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Further Properties on Functional SDEs

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Submitted to the University of Wales in fulfilment of the requirements for

the Degree of Doctor of Philosophy

Department of Mathematics

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Abstract

In this work, we aim to study some fine properties for functional stochastic differential equation. The results consist of five main parts. In the second chapter, by constructing successful couplings, the derivative formula, gradient estimates and Harnack inequalities are established for the semigroup associated with a class of degenerate functional stochastic differential equations. In the third chapter, by using Malliavin calculus, explicit derivative formulae are established for a class of semi-linear functional stochastic partial differential equations with additive or multiplicative noise. As applications, gradient estimates and Harnack inequalities are derived for the semigroup of the associated segment process. In the forth chapter, we apply the weak convergence approach to establish a large deviation principle for a class of neutral functional stochastic differential equations with jumps. In particular, we discuss the large deviation principle for neutral stochastic differential delay equations which allow the coefficients to be highly nonlinear with respect to the delay argument. In the fifth chapter, we discuss the convergence of Euler-Maruyama scheme for a class of *neutral* stochastic partial differential equations driven by *α-stable processes*, where the numerical scheme is based on spatial discretization and time discretization. In the last chapter, we discuss (i) the existence and uniqueness of the stationary distribution of explicit Euler-Maruyama scheme both in time and in space for a class of stochastic partial differential equations whenever the stepsize is sufficiently small, and (ii) show that the stationary distribution of the Euler-Maruyama scheme converges weakly to the counterpart of the stochastic partial differential equation.

Keywords: functional stochastic partial differential equation, derivative formula, Harnack inequality, gradient estimate, large deviation, α-stable process, numerical analysis, stationary distribution, limit distribution.

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Chapter 1

Introduction

A stochastic differential equation (SDE), developed in the framework of Itô's [45, 46], is a differential equation in which one or more of the terms is a stochastic process, resulting in a solution which is itself a stochastic process. The theory of stochastic differential equations (SDEs), which play an important role in many branches of science and industry, is one of the most beautiful and fruitful areas in the theory of stochastic processes. There are several excellent books on SDEs, e.g., Ikeda and Watanabe [44], Mao [55], Mao and Yuan [56], Øksendal [61], Protter [70], and Yin and Zhu [98].

In many applications, one assumes that the system under consideration is governed by a principle of causality; that is, the future state of the system is independent of the past states and is determined solely by the present. However, under closer scrutiny, it becomes apparent that the principle of causality is often only a first approximation to the true situation and that a more realistic model would include some of the past states of the system. Functional differential equations give a mathematical formulation for such system, e.g., Hale and Lunel [37] and Mao [55, Chapter 5]. Examples of such application domains include biochemical reactions in gene regulation, where lengthy transcription and translation operations have been modeled with delayed dynamics, e.g., [1, 10, 58], neuronal models, where the spatial distribution of neurons can result in delayed dynamics, epidemiological models, where incubation periods result in delayed transmission of disease, e.g., [11], packet level models of Internet rate control, where the finiteness of transmission times leads to delay in receipt of congestion signals or prices, e.g., [64, 82]. The study of functional SDEs is also motivated by the fact that when one wants to model some evolution phenomena arising in physics, biology and engineering, etc., some hereditary characteristics such as after-effect, time-lag and time-delay can appear in the variables. For more details on functional SDEs, we refer to the monographs, e.g., Mao [55] and Mohammed [59]. Moreover, stochastic equations, which not only depend on the past and the present values but also involve derivatives with delays as well as the function itself, have also been applied to model some evolution phenomena arising in physics, biology and engineering. Such equations historically have been referred to as neutral functional SDEs, or neutral stochastic differential delay equations (SDDEs), e.g., [55, Chapter 6].

Moreover, from 1960s, there is also enormous research activity on Stochastic Partial Differential Equations (SPDEs) of evolutionary type, which can be applied to model a wide range of dynamics with stochastic influence in nature or man-made complex systems, e.g., Da Prato and Zabczyk [19, 20], Peszat, J. Zabczyk [66], Prévôt and Röckner [67] and Walsh [84].

For a Hilbert space U, let $W(t)$ be a U-valued noise process (e.g., Wiener process, Poisson jump process or α-stable process) defined on some probability space $(\Omega, \mathscr{F}_t, \mathscr{F}, \mathbb{P})$ satisfying the usual condition. For a fixed time delay $\tau > 0$ and a Hilbert space $(H, \langle\cdot,\cdot\rangle, \|\cdot\|_H)$, denote $\mathscr{D} := D([-\tau, 0]; H)$ by the family of cádlág functions $f : [-\tau, 0] \to H$ endowed with the uniform norm $\|f\|_\infty := \sup_{-\tau\leq\theta\leq 0}\|f(\theta)\|_H$. Let $\mathcal{L}_{HS}(U, H)$ be the set of Hilbert-Schmidt operators from U to H. For a map $h : [-\tau, \infty) \to H$ and $t \geq 0$, let $h_t \in \mathscr{D}$ be the segment of $h(t)$, i.e., $h_t(\theta) = h(t+\theta), \theta \in [-\tau, 0]$.

As described above, there is natural motivation for considering SDE on H

$$\mathrm{d}\{X(t) - G(X_t)\} = F(X_t)\mathrm{d}t + \Phi(X_t)\mathrm{d}W(t), \quad X_0 = \xi \in \mathscr{D}, \tag{1.0.1}$$

where $F, G : \mathscr{D} \to H$ and $\Phi : \mathscr{D} \to \mathcal{L}_{HS}(U, H)$.

Let $n \in \mathbb{Z}_+$, $\mathscr{C} := C([-\tau, 0]; \mathbb{R}^m)$ and, in (1.0.1), $H \equiv \mathbb{R}^n$, $G \equiv 0$, $\Phi(\varphi) \equiv \Phi(\varphi(0))$ for $\varphi \in \mathscr{C}$ and $W(t)$ be a d-dimensional Brwonian motion defined on the probability space $(\Omega, \mathscr{F}_t, \mathscr{F}, \mathbb{P})$. Then (1.0.1) reduces to

$$\mathrm{d}X(t) = F(X_t)\mathrm{d}t + \Phi(X(t))\mathrm{d}W(t), \quad X_0 = \xi \in \mathscr{C}, \tag{1.0.2}$$

where $F : \mathscr{C} \to \mathbb{R}^n$ and $\Phi : \mathbb{R}^n \to \mathbb{R}^{n\times d}$. For $m \in \mathbb{Z}_+$ and $d \in \mathbb{Z}$, in Chapter 2 we shall establish by the coupling method the derivative formulae for the diffusion semigroup of the degenerate case of (1.0.2) on $\mathbb{R}^m \times \mathbb{R}^d$ in the form

$$\begin{cases} \mathrm{d}X(t) = \{AX(t) + MY(t)\}\mathrm{d}t, \\ \mathrm{d}Y(t) = \{Z(X(t), Y(t)) + b(X_t, Y_t)\}\mathrm{d}t + \sigma \mathrm{d}B(t), \end{cases}$$

where $B(t)$ is a d-dimensional Brownian motion, σ is an invertible $d \times d$-matrix, A is an $m\times m$-matrix, M is an $m\times d$-matrix, $Z : \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}^d$ and $b : \mathscr{C} \to \mathbb{R}^d$ are locally Lipschitz continuous.

Let A be a unbounded linear operator generating a contractive C_0-semigroup $\{\mathrm{e}^{tA}\}_{t\geq 0}$ on a Hilbert space H, $\mathscr{C} := C([-\tau, 0]; H)$, $b : \mathscr{C} \to H$ and $\sigma : H \to \mathcal{L}_{HS}(H, H)$. Assume that $G \equiv 0$, $F(\varphi) = A\varphi(0) + b(\varphi)$ and $\Phi(\varphi) = \sigma(\varphi(0)), \varphi \in \mathscr{C}$ in (1.0.1). Then (1.0.1) becomes a semilinear functional SPDE

$$\begin{cases} \mathrm{d}X(t) = \{AX(t) + F(X_t)\}\mathrm{d}t + \sigma(X(t))\mathrm{d}W(t), \\ X_0 = \xi \in \mathscr{C}. \end{cases} \tag{1.0.3}$$

In Chapter 3, we shall investigate by utilizing the Malliavin calculus the Bismut-type derivative formulae and their applications for the semigroup generated by the segment process of (1.0.3) with additive noise and multiplicative noise respectively.

Let $\mathscr{C} := C([-\tau, 0]; \mathbb{R}^n)$, $H \equiv \mathbb{R}^n$, $\Phi \equiv \sigma : \mathscr{C} \to \mathbb{R}^{n\times m}$, and $W(t)$ be an m-dimensional Brwomian motion defined on the classical Wiener space $(\Omega, \mathscr{F}, \mathbb{P})$ in (1.0.1). In Chapter 4, we shall discuss by a weak convergence approach due to [16, Theorem 4.4] a Large Deviation Principle (LDP) for (1.0.1) with a small multiplicative noise in the form

$$\begin{cases} \mathrm{d}[X^\epsilon(t) - G(X_t^\epsilon)] = b(X_t^\epsilon)\mathrm{d}t + \sqrt{\epsilon}\sigma(X_t^\epsilon)\mathrm{d}W(t), \quad t \in [0, T], \ \epsilon \in (0, 1), \\ X_0^\epsilon = \xi \in \mathscr{C}. \end{cases}$$

In Chapter 4, under muck weaker conditions for the delay arguments we shall also study the LDP for neutral Stochastic Differential Delay Equation (SDDE) on $\mathbb{R}^n$

$$\begin{cases} \mathrm{d}[Y^\epsilon(t) - G(Y^\epsilon(t-\tau))] = b(Y^\epsilon(t), Y^\epsilon(t-\tau))\mathrm{d}t \\ \qquad\qquad\qquad\qquad\qquad\qquad + \sqrt{\epsilon}\sigma(Y^\epsilon(t), Y^\epsilon(t-\tau))\mathrm{d}W(t), \\ Y^\epsilon(\theta) = \xi(\theta), \quad \theta \in [-\tau, 0]. \end{cases}$$

Moreover, in Chapter 4, we shall also discuss by the variational representation of functionals of Poisson random measure [15] the LDP for a class of neutral functional SDEs driven by jump processes

$$\begin{cases} \mathrm{d}[Z^\epsilon(t) - G(Z^\epsilon_t)] = b(Z^\epsilon_t)\mathrm{d}t + \sqrt{\epsilon}\sigma(Z^\epsilon_t)\mathrm{d}W(t) \\ \qquad\qquad + \int_{\mathbb{Z}} \Phi(Z^\epsilon_t, x)(\epsilon N^{\epsilon^{-1}}(\mathrm{d}t\mathrm{d}x) - \nu_T(\mathrm{d}t\mathrm{d}x)), \quad t \in [0, T], \\ Z^\epsilon_0 = \xi \in \mathscr{C}. \end{cases}$$

Let A be a unbounded linear operator generating a contractive C_0-semigroup $\{\mathrm{e}^{tA}\}_{t\geq 0}$ on a Hilbert space H, $b : H \times H \to H$, $\Phi(\varphi) = \mathbf{1}$, the identity operator on H and $Z(t) := W(t)$ a cylindrical α-stable process with $\alpha \in (1, 2)$. Assume further that $G(\varphi) \equiv G(\varphi(-\tau))$, $F(\varphi) \equiv A\varphi(0) + b(\varphi(0), \varphi(-\tau))$ in (1.0.1). Then we can rewrite (1.0.1) as

$$\begin{cases} \mathrm{d}\{X(t) - G(X(t-\tau))\} = \{AX(t) + b(X(t), X(t-\tau))\}\mathrm{d}t + \mathrm{d}Z(t), \\ X(\theta) = \xi(\theta) \in H, \quad \theta \in [-\tau, 0]. \end{cases} \tag{1.0.4}$$

In Chapter 1.0.4, we shall discuss by the semigroup approach the strong convergence of an explicit Euler-Maruyama (EM) of (1.0.4) based on time-discretization and spatial discretization.

The last Chapter is devoted to investigating the long-term behavior of an explicit EM, which is also based on time-discretization and spatial discretization, associated with the following SPDE

$$\mathrm{d}X(t) = \{AX(t) + b(X(t))\}\mathrm{d}t + \sigma(X(t))\mathrm{d}W(t), \qquad X(0) = x \in H,$$

where $\sigma(x) := \sigma^0 + \sigma^1(x), x \in H$ with $\sigma^0 \in \mathscr{L}(H)$ and $\sigma^1 : H \to \mathscr{L}_{HS}(H)$. That is, in (1.0.3), $\tau \equiv 0$ and $F \equiv b$.

Chapter 2

Derivative Formula and Harnack Inequality for Degenerate Functional SDEs

In this chapter, by constructing successful couplings, the derivative formula, gradient estimates and Harnack inequalities are established for the semigroup associated with a class of degenerate functional SDEs.

2.1 Introduction

In recent years, the coupling argument developed in [2] for establishing dimension-free Harnack inequality in the sense of [85] has been intensively applied to the study of Markov semigroups associated with a number of stochastic (partial) differential equations, see e.g. [18, 29, 52, 54, 62, 63, 86, 88, 90, 91, 92, 101] and references within. In particular, the Harnack inequalities have been established in [29, 91] for a class of non-degenerate functional stochastic differential equations (SDEs), while the (Bismut-Elworthy-Li type) derivative formula and applications have been investigated in [36] for a class of degenerate SDEs (see also [93, 102] for the study by using Malliavin calculus). The aim of this chapter is to establish the derivative formula and (log-)Harnack inequalities for degenerate functional SDEs. The derivative formula implies explicit gradient estimates of the associated semigroup, while a number of applications of the (log-)Harnack inequalities have been summarized in [89, §4.2] on heat kernel estimates, entropy-cost inequalities, characterizations of invariant measures and contractivity properties of the semigroup.

Let $m \in \mathbb{Z}_+$ and $d \in \mathbb{N}$. For $r_0 > 0$, let $\mathscr{C} := C([-r_0, 0]; \mathbb{R}^m \times \mathbb{R}^d)$ be the space of continuous functions from $[-r_0, 0]$ into $\mathbb{R}^m \times \mathbb{R}^d$, which is a Banach space with the uniform norm $\|\cdot\|_\infty$. Consider the following functional SDE on $\mathbb{R}^m \times \mathbb{R}^d$:

$$\begin{cases} \mathrm{d}X(t) = \{AX(t) + MY(t)\}\mathrm{d}t, \\ \mathrm{d}Y(t) = \{Z(X(t), Y(t)) + b(X_t, Y_t)\}\mathrm{d}t + \sigma \mathrm{d}B(t), \end{cases} \tag{2.1.1}$$

where $B(t)$ is a d-dimensional Brownian motion, σ is an invertible $d \times d$-matrix, A is an $m \times m$-matrix, M is an $m \times d$-matrix, $Z : \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}^d$ and $b : \mathscr{C} \to \mathbb{R}^d$ are locally Lipschitz continuous (i.e. Lipschitzian on compact sets), $(X_t, Y_t)_{t \geq 0}$ is a process on $\mathscr{C}$ with $(X_t, Y_t)(\theta) := (X(t+\theta), Y(t+\theta)), \theta \in [-r_0, 0]$. To ensure that $P_t f$ is differentiable for any bounded measurable function f and any $t > 0$, we will need a rank assumption on A and M such that the noise part of Y_t can also smooth the distribution of X_t via the linear drift terms. More precisely, we will make use of the following Hörmander type rank condition: there exists an integer number $0 \leq k \leq m-1$ such that

$$\mathrm{Rank}[M, AM, \cdots, A^k M] = m. \tag{2.1.2}$$

When $m = 0$ this condition automatically holds by convention. Note that when $m \geq 1$, this rank condition holds for some $k > m-1$ if and only if it holds for $k = m-1$.

Let $\nabla, \nabla^{(1)}$ and $\nabla^{(2)}$ denote the gradient operators on $\mathbb{R}^m \times \mathbb{R}^d, \mathbb{R}^m$ and $\mathbb{R}^d$ respectively, and let

$$\begin{aligned} Lf(x,y) :=& \langle Ax + My, \nabla^{(1)} f(x,y)\rangle + \langle Z(x,y), \nabla^{(2)} f(x,y)\rangle \\ &+ \frac{1}{2}\sum_{i,j=1}^d (\sigma\sigma^*)_{ij} \frac{\partial^2}{\partial y_i \partial y_j} f(x,y), \quad (x,y) \in \mathbb{R}^m \times \mathbb{R}^d, f \in C^2(\mathbb{R}^m \times \mathbb{R}^d). \end{aligned}$$

Since both Z and b are locally Lipschitz continuous, due to [79] the equation (2.1.1) has a unique local solution for any initial data $(X_0, Y_0) \in \mathscr{C}$. To ensure the non-explosion and further regular properties of the solution, we make use of the following assumptions:

(A) *There exist constants* $\lambda, l > 0$ *and* $W \in C^2(\mathbb{R}^m \times \mathbb{R}^d)$ *of compact level sets with* $W \geq 1$ *such that*

$(A1)$ $LW \leq \lambda W, \quad |\nabla^{(2)} W| \leq \lambda W;$

$(A2)$ $\langle b(\xi), \nabla^{(2)} W(\xi(0)) \rangle \leq \lambda \|W(\xi)\|_\infty, \quad \xi \in \mathscr{C};$

$(A3)$ $|Z(z) - Z(z')| \leq \lambda |z - z'| W(z')^l, \quad z, z' \in \mathbb{R}^m \times \mathbb{R}^d, |z - z'| \leq 1;$

$(A4)$ $|b(\xi) - b(\xi')| \leq \lambda \|\xi - \xi'\|_\infty \|W(\xi')\|_\infty^l, \xi, \xi' \in \mathscr{C}, \|\xi - \xi'\|_\infty \leq 1.$

Comparing with the framework investigated in [36, 102], where $b = 0, A = 0$ and $\text{Rank}[M] = m, d \geq m$, are assumed, the present model is more general and the segment process we are going to investigate is an infinite-dimensional Markov process. On the other hand, unlike in [36] in which the condition $|\nabla^{(2)} W| \leq \lambda W$ is not used, in the present setting this condition seems essential in order to derive moment estimates of the segment process (see the proof of Lemma 2.2.1 below). Moreover, if $|\nabla W| \leq cW$ holds for some constant $c > 0$, then $(A3)$ and $(A4)$ hold for some $\lambda > 0$ if and only if there exists a constant $\lambda' > 0$ such that $|\nabla Z| \leq \lambda' W^l$ and $|\nabla b| \leq \lambda' \|W\|_\infty^l$ hold on $\mathbb{R}^m \times \mathbb{R}^d$ and $\mathscr{C}$ respectively.

It is easy to see that **(A)** holds for $W(z) = 1 + |z|^2$, $l = 1$ and some constant $\lambda > 0$ provided that Z and b are globally Lipschitz continuous on $\mathbb{R}^m \times \mathbb{R}^d$ and $\mathscr{C}$ respectively. It is clear that $(A1)$ and $(A2)$ imply the non-explosion of the solution (see Lemma 2.2.1 below). In this chapter we shall investigate regularity properties of the Markov semigroup associated with the segment process:

$$P_t f(\xi) = \mathbb{E}^\xi f(X_t, Y_t), \quad f \in \mathscr{B}_b(\mathscr{C}), \xi \in \mathscr{C},$$

where $\mathscr{B}_b(\mathscr{C})$ is the class of all bounded measurable functions on $\mathscr{C}$ and $\mathbb{E}^\xi$ stands for the expectation for the solution starting at the point $\xi \in \mathscr{C}$. When $m = 0$ we have $X_t \equiv 0$ and $\mathscr{C} = \{0\} \times \mathscr{C}_2 \equiv \mathscr{C}_2 := C([-r_0, 0]; \mathbb{R}^d)$, so that $P_t f$

can be simply formulated as $P_t f(\xi) = \mathbb{E}^\xi f(Y_t)$ for $f \in \mathscr{B}_b(\mathscr{C}_2), \xi \in \mathscr{C}_2$. Thus, (2.1.1) also includes non-degenerate functional SDEs. For any $h = (h_1, h_2) \in \mathscr{C}$ and $z \in \mathbb{R}^m \times \mathbb{R}^d$, let ∇_h and ∇_z be the directional derivatives along h and z respectively. The following result provides an explicit derivative formula for $P_T, T > r_0$.

Theorem 2.1.1. *Assume* **(A)** *and let $T > r_0$ and $h = (h_1, h_2) \in \mathscr{C}$ be fixed. Let $v : [0,T] \to \mathbb{R}$ and $\alpha : [0,T] \to \mathbb{R}^m$ be differentiable functions such that $v(0) = 1, \alpha(0) = 0, v(s) = 0, \alpha(s) = 0$ for $s \geq T - r_0$, and*

$$h_1(0) + \int_0^t \mathrm{e}^{-sA} M\phi(s)\mathrm{d}s = 0, \quad t \geq T - r_0, \tag{2.1.3}$$

where $\phi(s) := v(s)h_2(0) + \alpha(s)$. Then for $f \in \mathscr{B}_b(\mathscr{C})$,

$$\nabla_h P_T f(\xi) = \mathbb{E}^\xi \Big\{ f(X_T, Y_T) \int_0^T \langle N(s), (\sigma^*)^{-1} \mathrm{d}B(s) \rangle \Big\}, \quad \xi \in \mathscr{C} \tag{2.1.4}$$

holds for

$$N(s) := (\nabla_{\Theta(s)} Z)(X(s), Y(s)) + (\nabla_{\Theta_s} b)(X_s, Y_s) - v'(s)h_2(0) - \alpha'(s), \quad s \in [0,T],$$

where

$$\Theta(s) = (\Theta^{(1)}(s), \Theta^{(2)}(s)) := \begin{cases} h(s), & \text{if } s \leq 0, \\ \left(\mathrm{e}^{As} h_1(0) + \int_0^s \mathrm{e}^{(s-r)A} M\phi(r)\mathrm{d}r,\ \phi(s)\right), & \text{if } s > 0. \end{cases}$$

A simple choice of v is

$$v(s) = \frac{(T - r_0 - s)^+}{T - r_0}, \quad s \geq 0.$$

To present a specific choice of α, let

$$Q_t := \int_0^t \frac{s(T - r_0 - s)^+}{(T - r_0)^2} \mathrm{e}^{-sA} M M^* \mathrm{e}^{-sA^*} \mathrm{d}s, \quad t > 0.$$

According to [78] (see also [93, Proof of Theorem 4.2(1)]), for $m \geq 1$ the condition (2.1.2) implies that Q_t is invertible with

$$\|Q_t^{-1}\| \leq c(T - r_0)(t \wedge 1)^{-2(k+1)}, \quad t > 0 \tag{2.1.5}$$

for some constant $c > 0$.

Corollary 2.1.2. *Assume* **(A)** *and let* $T > r_0$. *If* (2.1.2) *holds for some* $0 \le k \le m-1$, *then* (2.1.4) *holds for* $v(s) = \frac{(T-r_0-s)^+}{T-r_0}$ *and*

$$\alpha(s) = -\frac{s(T-r_0-s)^+}{(T-r_0)^2} M^* \mathrm{e}^{-sA^*} Q_{T-r_0}^{-1} \bigg(h_1(0) + \int_0^{T-r_0} \frac{(T-r_0-r)^+}{T-r_0} \mathrm{e}^{-rA} M h_2(0) \mathrm{d}r \bigg),$$

where by convention $M = 0$ *(hence,* $\alpha = 0$*) if* $m = 0$.

The following gradient estimates are direct consequences of Theorem 2.1.1.

Corollary 2.1.3. *Assume* **(A)**. *If* (2.1.2) *holds for some* $0 \le k \le m-1$, *then:*

(1) *There exists a constant* $C \in (0, \infty)$ *such that*

$$\begin{aligned} |\nabla_h P_T f(\xi)| \le C\sqrt{P_T f^2(\xi)} \bigg\{ & |h(0)| \Big(1 + \frac{\|M\|}{(T-r_0)^{2k+1} \wedge 1}\Big) \\ & + \|W(\xi)\|_\infty^l \sqrt{T \wedge (1+r_0)} \Big(\|h\|_\infty + \frac{\|M\| \cdot |h(0)|}{(T-r_0)^{2k+1} \wedge 1} \Big) \bigg\} \end{aligned}$$

holds for all $T > r_0, \xi, h \in \mathscr{C}$ *and* $f \in \mathscr{B}_b(\mathscr{C})$;

(2) *Let* $|\nabla^{(2)} W|^2 \le \delta W$ *hold for some constant* $\delta > 0$. *If* $l \in [0, 1/2)$ *then there exists a constant* $C \in (0, \infty)$ *such that*

$$\begin{aligned} |\nabla_h P_T f(\xi)| \le\, & r\{P_T f \log f - (P_T f) \log P_T f\}(\xi) \\ & + \frac{C P_T f(\xi)}{r} \bigg\{ |h(0)|^2 \Big(\frac{1}{(T-r_0) \wedge 1} + \frac{\|M\|^2}{\{(T-r_0) \wedge 1\}^{4k+3}} \Big) \\ & + \|h\|_\infty^2 \|W(\xi)\|_\infty + \Big(\|h\|_\infty^2 + \frac{|h(0)|^2 \|M\|^2}{\{(T-r_0) \wedge 1\}^{4k+2}} \Big)^{\frac{1}{1-2l}} \Big(\frac{r^2}{\|h\|_\infty^2} \vee 1 \Big)^{\frac{2l}{1-2l}} \bigg\} \end{aligned}$$

holds for all $r > 0, T > r_0, \xi, h \in \mathscr{C}$ *and positive* $f \in \mathscr{B}_b(\mathscr{C})$;

(3) *Let* $|\nabla^{(2)} W|^2 \le \delta W$ *hold for some constant* $\delta > 0$. *If* $l = \frac{1}{2}$ *then there exist constants* $C, C' \in (0, \infty)$ *such that*

$$\begin{aligned} |\nabla_h P_T f(\xi)| \le\, & r\{P_T f \log f - (P_T f) \log P_T f\}(\xi) \\ & + \frac{C P_T f(\xi)}{r} \bigg\{ |h(0)|^2 \Big(\frac{1}{(T-r_0) \wedge 1} + \frac{\|M\|^2}{\{(T-r_0) \wedge 1\}^{4k+3}} \Big) \\ & + \|W(\xi)\|_\infty \Big(\|h\|_\infty^2 + \frac{\|M\|^2 |h(0)|^2}{\{(T-r_0) \wedge 1\}^{4k+2}} \Big) \bigg\} \end{aligned}$$

holds for

$$r \ge C' \bigg(\|h\|_\infty + \frac{\|M\| \cdot |h(0)|}{\{(T-r_0) \wedge 1\}^{2k+1}} \bigg),$$

all $T > r_0, \xi, h \in \mathscr{C}$ *and positive* $f \in \mathscr{B}_b(\mathscr{C})$.

When $m = 0$ the above assertions hold with $\|M\| = 0$.

According to [3], the entropy gradient estimate implies the Harnack inequality with power, we have the following result which follows immediately from Corollary 2.1.3 (2) and [36, Proposition 4.1]. Similarly, Corollary 2.1.3 (3) implies the same type Harnack inequality for smaller $\|h\|_\infty$ comparing to $T - r_0$.

Corollary 2.1.4. *Assume* **(A)** *and that* (2.1.2) *holds for some* $0 \le k \le m-1$. *Let* $|\nabla^{(2)}W|^2 \le \delta W$ *hold for some constant* $\delta > 0$. *If* $l \in [0, \frac{1}{2})$ *then there exists a constant* $C \in (0, \infty)$ *such that*

$$\begin{aligned}(P_T f)^p(\xi + h) \le & P_T f^p(\xi) \exp\left[\frac{Cp}{p-1}\left\{\|h\|_\infty^2 \int_0^1 \|W(\xi + sh)\|_\infty \mathrm{d}s\right.\right. \\ &\left.\left. + \left(\|h\|_\infty^2 + \frac{\|M\|^2|h(0)|^2}{\{(T-r_0)\wedge 1\}^{4k+2}}\right)^{\frac{1}{1-2l}}\left(\frac{(p-1)^2}{\|h\|_\infty^2} \vee 1\right)^{\frac{2l}{1-2l}}\right\}\right]\end{aligned}$$

holds for all $T > r_0, p > 1, \xi, h \in \mathscr{C}$ *and positive* $f \in \mathscr{B}_b(\mathscr{C})$. *If* $m = 0$ *then the assertion holds for* $\|M\| = 0$.

Finally, we consider the log-Harnack inequality introduced in [74, 87]. To this end, as in [36], we slightly strengthen $(A3)$ and $(A4)$ as follows: there exists an increasing function U on $[0, \infty)$ such that

$(A3')$ $|Z(z) - Z(z')| \le \lambda|z - z'|\{W(z')^l + U(|z - z'|)\}, \quad z, z' \in \mathbb{R}^m \times \mathbb{R}^d;$

$(A4')$ $|b(\xi) - b(\xi')| \le \lambda\|\xi - \xi'\|_\infty\{\|W(\xi')\|_\infty^l + U(\|\xi - \xi'\|_\infty)\}, \ \xi, \xi' \in \mathscr{C}.$

Obviously, if

$$W(z)^l \le c\{W(z')^l + U(|z - z'|)\}, \quad z, z' \in \mathbb{R}^m \times \mathbb{R}^d$$

holds for some constant $c > 0$, then $(A3)$ and $(A4)$ imply $(A3')$ and $(A4')$ respectively with possibly different λ.

Theorem 2.1.5. *Assume* $(A1), (A2), (A3')$ *and* $(A4')$. *If* (2.1.2) *holds for some* $0 \le k \le m-1$, *then there exists a constant* $C \in (0, \infty)$ *such that for any positive* $f \in \mathscr{B}_b(\mathscr{C}), T > r_0$ *and* $\xi, h \in \mathscr{C}$,

$$\begin{aligned}P_T \log f(\xi + h) - \log P_T f(\xi) \le C\Big\{&\Big[\|W(\xi + h)\|_\infty^{2l} + U^2\Big(C\|h\|_\infty + \frac{C\|M\| \cdot |h(0)|}{(T-r_0)\wedge 1}\Big)\Big]\|h\|_\infty^2 \\ &+ \frac{|h(0)|^2}{(T-r_0)\wedge 1} + \frac{\|M\|^2|h(0)|^2}{\{(T-r_0)\wedge 1\}^{4k+3}}\Big\}.\end{aligned}$$

If $m = 0$ *then the assertion holds for* $\|M\| = 0$.

For applications of the Harnack and log-Harnack inequalities we are referred to [89, §4.2]. The remainder of this chapter is organized as follows: Theorem 2.1.1 and Corollary 2.1.2 are proved Section 2, while Corollary 2.1.3 and Theorem 2.1.5 are proved in Section 3; in Section 4 the assumption **(A)** is weakened for the discrete time delay case, and two examples are presented to illustrate our results.

2.2 Proofs of Theorem 2.1.1 and Corollary 2.1.2

Lemma 2.2.1. *Assume (A1) and (A2). Then for any $k > 0$ there exists a constant $C > 0$ such that*

$$\mathbb{E}^{\xi} \sup_{-r_0 \le s \le t} W(X(s), Y(s))^k \le 3\|W(\xi)\|_{\infty}^k e^{Ct}, \quad t \ge 0, \ \xi \in \mathscr{C}$$

holds. Consequently, the solution is non-explosive.

Proof. For any $n \ge 1$, let

$$\tau_n := \inf\{t \in [0, T] : |X(t)| + |Y(t)| \ge n\}.$$

Moreover, let

$$\ell(s) := W(X, Y)(s), \quad s \ge -r_0.$$

By the Itô formula and using the first inequality in $(A1)$ and $(A2)$, we may find a constant $C_1 > 0$ such that

$$\begin{aligned}
\ell(t \wedge \tau_n)^k &= \ell(0)^k + k \int_0^{t \wedge \tau_n} \ell(s)^{k-1} \langle \nabla^{(2)} W(X, Y)(s), \sigma \mathrm{d}B(s) \rangle \\
&\quad + k \int_0^{t \wedge \tau_n} \ell(s)^{k-1} \Big\{ LW(X, Y)(s) + \langle b(X_s, Y_s), \nabla^{(2)} W(X, Y)(s) \rangle \\
&\quad + \frac{1}{2}(k-1)\ell(s)^{-1} |\sigma^* \nabla^{(2)} W(X, Y)(s)|^2 \Big\} \mathrm{d}s \\
&\le l(0)^k + k \int_0^{t \wedge \tau_n} \ell(s)^{k-1} \langle \nabla^{(2)} W(X, Y)(s), \sigma \mathrm{d}B(s) \rangle + C_1 \int_0^{t \wedge \tau_n} \sup_{r \in [-r_0, s]} \ell(r)^k \mathrm{d}s.
\end{aligned} \tag{2.2.1}$$

Noting from the second inequality in $(A1)$ and the Burkholder-Davis-Gundy inequality, we obtain that

$$k\mathbb{E}^\xi \sup_{s\in[0,t]} \Big| \int_0^{s\wedge\tau_n} \ell(r)^{k-1} \langle \nabla^{(2)} W(X,Y)(s), \sigma \mathrm{d}B(r)\rangle \Big| \le C_2 \mathbb{E}^\xi \Big(\int_0^t \ell(s\wedge\tau_n)^{2k} \mathrm{d}s\Big)^{1/2}$$
$$\le C_2 \mathbb{E}^\xi \Big\{ \Big(\sup_{s\in[0,t]} \ell(s\wedge\tau_n)^k\Big)^{1/2} \Big(\int_0^t \ell(s\wedge\tau_n)^k \mathrm{d}s\Big)^{1/2}\Big\}$$
$$\le \frac{1}{2}\mathbb{E}^\xi \sup_{s\in[0,t]} \ell(s\wedge\tau_n)^k + \frac{C_2^2}{2}\mathbb{E}^\xi \int_0^t \sup_{r\in[0,s]} \ell(r\wedge\tau_n)^k \mathrm{d}s$$

for some constant $C_2 > 0$. Combining this with (2.2.1) and noting that $(X_0, Y_0) = \xi$, we conclude that there exists a constant $C > 0$ such that

$$\mathbb{E}^\xi \sup_{-r_0\le s\le t} \ell(s\wedge\tau_n)^k \le 3\|W(\xi)\|_\infty^k + C\mathbb{E}^\xi \int_0^t \sup_{s\in[-r_0,t]} \ell(s)^k \mathrm{d}s, \quad t\ge 0.$$

Due to the Gronwall lemma, this implies that

$$\mathbb{E}^\xi \sup_{-r_0\le s\le t} \ell(s\wedge\tau_n)^k \le 3\|W(\xi)\|_\infty^k \mathrm{e}^{Ct}, \quad t\ge 0, n\ge 1.$$

Consequently, we have $\tau_n \uparrow \infty$ as $n\uparrow\infty$, and thus the desired inequality follows by letting $n\to\infty$. □

To establish the derivative formula, we first construct couplings for solutions starting from ξ and $\xi + \varepsilon h$ for $\varepsilon \in (0,1]$, then let $\varepsilon \to 0$. For fixed $\xi = (\xi_1, \xi_2), h = (h_1, h_2) \in \mathscr{C}$, let $(X(t), Y(t))$ solve (2.1.1) with $(X_0, Y_0) = \xi$; and for any $\varepsilon \in (0,1]$, let $(X^\varepsilon(t), Y^\varepsilon(t))$ solve the equation

$$\begin{cases} \mathrm{d}X^\varepsilon(t) = \{AX^\varepsilon(t) + MY^\varepsilon(t)\}\mathrm{d}t, \\ \mathrm{d}Y^\varepsilon(t) = \{Z(X(t),Y(t)) + b(X_t, Y_t)\}\mathrm{d}t + \sigma\mathrm{d}B(t) \\ \qquad\qquad +\varepsilon\{v'(t)h_2(0) + \alpha'(t)\}\mathrm{d}t \end{cases} \tag{2.2.2}$$

with $(X_0^\varepsilon, Y_0^\varepsilon) = \xi + \varepsilon h$. By Lemma 2.2.1 and (2.2.3) below, the solution to (2.2.2) is non-explosive as well.

Proposition 2.2.2. *Let $\phi(s) := v(s)h_2(0) + \alpha(s),\ s\in[0,T]$, and the conditions of Theorem 2.1.1 hold. Then*

$$(X^\varepsilon(t), Y^\varepsilon(t)) = (X(t), Y(t)) + \varepsilon\Theta(t), \quad \varepsilon, t \ge 0 \tag{2.2.3}$$

holds for

$$\Theta(t) := (\Theta^{(1)}(t), \Theta^{(2)}(t)) := \begin{cases} h(t), & \textit{if } t \le 0, \\ \left(\mathrm{e}^{At}h_1(0) + \int_0^t \mathrm{e}^{(t-r)A}M\phi(r)\mathrm{d}r,\ \phi(t)\right), & \textit{if } t > 0. \end{cases}$$

In particular, $(X_T^\varepsilon, Y_T^\varepsilon) = (X_T, Y_T)$.

Proof. By (2.2.2) and noting that $v(0) = 1$ and $v(s) = 0$ for $s \ge T - r_0$, we have $Y^\varepsilon(t) = Y(t) + \varepsilon\phi(t)$ and

$$X^\varepsilon(t) = X(t) + \varepsilon \mathrm{e}^{At}h_1(0) + \varepsilon \int_0^t \mathrm{e}^{(t-s)A}M\phi(s)\mathrm{d}s, \quad t \ge 0.$$

Thus, (2.2.3) holds. Moreover, since $\alpha(s) = v(s) = 0$ for $s \ge T - r_0$, we have $\Theta^{(2)}(s) = \phi(s) = 0$ for $s \ge T - r_0$. Moreover, by (2.1.3) we have $\Theta^{(1)}(s) = 0$ for $s \ge T - r_0$. Therefore, the proof is finished. $\square$

According to Proposition 2.2.2, we have $(X_T^\varepsilon, Y_T^\varepsilon) = (X_T, Y_T)$. Noting that $(X_0^\varepsilon, Y_0^\varepsilon) = \xi + \varepsilon h$, if (2.2.2) can be formulated as (2.1.1) using a different Brownian motion, then we are able to link $P_T f(\xi)$ to $P_T f(\xi + \varepsilon h)$ and furthermore derive the derivative formula by taking derivative w.r.t. ε at $\varepsilon = 0$. To this end, let

$$\begin{aligned} \Phi^\varepsilon(s) = {} & Z(X(s), Y(s)) - Z(X^\varepsilon(s), Y^\varepsilon(s)) \\ & + b(X_s, Y_s) - b(X_s^\varepsilon, Y_s^\varepsilon) + \varepsilon\{v'(s)h_2(0) + \alpha'(s)\}. \end{aligned}$$

Set

$$R^\varepsilon(s) = \exp\left[-\int_0^s \langle \sigma^{-1}\Phi^\varepsilon(r), \mathrm{d}B(r)\rangle - \frac{1}{2}\int_0^s |\sigma^{-1}\Phi^\varepsilon(r)|^2 \mathrm{d}r\right],$$

and

$$B^\varepsilon(s) = B(s) + \int_0^s \sigma^{-1}\Phi^\varepsilon(r)\mathrm{d}r.$$

Then (2.2.2) reduces to

$$\begin{cases} \mathrm{d}X^\varepsilon(t) = \{AX^\varepsilon(t) + MY^\varepsilon(t)\}\mathrm{d}t, \\ \mathrm{d}Y^\varepsilon(t) = \{Z(X^\varepsilon(t), Y^\varepsilon(t)) + b(X_t^\varepsilon, Y_t^\varepsilon)\}\mathrm{d}t + \sigma \mathrm{d}B^\varepsilon(t). \end{cases} \tag{2.2.4}$$

According to the Girsanov theorem, to ensure that $B^\varepsilon(t)$ is a Browanian motion under $\mathbb{Q}_\varepsilon := R^\varepsilon(T)\mathbb{P}$, we first prove that $R^\varepsilon(t)$ is an exponential martingale.

Moreover, to obtain the derivative formula using the dominated convergence theorem, we also need $\{\frac{R^\varepsilon(T)-1}{\varepsilon}\}_{\varepsilon\in(0,1)}$ to be uniformly integrable. Therefore, we will need the following two lemmas.

