quad (2.15)$$

a simple calculation shows that:

$$\sqrt{1 - \frac{4}{\nu}} = \alpha + i\beta. \quad (2.16)$$

Substituting (2.15) into (2.16) yields:

$$\alpha + i\beta = \sqrt{1 - \frac{4\lambda^2}{r - \frac{x}{e} - \frac{iy\sqrt{1-e^2}}{e}}},$$

so we see that as  $e \rightarrow 1$ :

$$\alpha + i\beta \rightarrow \sqrt{1 - \frac{4\lambda^2}{r - x}}. \quad (2.17)$$

We will deal with the limit of  $\lambda^2$  later, however it is evident from (2.14) that  $\lambda^2 \rightarrow \infty$  as  $e \rightarrow 1$ . Therefore the imaginary part of (2.17) is going to be dominant in the limit, forcing the real part  $\alpha$  to be 0. Then using relation (2.14), we see that the imaginary part must tend to:

$$\sqrt{\frac{4\lambda^2}{r - x}} = \frac{2\lambda}{\sqrt{r - x}} = \frac{2\sqrt{l}}{\sqrt{1 - e^2}\sqrt{r - x}}.$$

We are now able to substitute these limits for  $\alpha$  and  $\beta$  into the expressions for the components of the elliptic drift  $b_x^e(X(t))$  and  $b_y^e(X(t))$  (equations (2.11) and (2.12)), by setting:

$$\alpha = 0 \quad \text{and} \quad \beta = \frac{2\sqrt{l}}{\sqrt{1 - e^2}\sqrt{r - x}}, \quad (2.18)$$

which yields:

$$\begin{aligned} b_x^e(X(t)) &= \frac{\sqrt{1 - e^2}}{2\sqrt{l}} \left( \left( \frac{2\sqrt{l}}{\sqrt{1 - e^2}\sqrt{r - x}} - 1 \right) \frac{1}{e} \right. \\ &\quad \left. - \left( \frac{2\sqrt{l}}{\sqrt{1 - e^2}\sqrt{r - x}} + 1 \right) \frac{x}{r} \right) \\ &= \frac{1}{e\sqrt{r - x}} - \frac{\sqrt{1 - e^2}}{2e\sqrt{l}} - \frac{x}{r\sqrt{r - x}} - \frac{x\sqrt{1 - e^2}}{2r\sqrt{l}}. \end{aligned}$$

Finally, taking the limit as  $e \rightarrow 1$ :

$$b_x^e(X(t)) \rightarrow \frac{1}{\sqrt{r - x}} - \frac{x}{r\sqrt{r - x}} = \frac{\sqrt{r - x}}{r}. \quad (2.19)$$

Again setting:

$$\alpha = 0 \quad \text{and} \quad \beta = \frac{2\sqrt{l}}{\sqrt{1 - e^2}\sqrt{r - x}},$$

we have:

$$\begin{aligned}
b_y^e(X(t)) &= \frac{\sqrt{1-e^2}}{2\sqrt{l}} \left( \left( \frac{-2\sqrt{l}}{\sqrt{1-e^2}\sqrt{r-x}} - 1 \right) \frac{\sqrt{1-e^2}}{e} \right. \\
&\quad \left. - \left( \frac{2\sqrt{l}}{\sqrt{1-e^2}\sqrt{r-x}} + 1 \right) \frac{x}{r} \right) \\
&= \frac{-\sqrt{1-e^2}}{e\sqrt{r-x}} - \frac{1-e^2}{2e\sqrt{l}} - \frac{y}{r\sqrt{r-x}} - \frac{r\sqrt{1-e^2}}{2r\sqrt{l}}.
\end{aligned}$$

Taking the limit as  $e \rightarrow 1$ :

$$b_y^e(X(t)) \rightarrow -\frac{y}{r\sqrt{r-x}}. \quad (2.20)$$

For the two-dimensional restriction of the limiting elliptic diffusion discussed in Section 2.1.4,  $b^e = \epsilon^2 \nabla(R_\epsilon + S_\epsilon)$ , where, given (2.8) with  $\mu = 1$ :

$$\nabla R_\epsilon = -\frac{1}{2e\lambda\epsilon^2} \left( \frac{ex}{r} + 1, \frac{ey}{r} + \beta\sqrt{1-e^2} \right).$$

Once again using the limits of  $\alpha$  and  $\beta$  given by (2.18), and the relation defining  $\lambda$ , (2.14), we have:

$$\nabla R_\epsilon = -\frac{\sqrt{1-e^2}}{2e\lambda\epsilon^2\sqrt{l}} \left( \frac{ex}{r} + 1, \frac{ey}{r} + \frac{2\sqrt{l}}{\sqrt{r-x}} \right) \rightarrow (0,0) \quad \text{as } e \rightarrow 1, \quad (2.21)$$

evidently.

We may also write  $b = \epsilon^2 \nabla(R+S)$  where  $R$  and  $S$  are the respective limits of (2.8) and (2.9) for the two-dimensional restriction. Since from (2.21), the limit of  $\nabla R$  is the zero vector, we must have that  $b = \epsilon^2 \nabla S$ , for some  $S : \mathbb{R}^2 \rightarrow \mathbb{R}$ . We return to this in Section 2.3.1.

The expressions (2.19) and (2.20) will henceforth be referred to as  $b_x(X(t))$  and  $b_y(X(t))$  respectively, the components of a drift  $b(X(t))$  of the stochastic differential equation which we will investigate throughout this work:

$$dX(t) = b(X(t)) dt + \epsilon dB(t) \quad (2.22)$$

where  $X(t) = (x(t), y(t))$  and  $B(t) = (B_x(t), B_y(t))$ , a two-dimensional Brownian motion.



It is important to note at this stage that due to the ambiguity of the square root throughout it is necessary to multiply both  $b_x(X(t))$  and  $b_y(X(t))$  by  $\text{sgn}(y)$  which denotes the signum function:

$$\text{sgn}(y) = \begin{cases} -1 & \text{if } y < 0, \\ 0 & \text{if } y = 0, \\ 1 & \text{if } y > 0. \end{cases} \quad (2.23)$$

Thus we have the drift of the Nelson diffusion fully defined by:

$$b = (b_x, b_y) = \left( \text{sgn}(y) \frac{\sqrt{r-x}}{r}, -\frac{y \text{sgn}(y)}{r\sqrt{r-x}} \right), \quad (2.24)$$

and the stochastic differential equations for the Cartesian coordinates are given by:

$$dx(t) = \left( \text{sgn}(y) \frac{\sqrt{r-x}}{r} \right) dt + \epsilon dB_x(t)$$

and:

$$dy(t) = \left( -\frac{y \text{sgn}(y)}{r\sqrt{r-x}} \right) dt + \epsilon dB_y(t).$$

In the subsequent section we will see that in the absence of the noise term, the above Nelson drift determines a parabolic trajectory. Following the approach of Wentzell and Friedlin [13], we may view the random process (2.22) as a result of random perturbations of the classical system  $dX(t) = b(X(t)) dt$ . Hence we would expect that the Nelson diffusion would characterise a parabolic motion with small random perturbations due to the infinitesimal parameter  $\epsilon$ . (From [9], we have that  $\epsilon = \sqrt{\hbar}$ , with  $\hbar$  being the reduced Planck's constant.) The simulations shown by Figure 2.7 support this conjecture.

As in [13], we will throughout represent the solution  $X(t) = (x(t), y(t))$  of (2.22) as an asymptotic series in the parameter  $\epsilon$ :

$$X(t) = X_t^0 + \epsilon X_t^1 + \epsilon^2 X_t^2 + \dots$$

where we also have  $x(t) = x_t^0 + \epsilon x_t^1 + \epsilon^2 x_t^2 + \dots$ , and  $y(t) = y_t^0 + \epsilon y_t^1 + \epsilon^2 y_t^2 + \dots$  with  $X_t^i = (x_t^i, y_t^i)$  for  $i = 0, 1, 2, \dots$ . The zeroth order approximation  $X_t^0 = (x_t^0, y_t^0)$  corresponds to the classical system which will be discussed later in this chapter.

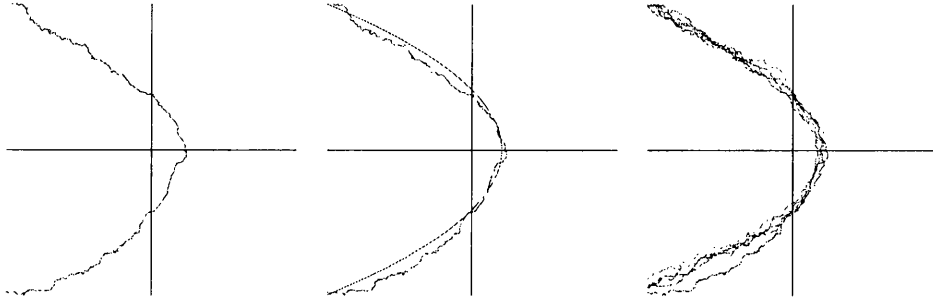


Figure 2.7: Simulations of: the semiclassical parabolic diffusion; the semiclassical diffusion compared with the classical trajectory; multiple sample paths of the diffusion process.

### 2.2.1 Analysis of the Parabolic Nelson Diffusion

On inspection, we see that both components  $b_x$  and  $b_y$  of the above drift (2.24) are singular at the origin, and it appears that  $b_y$  is also singular on the whole positive  $x$ -axis due to the term  $\sqrt{r-x}$  in the denominator. However, writing  $b_y$  in polar coordinates:

$$b_y = -\frac{\operatorname{sgn}(y) \sin \theta}{\sqrt{r} \sqrt{1 - \cos \theta}},$$

then using the identities  $\sin \theta = 2 \sin(\frac{\theta}{2}) \cos(\frac{\theta}{2})$  and  $\cos \theta = 2 \cos^2(\frac{\theta}{2}) - 1$  respectively in the numerator and denominator of the above:

$$b_y = -\frac{\operatorname{sgn}(y) \sqrt{2} \cos(\frac{\theta}{2})}{\sqrt{r}},$$

which shows  $b_y$  only has a singularity where  $r = 0$ . Therefore the drift of the parabolic Nelson diffusion is only locally unbounded at the origin. It is exactly this singularity that presents us with problems when attempting to show existence and uniqueness of solutions to (2.22).

Since we have seen above that the Nelson drift has a singularity, we look to some known results on existence and uniqueness of solutions to stochastic differential equations with locally unbounded drift. The relevant literature refers to equations of the form:

$$dX(t) = b(t, X(t)) dt + \sigma(t, X(t)) dB(t) \quad (2.25)$$

We first consider a result of Portenko [28] which proves the existence and uniqueness in law of solutions to the above if we have  $b \in L^p([0, T] \times \mathbb{R}^n)$  for some  $p > n + 2$ , in addition to some assumptions on  $\sigma$ .

Firstly note that, from (2.24), we may express  $b_x$  as follows:

$$b_x = \frac{\operatorname{sgn}(y)\sqrt{1 - \cos \theta}}{\sqrt{r}}.$$

Given this and the similar expression for  $b_y$ :

$$|b(X(t))| = \sqrt{\frac{1 - \cos \theta}{r} + \frac{2 \cos(\frac{\theta}{2})}{r}} \leq \frac{2}{\sqrt{r}}. \quad (2.26)$$

To apply the result of Portenko we require that  $b \in L^4$ . However:

$$\int_0^{2\pi} \int_\delta^R |b(r, \theta)|^4 r \, dr \, d\theta = 7\pi \ln\left(\frac{R}{\delta}\right),$$

which is clearly unbounded as  $\delta \downarrow 0$ , so the required condition is not satisfied. This result is generalized by Gyöngy and Martínez in [17], where for equations of the form (2.22) it becomes:

**Theorem 2.2.1.** *If  $|b(X(t))| \leq K + F(X(t))$  for a constant  $K > 0$  and some non-negative function  $F \in L^{n+1}(\mathbb{R}_+ \times \mathbb{R}^n)$ , then a weak solution to (2.22) exists.*

Fixing  $R > 1$ , note that if  $r \leq R$ , then:

$$|b(X(t))| \leq \frac{2}{\sqrt{r}} \leq 2 + \frac{2}{\sqrt{r}},$$

by the above bound (2.26) and for  $r > R > 1$ , we have:

$$|b(X(t))| \leq \frac{2}{\sqrt{r}} \leq 2,$$

Thus we may take  $K = 2$  and:

$$F(X(t)) = \frac{2}{\sqrt{r}} \chi_{B_R(0)}, \quad (2.27)$$

where  $\chi_{B_R(0)}$  denotes the characteristic function of the ball centred at the origin with a fixed radius  $R > 1$ . This guarantees  $|b(X(t))| < K + F(X(t))$  in all cases. In the two-dimensional case  $L^{n+1}$  becomes  $L^3$ .  $F(X(t)) \in L^3$  is guaranteed by:

$$\int_0^\infty |F(X(t))|^3 r \, dr = \int_0^R r^{-1/2} \, dr = R^{1/2} < \infty.$$

Hence a solution to the SDE governing the parabolic Nelson diffusion (2.22) exists in the weak sense. However the corresponding result for existence and uniqueness of the strong solution requires that the Nelson drift  $b \in L_{loc}^{2n+2}$ , and so obviously does not hold.

We now consider the  $n$ -dimensional Bessel process  $R_t$ , the real-valued process given by the Euclidean length of an  $n$ -dimensional Brownian motion  $B(t) = (B_1(t), B_2(t), \dots, B_n(t))$  where  $B_1, \dots, B_n$  are independent one-dimensional Brownian motions. That is:

$$R_t = \|B_t\|_2 = \left( \sum_{i=1}^n B_i^2 \right)^{\frac{1}{2}}.$$

As defined above,  $R_t$  satisfies the stochastic differential equation:

$$dR_t = \frac{n-1}{2R_t} dt + d\beta_t,$$

$\beta_t$  being an arbitrary one-dimensional Brownian path. Thus in two dimensions, we may use the Bessel process to describe the diffusion of a particle on a plane just in terms of its distance  $R_t$  from the origin, which satisfies:

$$dR_t = \frac{1}{2R_t} dt + d\beta_t.$$

It is clear to see that the drift of the above SDE has a singularity where  $R_t = 0$ , at the origin. However it can be shown (McKean [23], Karatzas and Shreve [21]) that this process reaches the point  $R_t = 0$  in finite time with probability 0. This suggests that we should attempt to determine the parabolic Nelson diffusion in terms of its radial parameter by finding an appropriate SDE, in order to investigate whether the process reaches the origin in finite time.

### Nonattainability of the Origin by the Parabolic Nelson Diffusion

From (2.22) and (2.24) we have that the parabolic Nelson diffusion in the correspondence limit  $X(t) = (x(t), y(t))$  is completely defined by some initial condition  $x(0), y(0)$  and:

$$dx(t) = b_x(X(t)) dt + \epsilon dB_x(t), \quad (2.28)$$

$$dy(t) = b_y(X(t)) dt + \epsilon dB_y(t). \quad (2.29)$$

We begin by defining a new stochastic process  $r(t)$  governing the time evolution of the polar radius  $r = \sqrt{x^2 + y^2}$ . Using Itô's formula:

$$dr(t) = \left( \frac{xb_x + yb_y}{r} + \frac{1}{2r} \right) dt + \frac{x dB_x(t) + y dB_y(t)}{r}, \quad (2.30)$$

where factors of  $\epsilon$  have been omitted for clarity since the subsequent analysis does not depend on their influence. Given the parabolic drift components in polar coordinates:

$$b_x = \sqrt{\frac{2}{r}} \sin\left(\frac{\theta}{2}\right) \operatorname{sgn}(y) \quad \text{and} \quad b_y = -\sqrt{\frac{2}{r}} \cos\left(\frac{\theta}{2}\right) \operatorname{sgn}(y),$$

a simple calculation yields:

$$xb_x + yb_y = \sqrt{2r} \left| \sin\left(\frac{\theta}{2}\right) \right| \operatorname{sgn}(y).$$

This may be substituted into (2.30) to give:

$$dr(t) = \left( \sqrt{\frac{2}{r}} \left| \sin\left(\frac{\theta}{2}\right) \right| \operatorname{sgn}(y) + \frac{1}{2r} \right) dt + dB_r(t) \quad (2.31)$$

where  $B_r(t)$  is a one-dimensional Brownian motion with:

$$B_r(t) = \frac{xB_x(t) + yB_y(t)}{r}.$$

Given the SDE for  $r(t)$ , (2.31) we are now able to show that the process does not reach the origin in finite time, thanks to a result by David Williams [33]. We first require a preliminary result.

**Lemma 2.2.2.** The stochastic differential equation:

$$d\rho(t) = c(\rho(t)) dt + dB(t)$$

with:

$$c(\rho) = -\sqrt{\frac{2}{\rho}} + \frac{1}{2\rho}$$

and  $\rho(0)$  a deterministic point, has a pathwise unique solution with:

$$\mathbb{P}[\rho(t) > 0, \quad \forall t] = 1.$$

*Proof.* Let  $\rho(0) \in (\alpha, \beta) \subset (0, \infty)$ , then we have  $\rho(t)$  defined up to  $\tau[\alpha, \beta] := \inf\{t > 0 : \rho(t) \notin (\alpha, \beta)\}$ .

Due to a result from Gihman and Skorohod, [14], the probability that the process  $\rho(t)$  starting at  $\rho(0) \in (\alpha, \beta)$  reaches the point  $\beta$  before the point  $\alpha$  is given by:

$$\mathbb{P}[\rho(\tau[\alpha, \beta]) = \beta] = \frac{u(\rho(0)) - u(\alpha)}{u(\beta) - u(\alpha)},$$

for  $u$  satisfying:

$$\frac{1}{2}u''(x) + c(x)u'(x) = 0.$$

First defining:

$$I_\alpha(x) := C \int_\alpha^x \frac{e^{4\sqrt{2}s}}{s} ds,$$

then solving the above equation for  $u'$  via the integrating factor method we have:

$$\frac{u(\rho(0)) - u(\alpha)}{u(\beta) - u(\alpha)} = \frac{I_\alpha(\rho(0))}{I_\alpha(\beta)}.$$

for some finite multiplicative constant  $C$ . Letting  $\alpha \downarrow 0$ , both integrals in the above tend to infinity at the same rate, and so:

$$\mathbb{P}[\rho(\tau[0, \beta]) = \beta] = 1,$$

for an arbitrary value of  $\beta$ . □

**Theorem 2.2.3.** *With probability 1, the radius  $r(t)$  of the parabolic Nelson diffusion satisfies  $r(t) > 0$  for all  $t$ .*

*Proof.* Consider the radial equation for the parabolic Nelson diffusion given above:

$$dr(t) = \left( \sqrt{\frac{2}{r}} \left| \sin\left(\frac{\theta}{2}\right) \right| \operatorname{sgn}(y) + \frac{1}{2r} \right) dt + dB(t) =: b_r(r(t)) dt + dB(t).$$

Compare this with the equation for  $\rho(t)$  from the above lemma. With the same starting point  $\rho(0) = r(0)$ , and the same driving Brownian motion, we have:

$$r(t) - \rho(t) = \int_0^t (b_r(r(s)) - c(\rho(s))) ds.$$

Also note that when  $\rho(t) = r(t)$ ,  $b_r(r(t)) > c(\rho(t))$ , so that for some small  $\delta$ , we have  $r(u) > \rho(u)$  for  $0 \leq u < \delta$ .

Now suppose that there exists a time  $v > \delta$  where  $\rho(v) > r(v)$ . Let  $T$  be the infimum of such times so that  $\rho(T) = r(T)$ . For small  $\gamma > 0$  with  $T - \gamma < u < T$ ,  $\rho(u)$  and  $r(u)$  are close to  $\rho(T)$  and  $r(T)$  respectively, and for such times  $u$ ,  $b_r(r(u)) > c(\rho(u))$ . Hence:

$$(r(T) - \rho(T)) - (r(T - \gamma) - \rho(T - \gamma)) = \int_{T-\gamma}^T (b_r(r(u)) - c(\rho(u))) du > 0.$$

However, by definition of  $T$ ,  $r(T) = \rho(T)$  and  $r(T - \gamma) > \rho(T - \gamma)$ .

The above contradiction establishes that  $r(t) > \rho(t)$  for all  $t > 0$ . The theorem follows, given the previous lemma. □

## 2.3 The Classical Case

We now focus on the classical analogue of the problem outlined in Section 2.2, that is, equation (2.22) in the absence of the noise term. By utilising the usual conservation laws of energy and angular momentum, we are able to extract information about this dynamical system from the aforementioned differential equations.

### 2.3.1 Investigating the Motion

As in the previous section, for  $X(t) = (x(t), y(t))$ , we have the drift field defined by:

$$\dot{x}(t) = b_x(x, y) = \operatorname{sgn}(y) \frac{\sqrt{r-x}}{r} \tag{2.32}$$

and:

$$\dot{y}(t) = b_y(x, y) = -\frac{y \operatorname{sgn}(y)}{r\sqrt{r-x}}, \tag{2.33}$$

for  $r = \sqrt{x^2 + y^2}$ .

From equation (2.23), we first note that for  $y \neq 0$ :

$$\frac{\partial}{\partial x} \operatorname{sgn}(y) = \frac{\partial}{\partial y} \operatorname{sgn}(y) = 0.$$

Therefore it is sufficient to treat  $\text{sgn}(y)$  as a constant in the following:

$$\frac{\partial b_x}{\partial y} = \frac{\partial}{\partial y} \left( \text{sgn}(y) \frac{\sqrt{r-x}}{r} \right) = \frac{y \text{sgn}(y)(2x-r)}{2r^3 \sqrt{r-x}}.$$

Similarly:

$$\frac{\partial b_y}{\partial x} = \frac{\partial}{\partial x} \left( -\frac{y \text{sgn}(y)}{r \sqrt{r-x}} \right) = \frac{y \text{sgn}(y)(2x-r)}{2r^3 \sqrt{r-x}} = \frac{\partial b_x}{\partial y}. \quad (2.34)$$

This implies that the drift  $b$  should be the gradient of some function  $S$ .

**Lemma 2.3.1.** The vector  $b(x, y) = (b_x(x, y), b_y(x, y))$  defined according to equations (2.32) and (2.33) is a gradient, that is  $b = \nabla S$  for:

$$S(x, y) = -\text{sgn}(y) \sqrt{r-x},$$

where  $y \neq 0$ .

*Proof.* Partial differentiation of  $S$  yields:

$$\frac{\partial S}{\partial x} = \text{sgn}(y) \frac{\sqrt{r-x}}{r},$$

and:

$$\frac{\partial S}{\partial y} = -\frac{y \text{sgn}(y)}{r \sqrt{r-x}}.$$

□

**Proposition 2.3.2.** With the drift  $b(X(t))$  defined according to equations (2.32) and (2.33), the solution  $X(t)$  to the ordinary differential equation  $dX(t) = b(X(t)) dt$  satisfies:

$$\ddot{X}(t) = \nabla \left( \frac{1}{|X(t)|} \right),$$

with energy  $E = 0$  and therefore the trajectory of  $X(t)$  corresponds to Keplerian motion on a parabola.

*Proof.* Since  $\dot{X}(t) = b(X(t))$ , we have:

$$\ddot{X}(t) = \left( \frac{db_x}{dt}, \frac{db_y}{dt} \right).$$

By the chain rule for differentiation we see that:



$$\frac{db_x}{dt} = \frac{\partial b_x}{\partial x} \frac{dx}{dt} + \frac{\partial b_x}{\partial y} \frac{dy}{dt}.$$

Firstly using the definitions of  $b_x$  and  $b_y$ , and then observation (2.34) we obtain from the above that:

$$\frac{db_x}{dt} = \frac{\partial b_x}{\partial x} b_x + \frac{\partial b_x}{\partial y} b_y = \frac{\partial b_x}{\partial x} b_x + \frac{\partial b_y}{\partial x} b_y.$$

Now given that  $|b|^2 = b_x^2 + b_y^2$ , observe that we have:

$$\frac{db_x}{dt} = \frac{\partial}{\partial x} \left( \frac{|b|^2}{2} \right) \quad \text{and} \quad \frac{db_y}{dt} = \frac{\partial}{\partial y} \left( \frac{|b|^2}{2} \right)$$

by a similar computation to the above. Hence:

$$\ddot{X}(t) = -\nabla V \quad \text{for a potential function} \quad V = -\frac{|b|^2}{2}.$$

Using the explicit expressions for  $b_x$  and  $b_y$  given by equations (2.32) and (2.33), we find:

$$V = -\frac{\dot{x}^2 + \dot{y}^2}{2} = -\frac{r-x}{2r^2} - \frac{y^2}{2r^2(r-x)} = -\frac{(r-x)^2 + y^2}{2r^2(r-x)} = -\frac{1}{r} = -\frac{1}{|X(t)|}.$$

Thus showing that the trajectory of  $X(t)$  satisfies Newton's laws in the presence of the Coulomb potential function. Recalling from Chapter 1, the equation for an orbit's eccentricity:

$$e = \sqrt{1 + \frac{2L^2 E}{\mu^2}},$$

we see that  $e = 1 \Leftrightarrow E = 0$ . Note that the system's kinetic energy:

$$T = \frac{|b|^2}{2} = -V.$$

Thus, by energy conservation we have  $E = T + V = 0$ , the zero energy state which corresponds to motion on a parabola.  $\square$

Knowing that we have motion on a parabola, we now replace the parameter  $e$  by 1 in the polar coordinate equation for a conic section, so that:

$$r = \frac{l}{1 + e \cos \theta} \quad \text{becomes} \quad r = \frac{l}{1 + \cos \theta}.$$

Using the identity  $1 + \cos \theta = 2 \cos^2 \left(\frac{\theta}{2}\right)$ , we arrive at:

$$r = \frac{l}{2} \sec^2 \left(\frac{\theta}{2}\right) = d \sec^2 \left(\frac{\theta}{2}\right), \quad (2.35)$$

where  $d = \frac{l}{2}$  is known as the apsidal distance. Substituting (2.35) into  $x = r \cos \theta$  and  $y = r \sin \theta$  we respectively obtain:

$$x = d \sec^2 \left(\frac{\theta}{2}\right) \cos \theta, \quad (2.36)$$

and:

$$y = d \sec^2 \left(\frac{\theta}{2}\right) \sin \theta. \quad (2.37)$$

Using the identity  $\sin \theta = 2 \sin \left(\frac{\theta}{2}\right) \cos \left(\frac{\theta}{2}\right)$  in (2.37):

$$y = 2d \frac{\sin \left(\frac{\theta}{2}\right)}{\cos \left(\frac{\theta}{2}\right)} = 2d \tan \left(\frac{\theta}{2}\right) = 2d\tau, \quad (2.38)$$

where we have defined  $\tau = \tan \left(\frac{\theta}{2}\right)$ . To obtain a similar expression for  $x$  in terms of  $\tau$ , we replace the secant and cosine terms in (2.36) by the the identities:

$$\sec^2 \theta = 1 + \tan^2 \theta \quad \text{and} \quad \cos \theta = \frac{1 - \tan^2 \left(\frac{\theta}{2}\right)}{1 + \tan^2 \left(\frac{\theta}{2}\right)}$$

respectively, yielding:

$$x = d(1 + \tau^2) \frac{1 - \tau^2}{1 + \tau^2} = d(1 - \tau^2). \quad (2.39)$$

Rearranging (2.38) gives:

$$\tau = \frac{y}{2d}.$$

We then substitute the above into (2.39) to obtain:

$$x = d \left(1 - \frac{y^2}{4d^2}\right).$$

Rearranging this yields:

$$y^2 = 4d(d - x),$$

which is the equation of the parabola on which this motion occurs in terms of the Cartesian coordinates and the apsidal distance  $d$ .

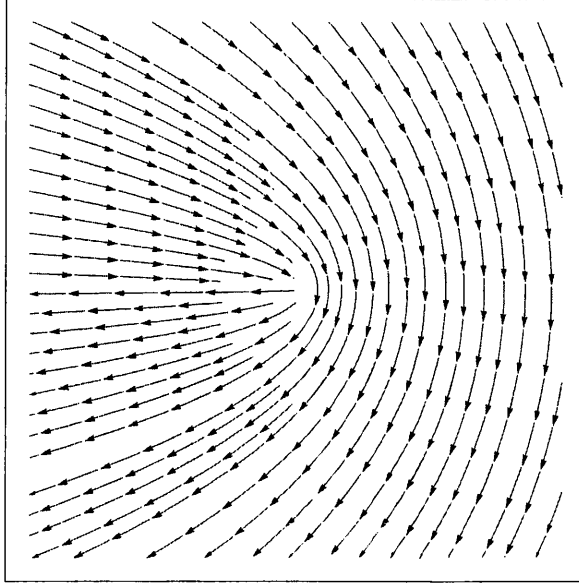


Figure 2.8: Streamlines of the parabolic drift field.

### 2.3.2 Calculating Explicit Time Dependence

To attempt to find the parameters  $r$ ,  $\theta$ ,  $x$  and  $y$  as functions of time, we begin by using the fact that the angular momentum  $L$  of this dynamical system is constant, and that in this general case  $L = r^2\dot{\theta}$ . Also, for motion on a parabolic orbit we have  $l = \frac{L^2}{\mu}$ , where once again for clarity we equate  $\mu$  to 1. Since Proposition 2.3.2 shows that we have motion on a parabola, we use the polar coordinate equation for a conic section in the form  $r = \frac{l}{1+\cos\theta}$ . As shown by (2.35), by using an appropriate identity we find  $r = \frac{l}{2} \sec^2\left(\frac{\theta}{2}\right)$ , and therefore:

$$L = \dot{\theta} \frac{l^2}{4} \sec^4\left(\frac{\theta}{2}\right)$$

which implies an ordinary differential equation for  $\theta$ :

$$\dot{\theta} = \frac{4L}{l^2} \cos^4\left(\frac{\theta}{2}\right).$$

Separation of variables in the above ODE yields:

$$\int_{\theta_0}^{\theta} \sec^4\left(\frac{\phi}{2}\right) d\phi = \int_0^t \frac{4L}{l^2} ds, \quad (2.40)$$

where  $\theta_0$  is an initial condition; one of the polar coordinates specifying the position of a particle at time 0. By virtue of the identity  $\sec^2 \theta = 1 + \tan^2 \theta$ , the above equation (2.40) becomes:

$$\frac{4Lt}{l^2} = \int_{\theta_0}^{\theta} \sec^2 \left( \frac{\phi}{2} \right) d\phi + \int_{\theta_0}^{\theta} \sec^2 \left( \frac{\phi}{2} \right) \tan^2 \left( \frac{\phi}{2} \right) d\phi.$$

Integration yields:

$$\frac{4Lt}{l^2} = 2 \tan \left( \frac{\theta}{2} \right) + \frac{2}{3} \tan^3 \left( \frac{\theta}{2} \right) - 2 \tan \left( \frac{\theta_0}{2} \right) - \frac{2}{3} \tan^3 \left( \frac{\theta_0}{2} \right). \quad (2.41)$$

Now let  $T(t, \theta_0)$  be a function such that:

$$\frac{4LT}{l^2} = \frac{4Lt}{l^2} + 2 \tan \left( \frac{\theta_0}{2} \right) + \frac{2}{3} \tan^3 \left( \frac{\theta_0}{2} \right), \quad (2.42)$$

then it is clear from (2.41) that in terms of  $\theta$ :

$$T = \frac{l^2}{4L} \left( 2 \tan \left( \frac{\theta}{2} \right) + \frac{2}{3} \tan^3 \left( \frac{\theta}{2} \right) \right).$$

From above, we see that  $L^2 = l$  and the apsidal distance  $d = \frac{l}{2}$ , as given in Section 2.3.1. By making the appropriate substitutions in the above equation we see that:

$$T = \sqrt{2d^3} \left( \tan \left( \frac{\theta}{2} \right) + \frac{1}{3} \tan^3 \left( \frac{\theta}{2} \right) \right).$$

Rearranging yields a reduced monic cubic in  $\tan \left( \frac{\theta}{2} \right)$ :

$$\tan^3 \left( \frac{\theta}{2} \right) + 3 \tan \left( \frac{\theta}{2} \right) - \frac{3T}{\sqrt{2d^3}} = 0 \quad (2.43)$$

which may be solved using Cardan's Solution as outlined in Ferrar [12]. Due to the absence of a quadratic term, we already have a cubic equation in the form  $z^3 + 3Hz + G = 0$  as necessary, where  $z = \tan \frac{\theta}{2}$ . Since we need not concern ourselves with complex roots, the real root of (2.43) is given by  $z = p + q$  where:

$$p = \sqrt[3]{\frac{1}{2}(-G + \sqrt{G^2 + 4H^3})} \quad (2.44)$$

and

$$q = \frac{-H}{p}. \quad (2.45)$$

By inspection, we see that in the above cubic equation (2.43):

$$H = 1 \quad \text{and} \quad G = \frac{-3T}{\sqrt{2d^3}}.$$

Substituting these values into the above expression for  $p$ , (2.44) yields:

$$p = \sqrt[3]{\frac{3T}{2\sqrt{2d^3}} + \frac{1}{2}\sqrt{\left(\frac{3T}{\sqrt{2d^3}}\right)^2 + 4}} = \sqrt[3]{\frac{3T^2}{2\sqrt{2d^3}} + \sqrt{\frac{9T^2}{8d^3} + 1}}. \quad (2.46)$$

Since  $H = 1$ , we have from (2.45) that  $q = -\frac{1}{p}$ . By writing  $p = (a + b)^{\frac{1}{3}}$  where:

$$a = \sqrt{\frac{9T^2}{8d^3} + 1} \quad \text{and} \quad b = \frac{3T}{2\sqrt{2d^3}},$$

we are able to write  $\frac{1}{p}$  in the form:

$$\frac{1}{p} = \left( \frac{(a - b)}{(a - b)(a + b)} \right)^{\frac{1}{3}}.$$

From the above definitions of  $a$  and  $b$ , direct computation gives:

$$(a + b)(a - b) = a^2 - b^2 = \frac{9T^2}{8d^3} + 1 - \frac{9T^2}{8d^3} = 1,$$

thus simplifying the above expression for  $\frac{1}{p}$  to:

$$\frac{1}{p} = (a - b)^{\frac{1}{3}} = \sqrt[3]{\frac{-3T^2}{2\sqrt{2d^3}} + \sqrt{\frac{9T^2}{8d^3} + 1}}. \quad (2.47)$$

Therefore substituting (2.46) and (2.47) into  $z = p + q$ , the solution to the original cubic equation (2.43) is given by:

$$\tan\left(\frac{\theta}{2}\right) = \sqrt[3]{\frac{3T^2}{2\sqrt{2d^3}} + \sqrt{\frac{9T^2}{8d^3} + 1}} - \sqrt[3]{\frac{-3T^2}{2\sqrt{2d^3}} + \sqrt{\frac{9T^2}{8d^3} + 1}}$$

where from (2.42) we see that as a function of  $t$  and  $\theta_0$ :

$$T(t, \theta_0) = t + \sqrt{2d^3} \left( \tan\left(\frac{\theta_0}{2}\right) + \frac{1}{3} \tan^3\left(\frac{\theta_0}{2}\right) \right). \quad (2.48)$$

This allows us to state the following theorem:

**Theorem 2.3.3.** For  $T(t, \theta_0)$  from (2.48), the polar and Cartesian coordinates of the classical parabolic orbit as functions of time and the initial condition are given by:

$$\theta(t, \theta_0) = 2 \tan^{-1} \left( \sqrt[3]{\frac{3T^2}{2\sqrt{2d^3}} + \sqrt{\frac{9T^2}{8d^3} + 1}} - \sqrt[3]{\frac{-3T^2}{2\sqrt{2d^3}} + \sqrt{\frac{9T^2}{8d^3} + 1}} \right),$$

$$r(t, \theta_0) = \frac{1}{1 + \cos \theta(t, \theta_0)},$$

$$x(t, \theta_0) = r(t, \theta_0) \cos \theta(t, \theta_0)$$

and:

$$y(t, \theta_0) = r(t, \theta_0) \sin \theta(t, \theta_0).$$

Note that, if required, the initial coordinate  $r_0$  is given by  $\frac{1}{1 + \cos \theta_0}$ . Initial Cartesian coordinates follow from this.