Lemma 2.2.3. *Let* **(A)** *hold. Then there exists* $\varepsilon_0 > 0$ *such that*

$$\sup_{s\in[0,T],\varepsilon\in(0,\varepsilon_0)} \mathbb{E}[R^\varepsilon(s)\log R^\varepsilon(s)] < \infty,$$

so that for each $\varepsilon \in (0,1)$, $(R^\varepsilon(s))_{s\in[0,T]}$ *is a uniformly integrable martingale.*

Proof. By (2.2.3), there exists $\varepsilon_0 > 0$ such that

$$\varepsilon_0|\Theta(t)| \le 1, \quad t \in [-r_0, T]. \tag{2.2.5}$$

For any $\varepsilon \in [0, \varepsilon_0]$, define

$$\tau_n := \inf\{t \ge 0 : |X(t)| + |Y(t)| + |X^\varepsilon(t)| + |Y^\varepsilon(t)| \ge n\},\ n \ge 1.$$

We have $\tau_n \uparrow \infty$ as $n \uparrow \infty$ due to the non-explosion. By the Girsanov theorem, the process $\{R^\varepsilon(s\wedge\tau_n)\}_{s\in[0,T]}$ is a martingale and $\{B^\varepsilon(s)\}_{s\in[0,T\wedge\tau_n]}$ is a Brownian motion under the probability measure $\mathbb{Q}_{\varepsilon,n} := R^\varepsilon(T\wedge\tau_n)\mathbb{P}$. By the definition of $R^\varepsilon(s)$, we have

$$\begin{aligned}\mathbb{E}[R^\varepsilon(s\wedge\tau_n)\log R^\varepsilon(s\wedge\tau_n)] &= \mathbb{E}_{\mathbb{Q}_{\varepsilon,n}}[\log R^\varepsilon(s\wedge\tau_n)]\\ &\le \frac{1}{2}\mathbb{E}_{\mathbb{Q}_{\varepsilon,n}}\int_0^{T\wedge\tau_n} |\sigma^{-1}\Phi^\varepsilon(r)|^2\mathrm{d}r.\end{aligned} \tag{2.2.6}$$

By (2.2.5), $(A3)$ and $(A4)$,

$$|\sigma^{-1}\Phi^\varepsilon(s)|^2 \le c\varepsilon^2\|W(X_s^\varepsilon, Y_s^\varepsilon)\|_\infty^{2l}, \tag{2.2.7}$$

holds for some constant c independent of ε. By the weak uniqueness of the solution to (2.1.1) and (2.2.4), the distribution of $(X^\varepsilon(s), Y^\varepsilon(s))_{s\in[0,T\wedge\tau_n]}$ under $\mathbb{Q}_{\varepsilon,n}$ coincides with that of the solution to (2.1.1) with $(X_0, Y_0) = \xi + \varepsilon h$ up to time $T\wedge\tau_n$, we therefore obtain from Lemma 2.2.1 that

$$\mathbb{E}[R^\varepsilon(s\wedge\tau_n)\log R^\varepsilon(s\wedge\tau_n)] \le c\|W(\xi+\varepsilon h)\|_\infty^{2l}\int_0^T \mathrm{e}^{Ct}\mathrm{d}t < \infty, \quad n\ge 1, \varepsilon\in(0,\varepsilon_0).$$

Then the required assertion follows by letting $n \to \infty$. □

Lemma 2.2.4. *If* (**A**) *holds, then there exists* $\varepsilon_0 > 0$ *such that*

$$\sup_{\varepsilon\in(0,\varepsilon_0)} \mathbb{E}\left(\frac{R^\varepsilon(T)-1}{\varepsilon}\log\frac{R^\varepsilon(T)-1}{\varepsilon}\right) < \infty.$$

Moreover,

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{R^\varepsilon(T)-1}{\varepsilon} = \int_0^T &\big\langle(\nabla_{\Theta(s)}Z)(X(s),Y(s)) \\ &+ (\nabla_{\Theta_s}b)(X_s,Y_s) - v'(s)h_2(0) - \alpha'(s), (\sigma^*)^{-1}\mathrm{d}B(s)\big\rangle.\end{aligned} \tag{2.2.8}$$

Proof. Let $\varepsilon_0 > 0$ be such that (2.2.5) holds. Since (2.2.8) is a direct consequence of (2.2.3) and the definition of $R^\varepsilon(T)$, we only prove the first assertion. By [36] we know that

$$\frac{R^\varepsilon(T)-1}{\varepsilon}\log\frac{R^\varepsilon(T)-1}{\varepsilon} \le 2R^\varepsilon(T)\left(\frac{\log R^\varepsilon(T)}{\varepsilon}\right)^2.$$

Since, due to Lemma 2.2.3, $\{B^\varepsilon(t)\}_{t\in[0,T]}$ is a Brownian motion under the probability measure $\mathbb{Q}_\varepsilon := R^\varepsilon(T)\mathbb{P}$, and

$$\begin{aligned}\log R^\varepsilon(T) &= -\int_0^T \langle\sigma^{-1}\Phi^\varepsilon(r), \mathrm{d}B(r)\rangle - \frac{1}{2}\int_0^T |\sigma^{-1}\Phi^\varepsilon(r)|^2\mathrm{d}r \\ &= -\int_0^T \langle\sigma^{-1}\Phi^\varepsilon(r), \mathrm{d}B^\varepsilon(r)\rangle + \frac{1}{2}\int_0^T |\sigma^{-1}\Phi^\varepsilon(r)|^2\mathrm{d}r,\end{aligned}$$

it follows from (2.2.7) that

$$\begin{aligned}&\mathbb{E}\left(\frac{R^\varepsilon(T)-1}{\varepsilon}\log\frac{R^\varepsilon(T)-1}{\varepsilon}\right) \\ &\le \mathbb{E}\left(2R^\varepsilon(T)\left(\frac{\log R^\varepsilon(T)}{\varepsilon}\right)^2\right) = 2\mathbb{E}_{\mathbb{Q}_\varepsilon}\left(\frac{\log R^\varepsilon(T)}{\varepsilon}\right)^2 \\ &\le \frac{4}{\varepsilon^2}\mathbb{E}_{\mathbb{Q}_\varepsilon}\left(\int_0^T \langle\sigma^{-1}\Phi^\varepsilon(r), \mathrm{d}B^\varepsilon(r)\rangle\right)^2 + \frac{1}{\varepsilon^2}\mathbb{E}_{\mathbb{Q}_\varepsilon}\left(\int_0^T |\sigma^{-1}\Phi^\varepsilon(r)|^2\mathrm{d}r\right)^2 \\ &\le \frac{4}{\varepsilon^2}\int_0^T \mathbb{E}_{\mathbb{Q}_\varepsilon}|\sigma^{-1}\Phi^\varepsilon(r)|^2\mathrm{d}r + \frac{T}{\varepsilon^2}\int_0^T \mathbb{E}_{\mathbb{Q}_\varepsilon}|\sigma^{-1}\Phi^\varepsilon(r)|^4\mathrm{d}r \\ &\le c\int_0^T \mathbb{E}_{\mathbb{Q}_\varepsilon}\|W(X_r^\varepsilon, Y_r^\varepsilon)\|_\infty^{4l}\mathrm{d}r\end{aligned}$$

holds for some constant $c > 0$. As explained in the proof of Lemma 2.2.3 the distribution of $(X_s^\varepsilon, Y_s^\varepsilon)_{s\in[0,T]}$ under $\mathbb{Q}_\varepsilon$ coincides with that of the segment process of the solution to (2.1.1) with $(X_0, Y_0) = \xi + \varepsilon h$, the first assertion follows by Lemma 2.2.1. □

Proof of Theorem 2.1.1. Since Lemma 2.2.3, together with the Girsanov theorem, implies that $\{B^\varepsilon(s)\}_{s\in[0,T]}$ is a Brownian motion with respect to $\mathbb{Q}_\varepsilon := R^\varepsilon(T)\mathbb{P}$, by (2.2.4) and $(X_T, Y_T) = (X_T^\varepsilon, Y_T^\varepsilon)$ we obtain

$$P_T f(\xi + \varepsilon h) = \mathbb{E}_{\mathbb{Q}_\varepsilon} f(X_T^\varepsilon, Y_T^\varepsilon) = \mathbb{E}\{R^\varepsilon(T) f(X_T, Y_T)\}. \tag{2.2.9}$$

Thus,

$$\begin{aligned} P_T f(\xi + \varepsilon h) - P_T f(\xi) &= \mathbb{E}R^\varepsilon(T) f(X_T, Y_T) - \mathbb{E}f(X_T, Y_T) \\ &= \mathbb{E}[(R^\varepsilon(T) - 1) f(X_T, Y_T)]. \end{aligned}$$

Combining this with Lemma 2.2.4 and using the dominated convergence theorem, we arrive at

$$\begin{aligned} \nabla_h P_T f(\xi, \eta) &= \lim_{\varepsilon\to 0} \frac{P_T f(\xi + \varepsilon h) - P_T f(\xi)}{\varepsilon} \\ &= \lim_{\varepsilon\to 0} \frac{\mathbb{E}[(R^\varepsilon(T) - 1) f(X_T, Y_T)]}{\varepsilon} \\ &= \mathbb{E}\left\{ f(X_T, Y_T) \int_0^T \langle N(s), (\sigma^*)^{-1} \mathrm{d}B(s)\rangle \right\}. \end{aligned}$$

□

Proof of Corollary 2.1.2. It suffices to verify (2.1.3) for the specific v and α. Since when $m = 0$ we have $h_1 = M = 0$ so that (2.1.3) trivially holds, we only consider $m \geq 1$. In this case, (2.1.3) is satisfied since according to the definition of $\phi(s)$ and $\alpha(s)$ we have for $t \geq T - r_0$,

$$\begin{aligned} &\int_0^t \mathrm{e}^{-sA} M\phi(s)\mathrm{d}s = \int_0^{T-r_0} \mathrm{e}^{-sA} M\phi(s)\mathrm{d}s \\ &= \int_0^{T-r_0} v(s)\mathrm{e}^{-sA} M h_2(0)\mathrm{d}s - Q_{T-r_0} Q_{T-r_0}^{-1} \left(h_1(0) + \int_0^{T-r_0} v(s)\mathrm{e}^{-sA} M h_2(0)\mathrm{d}s \right) \\ &= -h_1(0). \end{aligned}$$

□

2.3 Proofs of Corollary 2.1.3 and Theorem 2.1.5

To prove the entropy-gradient estimates in Corollary (2) and (3), we need the following simple lemma which seems new and might be interesting by itself.

Lemma 2.3.1. *Let $\ell(t)$ be a non-negative continuous semi-martingale and let $\mathscr{M}(t)$ be a continuous martingale with $\mathscr{M}(0)=0$ such that*

$$\mathrm{d}\ell(t) \le \mathrm{d}\mathscr{M}(t) + c\bar{\ell}_t \mathrm{d}t,$$

where $c \ge 0$ is a constant and $\bar{\ell}_t := \sup_{s\in[0,t]} \ell(s)$. Then

$$\mathbb{E}\exp\left[\frac{\varepsilon}{T\mathrm{e}^{1+cT}}\int_0^T \bar{\ell}_t \mathrm{d}t\right] \le \mathrm{e}^{\varepsilon\ell(0)+1}\big(\mathbb{E}\mathrm{e}^{2\varepsilon^2\langle\mathscr{M}\rangle(T)}\big)^{1/2}, \quad T,\varepsilon \ge 0.$$

Proof. Let $\bar{\mathscr{M}}_t := \sup_{s\in[0,t]}\mathscr{M}(t)$. We have

$$\bar{\mathscr{M}}_t + c\int_0^t \bar{\ell}_s \mathrm{d}s \ge \bar{\ell}_t - \ell(0).$$

Thus,

$$\begin{aligned}
&\frac{\ell_T}{\mathrm{e}^{1+cT}} - \ell(0) \le \frac{\bar{\mathscr{M}}_T + c\int_0^T \bar{\ell}_t \mathrm{d}t}{\mathrm{e}^{1+cT}} - \big(1-\mathrm{e}^{-(1+cT)}\big)\ell(0)\\
&= \int_0^T \mathrm{d}\Big\{\mathrm{e}^{-(c+T^{-1})t}\Big(\bar{\mathscr{M}}_t + c\int_0^t \bar{\ell}_s \mathrm{d}s\Big)\Big\} - \big(1-\mathrm{e}^{-(1+cT)}\big)\ell(0)\\
&= \int_0^T \mathrm{e}^{-(T^{-1}+c)t}\mathrm{d}\bar{\mathscr{M}}_t + \int_0^T \mathrm{e}^{-(c+T^{-1})t}\Big\{c\bar{\ell}_t - (T^{-1}+c)\Big(\bar{\mathscr{M}}_t + c\int_0^t \bar{\ell}_s \mathrm{d}s\Big)\Big\}\mathrm{d}t\\
&\qquad - \big(1-\mathrm{e}^{-(1+cT)}\big)\ell(0)\\
&\le \bar{\mathscr{M}}_T + \int_0^T \mathrm{e}^{-(c+T^{-1})t}\Big\{c\bar{\ell}_t - (T^{-1}+c)\big(\bar{\ell}_t - \ell(0)\big)\Big\}\mathrm{d}t - \big(1-\mathrm{e}^{-(1+cT)}\big)\ell(0)\\
&\le \bar{\mathscr{M}}_T - \frac{1}{T\mathrm{e}^{1+cT}}\int_0^T \bar{\ell}_t \mathrm{d}t.
\end{aligned}$$

Combining this with

$$\mathbb{E}\mathrm{e}^{\varepsilon\bar{\mathscr{M}}_t} \le \mathbb{E}\mathrm{e}^{1+\varepsilon\mathscr{M}(T)} \le \mathrm{e}\big(\mathbb{E}\mathrm{e}^{2\varepsilon^2\langle\mathscr{M}\rangle(T)}\big)^{1/2},$$

we complete the proof. □

Corollary 2.3.2. *Assume* **(A)** *and let $|\nabla^{(2)}W|^2 \le \delta W$ hold for some constant $\delta > 0$. Then there exists a constant $c > 0$ such that*

$$\begin{aligned}
&\mathbb{E}^{\xi}\exp\left[\frac{1}{2\|\sigma\|^2\delta T^2\mathrm{e}^{2+2cT}}\int_0^T \|W(X_t,Y_t)\|_\infty \mathrm{d}t\right]\\
&\le \exp\left[2 + \frac{W(\xi(0))}{\|\sigma\|^2\delta T\mathrm{e}^{1+cT}} + \frac{r_0\|W(\xi)\|_\infty}{2\|\sigma\|^2\delta T^2\mathrm{e}^{2+2cT}}\right], \quad T > r_0.
\end{aligned}$$

Proof. By **(A)** and the Itô formula, there exists a constant $c > 0$ such that

$$\mathrm{d}W(X,Y)(s) \le \langle \nabla^{(2)} W(X,Y)(s), \sigma \mathrm{d}B(s)\rangle + c\|W(X_s,Y_s)\|_\infty \mathrm{d}s.$$

Let

$$\mathscr{M}(t) := \int_0^t \langle \nabla^{(2)} W(X,Y)(s), \sigma \mathrm{d}B(s)\rangle, \quad l(t) := W(X,Y)(t),$$

and let $\varepsilon = (2\|\sigma\|^2 \delta T \mathrm{e}^{1+cT})^{-1}$ such that

$$\frac{\varepsilon}{T\mathrm{e}^{1+cT}} = 2\|\sigma\|^2 \varepsilon^2.$$

Then by Lemma 2.3.1 and $|\nabla^{(2)} W|^2 \le \delta W$, we have

$$\begin{aligned} &\mathbb{E}^\xi \exp\left[\frac{\varepsilon}{T\mathrm{e}^{1+cT}} \int_0^T \bar{l}_t \mathrm{d}t\right] \le \mathrm{e}^{\varepsilon l(0)+1} \left(\mathbb{E}^\xi \mathrm{e}^{2\varepsilon^2 \langle \mathscr{M}\rangle(T)}\right)^{1/2} \\ &\le \mathrm{e}^{1+\varepsilon l(0)} \left(\mathbb{E}^\xi \mathrm{e}^{2\varepsilon^2 \|\sigma\|^2 \delta \int_0^T \bar{l}_t \mathrm{d}t}\right)^{1/2} = \mathrm{e}^{1+\varepsilon l(0)} \left(\mathbb{E}^\xi \mathrm{e}^{\frac{\varepsilon}{T\mathrm{e}^{1+cT}} \int_0^T \bar{l}_t \mathrm{d}t}\right)^{1/2}. \end{aligned}$$

By using stopping times as in the proof of Lemma 2.2.1 we may assume that

$$\mathbb{E}^\xi \exp\left[\frac{\varepsilon}{T\mathrm{e}^{1+cT}} \int_0^T \bar{l}_t \mathrm{d}t\right] < \infty$$

so that

$$\mathbb{E}^\xi \exp\left[\frac{\varepsilon}{T\mathrm{e}^{1+cT}} \int_0^T \bar{l}_t \mathrm{d}t\right] \le \mathrm{e}^{2+2\varepsilon l(0)}.$$

This completes the proof by noting that

$$\frac{1}{2\|\sigma\|^2 \delta T^2 \mathrm{e}^{2+2cT}} \int_0^T \|W(X_t,Y_t)\|_\infty \mathrm{d}t \le \frac{r_0 \|W(\xi)\|_\infty}{2\|\sigma\|^2 \delta T^2 \mathrm{e}^{2+2cT}} + \frac{\varepsilon}{T\mathrm{e}^{1+cT}} \int_0^T \bar{l}_t \mathrm{d}t.$$

□

Proof of Corollary 2.1.3. Let v and α be given in Corollary 2.1.2. By the semigroup property and the Jensen inequality, we will only consider $T - r_0 \in (0,1]$.

(1) By (2.1.5) and the definitions of α and v, there exists a constant $C > 0$ such that

$$\begin{aligned} &|v'(s)h_2(0) + \alpha'(s)| \\ &\le C 1_{[0,T-r_0]}(s) |h(0)| \Big(\frac{1}{T-r_0} + \frac{\|M\|}{(T-r_0)^{2(k+1)}}\Big), \quad s \in [0,T], \\ &|\Theta(s)| \le C|h(0)| \Big(1 + \frac{\|M\|}{(T-r_0)^{2k+1}}\Big), \quad s \in [0,T], \\ &\|\Theta_s\|_\infty \le C\Big(\|h\|_\infty + \frac{\|M\| \cdot |h(0)|}{(T-r_0)^{2k+1}}\Big), \quad s \in [0,T]. \end{aligned} \tag{2.3.1}$$

Therefore, it follows from $(A3)$ and $(A4)$ that

$$\begin{aligned} |N(s)| \leq & C1_{[0,T-r_0]}(s)|h(0)|\Big(\frac{1}{T-r_0}+\frac{\|M\|}{(T-r_0)^{2(k+1)}}\Big) \\ & + C\Big(\|h\|_\infty + \frac{\|M\|\cdot|h(0)|}{(T-r_0)^{2k+1}}\Big)\|W(X_s,Y_s)\|_\infty^l \end{aligned} \tag{2.3.2}$$

holds for some constant $C > 0$. Combining this with Theorem 2.1.1 we obtain

$$\begin{aligned} |\nabla_h P_T f(\xi)| &\leq C\sqrt{P_T f^2(\xi)}\Big(\mathbb{E}^\xi \int_0^T |N(s)|^2 \mathrm{d}s\Big)^{1/2} \\ &\leq C\sqrt{P_T f^2(\xi)}\Big\{|h(0)|\Big(1+\frac{\|M\|}{(T-r_0)^{2k+1}}\Big) \\ &\quad + \Big(\|h\|_\infty + \frac{\|M\|\cdot|h(0)|}{(T-r_0)^{2k+1}}\Big)\Big(\int_0^T \mathbb{E}^\xi\|W(X_s,Y_s)\|_\infty^{2l}\mathrm{d}s\Big)^{1/2}\Big\}, \end{aligned}$$

This completes the proof of (1) since due to Lemma 2.2.1 one has

$$\mathbb{E}^\xi\|W(X_s,Y_s)\|_\infty^{2l} \leq 3\|W(\xi)\|_\infty^{2l}\mathrm{e}^{Cs}, \quad s\in[0,T]$$

for some constant $C > 0$.

(2) By Theorem 2.1.1 and the Young inequality (cf. [3, Lemma 2.4]), we have

$$\begin{aligned} |\nabla_h P_T f|(\xi) \leq & r\big\{P_T f\log f - (P_T f)\log P_T f\big\}(\xi) \\ & + rP_T f(\xi)\log \mathbb{E}^\xi \mathrm{e}^{\frac{1}{r}\int_0^T \langle N(s),(\sigma^*)^{-1}\mathrm{d}B(s)\rangle}, \quad r>0. \end{aligned} \tag{2.3.3}$$

Next, it follows from (2.3.2) that

$$\begin{aligned} & \Big(\mathbb{E}^\xi \exp\Big[\frac{1}{r}\int_0^T \langle N(s),(\sigma^*)^{-1}\mathrm{d}B(s)\rangle\Big]\Big)^2 \\ & \leq \mathbb{E}^\xi \exp\Big[\frac{2\|\sigma^{-1}\|^2}{r^2}\int_0^T |N(s)|^2\mathrm{d}s\Big] \\ & \leq \exp\Big[\frac{C_1|h(0)|^2}{r^2}\Big(\frac{1}{T-r_0}+\frac{\|M\|^2}{(T-r_0)^{4k+3}}\Big)\Big] \\ & \quad \times \mathbb{E}^\xi \exp\Big[\frac{C_1}{r^2}\Big(\|h\|_\infty^2 + \frac{\|M\|^2|h(0)|^2}{(T-r_0)^{4k+2}}\Big)\int_0^T \|W(X_s,Y_s)\|_\infty^{2l}\mathrm{d}s\Big] \end{aligned} \tag{2.3.4}$$

for $T\in(r_0, 1+r_0]$ holds for some constant $C_1\in(0,\infty)$. Since $2l\in[0,1)$ and $T\leq 1+r_0$, there exists a constant $C_2\in(0,\infty)$ such that

$$\beta\|W(X_s,Y_s)\|_\infty^{2l} \le \frac{(\frac{\|h\|_\infty^2}{r^2}\wedge 1)\|W(X_s,Y_s)\|_\infty}{2\|\sigma\|^2\delta T^2 e^{2+2cT}} + C_2\beta^{\frac{1}{1-2l}}\Big(\frac{\|h\|_\infty^2}{r^2}\wedge 1\Big)^{-\frac{2l}{1-2l}},\ \beta>0.$$

Taking

$$\beta = \frac{C_1}{r^2}\Big(\|h\|_\infty^2 + \frac{\|M\|^2|h(0)|^2}{(T-r_0)^{4k+2}}\Big),$$

and applying Corollary 2.3.2, we arrive at

$$\begin{aligned}
\mathbb{E}^\xi \exp\Big[\beta\int_0^T \|W(X_s,Y_s)\|_\infty^{2l}\mathrm{d}s\Big] &\le \exp\Big[C_2\beta^{\frac{1}{1-2l}}\Big(\frac{\|h\|_\infty^2}{r^2}\wedge 1\Big)^{-\frac{2l}{1-2l}}\Big]\\
&\quad\times\Big(\mathbb{E}^\xi \exp\Big[\frac{1}{2\|\sigma\|^2\delta T^2 e^{2+2cT}}\int_0^T \|W(X_s,Y_s)\|_\infty \mathrm{d}s\Big]\Big)^{\frac{\|h\|_\infty^2}{r^2}\wedge 1}\\
&\le \exp\Big[\frac{C_3}{r^2}\Big\{\|h\|_\infty^2\|W(\xi)\|_\infty + \Big(\|h\|_\infty^2 + \frac{\|M\|^2|h(0)|^2}{(T-r_0)^{4k+2}}\Big)^{\frac{1}{1-2l}}\Big(\frac{r^2}{\|h\|_\infty^2}\vee 1\Big)^{\frac{2l}{1-2l}}\Big\}\Big]
\end{aligned}$$

for some constant $C_3\in(0,\infty)$ and all $T\in(r_0,1+r_0]$. Therefore, the desired entropy-gradient estimate follows by combining this with (2.3.3) and (2.3.4).

(3) Let $C'>0$ be such that $r \ge C'\Big(\|h\|_\infty + \frac{\|M\|\cdot|h(0)|}{(T-r_0)^{2k+1}}\Big)$ implies

$$\frac{C_1}{r^2}\Big(\|h\|_\infty^2 + \frac{\|M\|^2|h(0)|^2}{(T-r_0)^{4k+2}}\Big) \le \frac{1}{2\|\sigma\|^2\delta T^2 e^{2+2cT}},$$

so that by Corollary 2.3.2

$$\begin{aligned}
&\mathbb{E}^\xi \exp\Big[\frac{C_1}{r^2}\Big(\|h\|_\infty^2 + \frac{\|M\|^2|h(0)|^2}{(T-r_0)^{4k+2}}\Big)\int_0^T \|W(X_s,Y_s)\|_\infty^{2l}\mathrm{d}s\Big]\\
&\le \Big(\mathbb{E}^\xi \exp\Big[\frac{1}{2\|\sigma\|^2\delta T^2 e^{2+2cT}}\int_0^T \|W(X_s,Y_s)\|_\infty \mathrm{d}s\Big]\Big)^{\frac{2C_1\|\sigma\|^2\delta T^2 e^{2+2cT}}{r^2}\Big(\|h\|_\infty^2+\frac{\|M\|^2|h(0)|^2}{(T-r_0)^{4k+2}}\Big)}\\
&\le \exp\Big[\frac{C\|W(\xi)\|_\infty}{r^2}\Big(\|h\|_\infty^2 + \frac{\|M\|^2|h(0)|^2}{(T-r_0)^{4k+2}}\Big)\Big]
\end{aligned}$$

holds for some constant $C>0$. Then proof is finished by combining this with (2.3.3) and (2.3.4). □

Proof of Theorem 2.1.5. Again, we only prove for $T\in(r_0,1+r_0]$. Applying (2.2.9) to $\varepsilon=1$ and using $\log f$ to replace f, we obtain

$$\begin{aligned}
P_T\log f(\xi+h) &= \mathbb{E}\{R^1(T)\log f(X_T,Y_T)\}\\
&\le \log P_T f(\xi) + \mathbb{E}(R^1\log R^1)(T).
\end{aligned}\tag{2.3.5}$$

Next, taking $\varepsilon = 1$ in (2.2.6) and letting $n \uparrow \infty$, we arrive at

$$\mathbb{E}(R^1 \log R^1)(T) \le \frac{1}{2}\mathbb{E}_{\mathbb{Q}_1} \int_0^T |\sigma^{-1}\Phi^1(r)|^2 \mathrm{d}r. \tag{2.3.6}$$

By $(A3'), (A4')$, (2.3.1) and the definition of Φ^1, we have

$$\begin{aligned}|\sigma^{-1}\Phi^1(s)|^2 \le & C_1\Big\{ \|W(X_s^1, Y_s^1)\|_\infty^{2l} + U^2\Big(C_1\|h\|_\infty + \frac{C_1\|M\|\cdot|h(0)|}{(T-r_0)^{2k+1}}\Big)\Big\}\|h\|_\infty^2 \\ & + C_1|h(0)|^2\Big(\frac{1}{(T-r_0)^2} + \frac{\|M\|^2}{(T-r_0)^{4(k+1)}}\Big)1_{[0,T-r_0]}(s)\end{aligned}$$

for some constant $C_1 > 0$. Then the proof is completed by combining this with (2.3.5), (2.3.6) and Lemma 2.2.1 (note that $(X^1(s), Y^1(s))$ under $\mathbb{Q}_1$ solves the same equation as (X_s, Y_s) under $\mathbb{P}$). □

2.4 Discrete Time Delay Case and Examples

In this section we first present a simple example to illustrate our main results presented in Section 1, then relax assumption **(A)** for the discrete time delay case in order to cover some highly non-linear examples.

Example 2.4.1. For $\alpha \in C([-r_0, 0]; \mathbb{R})$, consider functional SDE on $\mathbb{R}^2$

$$\begin{cases} \mathrm{d}X(t) = -\{X(t) + Y(t)\}\mathrm{d}t \\ \mathrm{d}Y(t) = \mathrm{d}B(t) + \Big\{ -\varepsilon Y^3(t) + Y(t-r_0) + \int_{-r_0}^0 \alpha(\theta)X(t+\theta)\mathrm{d}\theta \Big\}\mathrm{d}t \end{cases}$$

with initial data $\xi = (\xi_1, \xi_2) \in C([-r_0, 0]; \mathbb{R}^2)$, where $\varepsilon \ge 0$ and $n \in \mathbb{N}$ are constants. For $z = (x, y) \in \mathbb{R}^2$, let $W(x, y) = 1 + |x|^2 + |y|^2$ and set $Z(z) = -y^3$ and $b(\xi) = \int_{-r_0}^0 \alpha(\theta)\xi_1(\theta)\mathrm{d}\theta + \xi_2(-r_0)$. By a straightforward computation one has for $x, y \in \mathbb{R}$

$$LW(x, y) = 1 - 2x(x+y) - 2\varepsilon y^{2n} \le 3W(x, y)$$

and for $\xi \in C([-r_0, 0]; \mathbb{R}^2)$

$$\begin{aligned}\langle b(\xi), \nabla^{(2)}W(\xi(0))\rangle &\le 2\Big|\int_{-r_0}^0 \alpha(\theta)\xi_1(\theta)\mathrm{d}\theta + \xi_2(-r_0)\Big||\xi_2(0)| \\ &\le 2\Big(1 + \int_{-r_0}^0 \alpha(\theta)\mathrm{d}\theta\Big)\|\xi\|_\infty^2.\end{aligned}$$

Then conditions (A1) and (A2) hold. Next, there exists a constant $c > 0$ such that for any $z = (x, y)$ and $z' = (x', y') \in \mathbb{R}^2$,

$$|Z(z) - Z(z')| = \varepsilon|y^3 - y'^3| \leq c|y - y'|(|y'|^2 + |y - y'|^2).$$

Finally, for $\xi = (\xi_1, \xi_2), \xi' = (\xi_1', \xi_2') \in C([-r_0, 0]; \mathbb{R}^2)$,

$$|b(\xi) - b(\xi')| \leq \sqrt{2}\Big(\int_{-r_0}^{0} |\alpha(\theta)|\mathrm{d}\theta \vee 1\Big)\|\xi - \xi'\|_\infty.$$

So, $(A3)$ holds for $l = 1$ whenever $|y - y'| \leq 1$ and $(A4)$ holds for any $l \geq 0$. Moreover, $(A3')$ and $(A4')$ hold for $U(|z|) = |z|^2, z \in \mathbb{R}^2$. Therefore, Theorem 2.1.1, Theorem 2.1.5 and Corollary 2.1.3 hold.

To derive the entropy-gradient estimate and the Harnack inequality as in Corollary 2.1.4, we need to weaken the assumption **(A)**. To this end, we consider a simpler setting where the delay is time discrete. Consider

$$\begin{cases} \mathrm{d}X(t) = \{AX(t) + MY(t)\}\mathrm{d}t, \\ \mathrm{d}Y(t) = Z(X(t), Y(t)) + \tilde{b}(X(t - r_0), Y(t - r_0))\mathrm{d}t + \sigma \mathrm{d}B(t), \end{cases} \tag{2.4.1}$$

with initial data $\xi \in \mathscr{C}$, where $Z, \tilde{b} : \mathbb{R}^{m+d} \to \mathbb{R}^d$. If we define $b(\xi) = \tilde{b}(\xi(-r_0))$ for $\xi = (\xi_1, \xi_2) \in \mathscr{C}$, then equation (2.4.1) can be written as equation (2.1.1). For $(x, y), (x', y') \in \mathbb{R}^m \times \mathbb{R}^d$, define the diffusion operator associated with (2.4.1) by

$$\mathscr{L}W(x, y; x', y') = LW(x, y) + \langle \tilde{b}(x', y'), \nabla^{(2)}W(x, y)\rangle.$$

Theorem 2.4.2. *Assume that there exist constants $\alpha, \beta, \gamma > 0$ with $\beta \geq \gamma$, functions $W \in C^2(\mathbb{R}^{m+d})$ with $W \geq 1$ and $U \in C(\mathbb{R}^{m+d}; \mathbb{R}_+)$ such that for $(x, y), (x', y') \in \mathbb{R}^m \times \mathbb{R}^d$*

$$\mathscr{L}W(x, y; x', y') \leq \alpha\{W(x, y) + W(x', y')\} - \beta U(x, y) + \gamma U(x', y'). \tag{2.4.2}$$

Assume further that there exists $\nu > 0$ such that for $z = (x, y), z' = (x', y') \in \mathbb{R}^{m+d}$ with $|z - z'| \leq 1$

$$|Z(z) - Z(z')|^2 \vee |\tilde{b}(z) - \tilde{b}(z')|^2 \leq \nu|z - z'|^2 W(z'). \tag{2.4.3}$$

Then for $\delta := (\alpha r_0 + 1)\|W(\xi)\|_\infty + \gamma r_0 \|U(\xi)\|_\infty$ *and* $t \geq 0$

$$\mathbb{E}^\xi W(X(t), Y(t)) \leq \delta e^{2\alpha t}, \tag{2.4.4}$$

and

$$\begin{aligned} |\nabla_h P_T f(\xi)| \leq C\sqrt{P_T f^2(\xi)}\Big\{ & |h(0)|\Big(1 + \frac{\|M\|}{(T-r_0)^{2k+1} \wedge 1}\Big) + r_0^{\frac{1}{2}} \|W(\xi)\|_\infty^{\frac{1}{2}} \|h\|_\infty \\ & + |h(0)|\sqrt{\delta(T \wedge (1+r_0))}\Big(1 + \frac{\|M\|}{(T-r_0)^{2k+1}}\Big)\Big\} \end{aligned} \tag{2.4.5}$$

for all $T > r_0, \xi, h \in \mathscr{C}$ *and* $f \in \mathscr{B}_b(\mathscr{C})$, *where* $C > 0$ *is some constant. If moreover there exist constants* $K, \lambda_i \geq 0, i = 1, 2, 3, 4$, *with* $\lambda_1 \geq \lambda_2$ *and* $\lambda_3 \geq \lambda_4$, *functions* $\tilde{W} \in C^2(\mathbb{R}^{m+d})$ *with* $\tilde{W} \geq 1$ *and* $\tilde{U} \in C(\mathbb{R}^{m+d}; \mathbb{R}_+)$ *such that for* $(x, y), (x', y') \in \mathbb{R}^m \times \mathbb{R}^d$

$$\begin{aligned} \frac{\mathscr{L}\tilde{W}(x, y; x', y')}{\tilde{W}(x, y)} \leq & K - \lambda_1 W(x, y) + \lambda_2 W(x', y') \\ & - \lambda_3 \tilde{U}(x, y) + \lambda_4 \tilde{U}(x', y'), \end{aligned} \tag{2.4.6}$$

then there exist constants $\delta_0, C > 0$ *such that for* $r \geq \delta_0/(T - r_0)^{2k+1}, \xi, h \in \mathscr{C}$ *and positive* $f \in \mathscr{B}_b(\mathscr{C})$

$$\begin{aligned} & |\nabla_h P_T f|(\xi) \\ & \leq r\{P_T f \log f - (P_T f)\log P_T f\}(\xi) \\ & + \frac{C P_T f}{2r}\Big\{\|h\|_\infty^2\Big(\frac{1}{(T-r_0)\wedge 1} + \frac{\|M\|^2}{\{(T-r_0)\wedge 1\}^{4k+3}} + \|W(\xi)\|_\infty r_0\Big) \\ & + \frac{(1+\|M\|^2)|h(0)|^2}{\{(T-r_0)\wedge 1\}^{4k+2}}\Big(\lambda_2 r_0 \|W(\xi)\|_\infty + \lambda_4 r_0 \|\tilde{U}(\xi)\|_\infty \\ & \quad + KT + \log \tilde{W}(\xi(0))\Big)\Big\}. \end{aligned} \tag{2.4.7}$$

Proof. By the Itô formula one has for any $t \geq 0$

$$\begin{aligned} \mathbb{E}^\xi W(X(t), Y(t)) \leq & W(\xi(0)) + \alpha \mathbb{E}^\xi \int_0^t \{W(X(s), Y(s)) + W(X(s-r_0), Y(s-r_0))\} ds \\ & - \beta \mathbb{E}^\xi \int_0^t U(X(s), Y(s)) ds + \gamma \mathbb{E}^\xi \int_0^t U(X(s-r_0), Y(s-r_0)) ds \\ \leq & W(\xi(0)) + \alpha \int_{-r_0}^0 W(X(s), Y(s)) ds + \gamma \int_{-r_0}^0 U(X(s), Y(s)) ds \\ & + 2\alpha \mathbb{E}^\xi \int_0^t W(X(s), Y(s)) ds \\ \leq & \delta + 2\alpha \mathbb{E}^\xi \int_0^t W(X(s), Y(s) ds. \end{aligned}$$

Then (2.4.4) follows from the Gronwall inequality.

By Theorem 2.1.1, for $T - r_0 \in (0, 1]$ and some $C > 0$ we can deduce that

$$|\nabla_h P_T f(\xi)| \leq C\sqrt{P_T f^2(\xi)} \Big(\mathbb{E}^\xi \int_0^T |N(s)|^2 \mathrm{d}s \Big)^{1/2},$$

where for $s \in [0, T]$

$$N(s) := (\nabla_{\Theta(s)} Z)(X(s), Y(s)) + (\nabla_{\Theta(s-r_0)} \tilde{b})(X(s-r_0), Y(s-r_0)) - v'(s)h_2(0) - \alpha'(s).$$

Recalling the first two inequalities in (2.3.1) and combining (2.4.3) yields that for some $C > 0$

$$\begin{aligned}
|\nabla_h P_T f(\xi)| &\leq C\sqrt{P_T f^2(\xi)} \bigg\{ \Big(\int_0^T |v'(s)h_2(0) + \alpha'(s)|^2 ds \Big)^{1/2} \\
&\quad + \Big(\mathbb{E}^\xi \int_0^T |\Theta(s)|^2 W(X(s), Y(s)) \mathrm{d}s \Big)^{1/2} \\
&\quad + \Big(\mathbb{E}^\xi \int_0^T |\Theta(s-r_0)|^2 W(X(s-r_0), Y(s-r_0)) \mathrm{d}s \Big)^{1/2} \bigg\} \\
&\leq C\sqrt{P_T f^2(\xi)} \bigg\{ |h(0)| \Big(1 + \frac{\|M\|}{(T-r_0)^{2k+1}} \Big) + r_0^{\frac{1}{2}} \|W(\xi)\|_\infty^{\frac{1}{2}} \|h\|_\infty \\
&\quad + |h(0)| \Big(1 + \frac{\|M\|}{(T-r_0)^{2k+1}} \Big) \Big(\int_0^T \mathbb{E}^\xi W(X(s), Y(s)) \mathrm{d}s \Big)^{1/2} \bigg\}.
\end{aligned}$$

This, together with (2.4.4), leads to (2.4.5).

Due to (2.3.3) and (2.3.4) we can deduce that there exists $C > 0$ such that for arbitrary $r > 0$ and $T - r_0 \in (0, 1]$

$$\begin{aligned}
&|\nabla_h P_T f|(\xi) \leq r\{P_T f \log f - (P_T f) \log P_T f\}(\xi) \\
&+ \frac{r P_T f(\xi)}{2} \bigg\{ \frac{C|h(0)|^2}{r^2} \Big(\frac{1}{T-r_0} + \frac{\|M\|^2}{(T-r_0)^{4k+3}} \Big) + \frac{C\|h\|_\infty^2 \|W(\xi)\|_\infty r_0}{r^2} \\
&+ \log \mathbb{E}^\xi \exp\Big[\frac{C(1+\|M\|^2)|h(0)|^2}{r^2(T-r_0)^{4k+2}} \int_0^T W(X(s), Y(s)) \mathrm{d}s \Big] \bigg\}.
\end{aligned} \tag{2.4.8}$$

Moreover, since for $s \in [0, T]$

$$\tilde{W}(X(s), Y(s)) \exp\Big(- \int_0^s \frac{\mathscr{L}\tilde{W}(X(r), Y(r), X(r-r_0), Y(r-r_0))}{\tilde{W}(X(r), Y(r))} \mathrm{d}r \Big)$$

is a local martingale by the Itô formula, in addition to $\tilde{W} \geq 1$, we obtain from

(2.4.6) that

$$
\begin{aligned}
&\mathbb{E}^\xi \exp\left[(\lambda_1-\lambda_2)\int_0^T W(X(s),Y(s))\mathrm{d}s - \lambda_2 r_0\|W(\xi)\|_\infty\right]\\
&\le \mathbb{E}^\xi \exp\left[\int_0^T \Big(\lambda_1 W(X(s),Y(s)) - \lambda_2 W(X(s-r_0),Y(s-r_0))\Big)\mathrm{d}s\right]\\
&\le \mathbb{E}^\xi \exp\bigg[KT - \int_0^T \frac{\mathscr{L}\tilde{W}(X(s),Y(s);X(s-r_0),Y(s-r_0))}{\tilde{W}(X(s),Y(s))}\mathrm{d}s\\
&\quad - \lambda_3\int_0^T \tilde{U}(X(s),Y(s))\mathrm{d}s + \lambda_4\int_0^T \tilde{U}(X(s-r_0),Y(s-r_0))ds\bigg] \qquad (2.4.9)\\
&\le \exp(\lambda_4 r_0\|\tilde{U}(\xi)\|_\infty + KT)\mathbb{E}^\xi\bigg[\tilde{W}(X(T),Y(T))\\
&\quad \times \exp\bigg(-\int_0^T \frac{\mathscr{L}\tilde{W}(X(s),Y(s);X(s-r_0),Y(s-r_0))}{\tilde{W}(X(s),Y(s))}\mathrm{d}s\bigg)\bigg]\\
&\le \exp(\lambda_4 r_0\|\tilde{U}(\xi)\|_\infty + KT)\tilde{W}(\xi(0)).
\end{aligned}
$$

Combining (2.4.8) and (2.4.9), together with the Hölder inequality, yields (2.4.7).

□

The next example shows that Theorem 2.4.2 applies to the equation (2.4.1) with a highly non-linear drift.

Example 2.4.3. Consider delay SDE on $\mathbb{R}^2$

$$
\begin{cases}
\mathrm{d}X(t) = -\{X(t)+Y(t)\}\mathrm{d}t\\
\mathrm{d}Y(t) = \mathrm{d}B(t) + \Big\{-Y^3(t) + \frac{1}{4}Y^3(t-r_0) + \frac{1}{2}X(t) - Y(t)\Big\}\mathrm{d}t
\end{cases}
$$

with initial data $\xi \in C([-r_0,0];\mathbb{R}^2)$. In this example for $z=(x,y), z'=(x',y') \in \mathbb{R}^2$ let $Z(z) = \frac{1}{2}x - y - y^3$ and $b(z') = \frac{1}{4}y'^3$. For $W(x,y) = 1+x^2+y^4$ it is easy to see that

$$
\begin{aligned}
\mathscr{L}W(x,y;x',y') &= -2x(x+y) + 4y^3\Big(\frac{1}{2}x - y - y^3 + \frac{1}{4}y'^3\Big)\\
&\le -x^2 + y^2 - 4y^4 - 4y^6 + y^3y'^3 + 2y^3x\\
&\le y^2 - 4y^4 - \frac{5}{2}y^6 + \frac{1}{2}y'^6.
\end{aligned}
$$

Then (2.4.2) holds for $\beta = \frac{5}{2}, \gamma = \frac{1}{2}$ and $U(x,y) = y^6$. Moreover for $z = (x,y), z' = (x',y') \in \mathbb{R}^2$ there exists $c>0$ such that

$$
|Z(z) - Z(z')|^2 \vee |b(z) - b(z')|^2 \le c|z-z'|^2(|y-y'|^4 + |y'|^4).
$$

Thus condition (2.4.3) holds, Therefore, by Theorem 2.4.2 we obtain (2.4.5).

To derive (2.4.7), we take $w(x,y) = \frac{1}{4}(x^2+y^4) + \frac{1}{10}xy$ and set $\tilde{W}(x,y) = \exp(w(x,y) - \inf w)$. Compute for $(x,y,x',y') \in \mathbb{R}^4$

$$\begin{aligned}\frac{\mathscr{L}\tilde{W}}{\tilde{W}}(x,y,x',y') &= \mathscr{L}\log\tilde{W}(x,y) + \frac{1}{2}|\partial_y \log\tilde{W}|^2(x,y)\\ &\le -\Big(\frac{1}{2}x + \frac{1}{10}y\Big)(x+y) + \Big(y^3 + \frac{1}{10}x\Big)\Big(\frac{1}{2}x - y - y^3 + \frac{1}{4}y'^3\Big)\\ &\quad + \frac{3}{2}y^2 + \frac{1}{2}\Big(y^3 + \frac{1}{10}x\Big)^2\\ &\le 0.5((0.35)^2/\epsilon + 1.4)^2 - (0.2325-\epsilon)x^2\\ &- 0.5y^4 - 0.175y^6 + 0.1375y'^6,\end{aligned}$$

where $\epsilon > 0$ is some constant such that $0.2325 - \epsilon > 0$. Then condition (2.4.6) holds. Therefore, by Theorem 2.4.2 we obtain (2.4.7), which implies the Harnack inequality as in Corollary 2.1.4 according to [36, Proposition 4.1].

Chapter 3

Bismut Formulae and Applications for Functional SPDEs

In this chapter, by using Malliavin calculus, explicit derivative formulae are established for a class of semi-linear functional SPDEs with additive or multiplicative noise. As applications, gradient estimates and Harnack inequalities are derived for the semigroup of the associated segment process.

3.1 Introduction

The Bismut-type formulae, initiated in [7], are powerful tools to derive regularity estimates for the underlying Markov semigroups. The formulae have been developed and applied in various settings, e.g., in [20] for SPDEs driven by cylindrical Wiener processes and [25] for semi-linear SPDEs with Lévy noise, using a simple martingale approach proposed by Elworthy-Li [28]; in [89] for linear stochastic differential equations (SDEs) driven by (purely jump) Lévy processes in terms of lower bound conditions of Lévy measures; in [4, 36] for degenerate SDEs with additive noise, using a coupling technique; in [31, 69, 93, 102] for degenerate SDEs using Malliavin calculus.

However, there are few analogues for functional SPDEs (even for finite-dimensional functional SDEs) with multiplicative noise. In this chapter we aim to establish explicit Bismut-type formulae for a class of functional SPDEs with additive or multiplicative noise. Noting that for functional SDEs the martingale method used in [28] does not work due to the lack of backward Kolmogorov equation for the segment process, and the coupling method developed in [2, 4, 36, 92] seems not easy to apply provided the noise is multiplicative, we will mainly make use of Malliavin calculus.

Let $(H, \langle\cdot,\cdot\rangle, \|\cdot\|_H)$ be a real separable Hilbert space, and $(W(t))_{t\geq 0}$ a cylindrical Wiener process on H with respect to a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with the natural filtration $\{\mathscr{F}_t\}_{t\geq 0}$. Let $\mathscr{L}(H)$ and $\mathscr{L}_{HS}(H)$ be the spaces of all linear bounded operators and Hilbert-Schmidt operators on H respectively. Denote by $\|\cdot\|$ and $\|\cdot\|_{HS}$ the operator norm and the Hilbert-Schmidt norm respectively. Let $\tau > 0$ be fixed and let $\mathscr{C} := C([-\tau, 0] \to H)$, the space of all H-valued continuous functions $f : [-\tau, 0] \to H$, equipped with the uniform norm $\|f\|_\infty := \sup_{-\tau\leq\theta\leq 0} \|f(\theta)\|_H$. For a map $h : [-\tau, \infty) \to H$ and $t \geq 0$, let $h_t \in \mathscr{C}$ be the segment of $h(t)$, i.e. $h_t(\theta) = h(t+\theta), \theta \in [-\tau, 0]$.

Consider the following semi-linear functional SPDE

$$\begin{cases} \mathrm{d}X(t) = \{AX(t) + F(X_t)\}\mathrm{d}t + \sigma(X(t))\mathrm{d}W(t), \\ X_0 = \xi \in \mathscr{C}, \end{cases} \tag{3.1.1}$$

where

(A1) A is a linear operator on H generating a contractive C_0-semigroup $(\mathrm{e}^{tA})_{t\geq 0}$.

(A2) $F : \mathscr{C} \to H$ is Gâteaux differentiable such that $\nabla_\cdot F : \mathscr{C} \times \mathscr{C} \to H$ is bounded on $\mathscr{C} \times \mathscr{C}$ and uniformly continuous on bounded sets.

(A3) $\sigma : H \to \mathscr{L}(H)$ is bounded and Gâteaux differentiable such that $\nabla_\cdot\sigma : H \times H \to \mathscr{L}_{HS}(H)$ is bounded on $H \times H$ and uniformly continuous on bounded sets, and $\sigma(x)$ is invertible for each $x \in H$.

(A4) $\int_0^t s^{-2\alpha} \|e^{sA}\sigma(0)\|_{HS}^2 \mathrm{d}s < \infty$ holds for some constant $\alpha \in (0, \frac{1}{2})$ and all $t > 0$.

For specific examples of A and F satisfying (A1), (A2) and (A4) we may take e.g. A a negative definite self-adjoint operator with discrete spectrum $\{-\lambda_n\}_{n\geq 1}$ such that $\sum_{n=1}^{\infty} \frac{1}{\lambda_n^{\delta}} < \infty$ holds for some constant $\delta \in (0,1)$, $\sigma = I$ and the Nemytskii type non-linear drift (see [73, Section 9.3.4]) $F(\xi) = \int_{-\tau}^{0} f \circ \xi(s) \mathrm{d}s$ for some $f \in C_b^1(\mathbb{R})$.

Recall that a mild solution is a continuous adapted process $(X(t))_{t\geq -\tau}$ on H such that

$$X(t) = \mathrm{e}^{tA}\xi(0) + \int_0^t \mathrm{e}^{(t-s)A}F(X_s)\mathrm{d}s + \int_0^t \mathrm{e}^{(t-s)A}\sigma(X(s))\mathrm{d}W(s), \quad t \geq 0$$

with the initial condition $X_0(\theta) = X(\theta) = \xi(\theta)$ for $\theta \in [-\tau, 0]$. By $(A1) - (A4)$, equation (3.1.1) has a unique mild solution (see Theorem 3.4.1), denoted by $(X^{\xi}(t))_{t\geq 0}$, the solution with $X_0 = \xi \in \mathscr{C}$. Let

$$P_t f(\xi) := \mathbb{E} f(X_t^{\xi}), \qquad t \geq 0, \xi \in \mathscr{C}, f \in \mathscr{B}_b(\mathscr{C}),$$

where $\mathscr{B}_b(\mathscr{C})$ is the class of all bounded measurable functions on $\mathscr{C}$. We remark that due to the time-delay the solution $(X^{\xi}(t))_{t\geq 0}$ is not Markovian, but its segment process $(X_t^{\xi})_{t\geq 0}$ admits the strong Markov property, so that P_t is a Markov semigroup on $\mathscr{B}_b(\mathscr{C})$.

The following two theorems are the main results of this chapter, which provide derivative formulae for P_t with additive and multiplicative noise respectively.

Theorem 3.1.1 (Additive Noise). *Assume that (A1)-(A4) hold with constant $\sigma \in \mathscr{L}(H)$. Then for any $T > \tau$ and C^1-function $u : [0, \infty) \to \mathbb{R}$ such that $u(0) = 1$ and $u(t) = 0$ for $t \geq T - \tau$,*

$$(\nabla_{\eta} P_T f)(\xi) = \mathbb{E}\Big(f(X_T^{\xi}) \int_0^T \langle \sigma^{-1}\{(\nabla_{\Upsilon_t})F(X_t^{\xi}) - \dot{u}(t)\mathrm{e}^{tA}\eta(0)\}, \mathrm{d}W(t)\rangle \Big) \tag{3.1.2}$$

holds for all $\xi, \eta \in \mathscr{C}$ and $f \in C_b^1(\mathscr{C})$, where

$$\Upsilon(t) := \begin{cases} u(t)\mathrm{e}^{tA}\eta(0), & t > 0, \\ \eta(t), & t \in [-\tau, 0]. \end{cases}$$

Theorem 3.1.2 (Multiplicative Noise). *Assume that* (A1)-(A4) *hold. Let* $T > \tau$ *and* C^1*-function* $u : [0,\infty) \to \mathbb{R}$ *be such that* $u(t) > 0$ *for* $t \in [0, T-\tau)$, $u(t) = 0$ *for* $t \geq T-\tau$, *and*

$$\theta_p := \inf_{t\in[0,T-\tau]} \{p + (p-1)u'(t)\} > 0$$

hold for some $p > 1$. *Then for any* $\xi, \eta \in \mathscr{C}$:

(1) *The equation*

$$\begin{cases} \mathrm{d}Z(t) = \Big\{AZ(t) + \Big((\nabla_{Z_t})F(X_t^\xi) - \frac{Z(t)}{u(t)}\Big)1_{[0,T-\tau)}(t)\Big\}\mathrm{d}t \\ \qquad\qquad +((\nabla_{Z(t)}\sigma)(X^\xi(t)))\mathrm{d}W(t), \\ Z_0 = \eta, \end{cases} \tag{3.1.3}$$

has a unique solution such that $Z(t) = 0$ *for* $t \geq T-\tau$.

(2) *If* $\|\sigma^{-1}(\cdot)\| \leq c(1 + \|\cdot\|_H^q)$ *holds for some constants* $c, q > 0$, *then*

$$\begin{aligned} \nabla_\eta P_T f(\xi) = \mathbb{E}\Big(f(X_T^\xi) \int_0^T \Big\langle \sigma^{-1}(X^\xi(t))\Big\{\frac{Z(t)}{u(t)} 1_{[0,T-\tau)}(t) \\ + (\nabla_{Z_t}F)(X_t^\xi)1_{[T-\tau,T]}(t)\Big\}, \mathrm{d}W(t)\Big\rangle\Big) \end{aligned} \tag{3.1.4}$$

holds for $f \in C_b^1(\mathscr{C})$.

A simple choice of u for Theorem 3.1.1 is $u(t) = \frac{(T-\tau-t)^+}{T-\tau}$, while for Theorem 3.1.2 one may take $u(t) = (T-\tau-t)^+$ such that $\theta_p = 1$ for all $p > 1$. Both theorems will be proved in Section 3.2. In Section 3.3 these results are applied to derive explicit gradient estimates and Harnack inequalities of P_t. Finally, for completeness, in Section 3.4 we address the existence and uniqueness of mild solution to equation (3.1.1) under (A1)-(A4), and the existence of Malliavin derivative $D_h X^\xi(t)$ along direction h and derivative process $\nabla_\eta X^\xi(t)$ along direction η as solutions of SPDEs on H.

3.2 Proofs of Theorems 3.1.1 and 3.1.2

For the readers' convenience, let us first explain the main idea of establishing Bismut formula using Malliavin calculus. Let H_a^1 be the class of all adapted process $h = (h(t))_{t\geq 0}$ on H such that $\mathbb{P}$-a.s.

$$h(t) = \int_0^t \dot{h}(s)\mathrm{d}s, \qquad t \in [0,T]$$

holds for some bounded $L^2([0,T] \to H; \mathrm{d}t)$-valued random variable $\dot{h}$. For $\epsilon > 0$ and $h \in H_a^1$, let $X^{\xi,\epsilon h}(t)$ solve (3.1.1) with $W(t)$ replaced by $W(t) + \epsilon h(t)$, i.e.,

$$\begin{cases} \mathrm{d}X^{\xi,\epsilon h}(t) = \{AX^{\xi,\epsilon h}(t) + F(X_t^{\xi,\epsilon h}) + \epsilon\sigma(X^{\xi,\epsilon h}(t))\dot{h}(t)\}\mathrm{d}t \\ \qquad\qquad +\sigma(X^{\xi,\epsilon h}(t))\mathrm{d}W(t), \\ X_0^{\xi,\epsilon h} = \xi \in \mathscr{C}. \end{cases} \tag{3.2.1}$$

If for $h \in H_a^1$

$$D_h X_t^\xi := \frac{\mathrm{d}}{\mathrm{d}\epsilon} X_t^{\xi,\epsilon h}\Big|_{\epsilon=0}$$

exists in $L^{1+r}(\Omega \to \mathscr{C}; \mathbb{P})$ for some $r > 0$, we call it the Malliavin derivative of X_t^ξ along direction h. Next, let

$$\nabla_\eta X_t^\xi := \frac{\mathrm{d}}{\mathrm{d}\epsilon} X_t^{\xi+\epsilon\eta}\Big|_{\epsilon=0}$$

be the derivative process of X_t^ξ along direction $\eta \in \mathscr{C}$. If

$$D_h X_T^\xi = \nabla_\eta X_T^\xi, \quad \text{a.s.}, \tag{3.2.2}$$

then for any $f \in C_b^1(\mathscr{C})$

$$\begin{aligned}(\nabla_\eta P_T)f(\xi) &= \mathbb{E}(\nabla_\eta f)(X_T^\xi) = \mathbb{E}\Big(\nabla_{\nabla_\eta X_T^\xi} f\Big)(X_T^\xi) \\ &= \mathbb{E}\Big(\nabla_{D_h X_T^\xi} f\Big)(X_T^\xi) = \mathbb{E} D_h f(X_T^\xi).\end{aligned}$$

Since by the Girsanov theorem the distribution of $X_T^{\xi,\varepsilon h}$ under $R_\varepsilon\mathbb{P}$ coincides with that of X_T^ξ under $\mathbb{P}$, where

$$R_\varepsilon := \exp\bigg[-\varepsilon\int_0^T \langle \dot{h}(t), \mathrm{d}W(t)\rangle - \frac{\varepsilon^2}{2}\int_0^T \|\dot{h}(t)\|_H^2 \mathrm{d}t\bigg],$$

we have

$$P_T f(\xi) = \mathbb{E}\big[R_\varepsilon f(X_T^{\xi,\varepsilon h})\big]$$

and that $(R_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $L^p(\mathbb{P})$ for any $p > 1$. So,

$$\begin{aligned}0 = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} P_T f(\xi)\Big|_{\varepsilon=0} &= \mathbb{E}\bigg[\Big(\frac{\mathrm{d}}{\mathrm{d}\varepsilon} R_\varepsilon\Big|_{\varepsilon=0}\Big) f(X_T^\xi) + D_h f(X_T^\xi)\bigg] \\ &= \mathbb{E}\big[D_h f(X_T^\xi)\big] - \mathbb{E}\bigg[f(X_T^\xi)\int_0^T \langle \dot{h}(t), \mathrm{d}W(t)\rangle\bigg].\end{aligned}$$

That is

$$\mathbb{E}\big[D_h f(X_T^\xi)\big] = \mathbb{E}\bigg[f(X_T^\xi)\int_0^T \langle \dot{h}(t), \mathrm{d}W(t)\rangle\bigg].$$

Combining this with the integration by parts formula for D_h, we obtain

$$(\nabla_\eta P_T)f(\xi) = \mathbb{E}\Big(f(X_T^\xi)\int_0^T \langle \dot{h}(t), \mathrm{d}W(t)\rangle\Big).$$

In conclusion, the key point of the proof is, for given $T > \tau$, $\xi, \eta \in \mathscr{C}$ and $f \in C_b^1(\mathscr{C})$, to construct an $h \in H_a^1$ such that (3.2.2) holds.

We are now in a position to complete the proofs of Theorems 3.1.1 and 3.1.2.

Proof of Theorem 3.1.1. Let $h(0) = 0$ and

$$\dot{h}(t) = \sigma^{-1}\{(\nabla_{\Upsilon_t}F)(X_t^\xi) - \dot{u}(t)\mathrm{e}^{tA}\eta(0)\}, \quad t \geq 0.$$

By $(A1)$ and $u \in C^1([0, T-\tau])$, we see that $h \in H_a^1$. Moreover, $\Upsilon(t)$ solves the equation

$$\begin{cases} \mathrm{d}\Upsilon(t) = \{A\Upsilon(t) + (\nabla_{\Upsilon_t}F)(X_t^\xi) - \sigma\dot{h}(t)\}\mathrm{d}t, & t \geq 0, \\ \Upsilon_0 = \eta. \end{cases} \tag{3.2.3}$$

On the other hand, by Theorem 3.4.2, when $\nabla\sigma = 0$, $\nabla_\eta X^\xi(t) - D_h X^\xi(t)$ also solves this equation. Since it is trivial that (3.2.3) has a unique solution, we conclude that

$$\nabla_\eta X^\xi(t) - D_h X^\xi(t) = \Upsilon(t), \quad t \geq 0.$$

Thus, $\nabla_\eta X_T^\xi = D_h X_T^\xi$ as $\Upsilon_T = 0$ according to the choice of u. Therefore, the desired derivative formula holds as explained above. □

To prove Theorem 3.1.2, we need the following lemma. Since $(\nabla_{.}F)(X_t^\xi) : \mathscr{C} \to H$ and $(\nabla_{.}\sigma)(X^\xi(t)) : H \to \mathscr{L}_{HS}(H)$ are linear and bounded, (3.1.3) has a unique strong (variational) solution for $t \in [0, T-\tau)$.

Lemma 3.2.1. *In the situation of Theorem* 3.1.2, *let* $(Z(t))_{t\in[0,T-\tau)}$ *solve* (3.1.3). *Then for any* $p > 0$ *there exists a constant* $C > 0$ *such that*

$$\mathbb{E}\sup_{t\in[0,T-\tau)} \|Z_t\|_\infty^p < C\|\eta\|_\infty^p, \quad \eta \in \mathscr{C}.$$

Proof. It suffices to prove for $p > 4$. By Itô's formula and the boundedness of ∇F and $\nabla\sigma$, there exists a constant $c_1 > 0$ such that

$$\begin{aligned}
\mathrm{d}\|Z(t)\|_H^2 &= \Big\{2\Big\langle Z(t), AZ(t) + (\nabla_{Z_t}F)(X_t^\xi) - \frac{Z(t)}{u(t)}\Big\rangle + \|(\nabla_{Z(t)}\sigma)(X^\xi(t))\|_{HS}^2\Big\}\mathrm{d}t \\
&\quad + 2\langle Z(t), (\nabla_{Z(t)}\sigma(X^\xi(t)))\mathrm{d}W(t)\rangle \\
&\le \Big\{c_1\|Z_t\|_\infty^2 - \frac{2\|Z(t)\|_H^2}{u(t)}1_{[0,T-\tau)}(t)\Big\}\mathrm{d}t \\
&\quad + 2\langle Z(t), ((\nabla_{Z(t)}\sigma)(X^\xi(t)))\mathrm{d}W(t)\rangle
\end{aligned}$$

holds for $t \in [0, T-\tau)$. So, for $p > 4$ there exists a constant $c_2 > 0$ such that

$$\begin{aligned}
\mathrm{d}\|Z(t)\|_H^p &= \mathrm{d}(\|Z(t)\|_H^2)^{\frac{p}{2}} \\
&= \Big\{\frac{p}{2}\|Z(t)\|_H^{p-2}\mathrm{d}\|Z(t)\|_H^2 \\
&\quad + \frac{p}{2}(p-2)\|Z(t)\|_H^{p-4}\Big\|\big((\nabla_{Z(t)}\sigma)(X^\xi(t))\big)^* Z(t)\Big\|_H^2\Big\}\mathrm{d}t \\
&\quad + p\|Z(t)\|_H^{p-2}\big\langle Z(t), \big((\nabla_{Z(t)}\sigma)(X^\xi(t))\big)\mathrm{d}W(t)\big\rangle \\
&\le \Big\{c_2\|Z_t\|_\infty^p - \frac{p\|Z(t)\|_H^p}{u(t)}\Big\}\mathrm{d}t \\
&\quad + p\|Z(t)\|_H^{p-2}\big\langle Z(t), \big((\nabla_{Z(t)}\sigma)(X^\xi(t))\big)\mathrm{d}W(t)\big\rangle
\end{aligned} \tag{3.2.4}$$

holds for $t \in [0, T-\tau)$. Since $\|(\nabla_{Z(t)}\sigma)(X^\xi(t))\|_{HS} \le c\|Z(t)\|_H$ holds for some constant $c > 0$, combining this with the Burkhold-Davis-Gundy inequality, we arrive at

$$\mathbb{E}\sup_{s\in[-\tau,t]}\|Z(s)\|_H^p \le \|\eta\|_\infty^p + c_3\int_0^t \mathbb{E}\sup_{s\in[-\tau,\theta]}\|Z(s)\|_H^p\mathrm{d}\theta, \quad t\in[0,T-\tau)$$

for some constant $c_3 > 0$. The proof is then completed by the Gronwall lemma. □

Proof of Theorem 3.1.2. (1) Due to $(A1) - (A4)$, it is easy to see that (3.1.3) has a unique solution for $t \in [0, T-\tau)$. Let

$$\tilde{Z}(t) = Z(t)1_{[-\tau,T-\tau)}(t), \qquad t \ge -\tau.$$

If

$$\lim_{t\uparrow T-\tau} Z(t) = 0, \tag{3.2.5}$$

then it is easy to see that $(\tilde{Z}(t))_{t\geq 0}$ solves (3.1.3) and hence, the proof is finished.

By Itô's formula and (3.2.4) we can deduce that

$$\begin{aligned}
\mathrm{d}\frac{\|Z(t)\|_H^p}{u^{p-1}(t)} &= \frac{1}{u^{p-1}(t)}\mathrm{d}\|Z(t)\|_H^p - (p-1)\frac{\dot{u}(t)\|Z(t)\|_H^p}{u^p(t)}\mathrm{d}t \\
&\leq -\theta_p\frac{\|Z(t)\|_H^p}{u^p(t)}\mathrm{d}t + C_1\|Z(t)\|_\infty^p\mathrm{d}t \\
&\quad + \frac{p}{u^{p-1}(t)}\|Z(t)\|_H^{p-2}\langle Z(t), ((\nabla_{Z(t)}\sigma)(X^\xi(t)))\mathrm{d}W(t)\rangle
\end{aligned}$$

for some constant $C_1 > 0$. Combining this with Lemma 3.2.1 we obtain

$$\mathbb{E}\int_0^{T-\tau}\frac{\|Z(t)\|_H^p}{u^p(t)}\mathrm{d}t \leq C_2\Big(\|\eta\|_\infty^p + \frac{\|\eta(0)\|_H^p}{u^{p-1}(0)}\Big) \tag{3.2.6}$$

for some constant $C_2 > 0$, and due to the Burkhold-Davis-Gundy inequality

$$\mathbb{E}\sup_{s\in[0,T-\tau)}\frac{\|Z(s)\|_H^p}{u^{p-1}(s)} < \infty.$$

Since $u(s)\downarrow 0$ as $s\uparrow T-\tau$, the latter implies (3.2.5).

(2) Let

$$h(t) = \int_0^t \sigma^{-1}(X^\xi(s))\Big\{\frac{Z(s)}{u(s)}1_{[0,T-\tau)}(s) + (\nabla_{Z_s}F)(X_s^\xi)1_{[T-\tau,T]}(s)\Big\}\mathrm{d}s, \quad t\geq 0.$$

We first prove that $h\in H_a^1$. According to the boundedness of $\|\nabla F\|$ and using Hölder's inequality, we arrive at

$$\begin{aligned}
\mathbb{E}\int_0^T\|\dot{h}(t)\|_H^2\mathrm{d}t &\leq \mathbb{E}\int_0^{T-\tau}\|\sigma^{-1}(X^\xi(t))\|^2\frac{\|Z(t)\|_H^2}{u^2(t)}\mathrm{d}t \\
&\quad + C\mathbb{E}\int_{T-\tau}^T\|\sigma^{-1}(X^\xi(t))\|^2\|Z_s\|_\infty^2\mathrm{d}t \\
&\leq \Big(\mathbb{E}\int_0^T\|\sigma^{-1}(X^\xi(t))\|^{\frac{2p}{p-2}}\mathrm{d}t\Big)^{\frac{p-2}{p}} \\
&\quad \times\Big\{\Big(\mathbb{E}\int_0^{T-\tau}\frac{\|Z(t)\|_H^p}{u^p(t)}\mathrm{d}t\Big)^{\frac{2}{p}} + C\Big(\mathbb{E}\int_{T-\tau}^T\|Z_t\|_\infty^p\mathrm{d}t\Big)^{\frac{2}{p}}\Big\}
\end{aligned}$$

for some constant $C > 0$. Combining this with (3.2.6), $\|\sigma^{-1}(x)\| \leq c(1+\|x\|_H^q)$, Lemma 3.2.1 and Theorem 3.4.1, we conclude that $\mathbb{E}\int_0^T\|\dot{h}(t)\|_H^2\mathrm{d}t < \infty$; that is, $h\in H_a^1$.

Next, we intend to show that $\nabla_\eta X_T^\xi = D_h X_T^\xi$, which implies the desired derivative formula as explained in the beginning of this section. It is easy to see from Theorem 3.4.2 and the definition of h that $\Gamma(t) := \nabla_\eta X^\xi(t) - D_h X^\xi(t)$ solves the equation

$$\begin{cases} \mathrm{d}\Gamma(t) = \Big\{A\Gamma(t) + (\nabla_{\Gamma_t} F)(X_t^\xi) - \frac{Z(t)}{u(t)} 1_{[0,T-\tau)}(t) - (\nabla_{Z_t} F)(X_t^\xi) 1_{[T-\tau,T]}(t)\Big\}\mathrm{d}t \\ \qquad\qquad +((\nabla_{\Gamma(t)}\sigma)(X^\xi(t)))\mathrm{d}W(t), \quad t \in [0,T] \\ \Gamma_0 = \eta. \end{cases}$$

Then for $t \in [0,T]$,

$$\begin{cases} \mathrm{d}(\Gamma(t) - Z(t)) = \Big\{A(\Gamma(t) - Z(t)) + (\nabla_{\Gamma_t - Z_t}) F(X_t^\xi)\Big\}\mathrm{d}t \\ \qquad\qquad\qquad +((\nabla_{\Gamma(t)-Z(t)}\sigma)(X^\xi(t)))\mathrm{d}W(t), \\ \Gamma_0 - Z_0 = 0. \end{cases}$$

By Itô's formula and using (A1)-(A3), we obtain

$$\mathrm{d}\|\Gamma(t) - Z(t)\|_H^2 \le C\|\Gamma_t - Z_t\|_\infty^2 \mathrm{d}t + 2\langle \Gamma(t) - Z(t), ((\nabla_{\Gamma(t)-Z(t)}\sigma)(X^\xi(t)))\mathrm{d}W(t)\rangle$$

for some constant $C > 0$ and all $t \in [0,T]$. By the boundedness of $\|\nabla\sigma\|_{HS}$ and applying the Burkhold-Davis-Gundy inequality, we obtain

$$\mathbb{E} \sup_{s\in[0,t]} \|\Gamma(s) - Z(s)\|_H^2 \le C' \int_0^t \mathbb{E} \sup_{s\in[0,r]} \|\Gamma(s) - Z(s)\|_H^2 \mathrm{d}r, \quad t \in [0,T]$$

for some constant $C' > 0$. Therefore $\Gamma(t) = Z(t)$ for all $t \in [0,T]$. In particular, $\Gamma_T = Z_T$. Since $Z_T = 0$, we obtain $\nabla_\eta X_T^\xi = D_h X_T^\xi$. □

Remark 2.1. Our main results, Theorem 3.1.1 and Theorem 3.1.2, are established under the assumption that the infinitesimal generator A generates a contractive C_0-semigroup. Replacing A and $F(x)$ by $A - \alpha$ and $F(x) + \alpha x$ for a positive constant $\alpha > 0$, they also work for A generating a pesudo-contractive C_0-semigroup, i.e., $\|\mathrm{e}^{tA}\| \le \mathrm{e}^{\alpha t}$.

3.3 Gradient Estimate and Harnack Inequality

In this section we give some applications of Bismut formulae for P_t with additive and multiplicative noise respectively.

Theorem 3.3.1 (Additive Noise). *Assume that* $(A1)-(A4)$ *hold with constant* $\sigma \in \mathscr{L}(H)$. *Then there exists a constant* $C>0$ *such that*

(1) *For any* $T>\tau, \xi, \eta \in \mathscr{C}$ *and* $f \in \mathscr{B}_b(\mathscr{C})$,

$$|(\nabla_\eta P_T f)(\xi)|^2 \leq \frac{C\|\eta\|_\infty^2}{(T-\tau)^2 \wedge 1} P_T f^2(\xi).$$

(2) *For any* $T>\tau, \xi, \eta \in \mathscr{C}$ *and positive* $f \in \mathscr{B}_b(\mathscr{C})$,

$$\begin{aligned} |(\nabla_\eta P_T f)(\xi)| &\leq \delta\{P_T(f\log f)-(P_T f)\log P_T f\}(\xi) \\ &\quad + \frac{\|\eta\|_\infty^2}{\delta\{(T-\tau)\wedge 1\}^2} P_T f(\xi), \ \delta>0. \end{aligned} \tag{3.3.1}$$

Proof. By the Jensen inequality and the semigroup property of P_t, it suffices to prove for $T-\tau \in (0,1]$. Let $u(t)=\frac{T-\tau-t}{T-\tau}$. By Theorem 3.1.1, the proof is then standard and similar to that of [36, Theorem 4.2]. We include it below for completeness.

(1) Note that $\dot{u}(t)=-\frac{1}{T-\tau}$. Due to the definition of $\Upsilon(t)$ and the boundedness of $\|\nabla F\|$ it follows that

$$\|(\nabla_{\Upsilon_t}F)(X_t^\xi)\|_H^2 \leq C\|\eta\|_\infty^2$$

for some constant $C>0$. By (3.1.2), Hölder's inequality and the boundedness of $\|\sigma^{-1}\|$ we have

$$\begin{aligned} |(\nabla_\eta P_T f)(\xi)\|^2 &\leq 2P_T f^2(\xi)\mathbb{E}\int_0^T \Big\{\|\sigma^{-1}((\nabla_{\Upsilon_t}F)(X_t^\xi)\|_H^2 \\ &\quad + \frac{1}{(T-\tau)^2}\|e^{tA}\eta(0))\|_H^2\Big\}dt \\ &\leq \frac{C}{(T-\tau)^2}\|\eta\|_\infty^2 P_T f^2(\xi) \end{aligned} \tag{3.3.2}$$

for some constant $C>0$ and all $T \in (\tau, \tau+1]$.

(2) For $t \in [0, T-\tau]$, let

$$M(t) := \int_0^t \Big\langle \sigma^{-1}\Big((\nabla_{\Upsilon_s}F)(X_s^\xi)+\frac{1}{T-\tau}e^{sA}\eta(0)\Big), dW(s)\Big\rangle,$$

which is a mean-square integrable martingale, with quadratic variation process

$$\langle M\rangle(t) := \int_0^t \Big\|\sigma^{-1}\Big((\nabla_{\Upsilon_s}F)(X_s^\xi)+\frac{1}{T-\tau}e^{sA}\eta(0)\Big)\Big\|_H^2 ds \leq C\|\eta\|_\infty^2$$

for some constant $C > 0$. In the light of (3.1.2) and Young's inequality [3, Lemma 2.4], we have that for any $\delta > 0$ and positive $f \in \mathscr{B}_b(\mathscr{C})$

$$
\begin{aligned}
|(\nabla_\eta P_T f)(\xi)| \le \delta \{ P_T(f \log f) - (P_T f) \log P_T f\}(\xi) \\
+ \delta P_T f(\xi) \log \mathbb{E} \exp\Big(\frac{1}{\delta} M(T-\tau)\Big).
\end{aligned}
$$

Moreover, by the exponential martingale inequality, the boundedness of $\|\nabla F\|$ and the definition of Υ_s,

$$
\begin{aligned}
\mathbb{E}\exp\Big(\frac{1}{\delta}M(T-\tau)\Big) &\le \Big(\mathbb{E}\exp\Big(\frac{2}{\delta^2}\int_0^T \Big\|\sigma^{-1}\Big(\nabla_{\Upsilon_s} F(X_s^\xi) + \frac{1}{T-\tau} \mathrm{e}^{sA}\eta(0)\Big)\Big\|^2 \mathrm{d}t\Big)\Big)^{\frac{1}{2}} \\
&\le \exp\Big(\frac{C}{\delta^2 (T-\tau)^2} \|\eta\|_\infty^2\Big)
\end{aligned}
$$

holds for some constant $C > 0$ and all $T \in (\tau, \tau+1]$. Therefore, the proof is finished. □

According to [36, Proposition 4.1], (3.3.1) implies the following Harnack inequality. Applications of these inequalities to heat kernel estimates, invariant probability measure and Entropy-cost inequalities can be found in e.g. [74, 87, 89].

Corollary 3.3.2. *Assume that $(A1)-(A4)$ hold with constant $\sigma \in \mathscr{L}(H)$. Then there exists a constant $C > 0$ such that*

$$
\begin{aligned}
&|P_T f|^\alpha(\xi) \le \exp\Big[\frac{\alpha C \|\eta\|_\infty^2}{(\alpha-1)\{(T-\tau)^2 \wedge 1\}}\Big] P_T |f|^\alpha(\xi+\eta), \quad f \in \mathscr{B}_b(\mathscr{C}), \\
&\qquad T > \tau, \xi, \eta \in \mathscr{C}
\end{aligned}
\tag{3.3.3}
$$

holds for any $\alpha > 1$.

Next, we consider the multiplicative noise case. For simplicity we only consider the case where $\|\sigma^{-1}\|_\infty := \sup_{x \in H} \|\sigma^{-1}(x)\| < \infty$. The case for σ^{-1} having algebraic growth is similar, where the resulting estimate of $\|\nabla P_t f\|$ will be no longer bounded for bounded f, but bounded above by a polynomial function of $\|\xi\|_\infty$.

Theorem 3.3.3 (Multiplicative Noise). *Let $(A1)$-$(A4)$ hold and assume further that $\|\sigma^{-1}\|_\infty < \infty$. Then for any $p \ge 1$ there exists a constant $C > 0$ such that*

$$|(\nabla_\eta P_T f)(\xi)|^p \le \frac{C\|\eta\|_\infty^p}{1\wedge (T-\tau)^2} P_T|f|^p(\xi), \quad f\in \mathscr{B}_b(\mathscr{C}), T>\tau, \xi,\eta\in\mathscr{C}.$$

In particular, P_t is strong Feller for $t > T-\tau$.

Proof. It suffices to prove for $T\in(\tau,\tau+1]$. Let $u(t)=(T-\tau-t)^+, t\ge 0$. We have $\theta_p=1$. Since σ^{-1} is bounded, for any $p>1$ and $\eta\in\mathscr{C}$, it follows from (3.1.4) that

$$\begin{aligned}
&\frac{|\nabla_\eta P_T f|^{\frac{p}{p-1}}(\xi)}{(P_T|f|^p)^{\frac{1}{p-1}}(\xi)}\\
&\le \mathbb{E}\Big|\int_0^T \Big\langle \sigma^{-1}(X^\xi(s))\Big\{\frac{Z(s)}{u(s)}1_{[0,T-\tau)}(s) + (\nabla_{Z_s}F)(X_s^\xi)1_{[T-\tau,T]}(s)\Big\}, \mathrm{d}W(s)\Big\rangle\Big|^{\frac{p}{p-1}}\\
&\le C_1\mathbb{E}\Big(\int_0^T \Big(\frac{\|Z(t)\|_H^2}{u^2(t)}1_{[0,T-\tau)}(t) + \|Z_t\|_\infty^2 1_{[T-\tau,T]}(t)\Big)\mathrm{d}t\Big)^{\frac{p}{2(p-1)}}
\end{aligned}$$

holds for some constants $C_1, C_2>0$ and all $T\in(\tau,\tau+1]$, where the second inequality follows from the Burkholder-Davis-Gundy inequality: for any $q>1$ there exists a constant $C_q>0$ such that

$$\mathbb{E}\sup_{t\in[0,T]}|M(t)|^q \le C_q\mathbb{E}\langle M\rangle^{\frac{q}{2}}(T)$$

holds for any continuous martingale $M(t)$ and $T>0$. Then the proof is completed by combining this with (3.2.6) with $u(0)=T-\tau$ and Lemma 3.2.1. □

Remark 3.1. From Corollary 3.3.2 and [36, Proposition 4.1], we know that entropy estimation (3.3.1) plays a key role in establishing the Harnack inequality. However, the entropy estimation seems to be difficult to obtain for the multiplicative noise case. Hence we can not adopt the same method as in the additive noise case to derive the Harnack inequality. In order to establish the Harnack inequality for the multiplicative noise case, one may use coupling method as in Wang [88], and Wang and Yuan [91]. Since the derivation of the Harnack inequality for functional SPDEs with multiplicative noise is very similar to that of [91], we omit it here.

3.4 Appendix

In this section we give two auxiliary lemmas, where one concerns the existence and uniqueness of solution of equation (3.1.1) under (A1)-(A4), and the other one discusses not only the existence of Malliavin directional derivative but also the derivative process with respect to the initial data. To make the content self-contained, we sketch their proofs.

Theorem 3.4.1. *Let (A1), (A4) hold, and let $F : \mathscr{C} \to H, \sigma : H \to \mathscr{L}(H)$ be Lipschitz continuous. Then for any $p > 2$ and initial data $\xi \in L^p(\Omega \to \mathscr{C}, \mathscr{F}_0, \mathbb{P})$, equation (3.1.1) has a unique mild solution $(X^\xi(t))_{t\geq 0}$, and the solution satisfies*

$$\mathbb{E} \sup_{t\in[0,T]} \|X_t^\xi\|_\infty^p < \infty, \quad T > 0.$$

Proof. Obviously, (A4) remains true by replacing α with a smaller positive number. So, we may take in (A4) $\alpha \in (0, \frac{1}{p})$. Then, by [19, Proposition 7.9] with $r = \frac{p}{2} \in (1, \frac{1}{2\alpha})$, for any $T_0 > 0$ there exists a constant $C_0 > 0$ such that for any continuous adapted process $Y(s)$ on H,

$$\mathbb{E} \sup_{t\in[0,T]} \left\| \int_0^t \mathrm{e}^{(t-s)A}\sigma(Y(s))\mathrm{d}W(s) \right\|_H^p \leq C_0 \mathbb{E} \int_0^T \|\sigma(Y(s)\|^p \mathrm{d}s \tag{3.4.1}$$

for any $T \in [0, T_0]$. Using this inequality, the desired assertions follow from the classical fixed point theorem for contractions. Denote by $\mathscr{H}_p$ the Banach space of all the H-valued continuous adapted processes Y defined on the time interval $[-\tau, T]$ such that $Y(t) = \xi(t), t \in [-\tau, 0]$, and

$$\|Y\|_p := \left(\mathbb{E} \sup_{t\in[-\tau,T]} \|Y(t)\|_H^p\right)^{\frac{1}{p}} < \infty.$$

Let

$$\mathscr{K}(Y)(t) = \begin{cases} \xi(t), & \text{if } t \in [-\tau, 0], \\ \mathrm{e}^{tA}\xi(0) + \int_0^t \mathrm{e}^{(t-s)A}F(Y_s)\mathrm{d}s + \int_0^t \mathrm{e}^{(t-s)A}\sigma(Y(s))\mathrm{d}W(s), & \text{if } t \in (0, T]. \end{cases}$$

By (3.4.1) and the linear growth of F and σ, we conclude that $\mathscr{K}$ maps $\mathscr{H}_p$ into $\mathscr{H}_p$. For the existence and uniqueness of solutions, it suffices to show that the map $\mathscr{K}$ is contractive for small $T > 0$. By the Lipschitz continuity of F and σ, and applying (3.4.1) for $\sigma(Y_1(s)) - \sigma(Y_2(s))$ in place of $\sigma(Y(s))$, we obtain

$$\|\mathscr{K}(Y^1) - \mathscr{K}(Y^2)\|_p \leq CT\|Y^1 - Y^2\|_p^p, \quad Y^1, Y^2 \in \mathscr{H}_p.$$

for some constant $C > 0$ and all $T \in [0, T_0]$. Choosing sufficiently small $\tilde{T}$ such that $C\tilde{T} < 1$ we can conclude that $\mathscr{K}$ is contractive. In what follows, we consider (3.1.1) on intervals $[\tilde{T}, 2\tilde{T}], [2\tilde{T}, 3\tilde{T}] \cdots [\lfloor T/\tilde{T} \rfloor \tilde{T}, (\lfloor T/\tilde{T} \rfloor + 1)\tilde{T}]$ respectively, where $\lfloor T/\tilde{T} \rfloor$ denotes the integer part of $T/\tilde{T}$, and then (3.1.1) admits a global mild solution on the time interval $[0, T]$. □

Theorem 3.4.2. *Assume that (A1), (A2) and (A3) hold, and let $\xi, \eta \in \mathscr{C}$ and $h \in H_a^1$.*

(1) $(D_h X(t))_{t \geq 0}$ exists and is the unique solution to the equation

$$\begin{cases} \mathrm{d}\alpha(t) = \{A\alpha(t) + (\nabla_{\alpha_t} F)(X_t^\xi) + \sigma(X^\xi(t))\dot{h}(t)\}\mathrm{d}t \\ \qquad\qquad + ((\nabla_{\alpha(t)}\sigma)(X^\xi(t)))\mathrm{d}W(t), \\ \alpha_0 = 0. \end{cases}$$

(2) $(\nabla_\eta X(t))_{t \geq 0}$ exists and is the unique solution to the equation

$$\begin{cases} \mathrm{d}\beta(t) = \{A\beta(t) + (\nabla_{\beta_t} F)(X_t^\xi)\}\mathrm{d}t + ((\nabla_{\beta(t)}\sigma)(X^\xi(t)))\mathrm{d}W(t), \\ \beta_0 = \eta. \end{cases}$$

Proof. We only prove (1) since (2) can be proved in a similar way. The argument of the proof is standard in the setting of semi-linear SPDEs without delay. The only difference for the present setting is that one has to estimate the sup over time for the norm of the error process for small $\epsilon \in (0, 1)$

$$\Lambda^\epsilon(t) := X^{\xi, \epsilon h}(t) - X^\xi(t) - \epsilon\alpha(t), \quad t \geq 0,$$

where $X^{\xi, \epsilon h}$ is the mild solution to (3.2.1).