### 2.3.3 The Burgers Velocity Field for the Parabolic Motion

In vector form:

$$b(X(t)) = \left( \operatorname{sgn}(y) \frac{\sqrt{r-x}}{r}, -\frac{y \operatorname{sgn}(y)}{r\sqrt{r-x}} \right). \quad (2.49)$$

We proceed to show that the above drift field, obtained by taking the limit of a the semiclassical elliptic drift has zero divergence and satisfies the inviscid form of Burgers' equation, [4]. These results may then be used to show that a fluid moving with this velocity field has a constant density and is therefore incompressible. Moreover, the function  $S$  of which  $b$  is the gradient (Lemma 2.3.1) may be shown to satisfy the classical mechanical Hamilton-Jacobi equation.

**Lemma 2.3.4.** For the vector field  $b(X(t))$  defined by (2.49),  $\operatorname{div}(b) = 0$ .

*Proof.* As in the proof of Lemma 2.3.1, where  $y \neq 0$ , we treat  $\operatorname{sgn}(y)$  as a constant for the purposes of the following, and so we have:

$$\frac{\partial b_x}{\partial x} = \frac{\partial}{\partial x} \left( \operatorname{sgn}(y) \frac{\sqrt{r-x}}{r} \right) = -\operatorname{sgn}(y) \frac{(r+2x)(r-x)}{2r^3\sqrt{r-x}}.$$

Similarly:

$$\frac{\partial b_y}{\partial y} = \frac{\partial}{\partial y} \left( -\frac{y \operatorname{sgn}(y)}{r\sqrt{r-x}} \right) = \frac{(r-x)(r-x)}{r\sqrt{r-x}} = -\frac{\partial b_x}{\partial x},$$

and we thus have:

$$\operatorname{div}(b) = \frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y} = 0.$$

□

From Proposition 2.3.2, we have already seen that for the potential energy function:

$$V = -\frac{1}{|X(t)|}$$

and Nelson drift  $b(X(t))$  (defined according to (2.49)):

$$\frac{|b|^2}{2} + V = 0. \quad (2.50)$$

Differentiating the above gives:

$$(b \cdot \nabla)b + \nabla V = 0. \quad (2.51)$$

Consider the inviscid Burgers equation, that is Burgers equation with no viscosity coefficient:

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v = -\nabla V. \quad (2.52)$$

By setting the Burgers fluid velocity  $v(X(t)) = b(X(t))$ , given (2.51), we see that  $b(X(t))$  is a stationary state solution of the inviscid Burgers equation (2.52). That is,  $b(X(t))$  is a solution of (2.52) since  $\frac{\partial b}{\partial t} = 0$ . Moreover, using the fact that  $\operatorname{div}(b) = 0$  from Lemma 2.3.4, we may prove that this Burgers fluid is incompressible. We will begin the proof as for a three dimensional fluid, and then restrict motion to a plane to give the two-dimensional result.

**Theorem 2.3.5.** *A fluid moving in two dimensions with the parabolic Burgers velocity field  $b(X(t))$  has a constant density, hence the field  $b(X(t))$  determines an incompressible flow.*

*Proof.* Consider a fluid moving in three dimensions with vector-valued velocity  $v(X(t))$  and scalar density  $\rho(X(t))$ . For an arbitrary volume  $\Gamma$ , the total fluid mass in that volume is given by:

$$M(t) = \int_{\Gamma} \rho(X(t)) \, d\gamma$$

where  $d\gamma$  denotes integration with respect to the volume element. The rate of change of this mass is obviously:

$$\dot{M}(t) = \int_{\Gamma} \frac{\partial \rho}{\partial t} \, d\gamma. \quad (2.53)$$

This flow of mass per unit time is also given by the flow across the boundary  $\partial\Gamma$ :

$$\dot{M}(t) = - \int_{\partial\Gamma} \rho v \, d\Sigma = - \int_{\Gamma} \operatorname{div}(\rho v) \, d\gamma, \quad (2.54)$$

where integration  $d\Sigma$  is with respect to the outward normal unit, and the latter equality is due to the divergence theorem. Equating (2.53) and (2.54) gives:

$$\int_{\Gamma} \left( \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho v) \right) \, d\gamma = 0.$$

Since the volume  $\Gamma$  was arbitrary, we have for all fluid velocities and densities:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho v) = 0,$$

the continuity equation for fluids, which is equivalent to:

$$\frac{\partial \rho}{\partial t} + \nabla \rho \cdot v + \rho \operatorname{div}(v) = 0.$$

For the density  $\rho$  to be constant we require  $\frac{\partial \rho}{\partial t} = 0$ , hence requiring that  $\operatorname{div}(v) = 0$ . In particular, when the position vector  $X(t)$  of an element of the fluid is restricted to a two-dimensional plane, the condition retains its validity. Setting  $v(X(t)) = b(X(t))$ , the above stationary state solution to the inviscid Burgers equation satisfies the condition required for constant density, since  $\operatorname{div}(b) = 0$  by Lemma 2.3.4. □

We may further prove that the function  $S$  given by Lemma 2.3.1 satisfies the Hamilton-Jacobi equation with the Coulomb potential.

**Proposition 2.3.6.** The function  $S = -\operatorname{sgn}(y)\sqrt{r-x}$  is a Hamilton-Jacobi function satisfying the Hamilton-Jacobi equation:



$$\frac{\partial S(q, t)}{\partial t} + H(q, \nabla S, t) = 0.$$

for the Hamiltonian:

$$H(q, \nabla S) = \frac{|\nabla S|^2}{2} + V(q)$$

with the Coulomb potential  $V$ , where  $q = (x(t), y(t))$  denote generalised position coordinates.

*Proof.* From equation (2.50):

$$\frac{|b|^2}{2} + V = 0.$$

Since  $b = \nabla S$  (Lemma 2.3.1), the above becomes:

$$\frac{|\nabla S|^2}{2} + V = 0,$$

which is exactly the above Hamilton-Jacobi equation given that:

$$\frac{\partial S}{\partial t} = 0.$$

□

# Chapter 3

## The First Order Correction to the Parabolic Diffusion

Following the methods and results of Wentzell and Friedlin, [13] we attempt to find methods for solving general stochastic differential equations of the form:

$$dX_t = b(X_t) dt + \epsilon dB(t) \quad (3.1)$$

in both one and two dimensions for some small parameter  $\epsilon$ . We initially consider equations with arbitrarily differentiable drift fields. Since the two-dimensional semiclassical parabolic Nelson diffusion discussed in Chapter 2 is of the above form, we may use the resulting methods to find its solution correct to first order in  $\epsilon$  on domains where its drift is suitably differentiable.

The approach we take is to expand the solution  $X_t$  as an asymptotic series in  $\epsilon$ :

$$X_t = X_t^0 + \epsilon X_t^1 + \epsilon^2 X_t^2 + \dots$$

where methods are provided in [13] for finding the coefficients  $X_t^0, X_t^1$  etc. The convergence of the asymptotic series is ensured by Theorem 2.2 from Chapter 2 of Wentzell and Friedlin, [13] for the  $n$ -dimensional case of (3.1):

**Theorem 3.0.1.** *Suppose the drift coefficients  $(b_1, b_2, \dots, b_n)$  of (3.1) have bounded partial derivatives up to order  $k+1$  inclusive. Then for the solution  $X_t$ , we have the series expansion  $X_t = X_t^0 + \epsilon X_t^1 + \dots + \epsilon^k X_t^k + R_{k+1}(t)$  and the remainder term satisfies the inequality:*

$$\sup_{0 \leq t \leq T} \mathbb{E}(|R_{k+1}(t)|^2)^{\frac{1}{2}} \leq C \epsilon^{k+1} \quad C < \infty.$$

### 3.1 One-Dimensional Stochastic Differential Equations

Given a one-dimensional stochastic differential equation of the form:

$$dX_t = b(X_t) dt + \epsilon dB(t), \quad (3.2)$$

we may attempt to solve it by representing the solution  $X_t$  as the following asymptotic series in powers of  $\epsilon$ :

$$X_t = X_t^0 + \epsilon X_t^1 + \epsilon^2 X_t^2 + \dots \quad (3.3)$$

where in both (3.2) and (3.3),  $\epsilon$  represents some small quantity. We replace  $X_t$  with its asymptotic expansion in (3.2), formally yielding:

$$d(X_t^0 + \epsilon X_t^1 + \dots) = b(X_t^0 + \epsilon X_t^1 + \dots) dt + \epsilon dB(t). \quad (3.4)$$

By assuming the differentiability of the drift function  $b$ , we may expand the  $b(X_t^0 + \epsilon X_t^1 + \dots)$  term in (3.4) as a Taylor series around the zeroth order term  $X_t^0$ :

$$b(X_t) = b(X_t^0) + \frac{b'(X_t^0)}{1!}(X_t - X_t^0) + \frac{b''(X_t^0)}{2!}(X_t - X_t^0)^2 + \dots \quad (3.5)$$

We may also write  $X_t - X_t^0 = \epsilon X_t^1 + \epsilon^2 X_t^2 + \dots$  due to (3.3), so by substituting the Taylor expansion (3.5) into (3.4), we have:

$$\begin{aligned} d(X_t^0 + \epsilon X_t^1 + \epsilon^2 X_t^2 \dots) &= b(X_t^0) dt \\ &+ \frac{b'(X_t^0)}{1!}(\epsilon X_t^1 + \epsilon^2 X_t^2 + \dots) dt \\ &+ \frac{b''(X_t^0)}{2!}(\epsilon X_t^1 + \epsilon^2 X_t^2 + \dots)^2 dt \\ &+ \frac{b^{(3)}(X_t^0)}{3!}(\epsilon X_t^1 + \epsilon^2 X_t^2 + \dots)^3 dt \\ &+ \frac{b^{(4)}(X_t^0)}{4!}(\epsilon X_t^1 + \epsilon^2 X_t^2 + \dots)^4 dt \\ &+ \dots \\ &+ \epsilon dB(t). \end{aligned} \quad (3.6)$$

We are now able to obtain separate first order differential equations in  $X_t^0, X_t^1, X_t^2$  and so on, by comparing coefficients of  $\epsilon^0, \epsilon^1, \epsilon^2$  and higher order

terms respectively in the above expression (3.6). On inspection, we arrive at the trivial equation for the zeroth order term  $X_t^0$ :

$$dX_t^0 = b(X_t^0) dt \quad (3.7)$$

which may be solved by simple integration. Equating coefficients of  $\epsilon$  in (3.6), we can easily see that:

$$dX_t^1 = b'(X_t^0)X_t^1 dt + dB(t). \quad (3.8)$$

Since  $X_t^0$  is calculable from (3.7) and  $b$  is the known drift function, we may find a solution to the above equation by means of the integrating factor method, which will be detailed later. At present, we look for a methodical approach to finding the ODEs defining the terms of the asymptotic expansion, (3.3) of the solution to the SDE (3.2). If we attempt to find such an ODE for  $X_t^2$ , we must equate coefficients of  $\epsilon^2$ . By looking at expression (3.6), we see that we have an  $\epsilon^2$  term arises in both the second and third lines resulting from the quadratic term  $(\epsilon X_t^1)^2$ . Thus the ODE for  $X_t^2$  becomes:

$$dX_t^2 = \left( b'(X_t^0)X_t^2 + \frac{b''(X_t^0)}{2}(X_t^1)^2 \right) dt.$$

However, we can see from (3.6) that in attempting to find coefficients of  $\epsilon^3$  on the right hand side, cross terms from the expansion of  $(\epsilon X_t^1 + \epsilon^2 X_t^2 + \dots)^3$  will occur. Anticipating that as the power of  $\epsilon$  increases, the coefficients will become more complex, we must be able to calculate them in order to set up the recursive sequence of ordinary differential equations defining  $X_t^n$  for  $n \geq 2$ . This results in an excursion in combinatorics.

### 3.1.1 A Combinatorial Problem

By simply observing (3.6) and attempting to find all coefficients of relevant powers of  $\epsilon$ , we have:

$$\frac{dX_t^2}{dt} = b'(X_t^0)X_t^2 + \frac{b''(X_t^0)}{2}(X_t^1)^2, \quad (3.9)$$

$$\frac{dX_t^3}{dt} = b'(X_t^0)X_t^3 + \frac{b''(X_t^0)}{2}(2X_t^1X_t^2) + \frac{b^{(3)}(X_t^0)}{6}(X_t^1)^3, \quad (3.10)$$

and:

$$\begin{aligned}
\frac{dX_t^4}{dt} &= b'(X_t^0)X_t^4 \\
&+ \frac{b''(X_t^0)}{2}((X_t^2)^2 + 2X_t^1X_t^3) \\
&+ \frac{b^{(3)}(X_t^0)}{6}(3(X_t^1)^2X_t^2) \\
&+ \frac{b^{(4)}(X_t^0)}{24}(X_t^1)^4.
\end{aligned} \tag{3.11}$$

Notice that in each differential equation for  $X_t^n$ , the coefficient of each term  $\frac{b^{(j)}(X_t^0)}{j!}$  for  $j = 1, 2, \dots, n$  are sums of multiplicative combinations of  $X_t^1, \dots, X_t^n$ . In each of these combinations, the indices of  $X_t^1, \dots, X_t^n$  form a partition of  $n$  with the powers  $m_1, \dots, m_n$  determining how many times each index appears as a summand in the partitions. Also, terms in the coefficient of each  $\frac{b^{(j)}(X_t^0)}{j!}$  have powers such that  $m_1 + \dots + m_n = j$ , and so correspond to partitions of  $n$  with  $j$  parts. The numerical coefficients are consistent with the number of ways in which each  $n$  can be partitioned into the particular combinations of  $j$  summands. To make this approach more rigorous we can use the multinomial theorem, which states:

$$(x_1 + x_2 + \dots + x_n)^j = \sum_{m_1+m_2+\dots+m_n=j} j! \frac{x_1^{m_1} x_2^{m_2} \dots x_n^{m_n}}{m_1! m_2! \dots m_n!}, \tag{3.12}$$

where  $j, n, m_1, \dots, m_n \in \mathbb{N}$ . This may be applied to expanding brackets of the form  $(\epsilon X_t^1 + \epsilon^2 X_t^2 + \dots)^j$  in (3.6) as follows:

$$(\epsilon X_t^1 + \epsilon^2 X_t^2 + \dots)^j = \sum_{m_1+m_2+\dots=j} j! \frac{(\epsilon X_t^1)^{m_1} (\epsilon^2 X_t^2)^{m_2} \dots}{m_1! m_2! \dots}. \tag{3.13}$$