(a) There exists a constant $C > 0$ such that

$$\mathbb{E} \sup_{t \in [0, T]} \|X_t^{\xi, \epsilon h} - X_t^\xi\|_\infty^2 \leq \epsilon^2 \mathrm{e}^{C(T+1)} \mathbb{E} \int_0^T \|\dot{h}(t)\|_H^2 \mathrm{d}t, \quad T \geq 0. \tag{3.4.2}$$

Indeed, by $(A1)$, $(A2)$ and $(A3)$ we have the following Itô's formula for $\|X^{\xi,\epsilon h}(t) - X^{\xi}(t)\|_H^2$:

$$\begin{aligned}\mathrm{d}\|X^{\xi,\epsilon h}(t) - X^{\xi}(t)\|_H^2 = 2\langle X^{\xi,\epsilon h}(t) - X^{\xi}(t), &A(X^{\xi,\epsilon h}(t) - X^{\xi}(t)) \\ &+ F(X_t^{\xi,\epsilon h}) - F(X_t^{\xi}) + \epsilon\sigma(X^{\xi,\epsilon h}(t))\dot{h}(t)\rangle \mathrm{d}t \\ &+ \|\sigma(X^{\xi,\epsilon h}(t)) - \sigma(X^{\xi}(t))\|_{HS}^2 \mathrm{d}t \\ &+ 2\langle X^{\xi,\epsilon h}(t) - X^{\xi}(t), (\sigma(X^{\xi,\epsilon h}(t)) - \sigma(X^{\xi}(t)))\mathrm{d}W(t)\rangle.\end{aligned}$$

Noting from $(A1)$, $(A2)$ and $(A3)$ that

$$\begin{aligned}&\langle X^{\xi,\epsilon h}(t) - X^{\xi}(t), A(X^{\xi,\epsilon h}(t) - X^{\xi}(t))\rangle \le 0, \\ &\|F(X_t^{\xi,\epsilon h}) - F(X_t^{\xi}) + \epsilon\sigma(X^{\xi,\epsilon h}(t))\dot{h}(t)\|_H \le C_1(\|X_t^{\xi,\epsilon h} - X_t^{\xi}\|_\infty + \epsilon\|\dot{h}(t)\|_H),\end{aligned}$$

and by the Burkhold-Davis-Gundy inequality

$$\begin{aligned}&\mathbb{E}\sup_{t\in[0,T]}\left\|\int_0^t \langle X^{\xi,\epsilon h}(s) - X^{\xi}(s), (\sigma(X^{\xi,\epsilon h}(s)) - \sigma(X^{\xi}(s)))\mathrm{d}W(s)\rangle\right\|_H \\ &\le C_1\mathbb{E}\left(\int_0^T \|X^{\xi,\epsilon h}(s) - X^{\xi}(s)\|_H^4 \mathrm{d}s\right)^{\frac{1}{2}} \\ &\le \frac{1}{2}\mathbb{E}\sup_{t\in[0,T]}\|X^{\xi,\epsilon h}(t) - X^{\xi}(t)\|_H^2 + \frac{C_1}{2}\mathbb{E}\int_0^T \|X^{\xi,\epsilon h}(s) - X^{\xi}(s)\|_H^2 \mathrm{d}s\end{aligned}$$

for some constant $C_1 > 0$, we obtain

$$\mathbb{E}\sup_{t\in[0,T]}\|X_t^{\xi,\epsilon h} - X_t^{\xi}\|_\infty^2 \le C_2\epsilon^2\int_0^T \|\dot{h}(t)\|_H^2 \mathrm{d}t + C_2\int_0^T \mathbb{E}\sup_{s\in[0,t]}\|X_s^{\xi,\epsilon h} - X_s^{\xi}\|_\infty^2 \mathrm{d}t$$

for some constant $C_2 > 0$. This implies (3.4.2).

(b) To prove $D_h X^{\xi}(t) = \alpha(t)$ it suffices to show

$$\lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\mathbb{E}\sup_{t\in[0,T]}\|\Lambda^{\epsilon}(t\wedge\tau_n)\|_H = 0, \quad n \ge 1, \tag{3.4.3}$$

where $\tau_n := \inf\{t \ge 0, \|X_t^{\xi}\|_\infty \ge n\} \uparrow \infty$ as $n \uparrow \infty$. Indeed, by (3.4.2), (3.4.3) and the definition of Λ^{ε} we have

$$\limsup_{\varepsilon\downarrow 0}\frac{1}{\varepsilon^{\frac{3}{2}}}\mathbb{E}\|\Lambda^{\varepsilon}(t\wedge\tau_n)\|_H^{\frac{3}{2}} \le \limsup_{\epsilon\downarrow 0}\frac{1}{\varepsilon^{\frac{3}{2}}}\left(\mathbb{E}\|\Lambda^{\varepsilon}(t\wedge\tau_n)\|_H\right)^{\frac{1}{2}}\left(\mathbb{E}\|\Lambda^{\varepsilon}(t\wedge\tau_n)\|_H^2\right)^{\frac{1}{2}} = 0,$$

So that by (3.4.2),

$$\begin{aligned}\limsup_{\varepsilon\downarrow 0}\frac{1}{\varepsilon^{\frac{3}{2}}}\mathbb{E}\|\Lambda^{\varepsilon}(t)\|_H^{\frac{3}{2}} &\le \limsup_{\varepsilon\downarrow 0}\frac{1}{\varepsilon^{\frac{3}{2}}}\mathbb{E}\left[\|\Lambda^{\varepsilon}(t)\|_H^{\frac{3}{2}} 1_{\{\tau_n \le t\}}\right] \\ &\le \limsup_{\epsilon\downarrow 0}\frac{1}{\varepsilon^{\frac{3}{2}}}\left(\mathbb{E}\|\Lambda^{\varepsilon}(t)\|_H^2\right)^{\frac{3}{4}}\mathbb{P}(\tau_n \le t)^{\frac{1}{4}} \le c(t)\mathbb{P}(\tau_n \le t)^{\frac{1}{4}}\end{aligned}$$

holds for some constant $c(t) > 0$ and all $n \geq 1$. Letting $n \to \infty$ we arrive at

$$\lim_{\varepsilon \to 0} \left\| \frac{X^{\xi,\epsilon h}(t) - X^{\xi}(t)}{\varepsilon} - \alpha(t) \right\|_H^{\frac{3}{2}} = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^{\frac{3}{2}}} \mathbb{E} \|\Lambda^{\varepsilon}(t)\|_H^{\frac{3}{2}} = 0,$$

that is, $D_h X^{\xi}(t) = \alpha(t)$.

To prove (3.4.3), we observe that

$$\begin{aligned} \Lambda^{\epsilon}(t \wedge \tau_n) = \int_0^{t \wedge \tau_n} & \mathrm{e}^{(t-s)A} \{ F(X_s^{\xi,\epsilon h}) - F(X_s^{\xi}) - \epsilon \nabla_{\alpha_s} F(X_s^{\xi}) \\ & + \epsilon(\sigma(X^{\xi,\epsilon h}(s)) - \sigma(X^{\xi}(s))) \dot{h}(s) \} \mathrm{d}s \\ & + \int_0^{t \wedge \tau_n} \mathrm{e}^{(t-s)A} (\sigma(X^{\xi,\epsilon h}(s)) - \sigma(X^{\xi}(s)) \\ & - \epsilon \nabla_{\alpha(s)} \sigma(X^{\xi}(s))) \mathrm{d}W(s). \end{aligned} \tag{3.4.4}$$

Let

$$\gamma_n(s) := \sup_{\|\xi\|_\infty \leq n, \|\xi - \eta\|_\infty \leq s} \|\nabla F(\xi) - \nabla F(\eta)\|_\infty + \sup_{\|x\| \leq n, \|x-y\| \leq s} \|\nabla \sigma(x) - \nabla \sigma(y)\|_{HS}.$$

By $(A2)$ and $(A3)$ we have $\gamma_n(s) \downarrow 0$ as $s \downarrow 0$ and $\gamma_n(\infty) < \infty$. Then

$$s\gamma_n(s) \leq \gamma_n(\sqrt{\epsilon}) s + \frac{s^2 \gamma_n(\infty)}{\sqrt{\epsilon}}, \qquad s \geq 0.$$

Therefore, there exists a constant $C_1 > 0$ such that

$$\begin{aligned} & \|F(X_s^{\xi,\epsilon h}) - F(X_s^{\xi}) - \epsilon(\nabla_{\alpha_s} F)(X_s^{\xi})\|_\infty \\ & \leq \|\nabla F\|_\infty \|\Lambda_s^{\epsilon}\|_\infty + \|X_s^{\xi,\epsilon h} - X_s^{\xi}\|_\infty \gamma_n(\|X_s^{\xi,\epsilon h} - X_s^{\xi}\|_\infty) \\ & \leq C_1 \|\Lambda_s^{\epsilon}\|_\infty + \gamma(\sqrt{\epsilon}) \|X_s^{\xi,\epsilon h} - X_s^{\xi}\|_\infty + \frac{\gamma_n(\infty)}{\sqrt{\epsilon}} \|X_s^{\xi,\epsilon h} - X_s^{\xi}\|_\infty^2, \\ & \epsilon \|(\sigma(X^{\xi,\epsilon h}(s)) - \sigma(X^{\xi}(s))) \dot{h}(s)\|_H \leq \epsilon^2 \|\dot{h}(s)\|_H^2 + C_1 \|X^{\xi,\epsilon h}(s) - X^{\xi}(s)\|_H^2, \end{aligned}$$

and by the Burkhold-Davis-Gundy inequality

$$\begin{aligned} & \mathbb{E} \sup_{t \in [0,T]} \left\| \int_0^{t \wedge \tau_n} \mathrm{e}^{(t-s)A} (\sigma(X^{\xi,\epsilon h}(s)) - \sigma(X^{\xi}(s)) - \epsilon(\nabla_{\alpha(s)} \sigma)(X^{\xi}(s))) \mathrm{d}W(s) \right\|_H \\ & \leq \frac{1}{2} \mathbb{E} \sup_{t \in [0,T]} \|\Lambda^{\epsilon}(t \wedge \tau_n)\|_H + \gamma_n(\sqrt{\epsilon}) \mathbb{E} \sup_{t \in [0,T]} \|X^{\xi,\epsilon h}(t \wedge \tau_n) - X^{\xi}(t \wedge \tau_n)\|_H \\ & \quad + \frac{\gamma_n(\infty)}{\sqrt{\epsilon}} \mathbb{E} \sup_{t \in [0,T]} \|X^{\xi,\epsilon h}(t \wedge \tau_n) - X^{\xi}(t \wedge \tau_n)\|_H^2 \\ & \quad + C_1 \mathbb{E} \int_0^{T \wedge \tau_n} \Big(\|\Lambda^{\epsilon}(s)\|_H +_H \gamma_n(\sqrt{\epsilon}) \|X^{\xi,\epsilon h}(s) - X^{\xi}(s)\|_H \\ & \quad + \frac{\gamma_n(\infty)}{\sqrt{\epsilon}} \|X^{\xi,\epsilon h}(s) - X^{\xi}(s)\|_H^2 \Big) \mathrm{d}s. \end{aligned}$$

Combining this with (3.4.2) and (3.4.4) we obtain

$$\mathbb{E} \sup_{t\in[0,T]} \|\Lambda^\epsilon(t\wedge\tau_n)\|_H \le C_2 \int_0^T \mathbb{E} \sup_{s\in[0,t]} \|\Lambda^\epsilon(s\wedge\tau_n)\|_H \mathrm{d}s$$
$$+ C(T)\Big(\gamma_n(\sqrt{\epsilon})\epsilon + \frac{\gamma_n(\infty)\epsilon^2}{\sqrt{\epsilon}}\Big)$$

for some constant $C_2 > 0$ and

$$C(T) := \mathrm{e}^{C_2(1+T)}\Big(1 + \mathbb{E}\int_0^T \|\dot{h}(t)\|_H^2 \mathrm{d}t\Big), \quad T \ge 0.$$

Due to the Gronwall inequality, this implies (3.4.3). □

Chapter 4

Large Deviation for Neutral Functional SDEs with Jumps

In this chapter, we apply the weak convergence approach to establish a LDP for a class of neutral functional SDEs with jumps. In particular, we discuss the LDP for neutral SDDEs which allow the coefficients to be highly nonlinear with respect to the delay argument.

4.1 Introduction

Large deviation principle (LDP), concerning with the asymptotic computation of small probability events on an exponential scale, has being extensively studied beginning with the fundamental formulation of Donsker and Varadhan's [26], and been applied to stochastic differential equations (SDEs), e.g., [8, 23, 30, 83]. The weak convergence approach due to Dupuis and Ellis [27] has also been proved to be effective in establishing the LDP for various stochastic dynamics driven by Brownian motions, e.g., in [71, 104] for finite-dimensional SDEs, and in [53, 72, 76] for infinite-dimensional stochastic partial differential equations (SPDEs). The main advantage of this approach is that time-discretization and large calculations in showing the exponential-type estimates can be avoided. For SDEs and SPDEs driven by jump processes, most work focuses on the cases of additive Lévy noise, e.g., in [21] for finite-dimensional jump diffusions with bounded coefficients through a time-discretization argument, in [75] for stochastic evolution equations, and in [96] for 2-D stochastic Navier-Stokes equations. However, it seems hard to apply the approach introduced in [75] to SDEs and SPDEs with multiplicative Lévy noise. Recently, [15] gave a variational representation for functionals of Poisson measure plus an independent Brownian motion, which cover many continuous time models. Just like the variational representation of functionals of Brownian motion [16], this representation also proves to be effective in the study of LDP for stochastic models with jumps, e.g., [15] and [94] discussed the LDP for SDEs and multivalued SDEs with multiplicative Lévy noise respectively, where the drift and diffusion coefficients are imposed to be bounded. For the variational representation of functionals of Poisson random measures, we can also refer to [57, 103].

For stochastic systems with memory, there are only a few results on LDP. For example, Scheutzow [80] studied the topic of LDP for SDDEs with additive noise, and Mohammed and Zhang [60] provided the upper and lower large deviation estimates for SDDEs driven by multiplicative Brownian motion noise. We also point out that [60] and [80] adopted time-discretization arguments. However, for functional SDEs, time-discretization schemes, even the simplest Euler-Muruyama scheme, are relatively complicated, which thus brings a lot of troubles for the exponential-type estimates.

While there are few results on LDP for functional SDEs (FSDEs), in particular for neutral FSDEs, where a differential equation is called neutral if, besides the derivatives of the present state of the system, those of the past history are also involved. Motivated by the previous literature, in this chapter we shall apply the weak convergence approach [15, 16] to establish the LDP for a class of *neutral* FSDEs driven by *multiplicative noise*. In particular, we discuss the LDP for *neutral* stochastic differential *delay* equations which allow the coefficients to be *highly nonlinear* with respect to the delay argument, where the global Lipschitz condition, especially with respect to the delay argument, is imposed in [60] and [80]. Also, the boundedness of drift and diffusion coefficients imposed in [15, 94] is relaxed. Moreover, some tricks are adopted to overcome the difficulties caused by the neutral part and functional coefficients.

The organization of this chapter is as follows: We establish the LDP for a class of neutral FSDEs in Section 4.2, discuss the LDP for neutral SDDEs in Section 4.3, which in particular allow the coefficients to be highly nonlinear with respect to the delay argument, and in Section 4.4 we generalize the results in Section 4.2 to the case of neutral FSDEs with jumps.

4.2 LDP for Neutral FSDEs Driven by Brownian Motions

Let $(\mathbb{R}^n, \langle\cdot,\cdot\rangle, |\cdot|)$ be the Euclidean space and $\|A\|_{HS} := \sqrt{\text{trace}(A^*A)}$, the Hilbert-Schmidt norm for a matrix A, where A^* is its transpose. Fix $\tau > 0$, which is referred to as the delay, and $T > 0$, a finite time horizon. Let $W(t)$ be an m-dimensional Brownian motion defined on the classical Wiener space $(\Omega, \mathcal{F}, \mathbb{P})$, i.e., $\Omega := C_0(\mathbb{R}_+; \mathbb{R}^m)$, the space of $\mathbb{R}^m$-valued continuous functions ω on $\mathbb{R}_+$ vanishing at time 0, with the locally uniform convergence topology, $\mathcal{F}$ is the σ-algebra generated by coordinate mappings $W(t,\omega) = \omega(t)$, $\mathbb{P}$ is the Wiener measure on Ω. Let $\mathscr{C} := C([-\tau, 0]; \mathbb{R}^n)$, the space of continuous functions $f : [-\tau, 0] \mapsto \mathbb{R}^n$, endowed with a uniform norm $\|f\|_\infty := \sup_{-\tau\le\theta\le 0} |f(\theta)|$. For a map $\varphi : [-\tau, \infty) \mapsto \mathbb{R}^n$, let $\varphi_t \in \mathscr{C}$ be the segment of $\varphi(t)$ such that $\varphi_t(\theta) = \varphi(t+\theta), \theta \in [-\tau, 0], t \ge 0$.

Consider a neutral FSDE on $\mathbb{R}^n$

$$\begin{cases} \mathrm{d}[X(t) - G(X_t)] = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W(t), \\ X_0 = \xi \in \mathscr{C}. \end{cases} \tag{4.2.1}$$

Throughout this chapter we shall assume that

(H1) $G : \mathscr{C} \mapsto \mathbb{R}^n$ with $\alpha := |G(0)| < \infty$ and there exists $\kappa \in (0,1)$ such that

$$|G(\xi) - G(\eta)| \le \kappa \|\xi - \eta\|_\infty, \quad \xi, \eta \in \mathscr{C}.$$

(H2) $b : \mathscr{C} \mapsto \mathbb{R}^n$, satisfying a local Lipschitz condition, $\sigma : \mathscr{C} \mapsto \mathbb{R}^{n\times m}$ with

$\beta := \|\sigma(0)\|_{HS} < \infty$, and there exists $\lambda > 0$ such that for $\xi, \eta \in \mathscr{C}$

$$\langle (\xi(0) - \eta(0)) - (G(\xi) - G(\eta)), b(\xi) - b(\eta) \rangle \vee \|\sigma(\xi) - \sigma(\eta)\|_{HS}^2 \le \lambda \|\xi - \eta\|_\infty^2$$

and

$$\langle \xi(0) - G(\xi), b(\xi) \rangle \le \lambda(1 + \|\xi\|_\infty^2).$$

Remark 4.2.1. Note from (H1) and (H2) that

$$|G(\xi)| + \|\sigma(\xi)\|_{HS} \le \{(\kappa + \lambda) \vee (\alpha + \beta)\}(1 + \|\xi\|_\infty), \quad \xi \in \mathscr{C}. \tag{4.2.2}$$

Under (H1) and (H2), for any initial data $\xi \in \mathscr{C}$ Eq. (4.2.1) admits a unique solution $\{X(t)\}_{t \in [0,T]}$.

To establish the LDP for the law of small perturbation associated with Eq. (4.2.1), we need to recall some notions and notation in a general framework.

Let $\mathbb{S}$ be a Polish space (i.e., a separable complete metrizable topological space), and $\{Y^\epsilon, \epsilon \in (0,1)\}$ a family of $\mathbb{S}$-valued random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 4.2.1. A function $I : \mathbb{S} \mapsto [0, \infty]$ is called a rate function if it is lower semicontinuous. A rate function I is called a good rate function if the level set $\{f \in \mathbb{S} : I(f) \le a\}$ is compact for each $a < \infty$.

Definition 4.2.2. Let μ^ϵ be the law of $\{Y^\epsilon, \epsilon \in (0,1)\}$ in $\mathbb{S}$. The sequence $\{Y^\epsilon, \epsilon \in (0,1)\}$ is said to satisfy the LDP with rate function I if for each $A \in \mathscr{B}(\mathbb{S})$ (Borel σ-algebra generated by all open sets in $\mathbb{S}$)

$$-\inf_{f \in A^\circ} I(f) \le \liminf_{\epsilon \to 0} \epsilon \log \mu^\epsilon(A) \le \limsup_{\epsilon \to 0} \epsilon \log \mu^\epsilon(A) \le -\inf_{f \in \bar{A}} I(f),$$

where the interior A° and closure $\bar{A}$ are taken in $\mathbb{S}$.

The starting point of the weak convergence approach for stochastic dynamics driven by Brownian motion is the fact that the LDP is equivalent to a Laplace principle (LP) if the underlying space $\mathbb{S}$ is Polish.

Definition 4.2.3. The sequence $\{Y^\epsilon, \epsilon \in (0,1)\}$ is said to satisfy the LP on $\mathbb{S}$ with rate function I if for each bounded continuous mapping $g : \mathbb{S} \mapsto \mathbb{R}$

$$\lim_{\epsilon \to 0} \epsilon \log \mathbb{E}\Big(\exp\Big[-\frac{g(Y^\epsilon)}{\epsilon} \Big] \Big) = -\inf_{f \in \mathbb{S}} \{g(f) + I(f)\}.$$

To give sufficient conditions such that the LP holds, we also need to introduce some additional notation. Define the Cameron-Martin space $\mathbb{H}$ by

$$\begin{aligned}\mathbb{H} := \Big\{h : [0,T] \mapsto \mathbb{R}^m \Big| h(t) = \int_0^t \dot{h}(s)\mathrm{d}s,\ t \in [0,T], \\ \text{and } \int_0^T |\dot{h}(s)|^2 \mathrm{d}s < \infty\Big\},\end{aligned} \tag{4.2.3}$$

which is a Hilbert space, equipped with the norm $\|f\|_{\mathbb{H}} := (\int_0^T |\dot{f}(s)|^2 \mathrm{d}s)^{\frac{1}{2}}$, where the dot denotes the generalized derivative. Set

$$S_N := \{h \in \mathbb{H} : \|h\|_{\mathbb{H}} \leq N\}, \quad N > 0, \tag{4.2.4}$$

i.e., the ball in $\mathbb{H}$ with radius N, and

$$\begin{aligned}\mathcal{A}_N := \{h : [0,T] \mapsto \mathbb{R}^m | h \text{ is an } \mathcal{F}_t-\text{predictable process such that} \\ h(\cdot,\omega) \in S_N, \mathbb{P}-\text{a.s.}\}.\end{aligned}$$

Remark 4.2.2. By [51, Theorem III.1], S_N is metrizable as a compact Polish space under the weak topology in $\mathbb{H}$.

Let $\{\mathcal{G}^\epsilon : \mathcal{C} := C([0,T];\mathbb{R}^m) \mapsto \mathbb{S}, \epsilon \in (0,1)\}$ be a family of measurable mappings and

$$Z^\epsilon := \mathcal{G}^\epsilon(\sqrt{\epsilon}W). \tag{4.2.5}$$

Assume that there exists a measurable mapping $Z^0 : \mathbb{H} \mapsto \mathbb{S}$ such that

(i) For any $N > 0$, if the family $\{h^\epsilon, \epsilon \in (0,1)\} \subset \mathcal{A}_N$ (as S_N-valued random variables) converge in distribution to an $h \in \mathcal{A}_N$ (as S_N-valued random variable), then $\mathcal{G}^\epsilon(\sqrt{\epsilon}W + h^\epsilon) \to Z^0(h)$ in distribution in $\mathbb{S}$ as $\epsilon \to 0$.

(ii) For any $N > 0$, the set $\mathcal{K}_N := \{Z^0(h) : h \in S_N\}$ is a compact subset of $\mathbb{S}$.

We next recall an equivalent relationship due to [16, Theorem 4.4] between the LDP and the LP whenever the underlying space $\mathbb{S}$ is Polish.

Lemma 4.2.1. Let Z^ϵ be defined by (4.2.5) and assume that $\{\mathcal{G}^\epsilon, \epsilon \in (0,1)\}$ satisfy (i) and (ii). Then the family $\{Z^\epsilon, \epsilon \in (0,1)\}$ satisfy the LP (hence LDP) on $\mathbb{S}$ with the good rate function defined by

$$I(f) := \frac{1}{2} \inf_{\{h \in \mathbb{H}, f = Z^0(h)\}} \|h\|_{\mathbb{H}}^2, \quad f \in \mathbb{S}, \tag{4.2.6}$$

where the infimum over the empty set is taken to be ∞.

Remark 4.2.3. For a Polish metric space $\mathbb{S}$, $\mathcal{K}_N$ is a compact subset of $\mathbb{S}$ if for a sequence $h_n \in \mathbb{S}$ there exists a convergent subsequence $Z^0(h_{n_k})$. On the other hand, the rate function I defined in Lemma 4.2.1 is a good rate function if $\mathcal{K}_N$ is a compact subset of $\mathbb{S}$.

For $t \in [0,T], \epsilon \in (0,1)$, consider the small perturbation of Eq. (4.2.1)

$$\begin{cases} \mathrm{d}[X^\epsilon(t) - G(X_t^\epsilon)] = b(X_t^\epsilon)\mathrm{d}t + \sqrt{\epsilon}\sigma(X_t^\epsilon)\mathrm{d}W(t), \\ X_0^\epsilon = \xi \in \mathscr{C}, \end{cases} \tag{4.2.7}$$

which admits a unique solution $X^\epsilon := \{X^\epsilon(t)\}_{t\in[0,T]}$. By Lemma 4.2.1, to establish the LDP for the law of $\{X^\epsilon, \epsilon \in (0,1)\}$, it is sufficient to choose the Polish space $\mathbb{S}$, construct measurable mappings $\mathcal{G}^\epsilon : \mathcal{C} \mapsto \mathbb{S}$ and $Z^0 : \mathbb{H} \mapsto \mathbb{S}$ respectively, and then show that (i) and (ii) are satisfied for the measurable mapping $\mathcal{G}^\epsilon$.

In the sequel, let $\mathbb{S} := C([0,T];\mathbb{R}^n)$ be the family of continuous functions $f : [0,T] \mapsto \mathbb{R}^n$, which is a Polish space under the uniform topology. By the Yamada-Watanabe theorem there exists a unique measurable functional $\mathcal{G}^\epsilon$: $\mathcal{C} \mapsto \mathbb{S}$ such that

$$X^\epsilon(t) = \mathcal{G}^\epsilon(\sqrt{\epsilon}W)(t), \ t \in [0,T]. \tag{4.2.8}$$

Then, for $h^\epsilon \in \mathcal{A}_N$ (as S_N-valued random variables), by the Girsanov theorem, (4.2.7) and (4.2.8), we conclude that

$$X^{\epsilon,h^\epsilon}(t) := \mathcal{G}^\epsilon(\sqrt{\epsilon}W + h^\epsilon)(t), \quad t \in [0,T],$$

solves the following equation

$$\begin{cases} \mathrm{d}[X^{\epsilon,h^\epsilon}(t) - G(X_t^{\epsilon,h^\epsilon})] = b(X_t^{\epsilon,h^\epsilon})\mathrm{d}t + \sigma(X_t^{\epsilon,h^\epsilon})\dot{h}^\epsilon(t)\mathrm{d}t \\ \qquad\qquad\qquad\qquad\qquad + \sqrt{\epsilon}\sigma(X_t^{\epsilon,h^\epsilon})\mathrm{d}W(t), \\ X_0^{\epsilon,h^\epsilon} = \xi \in \mathscr{C}. \end{cases} \tag{4.2.9}$$

For any $h \in \mathbb{H}$, we also introduce the skeleton equation associated with Eq. (4.2.1)

$$\begin{cases} \mathrm{d}[X^h(t) - G(X_t^h)] = b(X_t^h)\mathrm{d}t + \sigma(X_t^h)\dot{h}(t)\mathrm{d}t, \\ X_0^h = \xi \in \mathscr{C}. \end{cases} \tag{4.2.10}$$

and define

$$X^0(h) := X^h, \quad h \in \mathbb{H}.$$

To verify that $\mathcal{G}^\epsilon$ defined by (4.2.8) satisfies (i) and (ii), we need to prepare the following several auxiliary lemmas. Throughout this chapter, $C > 0$ is a generic constant whose values may change from line to line.

Lemma 4.2.2. Let (H1) and (H2) hold. For $h^\epsilon \in \mathcal{A}_N$ (as S_N-valued random variables), $h \in S_N$ and $p \geq 2$, there exists $C > 0$ such that

$$\sup_{-\tau \leq t \leq T} |X^h(t)|^p \vee \mathbb{E}\Big(\sup_{-\tau \leq t \leq T} |X^{\epsilon,h^\epsilon}(t)|^p \Big) \leq C.$$

Proof. Recall the fundamental inequality: for any $q > 1, \delta > 0$ and $a, b \in \mathbb{R}$

$$|a+b|^q \leq [1 + \delta^{\frac{1}{q-1}}]^{q-1}(|a|^q + |b|^q/\delta), \tag{4.2.11}$$

and set

$$M(t) := X^{\epsilon,h^\epsilon}(t) - G(X_t^{\epsilon,h^\epsilon}), \quad t \in [0,T]. \tag{4.2.12}$$

By the inequality $(a+b+c)^p \leq 3^{p-1}(|a|^p + |b|^p + |c|^p)$ for $a, b, c \in \mathbb{R}$ and (H1), we have

$$|M(t)|^p \leq 3^{p-1}(\alpha^p + (1+\kappa^p)\|X_t^{\epsilon,h^\epsilon}\|_\infty^p). \tag{4.2.13}$$

Applying (4.2.11) and taking into consideration (H1) yields that

$$\begin{aligned}
|X^{\epsilon,h^\epsilon}(s)|^p &\leq (1 + \delta_1^{\frac{1}{p-1}})^{p-1}\Big(\frac{|G(X_s^{\epsilon,h^\epsilon})|^p}{\delta_1} + |M(s)|^p\Big) \\
&\leq (1 + \delta_1^{\frac{1}{p-1}})^{p-1}\Big(\frac{1}{\delta_1}\Big\{(1 + \delta_2^{\frac{1}{p-1}})^{p-1}\Big(\frac{1}{\delta_2}|G(X_s^{\epsilon,h^\epsilon}) - G(0)|^p \\
&\quad + |G(0)|^p\Big)\Big\} + |M(s)|^p\Big) \\
&\leq (1 + \delta_1^{\frac{1}{p-1}})^{p-1}\Big(\frac{1}{\delta_1}\Big\{(1 + \delta_2^{\frac{1}{p-1}})^{p-1}\Big(\frac{\kappa^p}{\delta_2}\|X_s^{\epsilon,h^\epsilon}\|_\infty^p \\
&\quad + |G(0)|^p\Big)\Big\} + |M(s)|^p\Big).
\end{aligned}$$

Letting

$$\delta_1 = \Big(\frac{\kappa^{\frac{1}{2(p-1)}}}{1 - \kappa^{\frac{1}{2(p-1)}}}\Big)^{p-1} \quad \text{and} \quad \delta_2 = \Big(\frac{\kappa}{1-\kappa}\Big)^{p-1},$$

one has

$$|X^{\epsilon,h^\epsilon}(s)|^p \leq C + \sqrt{\kappa}\|X_s^{\epsilon,h^\epsilon}\|_\infty^p + (1 - \kappa^{\frac{1}{2(p-1)}})^{1-p}|M(s)|^p.$$

This further implies that

$$\mathbb{E}\Big(\sup_{-\tau\le s\le t}|X^{\epsilon,h^\epsilon}(t)|^p\Big)\le C+\delta(p)\Big(\sup_{0\le s\le t}|M(s)|^p\Big), \tag{4.2.14}$$

where $\delta(p):=(1-\kappa^{\frac{1}{2(p-1)}})^{1-p}/(1-\sqrt{\kappa})$. Next, by the Itô formula, (H1), (H2) and (4.2.13), we arrive at

$$\begin{aligned}\mathbb{E}\Big(\sup_{0\le t\le T}|M(t)|^p\Big)\le C+C\mathbb{E}\int_0^T\|X_t^{\epsilon,h^\epsilon}\|_\infty^p\mathrm{d}t\\ +\mathbb{E}I_1(T)+\mathbb{E}\Big(\sup_{0\le t\le T}|I_2(t)|\Big),\end{aligned} \tag{4.2.15}$$

where

$$I_1(T):=p\int_0^T|M(s)|^{p-2}|\langle M(s),\sigma(X_s^{\epsilon,h^\epsilon})\dot{h}^\epsilon(s)\rangle|\mathrm{d}s,$$

and

$$I_2(t):=p\int_0^t|M(s)|^{p-2}\langle M(s),\sigma(X_s^{\epsilon,h^\epsilon})\mathrm{d}W(s)\rangle.$$

Note from (4.2.2), (4.2.13) with p replaced by $p-1$, the Hölder inequality and the Young inequality that

$$\begin{aligned}\mathbb{E}I_1(T)&\le CN+C\mathbb{E}\Big(\int_0^T\|X_t^{\epsilon,h^\epsilon}\|_\infty^{2p}\mathrm{d}t\Big)^{\frac12}\mathbb{E}\Big(\int_0^T|\dot{h}^\epsilon(t)|^2\mathrm{d}t\Big)^{\frac12}\\ &\le CN+CN\mathbb{E}\Big(\sup_{-\tau\le t\le T}|X^{\epsilon,h^\epsilon}(t)|^p\int_0^T\sup_{-\tau\le s\le t}|X^{\epsilon,h^\epsilon}(s)|^p\mathrm{d}t\Big)^{\frac12}\\ &\le CN+\frac{1}{4\delta(p)}\mathbb{E}\Big(\sup_{-\tau\le t\le T}|X^{\epsilon,h^\epsilon}(t)|^p\Big)\\ &\quad+C^2N^2\delta(p)\int_0^T\mathbb{E}\Big(\sup_{-\tau\le s\le t}|X^{\epsilon,h^\epsilon}(s)|^p\Big)\mathrm{d}t,\end{aligned} \tag{4.2.16}$$

where we have used that $h^\epsilon\in S_N$, and by the Burkhold-Davis-Gundy inequality that

$$\begin{aligned}\mathbb{E}\Big(\sup_{0\le t\le T}|I_2(t)|\Big)&\le C\mathbb{E}\Big(\int_0^t|M(s)|^{2(p-1)}\|\sigma(X_s^{\epsilon,h^\epsilon})\|_{HS}^2\mathrm{d}s\Big)^{\frac12}\\ &\le C\mathbb{E}\Big(\sup_{0\le t\le T}|M(s)|^p\int_0^t|M(s)|^{p-2}\|\sigma(X_s^{\epsilon,h^\epsilon})\|_{HS}^2\mathrm{d}s\Big)^{\frac12}\\ &\le C+\frac12\mathbb{E}\Big(\sup_{0\le t\le T}|M(t)|^p\Big)\\ &\quad+C\int_0^T\mathbb{E}\Big(\sup_{-\tau\le s\le t}|X^{\epsilon,h^\epsilon}(s)|^p\Big)\mathrm{d}t.\end{aligned} \tag{4.2.17}$$

Substituting (4.2.16) and (4.2.17) into (4.2.15) leads to

$$\mathbb{E}\Big(\sup_{0\le t\le T}|M(t)|^p\Big) \le C + C\int_0^T \mathbb{E}\Big(\sup_{-\tau\le s\le t}|X^{\epsilon,h^\epsilon}(s)|^p\Big)\mathrm{d}t + \frac{1}{2\delta(p)}\mathbb{E}\Big(\sup_{-\tau\le t\le T}|X^{\epsilon,h^\epsilon}(t)|^p\Big).$$

Combining this with (4.2.14), we obtain that

$$\mathbb{E}\Big(\sup_{-\tau\le t\le T}|X^{\epsilon,h^\epsilon}(t)|^p\Big) \le C + C\int_0^T \mathbb{E}\Big(\sup_{-\tau\le s\le t}|X^{\epsilon,h^\epsilon}(s)|^p\Big)\mathrm{d}t.$$

Thus the desired assertion follows from the Gronwall inequality, and the uniform estimate of $X^h(t)$ can be obtained similarly. □

Lemma 4.2.3. Under (H1) and (H2), the set $\mathcal{K}_N := \{X^0(h) : h \in S_N\}$ is a compact subset of $\mathbb{S}$.

Proof. For any $h_n \in S_N$, there exists a subsequence, still denoted by h_n, which converges weakly in $\mathbb{H}$ to some $h \in S_N$, i.e., $\int_0^T |\dot{h}_n(s) - \dot{h}(s)|^2 \mathrm{d}s \to 0$, due to the fact that S_N is weakly compact by Remark 1.1. Denote by X^{h_n} and X^h the solutions of Eq. (4.2.10) with the controls $h_n \in S_N$ and $h \in S_N$ respectively. In view of Remark 4.2.3 it is sufficient to show that $X^{h_n} \to X^h$ in $\mathbb{S}$. Let

$$\bar{M}(t) := X^{h_n}(t) - X^h(t) - (G(X_t^{h_n}) - G(X_t^h)), \quad t \in [0,T]. \tag{4.2.18}$$

Note from (4.2.11), (H1) and $X_0^{h_n} = X_0^h$ that

$$\sup_{0\le t\le T}|X^{h_n}(t) - X^h(t)|^2 \le \frac{1}{(1-\kappa)^2}\sup_{0\le t\le T}|\bar{M}(t)|^2 \tag{4.2.19}$$

and

$$|\bar{M}(t)| \le |X^{h_n}(t) - X^h(t)| + |G(X_t^{h_n}) - G(X_t^h)| \le (1+\kappa)\|X_t^{h_n} - X_t^h\|_\infty. \tag{4.2.20}$$

From the chain rule, (H1), (H2), (4.2.20) and Lemma 4.2.2, we derive by following the argument of (4.2.16) that

$$
\begin{aligned}
|\bar{M}(t)|^2 &= 2\int_0^t \langle \bar{M}(s), b(X_s^{h_n}) - b(X_s^h)\rangle \mathrm{d}s \\
&\quad + 2\int_0^t \langle \bar{M}(s), \sigma(X_s^{h_n})(\dot{h}_n(s) - \dot{h}(s)) + (\sigma(X_s^{h_n}) - \sigma(X_s^h))\dot{h}(s)\rangle \mathrm{d}s \\
&\leq C\int_0^t \sup_{0\leq r\leq s} |X^{h_n}(r) - X^h(r)|^2 \mathrm{d}s \\
&\quad + C\int_0^t \sup_{0\leq r\leq s} |X^{h_n}(r) - X^h(r)||\dot{h}_n(s) - \dot{h}(s)|\mathrm{d}s \\
&\quad + C\int_0^t \sup_{0\leq r\leq s} |X^{h_n}(r) - X^h(r)|^2|\dot{h}(s)|\mathrm{d}s \\
&\leq C\int_0^t \sup_{0\leq r\leq s} |X^{h_n}(r) - X^h(r)|^2 \mathrm{d}s + \int_0^t |\dot{h}_n(s) - \dot{h}(s)|^2 \mathrm{d}s \\
&\quad + CN\Big(\int_0^t \sup_{0\leq r\leq s} |X^{h_n}(r) - X^h(r)|^4 \mathrm{d}s\Big)^{\frac{1}{2}} \\
&\leq C\int_0^t \sup_{0\leq r\leq s} |X^{h_n}(r) - X^h(r)|^2 \mathrm{d}s + \int_0^t |\dot{h}_n(s) - \dot{h}(s)|^2 \mathrm{d}s \\
&\quad + \frac{(1-\kappa)^2}{2} \sup_{0\leq s\leq t} |X^{h_n}(s) - X^h(s)|^2,
\end{aligned}
$$

where $X_0^{h_n} = X_0^h$ and $h \in S_N$ have also been utilized. Putting this into (4.2.19) leads to

$$
\sup_{0\leq t\leq T} |X^{h_n}(t) - X^h(t)|^2 \leq C\int_0^T \sup_{0\leq r\leq s} |X^{h_n}(r) - X^h(r)|^2 \mathrm{d}s + C\int_0^T |\dot{h}_n(s) - \dot{h}(s)|^2 \mathrm{d}s.
$$

Then the Gronwall inequality gives that

$$
\sup_{0\leq t\leq T} |X^{h_n}(t) - X^h(t)|^2 \leq C\int_0^T |\dot{h}_n(t) - \dot{h}(t)|^2 \mathrm{d}t,
$$

and the desired assertion follows from that h^n converges weakly to $h \in S_N$ in $\mathbb{H}$. □

Lemma 4.2.4. Let (H1) and (H2) hold. Assume further that the family $\{h^\epsilon, \epsilon \in (0,1)\} \subset \mathcal{A}_N$ (as S_N-valued random variables) converge almost surely in $\mathbb{H}$ to $h \in \mathcal{A}_N$ (as S_N-valued random variable). Then $X^{\epsilon,h^\epsilon} \to X^h$ converges in distribution in $\mathbb{S}$.

Proof. It is sufficient to show that $X^{\epsilon,h^\epsilon} \to X^h$ in probability in $\mathbb{S}$ since convergence in probability implies convergence in distribution. Let $\bar{M}(t)$ be defined by

(4.2.18) with X^{ϵ,h^ϵ} replaced by X^{h_n}. Note that (4.2.19) and (4.2.20) still hold with X^{h_n} replaced by X^{ϵ,h^ϵ}. Applying the Itô formula, we have

$$\begin{aligned}
|\bar{M}(t)|^2 &= 2\int_0^t \langle \bar{M}(s), b(X_s^{\epsilon,h^\epsilon}) - b(X_s^h)\rangle \mathrm{d}s \\
&\quad + 2\int_0^t \langle \bar{M}(s), \sigma(X_s^{\epsilon,h^\epsilon})(\dot{h}^\epsilon(s) - \dot{h}(s))\rangle \mathrm{d}s \\
&\quad + 2\int_0^t \langle \bar{M}(s), (\sigma(X_s^{\epsilon,h^\epsilon}) - \sigma(X_s^h))\dot{h}(s)\rangle \mathrm{d}s \qquad (4.2.21)\\
&\quad + \epsilon\int_0^t \|\sigma(X_s^{\epsilon,h^\epsilon})\|_{HS}^2 \mathrm{d}s + 2\sqrt{\epsilon}\int_0^t \langle \bar{M}(s), \sigma(X_s^{\epsilon,h^\epsilon})\mathrm{d}W(s)\rangle \\
&=: J_1(t) + J_2(t) + J_3(t) + J_4(t) + J_5(t).
\end{aligned}$$

In light of (H2)

$$\sup_{t\in[0,T]} |J_1(t)| \le C\int_0^T \|X_t^{\epsilon,h^\epsilon} - X_t^h\|_\infty^2 \mathrm{d}t, \qquad (4.2.22)$$

and, by the Hölder inequality and the Young inequality, for $h \in S_N$ one has

$$\begin{aligned}
\sup_{t\in[0,T]} |J_3(t)| &\le C\Big(\int_0^T \|X_t^{\epsilon,h^\epsilon} - X_t^h\|_\infty^4 \mathrm{d}t\Big)^{\frac{1}{2}}\Big(\int_0^T |\dot{h}(s)|^2 \mathrm{d}s\Big)^{\frac{1}{2}} \\
&\le \frac{(1-\kappa)^2}{2}\sup_{0\le t\le T} |X^{\epsilon,h^\epsilon}(t) - X^h(t)|^2 \qquad (4.2.23)\\
&\quad + C\int_0^T |X^{\epsilon,h^\epsilon}(t) - X^h(t)|^2 \mathrm{d}t.
\end{aligned}$$

Substituting (4.2.22) and (4.2.23) into (4.2.21) and recalling (4.2.19), we deduce from the Gronwall inequality that

$$\sup_{0\le t\le T} |X^{\epsilon,h^\epsilon}(t) - X^h(t)|^2 \le C\Big\{\sup_{t\in[0,T]} |J_2(t)| + \sup_{t\in[0,T]} |J_4(t)| + \sup_{t\in[0,T]} |J_5(t)|\Big\}.$$

Observe from Lemma 4.2.2, (4.2.2), (4.2.20) and the Burkhold-Davis-Gundy inequality that

$$\mathbb{E}\Big(\sup_{t\in[0,T]} |J_4(t)| + \sup_{t\in[0,T]} |J_5(t)|\Big) \to 0 \text{ as } \epsilon \downarrow 0.$$

Therefore

$$\sup_{t\in[0,T]} |J_4(t)| + \sup_{t\in[0,T]} |J_5(t)| \to 0 \text{ in probability as } \epsilon \downarrow 0.$$

To obtain the desired assertion, it is sufficient to show that

$$\sup_{t\in[0,T]} |J_2(t)| \to 0 \text{ in probability as } \epsilon \downarrow 0. \qquad (4.2.24)$$

For any $\delta > 0$, notice that

$$\begin{aligned}\mathbb{P}\Big(\sup_{t\in[0,T]}|J_2(t)|\geq\delta\Big)&=\mathbb{P}\Big(\sup_{t\in[0,T]}\Big|2\int_0^t\langle\bar{M}(s),\sigma(X_s^{\epsilon,h^\epsilon})(\dot{h^\epsilon}(s)-\dot{h}(s))\rangle\mathrm{d}s\Big|\geq\delta\Big)\\&\leq\mathbb{P}\Big(\int_0^T|\bar{M}(s)|\cdot\|\sigma(X_s^{\epsilon,h^\epsilon})\|_{HS}|\dot{h^\epsilon}(s)-\dot{h}(s)|\mathrm{d}s\geq\frac{\delta}{2}\Big)\\&\leq\mathbb{P}\Big(\Lambda(T)\int_0^T|\dot{h^\epsilon}(s)-\dot{h}(s)|^2\mathrm{d}s\geq\frac{\delta^2}{4}\Big),\end{aligned}$$

where $\Lambda(T):=\int_0^T(|\bar{M}(s)|\|\sigma(X_s^{\epsilon,h^\epsilon})\|_{HS})^2\mathrm{d}s$. On the other hand, for arbitrary $R>0$

$$\begin{aligned}&\mathbb{P}\Big(\Lambda(T)\int_0^T|\dot{h^\epsilon}(s)-\dot{h}(s)|^2\mathrm{d}s\geq\frac{\delta^2}{4}\Big)\\&\leq\mathbb{P}\Big(\int_0^T|\dot{h^\epsilon}(s)-\dot{h}(s)|^2\mathrm{d}s\geq\frac{\delta^2}{4R},\Lambda(T)\leq R\Big)+\mathbb{P}\Big(\Lambda(T)>R\Big)\\&\leq\mathbb{P}\Big(\int_0^T|\dot{h^\epsilon}(s)-\dot{h}(s)|^2\mathrm{d}s\geq\frac{\delta^2}{4R}\Big)+\mathbb{P}\Big(\Lambda(T)>R\Big).\end{aligned}$$

By (4.2.2), (4.2.20) and Lemma 4.2.2, it is easy to see that

$$\mathbb{E}\Lambda(T)\leq C\mathbb{E}\int_0^T\{\|X_t^{\epsilon,h^\epsilon}-X_t^h\|_\infty^2(\|X_s^{\epsilon,h^\epsilon}\|_\infty^2+1)\}\mathrm{d}s\leq C.$$

Next, using the Chebyshev inequality, for any $\bar{\epsilon}\in(0,1)$ we choose $R_0>0$ sufficiently large such that

$$\mathbb{P}\Big(\Lambda(T)>R_0\Big)\leq\frac{1}{R_0}\mathbb{E}\Lambda(T)\leq\frac{C}{R_0}\leq\bar{\epsilon}.$$

For fixed R_0, due to $h^\epsilon\to h$ a.s. in $\mathbb{H}$, it then follows that

$$\mathbb{P}\Big(\int_0^T|\dot{h^\epsilon}(s)-\dot{h}(s)|^2\mathrm{d}s\geq\frac{\delta^2}{4R_0}\Big)\to0\text{ as }\epsilon\downarrow0.$$

Consequently, (4.2.24) holds due to the arbitrariness of $\bar{\epsilon}\in(0,1)$. □

We now state our first main results.

Theorem 4.2.5. Under (H1) and (H2), X^ϵ satisfies the LDP on $\mathbb{S}$ with the good rate function $I(f)$ defined by (4.2.6), where $X^0(h)$ solves Eq. (4.2.10).

Proof. The proof is standard while we outline the argument for the sake of completeness. By Lemma 4.2.1 it is sufficient to show that (i) (in Lemma 4.2.1) holds since (ii) is true by Lemma 4.2.3. Assume that $\{h^\epsilon,\epsilon\in(0,1)\}\subset\mathcal{A}_N$

(as S_N-valued random variables) converge in distribution to an $h \in \mathcal{A}_N$ (as an S_N-valued random variable). Noting that $\{h^\epsilon, W\}$ is tight in $S_N \times C$ by recalling that S_N is compact from Remark 1.1 and the law of W is tight, we assume that the law of $\{h^\epsilon, W\}$ converges weakly to some μ on $S_N \times C$. Thus by the Skorohod representation theorem [19, Theorem 2.4, p33] there exist a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and stochastic processes $\{\tilde{h}^\epsilon, \tilde{W}^\epsilon\}$ and $\{\tilde{h}, \tilde{W}\}$ such that

$$\mathscr{L}(\{\tilde{h}^\epsilon, \tilde{W}^\epsilon\}) = \mathscr{L}(\{h^\epsilon, W\}), \quad \mathscr{L}(\{\tilde{h}, \tilde{W}\}) = \mu \tag{4.2.25}$$

and

$$\{\tilde{h}^\epsilon, \tilde{W}^\epsilon\} \to \{\tilde{h}, \tilde{W}\}, \quad \tilde{\mathbb{P}} - \text{a.s.} \tag{4.2.26}$$

Note that Lemma 4.2.4 still holds when W and h in Eq. (4.2.9) are replaced by $\tilde{W}$ and $\tilde{h}$ respectively. Hence we have

$$\mathcal{G}^\epsilon(\sqrt{\epsilon}\tilde{W} + \tilde{h}^\epsilon) \to X^{\tilde{h}} \text{ in distribution} \tag{4.2.27}$$

due to the fact that $\tilde{h}^\epsilon \to \tilde{h}$, $\tilde{\mathbb{P}}$-a.s. by (4.2.26). Moreover observing that $\{h^\epsilon, \epsilon \in (0,1)\}$ converge weakly to h and the law of $\{h^\epsilon, W\}$ converges weakly to some μ, we obtain from (4.2.25) that

$$\mathscr{L}(h) = \mathscr{L}(\tilde{h}) = \mu(\cdot, C). \tag{4.2.28}$$

Combining (4.2.25), (4.2.27) with (4.2.28) implies (i), as required. □

Remark 4.2.4. Applying a time-discretization argument, Mohammed and Zhang [60] establish a LDP for stochastic systems with constant delay. For functional differential equations, time-discretization schemes, even the simplest EM scheme, are relatively complicated, which bring a lot of troubles for the exponential-type estimates. While this has been avoided by the weak convergence approach.

4.3 LDP for Neutral SDDEs

In this section we discuss the LDP for a class of neutral SDDEs. Consider the following equation on $\mathbb{R}$:

$$\begin{cases} \mathrm{d}[Y(t) - 2Y^2(t-\tau)] = [\sin(Y(t)) + Y(t-\tau)]\mathrm{d}t \\ \qquad\qquad\qquad\qquad\qquad + Y^3(t-\tau)\mathrm{d}W(t), \\ Y(\theta) = \xi(\theta), \quad \theta \in [-\tau, 0], \end{cases} \tag{4.3.1}$$

where $W(t)$ is a scalar Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Since $G(\xi) = 2\xi^2(-\tau)$ and $\sigma(\xi) = 3\xi^3(-\tau)$ for $\xi \in \mathscr{C}$ do not satisfy (H1) and (H2), we can not apply Theorem 4.2.5 to Eq. (4.3.1), although which has a unique solution (see Remark 4.3.3). In order for Eq. (4.3.1) satisfies LDP, we shall establish a new theorem.

Consider a neutral SDDE on $\mathbb{R}^n$

$$\begin{cases} \mathrm{d}[Y(t) - G(Y(t-\tau))] = b(Y(t), Y(t-\tau))\mathrm{d}t \\ \qquad\qquad\qquad\qquad\qquad + \sigma(Y(t), Y(t-\tau))\mathrm{d}W(t), \\ Y(\theta) = \xi(\theta), \quad \theta \in [-\tau, 0], \end{cases} \tag{4.3.2}$$

where $G : \mathbb{R}^n \to \mathbb{R}^n$, $b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ $\sigma : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $W(t)$ is an $m-$dimensional Brownian motion. Assume that there exist $\lambda_3, \lambda_4 > 0$ such that

(A1) $|G(x) - G(y)| \le \lambda_3 V_1(x,y)|x-y|,\ x, y \in \mathbb{R}^n$.

(A2) For $x_i, y_i \in \mathbb{R}^n, i = 1, 2,$

$$\begin{aligned} &|b(x_1,y_1) - b(x_2,y_2)| \vee \|\sigma(x_1,y_1) - \sigma(x_2,y_2)\|_{HS} \\ &\le \lambda_4(|x_1 - x_2| + V_2(y_1,y_2)|y_1 - y_2|), \end{aligned}$$

where $V_i : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}_+$ such that

$$V_i(x,y) \le \lambda_i(1 + |x|^{q_i} + |y|^{q_i}), \quad x, y \in \mathbb{R}^n$$

for some $\lambda_i > 0$ and $q_i \ge 1, i = 1, 2.$

Remark 4.3.1. Eq. (4.3.1) satisfies (A1) and (A2).

Remark 4.3.2. By (A1) and (A2), there exists $C > 0$ such that

$$|G(x)| \le C(1 + |x|^{q_1+1}), \quad x \in \mathbb{R}^n,$$

and

$$|b(x, y)| + \|\sigma(x, y)\|_{HS} \le C(1 + |x| + |y|^{q_2+1}), \quad x, y \in \mathbb{R}^n.$$

Remark 4.3.3. For $t \in [0, \tau]$, Eq. (4.3.2) reduces to stochastic integral equation

$$Y(t) = \xi(0) + G(\xi(t - \tau)) - G(\xi(-\tau)) + \int_0^t b(Y(s), \xi(s - \tau))\mathrm{d}s + \int_0^t \sigma(Y(s), \xi(s - \tau))\mathrm{d}W(s).$$

Let $\tilde{Y}(t) := Y(t) - G(\xi(t - \tau))$. Then the previous integral equation can be rewritten as

$$\tilde{Y}(t) = Y(0) + \int_0^t b(\tilde{Y}(s) + G(\xi(s - \tau)), \xi(s - \tau))\mathrm{d}s + \int_0^t \sigma(\tilde{Y}(s) + G(\xi(s - \tau)), \xi(s - \tau))\mathrm{d}W(s).$$

This is an SDE without the delay argument and the neutral term, and (A2) guarantees the existence and uniqueness of the solution $\{\tilde{Y}(t)\}_{t\in[0,\tau]}$ such that $\mathbb{E}\Big(\sup_{0\le t\le\tau} |\tilde{Y}(t)|^q\Big) < \infty$ for any $q \ge 1$. Repeating this argument we deduce that Eq. (4.3.2) admits a unique global solution on $[0, T]$.

For $\epsilon \in (0, 1)$ consider the small perturbation of Eq. (4.3.2)

$$\begin{cases} \mathrm{d}[Y^\epsilon(t) - G(Y^\epsilon(t - \tau))] = b(Y^\epsilon(t), Y^\epsilon(t - \tau))\mathrm{d}t \\ \qquad\qquad + \sqrt{\epsilon}\sigma(Y^\epsilon(t), Y^\epsilon(t - \tau))\mathrm{d}W(t), \\ Y^\epsilon(\theta) = \xi(\theta), \quad \theta \in [-\tau, 0]. \end{cases} \tag{4.3.3}$$

By the Yamada-Watanabe theorem there exists a unique measurable functional $\mathcal{G}^\epsilon : \mathcal{C} \mapsto \mathbb{S}$ such that

$$Y^\epsilon(t) = \mathcal{G}^\epsilon(\sqrt{\epsilon}W)(t), \quad t \in [0, T]. \tag{4.3.4}$$

Then, for $h^\epsilon \in \mathcal{A}_N$ (as S_N-valued random variables), by the Girsanov theorem, (4.3.3) and (4.3.4)

$$Y^{\epsilon,h^\epsilon}(t) := \mathcal{G}^\epsilon(\sqrt{\epsilon}W + h^\epsilon)(t), \quad t \in [0, T]$$

solves the following equation

$$\begin{cases} \mathrm{d}[Y^{\epsilon,h^\epsilon}(t) - G(Y^{\epsilon,h^\epsilon}(t-\tau))] = b(Y^{\epsilon,h^\epsilon}(t), Y^{\epsilon,h^\epsilon}(t-\tau))\mathrm{d}t \\ \qquad\qquad +\sigma(Y^{\epsilon,h^\epsilon}(t), Y^{\epsilon,h^\epsilon}(t-\tau))\dot{h^\epsilon}(t)\mathrm{d}t \\ \qquad\qquad +\sqrt{\epsilon}\sigma(Y^{\epsilon,h^\epsilon}(t), Y^{\epsilon,h^\epsilon}(t-\tau))\mathrm{d}W(t), \\ Y^{\epsilon,h^\epsilon}(\theta) = \xi(\theta), \quad \theta \in [-\tau, 0]. \end{cases}$$

For any $h \in \mathbb{H}$, we introduce the skeleton equation associated with Eq. (4.3.2)

$$\begin{cases} \mathrm{d}[Y^h(t) - G(Y^h(t-\tau))] = b(Y^h(t), Y^h(t-\tau))\mathrm{d}t \\ \qquad\qquad +\sigma(Y^h(t), Y^h(t-\tau))\dot{h}(t)\mathrm{d}t, \\ Y^h(\theta) = \xi(\theta), \quad \theta \in [-\tau, 0]. \end{cases} \tag{4.3.5}$$

Define

$$Y^0(h) := Y^h, \quad h \in \mathbb{H}.$$

Following the argument of that of Theorem 4.2.5, we need to prepare the following lemmas.

Lemma 4.3.1. Let (A1) and (A2) hold. For any $h^\epsilon \in \mathcal{A}_N$ (as S_N-valued random variables), $h \in S_N$ and $p \geq 2$, there exists $C > 0$ such that

$$\sup_{-\tau \leq s \leq T} |Y^h(s)|^p \vee \mathbb{E}\Big(\sup_{-\tau \leq s \leq T} |Y^{\epsilon,h^\epsilon}(s)|^p\Big) \leq C.$$

Proof. Let $q := (q_1+1)\vee(q_2+1)$ and $M(t) := Y^{\epsilon,h^\epsilon}(t) - G(Y^{\epsilon,h^\epsilon}(t-\tau)), t \in [0,T]$. Applying the Itô formula, together with the Hölder inequality and the Young inequality, we obtain that

$$\begin{aligned} |M(t)|^p \leq & |M(0)|^p + C\int_0^t |M(s)|^p \mathrm{d}s \\ & + C\int_0^t \{|b(Y^{\epsilon,h^\epsilon}(s), Y^{\epsilon,h^\epsilon}(s-\tau))|^p \\ & + \|\sigma(Y^{\epsilon,h^\epsilon}(s), Y^{\epsilon,h^\epsilon}(s-\tau))\|_{HS}^p\}\mathrm{d}s \\ & + CN\Big(\int_0^t |M(s)|^{2(p-1)}\|\sigma(Y^{\epsilon,h^\epsilon}(s), Y^{\epsilon,h^\epsilon}(s-\tau))\|_{HS}^2 \mathrm{d}s\Big)^{\frac{1}{2}} \\ & + p\sqrt{\epsilon}\int_0^t |M(s)|^{p-2}\langle M(s), \sigma(Y^{\epsilon,h^\epsilon}(s), Y^{\epsilon,h^\epsilon}(s-\tau))\mathrm{d}W(s)\rangle, \end{aligned} \tag{4.3.6}$$

where we have also used $h^\epsilon \in \mathcal{A}_N$ (as S_N-valued random variables). Note from Remark 4.3.2 that

$$\begin{aligned}\sup_{0\le s\le t}|Y^{\epsilon,h^\epsilon}(s)|^p &\le 2^{p-1}\sup_{0\le s\le t}\Big(|M(s)|^p+|G(Y^{\epsilon,h^\epsilon}(s-\tau))|^p\Big)\\ &\le C\Big(1+\|\xi\|_\infty^{p(q_1+1)}+\sup_{0\le s\le t}|M(s)|^p \\ &\quad +\sup_{0\le s\le t-\tau}|Y^{\epsilon,h^\epsilon}(s)|^{p(q_1+1)}\Big).\end{aligned} \tag{4.3.7}$$

Thus, by (4.3.6), together with Remark 4.3.2, we derive from the Burkhold-Davis-Gundy inequality and the Young inequality that

$$\begin{aligned}\mathbb{E}\Big(\sup_{0\le s\le t}|M(t)|^p\Big) &\le \frac{1}{2}\mathbb{E}\Big(\sup_{0\le s\le t}|M(t)|^p\Big)+C\Big\{1+\mathbb{E}\int_0^t|Y^{\epsilon,h^\epsilon}(s)|^p\mathrm{d}s\\ &\quad +\mathbb{E}\int_0^t|Y^{\epsilon,h^\epsilon}(s-\tau)|^{p(q_1+1)}\mathrm{d}s+\mathbb{E}\int_0^t|Y^{\epsilon,h^\epsilon}(s-\tau)|^{p(q_2+1)}\mathrm{d}s\Big\},\end{aligned}$$

that is,

$$\begin{aligned}\mathbb{E}\Big(\sup_{0\le s\le t}|M(t)|^p\Big) &\le 2C\Big\{1+\mathbb{E}\int_0^t|Y^{\epsilon,h^\epsilon}(s)|^p\mathrm{d}s\\ &\quad +\mathbb{E}\int_0^t|Y^{\epsilon,h^\epsilon}(s-\tau)|^{p(q_1+1)}\mathrm{d}s+\mathbb{E}\int_0^t|Y^{\epsilon,h^\epsilon}(s-\tau)|^{p(q_2+1)}\mathrm{d}s\Big\}.\end{aligned}$$

This, together with (4.3.7), yields that

$$\mathbb{E}\Big(\sup_{0\le s\le t}|Y^{\epsilon,h^\epsilon}(s)|^p\Big)\le C\Big\{1+\mathbb{E}\Big(\sup_{0\le s\le t-\tau}|Y^{\epsilon,h^\epsilon}(s)|^{pq}\Big)+\int_0^t\mathbb{E}|Y^{\epsilon,h^\epsilon}(s)|^p\mathrm{d}s\Big\}.$$

Thus, by the Gronwall inequality, one has

$$\mathbb{E}\Big(\sup_{0\le s\le t}|Y^{\epsilon,h^\epsilon}(s)|^p\Big)\le C\Big\{1+\mathbb{E}\Big(\sup_{0\le s\le t-\tau}|Y^{\epsilon,h^\epsilon}(s)|^{pq}\Big)\Big\}. \tag{4.3.8}$$

In particular, for any $m\ge 1$ sufficiently large

$$\mathbb{E}\Big(\sup_{0\le s\le \tau}|Y^{\epsilon,h^\epsilon}(s)|^{pq^m}\Big)\le C,$$

due to the arbitrariness of $p\ge 2$ in (4.3.8). This further implies that

$$\mathbb{E}\Big(\sup_{0\le s\le 2\tau}|Y^{\epsilon,h^\epsilon}(s)|^{pq^{m-1}}\Big)\le C\Big\{1+\mathbb{E}\Big(\sup_{0\le s\le \tau}|Y^{\epsilon,h^\epsilon}(s)|^{pq^m}\Big)\Big\}\le C.$$

Consequently, the desired assertion follows from an induction argument. □

Lemma 4.3.2. Under (A1) and (A2), $\mathcal{K}_N := \{Y^0(h) : h \in S_N\}$ is a compact subset of $\mathbb{S}$.

Proof. It is sufficient to show $Y^{h_n} \to Y^h$ in $\mathbb{S}$, where Y^{h_n} and Y^h are the solutions of Eq. (4.3.5) with the controls $h_n \in S_N$ and $h \in S_N$ respectively. Let

$$\bar{M}(t) := (Y^{h_n}(t) - Y^h(t)) - (G(Y^{h_n}(t-\tau)) - G(Y^h(t-\tau))), \quad t \in [0,T].$$

By (A1), Lemma 4.3.1 and the property of V_1, we arrive at

$$\begin{aligned}
&\sup_{0\le s\le t} |Y^{h_n}(s) - Y^h(s)|^2 \\
&\le 2\Big\{\sup_{0\le s\le t} |\bar{M}(s)|^2 + \sup_{0\le s\le T-\tau} V_1^2(Y^{h_n}(t), Y^h(t)) \\
&\quad \times \sup_{0\le s\le t-\tau} |Y^{h_n}(s) - Y^h(s)|^2\Big\} \\
&\le C\Big\{\sup_{0\le s\le t} |\bar{M}(s)|^2 + \sup_{0\le s\le t-\tau} |Y^{h_n}(s) - Y^h(s)|^2\Big\}.
\end{aligned} \tag{4.3.9}$$

Applying the chain rule and Lemma 4.3.1, and using (A1) and (A2), we derive that

$$
\begin{aligned}
&|\bar{M}(t)|^2\\
&= 2\int_0^t \langle \bar{M}(s), b(Y^{h_n}(s), Y^{h_n}(s-\tau)) - b(Y^h(s), Y^h(s-\tau))\rangle \mathrm{d}s\\
&\quad + 2\int_0^t \langle \bar{M}(s), \sigma(Y^{h_n}(s), Y^{h_n}(s-\tau))(\dot{h_n}(s) - \dot{h}(s))\\
&\quad + (\sigma(Y^{h_n}(s), Y^{h_n}(s-\tau)) - \sigma(Y^h(s), Y^h(s-\tau)))\dot{h}(s)\rangle \mathrm{d}s\\
&\le C\int_0^t \{(|Y^{h_n}(s) - Y^h(s)| + V_1(Y^{h_n}(s-\tau), Y^h(s-\tau))|Y^{h_n}(s-\tau) - Y^h(s-\tau)|)\\
&\quad \times (|Y^{h_n}(s) - Y^h(s)| + V_2(Y^{h_n}(s-\tau), Y^h(s-\tau))|Y^{h_n}(s-\tau) - Y^h(s-\tau)|)\}\mathrm{d}s\\
&\quad + C\int_0^t (|Y^{h_n}(s) - Y^h(s)| + V_1(Y^{h_n}(s-\tau), Y^h(s-\tau))|Y^{h_n}(s-\tau) - Y^h(s-\tau)|)\\
&\quad \times \{(1 + |Y^{h_n}(s)| + |Y^{h_n}(s-\tau)|^{q_2+1})|\dot{h_n}(s) - \dot{h}(s)| + (|Y^{h_n}(s) - Y^h(s)| +\\
&\quad + V_2(Y^{h_n}(s-\tau), Y^h(s-\tau))|Y^{h_n}(s-\tau) - Y^h(s-\tau)|)|\dot{h}(s)|\}\mathrm{d}s\\
&\le C\int_0^t \sup_{0\le r\le s}|Y^{h_n}(r) - Y^h(r)|^2\mathrm{d}s + C\int_0^t \sup_{0\le r\le s}|Y^{h_n}(r) - Y^h(r)|\cdot|\dot{h_n}(s) - \dot{h}(s)|\mathrm{d}s\\
&\quad + C\int_0^t \sup_{0\le r\le s}|Y^{h_n}(r) - Y^h(r)|^2\cdot|\dot{h}(s)|\mathrm{d}s\\
&\le C\int_0^t \sup_{0\le r\le s}|Y^{h_n}(r) - Y^h(r)|^2\mathrm{d}s + \int_0^t |\dot{h_n}(s) - \dot{h}(s)|^2\mathrm{d}s\\
&\quad + \delta \sup_{0\le s\le t}|Y^{h_n}(s) - Y^h(s)|^2,
\end{aligned}
$$

where $\delta \in (0,1)$ is some constant sufficiently small such that $C\delta \in (0,1)$ with $C > 0$ appearing in (4.3.9). Substituting this into (4.3.9) gives that

$$
\begin{aligned}
\sup_{0\le s\le t}|Y^{h_n}(t) - Y^h(t)|^2 \le C\Big\{\sup_{0\le s\le t-\tau}|Y^{h_n}(s) - Y^h(s)|^2 + \int_0^t |\dot{h_n}(s) - \dot{h}(s)|^2\mathrm{d}s\Big\}\\
+ C\int_0^t \sup_{0\le r\le s}|Y^{h_n}(r) - Y^h(r)|^2\mathrm{d}s.
\end{aligned}
$$

Then the Gronwall inequality gives that

$$
\sup_{0\le s\le t}|Y^{h_n}(s) - Y^h(s)|^2 \le C\Big\{\sup_{0\le s\le t-\tau}|Y^{h_n}(s) - Y^h(s)|^2 + \int_0^T |\dot{h_n}(s) - \dot{h}(s)|^2\mathrm{d}s\Big\},
$$

and the desired assertion follows by an induction argument and noting that h^n converges weakly in $\mathbb{H}$ to $h \in S_N$. □

Carrying out similar arguments to those of Lemma 4.2.4 and Lemma 4.3.2 we can also deduce the following result.

Lemma 4.3.3. Assume that the family $\{h^\epsilon, \epsilon \in (0,1)\} \subset \mathcal{A}_N$ (as S_N-valued random variables) converge almost surely in $\mathbb{H}$ to $h \in \mathcal{A}_N$ (as S_N-valued random variables). Then $Y^{\epsilon,h^\epsilon} \to Y^h$ converges in distribution in $\mathbb{S}$.

Our second main result is:

Theorem 4.3.4. Under (A1) and (A2), $Y^\epsilon = \{Y^\epsilon(t)\}_{t\in[0,T]}$, the solution of Eq. (4.3.3), satisfies the LDP on $\mathbb{S}$ with the good rate function $I(f)$ defined by (4.2.6), where $Y^0(h)$ solves Eq. (4.3.5).

Proof. Since the proof is similar to that of Theorem 4.2.5, we omit the details here. □

Remark 4.3.4. Mohammed and Zhang [60] established the LDP for stochastic systems with memory, where the drift and diffusion coefficients need to satisfy a global Lipschitz condition, while our result allows the coefficients to be highly nonlinear with respect to the delay argument.

Remark 4.3.5. The theories established can also be generalized to the cases of neutral functional SDEs with infinite delay and neutral functional SPDEs in infinite-dimensions, which will be reported in the forthcoming papers.

4.4 LDP for Neutral FSDEs with Jumps

In the previous section, we discuss the LDP for neutral FSDEs driven by Brownian motion, while in this section we shall study the LDP for neutral FSDEs with jumps. To this end, we need to recall from [15, p727 and p735-736] some notions and notation.

For a locally compact Polish space $\mathbb{K}$, denote by $\mathcal{M}_F(\mathbb{K})$ the family of all measures ν on $(\mathbb{K}, \mathscr{B}(\mathbb{K}))$ such that $\nu(K) < \infty$ for every compact $K \subset \mathbb{K}$. Note from [15, p727] that $\mathcal{M}_F(\mathbb{S})$ is a Polish space under the weakest topology. For a locally compact space $\mathbb{X}$ and fixed $T > 0$, let $\mathbb{Y} := \mathbb{X} \times [0,\infty), \mathbb{Y}_T := [0,T] \times \mathbb{Y}, \bar{\mathbb{M}} := \mathcal{M}_F(\mathbb{Y}_T)$. Let $\mathbb{W} := C([0,T];\mathbb{R}^m)$, which is a Polish space under the uniform topology, and $\bar{\mathbb{V}} := \mathbb{W} \times \bar{\mathbb{M}}$.

Let $\bar{\mathbb{P}}$ be the unique probability measure on $(\bar{\mathbb{V}}, \mathscr{B}(\bar{\mathbb{V}}))$ such that

(i) The canonical map $W : \bar{\mathbb{V}} \mapsto \mathbb{W}$, $W(w,m) := w$ is a standard Brownian motion.

(ii) The canonical map $\bar{N} : \bar{\mathbb{V}} \mapsto \bar{\mathbb{M}}$, $\bar{N}(w,m) := m$ is a Poisson random measure with intensity measure $\bar{\nu}_T := \nu_T \times \lambda_\infty$ with $\nu_T := \lambda_T \times \nu$, where λ_T and λ_∞ are Lebesgue measures on $[0,T]$ and $[0,\infty)$ respectively, and $\nu \in \mathcal{M}_F(\mathbb{X})$.

(iii) For $A \in \mathscr{B}(\mathbb{Y})$ such that $(\nu \times \lambda_\infty)(A) < \infty$, $W(t)$ and $\bar{N}((0,t] \times A) - \bar{\nu}_T((0,t] \times A)$ are $\bar{\mathcal{G}}_t$-martingales, where

$$\bar{\mathcal{G}}_t := \sigma\{\bar{N}((0,s] \times A) : 0 \in (0,t], A \in \mathscr{B}(\mathbb{Y})\}.$$

Let $\bar{\mathcal{F}}_t$ be the completion under $\bar{\mathbb{P}}$. Denote by $\bar{\mathcal{P}}$ the predictable σ-field on $[0,T] \times \bar{\mathbb{V}}$ with the filtration $\{\bar{\mathcal{F}}_t\}_{t\in[0,T]}$ on $(\bar{\mathbb{V}}, \mathscr{B}(\bar{\mathbb{V}}))$. For $\mathbb{X}_T := [0,T] \times \mathbb{X}$, let

$$\bar{\mathcal{A}} := \{\varphi : \mathbb{X}_T \times \bar{\mathbb{V}} \to [0,\infty) | \varphi \text{ is } (\bar{\mathcal{P}} \otimes \mathscr{B}(\mathbb{X}))\backslash\mathscr{B}([0,\infty)) - \text{measurable}\},$$

Since $(\bar{\mathbb{V}}, \mathscr{B}(\bar{\mathbb{V}}), \bar{\mathbb{P}})$ is the underlying probability space, following the standard convention, we will suppress the dependence of $\varphi(t,x,\omega)$ on ω, and simply write $\varphi(t,x)$ for $(t,x,\omega) \in \mathbb{X}_T \times \bar{\mathbb{V}}$. For $\varphi \in \bar{\mathcal{A}}$, define a counting measure N^φ on $\mathbb{X}_T$ by

$$N^\varphi((0,t] \times U) := \int_0^t \int_U \int_0^\infty 1_{[0,\varphi(s,x)]}(r)\bar{N}(\mathrm{d}s\mathrm{d}x\mathrm{d}r), \quad t \in [0,T],\ U \in \mathscr{B}(\mathbb{X}). \tag{4.4.1}$$

N^φ is called a controlled random counting measure, with φ selecting the intensity for the points at location x and time s. Moreover, for some $\theta > 0$ and any $(s,x,\omega) \in \mathbb{X}_T \times \bar{\mathbb{V}}$, if $\varphi(s,x,\omega) \equiv \theta$, then we write N^φ as N^θ.

Remark 4.4.1. Recall that, under the probability measure $\bar{\mathbb{P}}$, $\bar{N}$ is a Poisson random measure with intensity measure $\bar{\nu}_T$. It then follows from (4.4.1) that $N^\theta((0,t] \times U) - \theta t\nu(U)$ is an $\bar{\mathcal{F}}_t$ -martingale for $U \in \mathscr{B}(\mathbb{X})$.

Let $\phi : (0,\infty) \mapsto [0,\infty)$ be a mapping such that

$$\phi(r) := r\log r - r + 1, \qquad r \in (0,\infty). \tag{4.4.2}$$

Let $\mathcal{U} := \mathbb{H} \times \bar{\mathcal{A}}$, where $\mathbb{H}$ is defined by (4.2.3). For $u := (h, \varphi) \in \mathcal{U}$, define

$$L_T(u) := \frac{1}{2}\|h\|^2_{\mathbb{H}} + \bar{L}_T(\varphi), \tag{4.4.3}$$

where

$$\bar{L}_T(\varphi)(\omega) := \int_{\mathbb{X}_T} \phi(\varphi(t,x,\omega))\nu_T(\mathrm{d}t\mathrm{d}x).$$

Note from [15, p728] that ϕ is the rate function for the standard Poisson process. For $N > 0$ define

$$\bar{\mathcal{S}}_N := \{\varphi : \mathbb{X}_T \mapsto [0,\infty) |\ \bar{L}_T(\varphi) \le N\}.$$

By [15, Line 1-4, p736], $\{\nu_T^{\varphi} : \varphi \in \bar{\mathcal{S}}_N\}$ is a compact subset of $\mathbb{M} := \mathcal{M}_F(\mathbb{X}_T)$ under the weak topology, where

$$\nu_T^{\varphi}(A) := \int_A \varphi(s,x)\nu_T(\mathrm{d}s\mathrm{d}x), \quad A \in \mathscr{B}(\mathbb{X}_T).$$

Moreover, since a function $\varphi \in \bar{\mathcal{S}}_N$ can be identified with a measure ν_T^{φ} defined above, $\bar{\mathcal{S}}_N$ is also a compact space through this identification. Let $\tilde{S}_N := S_N \times \bar{\mathcal{S}}_N$, where S_N is defined by (4.2.4), $\mathbb{S} := \bigcup_{N\ge 1} \tilde{S}_N$ and

$$\mathcal{U}^N := \{u = (h,\varphi) \in \mathcal{U} | u(\cdot,\omega) \in \tilde{S}_N, \quad \bar{\mathbb{P}} - \text{a.s.}\}.$$

Denote by $\{\mathcal{G}^{\epsilon}, \epsilon \in (0,1)\}$ the family of measurable functions from $\mathbb{V} := \mathbb{W} \times \mathbb{M}$ to $\mathbb{U}$, where $\mathbb{U}$ is a Polish space. Let $\{Z^{\epsilon}, \epsilon \in (0,1)\}$ be the space of all $\mathbb{U}$-valued random variables defined on the probability space $(\bar{\mathbb{V}}, \mathscr{B}(\bar{\mathbb{V}}), \bar{\mathbb{P}})$ by

$$Z^{\epsilon} := \mathcal{G}^{\epsilon}(\sqrt{\epsilon}W, \epsilon N^{\epsilon^{-1}}). \tag{4.4.4}$$

To establish the LDP for the family $\{Z^{\epsilon}, \epsilon \in (0,1)\}$, Budhiraja, Dupuis and Vasileios [15] formulate the following sufficient condition:

Assume that there exists a measurable mapping $\mathcal{G}^0 : \mathbb{V} \mapsto \mathbb{U}$ such that

(i) For any $N > 0$, if the family $\{u^{\epsilon} = (h^{\epsilon}, \varphi^{\epsilon}), \epsilon \in (0,1)\} \subset \mathcal{U}^N$ (as $\tilde{S}_N$-valued random variables) converge in distribution to $u = (h,\varphi) \in \mathcal{U}^N$ (as $\tilde{S}_N$-valued random variable), then $\mathcal{G}^{\epsilon}(\sqrt{\epsilon}W + h^{\epsilon}, \epsilon N^{\epsilon^{-1}\varphi^{\epsilon}}) \to \mathcal{G}^0(h, \nu_T^{\varphi})$ in distribution in $\mathbb{U}$ as $\epsilon \to 0$.

(ii) For any $N > 0$, the set $\mathcal{K}_N := \{\mathcal{G}^0(h, \nu_T^{\varphi}) : (h, \varphi) \in \tilde{S}_N\}$ is a compact subset of $\mathbb{U}$.

Lemma 4.4.1. ([15, Theorem 4.2]) Under (i) and (ii), the family $\{Z^\epsilon, \epsilon \in (0,1)\}$, defined by (4.4.4), satisfies the LDP with rate function given by

$$I(\phi) := \inf_{u=(h,\varphi)\in\mathbb{S}_\phi} L_T(u), \tag{4.4.5}$$

where L_T is defined by (4.4.3), and $\mathbb{S}_\phi := \{u = (h, \varphi) | \phi = \mathcal{G}^0(h, \nu_T^{\varphi})\}$.

In this section, we shall apply Lemma 4.4.1 to establish the LDP for neutral FSDEs driven by multiplicative Lévy noise. Consider a neutral functional differential equation on $\mathbb{R}^n$

$$\begin{cases} \mathrm{d}[Z(t) - G(Z_t)] = b(Z_t)\mathrm{d}t, \quad t \in [0,T], \\ Z_0 = \xi \in \mathscr{C}, \end{cases}$$

and the associated perturbed neutral FSDE

$$\begin{cases} \mathrm{d}[Z^\epsilon(t) - G(Z_t^\epsilon)] = b(Z_t^\epsilon)\mathrm{d}t + \sqrt{\epsilon}\sigma(Z_t^\epsilon)\mathrm{d}W(t) \\ \qquad\qquad + \int_{\mathbb{Z}} \Phi(Z_t^\epsilon, x)(\epsilon N^{\epsilon^{-1}}(\mathrm{d}t\mathrm{d}x) - \nu_T(\mathrm{d}t\mathrm{d}x)), \quad t \in [0,T], \\ Z_0^\epsilon = \xi \in \mathscr{C}, \end{cases}$$

where W and $N^{\epsilon^{-1}}$ are the Brownian motion and the Poisson process defined on the probability space $(\bar{\mathbb{V}}, \mathscr{B}(\bar{\mathbb{V}}), \bar{\mathbb{P}}, \bar{\mathcal{F}}_t)$ respectively.

In what follows, we still assume that $(H1)$ and $(H2)$ hold. For $\Phi : \mathbb{R}^n \times \mathbb{X} \mapsto \mathbb{R}^n$, we further assume that

(B1) $\Phi : \mathbb{R}^n \times \mathbb{X} \mapsto \mathbb{R}^n$ is bounded and there exists an $L > 0$ such that

$$|\Phi(z,x) - \Phi(z',x)| \le L|z - z'|, \quad z, z' \in \mathbb{R}^n, x \in \mathbb{X}.$$

(B2) For some compact subset $K \subset \mathbb{X}$, $\Phi(z,x) = 0$ for $(z,x) \in \mathbb{R}^n \times K^c$.