For a fixed value of  $n$  we are looking for the coefficient of  $\epsilon^n$  which will involve only the  $X_t^1, \dots, X_t^n$  terms. In particular the powers of any terms  $X_t^{n+1}, X_t^{n+2}, \dots$  will just be 0. By setting  $m_{n+1} = m_{n+2} = \dots = 0$  accordingly in (3.13), we may make the following refinement:

$$(\epsilon X_t^1 + \epsilon^2 X_t^2 + \dots + \epsilon^n X_t^n)^j = \sum_{m_1+m_2+\dots+m_n=j} j! \frac{(\epsilon X_t^1)^{m_1} (\epsilon^2 X_t^2)^{m_2} \dots (\epsilon^n X_t^n)^{m_n}}{m_1! m_2! \dots m_n!}.$$

where we have used the convention that  $0! = 1$ . From the above:

$$(\epsilon X_t^1 + \epsilon^2 X_t^2 + \dots + \epsilon^n X_t^n)^j = \sum_{m_1+m_2+\dots+m_n=j} j! \frac{\epsilon^{m_1+2m_2+\dots+nm_n} (X_t^1)^{m_1} (X_t^2)^{m_2} \dots (X_t^n)^{m_n}}{m_1! m_2! \dots m_n!}. \quad (3.14)$$

Given (3.14) we may find the coefficient of  $\epsilon^n$  by setting  $m_1 + 2m_2 + \dots + nm_n = n$ . Thus the coefficient of  $\frac{b^{(j)}(X_t^0)}{j!}$  in the ODE for  $X_t^n$  is given by:

$$\sum_{\substack{m_1+m_2+\dots+m_n=j \\ m_1+2m_2+\dots+nm_n=n}} j! \frac{(X_t^1)^{m_1} (X_t^2)^{m_2} \dots (X_t^n)^{m_n}}{m_1! m_2! \dots m_n!}.$$

The ODE for each  $n \geq 2$  contains a term for each  $j = 1, 2, \dots$ , so we may write it in the form:

$$\begin{aligned} \frac{dX_t^n}{dt} &= \sum_{j=1}^n \frac{b^{(j)}(X_t^0)}{j!} \sum_{\substack{m_1+m_2+\dots+m_n=j \\ m_1+2m_2+\dots+nm_n=n}} j! \frac{(X_t^1)^{m_1} (X_t^2)^{m_2} \dots (X_t^n)^{m_n}}{m_1! m_2! \dots m_n!} \\ &= \sum_{j=1}^n b^{(j)}(X_t^0) \sum_{\substack{m_1+m_2+\dots+m_n=j \\ m_1+2m_2+\dots+nm_n=n}} \frac{(X_t^1)^{m_1} (X_t^2)^{m_2} \dots (X_t^n)^{m_n}}{m_1! m_2! \dots m_n!}. \end{aligned} \quad (3.15)$$

The summation constraint  $m_1 + 2m_2 + \dots + nm_n = n$  forces the powers of  $X_t^1, X_t^2, \dots, X_t^n$  to form a partition of  $n$ ,  $(m_1, m_2, \dots, m_n)$ , where each  $m_k$  is the number of summands in the partition equal to  $k$ . Partitions with  $j$  parts, or summands correspond to terms in the coefficient of  $b^{(j)}(X_t^0)$  due to the constraint  $m_1 + m_2 + \dots + m_n = j$ . This is consistent with the above observations of equations (3.9) to (3.11). However it remains to find the partitions occurring in (3.15).

### The Partitioning Algorithm

When attempting to partition a natural number  $n$ , we are actually looking for all solutions to the equation:

$$m_1 + 2m_2 + 3m_3 + \dots + nm_n = n, \quad (3.16)$$

where  $m_1, m_2, m_3, \dots, m_n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ . The vector valued solution  $m = (m_1, m_2, m_3, \dots, m_n)$  then forms a partition of  $n$  since each  $m_k$  is the

number of summands in the partition equal to  $k$ . (For example, when  $n = 4$ ,  $m = (2, 1, 0, 0)$  corresponds to the partition  $4 = 1 + 1 + 2$ .) From (3.16) it is clear to see that theoretically, for any given  $k \leq n$ , the largest possible value of  $m_k$  would be given if  $m_j = 0$  for all  $j \neq k$ . This reduces (3.16) to:

$$km_k = n.$$

So we obviously have the inequality:

$$0 \leq m_k \leq \frac{n}{k}. \quad (3.17)$$

However since for all  $k \leq n$ ,  $m_k \in \mathbb{N}_0$ , the maximum value that  $m_k$  may assume is  $\lfloor \frac{n}{k} \rfloor$ , the integer part of  $\frac{n}{k}$ . We rely on this fact in the following algorithm for finding all possible partitions of  $n$ .

**Step 1.** First note that  $m_n \in \{0, 1\}$  from the above relation (3.17). In the case that  $m_n = 1$ , the equation (3.16) becomes:

$$m_1 + 2m_2 + 3m_3 + \dots + n = n$$

which implies that  $m_1 = m_2 = \dots = m_{n-1} = 0$ , giving the trivial solution to (3.16):

$$m = (0, 0, \dots, 1).$$

For the case that  $m_n = 0$  proceed to the next step.

**Step 2.** Since  $m_n = 0$ , (3.16) becomes:

$$m_1 + 2m_2 + 3m_3 + \dots + (n-1)m_{n-1} = n.$$

Assume for the purposes of demonstration that  $n \geq 2$ , so that  $\lfloor \frac{n}{n-1} \rfloor \leq 1$ , and  $m_{n-1} \in \{0, 1\}$ . The above equation can be written in the following form:

$$m_1 + 2m_2 + 3m_3 + \dots + (n-1)(m_{n-1} - 1) = 1.$$

Assume the case where  $m_{n-1} = 1$ , which automatically implies that  $m_2 = m_3 = \dots m_{n-2} = 0$  and  $m_1 = 1$ , giving the solution to (3.16):

$$m = (1, 0, \dots, 1, 0).$$

The case  $m_{n-1} = 0$  is then dealt with in the next step.

By the  $k$ th step, the equation (3.16) will have reduced to:

$$m_1 + 2m_2 + \dots + (n - k + 1)m_{n-k+1} = n$$

which may always be written in the form:

$$m_1 + 2m_2 + \dots + (n - k + 1)(m_{n-k+1} - k + 1) = k - 1.$$

Then, using the relation  $m_k \leq \lfloor \frac{n}{k} \rfloor$  to find all possible values of  $m_{n-k+1}$ , we may substitute these values into the above and solve each equation to give a different solution to the original equation (3.16). At the  $n$ th step, the equation simply becomes  $m_1 = n$ , corresponding to the solution:

$$m = (n, 0, \dots, 0).$$

This corresponds to the trivial partition:

$$\underbrace{1 + 1 + \dots + 1}_{n \text{ times}} = n.$$

### 3.1.2 The Integrating Factor Method

When representing the solution to the one-dimensional stochastic differential equation (3.2) as an asymptotic series (3.3), we are now able to define the terms up to  $n$ th order by the differential equations (3.7), (3.8) and (3.15). As previously stated, (3.7) may be solved by simple integration. It remains to outline a method for solving the equations for the higher order terms. To solve the SDE (3.8), we use the integrating factor method by first writing it as:

$$dX_t^1 - X_t^1 b'(X_t^0) dt = dB(t). \quad (3.18)$$

Define:

$$F(t) = - \int_0^t b'(X_s^0) ds$$

and then multiply (3.18) by  $e^{F(t)}$ , giving:

$$e^{F(t)} dX_t^1 + e^{F(t)} X_t^1 b'(X_t^0) dt = e^{F(t)} dB(t).$$

Observing that  $d(e^{F(t)}) = -e^{F(t)} b'(X_t^0) dt$ , the above becomes:

$$e^{F(t)} dX_t^1 - d(e^{F(t)}) = e^{F(t)} dB(t).$$



Then using the product rule for differentiation:

$$d(e^{F(t)} X_t^1) = e^{F(t)} dB(t),$$

or:

$$e^{F(t)} X_t^1 - X_0^1 = \int_0^t e^{F(s)} dB(s).$$

Finally, we are able to determine the first order term of the asymptotic expansion:

$$X_t^1 = e^{-F(t)} \left( X_0^1 + \int_0^t e^{F(s)} dB(s) \right)$$

where we look to the methods of McKean [23] to evaluate the stochastic integral. The same method may be employed in finding the higher order terms of the asymptotic expansion from the ordinary differential equations defined by (3.15) since each equation has a known coefficient of  $X_t^n$ .

## 3.2 Two-Dimensional Stochastic Differential Equations

We now consider two-dimensional stochastic differential equations of the form:

$$dX_t = b(X_t) dt + \epsilon dB(t), \quad (3.19)$$

where  $X_t^T = (x_t, y_t)$  and  $b^T = (b_x, b_y)$  is a two-dimensional drift with  $b_x = b_x(x_t, y_t)$  and  $b_y = b_y(x_t, y_t)$ . Here,  $B(t)$  denotes a two-dimensional Brownian motion,  $(B_1(t), B_2(t))^T$ . By writing (3.19) as two simultaneous one-dimensional SDEs in  $x_t$  and  $y_t$ , we may then attempt to find solutions as asymptotic expansions as in the one-dimensional case (3.2). In this case we have:

$$dx_t = b_x(x_t, y_t) dt + \epsilon dB_1(t), \quad (3.20)$$

and:

$$dy_t = b_y(x_t, y_t) dt + \epsilon dB_2(t). \quad (3.21)$$

Given the asymptotic representations  $x_t = x_t^0 + \epsilon x_t^1 + \epsilon^2 x_t^2 \dots$  and  $y_t = y_t^0 + \epsilon y_t^1 + \epsilon^2 y_t^2 \dots$ , (3.20) and (3.21) become:

$$d(x_t^0 + \epsilon x_t^1 + \epsilon^2 x_t^2 + \dots) = b_x(x_t^0 + \epsilon x_t^1 + \epsilon^2 x_t^2 + \dots, y_t^0 + \epsilon y_t^1 + \epsilon^2 y_t^2 + \dots) dt + \epsilon dB_1(t), \quad (3.22)$$

and:

$$d(y_t^0 + \epsilon y_t^1 + \epsilon^2 y_t^2 + \dots) = b_y(x_t^0 + \epsilon x_t^1 + \epsilon^2 x_t^2 + \dots, y_t^0 + \epsilon y_t^1 + \epsilon^2 y_t^2 + \dots) dt + \epsilon dB_1(t). \quad (3.23)$$

By assuming appropriate differentiability of the drift functions  $b_x$  and  $b_y$ , we may expand the relevant terms in the above as a Taylor series in two variables about the point  $(x_t^0, y_t^0)$ :

$$b_i(x_t^0, y_t^0) = b_i(x_t^0, y_t^0) + (x_t - x_t^0) \frac{\partial b_i(x_t^0, y_t^0)}{\partial x} + (y_t - y_t^0) \frac{\partial b_i(x_t^0, y_t^0)}{\partial y} + \dots \quad (3.24)$$

correct to first order, for  $i \in \{x, y\}$ . Note that  $x_t - x_t^0 = \epsilon x_t^1 + \epsilon^2 x_t^2 + \dots$ , and similarly  $y_t - y_t^0 = \epsilon y_t^1 + \epsilon^2 y_t^2 + \dots$ . Substituting the Taylor expansion (3.24) up to first order into (3.22) and (3.23) yields:

$$dx_t = \left( b_x(x_t^0, y_t^0) + (\epsilon x_t^1 + \dots) \frac{\partial b_x(x_t^0, y_t^0)}{\partial x} + (\epsilon y_t^1 + \dots) \frac{\partial b_x(x_t^0, y_t^0)}{\partial y} + \dots \right) dt + \epsilon dB_1(t), \quad (3.25)$$

and:

$$dy_t = \left( b_y(x_t^0, y_t^0) + (\epsilon x_t^1 + \dots) \frac{\partial b_y(x_t^0, y_t^0)}{\partial x} + (\epsilon y_t^1 + \dots) \frac{\partial b_y(x_t^0, y_t^0)}{\partial y} + \dots \right) dt + \epsilon dB_2(t). \quad (3.26)$$

Equating coefficients of  $\epsilon^0$  in the above gives the trivial equation for the zeroth order terms:

$$dX_t^0 = b(X_t^0) dt, \quad (3.27)$$

where  $X_t^0 = (x_t^0, y_t^0)$ . However we are mostly interested in the first order term  $X_t^1 = (x_t^1, y_t^1)$ . We may find differential equations for  $x_t^1$  and  $y_t^1$  by equating coefficients of  $\epsilon^1$  in equations (3.25) and (3.26) respectively:

$$dx_t^1 = \left( x_t^1 \frac{\partial b_x(x_t^0, y_t^0)}{\partial x} + y_t^1 \frac{\partial b_x(x_t^0, y_t^0)}{\partial y} \right) dt + dB_1(t),$$

and:

$$dy_t^1 = \left( x_t^1 \frac{\partial b_y(x_t^0, y_t^0)}{\partial x} + y_t^1 \frac{\partial b_y(x_t^0, y_t^0)}{\partial y} \right) dt + dB_2(t).$$

The above may be summarised as the following matrix equation:

$$dX_t^1 = \beta(t)X_t^1 dt + dB(t), \quad (3.28)$$

where the matrix  $\beta(t)$  is given by:

$$\beta(t) = \begin{pmatrix} \frac{\partial b_x(x_t^0, y_t^0)}{\partial x} & \frac{\partial b_x(x_t^0, y_t^0)}{\partial y} \\ \frac{\partial b_y(x_t^0, y_t^0)}{\partial x} & \frac{\partial b_y(x_t^0, y_t^0)}{\partial y} \end{pmatrix}. \quad (3.29)$$

We may attempt to solve the matrix equation (3.28) to find the first order term  $X_t^1$  using a method analogous to the integrating factor method used for differential equations of one variable, as in Section 3.1.2. This first requires a generalisation of the exponential function as a time-ordered product.

### 3.3 Time-Ordered Products

In this section we define the time-ordered product of some time-dependent matrix as the infinite sum of increasingly repeated integrals. From [6], we see that the time-ordered product (also called the time-ordered exponential) has equivalent representation as a product integral, ensuring its convergence in matrix norm. We are then able to use time-ordered products in solving matrix stochastic differential equations such as (3.28) in the previous section.

**Definition 3.3.1.** For a given time-dependent  $n \times n$  matrix  $D(t)$ , define the time-ordered product  $T_+$  as:

$$T_+ \left( \int_0^t D(s) ds \right) = I + \int_0^t D(t_1) dt_1 + \int_0^t \int_0^{t_1} D(t_2)D(t_1) dt_1 dt_2 + \dots$$

where  $I$  is the  $n \times n$  identity matrix.

#### 3.3.1 Product Integration

The time-ordered product defined above may be equivalently represented as the product integral of the function  $D : [a, b] \rightarrow \mathbb{C}_{n \times n}$  for some interval

$[a, b] \in \mathbb{R}$  and where  $\mathbb{C}_{n \times n}$  denotes the space of  $n \times n$  matrices with complex entries. Following the results of [6], we may define the product integral by a limiting procedure using step-functions analogous to those used in the definition of the ordinary integral.

Letting  $P = \{s_0, s_1, \dots, s_n\}$  be a partition of the interval  $[a, b]$ , the corresponding point-value approximant,  $D_P$  of  $D : [a, b] \rightarrow \mathbb{C}_{n \times n}$  is the (matrix-valued) step-function taking value  $D(s_1)$  on  $[s_0, s_1]$ ,  $D(s_2)$  on  $(s_1, s_2]$ , continuing until  $D(s_n)$  on  $(s_{n-1}, s_n]$ . For continuous (and therefore uniformly continuous)  $D$ , we have that the point-value approximant,  $D_P$  converges uniformly to  $D$  since:

$$\lim_{\mu(P) \rightarrow 0} D_P(x) = D(x) \quad \text{for} \quad x \in [a, b].$$

where  $\mu(P)$  denotes the mesh of the partition  $P$ .