Let $\mathbb{U} := D([0,T];\mathbb{R}^n)$, the space of all càdlàg paths from $[0,T]$ into $\mathbb{R}^n$, equipped with the uniform convergence topology. By the Yamada-Watanabe theorem, there exists a measurable map $\mathcal{G}^\epsilon : \mathbb{V} \mapsto \mathbb{U}$ such that

$$Z^\epsilon = \mathcal{G}^\epsilon(\sqrt{\epsilon}W, \epsilon N^{\epsilon^{-1}}).$$

Then for $u^\epsilon := (h^\epsilon, \varphi^\epsilon) \in \mathcal{U}^N$ (as $\tilde{S}_N$-valued random variables)

$$Z^{\epsilon,u^\epsilon} := \mathcal{G}^\epsilon(\sqrt{\epsilon}W + h^\epsilon, \epsilon N^{\epsilon^{-1}\varphi^\epsilon})$$

uniquely solves the following equation

$$\begin{cases} \mathrm{d}[Z^{\epsilon,u^\epsilon}(t) - G(Z_t^{\epsilon,u^\epsilon})] = [b(Z_t^{\epsilon,u^\epsilon}) + \sigma(Z_t^{\epsilon,u^\epsilon})\dot{h}^\epsilon(t)]\mathrm{d}t + \sqrt{\epsilon}\sigma(Z_t^{\epsilon,u^\epsilon})\mathrm{d}W(t) \\ \qquad\qquad + \int_{\mathbf{X}} \Phi(Z_t^{\epsilon,u^\epsilon}, x)(\epsilon N^{\epsilon^{-1}\varphi^\epsilon}(\mathrm{d}t\mathrm{d}x) - \nu_T(\mathrm{d}t\mathrm{d}x)), \\ Z_0^{\epsilon,u^\epsilon} = \xi \in \mathscr{C}. \end{cases} \tag{4.4.6}$$

For $u = (h, \varphi) \in \mathbb{S}$, we introduce the skeleton equation for $t \in [0, T]$

$$\begin{cases} \mathrm{d}[Z^u(t) - G(Z_t^u)] = [b(Z_t^u) + \sigma(Z_t^u)\dot{h}(t)]\mathrm{d}t \\ \qquad\qquad + \int_{\mathbf{X}} \Phi(Z_t^u, x)(\varphi(t,x) - 1)\nu_T(\mathrm{d}t\mathrm{d}x), \\ Z_0^u = \xi \in \mathscr{C}, \end{cases} \tag{4.4.7}$$

and define

$$\mathcal{G}^0(h, \nu_T^\varphi) := Z^u. \tag{4.4.8}$$

Lemma 4.4.2. Under (H1), (H2) and (B1), for $u^\epsilon := (h^\epsilon, \varphi^\epsilon) \in \mathcal{U}^N$ (as $\tilde{S}_N$-valued random variables) and $u := (h, \varphi) \in \tilde{S}_N$, there exists $C > 0$ such that

$$\sup_{-\tau \le t \le T} |Z^u(t)|^4 \vee \bar{\mathbb{E}}\Big(\sup_{-\tau \le t \le T} |Z^{u^\epsilon}(t)|^4\Big) \le C.$$

Proof. Let $M(t)$ be defined by (4.2.12) with Z^{ϵ,h^ϵ} replaced by Z^{ϵ,u^ϵ} and note that (4.2.14) still hold. By the Itô formula, one has

$$\begin{aligned} |M(t)|^2 = |M(0)|^2 &+ \int_0^t \{2\langle M(s), b(Z_s^{\epsilon,u^\epsilon}) + \sigma(Z_s^{\epsilon,u^\epsilon})\dot{h}^\epsilon(s)\rangle + \epsilon\|\sigma(Z_s^{\epsilon,u^\epsilon})\|_{HS}^2\}\mathrm{d}s \\ &+ \int_0^t \int_{\mathbf{X}} \{(\epsilon|\Phi(Z_s^{\epsilon,u^\epsilon}, x)|^2 + 2\langle M(s), \Phi(Z_s^{\epsilon,u^\epsilon}, x)\rangle)\varphi^\epsilon(s,x) \\ &- 2\langle M(s), \Phi(Z_s^{\epsilon,u^\epsilon}, x)\rangle\}\nu_T(\mathrm{d}s\mathrm{d}x) + \bar{J}_1(t) + \bar{J}_2(t), \end{aligned}$$

where $\bar{J}_1(t) := 2\sqrt{\epsilon}\int_0^t \langle M(s), \sigma(Z_s^{\epsilon,u^\epsilon})\mathrm{d}W(s)\rangle$ and

$$\begin{aligned} \bar{J}_2(t) := \int_0^t \int_{\mathbf{X}} &\{\epsilon^2|\Phi(Z_s^{\epsilon,u^\epsilon}, x)|^2 + 2\epsilon\langle M(s), \Phi(Z_s^{\epsilon,u^\epsilon}, x)\rangle\} \\ &(N^{\epsilon^{-1}\varphi^\epsilon}(\mathrm{d}s\mathrm{d}x) - \epsilon^{-1}\varphi^\epsilon(s,x)\nu_T(\mathrm{d}s\mathrm{d}x)). \end{aligned}$$

In light of (H1), (4.2.2), (4.2.13) and (B1)

$$\begin{aligned}\sup_{0\le t\le T}|M(t)|^2 \le C + C\int_0^T \{|\dot h^\epsilon(t)| + (1+|\dot h^\epsilon(t)|)\|Z_t^{\epsilon,u^\epsilon}\|_\infty^2\}\mathrm{d}t \\ + C\int_0^T\int_{\mathbb{X}}(1+\|Z_t^{\epsilon,u^\epsilon}\|_\infty^2)\varphi^\epsilon(t,x)\nu_T(\mathrm{d}t\mathrm{d}x) \\ + \sup_{0\le t\le T}|\bar J_1(t)| + \sup_{0\le t\le T}|\bar J_2(t)|.\end{aligned}$$

Note from the definition of $\mathcal{M}_F(\mathbb{X})$ and the compact property of $\mathbb{X}$ that $\nu(\mathbb{X}) < \infty$. Due to u^ϵ are $\tilde S_N$-valued random variables and $\phi(r) \ge r + 1 - e, r \in [0,\infty)$, with ϕ defined in (4.4.2), we have

$$\|\dot h^\epsilon\|_{\mathbb{H}} \le N \text{ and } \int_{\mathbb{X}_T}\varphi^\epsilon(t,x)\nu_T(\mathrm{d}t\mathrm{d}x) \le N + (e-1)\nu(\mathbb{X}). \tag{4.4.9}$$

It then follows that

$$\begin{aligned}\sup_{0\le t\le T}|M(t)|^2 \le C + \sup_{0\le t\le T}|\bar J_1(t)| + \sup_{0\le t\le T}|\bar J_2(T)| \\ + C\int_0^T\Big(1+|\dot h^\epsilon(t)| + \int_{\mathbb{X}}\varphi^\epsilon(t,x)\Big)\sup_{0\le s\le t}|Z^{\epsilon,u^\epsilon}(s)|^2\mathrm{d}t.\end{aligned} \tag{4.4.10}$$

Now substituting (4.2.14) with $p=2$ into (4.4.10) and using (4.4.9) leads to

$$\begin{aligned}\sup_{0\le t\le T}|M(t)|^2 \le C + \sup_{0\le t\le T}|\bar J_1(t)| + \sup_{0\le t\le T}|\bar J_2(t)| \\ + C\int_0^T\Big(1+|\dot h^\epsilon(t)| + \int_{\mathbb{X}}\varphi^\epsilon(t,x)\Big)\sup_{0\le s\le t}|M(s)|^2\mathrm{d}t.\end{aligned}$$

Then, by the Gronwall inequality, we arrive at

$$\begin{aligned}\sup_{0\le t\le T}|M(t)|^2 \le \Big(C + \sup_{0\le t\le T}|\bar J_1(t)| + \sup_{0\le t\le T}|\bar J_2(T)|\Big) \\ \times \exp\Big(C\int_0^T\Big(1+|\dot h^\epsilon(t)| + \int_{\mathbb{X}}\varphi^\epsilon(t,x)\nu(\mathrm{d}x)\Big)\mathrm{d}t\Big).\end{aligned}$$

This, together with (4.4.9), gives

$$\sup_{0\le t\le T}|M(t)|^2 \le C\Big(1 + \sup_{0\le t\le T}|\bar J_1(t)| + \sup_{0\le t\le T}|\bar J_2(T)|\Big).$$

Hence

$$\bar{\mathbb{E}}\Big(\sup_{0\le t\le T}|M(t)|^4\Big) \le C\Big(1+\bar{\mathbb{E}}\Big(\sup_{0\le t\le T}|\bar J_1(t)|^2\Big) + \bar{\mathbb{E}}\Big(\sup_{0\le t\le T}|\bar J_2(T)|^2\Big)\Big). \tag{4.4.11}$$

Next taking into account the Doob inequality and the Itô isometry, we obtain from (4.2.2) and (4.2.13) that

$$
\begin{aligned}
\bar{\mathbb{E}}\Big(\sup_{0\le t\le T}|\bar{J}_1(t)|^2\Big) &\le 16\epsilon\bar{\mathbb{E}}\Big|\int_0^T\langle M(s),\sigma(Z_s^{\epsilon,u^\epsilon})\mathrm{d}W(s)\rangle\Big|^2\\
&\le 16\epsilon\bar{\mathbb{E}}\int_0^T|M(s)|^2\|\sigma(Z_s^{\epsilon,u^\epsilon})\|_{HS}^2\mathrm{d}s \qquad (4.4.12)\\
&\le C\epsilon\bar{\mathbb{E}}\int_0^T(1+\|Z_s^{\epsilon,u^\epsilon}\|_\infty^4)\mathrm{d}s,
\end{aligned}
$$

and by (4.2.7) and (B1) that

$$
\begin{aligned}
&\bar{\mathbb{E}}\Big(\sup_{0\le t\le T}|\bar{J}_2(t)|^2\Big)\\
&\le 4\bar{\mathbb{E}}\Big|\int_0^T\int_{\mathbf{X}}\{\epsilon^2|\Phi(Z_s^{\epsilon,u^\epsilon},x)|^2+2\epsilon\langle M(s),\Phi(Z_s^{\epsilon,u^\epsilon},x)\rangle\}\\
&\quad\times(N^{\epsilon^{-1}\varphi^\epsilon}(\mathrm{d}s\mathrm{d}x)-\epsilon^{-1}\varphi^\epsilon(s,x)\nu_T(\mathrm{d}s\mathrm{d}x))\Big|^2 \qquad (4.4.13)\\
&=4\epsilon\bar{\mathbb{E}}\int_0^T\int_{\mathbf{X}}\{\epsilon|\Phi(Z_s^{\epsilon,u^\epsilon},x)|^2\\
&\quad+2\langle M(s),\Phi(Z_s^{\epsilon,u^\epsilon},x)\rangle\}^2\varphi^\epsilon(s,x)\nu_T(\mathrm{d}s\mathrm{d}x)\\
&\le C\epsilon\bar{\mathbb{E}}\int_0^T\int_{\mathbf{X}}(1+\|Z_s^{\epsilon,u^\epsilon}\|_\infty^4)\varphi^\epsilon(s,x)\nu_T(\mathrm{d}s\mathrm{d}x).
\end{aligned}
$$

Putting (4.4.12) and (4.4.13) into (4.4.11), and using (4.2.14) with $p=4$ and (4.4.9) implies that

$$
\begin{aligned}
\bar{\mathbb{E}}\Big(\sup_{0\le t\le T}|Z^{\epsilon,u^\epsilon}(t)|^4\Big) &\le C+C\epsilon\bar{\mathbb{E}}\int_0^T\|Z_s^{\epsilon,u^\epsilon}\|_\infty^4\mathrm{d}s\\
&\quad+C\epsilon\bar{\mathbb{E}}\int_0^T\int_{\mathbf{X}}\|Z_s^{\epsilon,u^\epsilon}\|_\infty^4\varphi^\epsilon(s,x)\nu_T(\mathrm{d}s\mathrm{d}x)\\
&\le C+C\epsilon\bar{\mathbb{E}}\Big\{\Big(T+\int_0^T\int_{\mathbf{X}}\varphi^\epsilon(s,x)\nu_T(\mathrm{d}s\mathrm{d}x)\Big)\\
&\quad\times\Big(\sup_{0\le t\le T}|Z^{u^\epsilon}(t)|^4\Big)\Big\}\\
&\le C+C(T+N)\epsilon\bar{\mathbb{E}}\Big(\sup_{0\le t\le T}|Z^{u^\epsilon}(t)|^4\Big).
\end{aligned}
$$

Consequently, the second assertion follows by choosing $\epsilon>0$ sufficiently small, and the first assertion holds by applying the chain rule and following the argument of that of (4.4.10). □

Remark 4.4.2. In fact, following the argument of Lemma 4.4.2 we can also verify that

$$\sup_{-\tau\le t\le T}|Z^u(t)|^p \vee \bar{\mathbb{E}}\Big(\sup_{-\tau\le t\le T}|Z^{u^\epsilon}(t)|^p\Big) \le C$$

for any $p \ge 2$. While we only show the case $p = 4$ in Lemma 4.4.2, which is enough for later purpose.

Lemma 4.4.3. Under (H1), (H2), (B1) and (B2), the set $\mathcal{K}_N := \{\mathcal{G}^0(h, \nu_T^{\varphi}) : (h,\varphi) \in \tilde{S}_N\}$ is a compact subset of $\mathbb{U}$.

Proof. The argument is similar to that of Lemma 4.2.3, while we give an outline to point out some differences. For $u^n := (h^n, \varphi^n)$ and $u := (h,\varphi)$, denote by Z^{u_n} and Z^u the solutions of Eq. (4.4.7) with the controls $u_n \in \tilde{S}_N$ and $u \in \tilde{S}_N$ respectively. Let $\bar{M}(t)$ be defined by (4.2.18) with h^n and h replaced by u^n and u respectively. By the chain rule, we have from Eq. (4.4.7) that

$$\begin{aligned}
|\bar{M}(t)|^2 &= 2\int_0^t \langle \bar{M}(s), b(Z_s^{u_n}) - b(Z_s^u)\rangle \mathrm{d}s \\
&\quad + 2\int_0^t \langle \bar{M}(s), \sigma(Z_s^{h_n})(\dot{h^n}(s) - \dot{h}(s)) + (\sigma(Z_s^{h_n}) - \sigma(Z_s^h))\dot{h}(s)\rangle \mathrm{d}s \\
&\quad + 2\int_0^t\int_{\mathbf{X}} \langle \bar{M}(s), \Phi(Z_s^u, x) - \Phi(Z_s^{u_n}, x)\rangle \nu(\mathrm{d}x)\mathrm{d}s \\
&\quad + 2\int_0^t\int_{\mathbf{X}} \langle \bar{M}(s), (\Phi(Z_s^{u_n}, x) - \Phi(Z_s^u, x))\rangle \varphi(s,x)\nu(\mathrm{d}x)\mathrm{d}s \\
&\quad + 2\int_0^t\int_{\mathbf{X}} \langle \bar{M}(s), \Phi(Z_s^{u_n}, x)\rangle (\varphi^n(s,x) - \varphi(s,x))\nu(\mathrm{d}x)\mathrm{d}s \\
&=: \bar{I}_1(t) + \bar{I}_2(t) + \bar{I}_3(t) + \bar{I}_4(t) + \bar{I}_5(t).
\end{aligned}$$

A computation shows from (H1) and (B1) that

$$|\bar{I}_3(t)| + |\bar{I}_4(t)| \le C\int_0^t \Big(1 + \int_{\mathbf{X}} \varphi(s,x)\nu(\mathrm{d}x)\Big)\|Z_s^{u_n} - Z_s^u\|_\infty^2 \mathrm{d}s.$$

Following the argument of that of Lemma 4.2.3, we obtain that

$$\begin{aligned}
\sup_{0\le t\le T}|Z^{u_n}(t) - Z^u(t)|^2 &\le \sup_{0\le t\le T}|\bar{I}_5(t)| + C\int_0^T |\dot{h^n}(t) - \dot{h}(t)|^2 \mathrm{d}t \\
&\quad + C\int_0^T \Big(1 + |\dot{h}(t)| + \int_{\mathbf{X}} \varphi(t,x)\nu(\mathrm{d}x)\Big) \\
&\quad \times \sup_{0\le s\le t}|Z^{u_n}(s) - Z^u(s)|^2 \mathrm{d}t.
\end{aligned}$$

By the Gronwall inequality and (4.4.9), one has

$$\sup_{0\le t\le T}|Z^{u_n}(t)-Z^u(t)|^2\le C\Big(\sup_{0\le t\le T}|\bar I_5(t)|+\int_0^T|\dot{h^n}(t)-\dot h(t)|^2\mathrm{d}t\Big).$$

Moreover, due to (4.2.13), (B1) and Lemma 4.4.2, $\langle \bar M(s),\Phi(Z_s^{u_n},x)\rangle, s\in[0,T]$, is uniformly bounded. Then, by [94, Lemma 3.4], $\sup_{0\le t\le T}|\bar I_5(t)|\to 0$ as $n\to\infty$. Then the conclusion follows from that $h^n\to h, n\to\infty$, in $\mathbb{H}$, as required. □

Lemma 4.4.4. Let (H1), (H2), (B1) and (B2) hold. Assume further that the family $\{u^\epsilon=(h^\epsilon,\varphi^\epsilon),\epsilon\in(0,1)\}\subset\mathcal{U}_N$ (as $\tilde S_N$-valued random variables) converge almost surely in $\mathbb{H}$ to $u=(h,\varphi)\in\mathcal{U}_N$ (as $\tilde S_N$-valued random variable). Then $Z^{\epsilon,u^\epsilon}\to Z^u$ converges in distribution in $\mathbb{S}$.

Proof. It is sufficient to show that $Z^{\epsilon,u^\epsilon}\to Z^u$ in probability in $\mathbb{S}$. Let $\bar M(t)$ be defined by (4.2.18) with Z^{ϵ,u^ϵ} and Z^u replaced by Z^{h_n} and Z^h respectively. Applying the Itô formula, we obtain from (4.4.6) and (4.4.7) that

$$\begin{aligned}
|\bar M(t)|^2&=2\int_0^t\langle\bar M(s),b(Z_s^{\epsilon,u^\epsilon})-b(Z_s^u)\rangle\mathrm{d}s\\
&\quad+2\int_0^t\int_{\mathbf{X}}\langle\bar M(s),\{\Phi(Z_s^u,x)-\Phi(Z_s^{\epsilon,u^\epsilon},x)\}\rangle\nu_T(\mathrm{d}x\mathrm{d}s)\\
&\quad+2\int_0^t\langle\bar M(s),(\sigma(Z_s^{\epsilon,u^\epsilon})-\sigma(Z_s^u))\dot h(s)\rangle\mathrm{d}s\\
&\quad+2\int_0^t\int_{\mathbf{X}}\langle\bar M(s),(\Phi(Z_s^{\epsilon,u^\epsilon},x)-\Phi(Z_s^u,x))\rangle\varphi(s,x)\nu_T(\mathrm{d}x\mathrm{d}s)\\
&\quad+\epsilon\int_0^t\|\sigma(Z_s^{\epsilon,u^\epsilon})\|_{HS}^2\mathrm{d}s+2\sqrt{\epsilon}\int_0^t\langle\bar M(s),\sigma(Z_s^{\epsilon,u^\epsilon})\mathrm{d}W(s)\rangle\\
&\quad+\epsilon\int_0^t\int_{\mathbf{X}}|\Phi(Z_s^{\epsilon,u^\epsilon},x)|^2\varphi^\epsilon(s,x)\nu_T(\mathrm{d}x\mathrm{d}s)\qquad(4.4.14)\\
&\quad+\int_0^t\int_{\mathbf{X}}\{\epsilon^2|\Phi(Z_s^{\epsilon,u^\epsilon},x)|^2+2\epsilon\langle\bar M(s),\Phi(Z_s^{\epsilon,u^\epsilon},x)\rangle\}\\
&\quad\times(N^{\epsilon^{-1}\varphi^\epsilon}(\mathrm{d}x\mathrm{d}s)-\epsilon^{-1}\varphi^\epsilon\nu_T(\mathrm{d}x\mathrm{d}s))\\
&\quad+2\int_0^t\langle\bar M(s),\sigma(Z_s^{\epsilon,u^\epsilon})(\dot{h^\epsilon}(s)-\dot h(s))\rangle\mathrm{d}s\\
&\quad+2\int_0^t\int_{\mathbf{X}}\langle\bar M(s),\Phi(Z_s^{\epsilon,u^\epsilon},x)(\varphi^\epsilon(s,x)-\varphi(s,x))\rangle\nu_T(\mathrm{d}x\mathrm{d}s)\\
&=:\tilde J_1(t)+\tilde J_2(t)+\tilde J_3(t)+\tilde J_4(t)+\tilde J_5(t)+\tilde J_6(t)\\
&\quad+\tilde J_7(t)+\tilde J_8(t)+\tilde J_9(t).
\end{aligned}$$

By (H2) and (B1), it is easy to see that

$$|\tilde{J}_1(t)| + |\tilde{J}_2(t)| + |\tilde{J}_3(t)| + |\tilde{J}_4(t)| \leq C\int_0^t \Big(1 + |\dot{h}(s)| + \int_{\mathbb{X}} \varphi(s,x)\nu(\mathrm{d}x)\Big) \times \|Z_s^{\epsilon,u^\epsilon} - Z_s^u\|_\infty^2 \mathrm{d}s.$$

This, combining (4.2.19) with (4.4.14) and applying the Gronwall inequality, yields from (4.4.9) that

$$\sup_{0\leq t\leq T} |Z^{\epsilon,u^\epsilon}(s) - Z^u(s)|^2 \leq C\Big(\sup_{0\leq t\leq T}|\tilde{J}_5(t)| + \sup_{0\leq t\leq T}|\tilde{J}_6(t)| + \sup_{0\leq t\leq T}|\tilde{J}_7(t)| + \sup_{0\leq t\leq T}|\tilde{J}_8(t)| + \sup_{0\leq t\leq T}|\tilde{J}_9(t)|\Big).$$

Furthermore, due to Lemma 4.4.2 and the Burkhold-Davis-Gundy inequality, we have

$$\bar{\mathbb{E}}\Big(\sup_{0\leq t\leq T}|\tilde{J}_5(t)|^2\Big) \to 0 \text{ as } \epsilon \downarrow 0.$$

For arbitrary $R > 0$ and $\delta > 0$, note that

$$\bar{\mathbb{P}}\Big(\sup_{0\leq t\leq T}\tilde{J}_6(t) \geq \delta\Big) \leq \bar{\mathbb{P}}\Big(\epsilon\int_0^T\int_{\mathbb{X}}\varphi^\epsilon(s,x)\nu_T(\mathrm{d}x\mathrm{d}s) \geq \frac{\delta}{R}\Big) + \bar{\mathbb{P}}\Big(\sup_{0\leq t\leq T}|\Phi(Z_s^{\epsilon,u^\epsilon},x)|^2 > R\Big).$$

In view of (4.4.9), (B1) and Lemma 4.4.2, it thus follows that

$$\sup_{0\leq t\leq T}|\tilde{J}_6(t)| \to 0 \text{ in probability as } \epsilon \downarrow 0.$$

Observe by the Chebyshev inequality, (4.4.13) and (4.4.9) that for any $\delta > 0$

$$\begin{aligned}\bar{\mathbb{P}}\Big(\sup_{0\leq t\leq T}|\tilde{J}_7(t)| \geq \delta\Big) &\leq \frac{1}{\delta^2}\bar{\mathbb{E}}\Big(\sup_{0\leq t\leq T}|\tilde{J}_7(t)|^2\Big)\\ &\leq \frac{C\epsilon}{\delta^2}\bar{\mathbb{E}}\int_0^T\int_{\mathbb{X}}(1 + \|Z_s^{\epsilon,u^\epsilon}\|_\infty^4)\varphi^\epsilon(s,x)\nu_T(\mathrm{d}s\mathrm{d}x)\\ &\leq \frac{C(N + (e-1)\nu(\mathbb{X}))\epsilon}{\delta^2}\Big(1 + \bar{\mathbb{E}}\Big(\sup_{0\leq t\leq T}\|Z_t^{\epsilon,u^\epsilon}\|_\infty^4\Big)\Big).\end{aligned}$$

Hence

$$\sup_{0\leq t\leq T}|\tilde{J}_7(t)| \to 0 \text{ in probability as } \epsilon \downarrow 0.$$

Recall from (4.2.24) that

$$\sup_{0\leq t\leq T}|\tilde{J}_8(t)| \to 0 \text{ in probability as } \epsilon \downarrow 0.$$

Then the desired assertion follows from [94, (23)]. □

Theorem 4.4.5. Let (H1), (H2), (B1) and (B2) hold. Then Z^ϵ satisfies the LDP on $\mathbb{S}$ with the good rate function $I(f)$ defined by (4.4.5), where $\mathcal{G}^0$ is defined by (4.4.8).

Proof. The proof can be complete by following the argument of that of Theorem 4.2.5. □

Remark 4.4.3. The boundedness of drift and diffusion coefficients are imposed in [15, 94] to study the LDP, while in this chapter this condition has been relaxed.

Remark 4.4.4. Following the approach adopted in Section 3, we can also generalize Theorem 4.3.4 to the cases of neutral SDDEs with jumps, which allow the coefficients to be highly nonlinear with respect to the delay argument.

Chapter 5

Numerical Analysis for Neutral SPDEs Driven by α-stable Processes

In this chapter, we discuss the convergence of EM scheme for a class of *neutral* SPDEs driven by *α-stable processes*, where the numerical scheme is based on spatial discretization and time discretization.

5.1 Introduction

Numerical approximations of SPDEs driven by the Gaussian or Poisson-jump noise is well understood, e.g., in [33, 34, 41, 47, 48, 49, 50] for Gaussian case, in [41, 42] for Poisson-jump case. However, most of the existing papers cannot cover an important class of SPDEs driven by α-stable Lévy motion with $\alpha \in (0, 2)$. Note that Wiener noise and Poisson-jump noise have arbitrary finite moments, while α-stable noise only has finite p-th moment for $p \in (0, \alpha)$. This therefore brings a lot of difficulties in discussing SPDEs driven by α-stable processes. For additive cases, [68] investigated the structural properties of mild solutions such as strong Feller properties, irreducibility and invariant measure, [24] discussed Markovian solutions of stochastic 2D Navier-Stokes equations, and [95] studied trajectory property of stochastic Burgers equations. On the other hand, there are few papers on the numerical analysis of *explicit schemes* of neutral SPDEs, although there are some papers on successive approximations of neutral SPDEs, e.g., [6, 9]. Motivated by the previous papers, in this chapter we shall discuss the convergence of EM scheme for a class of *neutral* SPDEs driven by *α-stable processes*, where the numerical scheme is based on spatial discretization and time discretization.

Let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ be a real separable Hilbert space, and $Z(t)$ a cylindrical α-stable process, $\alpha \in (0, 2)$, defined by

$$Z(t) := \sum_{m=1}^{\infty} \beta_m Z_m(t) e_m,$$

Here $\{e_m\}_{m\geq1}$ is an orthogonal basis of H, $\{Z_m(t)\}_{m\geq1}$ are independent, real-valued, normalized, symmetric α-stable Lévy processes defined on stochastic basis $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq0}, \mathbb{P})$, and $\{\beta_m\}_{m\geq1}$ is a sequence of positive numbers. Recall that a random variable η is said to be stable with stability index $\alpha \in (0, 2)$, scale parameter $\sigma \in (0, \infty)$, skewness parameter $\beta \in [-1, 1]$, and location parameter $\mu \in (-\infty, \infty)$ if its characteristic function is

$$\phi_\eta(u) = \mathbb{E}\exp(iu\eta) = \exp\{-\sigma^\alpha|u|^\alpha(1 - i\beta\text{sgn}(u)\Phi + i\mu u\}, \quad u \in \mathbb{R},$$

where $\Phi = \tan(\pi\alpha/2)$ for $\alpha \neq 1$ and $\Phi = -(2/\pi)\log|u|$ for $\alpha = 1$. We call η is strictly α-stable whenever $\mu = 0$, and if, in addition, $\beta = 0$, η is said to be symmetric α-stable. For a real-valued normalized (standard) symmetric α-stable Lévy process $z(t)$, $\alpha \in (0, 2)$, it has the characteristic function

$$\mathbb{E}\exp(iuz(t)) = e^{-t|u|^\alpha}, \quad u \in \mathbb{R}, \tag{5.1.1}$$

and the Lévy measure $\lambda_\alpha(\mathrm{d}x) := \frac{c_\alpha}{|x|^{1+\alpha}}, x \in \mathbb{R} - \{0\}$, where c_α is some constant.

For fixed $\tau > 0$, consider the following neutral SPDE driven by α-stable process

$$\begin{cases} \mathrm{d}\{X(t) - G(X(t-\tau))\} = \{AX(t) + b(X(t), X(t-\tau))\}\mathrm{d}t + \mathrm{d}Z(t), \\ X(\theta) = \xi(\theta) \in H, \quad \theta \in [-\tau, 0], \end{cases} \tag{5.1.2}$$

where $G : H \to H$ with $G(0) = 0$ and $b : H \times H \to H$.

Throughout this chapter we assume that

(A1) A is a self-adjoint operator on H such that $-A$ has discrete spectrum $0 < \lambda_1 \le \lambda_2 \le \cdots \lambda_m \uparrow \infty$ with corresponding eigenbasis $\{e_m\}_{m\ge1}$ of H. In this case A generates a compact C_0-semigroup $e^{tA}, t \ge 0$, such that $\|e^{tA}\| \le e^{-\lambda_1 t}$.

(A2) There exists $L_1 > 0$ such that

$$\|b(x_1,y_1) - b(x_2,y_2)\|_H \le L_1(\|x_1 - x_2\|_H + \|y_1 - y_2\|_H), \quad x_1, x_2, y_1, y_2 \in H.$$

(A3) There exist $\kappa \in (0,1)$ and $L_2 > 0$ such that

$$\|(-A)^\kappa (G(x) - G(y))\|_H \le L_2 \|x - y\|_H, \quad x, y \in H.$$

(A4) There exists $\theta \in (0, \kappa)$ such that $\alpha\theta \in (0,1)$ and $\delta := \sum_{m=1}^{\infty} \frac{\beta_m^\alpha}{\lambda_m^{1-\alpha\theta}} < \infty$.

Note from (A4) that $\sum_{m=1}^{\infty} \frac{\beta_m^\alpha}{\lambda_m} = \sum_{m=1}^{\infty} \frac{\beta_m^\alpha}{\lambda_m^{1-\alpha\theta}} \frac{1}{\lambda_m^{\alpha\theta}} \le \frac{\delta}{\lambda_1^{\alpha\theta}}$ due to the non-decreasing property of $\{\lambda_m\}_{m\ge1}$. For $t \in [0, \tau]$, (5.1.2) reduces to an SPDE without the delay argument and the neutral term. By [68, Theorem 5.3], under (A1)-(A3), (5.1.2) has a unique mild solution $X(t)$ on the time interval $[0, \tau]$. Carrying out a similar procedure, (5.1.2) has a unique global mild solution $X(t)$ on $[0, T]$, that is, there exists a predictable H-valued stochastic process $X(t)$ such that

$$\begin{aligned} X(t) &= e^{tA}\{\xi(0) - G(\xi(-\tau))\} + G(X(t-\tau)) \\ &\quad + \int_0^t A e^{(t-s)A} G(X(s-\tau)) \mathrm{d}s \\ &\quad + \int_0^t e^{(t-s)A} b(X(s), X(s-\tau)) \mathrm{d}s + \int_0^t e^{(t-s)A} \mathrm{d}Z(s). \end{aligned} \tag{5.1.3}$$

Example 5.1.1. Let $H = L^2(0, \pi)$ and A be given by

$$\begin{cases} Ax := \triangle_\xi x, \\ \mathcal{D}(A) := H^2(0,\pi) \cap H_0^1(0,\pi), \end{cases}$$

where $\triangle_\xi$ represents the Laplace operator, $H^k(0,\pi), k = 1, 2$, represents Sobolev spaces, and $H_0^1(0,\pi)$ is the subspace of $H^1(0,\pi)$ of all functions vanishing at 0 and π. Note that A is a self-adjoint negative operator in H, and $Ae_m =$

$-m^2 e_m, m \in \mathbb{N}$, where $e_m(\xi) = (2/\pi)^{1/2} \sin m\xi, m \in \mathbb{N}, \xi \in [0, \pi]$. Let $Z(t, \xi) := \sum_{m=1}^{\infty} m^{\rho} Z_m(t) e_m(\xi)$, where $\alpha\rho \in (0,1)$ and $\{Z_m(t)\}_{m \geq 1}$ is an independent, real-valued, normalized, symmetric α-stable process sequence. It is trivial to see that $\delta = \sum_{m=1}^{\infty} \frac{1}{m^{2(1-\alpha\theta)-\rho\alpha}}$. Due to $\alpha\rho \in (0,1)$, one has $1 < 2 - \rho\alpha < 2$. Hence there exists $\theta \in (0, \kappa)$ such that $1 < 2(1 - \alpha\theta) - \rho\alpha < 2$, and therefore $\delta = \sum_{m=1}^{\infty} \frac{1}{m^{2(1-\alpha\theta)-\rho\alpha}} < \infty$. In other words, (A4) holds for such case.

Remark 5.1.1. We remark that by [68, Theorem 5.3] (5.1.2) has a unique mild solution under $\sum_{m=1}^{\infty} \frac{\beta_m^{\alpha}}{\lambda_m} < \infty$, which is weaker than (A4). While the a little bit stronger condition (A4) is just imposed for later numerical analysis. On the other hand, [43, Example 4.1] satisfies (A3) with $\kappa = \frac{1}{2}$.

For any $n \geq 1$, let $\pi_n : H \to H_n := \text{span}\{e_1, \cdots, e_n\}$ be the orthogonal projection, that is, $\pi_n x = \sum_{i=1}^{n} \langle x, e_i \rangle_H e_i$ for $x \in H$. Define $G_n := \pi_n G$, $A_n := \pi_n A, b_n := \pi_n b$, and $Z^n := \pi_n Z$.

Consider the following finite-dimensional SDE on H_n

$$\begin{cases} \mathrm{d}\{X^n(t) - G_n(X^n(t-\tau))\} = \{A_n X^n(t) \\ \qquad\qquad +b_n(X^n(t), X^n(t-\tau))\}\mathrm{d}t + \mathrm{d}Z^n(t), \\ X^n(\theta) = \pi_n \xi(\theta), \quad \theta \in [-\tau, 0], \end{cases} \tag{5.1.4}$$

which admits a unique solution $X^n(t)$ on H_n due to the fact that $A_n x + b_n(x, y), x, y \in H_n$, satisfies Lipschitz condition.

Let $\triangle := \frac{\tau}{N} \in (0,1)$ for some sufficiently large integer N. For any integer $k \geq 0$, compute the discrete EM approximations $\bar{Y}^n(k\triangle) \approx X^n(k\triangle)$ by forming

$$\begin{cases} \bar{Y}^n((k+1)\triangle) := G_n(\bar{Y}^n((k+1)\triangle - \tau)) + e^{\triangle A_n}\{\bar{Y}^n(k\triangle) \\ \qquad -G_n(\bar{Y}^n(k\triangle - \tau)) + A_n G_n(\bar{Y}^n(k\triangle - \tau))\triangle \\ \qquad +b_n(\bar{Y}^n(k\triangle), \bar{Y}^n(k\triangle - \tau))\triangle + \triangle Z_k^n\}, \\ \bar{Y}^n(\theta) := \pi_n \xi(\theta), \ \theta \in [-\tau, 0], \end{cases}$$

where $\triangle Z_k^n := Z((k+1)\triangle) - Z(k\triangle)$, and define the continuous EM scheme associated with (5.1.4) by

$$\begin{cases} Y^n(t) := G_n(Y^n(t-\tau)) + e^{tA_n}\{Y^n(0) - G_n(Y^n(-\tau))\} \\ \qquad + \int_0^t e^{(t-\lfloor s\rfloor)A_n} A_n G_n(Y^n(\lfloor s\rfloor - \tau))\mathrm{d}s \\ \qquad + \int_0^t e^{(t-\lfloor s\rfloor)A_n} b_n(Y^n(\lfloor s\rfloor), Y^n(\lfloor s\rfloor - \tau))\mathrm{d}s \\ \qquad + \int_0^t e^{(t-\lfloor s\rfloor)A_n}\mathrm{d}Z^n(s), \\ Y^n(\theta) := \pi_n\xi(\theta),\ \theta \in [-\tau, 0], \end{cases} \tag{5.1.5}$$

where $\lfloor t\rfloor := [t/\triangle]\triangle$ with $[t/\triangle]$ denoting the integer part of $t/\triangle$. By a straightforward computation, we have $Y^n(k\triangle) = \bar{Y}^n(k\triangle), k \geq 0$, that is, the continuous scheme $Y(t)$ coincides with the discrete approximate solution at the gridpoints.

Throughout this chapter, $C > 0$ is a generic constant whose values may change from line to line. Our main result in this chapter is as follows.

Theorem 5.1.2. Let (A1)-(A4) hold and $\alpha \in (1,2)$. Assume further that there exists $L_3 > 0$ such that

$$\|\xi(\theta_1) - \xi(\theta_2)\|_H \leq L_3|\theta_1 - \theta_2|, \quad \theta_1, \theta_2 \in [-\tau, 0]. \tag{5.1.6}$$

Then, for arbitrary $T > 0$ and $p \in (1, \alpha)$,

$$\sup_{0\leq t\leq T} (\mathbb{E}\|X(t) - Y^n(t)\|_H^p)^{1/p} \leq C\{\lambda_n^{-((\kappa-\theta)\wedge\theta)} + \triangle^{(1-\theta)\wedge\theta}\}, \tag{5.1.7}$$

where $C > 0$ is a constant dependent on p, α, θ, T, but independent of n and $\triangle$.

Remark 5.1.2. Kloeden et al. [50] presented an error analysis of EM scheme for semilinear stochastic evolution equation on $H = L^2([a,b]^d)$

$$\mathrm{d}X(t) = [-AX(t) + F(X(t))]\mathrm{d}t + \mathrm{d}W(t), \quad X(0) = x, \tag{5.1.8}$$

where the Burkhold-Davis-Gundy inequality, which is controlled by quadratic processes, is one of the vital tools to cope with the noise term. The approach in [50] does not work for (5.1.2), even for the case $G \equiv 0$ and $\tau \equiv 0$, since α-stable noise only has finite p-th moment for $p \in (0, \alpha)$, where $\alpha \in (0, 2)$.

Remark 5.1.3. For SPDEs driven by multiplicative H-valued α-stable Lévy motions, there are few papers on this topic as stated in [68]. Therefore, in this chapter we only discuss the additive noise cases. On the other hand, some tricks have been utilized in dealing with the neutral term, and, by following these tricks, Theorem 5.1.2 can be extended to the cases of neutral SPDEs driven by additive Wiener processes and multiplicative Poisson jumps.

Remark 5.1.4. In this chapter, we only discuss the convergence of numerical scheme for the case $p \in (1,2)$, while the counterpart for the case $p \in (0,1]$ is still open even for the case $G \equiv 0$ and $\tau \equiv 0$. Priola and Zabczyk [68] discussed the existence and uniqueness of (5.1.8) with $W(t)$ replaced by H-valued α-stable process $Z(t)$ whenever F is bounded. Even for such simple case, the invariant measure of numerical scheme is still open. Therefore, there are still a lot of work to do to investigate the numerical analysis of SPDEs driven by α-stable processes.

In Section 2 we give two auxiliary lemmas, and in Section 3 we complete the proof of Theorem 5.1.2.

5.2 Auxiliary Lemmas

To end the proof of Theorem 5.1.2, we prepare the following two lemmas.

Lemma 5.2.1. Under the assumptions of Theorem 5.1.2,

$$\sup_{0\le t\le T}(\mathbb{E}\|X(t)-X(\lfloor t\rfloor)\|_H^p)^{1/p} \le C\triangle^{(1-\theta)\wedge\theta}, \quad p\in(1,\alpha), \tag{5.2.1}$$

where $C>0$ is a constant dependent on p,α,θ,T but independent of $\triangle$.