For some general step function  $B$ , we let  $B_k$  denote the value of  $B$  on  $(s_{k-1}, s_k]$  and  $\Delta s_k := s_k - s_{k-1}$ . (Although  $\Delta$  usually denotes the forward difference operator, we use notation consistent with Dollard and Friedman, [6].) We may define  $E_B$  as follows:

$$E_B(x) = \begin{cases} e^{(x-s_0)B_1} & \text{if } x \in [s_0, s_1], \\ e^{(x-s_1)B_2} e^{\Delta s_1 B_1} & \text{if } x \in (s_1, s_2], \\ \vdots \\ e^{(x-s_n)B_n} \dots e^{\Delta s_2 B_2} e^{\Delta s_1 B_1} & \text{if } x \in (s_{n-1}, s_n], \end{cases}$$

which allows us to define the product integral according to the following theorem from [6]:

**Theorem 3.3.2.** *Let  $D : [a, b] \rightarrow \mathbb{C}_{n \times n}$  be continuous and let  $\{D_n\}$  be any sequence of step-functions converging to  $D$ . Then the sequence  $\{E_{D_n}\}$  converges uniformly on  $[a, b]$  to a matrix called the product integral of  $D$  over  $[a, x]$ , denoted:*

$$\prod_a^x e^{D(s)ds}.$$

Since for continuous  $D$  we have a sequence of point-value approximants converging to  $D$ , we have in the previous notation:

$$\prod_a^b e^{D(s)ds} = \lim_{\mu(P) \rightarrow 0} \prod_{k=1}^n e^{\Delta(s_k)D(s_k)}.$$

Having suitably defined the product integral, we now concentrate on functions  $D$  mapping some interval  $[0, t]$  onto the complex-valued matrices. Still following the notation of [6], for such  $D$  we recursively define a sequence of functions  $\{J_n(0, t)\}$  as follows:

$$J_0(0, t) = I$$

and:

$$J_n(0, t) = \int_0^t D(s)J_{n-1}(0, s) ds \quad \text{for } n \geq 1.$$

In this notation we may then write the time-ordered product from Definition 3.3.1 as:

$$T_+ \left( \int_0^t D(s) ds \right) = \sum_{k=0}^{\infty} J_k(0, t).$$

From the following theorem (proved in [6]), we see that the this time-ordered product is equivalent to the product integral of the matrix  $D$  over the interval  $[0, t]$ , therefore ensuring the convergence of the series representation in the matrix norm.

**Theorem 3.3.3.** *Let  $D : [0, t] \rightarrow \mathbb{C}_{n \times n}$  be continuous. Then:*

$$\prod_0^t e^{D(s)ds} = \sum_{k=0}^{\infty} J_k(0, t),$$

*the series on the right converging uniformly.*

### 3.3.2 Using Time-Ordered Products

Note that in the above definition of the time-ordered product, Definition 3.3.1,  $0 \leq t_1 \leq t_2 \leq \dots \leq t$ , so that the arguments of the matrices in products of the form  $D(t_n)\dots D(t_1)$  are in decreasing order. Since in general such matrices will not commute, we must consider a similar notion where the time arguments are arranged increasingly:

**Definition 3.3.4.** Define the reversed time-ordered product  $T_-$  as:

$$T_- \left( \int_0^t D(s) ds \right) = I + \int_0^t D(t_1) dt_1 + \int_0^t \int_0^{t_2} D(t_1)D(t_2) dt_1 dt_2 + \dots$$

Then by definition, we have:

$$T_+ \left( - \int_0^t D(s) ds \right) = I - \int_0^t D(t_1) dt_1 + \int_0^t \int_0^{t_2} D(t_2)D(t_1) dt_1 dt_2 - \dots$$

and:

$$T_- \left( - \int_0^t D(s) ds \right) = I - \int_0^t D(t_1) dt_1 + \int_0^t \int_0^{t_2} D(t_1)D(t_2) dt_1 dt_2 - \dots$$

Using these definitions and observations we may now find solutions to matrix SDEs. We first require a preliminary lemma.

**Lemma 3.3.5.** The multiplicative matrix inverse of  $T_-(-\int_0^t D(s) ds)$  is  $T_+(\int_0^t D(s) ds)$ .

*Proof.* First, note that:

$$\begin{aligned} \frac{d}{dt} T_+ \left( \int_0^t D(s) ds \right) &= D(t) + D(t) \left( \int_0^t D(t_1) dt_1 \right) + \dots \\ &= D(t) T_+ \left( \int_0^t D(s) ds \right). \end{aligned}$$

Similarly:

$$\frac{d}{dt} T_- \left( - \int_0^t D(s) ds \right) = -T_- \left( - \int_0^t D(s) ds \right) D(t).$$

Define  $T_-(-) := T_-(-\int_0^t D(s) ds)$ , and  $T_+(+) := T_+(\int_0^t D(s) ds)$ . Using the matrix product rule for differentiation:

$$\begin{aligned} \frac{d}{dt} T_-(-) T_+(+) &= -T_-(-) D(t) T_+(+) + T_-(-) D(t) T_+(+) \\ &= T_-(-) (-D(t) + D(t)) T_+(+) \\ &= 0. \end{aligned}$$

Hence the product  $T_-(-\int_0^t D(s) ds) T_+(\int_0^t D(s) ds)$  is constant with respect to time. Setting  $t = 0$  gives  $T_+(0) = T_-(0) = I$ , hence for all  $t \geq 0$ :

$$T_- \left( - \int_0^t D(s) ds \right) T_+ \left( \int_0^t D(s) ds \right) = I.$$

□

We now use the above defined time-ordered products analogously to exponentials in solving matrix SDEs using a generalisation of the integrating factor method. The solution is then given in terms of time-ordered products.

**Proposition 3.3.6.** Given the matrix stochastic differential equation:

$$dX_t = D(t)X_t dt + dB(t),$$

the solution in terms of time-ordered products is:

$$X_t = T_+ \left( \int_0^t D(s) ds \right) \int_0^t T_- \left( - \int_0^u D(s) ds \right) dB(u).$$

*Proof.* Premultiply the above SDE by  $T_-(-\int_0^t D(s) ds)$  to give (using the above shorthand):

$$T_-(-) dX_t - T_-(-)D(t)X_t = T_-(-) dB(t).$$

Again using that  $\frac{d}{dt}T_-(-) = -T_-(-)D(t)$ , and the product rule for matrix differentiation, this becomes:

$$d(T_-(-)X_t) = T_-(-) dB(t),$$

or:

$$T_-(-)X_t = \int_0^t T_-(-) dB(s).$$

To find the solution  $X_t$  we must premultiply by the inverse of  $T_-(-)$ , giving:

$$X_t = \left( T_- \left( - \int_0^t D(s) ds \right) \right)^{-1} \int_0^t T_- \left( - \int_0^u D(s) ds \right) dB(u).$$

Finally we may invert the time-ordered product using the result from Lemma (3.3.5), giving the solution:

$$X_t = T_+ \left( \int_0^t D(s) ds \right) \int_0^t T_- \left( - \int_0^u D(s) ds \right) dB(u).$$

□

### 3.4 The First Order Correction to the Parabolic Orbit

Consider equation (3.28) from the previous section. We are now able to calculate the first order term in the asymptotic expansion of its solution in terms of time-ordered products. From Proposition 3.3.6, the first order term is given by:

$$X_t^1 = T_+ \left( \int_0^t \beta(s) ds \right) \int_0^t T_- \left( - \int_0^u \beta(s) ds \right) dB(u), \quad (3.30)$$

where the matrix  $\beta(t)$  is given above by (3.29). Also, from (3.27),  $dX_t^0 = b(X_t^0) dt$ . Thus the solution to (3.28) correct to first order is given by:

$$X_t = \int_0^t b(X_0^s) ds + \epsilon T_+ \left( \int_0^t \beta(s) ds \right) \int_0^t T_- \left( - \int_0^u \beta(s) ds \right) dB(u) + O(\epsilon^2). \quad (3.31)$$

We may now apply this method to finding the first order correction to the semiclassical parabolic Nelson diffusion from the previous chapter, defined by equations (2.22) and (2.24). Given calculations from Section 2.3, the matrix of derivatives of the drift  $b = (b_x, b_y)$  is:

$$\begin{pmatrix} \frac{\partial b_x(x, y)}{\partial x} & \frac{\partial b_x(x, y)}{\partial y} \\ \frac{\partial b_y(x, y)}{\partial x} & \frac{\partial b_y(x, y)}{\partial y} \end{pmatrix} = \text{sgn}(y) \begin{pmatrix} \frac{-(r+2x)(r-x)}{2r^3\sqrt{r-x}} & \frac{y(2x-r)}{2r^3\sqrt{r-x}} \\ \frac{y(2x-r)}{2r^3\sqrt{r-x}} & \frac{(r+2x)(r-x)}{2r^3\sqrt{r-x}} \end{pmatrix},$$

where we work on domains avoiding its singularity at the origin and jump discontinuity at  $y = 0$  due to the signum function.

Given the zeroth order terms  $x_0$  and  $y_0$  defined by equations (2.3.3) and (2.3.3) respectively, for the parabolic diffusion:

$$\beta(t) = \text{sgn}(y_0) \begin{pmatrix} \frac{-(r_0+2x_0)(r_0-x_0)}{2r_0^3\sqrt{r_0-x_0}} & \frac{y_0(2x_0-r_0)}{2r_0^3\sqrt{r_0-x_0}} \\ \frac{y_0(2x_0-r_0)}{2r_0^3\sqrt{r_0-x_0}} & \frac{(r_0+2x_0)(r_0-x_0)}{2r_0^3\sqrt{r_0-x_0}} \end{pmatrix}.$$

Hence the first order correction to the parabolic diffusion is:



$$X_t^1 = T_+ \left( \int_0^t \beta(s) ds \right) \int_0^t T_- \left( - \int_0^u \beta(s) ds \right) dB(u),$$

in terms of time-ordered products of the matrix  $\beta(t)$  defined in Section 3.3.

### 3.4.1 The Radial Equation for the Parabolic Orbit

An alternative approach to finding the first order correction to the parabolic Nelson diffusion is by firstly finding a stochastic differential equation governing the time evolution of the polar radius of the parabolic orbit, ( $r(t) = \sqrt{x^2(t) + y^2(t)}$  in Cartesian coordinates) as in Section 2.2.1. Given equations (2.22) and (2.24), the parabolic Nelson diffusion  $X(t) = (x(t), y(t))$  may be completely defined by:

$$dx(t) = b_x(X(t)) dt + \epsilon dB_x(t),$$

and:

$$dy(t) = b_y(X(t)) dt + \epsilon dB_y(t).$$

Since  $r = r(x, y)$ , Itô's formula gives:

$$dr(t) = \frac{\partial r}{\partial x} dx + \frac{\partial r}{\partial y} dy + \epsilon^2 \frac{\Delta r}{2} dt.$$

After evaluating derivatives of  $r$  and substituting in the above expressions for  $dx(t)$  and  $dy(t)$ , we have:

$$dr(t) = \left( \frac{xb_x + yb_y}{r} + \epsilon^2 \frac{1}{2r} \right) dt + \epsilon \frac{x dB_x(t) + y dB_y(t)}{r}.$$

A simple calculation using (2.24) shows that:

$$xb_x + yb_y = -\frac{\sqrt{r-x}}{r} \text{sgn}(y).$$

Therefore the radial equation becomes:

$$dr(t) = \left( \sqrt{\frac{2}{r}} \sin\left(\frac{\theta}{2}\right) \text{sgn}(y) + \epsilon^2 \frac{1}{2r} \right) dt + \epsilon dB_r(t),$$

where  $B_r(t)$  is again a one-dimensional Brownian motion.

Firstly, it is important to note that:

$$dx(t) + dr(t) = \frac{\epsilon^2}{2r} dt + \epsilon (dB_x(t) + dB_r(t)).$$

Then defining a new variable,  $C(t) = x(t) + r(t)$ :

$$\nabla C = \left( \frac{x}{r} + 1, \frac{y}{r} \right),$$

since:

$$\frac{\partial r}{\partial x} = \frac{x}{r} \quad \text{and} \quad \frac{\partial r}{\partial y} = \frac{y}{r}.$$

Conveniently for the two-dimensional Brownian motion  $B(t) = (B_x(t), B_y(t))$ ,  $\nabla C \cdot B(t) = B_x(t) + B_r(t)$ , so setting  $B_C(t) = \hat{\nabla} C \cdot B(t)$  for  $\hat{\nabla} C$  a unit vector in the direction of  $\nabla C$ , we have the following SDE for  $C(t)$ :

$$dC(t) = \epsilon \sqrt{\frac{2C}{r}} dB_C(t) + \frac{\epsilon^2}{2r} dt.$$

Expanding  $C(t)$  as an asymptotic series in  $\epsilon$  as follows:

$$C(t) = C_t^0 + \epsilon C_t^1 + \epsilon^2 C_t^2 + \dots$$

we see that the zeroth order term  $C_t^0$  is constant, and:

$$dC_t^1 = \sqrt{\frac{2C_t^0}{r_t^0}} dB_C(t),$$

for  $r_t^0$  the zeroth order (classical) value of  $r(t)$ . The above equation implies:

$$C_t^1 = \sqrt{2C_t^0} \int_0^t \frac{dB_C(s)}{\sqrt{r_s^0}}.$$

Therefore correct to first order in  $\epsilon$ :

$$C(t) = C_t^0 + \epsilon \sqrt{2C_t^0} W_C \left( \int_0^t \frac{ds}{r_s^0} \right) + o(\epsilon^2),$$

for a time-changed Brownian motion  $W_C$ , which is possible due to a result from McKean [23]:

**Theorem 3.4.1.** *Consider a stochastic integral based upon a non-anticipating Brownian functional  $e$ :*

$$I(t) = \int_0^t e dB(t) \quad \text{with} \quad \tau(t) := \int_0^t e^2 dt < \infty \quad (t \geq 0).$$

Then  $I(t) = B_2(\tau(t))$  for a new Brownian motion  $B_2$ .

Analogously to the above, we may define  $D(t) = r(t) - x(t)$ , and attempt to find a corresponding SDE. Correct to first order in  $\epsilon$ :

$$dD(t) = -\frac{2\sqrt{r-x}}{r}\text{sgn}(y) dt + \epsilon(dB_r(t) - dB_x(t)).$$

Noting that :

$$\nabla D = \left(\frac{x}{r} - 1, \frac{y}{r}\right) \quad \text{with} \quad |\nabla D| = \sqrt{\frac{2D}{r}},$$

and then that  $\nabla D \cdot B(t) = B_r(t) - B_x(t)$ , we have:

$$dD(t) = -\frac{2\sqrt{r-x}}{r}\text{sgn}(y) dt + \epsilon\sqrt{\frac{2D}{r}} dB_D(t),$$

where  $B_D(t) = \hat{\nabla} D \cdot B(t)$ , and as above  $\hat{\nabla} D$  is a unit vector parallel to  $\nabla D$ .

Finding the first order corrections to both  $C(t)$  and  $D(t)$  would allow us to calculate explicitly the first order corrections to the Cartesian coordinates of the parabolic diffusion. However we may obtain such corrections via a more elegant method by using the Levi-Civita transformation, as detailed in the next chapter.

## Chapter 4

# The Levi-Civita Transformation for the Parabolic Orbit

We return to the parabolic Nelson diffusion which was obtained in Chapter 2 by taking the limit of the eccentricity parameter in the Nelson diffusion for the atomic elliptic state from [9]:

$$dX(t) = b(X(t)) dt + \epsilon dB(t). \quad (4.1)$$

As usual,  $B(t) = (B_x(t), B_y(t))$  is a two-dimensional Brownian motion and the drift  $b$  is given by:

$$b = (b_x, b_y) = \left( \operatorname{sgn}(y) \frac{\sqrt{r-x}}{r}, -\frac{y \operatorname{sgn}(y)}{r\sqrt{r-x}} \right). \quad (4.2)$$

From Chapter 2, we have seen that in the absence of the noise term, the above Nelson drift determines a parabolic trajectory. After some analysis of the Nelson diffusion, we follow the approach of Wentzell and Friedlin [13], using asymptotic expansions in  $\epsilon$  to calculate the first order corrections by applying the Levi-Civita transformation detailed in Chapter 1 to the stochastic parabolic orbit.

By letting the coordinates of the parabolic Nelson diffusion  $(x(t), y(t))$  be the respective real and imaginary parts of a complex variable  $z(t) = x(t) + iy(t)$ , we may make the Levi-Civita transformation to new coordinates  $(u(t), v(t))$  where  $w(t) = u(t) + iv(t)$  and  $z(t) = w^2(t)$ . First note that:

$$w = u + iv = (x + iy)^{1/2} = (re^{i\theta})^{1/2} = r^{1/2} e^{i\frac{\theta}{2}} = r^{1/2} \left( \cos\left(\frac{\theta}{2}\right) + i \sin\left(\frac{\theta}{2}\right) \right).$$

Hence in polar coordinates we have:

$$u = r^{1/2} \cos\left(\frac{\theta}{2}\right) \quad \text{and} \quad v = r^{1/2} \sin\left(\frac{\theta}{2}\right). \quad (4.3)$$

Since  $w = z^{1/2}$  is a regular function of  $z$  with a simple branch point at the origin, the Cauchy-Riemann equations imply that  $\Delta u = \Delta v = 0$ . Applying Itô's formula to equation (4.1) gives:

$$du(X(t)) = \nabla u \cdot b(X(t)) dt + \epsilon \nabla u \cdot dB(t), \quad (4.4)$$

and similarly for  $v$ :

$$dv(X(t)) = \nabla v \cdot b(X(t)) dt + \epsilon \nabla v \cdot dB(t). \quad (4.5)$$

Recall for standard polar coordinates we have:

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial \theta}{\partial x} = -\frac{y}{r^2} \quad \text{and} \quad \frac{\partial \theta}{\partial y} = \frac{x}{r^2},$$

so given (4.3), we may calculate the gradient of  $u$  as follows:

$$\begin{aligned} \nabla u &= \left( \frac{1}{2r^{1/2}} \frac{x}{r} \cos\left(\frac{\theta}{2}\right) + \frac{r^{1/2}}{2} \frac{y}{r^2} \sin\left(\frac{\theta}{2}\right), \frac{1}{2r^{1/2}} \frac{y}{r} \cos\left(\frac{\theta}{2}\right) - \frac{r^{1/2}}{2} \frac{x}{r^2} \sin\left(\frac{\theta}{2}\right) \right) \\ &= \frac{1}{2r^{3/2}} \left( x \cos\left(\frac{\theta}{2}\right) + y \sin\left(\frac{\theta}{2}\right), y \cos\left(\frac{\theta}{2}\right) - x \sin\left(\frac{\theta}{2}\right) \right) \\ &= \frac{1}{2r^{1/2}} \left( \cos\left(\frac{\theta}{2}\right), \sin\left(\frac{\theta}{2}\right) \right). \end{aligned} \quad (4.6)$$

Similarly for  $v$ :

$$\nabla v = \frac{1}{2r^{1/2}} \left( -\sin\left(\frac{\theta}{2}\right), \cos\left(\frac{\theta}{2}\right) \right), \quad (4.7)$$

also noting that:

$$|\nabla u| = |\nabla v| = \frac{1}{2r^{1/2}}.$$

In polar coordinates, the drift of the parabolic Nelson diffusion (4.2) is given by:

$$b = (b_x, b_y) = \left( \sqrt{\frac{2}{r}} \operatorname{sgn}(y) \sin\left(\frac{\theta}{2}\right), -\sqrt{\frac{2}{r}} \operatorname{sgn}(y) \cos\left(\frac{\theta}{2}\right) \right).$$

It follows that:

$$\nabla u \cdot b = \frac{\operatorname{sgn}(y)}{\sqrt{2r}} \left( \sin\left(\frac{\theta}{2}\right), -\cos\left(\frac{\theta}{2}\right) \right) \cdot \left( \cos\left(\frac{\theta}{2}\right), \sin\left(\frac{\theta}{2}\right) \right) = 0.$$

Substituting the above in the SDE for  $u$ , (4.4) gives:

$$du(t) = \frac{\epsilon}{2r^{1/2}(t)} dB_u(t),$$

where  $B_u(t) = \hat{\nabla}u \cdot B(t)$ , is a one-dimensional Brownian motion and  $\hat{\nabla}u$  is a unit vector in the direction of  $\nabla u$ . For a time-changed Brownian motion  $W_u$ , this is equivalent to:

$$du(t) = \frac{\epsilon}{2} dW_u \left( \int_0^t \frac{ds}{r(s)} \right), \quad (4.8)$$

due to the result from McKean [23] stated previously (Theorem 3.4.1).

We may obtain a similar SDE for the coordinate  $v$  by first noting that:

$$\nabla v \cdot b = -\frac{\operatorname{sgn}(y(t))}{\sqrt{2r(t)}}.$$

Substituting this into the above SDE for  $v$ , (4.5), yields:

$$dv(t) = -\frac{\operatorname{sgn}(y)}{\sqrt{2r}} dt + \frac{\epsilon}{2r^{1/2}} dB_v(t),$$

where using similar conventions as for  $u$ ,  $B_v(t) = \hat{\nabla}v \cdot B(t)$ . Again, using Theorem 3.4.1, for a time-changed Brownian motion  $W_v$ :

$$dv(t) = -\frac{\operatorname{sgn}(y(t))}{\sqrt{2r(t)}} dt + \frac{\epsilon}{2} dW_v \left( \int_0^t \frac{ds}{r(s)} \right). \quad (4.9)$$

From Theorem 2.2.3, we know that the two-dimensional parabolic Nelson diffusion does not reach its singular point at the origin in finite time. Hence the time change from physical time  $t$  to fictitious time  $\tau$ :

$$\tau(t) = \int_0^t \frac{ds}{r(s)} \quad (4.10)$$

is well-defined almost surely. From equations (4.8) and (4.9), in terms of the time variable  $\tau$ , we now have:

$$du(\tau) = \frac{\epsilon}{2} dW_u(\tau) \quad \text{and} \quad dv(\tau) = -\frac{\operatorname{sgn}(y)}{\sqrt{2}} d\tau + \frac{\epsilon}{2} dW_v(\tau), \quad (4.11)$$

where  $W_u$  and  $W_v$  are independent one-dimensional Brownian motions.

The above stochastic differential equations (4.11) define the Levi-Civita transform of the parabolic Nelson diffusion process. Although the drift of the equation for  $v(\tau)$  rapidly oscillates due to the  $\text{sgn}(y)$  term, it is bounded and hence both equations have pathwise unique solutions as a result of the random time change.

In what follows we shall assume that  $y(0) < 0$  (the initial  $y$  coordinate of the diffusion) and let  $\hat{\tau} = \inf\{s > 0 : y(s) > 0\}$ . Clearly for all times  $t < \hat{\tau}$ ,  $y(t) < 0$ . Since  $y(t)$  has a definite sign for this time interval, we may simplify our SDE for  $v(\tau)$ :

$$dv(\tau) = \frac{d\tau}{\sqrt{2}} + \frac{\epsilon}{2} dW_v(\tau). \quad (4.12)$$

## 4.1 The Quantum Correction to the Levi-Civita Transformation

We now look at the quantum corrections to the  $u$  and  $v$  coordinates by first calculating the correction to fictitious time  $\tau$ . We begin by writing the variables as asymptotic series in increasing powers of  $\epsilon$ ,  $u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \dots$ ,  $v = v_0 + \epsilon v_1 + \epsilon^2 v_2 + \dots$  and  $\tau = \tau_0 + \epsilon \tau_1 + \epsilon^2 \tau_2 + \dots$ . Here, the values  $u_0$ ,  $v_0$  and  $\tau_0$  correspond to classical values of each variable. The cancellation of the drift terms in the SDE for  $u$  in the previous section implies that the classical value  $u_0$  must be constant, which we return to later.

We define the quantum correction to  $u$ ,  $\delta u := \epsilon u_1 + \epsilon^2 u_2 + \dots$ , the difference between the classical and quantum values. Similarly, the quantum corrections to  $v$  and  $\tau$  will be denoted  $\delta v$  and  $\delta \tau$  respectively.

Integrating the SDEs for  $u(\tau)$  and  $v(\tau)$ , (4.11) and (4.12), gives:

$$u(\tau) = u(0) + \frac{\epsilon}{2} W_u(\tau) \quad \text{and} \quad v(\tau) = v(0) + \frac{\tau}{\sqrt{2}} + \frac{\epsilon}{2} W_v(\tau), \quad (4.13)$$

for deterministic values of  $u(0)$  and  $v(0)$ . The quantum corrections are then given by:

$$\delta u(\tau) = \frac{\epsilon}{2} W_u(\tau) \quad \text{and} \quad \delta v(\tau) = \frac{\delta \tau}{\sqrt{2}} + \frac{\epsilon}{2} W_v(\tau),$$

where we allow  $\delta u$  and  $\delta v$  both to be  $O(\epsilon)$  to obtain the first order correction to  $\tau$ .

Given that  $(u + iv)^2 = x + iy$ , we have  $x = u^2 - v^2$  and  $y = 2uv$ . A simple calculation then gives for the polar radius  $r = u^2 + v^2$ . Substituting this into the above definition of  $\tau$  (4.10):

$$\tau(t) = \int_0^t \frac{ds}{u^2(s) + v^2(s)}, \quad \text{with} \quad \tau_0(t) = \int_0^t \frac{ds}{u_0^2(s) + v_0^2(s)}.$$

The quantum correction to  $\tau$  is then given by:

$$\delta\tau = \tau - \tau_0 = \int_0^t \left( \frac{1}{u^2(s) + v^2(s)} - \frac{1}{u_0^2(s) + v_0^2(s)} \right) ds.$$

Correct to first order in  $\epsilon$ :

$$\delta\tau = -2 \int_0^t \frac{u_0 \delta u + v_0 \delta v}{r_0^2} ds,$$

$\delta\tau$  being  $O(\epsilon)$ . The quantum correction to  $\tau$ , correct to first order in  $\epsilon$  is then given by:

$$\delta\tau = -2 \int_0^t \frac{1}{r_0^2} \left( \frac{\epsilon}{2} u_0 W_u(\tau) + v_0 \left( \frac{\delta\tau}{\sqrt{2}} + \frac{\epsilon}{2} W_v(\tau) \right) \right) ds.$$

Equivalently:

$$\dot{\delta\tau} + \frac{\sqrt{2}v_0}{r_0^2} \delta\tau = -\epsilon \frac{u_0 W_u(\tau) + v_0 W_v(\tau)}{r_0^2},$$

where  $\dot{\delta\tau}$  denotes differentiation with respect to the physical time  $t$ . This equation may be solved to find the quantum correction  $\delta\tau$  by first multiplying by the integrating factor:

$$\exp \left( \int_0^t \frac{\sqrt{2}v_0}{r_0^2} ds \right).$$

Finally correct to first order in  $\epsilon$ :

$$\begin{aligned} \delta\tau(t) = & \\ -\epsilon \exp \left( - \int_0^t \frac{\sqrt{2}v_0}{r_0^2} ds \right) & \int_0^t \frac{u_0 W_u(\tau_0) + v_0 W_v(\tau_0)}{r_0^2} \exp \left( \int_0^s \frac{\sqrt{2}v_0}{r_0^2} ds' \right) ds, \end{aligned} \quad (4.14)$$



where the Brownian motions  $W_u$  and  $W_v$  have been evaluated at the classical value  $\tau_0$ , the value of which may be found as follows:

$$\tau_0(t) = \int_0^t \frac{ds}{r_0(s)} = \int_0^t \frac{r_0(s)\dot{\theta}(s)}{r_0^2(s)\dot{\theta}(s)} ds = \int_0^t \frac{r_0(s)\dot{\theta}(s)}{L} ds = \frac{1}{L} \int_{\theta(0)}^{\theta(t)} r_0(\theta) d\theta, \quad (4.15)$$

where the penultimate equality is due to the fact that for a classical system, the angular momentum  $r_0^2\dot{\theta} = L$ , a constant. Given that the underlying classical system is a parabolic orbit, as discussed in Chapter 2:

$$\frac{l}{r_0} = 1 + \cos \theta \quad \text{and} \quad l = L^2,$$

for semi-latus rectum  $l$ , recalling that we work in suitable units allowing the force constant  $\mu$  to equal 1. From the above equations:

$$\frac{L^2}{r_0} = 2 \cos^2 \left( \frac{\theta}{2} \right) \implies r_0 = \frac{L^2}{2} \sec^2 \left( \frac{\theta}{2} \right), \quad (4.16)$$

which may be substituted into (4.15) to give:

$$\tau_0(t) = \frac{L}{2} \int_{\theta(0)}^{\theta(t)} \sec^2 \left( \frac{\theta}{2} \right) d\theta = L \left( \tan \left( \frac{\theta(t)}{2} \right) - \tan \left( \frac{\theta(0)}{2} \right) \right). \quad (4.17)$$

Further computation will require us to calculate the values of  $u_0$  and  $v_0$ . After the Levi-Civita transform, given equations (4.3), we have that:

$$u_0 = r_0^{1/2} \cos \left( \frac{\theta}{2} \right).$$

Again using the relations for the classical parabolic orbit (4.16):

$$\frac{L^2}{r_0} = 1 + \cos \theta \quad \implies \quad \frac{L^2}{2} = r_0 \cos^2 \left( \frac{\theta}{2} \right).$$

Therefore:

$$u_0 = \frac{L}{\sqrt{2}},$$

a constant. Given (4.13), note that the initial condition  $u(0) = u_0$ . Similarly:

$$v_0 = \frac{L}{\sqrt{2}} \tan \left( \frac{\theta}{2} \right).$$

Since  $v_0 = v(0) + \frac{\tau_0}{\sqrt{2}}$  from (4.13), we have:

$$v(0) = \frac{L}{\sqrt{2}} \tan\left(\frac{\theta(0)}{2}\right),$$

using the expression for  $\tau_0$ , (4.17).

We now evaluate the quantum correction  $\delta\tau$  by first evaluating the integrating factor term:

$$\begin{aligned} \exp\left(-\int_0^t \frac{\sqrt{2}v_0(s)}{r_0^2(s)} ds\right) &= \exp\left(-\int_0^t \frac{\sqrt{2}v_0(s)\dot{\theta}(s)}{L} ds\right) \\ &= \exp\left(-\frac{\sqrt{2}}{L} \int_{\theta(0)}^{\theta(t)} v_0(\theta) d\theta\right). \end{aligned}$$

Therefore:

$$\begin{aligned} \exp\left(-\int_0^t \frac{\sqrt{2}v_0(s)}{r_0^2(s)} ds\right) &= \exp\left(-\int_{\theta(0)}^{\theta(t)} \tan\left(\frac{\theta}{2}\right) d\theta\right) \\ &= \left|\frac{\cos\left(\frac{\theta(t)}{2}\right)}{\cos\left(\frac{\theta(0)}{2}\right)}\right|^2 \end{aligned}$$

A similar calculation shows that:

$$\exp\left(\int_0^t \frac{\sqrt{2}v_0(s)}{r_0^2(s)} ds\right) = \left|\frac{\cos\left(\frac{\theta(0)}{2}\right)}{\cos\left(\frac{\theta(t)}{2}\right)}\right|^2$$

Given the above calculations, we may now write the quantum correction to  $\tau$ , equation (4.14), as:

$$\delta\tau(t) = -\epsilon \left|\frac{\cos\left(\frac{\theta(t)}{2}\right)}{\cos\left(\frac{\theta(0)}{2}\right)}\right|^2 \int_{\theta(0)}^{\theta(t)} f(\theta) d\theta,$$

with:

$$f(\theta) = \frac{u_0}{L} W_u \left( L \left( \tan \left( \frac{\theta}{2} \right) - \tan \left( \frac{\theta(0)}{2} \right) \right) \right) \left| \frac{\cos \left( \frac{\theta(0)}{2} \right)}{\cos \left( \frac{\theta}{2} \right)} \right|^2$$

$$+ \frac{1}{\sqrt{2}} \tan \left( \frac{\theta}{2} \right) W_v \left( L \left( \tan \left( \frac{\theta}{2} \right) - \tan \left( \frac{\theta(0)}{2} \right) \right) \right) \left| \frac{\cos \left( \frac{\theta(0)}{2} \right)}{\cos \left( \frac{\theta}{2} \right)} \right|^2.$$

Recalling that the classical value  $u_0 = \frac{L}{\sqrt{2}}$ , a constant:

$$\delta\tau(t) = -\epsilon \sqrt{\frac{L}{2}} \int_{\theta(0)}^{\theta(t)} \left( \left| \frac{\cos \left( \frac{\theta(t)}{2} \right)}{\cos \left( \frac{\theta}{2} \right)} \right|^2 W_u \left( \tan \left( \frac{\theta}{2} \right) - \tan \left( \frac{\theta(0)}{2} \right) \right) \right.$$

$$\left. + \left| \frac{\cos \left( \frac{\theta(t)}{2} \right)}{\cos \left( \frac{\theta}{2} \right)} \right|^2 \tan \left( \frac{\theta}{2} \right) W_v \left( \tan \left( \frac{\theta}{2} \right) - \tan \left( \frac{\theta(0)}{2} \right) \right) \right) d\theta.$$

We have  $u(t)$  correct to first order in  $\epsilon$  as follows:

$$u(t) = \frac{L}{\sqrt{2}} + \epsilon \frac{\sqrt{L}}{2} W_u \left( \tan \left( \frac{\theta(t)}{2} \right) - \tan \left( \frac{\theta(0)}{2} \right) \right). \quad (4.18)$$

Given the first order correction to  $\tau$ ,  $v(t)$  correct to first order is given by:

$$v(t) = \frac{L}{\sqrt{2}} \tan \left( \frac{\theta(t)}{2} \right) + \epsilon \frac{\sqrt{L}}{2} W_v \left( \tan \left( \frac{\theta(t)}{2} \right) - \tan \left( \frac{\theta(0)}{2} \right) \right)$$

$$- \epsilon \frac{\sqrt{L}}{2} \int_{\theta(0)}^{\theta(t)} \left( \left| \frac{\cos \left( \frac{\theta(t)}{2} \right)}{\cos \left( \frac{\theta}{2} \right)} \right|^2 \left( W_u \left( \tan \left( \frac{\theta}{2} \right) - \tan \left( \frac{\theta(0)}{2} \right) \right) \right.$$

$$\left. + \tan \left( \frac{\theta}{2} \right) W_v \left( \tan \left( \frac{\theta}{2} \right) - \tan \left( \frac{\theta(0)}{2} \right) \right) \right) d\theta. \quad (4.19)$$

To simplify subsequent calculation, we will assume that  $\theta(0) \sim 0$ . so that:

$$u(t) = \frac{L}{\sqrt{2}} + \epsilon \frac{\sqrt{L}}{2} W_u \left( \tan \left( \frac{\theta(t)}{2} \right) \right),$$

and:

$$v(t) = \frac{L}{\sqrt{2}} \tan\left(\frac{\theta(t)}{2}\right) + \epsilon \frac{\sqrt{L}}{2} W_v\left(\tan\left(\frac{\theta(t)}{2}\right)\right) - \epsilon \frac{\sqrt{L}}{2} \int_0^{\theta(t)} \left| \frac{\cos\left(\frac{\theta(t)}{2}\right)}{\cos\left(\frac{\theta}{2}\right)} \right|^2 \left( W_u\left(\tan\left(\frac{\theta}{2}\right)\right) + \tan\left(\frac{\theta}{2}\right) W_v\left(\tan\left(\frac{\theta}{2}\right)\right) \right) d\theta.$$

Setting a new variable  $T = \tan\left(\frac{\theta}{2}\right)$ , the classical value  $v_0$  becomes  $\frac{LT}{\sqrt{2}}$  and we have a simple expression for the quantum correction to  $u$ :

$$\delta u(T) = \epsilon \frac{\sqrt{L}}{2} W_u(T). \quad (4.20)$$

Observing that:

$$\frac{dT}{d\theta} = \frac{1}{2} \sec^2\left(\frac{\theta}{2}\right),$$

a remarkable simplification occurs since for the quantum correction to  $v$ :

$$\begin{aligned} \delta v(T) &= \epsilon \frac{\sqrt{L}}{2} W_v(T) - \epsilon \sqrt{L} \cos^2\left(\frac{\theta(t)}{2}\right) \int_0^T (W_u(s) + sW_v(s)) ds \\ &= \epsilon \frac{\sqrt{L}}{2} W_v(T) - \epsilon \frac{\sqrt{L}}{1+T^2} \int_0^T (W_u(s) + sW_v(s)) ds. \end{aligned} \quad (4.21)$$

Since  $W_u$  and  $W_v$  are two independent one-dimensional Brownian motions,  $\delta u$  and  $\delta v$  are obviously Gaussian by linearity, and have zero mean. We are now able to calculate the joint distribution of  $\delta u$  and  $\delta v$ , which in turn implies to joint distribution of  $\delta x$  and  $\delta y$ , the quantum corrections to the original Cartesian coordinates of the diffusion process.