Proof. For any $t\in[0,T]$, it is easy to see from (5.1.3) that

$$\begin{aligned}
X(t)-X(\lfloor t\rfloor) &= e^{\lfloor t\rfloor A}(e^{(t-\lfloor t\rfloor)A}-1)\{\xi(0)-G(\xi(-\tau))\} \\
&\quad + \int_0^{\lfloor t\rfloor}(e^{(t-\lfloor t\rfloor)A}-1)e^{(\lfloor t\rfloor-s)A}\mathrm{d}Z(s) + \int_{\lfloor t\rfloor}^t e^{(t-s)A}\mathrm{d}Z(s) \\
&\quad + G(X(t-\tau)) - G(X(\lfloor t\rfloor-\tau)) \\
&\quad + \int_0^{\lfloor t\rfloor}(e^{(t-\lfloor t\rfloor)A}-1)e^{(\lfloor t\rfloor-s)A}AG(X(s-\tau))\mathrm{d}s \\
&\quad + \int_0^{\lfloor t\rfloor}(e^{(t-\lfloor t\rfloor)A}-1)e^{(\lfloor t\rfloor-s)A}b(X(s),X(s-\tau))\mathrm{d}s
\end{aligned}$$

$$+ \int_{\lfloor t \rfloor}^{t} e^{(t-s)A} AG(X(s-\tau))\mathrm{d}s + \int_{\lfloor t \rfloor}^{t} e^{(t-s)A} b(X(s), X(s-\tau))\mathrm{d}s$$
$$=: \sum_{m=1}^{8} I_m(t).$$

Since $(\mathbb{E}\|\cdot\|_H^p)^{1/p}, p \in (1,\alpha)$, is a norm, it then follows that

$$(\mathbb{E}\|X(t) - X(\lfloor t \rfloor)\|_H^p)^{1/p} \le \sum_{m=1}^{8} (\mathbb{E}\|I_m(t)\|_H^p)^{1/p}. \tag{5.2.2}$$

Recall from [65, Theorem 6.13, p74] that

$$\|(-A)^{\delta_1} e^{tA}\| \le Ct^{-\delta_1}, \quad \|(-A)^{-\delta_2}(1 - e^{tA})\| \le Ct^{\delta_2}, \tag{5.2.3}$$

for arbitrary $\delta_1 \ge 0,\ \delta_2 \in [0,1]$, and that

$$(-A)^{\alpha+\beta} x = (-A)^{\alpha}(-A)^{\beta} x, \quad x \in \mathcal{D}((-A)^{\gamma}), \tag{5.2.4}$$

for any $\alpha, \beta \in \mathbb{R}$, where $\gamma := \max\{\alpha, \beta, \alpha+\beta\}$. In the sequel, $\theta \in (0,\kappa)$ is the constant such that (A4). By (5.2.3) and (5.2.4)

$$\begin{aligned}
&\|I_1(t)\|_H \\
&= \|(-A)^{-\theta} e^{\lfloor t \rfloor A} (-A)^{\theta-1}\{e^{(t-\lfloor t \rfloor)A} - 1\}(-A)\{\xi(0) - G(\xi(-\tau))\}\|_H \\
&\le \|(-A)^{-\theta} e^{\lfloor t \rfloor A}\| \cdot \|(-A)^{\theta-1}\{e^{(t-\lfloor t \rfloor)A} - 1\}\| \\
&\quad \times \|(-A)\{\xi(0) - G(\xi(-\tau))\}\|_H \\
&\le C\|(-A)^{-\theta}\| \cdot \|A\{\xi(0) - G(\xi(-\tau))\}\|_H \Delta^{(1-\theta)},
\end{aligned} \tag{5.2.5}$$

where we have also used the boundedness of $(-A)^{-\theta}$, e.g., [65, Lemma 6.3, p71], and the contractive property of e^{tA} due to (A1). We point out that the following idea is taken from that of [68, Theorem 4.4]. Let $\{r_m\}_{m\ge1}$ be a Rademacher sequence defined on a new probability space $(\Omega', \mathscr{F}', \{\mathscr{F}'_t\}_{t\ge0}, \mathbb{P}')$, i.e., $r_m : \Omega' \to \{1,-1\}$ are independent and identically distributed with $\mathbb{P}'(r_m = 1) = \mathbb{P}'(r_m = -1) = 1/2$. By the Khintchine inequality, i.e., for a sequence of real numbers $\{c_m\}_{m\ge1}$ and any $q > 0$, there exists $c_q > 0$ such that

$$\left(\sum_{m\ge1} c_m^2\right)^{1/2} \le c_q \left(\mathbb{E}'\left|\sum_{m\ge1} r_m c_m\right|^q\right)^{1/q}.$$

Set

$$I_9(t) := \int_0^{\lfloor t \rfloor} (-A)^\theta e^{(\lfloor t \rfloor - s)A} \mathrm{d}Z(s) = \sum_{i=1}^\infty \Lambda_i(t) e_i,$$

where $\Lambda_m(t) := \int_0^{\lfloor t \rfloor} \beta_m \lambda_m^\theta e^{-\lambda_m(\lfloor t \rfloor - s)} \mathrm{d}Z_m(s)$. By the Fubini theorem

$$\begin{aligned}
\mathbb{E}\|I_9(t)\|_H^p = \mathbb{E}(\|I_9(t)\|_H^2)^{p/2} &= \mathbb{E}\left(\sum_{m=1}^\infty \left(\int_0^{\lfloor t \rfloor} \beta_m \lambda_m^\theta e^{-\lambda_m(\lfloor t \rfloor - s)} \mathrm{d}Z_m(s)\right)^2\right)^{p/2} \\
&\le c_p^p \mathbb{E}\mathbb{E}' \left|\sum_{m=1}^\infty r_m \Lambda_m(t)\right|^p = c_p^p \mathbb{E}'\mathbb{E}\left|\sum_{m=1}^\infty r_m \Lambda_m(t)\right|^p.
\end{aligned}$$

Recall the inner clock property of stable stochastic integral, i.e., for deterministic function $f : [0, t] \to \mathbb{R}_+$ such that $\int_0^t f^\alpha(s)\mathrm{d}s < \infty$, there exists a symmetric α-stable process Z' such that $\mathscr{L}(Z') = \mathscr{L}(Z)$ and $\int_0^t f(s)\mathrm{d}Z(s) = Z'(\tau_t)$, where $\mathscr{L}(Z)$ denotes the law of Z and $\tau_t := \int_0^t f^\alpha(s)\mathrm{d}s$. Thus, for any $u \in \mathbb{R}$, by (5.1.1) and the independence of $\{Z_m(t)\}_{m\ge1}$

$$\begin{aligned}
\mathbb{E}\exp\left(iu\sum_{m=1}^\infty r_m\Lambda_m(t)\right) &= \prod_{m=1}^\infty \mathbb{E}\exp(iur_m\Lambda_m(t)) \\
&= \prod_{m=1}^\infty \mathbb{E}\exp(iu\mathrm{sgn}(r_m)\mathrm{sgn}(r_m)r_m\Lambda_m(t)) \\
&= \prod_{m=1}^\infty \exp\Big(-|u\mathrm{sgn}(r_m)|^\alpha \\
&\quad \times \int_0^{\lfloor t \rfloor} (\mathrm{sgn}(r_m) r_m \beta_m \lambda_m^\theta e^{-\lambda_m(\lfloor t \rfloor - s)})^\alpha \mathrm{d}s\Big) \\
&= \exp\Big(-|u|^\alpha \sum_{m=1}^\infty \beta_m^\alpha \lambda_m^{\alpha\theta} \int_0^{\lfloor t \rfloor} e^{-\alpha\lambda_m s}\mathrm{d}s\Big)
\end{aligned}$$

due to $\mathrm{sgn}(r_m)r_m = 1$. Thus, in view of [68, (3.2)] we arrive at

$$\mathbb{E}\left|\sum_{m=1}^\infty r_m\Lambda_m(t)\right|^p = c_{\alpha,p}\left(\sum_{m=1}^\infty \beta_m^\alpha \frac{1}{\alpha\lambda_m^{1-\alpha\theta}}(1 - e^{-\alpha\lambda_m\lfloor t \rfloor})\right)^{p/\alpha},$$

where the explicit form of $c_{\alpha,p}$ is given in [77, p18], and therefore

$$\mathbb{E}\|I_9(t)\|_H^p \le c_p^p c_{\alpha,p}(\delta/\alpha)^{p/\alpha}. \tag{5.2.6}$$

According to (5.2.3) and (5.2.6), one thus has

$$\begin{aligned}
\mathbb{E}\|I_2(t)\|_H^p &= \mathbb{E}\|(-A)^{-\theta}(e^{(t-\lfloor t \rfloor)A} - 1)I_9(t)\|_H^p \\
&\le C\triangle^{p\theta}.
\end{aligned} \tag{5.2.7}$$

Noting that, for arbitrary $\gamma \in (0,1)$ there exists $c_\gamma > 0$ such that

$$|e^{-x} - e^{-y}| \le c_\gamma |x-y|^\gamma, \quad x, y \ge 0, \tag{5.2.8}$$

and following the argument of that of (5.2.6), we deduce that

$$\begin{aligned}\mathbb{E}\|I_3(t)\|_H^p &\le C\Big(\sum_{m=1}^\infty \beta_m^\alpha \int_{\lfloor t\rfloor}^t e^{-\alpha\lambda_m s}\mathrm{d}s\Big)^{p/\alpha}\\ &= C\Big(\sum_{m=1}^\infty \frac{\beta_m^\alpha}{\alpha\lambda_m}(e^{-\alpha\lambda_m\lfloor t\rfloor} - e^{-\alpha\lambda_m t})\Big)^{p/\alpha}\\ &\le C\Big(\sum_{m=1}^\infty \frac{\beta_m^\alpha}{\alpha\lambda_m} c_\gamma(\alpha\lambda_m)^\gamma \triangle^\gamma\Big)^{p/\alpha}, \quad \gamma \in (0,1).\end{aligned}$$

In particular, taking $\gamma = \alpha\theta \in (0,1)$ gives

$$\mathbb{E}\|I_3(t)\|_H^p \le C\delta(c_\gamma\alpha^{\alpha\theta-1})^{p/\alpha}\triangle^{p\theta}. \tag{5.2.9}$$

Recall from [38, Theorem 202] the Minkowski integral inequality:

$$\Big(\mathbb{E}\Big|\int_0^t F(s)\mathrm{d}s\Big|^p\Big)^{1/p} \le \int_0^t \Big(\mathbb{E}|F(s)|^p\Big)^{1/p}\mathrm{d}s, \quad t \in [0,T], \tag{5.2.10}$$

where $F : [0,T]\times\Omega \to R$ is measurable. The previous inequality, combining (A1) with (5.2.3) and (5.2.4), gives that for $\theta \in (0,\kappa)$

$$\begin{aligned}&\sum_{m=5}^8 (\mathbb{E}\|I_m(t)\|_H^p)^{1/p}\\ &\le \int_0^{\lfloor t\rfloor} \|(-A)^{-\theta}(e^{(t-\lfloor t\rfloor)A} - 1)\| \cdot \|e^{(\lfloor t\rfloor - s)A}(-A)^{1-\kappa+\theta}\\ &\quad \times \|(\mathbb{E}\|(-A)^\kappa G(X(s-\tau))\|_H^p)^{1/p}\mathrm{d}s\\ &\quad + \int_0^{\lfloor t\rfloor} \|(-A)^{-\theta}(e^{(t-\lfloor t\rfloor)A} - 1)\|\\ &\quad \times \|(-A)^\theta e^{(\lfloor t\rfloor - s)A}\|(\mathbb{E}\|b(X(s))\|_H^p)^{1/p}\mathrm{d}s\\ &\quad + \int_{\lfloor t\rfloor}^t \|(-A)^{1-\kappa}e^{(t-s)A}\|(\mathbb{E}\|(-A)^\kappa G(X(s-\tau))\|_H^p)^{1/p}\mathrm{d}s\\ &\quad + \int_{\lfloor t\rfloor}^t (\mathbb{E}\|b(X(s))\|_H^p)^{1/p}\mathrm{d}s\\ &\le C\Big(1 + \sup_{0\le t\le T}(\mathbb{E}\|X(t)\|_H^p)^{1/p}\Big)\triangle^\theta,\end{aligned} \tag{5.2.11}$$

where we have also utilized (A2) and (A3). Substituting (5.2.5), (5.2.7), (5.2.9) and (5.2.11) into (5.2.2), we arrive at

$$(\mathbb{E}\|X(t)-X(\lfloor t\rfloor)\|_H^p)^{1/p} \le C\Big(1+\sup_{0\le t\le T}(\mathbb{E}\|X(t)\|_H^p)^{1/p}\Big)\triangle^{(1-\theta)\wedge\theta}+(\mathbb{E}\|I_4(t)\|_H^p)^{1/p}.$$

Furthermore, in the light of (A3)

$$\begin{aligned}(\mathbb{E}\|I_4(t)\|_H^p)^{1/p} &= (\mathbb{E}\|(-A)^{-\kappa}(-A)^{\kappa}(G(X(t-\tau))-G(X(\lfloor t\rfloor-\tau)))\|_H^p)^{1/p}\\ &\le \|(-A)^{-\kappa}\|L_2(\mathbb{E}\|X(t-\tau)-X(\lfloor t\rfloor-\tau)\|_H^p)^{1/p}.\end{aligned}$$

Hence we arrive at

$$\begin{aligned}(\mathbb{E}\|X(t)-X(\lfloor t\rfloor)\|_H^p)^{1/p} &\le C\Big(1+\sup_{0\le t\le T}(\mathbb{E}\|X(t)\|_H^p)^{1/p}\Big)\triangle^{(1-\theta)\wedge\theta}\\ &\quad+\|(-A)^{-\kappa}\|L_2(\mathbb{E}\|X(t-\tau)-X(\lfloor t\rfloor-\tau)\|_H^p)^{1/p}.\end{aligned}$$

By (5.1.6), one has

$$\sup_{0\le t\le\tau}(\mathbb{E}\|X(t)-X(\lfloor t\rfloor)\|_H^p)^{1/p} \le C\Big(1+\sup_{0\le t\le T}(\mathbb{E}\|X(t)\|_H^p)^{1/p}\Big)\triangle^{(1-\theta)\wedge\theta}.$$

In the sequel, an induction argument yields that

$$\sup_{t\in[0,T]}(\mathbb{E}\|X(t)-X(\lfloor t\rfloor)\|_H^p)^{1/p} \le C\Big(1+\sup_{0\le t\le T}(\mathbb{E}\|X(t)\|_H^p)^{1/p}\Big)\triangle^{(1-\theta)\wedge\theta}.$$

Thus, the desired assertion follows from Lemma 5.2.2 below. □

Lemma 5.2.2. Under (A1)-(A4),

$$\sup_{0\le t\le T}(\mathbb{E}\|X(t)\|_H^p)^{1/p}\vee\sup_{0\le t\le T}(\mathbb{E}\|Y^n(t)\|_H^p)^{1/p}\le C, \tag{5.2.12}$$

where $C>0$ is a constant dependent on p,α,θ,T but independent of $\triangle$.

Proof. Also, due to the fact that $(\mathbb{E}\|\cdot\|_H^p)^{1/p}, p\in(1,\alpha)$, is a norm, we derive from (A1)-(A3) and (5.2.10) that

$$\begin{aligned}(\mathbb{E}\|X(t)\|_H^p)^{1/p} &\le \|\xi(0)-G(\xi(-\tau))\|_H+\|(-A)^{-\kappa}L_2\|(\mathbb{E}\|X(t-\tau)\|_H^p)^{1/p}\\ &\quad+CL_2\int_0^t(t-s)^{-(1-\kappa)}(\mathbb{E}\|X(s-\tau)\|_H^p)^{1/p}\mathrm{d}s\\ &\quad+C\int_0^t\{1+(\mathbb{E}\|X(s)\|^p)^{1/p}+(\mathbb{E}\|X(s-\tau)\|^p)^{1/p}\}\mathrm{d}s\\ &\quad+\Big(\mathbb{E}\Big\|\int_0^t e^{(t-s)A}\mathrm{d}Z(s)\Big\|_H^p\Big)^{1/p}.\end{aligned}$$

Following the argument of (5.2.6), one has

$$\mathbb{E}\Big\|\int_0^t e^{(t-s)A}\mathrm{d}Z(s)\Big\|_H^p \le c_p\Big(\sum_{m\ge1}\beta_m^\alpha\frac{1-e^{-\alpha\lambda_m t}}{\alpha\lambda_m}\Big)^{p/\alpha} \le c_p(\delta/(\alpha\lambda_1^{\alpha\theta}))^{p/\alpha}.$$

Hence

$$\begin{aligned}(\mathbb{E}\|X(t)\|_H^p)^{1/p} \le C\Big\{1 &+ (\mathbb{E}\|X(t-\tau)\|_H^p)^{1/p} + \int_0^t (\mathbb{E}\|X(s)\|^p)^{1/p}\mathrm{d}s \\ &+ \int_0^t (t-s)^{-(1-\kappa)}(\mathbb{E}\|X(s-\tau)\|_H^p)^{1/p}\mathrm{d}s\Big\}, \quad t\in[0,T].\end{aligned}$$

Applying the Gronwall inequality leads to

$$\sup_{0\le t\le\tau}(\mathbb{E}\|X(t)\|_H^p)^{1/p} \le C.$$

Then, the first assertion of (5.2.12) follows by an induction argument and the second assertion also holds by noting that $Ax = A_n x$ for $x \in H_n$ and that

$$\begin{aligned}\|(-A)^\kappa G_n(x)\|_H &= \Big\|\sum_{m=1}^\infty \lambda_m^\kappa\langle G_n(x), e_m\rangle_H e_m\Big\|_H \\ &= \Big(\sum_{m=1}^n \lambda_m^{2\kappa}\langle G(x), e_m\rangle_H^2\Big)^{1/2} \qquad (5.2.13)\\ &\le \|(-A)^\kappa G(x)\|_H, \quad x\in H.\end{aligned}$$

□

Remark 5.2.1. For $p \in (1,\alpha)$, applying, in general, the Hölder inequality yields that

$$\begin{aligned}&\Big\|\int_0^t e^{(t-\lfloor s\rfloor)A_n}A_nG_n(Y^n(\lfloor s\rfloor-\tau))\mathrm{d}s\Big\|_H^p \\ &\le C\Big(\int_0^t (t-s)^{-\frac{p(1-\kappa)}{p-1}}\mathrm{d}s\Big)^{(p-1)/p}\int_0^t \|Y^n(\lfloor s\rfloor-\tau)\|_H^p\mathrm{d}s.\end{aligned}$$

Thus, to guarantee that the integral $\int_0^t (t-s)^{-\frac{p(1-\kappa)}{p-1}}\mathrm{d}s$ is finite, it is sufficient to require $1/\kappa < p < \alpha$. While by the argument of Lemma 5.2.2, (5.2.12) holds for all $p \in (1,\alpha)$.

5.3 Proof of Theorem 5.1.2

By Lemma 5.2.1 and Lemma 5.2.2, in the sequel we shall complete the proof of Theorem 5.1.2.

Proof of Theorem 5.1.2. By (5.1.3) and (5.1.5),

$$
\begin{aligned}
&X(t) - Y^n(t) \\
&= e^{tA}(1-\pi_n)\{\xi(0) - G(\xi(-\tau))\} + e^{tA}\{G_n(\pi_n\xi(-\tau)) - G_n(\xi(-\tau))\} \\
&\quad + G(Y^n(t-\tau)) - G_n(Y^n(t-\tau)) + G(X(t-\tau)) - G(Y^n(t-\tau)) \\
&\quad + \int_0^t e^{(t-s)A}A\{G(X(s-\tau)) - G_n(X(s-\tau))\}\mathrm{d}s \\
&\quad + \int_0^t e^{(t-s)A}\{b(X(s), X(s-\tau)) - b_n(X(s), X(s-\tau))\}\mathrm{d}s \\
&\quad + \int_0^t e^{(t-s)A}A\{G_n(X(s-\tau)) - G_n(X(\lfloor s\rfloor-\tau))\}\mathrm{d}s \\
&\quad + \int_0^t e^{(t-s)A}\{b_n(X(s), X(s-\tau)) - b_n(X(\lfloor s\rfloor), X(\lfloor s\rfloor-\tau))\}\mathrm{d}s \\
&\quad + \int_0^t e^{(t-s)A}A\{G_n(X(\lfloor s\rfloor-\tau)) - G_n(Y^n(\lfloor s\rfloor-\tau))\}\mathrm{d}s \\
&\quad + \int_0^t e^{(t-s)A}\{b_n(X(\lfloor s\rfloor), X(\lfloor s\rfloor-\tau)) - b_n(Y^n(\lfloor s\rfloor), Y^n(\lfloor s\rfloor-\tau))\}\mathrm{d}s \\
&\quad + \int_0^t e^{(t-s)A}A\{1 - e^{(s-\lfloor s\rfloor)A}\}G_n(Y^n(\lfloor s\rfloor-\tau))\mathrm{d}s \\
&\quad + \int_0^t e^{(t-s)A}\{1 - e^{(s-\lfloor s\rfloor)A}\}b_n(Y^n(\lfloor s\rfloor), Y^n(\lfloor s\rfloor-\tau))\mathrm{d}s \\
&\quad + \int_0^t \{e^{(t-s)A} - e^{(t-\lfloor s\rfloor)A}\}\mathrm{d}Z^n(s) + \int_0^t e^{(t-s)A}\mathrm{d}(Z(s) - Z^n(s)) \\
&=: \sum_{m=1}^{14} J_m(t).
\end{aligned}
$$

Noting that $(\mathbb{E}\|\cdot\|_H^p)^{1/p}$ is a norm, one has

$$
(\mathbb{E}\|X(t) - Y^n(t)\|_H^p)^{1/p} \le \sum_{m=1}^{14}(\mathbb{E}\|J_m(t)\|_H^p)^{1/p}. \tag{5.3.1}
$$

Due to (A1), a straightforward computation shows that

$$
\begin{aligned}
\|(1-\pi_n)x\|_H &= \Big(\sum_{m=n+1}^{\infty}\langle x, e_m\rangle_H^2\Big)^{1/2} \\
&= \Big(\sum_{m=n+1}^{\infty}\frac{1}{\lambda_m^2}\lambda_m^2\langle x, e_m\rangle_H^2\Big)^{1/2} \\
&\le \frac{1}{\lambda_n}\|Ax\|_H, \quad x \in H
\end{aligned} \tag{5.3.2}
$$

by recalling that $\{\lambda_m\}_{m\ge1}$ is nondecreasing and observing from (A1) that

$$
\|Ax\|_H^2 = \Big\|A\Big(\sum_{m=1}^{\infty}\langle x, e_m\rangle_H e_m\Big)\Big\|_H^2 = \sum_{m=1}^{\infty}\lambda_m^2\langle x, e_m\rangle_H^2, \quad x \in \mathcal{D}(A).
$$

By (A1), (A3) and (5.2.13)

$$\begin{aligned}&\|e^{tA}\{G_n(\xi(-\tau)) - G_n(\pi_n\xi(-\tau))\}\|_H \\ &\qquad \le \|(-A)^{-\kappa}\| \cdot \|(-A)^{\kappa}\{G(\xi(-\tau)) - G(\pi_n\xi(-\tau))\}\|_H \\ &\qquad \le L_2\|(-A)^{-\kappa}\| \cdot \|(1-\pi_n)\xi(-\tau)\|_H.\end{aligned} \tag{5.3.3}$$

Moreover notice that for $\mu \in [0,\kappa)$ and $x \in H_n$

$$\begin{aligned}&\|(-A)^{\mu}\{G(x) - G_n(x)\}\|_H \\ &= \Big(\sum_{m=n+1}^{\infty} \lambda_m^{-2(\kappa-\mu)}\lambda_m^{2\kappa}\langle G(x), e_m\rangle_H^2\Big)^{1/2} \\ &\le \lambda_n^{-(\kappa-\mu)}\Big(\sum_{m=n+1}^{\infty} \lambda_m^{2\kappa}\langle G(x), e_m\rangle_H^2\Big)^{1/2} \\ &\le \lambda_n^{-(\kappa-\mu)}\|(-A)^{\kappa}G(x)\|_H \le L_2\lambda_n^{-(\kappa-\mu)}\|x\|_H.\end{aligned} \tag{5.3.4}$$

Applying (5.3.4) with $\mu = 0$ and taking into consideration (5.3.2) and (5.3.3), we get

$$\sum_{m=1}^{3}(\mathbb{E}\|J_m(t)\|_H^p)^{1/p} \le \frac{C}{\lambda_n^{\kappa}}. \tag{5.3.5}$$

Thanks to (A3)

$$\|J_4(t)\|_H \le L_2\|(-A)^{-\kappa}\| \cdot \|X(t-\tau) - Y^n(t-\tau)\|_H.$$

Next, by (5.2.3), (5.2.10), (5.2.12), (5.2.13) and (5.3.4) it follows that

$$\begin{aligned}&\sum_{m=5}^{6}(\mathbb{E}\|J_m(t)\|_H^p)^{1/p} \\ &\le C\int_0^t (t-s)^{1-\theta}\Big(\mathbb{E}\Big\{\sum_{m=n+1}^{\infty}\lambda_m^{-2(\kappa-\theta)}\lambda_m^{2\kappa}\langle G(X(s-\tau)), e_m\rangle_H^2\Big\}^{p/2}\Big)^{1/p} \\ &\quad + \int_0^t e^{-\lambda_n(t-s)}\Big(\mathbb{E}\Big\{\sum_{m=n+1}^{\infty}\langle b(X(s), X(s-\tau)), e_m\rangle_H^2\Big\}^{p/2}\Big)^{1/p}\mathrm{d}s \\ &\le C\lambda_n^{-(\kappa-\theta)}\int_0^t (t-s)^{1-\theta}(\mathbb{E}\|(-A)^{\kappa}G(X(s-\tau))\|_H^p)\Big)^{1/p} \\ &\quad + \int_0^t e^{-\lambda_n(t-s)}(\mathbb{E}\|b(X(s), X(s-\tau))\|_H^p)^{1/p}\mathrm{d}s \\ &\le C\lambda_n^{-(\kappa-\theta)}\int_0^t (t-s)^{1-\theta}(\mathbb{E}\|X(s-\tau)\|_H^p)\Big)^{1/p} \\ &\quad + \int_0^t e^{-\lambda_n(t-s)}\{1 + (\mathbb{E}\|X(s)\|_H^p)^{1/p} + (\mathbb{E}\|X(s-\tau)\|_H^p)^{1/p}\}\mathrm{d}s \\ &\le C\lambda_n^{-(\kappa-\theta)}.\end{aligned}$$

Applying (A1)-(A3), (5.2.3) and (5.2.1) one has

$$\begin{aligned}
&\sum_{m=7}^{10}(\mathbb{E}\|J_m(t)\|_H^p)^{1/p}\\
&\leq C\int_0^t (t-s)^{1-\kappa}(\mathbb{E}\|X(s-\tau)-X(\lfloor s\rfloor-\tau)\|_H^p)^{1/p}\mathrm{d}s\\
&\quad + C\int_0^t \{(\mathbb{E}\|X(s)-X(\lfloor s\rfloor)\|_H^p)^{1/p}\\
&\quad + (\mathbb{E}\|X(s-\tau)-X(\lfloor s\rfloor-\tau)\|_H^p)^{1/p}\}\mathrm{d}s\\
&\quad + C\int_0^t (t-s)^{1-\kappa}(\mathbb{E}\|X(\lfloor s\rfloor-\tau)-Y^n(\lfloor s\rfloor-\tau)\|_H^p)^{1/p}\mathrm{d}s\\
&\quad + C\int_0^t \{(\mathbb{E}\|X(\lfloor s\rfloor)-Y^n(\lfloor s\rfloor)\|_H^p)^{1/p}\\
&\quad + (\mathbb{E}\|X(\lfloor s\rfloor-\tau)-Y^n(\lfloor s\rfloor-\tau)\|_H^p)^{1/p}\}\mathrm{d}s\\
&\leq C\triangle^{(1-\theta)\wedge\theta} + C\int_0^t (t-s)^{1-\kappa}(\mathbb{E}\|X(\lfloor s\rfloor-\tau)-Y^n(\lfloor s\rfloor-\tau)\|_H^p)^{1/p}\mathrm{d}s\\
&\quad + C\int_0^t \{(\mathbb{E}\|X(\lfloor s\rfloor)-Y^n(\lfloor s\rfloor)\|_H^p)^{1/p}\\
&\quad + (\mathbb{E}\|X(\lfloor s\rfloor-\tau)-Y^n(\lfloor s\rfloor-\tau)\|_H^p)^{1/p}\}\mathrm{d}s.
\end{aligned}$$

Carrying out a similar argument to that of (5.2.11), we derive from (5.2.12) that

$$(\mathbb{E}\|J_{11}(t)\|_H^p)^{1/p} + (\mathbb{E}\|J_{12}(t)\|_H^p)^{1/p} \leq C\triangle^{\theta}.$$

Furthermore, also following the procedure of (5.2.6) and utilizing (A3) gives that

$$\begin{aligned}
\mathbb{E}\|J_{13}(t)\|_H^p &\leq C\Big(\sum_{m=1}^n \beta_m^\alpha\int_0^t e^{-\alpha\lambda_m(t-s)}(1-e^{-\lambda_m(s-\lfloor s\rfloor)})^\alpha\mathrm{d}s\Big)^{p/\alpha}\\
&= C\Big(\sum_{m=1}^n \beta_m^\alpha\int_0^t (\lambda_m^{\alpha\theta}e^{-\alpha\lambda_m(t-s)})\lambda_m^{-\alpha\theta}(1-e^{-\lambda_m(s-\lfloor s\rfloor)})^\alpha\mathrm{d}s\Big)^{p/\alpha}\\
&\leq C\triangle^{p\theta}\Big(\sum_{m=1}^n \beta_m^\alpha\int_0^t \lambda_m^{\alpha\theta}e^{-\alpha\lambda_m(t-s)}\mathrm{d}s\Big)^{p/\alpha}\\
&\leq C\triangle^{p\theta}
\end{aligned}$$

where we have also used $1-e^{-\lambda_m(s-\lfloor s\rfloor)} \leq c_\theta\lambda_m^\theta(s-\lfloor s\rfloor)^\theta$ by (5.2.8), and that

$$\begin{aligned}
\mathbb{E}\|J_{14}(t)\|_H^p &\leq C\Big(\sum_{m=n+1}^\infty \beta_m^\alpha\int_0^t e^{-\alpha\lambda_m(t-s)}\mathrm{d}s\Big)^{p/\alpha} \leq C\Big(\sum_{m=n+1}^\infty \frac{\beta_m^\alpha}{\alpha\lambda_m}\Big)^{p/\alpha}\\
&\leq \frac{C}{\lambda_n^{p\theta}}\Big(\sum_{m=n+1}^\infty \frac{\beta_m^\alpha}{\alpha\lambda_m^{1-\alpha\theta}}\Big)^{p/\alpha} \qquad (5.3.6)\\
&\leq \frac{C}{\lambda_n^{p\theta}}.
\end{aligned}$$

As a results, putting (5.3.5)-(5.3.6) into (5.3.1) gives that

$$
\begin{aligned}
&(\mathbb{E}\|X(t) - Y^n(t)\|_H^p)^{1/p} \\
&\le C(\lambda_n^{-\theta} + \lambda_n^{-(\kappa-\theta)} + \triangle^{(1-\theta)\wedge\theta}) \\
&\quad + L_2\|(-A)^{-\kappa}\|(\mathbb{E}\|X(t-\tau) - Y^n(t-\tau)\|_H^p)^{1/p} \\
&\quad + C\int_0^t (t-s)^{1-\kappa}(\mathbb{E}\|X(\lfloor s\rfloor - \tau) - Y^n(\lfloor s\rfloor - \tau)\|_H^p)^{1/p}\mathrm{d}s \\
&\quad + C\int_0^t \{(\mathbb{E}\|X(\lfloor s\rfloor) - Y^n(\lfloor s\rfloor)\|_H^p)^{1/p} \\
&\quad + (\mathbb{E}\|X(\lfloor s\rfloor - \tau) - Y^n(\lfloor s\rfloor - \tau)\|_H^p)^{1/p}\}\mathrm{d}s.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\sup_{0\le s\le t}(\mathbb{E}\|X(s) - Y^n(s)\|_H^p)^{1/p} \\
&\le C\Big\{(\lambda_n^{-\theta} + \lambda_n^{-(\kappa-\theta)} + \triangle^{(1-\theta)\wedge\theta}) \\
&\quad + \sup_{-\tau\le s\le t-\tau}\|(\mathbb{E}\|X(s) - Y^n(s)\|_H^p)^{1/p} \\
&\quad + C\int_0^t (t-s)^{1-\kappa}\sup_{-\tau\le r\le s-\tau}(\mathbb{E}\|X(r) - Y^n(r)\|_H^p)^{1/p}\mathrm{d}s \\
&\quad + C\int_0^t\{\sup_{0\le r\le s}(\mathbb{E}\|X(r) - Y^n(r)\|_H^p)^{1/p} \\
&\quad + \sup_{-\tau\le r\le s-\tau}(\mathbb{E}\|X(r) - Y^n(r)\|_H^p)^{1/p}\}\mathrm{d}s\Big\}.
\end{aligned}
$$

Finally, (5.1.7) follows from an induction argument and the Gronwall inequality.

Chapter 6

Numerical Approximation of Stationary Distribution for SPDEs

In this chapter, we discuss (i) the existence and uniqueness of the stationary distribution of explicit EM scheme both in time and in space for a class of stochastic partial differential equations whenever the stepsize is sufficiently small, and (ii) show that the stationary distribution of the EM scheme converges weakly to the counterpart of the SPDEs.

6.1 Introduction

Numerical (approximate) schemes of stochastic partial differential equations (SPDEs) are becoming more and more popular nowadays since there are only a few SPDEs which have explicit formulae. There are extensive literature on approximate solutions of SPDEs. Under a dissipative condition, Caraballo and Kloeden [17] showed the pathwise convergence of finite-dimensional approximations for a class of reaction-diffusion equations. Applying the Malliavin calculus approach, Debussche [22] discussed the error of the Euler scheme applied to an SPDE. Greksch and Kloeden [32] investigated the approximation of parabolic SPDEs through eigenfunction argument. Gyöngy [35], Shardlow

[81], and Yoo [99] applied finite differences to approximate the mild solutions of parabolic SPDEs driven by space-time white noise. Hausenblas [39, 40] utilized space discretization and time discretization, including implicit Euler, explicit Euler scheme and Crank-Nicholson scheme, to approximate quasi linear evolution equations. Higher order pathwise numerical approximations of SPDEs with additive noise was considered in [47]. For the Taylor approximations of SPDEs, we refer to the monograph [48].

Most of the existing literature is concerned with the (strong or weak) convergence of numerical approximate solutions of SPDEs. We also point out that Bao et al. [5] investigated the existence and uniqueness of stationary distributions of analytic mild solutions for a class of SPDEs, while the stationary distribution (or stability) of numerical solutions of infinite-dimensional SPDEs is seldom discussed. Motivated by the papers above, for the explicit EM (6.2.7) based on the time-discretization and spatial discretization, two questions are natural to be put forward, i.e.,

- For what choices of the stepsize does the numerical scheme (6.2.7) have a unique stationary distribution;
- Will the stationary distribution of EM scheme converge weakly to some probability measure whenever the dimension of finite-dimensional approximation is sufficiently large and the stepsize is sufficiently small ? If so, what's the limit probability measure ?

In this chapter, we shall give the positive answers to these two questions one-by-one.

It is also worth pointing out that, for finite-dimensional case, Yuan and Mao [100] studied the invariant measure of EM numerical solutions for a class of SDEs, and Yevik and Zhao [97] discussed by the global attractor approach the existence of stationary distribution of EM scheme for SDEs generating random dynamical systems. While, for the mild solutions of SPDEs, the explicit EM schemes are much more complicated than those for finite-dimensional SDEs, and moreover the diffusion coefficient in our case is not Hilbert-Schmidt, which leads to be unavailable of the Itô formula. Therefore, our approaches are different from those of [97, 100]. What's more, Bréhier [12] investigated the existence of invariant measure for semi-implicit Euler scheme (in time), and discussed the numerical approximation of the invariant measure for a class of parabolic SPDEs driven by additive noise, where the drift coefficient is assumed to be bounded.

The organization of this chapter goes as follows: In Section 2, we give the explicit EM scheme both in time discretization and in spatial discretization, and show the existence and uniqueness of stationary distribution of EM scheme under the properties ($\mathbb{P}$1) and ($\mathbb{P}$2); To make the result more applicable, sufficient criteria for ($\mathbb{P}$1) and ($\mathbb{P}$2) are provided in Section 3; In the last section, we reveal that the limit probability measure of stationary distribution for EM scheme is in fact the stationary distribution of the exact mild solution.

6.2 Stationary Distribution for EM Scheme

Let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ be a real separable Hilbert space, and $W(t)$ an H-valued cylindrical Wiener process defined on some complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ equipped with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Denote by $(\mathscr{L}(H), \|\cdot\|)$ and $(\mathscr{L}_{HS}(H), \|\cdot\|_{HS})$ the family of bounded linear operators and Hilbert-Schmidt operators from H into H, respectively.

Consider SPDE on H

$$\mathrm{d}X(t) = [AX(t) + b(X(t))]\mathrm{d}t + \sigma(X(t))\mathrm{d}W(t) \tag{6.2.1}$$

with initial value $X(0) = x \in H$. Here $b : H \to H, \sigma(x) := \sigma^0 + \sigma^1(x), x \in H$, where $\sigma^0 \in \mathscr{L}(H)$ and $\sigma^1 : H \to \mathscr{L}_{HS}(H)$.

Throughout this chapter we impose the following assumptions:

(H1) A is a self-adjoint operator on H generating a C_0-semigroup $\{e^{tA}\}_{t\geq 0}$, such that $\|e^{tA}\| \leq e^{-\alpha t}$ for some $\alpha > 0$. In this case $-A$ has discrete spectrum $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lim_{i\to\infty}\lambda_i = \infty$ with corresponding eigenbasis $\{e_i\}_{i\geq 1}$ of H.

(H2) There exist $\theta_1 \in (0,1)$ and $\delta_1 \in (0,\infty)$ such that $\int_0^t \|(-A)^{\theta_1} e^{sA}\sigma^0\|_{HS}^2 \mathrm{d}s \leq \delta_1$ for any $t>0$, where $(-A)^{\theta_1} := \sum_{k\geq 1} \lambda_k^{\theta_1}(e_k \otimes e_k)$ denotes the fractional power of the operator $-A$.

(H3) There exist $L_1, L_2 > 0$ such that

$$\|b(x)-b(y)\|_H \leq L_1\|x-y\|_H \text{ and } \|\sigma^1(x)-\sigma^1(y)\|_{HS} \leq L_2\|x-y\|_H, \quad x,y \in H.$$

(H4) There exists $\gamma \in \mathbb{R}$ such that

$$2\langle x-y, b(x)-b(y)\rangle_H + \|\sigma^1(x)-\sigma^1(y)\|_{HS}^2 \leq -\gamma\|x-y\|_H^2, \quad x,y \in H.$$

(H1)-(H3) imply the existence and the uniqueness of the mild solution to (6.2.1), that is, for any $x \in H$ there exists a unique H-valued adapted process $X(t)$ such that

$$X(t) = e^{tA}x + \int_0^t e^{(t-s)A} b(X(s))\mathrm{d}s + \int_0^t e^{(t-s)A}\sigma(X(s))\mathrm{d}W(s). \tag{6.2.2}$$

Remark 6.2.1. In fact, under (H1), (H3) and $\int_0^t \|e^{sA}\sigma^0\|_{HS}^2 \mathrm{d}s \leq \delta_2$ for any $t>0$ and some $\delta_2 > 0$, (6.2.1) also admits a unique mild solution on H. While (H2) is just imposed for the later numerical analysis. Let $\sigma^0 = \mathbf{1}$, an identity operator, and $Ax := \partial_\xi^2 x$ for $x \in \mathcal{D}(A) := H^2(0,\pi) \cap H_0^1(0,\pi)$. Then A is a self-adjoint negative operator and $Ae_k = -k^2 e_k$, $k \in \mathbb{N}$, where $e_k(\xi) := (2/\pi)^{1/2}\sin k\xi$, $\xi \in [0,\pi]$, $k \in \mathbb{N}$. A simple computation shows that

$$\int_0^t \|(-A)^{\theta_1} e^{sA}\|_{HS}^2 \mathrm{d}s = \sum_{k=1}^\infty (k^2)^{2\theta_1} \int_0^t e^{-2k^2 s}\mathrm{d}s \leq \frac{1}{2}\sum_{k=1}^\infty (k^2)^{2\theta_1 - 1}.$$

Then (H2) holds with $\delta_1 = \frac{1}{2}\sum_{k=1}^\infty (k^2)^{2\theta_1-1}$ for $\theta_1 \in (0, 1/4)$.

Remark 6.2.2. By (H3), it is easy to see that

$$\|b(x)\|_H^2 + \|\sigma^1(x)\|_{HS}^2 \leq \bar{L}(1+\|x\|_H^2), \quad x \in H, \tag{6.2.3}$$

where $\bar{L} := 2((L_1^2 + L_2^2) \vee \mu)$ with $\mu := \|b(0)\|_H^2 + \|\sigma^1(0)\|_{HS}^2$. Moreover, by (H4) one has

$$\begin{aligned} &2\langle x, b(x)\rangle_H + \|\sigma^1(x)\|_{HS}^2 \\ &= 2\langle x, b(x)-b(0)\rangle_H + \|\sigma^1(x)-\sigma^1(0)\|_{HS}^2 \\ &\quad + 2\langle x, b(0)\rangle_H + 2\langle \sigma^1(x)-\sigma^1(0), \sigma^1(0)\rangle_{HS} + \|\sigma^1(0)\|_{HS}^2 \\ &\leq -(\gamma-\epsilon)\|x\|_H^2 + 2(L_2+1+\epsilon)\mu\epsilon^{-1}, \quad \epsilon \in (0,1), \quad x \in H, \end{aligned} \tag{6.2.4}$$

where $\langle T, S\rangle_{HS} := \sum_{i=1}^{\infty}\langle Te_i, Se_i\rangle_H$ for $S, T \in \mathscr{L}_{HS}(H)$.

For any $n \geq 1$, let $\pi_n : H \to H_n := \mathrm{span}\{e_1, \cdots, e_n\}$ be the orthogonal projection, that is, $\pi_n x = \sum_{i=1}^{n}\langle x, e_i\rangle_H e_i, x \in H$, $A_n := \pi_n A, b_n := \pi_n b$ and $\sigma_n := \pi_n\sigma$. Consider SDE on H_n

$$\begin{cases} \mathrm{d}X^n(t) = [A_n X^n(t) + b_n(X^n(t))]\mathrm{d}t + \sigma_n(X^n(t))\mathrm{d}W(t), \\ X^n(0) = \pi_n x. \end{cases} \tag{6.2.5}$$

Due to $\pi_n Ax = \pi_n A\Big(\sum_{i=1}^{n}\langle x, e_i\rangle_H e_i\Big) = -\sum_{i=1}^{n}\langle x, e_i\rangle_H \lambda_i e_i x \in H_n$, it follows that

$$A_n x = Ax, \quad e^{tA_n}x = e^{tA}x \text{ and } \langle x, b_n\rangle_H = \langle x, b\rangle_H, \quad x \in H_n. \tag{6.2.6}$$

By (H3) and the property of the projection operator, we have

$$\begin{aligned} &\|A_n(x-y) + b_n(x) - b_n(y)\|_H^2 + \|\sigma_n^1(x) - \sigma_n^1(y)\|_{HS}^2 \\ &\leq 2\|A_n(x-y)\|_H^2 + 2\|b_n(x) - b_n(y)\|_H^2 + \|\sigma_n^1(x) - \sigma_n^1(y)\|_{HS}^2 \\ &\leq 2\lambda_n^2\|x-y\|_H^2 + 2\|b(x) - b(y)\|_H^2 + \|\sigma^1(x) - \sigma^1(y)\|_{HS}^2 \\ &\leq 2(\lambda_n^2 + L_1^2 + L_2^2)\|x-y\|_H^2, \quad x, y \in H_n. \end{aligned}$$

Hence (6.2.5) admits a unique strong solution $\{X_n(t)\}_{t\geq 0}$ on H_n under (H1) and (H3).

For a stepsize $\triangle \in (0,1)$ and each integer $k \geq 0$, compute the discrete EM approximations $\bar{Y}^n(k\triangle) \approx X^n(k\triangle)$ by setting $\bar{Y}^n(0) := \pi_n x$ and forming

$$\bar{Y}^n((k+1)\triangle) := e^{\triangle A_n}\{\bar{Y}^n(k\triangle) + b_n(\bar{Y}^n(k\triangle))\triangle + \sigma_n(\bar{Y}^n(k\triangle))\triangle W_k\}, \tag{6.2.7}$$

where $\triangle W_k := W((k+1)\triangle) - W(k\triangle)$, and define the continuous EM approximate solution associated with (6.2.5) by

$$\begin{aligned} Y^n(t) :&= e^{tA_n}\pi_n x + \int_0^t e^{(t-\lfloor s\rfloor)A_n} b_n(Y^n(\lfloor s\rfloor))\mathrm{d}s \\ &\quad + \int_0^t e^{(t-\lfloor s\rfloor)A_n}\sigma_n(Y^n(\lfloor s\rfloor))\mathrm{d}W(s) \\ &= e^{tA}\pi_n x + \int_0^t e^{(t-\lfloor s\rfloor)A} b_n(Y^n(\lfloor s\rfloor))\mathrm{d}s \\ &\quad + \int_0^t e^{(t-\lfloor s\rfloor)A}\sigma_n(Y^n(\lfloor s\rfloor))\mathrm{d}W(s) \end{aligned} \tag{6.2.8}$$

due to (6.2.6), where $\lfloor t \rfloor := [t/\triangle]\triangle$ and $[t/\triangle]$ denotes the integer part of $t/\triangle$. For $0 \le s \le t$, it is easy to see from (6.2.8) that

$$\begin{aligned} Y^n(t) = e^{(t-s)A}Y^n(s) + \int_s^t e^{(t-\lfloor r \rfloor)A} b_n(Y^n(\lfloor r \rfloor))\mathrm{d}r \\ + \int_s^t e^{(t-\lfloor r \rfloor)A} \sigma_n(Y^n(\lfloor r \rfloor))\mathrm{d}W(r). \end{aligned} \tag{6.2.9}$$

By $Y^n(0) = \bar{Y}^n(0)$, we deduce from (6.2.7) and (6.2.8) that $Y^n(k\triangle) = \bar{Y}^n(k\triangle)$, that is, $Y^n(t)$ coincides with the discrete approximate solution at the gridpoints.

Remark 6.2.3. For the finite-dimensional case, the discrete EM scheme and the continuous EM scheme are standard, e.g., [56, (4.3) and (4.5), p113]. While the ideas of constructing schemes (6.2.7) and (6.2.8) go back to, e.g., [12, equation (13)], [22, equation (2.9)] and [50, equation (8)].

For each integer $k \ge 0, x \in H_n$ and $\Gamma \in \mathscr{B}(H_n)$, define

$$\mathbb{P}^{n,\triangle}(x,\Gamma) := \mathbb{P}(\bar{Y}^n(\triangle) \in \Gamma | \bar{Y}^n(0) = x)$$

and

$$\mathbb{P}_k^{n,\triangle}(x,\Gamma) := \mathbb{P}(\bar{Y}^n(k\triangle) \in \Gamma | \bar{Y}(0) = x).$$

Following the argument of that of [100, Theorem 1.2], we deduce that $\{\bar{Y}^n(k\triangle)\}_{k\ge 0}$ is a homogeneous Markov process.