Note that for  $\delta u$  and  $\delta v$  as given above by (4.20) and (4.21):

$$\mathbb{E}(\delta u) = \mathbb{E}(\delta v) = 0.$$

Thus the covariance matrix of the random variables  $\delta u$  and  $\delta v$  is given by:

$$\text{Cov}(\delta u, \delta v) = \begin{pmatrix} \mathbb{E}(\delta u^2) & \mathbb{E}(\delta u \delta v) \\ \mathbb{E}(\delta u \delta v) & \mathbb{E}(\delta v^2) \end{pmatrix}, \quad (4.22)$$

where:

$$\mathbb{E}(\delta u^2) = \frac{\epsilon^2 L}{4} \mathbb{E}(W_u^2(T)) = \frac{\epsilon^2 L}{4} T.$$

Similarly, due to independence of the Brownian motions  $W_u$  and  $W_v$ :

$$\mathbb{E}(\delta u \delta v) = -\frac{\epsilon^2 L}{2} C^2 \int_0^T \mathbb{E}(W_u(T)W_u(s)) ds = -\frac{\epsilon^2 L}{2} C^2 \int_0^T s ds,$$

using the notation  $C = \cos \frac{\theta(t)}{2}$ . We will similarly denote  $\sin \frac{\theta(t)}{2}$  by  $S$  in the following:

$$\mathbb{E}(\delta u \delta v) = -\frac{\epsilon^2 L}{4} C^2 T^2 = -\frac{\epsilon^2 L}{4} S^2. \quad (4.23)$$

Finally:

$$\begin{aligned} \mathbb{E}(\delta v^2) &= \frac{\epsilon^2 L}{4} \mathbb{E}(W_v^2(T)) - \epsilon^2 L C^2 \int_0^T s \mathbb{E}(W_v(T)W_v(s)) ds \\ &\quad + \epsilon^2 L C^4 \int_0^T \int_0^s \mathbb{E}(W_u(s)W_u(\sigma)) + s\sigma \mathbb{E}(W_v(s)W_v(\sigma)) d\sigma ds. \end{aligned}$$

Rewriting the third term as follows:

$$\begin{aligned} \mathbb{E}(\delta v^2) &= \frac{\epsilon^2 L}{4} \mathbb{E}(W_v^2(T)) - \epsilon^2 L C^2 \int_0^T s \mathbb{E}(W_v(T)W_v(s)) ds \\ &\quad + 2\epsilon^2 L C^4 \int_0^T \int_0^s \mathbb{E}(W_u(s)W_u(\sigma)) + s\sigma \mathbb{E}(W_v(s)W_v(\sigma)) d\sigma ds, \end{aligned}$$

allows us to evaluate the expectation terms since  $s \geq \sigma$ . After integration, we have:

$$\begin{aligned} \mathbb{E}(\delta v^2) &= \epsilon^2 L \left( \frac{T}{4} - \frac{C^2 T^3}{3} + \frac{C^4 T^3}{3} + \frac{2C^4 T^5}{15} \right) \\ &= \epsilon^2 L \left( \frac{T}{4} - \frac{TS^2}{3} + \frac{CS^3}{3} + \frac{2TS^4}{15} \right). \end{aligned}$$

## 4.2 The Quantum Correction to the Cartesian Coordinates

As with the  $(u, v)$  coordinates, we define for the Cartesian coordinates,  $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots$  and  $y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \dots$ . Recalling that under the Levi-Civita transformation,  $w = u + iv$ , where  $w^2 = z = x + iy$ :

$$z = (u + iv)^2 = (u^2 - v^2) + 2uvi,$$

so that:

$$x = u^2 - v^2 \quad \text{and} \quad y = 2uv.$$

Then:

$$x = (u_0 + \delta u)^2 - (v_0 + \delta v)^2 = u_0^2 + v_0^2 + 2u_0\delta u - 2v_0\delta v,$$

where the latter equality is correct to first order in  $\epsilon$ , given the first order corrections  $\delta u$  and  $\delta v$ . Similarly:

$$y = 2u_0v_0 + 2u_0\delta v + 2v_0\delta u,$$

finally giving the first order quantum corrections to  $x$  and  $y$  as the following zero mean Gaussian random variables:

$$\delta x = 2u_0\delta u - 2v_0\delta v \quad \text{and} \quad \delta y = 2u_0\delta v + 2v_0\delta u. \quad (4.24)$$

Therefore, given the joint distribution of  $\delta u$  and  $\delta v$ , we may compute the joint distribution of  $\delta x$  and  $\delta y$  as follows:

$$\mathbb{E}(\delta x^2) = 4(u_0^2 \mathbb{E}(\delta u^2) + v_0^2 \mathbb{E}(\delta v^2) - 2u_0v_0 \mathbb{E}(\delta u \delta v)), \quad (4.25)$$

$$\mathbb{E}(\delta y^2) = 4(u_0^2 \mathbb{E}(\delta v^2) + v_0^2 \mathbb{E}(\delta u^2) + 2u_0v_0 \mathbb{E}(\delta u \delta v)), \quad (4.26)$$

$$\mathbb{E}(\delta x \delta y) = 4(u_0v_0 \mathbb{E}(\delta u^2) - u_0v_0 \mathbb{E}(\delta v^2) + (u_0^2 - v_0^2) \mathbb{E}(\delta u \delta v)), \quad (4.27)$$

since  $u_0$  and  $v_0$  are deterministic, and:

$$u_0 = \frac{L}{\sqrt{2}} \quad \text{and} \quad v_0 = \frac{LT}{\sqrt{2}}.$$

After substituting the quantum corrections to the  $(u, v)$  coordinates given by equations (4.20) and (4.21) into the above equations (4.24), we may state the following theorem:

**Theorem 4.2.1.** *The first order quantum corrections to the Cartesian coordinates of the semiclassical parabolic orbit defined by equations (2.22) and (2.24) are zero mean Gaussian random variables given by:*

$$\delta x(t) = \epsilon \frac{L^{3/2}}{\sqrt{2}} \left( W_u(T) - TW_v(T) + \frac{2T}{1+T^2} \int_0^T (W_u(s) + sW_v(s)) ds \right),$$

and:

$$\delta y(t) = \epsilon \frac{L^{3/2}}{\sqrt{2}} \left( W_v(T) + TW_u(T) - \frac{2}{1+T^2} \int_0^T (W_u(s) + sW_v(s)) ds \right),$$

where  $T = \tan \frac{\theta(t)}{2}$  for the polar angle  $\theta$ ,  $L$  is the angular momentum constant of the corresponding classical parabolic orbit, and  $W_u$  and  $W_v$  are two independent one-dimensional Brownian motions.

Again the fact that:

$$\mathbb{E}(\delta x) = \mathbb{E}(\delta y) = 0$$

is guaranteed by the linear dependence of  $\delta x$  and  $\delta y$  on the Wiener processes  $W_u$  and  $W_v$ . We may then find the joint distribution of  $\delta x$  and  $\delta y$  in terms of its covariance matrix.

**Theorem 4.2.2.** *The covariance matrix of the quantum corrections  $\delta x$  and  $\delta y$ , as stated in Theorem 4.2.1 is given by:*

$$\text{Cov}(\delta x, \delta y) = \begin{pmatrix} \mathbb{E}(\delta x^2) & \mathbb{E}(\delta x \delta y) \\ \mathbb{E}(\delta x \delta y) & \mathbb{E}(\delta y^2) \end{pmatrix},$$

with:

$$\mathbb{E}(\delta x^2) = \epsilon^2 L^3 \left( TS^2 + \frac{T}{2C^2} + \frac{2TS^2}{3}(S^2 - T^2) + \frac{4T^3S^4}{15} \right),$$

$$\mathbb{E}(\delta y^2) = \epsilon^2 L^3 \left( -TS^2 + \frac{T}{2C^2} + \frac{2S^2}{3}(CS - T) + \frac{4TS^4}{15} \right),$$

and:

$$\mathbb{E}(\delta x \delta y) = \epsilon^2 L^3 \left( \frac{S^2}{2}(T^2 - 1) + \frac{2TS^2}{3}(T - S^2) + \frac{4TS^4}{15} \right),$$

where  $S$ ,  $C$ , and  $T$  denote the respective sine, cosine and tangent functions of  $\frac{\theta}{2}$ .

The above explicit forms of  $\mathbb{E}(\delta x^2)$ ,  $\mathbb{E}(\delta y^2)$  and  $\mathbb{E}(\delta x \delta y)$  have been found by substituting the entries of the covariance matrix of  $(\delta u, \delta v)$  into the above equations (4.25) to (4.27).

### 4.3 Quantum Areal Velocity

Given the above equations for  $du(t)$  and  $dv(t)$ , we may now begin to calculate the quantum correction to the areal velocity, which is equal to a constant,  $A_0$  in the classical setting, discussed in Chapter 1. For Cartesian coordinates  $(x, y)$ :

$$dA_0 = \frac{|x dy - y dx|}{2} = \frac{1}{2} \Im(\bar{z} dz)$$

for  $z = x + iy$ . Again, setting  $w^2 = z$ , implying  $\bar{z} = \bar{w}^2$  and  $dz = 2w dw$ . Therefore:

$$dA_0 = \frac{1}{2} \Im(\bar{w}^2 2w dw) = |w|^2 \Im(\bar{w} dw),$$

for a classical system.

We now turn our attention to the areal velocity for the stochastic mechanical system defined under the above Levi-Civita transformation. For  $w = u + iv$ :

$$w^2 = (u^2 - v^2) + 2uvi = z,$$

so that:

$$\frac{\partial z}{\partial u} = 2u + 2vi \quad \text{and} \quad \frac{\partial z}{\partial v} = -2v + 2ui.$$

From Itô's formula, we obtain:

$$dz = \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv + \frac{1}{2} \left( \frac{\partial^2 z}{\partial u^2} du du + 2 \frac{\partial^2 z}{\partial u \partial v} du dv + \frac{\partial^2 z}{\partial v^2} dv dv \right).$$

From the above expressions for  $u(t)$  and  $v(t)$ , equations (4.18) and (4.19), observe that  $du dv = 0$  and  $du du$  and  $dv dv$  are both of order  $\epsilon^2$ , and so:

$$dz = 2u du - 2v dv + (2u dv + 2v du)i = 2(u + iv)(du + i dv) = 2w dw.$$

Hence for quantum areal velocity  $A_Q$ :

$$dA_Q = |w|^2 \Im(\bar{w} dw), \tag{4.28}$$

as for the classical case, with no additional terms.

We may now find a stochastic differential equation for  $A_Q$  to first order in  $\epsilon$ . Again we set:



$$u(t) = u_0(t) + \delta u(t) \quad \text{and} \quad v(t) = v_0(t) + \delta v(t),$$

so that:

$$du(t) = (\dot{u}_0(t) + \delta \dot{u}(t)) dt \quad \text{and} \quad dv(t) = (\dot{v}_0(t) + \delta \dot{v}(t)) dt.$$

Given (4.28) from above:

$$dA_Q = (u^2 + v^2)(u dv - v du),$$

so omitting terms of order  $\epsilon^2$ ,

$$dA_Q = (u_0^2 + v_0^2 + 2u_0\delta u + 2v_0\delta v)(u_0\dot{v}_0 - v_0\dot{u}_0 + u_0\delta\dot{v} - v_0\delta\dot{u} + \delta u\dot{v}_0 - \delta v\dot{u}_0) dt,$$

where we formally write  $d(\delta u) = \delta \dot{u} dt$  etc.

Observing that  $dA_0 = (u_0^2 + v_0^2)(u_0\dot{v}_0 - v_0\dot{u}_0) dt$ , then:

$$dA_Q = dA_0 + r_0(u_0\delta\dot{v} - v_0\delta\dot{u}) dt + r_0(\delta u\dot{v}_0 - \delta v\dot{u}_0) dt + \frac{(2u_0\delta u + 2v_0\delta v)}{r_0} dA_0.$$

Finally since  $u_0$  is constant,  $\dot{u}_0 = 0$ , and  $dA_0 = \frac{L}{2} dt$  for the classical angular momentum  $L$ :

$$d(\delta A) := d(A_Q - A_0) = r_0(\delta u\dot{v}_0 + u_0\delta\dot{v} - v_0\delta\dot{u}) dt + \frac{L}{r_0}(u_0\delta u + v_0\delta v) dt,$$

or equivalently:

$$d(\delta A) = r_0 u_0 d(\delta v) - r_0 v_0 d(\delta u) + r_0 \delta u d(v_0) + \frac{L}{r_0}(u_0 \delta u + v_0 \delta v) dt. \quad (4.29)$$

Recalling from the above Section (4.1):

$$u_0 = \frac{L}{\sqrt{2}} \quad \text{and} \quad v_0 = \frac{L}{\sqrt{2}} \tan\left(\frac{\theta(t)}{2}\right) = \frac{LT}{\sqrt{2}},$$

since the most convenient time variable is  $T = \tan\left(\frac{\theta(t)}{2}\right)$ . Hence:

$$dv_0 = \frac{L}{\sqrt{2}} dT.$$

Given equation (4.20) from above:

$$\delta u = \epsilon \frac{\sqrt{L}}{2} W_u(T),$$

and from equation (4.21):

$$\delta v = \epsilon \frac{\sqrt{L}}{2} W_v(T) - \epsilon \frac{\sqrt{L}}{1+T^2} \int_0^T (W_u(s) + sW_v(s)) ds,$$

where we have replaced  $\cos^2\left(\frac{\theta(t)}{2}\right)$  by  $\frac{1}{1+T^2}$  given that  $1 + \tan^2 \psi = \sec^2 \psi$ . We may now calculate  $d(\delta u)$  and  $d(\delta v)$  as follows:

$$d(\delta u) = \epsilon \frac{\sqrt{L}}{2} W_u(dT)$$

and:

$$\begin{aligned} d(\delta v) = & \epsilon \frac{\sqrt{L}}{2} W_v(dT) - \epsilon \frac{\sqrt{L}}{1+T^2} (W_u(T) + TW_v(T)) dT \\ & + \epsilon \sqrt{L} \frac{2T}{(1+T^2)^2} \left( \int_0^T W_u(s) + sW_v(s) ds \right) dT. \end{aligned}$$

As discussed in Chapter 1, for the classical polar radius  $r_0$ :

$$r_0 = \frac{l}{1 + \cos \theta},$$

where the semilatus rectum may be written  $l = L^2$  since we have allowed the force constant  $\mu = 1$  throughout. Hence using the identity  $\cos \theta = 2 \cos^2\left(\frac{\theta}{2}\right) - 1$ :

$$r_0 = \frac{L^2}{2} \sec^2\left(\frac{\theta}{2}\right) = \frac{L^2}{2}(1+T^2) \quad (4.30)$$

in terms of  $T = \tan\left(\frac{\theta}{2}\right)$ .

We may now use the above expressions to evaluate the differential equation for  $\delta A$ , (4.29). For the first term:

$$\begin{aligned} r_0 u_0 d(\delta v) = & r_0 u_0 \epsilon \frac{\sqrt{L}}{2} W_v(dT) - r_0 u_0 \epsilon \frac{\sqrt{L}}{1+T^2} (W_u(T) + TW_v(T)) dT \\ & + r_0 u_0 \epsilon \sqrt{L} \frac{2T}{(1+T^2)^2} \left( \int_0^T (W_u(s) + sW_v(s)) ds \right) dT. \end{aligned}$$

Then observing that:

$$r_0 u_0 = \frac{L^3}{2\sqrt{2}}(1 + T^2),$$

we have:

$$\begin{aligned} r_0 u_0 d(\delta v) &= \epsilon \frac{L^{7/2}}{4\sqrt{2}}(1 + T^2)W_v(dT) - \epsilon \frac{L^{7/2}}{2\sqrt{2}}(W_u(T) + TW_v(T)) dT \\ &\quad + \epsilon \frac{L^{7/2}T}{\sqrt{2}(1 + T^2)} \left( \int_0^T (W_u(s) + sW_v(s)) ds \right) dT. \end{aligned} \quad (4.31)$$

The second and third terms from (4.29) are given by:

$$-r_0 v_0 d(\delta u) = -\epsilon \frac{L^{7/2}T}{4\sqrt{2}}(1 + T^2)W_u(dT), \quad (4.32)$$

and:

$$r_0 \delta u dv_0 = \epsilon \frac{L^{7/2}}{4\sqrt{2}}(1 + T^2)W_u(T) dT. \quad (4.33)$$

To evaluate the final terms of (4.29), first note that:

$$\frac{dT}{dt} = \frac{\dot{\theta}}{2} \sec^2\left(\frac{\theta}{2}\right) = \frac{L}{2r_0^2}(1 + T^2), \quad (4.34)$$

given that due to classical angular momentum conservation we have  $\dot{\theta} = \frac{L}{r_0^2}$ . Then given the above expression for  $r_0$  in terms of  $T$ , (4.30):

$$r_0^2 = \frac{L^4}{4}(1 + T^2)^2,$$

so that (4.34) becomes:

$$dt = \frac{L^3}{2}(1 + T^2) dT.$$

Since:

$$\frac{L}{r_0} dt = L^2 dT,$$

and:

$$u_0\delta u + v_0\delta v = \epsilon \frac{L^{3/2}}{2\sqrt{2}} W_u(T) + \epsilon \frac{LT}{\sqrt{2}} \left( \frac{\sqrt{L}}{2} W_v(T) - \frac{\sqrt{L}}{1+T^2} \int_0^T (W_u(s) + sW_v(s)) ds \right),$$

the final term in (4.29) becomes:

$$\epsilon \frac{L^{7/2}}{\sqrt{2}} \left( \frac{1}{2} W_u(T) + \frac{T}{2} W_v(T) - \frac{T}{1+T^2} \int_0^T (W_u(s) + sW_v(s)) ds \right) dT. \quad (4.35)$$

Adding (4.31), (4.32), (4.33) and (4.35), due to cancellation in the the first and final terms, the differential equation (4.29) for the quantum correction to the areal velocity becomes:

$$d(\delta A) = \epsilon \frac{L^{7/2}}{4\sqrt{2}} (1+T^2) (W_u(T) dT + W_v(dT) - TW_u(dT)).$$

We are now able to state the following theorem:

**Theorem 4.3.1.** *The first order quantum correction to the areal velocity of the semiclassical parabolic orbit corresponding to the Nelson diffusion defined by equations (2.22) and (2.24), is given by the zero mean Gaussian random variable:*

$$\delta A(t) = \epsilon \frac{L^{7/2}}{4\sqrt{2}} \int_0^T (1+s^2) (W_u(s) ds + W_v(ds) - sW_u(ds)) \quad (4.36)$$

where  $T = \tan\left(\frac{\theta(t)}{2}\right)$  for the polar angle  $\theta$ ,  $L$  is the angular momentum constant of the corresponding classical parabolic orbit, and  $W_u$  and  $W_v$  are two independent one-dimensional Brownian motions.

The linear dependence of  $\delta A$  on the Brownian motions  $W_u$  and  $W_v$  ensure its Gaussianity and hence it has the zero mean property. All that remains is to calculate its variance.

We begin by writing:

$$\delta A(t) = \epsilon \frac{L^{7/2}}{4\sqrt{2}} (\Gamma_1 + \Gamma_2 - \Gamma_3),$$

with:

$$\Gamma_1 = \int_0^T (1+s^2) W_u(s) ds,$$

and stochastic integrals:

$$\Gamma_2 = \int_0^T (1 + s^2) dW_v(s) \quad \text{and} \quad \Gamma_3 = \int_0^T s(1 + s^2) dW_u(s).$$

Due to the zero mean property of  $\delta A$  and independence of the Brownian motions  $W_u$  and  $W_v$ , the variance is given by:

$$\text{var}(\delta A(t)) = \mathbb{E}(\delta A^2) = \epsilon^2 \frac{L^7}{32} (\mathbb{E}(\Gamma_1^2) + \mathbb{E}(\Gamma_2^2) + \mathbb{E}(\Gamma_3^2) - 2\mathbb{E}(\Gamma_1\Gamma_3)). \quad (4.37)$$

Firstly:

$$\begin{aligned} \mathbb{E}(\Gamma_1^2) &= \int_0^T \int_0^T (1 + s^2)(1 + \sigma^2) \mathbb{E}(W_u(s)W_u(\sigma)) d\sigma ds \\ &= 2 \int_0^T \int_0^s (1 + s^2)(\sigma + \sigma^3) d\sigma ds \\ &= \frac{T^3}{3} + \frac{3T^5}{10} + \frac{T^7}{14}. \end{aligned} \quad (4.38)$$

Both  $\mathbb{E}(\Gamma_2^2)$  and  $\mathbb{E}(\Gamma_3^2)$  may be evaluated using the fundamental Itô isometry (Chapter 1) as follows:

$$\mathbb{E}(\Gamma_2^2) = \mathbb{E} \left( \left( \int_0^T (1 + s^2) dW_v(s) \right)^2 \right) = \int_0^T (1 + s^2)^2 ds = T + \frac{2T^3}{3} + \frac{T^5}{5}, \quad (4.39)$$

and:

$$\mathbb{E}(\Gamma_3^2) = \mathbb{E} \left( \left( \int_0^T s(1 + s^2) dW_u(s) \right)^2 \right) = \int_0^T s^2(1 + s^2)^2 ds = \frac{T^3}{3} + \frac{2T^5}{5} + \frac{T^7}{7}. \quad (4.40)$$

Evaluating  $\mathbb{E}(\Gamma_1\Gamma_3)$  requires us to use Theorem 14.4 from Simon [30], which implies that for suitably integrable deterministic functions  $f$  and  $g$ :

$$\mathbb{E} \left( \int_0^T f(\sigma) dB(\sigma) \int_0^T g(s) B(s) ds \right) = \int_0^T \int_0^s f(\sigma)g(s) d\sigma ds.$$

Therefore:



$$\begin{aligned}
\mathbb{E}(\Gamma_1\Gamma_3) &= \int_0^T \int_0^s \sigma(1+\sigma^2)(1+s^2) d\sigma ds, \\
&= \frac{T^3}{6} + \frac{3T^5}{20} + \frac{T^7}{28}.
\end{aligned} \tag{4.41}$$

Combining equations (4.37), (4.38), (4.39), (4.40), and (4.41), we may state the following theorem:

**Theorem 4.3.2.** *The variance of the first order quantum correction to the areal velocity of the semiclassical parabolic orbit given by Theorem 4.3.1, is given by:*

$$\text{var}(\delta A(t)) = \epsilon \frac{L^7}{32} \left( T + T^3 + \frac{3T^5}{5} + \frac{T^7}{7} \right).$$

where  $T = \tan\left(\frac{\theta(t)}{2}\right)$  for the polar angle  $\theta$ , and  $L$  is the angular momentum constant of the corresponding classical parabolic orbit.

Note that all of the above Theorems 4.2.1, 4.3.1 and 4.3.2 depend on the finiteness of the tangent function. The variable  $T = \tan\left(\frac{\theta}{2}\right)$  becomes infinite for  $\theta = \pi$ . However our earlier restriction to the lower half plane ensures  $\theta \neq \pi$  for  $t > 0$ , and therefore  $T$  remains finite.

# Appendix A

## Simulating Diffusion Processes

We have simulated the diffusion processes from Chapter 2 using *Mathematica* by first simulating their driving Brownian motions, and then using an Euler scheme [22]. For example, we simulate the parabolic diffusion by first defining the drift field as follows, with  $b_x \sim bx$  and  $b_y \sim by$ .

```
bx[x_, y_] := Sign[y] Sqrt[Sqrt[x^2 + y^2] - x]/Sqrt[x^2 + y^2]
by[x_, y_] := -y Sign[y]/(Sqrt[x^2 + y^2] Sqrt[Sqrt[x^2 + y^2] - x])
```

We then use the following Euler scheme with step length 0.05 to generate a table of the coordinates  $X$  and  $Y$  of a sample path of the diffusion at each discrete time between 0 and  $T$ .

```
SeedRandom[123456]
X[0] = -1; Y[0] = 1; h = 0.005; T = 30; k = T/h; s = 0.1;
Do[
  Z1[i] = RandomReal[NormalDistribution[0, 1]];
  Z2[i] = RandomReal[NormalDistribution[0, 1]];
  X[i + 1] = X[i] + bx[X[i], Y[i]] h + sSqrt[h] Z1[i];
  Y[i + 1] = Y[i] + by[X[i], Y[i]] h + sSqrt[h] Z2[i];
  {i,0,k}]
PD = Table[{X[i], Y[i]}, {i, 0, k}]
```

The command `ListLinePlot` then plots the coordinates, interpolating intermediate times linearly.

Sample paths of the driving Brownian motions of the parabolic diffusion may also be generated by the following, where  $B_x \sim W1$  and  $B_y \sim W2$ .

```
W1[0]=0; W2[0]=0
```

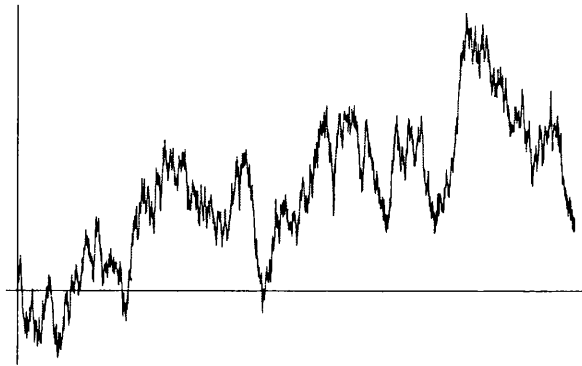


Figure A.1: A simulated Brownian motion.

```
Do[
W1[i+1]=W1[i] + Sqrt[h] Z1[i];
W2[i+1]=W2[i] + Sqrt[h] Z2[i];
{i,0,k}
BM=Table[{i, W1[i]},{i,0,k}]
```

Again the command `ListLinePlot` will generate a simulation of  $W1$  (shown by Figure A.1) by linearly interpolating between the numerical values tabulated by the above.



# List of Figures

1.1	The foci and directrices of an ellipse. . . . .	18
1.2	The focus and directrix of a parabola. . . . .	20
2.1	The probability density $ \Psi_{n,n-1,n-1} ^2$ of the atomic circular state in the plane $z = 0$ . . . . .	28
2.2	The probability density $ \Psi_{\epsilon,n} ^2$ for the atomic elliptic state in the plane $z = 0$ . . . . .	29
2.3	Simulation of the Nelson diffusion $X_{\epsilon,n}$ for the atomic elliptic state in three dimensions. . . . .	31
2.4	The probability density $ \Psi_{\epsilon} ^2$ of the limiting elliptic diffusion in the plane $z = 0$ . . . . .	33
2.5	Simulation of the limiting elliptic diffusion process $X_{\epsilon}$ in three-dimensions. . . . .	35
2.6	Simulation of the limiting elliptic diffusion process in two dimensions. . . . .	35
2.7	Simulations of: the semiclassical parabolic diffusion; the semiclassical diffusion compared with the classical trajectory; multiple sample paths of the diffusion process. . . . .	40
2.8	Streamlines of the parabolic drift field. . . . .	49
A.1	A simulated Brownian motion. . . . .	94

# Bibliography

- [1] V. I. Arnold, V. V. Kozlov & A. I. Neishtadt (2006), *Mathematical Aspects of Classical and Celestial Mechanics*, Springer-Verlag
- [2] A. Bommier, D. Delande & J.-C. Gay (1989), Atomic Quantum States with Maximum Localization on Classical Elliptic Orbits, *Physical Review Brief Reports*, Volume 39, 6587-6590
- [3] H. Buchholz (1969), *The Confluent Hypergeometric Function with Special Emphasis on its Applications*, Springer-Verlag
- [4] J. M. Burgers (1974), *The Nonlinear Diffusion Equation*, D. Reidel
- [5] C. Lena, D. Delande & J.-C. Gay (1991), Wave Functions of Atomic Elliptic States, *EPL*, Volume 15, 697-702
- [6] J. D. Dollard & C. N. Friedman (1979), *Product Integration with Applications to Differential Equations*, Addison-Wesley
- [7] R. M. Durran (1990), *Newtonian Diffusions*, PhD Thesis, University College of Swansea
- [8] R. Durran, A. Neate, A. Truman & F.-Y. Wang (2008), The Correspondence Limit of the Atomic Elliptic State, *EPL*, Volume 84(3):30003
- [9] R. Durran, A. Neate & A. Truman  
The Divine Clockwork: Bohr's Correspondence Principle and Nelson's Stochastic Mechanics for the Atomic Elliptic State  
*Journal of Mathematical Physics*, Volume 49(3):032102, 30
- [10] R. Durran, A. Neate, A. Truman & F.-Y. Wang, On the Divine Clockwork: The Spectral Gap for the Correspondence Limit of the Nelson Diffusion Generator for the Atomic Elliptic State, *Journal of Mathematical Physics*, Volume 49(10):102103, 22
- [11] A. Erdélyi (1956), *Asymptotic Expansions*, Dover

- [12] W. L. Ferrar (1943), *Higher Algebra*, Oxford University Press, 1943
- [13] M.I. Friedlin & A.D. Wentzell (1984), *Random Perturbations of Dynamical Systems*, Springer-Verlag
- [14] I. I. Gihman & A. V. Skorohod (1972), *Stochastic Differential Equations*, Springer-Verlag
- [15] B. A. Griffith & J. L. Synge (1959), *Principles of Mechanics*, McGraw-Hill, 1959
- [16] F. Guerra (1981), *Structural Aspects of Stochastic Mechanics and Stochastic Field Theory*, *Physics Reports*, Volume 77, 263-312
- [17] I. Gyöngy & T. Martínez (2001), *On Stochastic Differential Equations with Locally Unbounded Drift*, *Czechoslovak Mathematical Journal*, Volume 51, 763-783
- [18] K. Hannabuss (1997), *An Introduction to Quantum Theory*, Oxford University Press
- [19] W. Hausern (1965), *Introduction to the Principles of Mechanics*, Addison-Wesley
- [20] K. Itô, *Stochastic Integral*, *Proceedings of the Imperial Academy Tokyo*, Volume 20, 1944
- [21] I. Karatzas & S.E. Shreve (1991), *Brownian Motion and Stochastic Calculus*, Springer, second edition
- [22] P. E. Kloeden & E. Platen (1992), *Numerical Solution of Stochastic Differential Equations*, Springer-Verlag
- [23] H. P. McKean (1969), *Stochastic Integrals*, AMS Chelsea Publishing
- [24] E. Nelson (1966), *Derivation of the Schrödinger Equation from Newtonian Mechanics*, *Physical Review*, Volume 150, 1079-1085
- [25] E. Nelson (1967), *Dynamical Theories of Brownian Motion*, Princeton University Press
- [26] E. Nelson (1985), *Quantum Fluctuations*, Princeton University Press, 1985
- [27] B. Øksendal (1998), *Stochastic Differential Equations: An Introduction with Applications*, Springer-Verlag

- [28] N. I. Portenko (1990), *Generalized Diffusion Processes*, AMS
- [29] L. C. G. Rogers & D. Williams (1987), *Diffusions, Markov Processes and Martingales - Volume 2 Itô Calculus*, John Wiley & Sons
- [30] B. Simon (1979), *Functional Integration and Quantum Physics*, New York Academic Press
- [31] D. M. Y. Sommerville (1941), *Analytical Conics*, G. Bell & Sons
- [32] E. L. Stiefel & G. Scheifele (1971), *Linear and Regular Celestial Mechanics*, Springer-Verlag
- [33] D. Williams (2010), *Private Correspondence*