Lemma 6.2.1. $\{\bar{Y}^n(k\triangle)\}_{k\ge 0}$ is a homogeneous Markov process with the transition probability kernel $\mathbb{P}^{n,\triangle}(x,\Gamma)$.

We still need to introduce some additional notation and notions. For Hilbert space $(K, \|\cdot\|_K)$, denote by $\mathcal{P}(K)$ the family of all probability measures on K. For $P_1, P_2 \in \mathcal{P}(K)$ define the metric $\mathrm{d}_{\mathbb{L}}$ as follows:

$$\mathrm{d}_{\mathbb{L}}(P_1,P_2) := \sup \left| \int_K f(u)P_1(\mathrm{d}u) - \int_K f(u)P_2(\mathrm{d}u) \right|,$$

where $\mathbb{L} := \{f : K \to \mathbb{R} : |f(u) - f(v)| \le \|u-v\|_K \text{ and } |f(\cdot)| \le 1\}$.

Remark 6.2.4. It is known that the weak convergence of probability measures is a metric concept. In other words, a sequence of probability measures $\{P_k\}_{k\ge 1} \subset \mathcal{P}(K)$ converges weakly to a probability measure $P_0 \in \mathcal{P}(K)$ if and only if $\lim_{k\to\infty} d_{\mathbb{L}}(P_k, P_0) = 0$.

To highlight the initial value, denote by $\{\bar{Y}^{n,x}(k\triangle)\}_{k\geq 0}$ the EM approximate solution of (6.2.7), starting from the point x at the gridpoint 0.

Definition 6.2.1. For a given stepsize $\triangle$ and arbitrary $x \in H_n$, $\{\bar{Y}^{n,x}(k\triangle)\}_{k\geq 0}$ is said to have a stationary distribution $\pi^{n,\triangle} \in \mathcal{P}(H_n)$ if the k-step transition probability kernel $\mathbb{P}_k^{n,\triangle}(x,\cdot)$ converges weakly to $\pi^{n,\triangle}(\cdot)$ as $k \to \infty$, i.e., $\lim_{k\to\infty} \mathrm{d}_{\mathbb{L}}(\mathbb{P}_k^{n,\triangle}(x,\cdot), \pi^{n,\triangle}(\cdot)) = 0$.

Definition 6.2.2. $\{\bar{Y}^{n,x}(k\triangle)\}_{k\geq 0}$ is said to have Property ($\mathbb{P}1$) if

$$\sup_{k\geq 0} \mathbb{E}\|\bar{Y}^{n,x}(k\triangle)\|_H^2 < \infty, \quad x \in U,$$

while it is said to have Property ($\mathbb{P}2$) if

$$\lim_{k\to\infty} \mathbb{E}\|\bar{Y}^{n,x}(k\triangle) - \bar{Y}^{n,y}(k\triangle)\|_H^2 = 0 \text{ uniformly in } x, y \in U,$$

where U is a bounded subset of H_n.

Theorem 6.2.2. Under ($\mathbb{P}1$) and ($\mathbb{P}2$), for a given stepsize $\triangle$ and arbitrary $x \in H_n$, $\{\bar{Y}^{n,x}(k\triangle)\}_{k\geq 0}$ has a unique stationary distribution $\pi^{n,\triangle} \in \mathcal{P}(H_n)$.

Proof. Note that H_n is finite-dimensional. Following the argument of that of [100, Lemma 2.4 and Lemma 2.6], we deduce that

$$\lim_{k\to\infty} \mathrm{d}_{\mathbb{L}}(\mathbb{P}_k^{n,\triangle}(x,\cdot), \mathbb{P}_k^{n,\triangle}(y,\cdot)) = 0 \tag{6.2.10}$$

uniformly in $x, y \in H_n$, and that, together with Lemma 6.2.1, there exists $\pi^{n,\triangle} \in \mathcal{P}(H_n)$ such that

$$\lim_{k\to\infty} \mathrm{d}_{\mathbb{L}}(\mathbb{P}_k^{n,\triangle}(0,\cdot), \pi^{n,\triangle}(\cdot)) = 0. \tag{6.2.11}$$

Then the desired assertion follows from (6.2.10), (6.2.11) and the triangle inequality

$$\mathrm{d}_{\mathbb{L}}(\mathbb{P}_k^{n,\triangle}(x,\cdot), \pi^{n,\triangle}(\cdot)) \leq \mathrm{d}_{\mathbb{L}}(\mathbb{P}_k^{n,\triangle}(x,\cdot), \mathbb{P}_k^{n,\triangle}(0,\cdot)) + \mathrm{d}_{\mathbb{L}}(\mathbb{P}_k^{n,\triangle}(0,\cdot), \pi^{n,\triangle}(\cdot)).$$

□

6.3 Sufficient Condition for Properties ($\mathbb{P}1$) and ($\mathbb{P}2$)

In the previous section we show that $\{\bar{Y}^{n,x}(k\triangle)\}_{k\geq 0}$ has a unique stationary distribution under ($\mathbb{P}1$) and ($\mathbb{P}2$). To make Theorem 6.2.2 more applicable, in this section we intend to give some sufficient condition such that ($\mathbb{P}1$) and ($\mathbb{P}2$) hold. In what follows, $C > 0$ is a generic constant whose values may change from line to line. Let

$$\tilde{Y}^n(t) := \int_0^t e^{(t-\lfloor s\rfloor)A}\sigma_n^0 \mathrm{d}W(s) \quad \text{and} \quad Z^n(t) := Y^n(t) - \tilde{Y}^n(t).$$

Lemma 6.3.1. Under (H1)-(H3), then,

$$\mathbb{E}\|Z^n(t) - Z^n(\lfloor t\rfloor)\|_H^2 \leq \beta_1\triangle(1 + \mathbb{E}\|Z^n(\lfloor t\rfloor)\|_H^2), \quad t \geq 0, \tag{6.3.1}$$

where $\beta_1 := 3\{(\lambda_n^2 + 2\bar{L}) \vee (2\bar{L}(1 + \|(-A)^{-\theta_1}\|^2\delta_1))\}$.

Proof. Observe from (6.2.8) that

$$\begin{aligned} Z^n(t) = e^{tA}\pi_n x &+ \int_0^t e^{(t-\lfloor s\rfloor)A} b_n(Y^n(\lfloor s\rfloor))\mathrm{d}s \\ &+ \int_0^t e^{(t-\lfloor s\rfloor)A}\sigma_n^1(Y^n(\lfloor s\rfloor))\mathrm{d}W(s). \end{aligned} \tag{6.3.2}$$

This further gives

$$\begin{aligned} Z^n(t) = e^{(t-\lfloor t\rfloor)A}Z^n(\lfloor t\rfloor) &+ \int_{\lfloor t\rfloor}^t e^{(t-\lfloor s\rfloor)A} b_n(Y^n(\lfloor s\rfloor))\mathrm{d}s \\ &+ \int_{\lfloor t\rfloor}^t e^{(t-\lfloor s\rfloor)A}\sigma_n^1(Y^n(\lfloor s\rfloor))\mathrm{d}W(s). \end{aligned}$$

Then, by the Hölder inequality, the Itô isometry and (H1), one has

$$\begin{aligned} &\mathbb{E}\|Z^n(t) - Z^n(\lfloor t\rfloor)\|_H^2 \\ &\leq 3\Big\{\mathbb{E}\|(e^{(t-\lfloor t\rfloor)A} - 1)Z^n(\lfloor t\rfloor)\|_H^2 + \mathbb{E}\int_{\lfloor t\rfloor}^t \|b(Y^n(\lfloor s\rfloor))\|_H^2\mathrm{d}s \\ &\quad + \mathbb{E}\int_0^t \|\sigma^1(Y^n(\lfloor s\rfloor))\|_{HS}^2\mathrm{d}s\Big\} \\ &=: 3\{I_1(t) + I_2(t) + I_3(t)\}. \end{aligned} \tag{6.3.3}$$

Recalling the fundamental inequality $1 - e^{-y} \le y, y > 0$, we get from (H1) that

$$\begin{aligned} \|(e^{(t-\lfloor t\rfloor)A} - \mathbf{1})x\|_H^2 &= \left\|\sum_{i=1}^n (e^{-\lambda_i(t-\lfloor t\rfloor)} - 1)\langle x, e_i\rangle_H e_i\right\|_H^2 \\ &\le (1 - e^{-\lambda_n(t-\lfloor t\rfloor)})^2\|x\|_H^2 \\ &\le \lambda_n^2\triangle^2\|x\|_H^2, \quad x \in H_n. \end{aligned} \tag{6.3.4}$$

Therefore we arrive at

$$I_1(t) \le \lambda_n^2\triangle^2\mathbb{E}\|Z^n(\lfloor t\rfloor)\|_H^2. \tag{6.3.5}$$

Observe from the Itô isometry, (H1) and (H2) that

$$\begin{aligned} \mathbb{E}\|\tilde{Y}^n(t)\|_H^2 &= \int_0^t \|e^{(s-\lfloor s\rfloor)A}e^{(t-s)A}\sigma_n^0\|_{HS}^2\mathrm{d}s \\ &\le \int_0^t \|(-A)^{-\theta_1}(-A)^{\theta_1}e^{(t-s)A}\sigma_n^0\|_{HS}^2\mathrm{d}s \\ &\le \|(-A)^{-\theta_1}\|^2\int_0^t \|(-A)^{\theta_1}e^{(t-s)A}\sigma^0\|_{HS}^2\mathrm{d}s \le \|(-A)^{-\theta_1}\|^2\delta_1. \end{aligned} \tag{6.3.6}$$

Thus, by (6.2.3) and (6.3.6) it follows that

$$\begin{aligned} I_2(t) + I_3(t) &\le \triangle\mathbb{E}\{\|b(Y^n(\lfloor t\rfloor))\|_H^2 + \|\sigma^1(Y^n(\lfloor t\rfloor))\|_{HS}^2\} \\ &\le 2\bar{L}\triangle\{1 + \mathbb{E}\|Z^n(\lfloor t\rfloor)\|_H^2 + \mathbb{E}\|\tilde{Y}^n(\lfloor t\rfloor)\|_H^2\} \\ &\le 2\bar{L}\triangle\{1 + \|(-A)^{-\theta_1}\|^2\delta_1 + \mathbb{E}\|Z^n(\lfloor t\rfloor)\|_H^2\}. \end{aligned} \tag{6.3.7}$$

As a result, (6.3.1) follows by substituting (6.3.5) and (6.3.7) into (6.3.3). □

Theorem 6.3.2. Let (H1)-(H4) hold and assume further that $2\alpha + \gamma > 0$. If $\triangle < \min\{1, (2\alpha + \gamma)^2/(4\rho_1^2)\}$, then there exists $C > 0$ independent of $\triangle$ such that

$$\sup_{t\ge 0}\mathbb{E}\|Y^n(t)\|_H^2 \le C, \tag{6.3.8}$$

where $\rho_1 := 2 + (|14\alpha - \gamma|^2/64 + 2\bar{L} + |14\alpha - \gamma|/8)\beta_1 + 2(1 + \beta_1 + \lambda_n^2\bar{L})$. Hence Property ($\mathbb{P}$1) holds whenever the stepsize $\triangle$ is sufficiently small.

Proof. Note that (6.3.2) can be rewritten in the differential form

$$\begin{aligned} \mathrm{d}Z^n(t) = \{&AZ^n(t) + e^{(t-\lfloor t\rfloor)A}b_n(Y^n(\lfloor t\rfloor))\}\mathrm{d}t \\ &+ e^{(t-\lfloor t\rfloor)A}\sigma_n^1(Y^n(\lfloor t\rfloor))\mathrm{d}W(t) \end{aligned} \tag{6.3.9}$$

with $Z^n(0) = \pi_n x$. For any $\nu > 0$, applying the Itô formula we deduce from (6.3.9) and (H1) that

$$\begin{aligned}\mathbb{E}(e^{\nu t}\|Z^n(t)\|_H^2) &\le \|x\|_H^2 + \mathbb{E}\int_0^t e^{\nu s}\{\nu\|Z^n(s)\|_H^2 + 2\langle Z^n(s), AZ^n(s)\rangle_H \\ &\quad + 2\langle Z^n(s), e^{(s-\lfloor s\rfloor)A} b_n(Y^n(\lfloor s\rfloor))\rangle_H \\ &\quad + \|e^{(s-\lfloor s\rfloor)A}\sigma_n^1(Y^n(\lfloor s\rfloor))\|_{HS}^2\}\mathrm{d}s \\ &\le \|x\|_H^2 + \mathbb{E}\int_0^t e^{\nu s}\{-(2\alpha-\nu)\|Z^n(s)\|_H^2 \\ &\quad + 2\langle Z^n(s), e^{(s-\lfloor s\rfloor)A} b_n(Y^n(\lfloor s\rfloor))\rangle_H \\ &\quad + \|\sigma^1(Y^n(\lfloor s\rfloor))\|_{HS}^2\}\mathrm{d}s.\end{aligned} \tag{6.3.10}$$

Since

$$\begin{aligned}\|Z^n(t)\|_H^2 &= \|Z^n(\lfloor t\rfloor)\|_H^2 + 2\langle Z^n(\lfloor t\rfloor), Z^n(t) - Z^n(\lfloor t\rfloor)\rangle_H \\ &\quad + \|Z^n(t) - Z^n(\lfloor t\rfloor)\|_H^2,\end{aligned} \tag{6.3.11}$$

and

$$\begin{aligned}\langle Z^n(t), e^{(t-\lfloor t\rfloor)A} b_n(Y^n(\lfloor t\rfloor))\rangle_H &= \langle Y^n(\lfloor t\rfloor), b(Y^n(\lfloor t\rfloor))\rangle_H + \langle Z^n(t) - Z^n(\lfloor t\rfloor), b(Y^n(\lfloor t\rfloor))\rangle_H \\ &\quad - \langle \tilde{Y}^n(\lfloor t\rfloor), b(Y^n(\lfloor t\rfloor))\rangle_H \\ &\quad + \langle Z^n(t), (e^{(t-\lfloor t\rfloor)A} - \mathbf{1}) b_n(Y^n(\lfloor t\rfloor))\rangle_H,\end{aligned}$$

it follows from (6.3.10) that

$$\begin{aligned}&\mathbb{E}(e^{\nu t}\|Z^n(t)\|_H^2) \\ &\le \|x\|_H^2 + \mathbb{E}\int_0^t e^{\nu s}\{-(2\alpha-\nu)\|Z^n(\lfloor s\rfloor)\|_H^2 + \|\sigma^1(Y^n(\lfloor s\rfloor))\|_{HS}^2 \\ &\quad + 2\langle Y^n(\lfloor s\rfloor), b(Y^n(\lfloor s\rfloor))\rangle_H - 2(2\alpha-\nu)\langle Z^n(\lfloor s\rfloor), Z^n(s) - Z^n(\lfloor s\rfloor)\rangle_H \\ &\quad - (2\alpha-\nu)\|Z^n(s) - Z^n(\lfloor s\rfloor)\|_H^2 + 2\langle Z^n(s) - Z^n(\lfloor s\rfloor), b(Y^n(\lfloor s\rfloor))\rangle_H \\ &\quad - 2\langle \tilde{Y}^n(\lfloor s\rfloor)), b(Y^n(\lfloor s\rfloor))\rangle_H + 2\langle Z^n(s), (e^{(s-\lfloor s\rfloor)A} - \mathbf{1}) b_n(Y^n(\lfloor s\rfloor))\rangle_H\}\mathrm{d}s.\end{aligned}$$

This, together with (6.2.4), yields that

$$\begin{aligned}&\mathbb{E}(e^{\nu t}\|Z^n(t)\|_H^2) \\ &\le \|x\|_H^2 - (2\alpha+\gamma-\epsilon-\nu)\mathbb{E}\int_0^t e^{\nu s}\|Z^n(\lfloor s\rfloor)\|_H^2\mathrm{d}s \\ &\quad + \mathbb{E}\int_0^t e^{\nu s}\{-2(2\alpha-\nu)\langle Z^n(\lfloor s\rfloor), Z^n(s) - Z^n(\lfloor s\rfloor)\rangle_H \\ &\quad - (2\alpha-\nu)\|Z^n(s) - Z^n(\lfloor s\rfloor)\|_H^2 \\ &\quad + 2\langle Z^n(s) - Z^n(\lfloor s\rfloor), b(Y^n(\lfloor s\rfloor))\rangle_H\}\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&+ 2\mathbb{E}\int_0^t e^{\nu s}\langle Z^n(s), (e^{(s-\lfloor s\rfloor)A} - \mathbf{1})b_n(Y^n(\lfloor s\rfloor))\rangle_H \mathrm{d}s \\
&+ \mathbb{E}\int_0^t e^{\nu s}\{2(L+1+\epsilon^{-1})\mu\epsilon^{-1} - 2\langle \tilde{Y}^n(\lfloor s\rfloor), b(Y^n(\lfloor s\rfloor))\rangle_H \\
&- 2(\gamma-\epsilon)\langle \tilde{Y}^n(\lfloor s\rfloor), Z^n(\lfloor s\rfloor)\rangle_H - (\gamma-\epsilon)\|\tilde{Y}^n(\lfloor s\rfloor)\|_H^2\}\mathrm{d}s \\
&=: J_1(t) + J_2(t) + J_3(t) + J_4(t).
\end{aligned}
\tag{6.3.12}
$$

By the elemental inequality: $2ab \le \kappa a^2 + b^2/\kappa, a, b \in \mathbb{R}, \kappa > 0$, and (6.3.1), we arrive at

$$
\begin{aligned}
J_2(t) &\le \mathbb{E}\int_0^t e^{\nu s}\Big\{\Delta^{\frac{1}{2}}\|Z^n(\lfloor s\rfloor)\|_H^2 + 2^{-1}\bar{L}^{-1}\Delta^{\frac{1}{2}}\|b(Y^n(\lfloor s\rfloor))\|_H^2 \\
&\quad + \{(|2\alpha-\nu|^2 + 2\bar{L})\Delta^{-\frac{1}{2}} + |2\alpha-\nu|\}\|Z^n(s) - Z^n(\lfloor s\rfloor)\|_H^2\Big\}\mathrm{d}s \\
&\le \mathbb{E}\int_0^t e^{\nu s}\Big\{2\Delta^{\frac{1}{2}}\|Z^n(\lfloor s\rfloor)\|_H^2 + 2^{-1}\Delta^{\frac{1}{2}} + \Delta^{\frac{1}{2}}\|\tilde{Y}^n(\lfloor s\rfloor)\|_H^2 \\
&\quad + \{(|2\alpha-\nu|^2 + 2\bar{L})\Delta^{-\frac{1}{2}} + |2\alpha-\nu|\}\|Z^n(s) - Z^n(\lfloor s\rfloor)\|_H^2\Big\}\mathrm{d}s,
\end{aligned}
$$

where in the last step we have used (6.2.3). Combining (6.3.1) with (6.3.6), we thus obtain that

$$
\begin{aligned}
&J_2(t) \\
&\le \int_0^t e^{\nu s}\Big\{\{2 + (|2\alpha-\nu|^2 + 2\bar{L} + |2\alpha-\nu|)\beta_1\}\Delta^{\frac{1}{2}}\mathbb{E}\|Z^n(\lfloor t\rfloor)\|_H^2 \\
&+ \{1 + \|(-A)^{-\theta_1}\|^2\delta_1 + (|2\alpha-\nu|^2 + 2\bar{L} + |2\alpha-\nu|)\beta_1\}\Delta^{\frac{1}{2}}\Big\}\mathrm{d}s.
\end{aligned}
\tag{6.3.13}
$$

On the other hand, we deduce from (6.2.3), (6.3.4), (6.3.6) and (6.3.11) that

$$
\begin{aligned}
&J_3(t) \\
&\le \mathbb{E}\int_0^t e^{\nu s}\{\Delta^{\frac{1}{2}}\|Z^n(\lfloor t\rfloor)\|_H^2 + 2\Delta^{\frac{1}{2}}\langle Z^n(\lfloor t\rfloor), Z^n(t) - Z^n(\lfloor t\rfloor)\rangle_H \\
&+ \Delta^{\frac{1}{2}}\|Z^n(t) - Z^n(\lfloor t\rfloor)\|_H^2 + \Delta^{-\frac{1}{2}}\|(e^{(s-\lfloor s\rfloor)A} - \mathbf{1})b_n(Y^n(\lfloor s\rfloor))\|_H^2\}\mathrm{d}s \\
&\le \mathbb{E}\int_0^t e^{\nu s}\{2\Delta^{\frac{1}{2}}\|Z^n(\lfloor t\rfloor)\|_H^2 + 2\Delta^{\frac{1}{2}}\|Z^n(t) - Z^n(\lfloor t\rfloor)\|_H^2 \\
&+ \Delta^{-\frac{1}{2}}\|(e^{(s-\lfloor s\rfloor)A} - \mathbf{1})b_n(Y^n(\lfloor s\rfloor))\|_H^2\}\mathrm{d}s \\
&\le \int_0^t e^{\nu s}\{2(1 + \beta_1 + \lambda_n^2\bar{L})\Delta^{\frac{1}{2}}\mathbb{E}\|Z^n(\lfloor t\rfloor)\|_H^2 \\
&+ 2(\beta_1 + \lambda_n^2\bar{L}(1 + \|(-A)^{-\theta_1}\|^2\delta_1))\Delta^{\frac{1}{2}}\}\mathrm{d}s.
\end{aligned}
\tag{6.3.14}
$$

Furthermore, due to (6.2.3) and (6.3.6), for arbitrary $\kappa > 0$ one has

$$\begin{aligned}
J_4(t) &\le \mathbb{E}\int_0^t e^{\nu s}\{2(L+1+\epsilon^{-1})\mu\epsilon^{-1} + 2\|\tilde{Y}^n(\lfloor s\rfloor)\|_H\|b(Y^n(\lfloor s\rfloor))\|_H \\
&\quad + 2|\gamma-\epsilon|\cdot\|\tilde{Y}^n(\lfloor s\rfloor)\|_H\|Z^n(\lfloor s\rfloor)\|_H + |\gamma-\epsilon|\cdot\|\tilde{Y}^n(\lfloor s\rfloor)\|_H^2\}\mathrm{d}s \\
&\le \mathbb{E}\int_0^t e^{\nu s}\{2(L+1+\epsilon^{-1})\mu\epsilon^{-1} + \kappa^{-1}\|\tilde{Y}^n(\lfloor s\rfloor)\|_H^2 + \kappa\|b(Y^n(\lfloor s\rfloor))\|_H^2 \\
&\quad + |\gamma-\epsilon|^2\kappa^{-1}\|\tilde{Y}^n(\lfloor s\rfloor)\|_H + \kappa\|Z^n(\lfloor s\rfloor)\|_H^2 + |\gamma-\epsilon|\cdot\|\tilde{Y}^n(\lfloor s\rfloor)\|_H^2\}\mathrm{d}s \\
&\le \int_0^t e^{\nu s}\{\kappa\bar{L} + (\kappa^{-1} + 2\kappa\bar{L} + |\gamma-\epsilon|^2\kappa^{-1} + |\gamma-\epsilon|)\|(-A)^{-\theta_1}\|^2\delta_1 \\
&\quad + 2(L+1+\epsilon^{-1})\mu\epsilon^{-1} + (1+2\bar{L})\kappa\mathbb{E}\|Z^n(\lfloor s\rfloor)\|_H^2\}\mathrm{d}s.
\end{aligned}$$

In particular, taking $\epsilon = \nu = (2\alpha+\gamma)/8$ and $\kappa = (2\alpha+\gamma)/(4(1+2\bar{L}))$ yields that

$$J_4(t) \le \int_0^t e^{\nu s}\{4^{-1}(2\alpha+\gamma)\mathbb{E}\|Z^n(\lfloor s\rfloor)\|_H^2 + C\}\mathrm{d}s. \tag{6.3.15}$$

Putting (6.3.13)-(6.3.15) into (6.3.12), we deduce that

$$\mathbb{E}(e^{\nu t}\|Z^n(t)\|_H^2) \le \|x\|_H^2 - \frac{2\alpha+\gamma-2\rho_1\Delta^{\frac{1}{2}}}{2}\mathbb{E}\int_0^t e^{\nu s}\|Z^n(\lfloor s\rfloor)\|_H^2\mathrm{d}s + C\int_0^t e^{\nu s}\mathrm{d}s.$$

For $\Delta < (2\alpha+\gamma)^2/(4\rho_1^2)$, it is trivial to see that $2\alpha+\gamma-2\rho_1\Delta^{\frac{1}{2}} > 0$. Thus we have

$$\mathbb{E}(\|Z^n(t)\|_H^2) \le C.$$

Finally, (6.3.8) follows by recalling $Z^n(t) = Y^n(t) - \tilde{Y}^n(t)$ and (6.3.6). □

Let $Y^{n,x}(t)$ and $Y^{n,y}(t)$ be the continuous EM schemes defined by (6.2.8) and starting from the points x and y at time 0 respectively.

Theorem 6.3.3. Let the assumptions of Theorem 6.3.2 hold. If $\Delta \le \min\{1, (2\alpha+\gamma)^2/(4\rho_2^2)\}$, then

$$\lim_{t\to\infty}\mathbb{E}\|Y^{n,x}(t) - Y^{n,y}(t)\|_H^2 = 0 \quad \text{uniformly for } x, y \in U, \tag{6.3.16}$$

where $\rho_2 := 6(\lambda_n^2+\bar{L})(|2\alpha-\gamma|+1) + 3 + 7\bar{L} + \lambda_n^2\bar{L} + 6\lambda_n^2$ with $\bar{L} := L_1^2 + L_2^2$. Hence Property ($\mathbb{P}$2) holds whenever the stepsize Δ is sufficiently small.

Proof. Let

$$Z^{n,x,y}(t) := Y^{n,x}(t) - Y^{n,y}(t).$$

Note from (6.2.8) that

$$\begin{aligned}Z^{n,x,y}(t) - Z^{n,x,y}(\lfloor t\rfloor) &= (e^{(t-\lfloor t\rfloor)A} - \mathbf{1})Z^{n,x,y}(\lfloor t\rfloor)\\ &\quad + \int_{\lfloor t\rfloor}^t e^{(t-\lfloor s\rfloor)A}(b_n(Y^{n,x}(\lfloor s\rfloor)) - b_n(Y^{n,y}(\lfloor s\rfloor)))\mathrm{d}s\\ &\quad + \int_{\lfloor t\rfloor}^t e^{(t-\lfloor s\rfloor)A}(\sigma_n^1(Y^{n,x}(\lfloor s\rfloor)) - \sigma^1(Y^{n,y}(\lfloor s\rfloor)))\mathrm{d}W(s).\end{aligned}$$

Following the argument of that of (6.3.1), we derive that

$$\mathbb{E}\|Z^{n,x,y}(t) - Z^{n,x,y}(\lfloor t\rfloor)\|_H^2 \leq 3(\lambda_n^2 + \bar{L})\triangle\mathbb{E}\|Z^{n,x,y}(\lfloor t\rfloor)\|_H^2. \tag{6.3.17}$$

For $\nu := (2\alpha + \gamma)/2$, by the Itô formula it follows from (6.2.8), (H1) and (H4) that

$$\begin{aligned}&\mathbb{E}(e^{\nu t}\|Z^{n,x,y}(t)\|_H^2)\\ &\leq \|x-y\|_H^2 + \nu\mathbb{E}\int_0^t e^{\nu s}\|Z^{n,x,y}(s)\|_H^2\mathrm{d}s + \mathbb{E}\int_0^t e^{\nu s}\{2\langle Z^{n,x,y}(s), AZ^{n,x,y}(s)\rangle_H\\ &\quad + 2\langle Z^{n,x,y}(\lfloor s\rfloor), b(Y^{n,x}(\lfloor s\rfloor)) - b(Y^{n,y}(\lfloor s\rfloor))\rangle_H\\ &\quad + \|\sigma^1(Y^{n,x}(\lfloor s\rfloor)) - \sigma^1(Y^{n,y}(\lfloor s\rfloor))\|_{HS}^2\\ &\quad + 2\langle Z^{n,x,y}(s) - Z^{n,x,y}(\lfloor s\rfloor), b(Y^{n,x}(\lfloor s\rfloor)) - b(Y^{n,y}(\lfloor s\rfloor))\rangle_H\\ &\quad + 2\langle Z^{n,x,y}(s), (e^{(s-\lfloor s\rfloor)A} - \mathbf{1})(b_n(Y^{n,x}(\lfloor s\rfloor)) - b_n(Y^{n,y}(\lfloor s\rfloor)))\rangle_H\}\mathrm{d}s\\ &\leq \|x-y\|_H^2 - (2\alpha+\gamma-\nu)\mathbb{E}\int_0^t e^{\nu s}\|Z^{n,x,y}(\lfloor s\rfloor)\|_H^2\mathrm{d}s\\ &\quad + \mathbb{E}\int_0^t e^{\nu s}\{-2(2\alpha-\nu)\langle Z^{n,x,y}(\lfloor s\rfloor), Z^{n,x,y}(s) - Z^{n,x,y}(\lfloor s\rfloor)\rangle_H\\ &\quad - (2\alpha-\nu)\|Z^{n,x,y}(s) - Z^{n,x,y}(\lfloor s\rfloor)\|_H^2\\ &\quad + 2\langle Z^{n,x,y}(s) - Z^{n,x,y}(\lfloor s\rfloor), b(Y^{n,x}(\lfloor s\rfloor)) - b(Y^{n,y}(\lfloor s\rfloor))\rangle_H\}\mathrm{d}s\\ &\quad + 2\mathbb{E}\int_0^t e^{\nu s}\langle Z^{n,x,y}(s), (e^{(s-\lfloor s\rfloor)A} - \mathbf{1})(b_n(Y^{n,x}(\lfloor s\rfloor)) - b_n(Y^{n,y}(\lfloor s\rfloor)))\rangle_H\mathrm{d}s\\ &=: \bar{J}_1(t) + \bar{J}_2(t) + \bar{J}_3(t).\end{aligned}$$

where we have also used the (6.3.11) with $Z^n(t)$ replaced by $Z^{n,x,y}(t)$. By (H3)

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and (6.3.17), one has

$$\begin{aligned}\bar{J}_2(t) &\le \mathbb{E}\int_0^t e^{\nu s}\{\Delta^{\frac{1}{2}}\|Z^{n,x,y}(\lfloor s\rfloor)\|_H^2 + \Delta^{\frac{1}{2}}\|b(Y^{n,x}(\lfloor s\rfloor)) - b(Y^{n,y}(\lfloor s\rfloor))\|_H^2\\ &\quad + \{|2\alpha-\nu| + (|2\alpha-\nu|+1)\Delta^{-\frac{1}{2}}\}\|Z^{n,x,y}(s) - Z^{n,x,y}(\lfloor s\rfloor)\|_H^2\}\mathrm{d}s\\ &\le \{6(\lambda_n^2+\bar{L})(|2\alpha-\nu|+1) + 1 + \bar{L}\}\Delta^{\frac{1}{2}}\mathbb{E}\int_0^t e^{\nu s}\|Z^{n,x,y}(\lfloor s\rfloor)\|_H^2\mathrm{d}s.\end{aligned}$$

On the other hand, carrying out a similar argument to that of (6.3.14) leads to

$$\begin{aligned}\bar{J}_3(t) &\le 2\mathbb{E}\int_0^t e^{\nu s}\{\Delta^{\frac{1}{2}}\|Z^{n,x,y}(s) - Z^{n,x,y}(\lfloor s\rfloor)\|_H^2\\ &\quad + \Delta^{\frac{1}{2}}\langle Z^{n,x,y}(s) - Z^{x,y}(\lfloor s\rfloor), Z^{n,x,y}(\lfloor s\rfloor)\rangle_H\\ &\quad + \Delta^{\frac{1}{2}}\|Z^{n,x,y}(\lfloor s\rfloor)\|_H^2\\ &\quad + \Delta^{-\frac{1}{2}}\|(e^{(s-\lfloor s\rfloor)A} - \mathbf{1})(b_n(Y^{n,x}(\lfloor s\rfloor)) - b_n(Y^{n,y}(\lfloor s\rfloor)))\|_H^2\mathrm{d}s\\ &\le (2+\lambda_n^2\bar{L} + 6\lambda_n^2 + 6\bar{L})\Delta^{\frac{1}{2}}\mathbb{E}\int_0^t e^{\nu s}\|Z^{n,x,y}(\lfloor s\rfloor)\|_H^2\mathrm{d}s.\end{aligned}$$

Hence we arrive at

$$\mathbb{E}(e^{\nu t}\|Z^{n,x,y}(t)\|_H^2) \le \|x-y\|_H^2 - \frac{2\alpha+\gamma-2\rho_2\Delta^{\frac{1}{2}}}{2}\mathbb{E}\int_0^t e^{\nu s}\|Z^{n,x,y}(\lfloor s\rfloor)\|_H^2\mathrm{d}s,$$

and then the desired assertion (6.3.16) follows by $\Delta \le \min\{1, (2\alpha+\gamma)^2/(4\rho_2^2)\}$.

□

6.4 Limit Distribution

In the previous section we give some sufficient conditions such that (6.2.7) has a unique stationary distribution $\pi^{n,\Delta} \in \mathcal{P}(H_n)$ for fixed n and sufficiently small Δ. In this section we proceed to discuss the limit behavior of $\pi^{n,\Delta} \in \mathcal{P}(H_n)$ and give positive answers to the following questions:

- Will the stationary distribution $\pi^{n,\Delta}(\cdot)$ converge weakly to some probability measure in $\mathcal{P}(H)$ whenever $n \to \infty$ and $\Delta \to 0$?
- If yes, what is the limit probability measure ?

Denote by $\{X^x(t)\}_{t\ge 0}$ the mild solution of (6.2.1) starting from the point x at time $t=0$, which is a homogenous Markov process. For any subset $\Gamma \subset$

$\mathscr{B}(H)$, let $\mathbb{P}_t(x,\Gamma)$ be the probability measure induced by $X^x(t), t \geq 0$, that is, $\mathbb{P}_t(x,\Gamma) = \mathbb{P}(X^x(t) \in \Gamma)$.

Definition 6.4.1. $X^x(t)$ is said to have a stationary distribution $\pi(\cdot) \in \mathcal{P}(H)$ if $\mathbb{P}_t(x,\cdot)$ converges weakly to $\pi(\cdot) \in \mathcal{P}(H)$ as $t \to \infty$ for every $x \in U$, a bounded subset of H, that is, $\lim_{t\to\infty} d_{\mathbb{L}}(\mathbb{P}_t(x,\cdot), \pi(\cdot)) = 0$.

To reveal the limit behavior of $\pi^{n,\triangle}(\cdot)$, we first give several auxiliary lemmas.

Lemma 6.4.1. Let (H1)-(H4) hold and assume further that $2\alpha + \gamma > 0$. Then the mild solution $X^x(t)$ of (6.2.1) has a unique stationary distribution $\pi(\cdot) \in \mathcal{P}(H)$.

Proof. We remark that [5, Theorem 3.1] investigates the stationary distribution of (6.2.1) with $\sigma^0 = 0$, that is, the diffusion coefficient there is a Hilbert-Schmidt operator. For $\sigma^0 \neq 0$, note that σ is not Hilbert-Schmidt. Therefore [5, Theorem 3.1] is unavailable for (6.2.1). Let

$$\bar{Z}(t) := \int_0^t e^{(t-s)A}\sigma^0 \mathrm{d}W(s) \text{ and } \bar{X}(t) := X(t) - \bar{Z}(t).$$

Then (6.2.1) can be rewritten in the form

$$\mathrm{d}\bar{X}(t) = [A\bar{X}(t) + b(X(t))]\mathrm{d}t + \sigma^1(X(t))\mathrm{d}W(t). \tag{6.4.1}$$

To be precise, (6.4.1) is first meant in the mild sense. But under (H1)-(H3) it also has a unique variation solution, and therefore the Itô formula applies to $\|\bar{X}(t)\|_H^2$. Carrying out similar arguments to those of Theorem 6.3.2 and Theorem 6.3.3 respectively, we deduce that

$$\sup_{t\geq 0} \mathbb{E}\|X(t)\|^2 \leq C \tag{6.4.2}$$

and

$$\lim_{t\to\infty} \mathbb{E}\|X^x(t) - X^y(t)\|_H^2 = 0, \text{ uniformly for } x, y \in U.$$

Then [5, Theorem 3.1] yields the desired assertion. □

Lemma 6.4.2. Let (H1) and (H2) hold and assume further that there exists $\delta_2 > 0$ and $\theta_2 \in (0,1)$ such that

$$\int_0^\triangle \|e^{sA}\sigma^0\|_{HS}^2 \mathrm{d}s \leq \delta_2 \triangle^{\theta_2}. \tag{6.4.3}$$

Then

$$\sup_{t\geq 0}\mathbb{E}\|\bar{Z}(t)-\bar{Z}(\lfloor t\rfloor)\|_H^2 \leq C\triangle^{\theta_1\wedge\theta_2}, \tag{6.4.4}$$

where $C>0$ is a constant independent of $\triangle$.

Proof. Recall from [65, Theorem 6.13, p74] that there exists $C_1>0$ such that

$$\|(-A)^{\alpha_1}e^{tA}\|\leq C_1t^{-\alpha_1}, \quad \|(-A)^{-\alpha_2}(1-e^{tA})\|\leq C_1t^{\alpha_2}, \tag{6.4.5}$$

for arbitrary $\alpha_1\geq 0,\ \alpha_2\in[0,1]$, and that

$$(-A)^{\alpha_3+\alpha_4}x=(-A)^{\alpha_3}(-A)^{\alpha_4}x, \quad x\in\mathcal{D}((-A)^{\gamma}), \tag{6.4.6}$$

for any $\alpha_3,\alpha_4\in\mathbb{R}$, where $\gamma:=\max\{\alpha_3,\alpha_4,\alpha_3+\alpha_4\}$. In the light of Itô's isometry and (H1)

$$\begin{aligned}\mathbb{E}\|\bar{Z}(t)-\bar{Z}(\lfloor t\rfloor)\|_H^2 &\leq 2\int_0^{\lfloor t\rfloor}\|(e^{(t-\lfloor t\rfloor)A}-\mathbf{1})e^{(\lfloor t\rfloor-s)A}\sigma^0\|_{HS}^2\mathrm{d}s\\ &\quad+2\int_{\lfloor t\rfloor}^t\|e^{(t-s)A}\sigma^0\|_{HS}^2\mathrm{d}s.\end{aligned}$$

This, combining (H2), (6.4.3), (6.4.5) with (6.4.6), yields that

$$\begin{aligned}&\mathbb{E}\|\bar{Z}(t)-\bar{Z}(\lfloor t\rfloor)\|_H^2\\ &\leq 2\int_0^{\lfloor t\rfloor}\|(-A)^{-\theta_1}(e^{(t-\lfloor t\rfloor)A}-\mathbf{1})\|^2\cdot\|(-A)^{\theta_1}e^{(\lfloor t\rfloor-s)A}\sigma^0\|_{HS}^2\mathrm{d}s\\ &\quad+2\int_0^{\triangle}\|e^{sA}\sigma^0\|_{HS}^2\mathrm{d}s\\ &\leq 2C_1^2\triangle^{2\theta_1}\int_0^{\lfloor t\rfloor}\|(-A)^{\theta_1}e^{sA}\sigma^0\|_{HS}^2\mathrm{d}s+2\delta_2\triangle^{\theta_2}\\ &\leq 2(C_1^2\delta_1+\delta_2)\triangle^{\theta_1\wedge\theta_2},\end{aligned}$$

and therefore the desired assertion follows. □

Remark 6.4.1. Let $\sigma^0=\mathbf{1}$ and A be the Laplace operator defined in Remark 6.2.1. A straightforward computation shows that

$$\int_0^{\triangle}\|e^{sA}\|_{HS}^2\mathrm{d}s=\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k^2}(1-e^{-2k^2\triangle}). \tag{6.4.7}$$

Recall that for arbitrary $\delta\in(0,1)$ there exists $c_\delta>0$ such that

$$|e^{-x}-e^{-y}|\leq c_\delta|x-y|^{\delta}, \quad x,y\geq 0. \tag{6.4.8}$$

It then follows from (6.4.7) and (6.4.8) that

$$\int_0^{\triangle} \|e^{sA}\|_{HS}^2 \mathrm{d}s \le 2^{\delta-1} c_\delta \triangle^\delta \sum_{k=1}^{\infty} \frac{1}{k^{2(1-\delta)}}.$$

Hence, (6.4.3) holds with $\delta_2 = 2^{\delta-1} c_\delta \sum_{k=1}^{\infty} \frac{1}{k^{2(1-\delta)}}$ and $\theta_2 = \delta \in (0, 1/2)$.

Lemma 6.4.3. Let the assumptions of Lemma 6.4.1 hold and

$$\tau := \alpha^{-1} L_1 + (2\alpha)^{-1/2} L_2 \in (0,1). \tag{6.4.9}$$

Then

$$\sup_{t\ge 0} \mathbb{E}\|X(t) - Y^n(t)\|_H^2 \le C\{\lambda_n^{-(\theta_1 \wedge 1/2)} + \triangle^{\theta_1 \wedge \theta_2}\},$$

where $C > 0$ is a constant independent of n and $\triangle$.

Proof. By (6.2.3) and (6.4.2), it follows that

$$\sup_{t\ge 0} \mathbb{E}\|b(X(t))\|_H^2 + \sup_{t\ge 0} \mathbb{E}\|\sigma^1(X(t))\|_{HS}^2 \le C. \tag{6.4.10}$$

Note that $(\mathbb{E}\|\cdot\|_H^2)^{1/2}$ is a norm and recall from [38, Theorem 202] the Minkowski integral inequality:

$$\left(\mathbb{E}\left|\int_0^t F(s)\mathrm{d}s\right|^2\right)^{1/2} \le \int_0^t \left(\mathbb{E}|F(s)|^2\right)^{1/2} \mathrm{d}s, \quad t \ge 0,$$

where $F : [0,\infty) \times \Omega \to R$ is measurable and locally integrable. Then, applying the Itô isometry and using (H1), we obtain from (6.2.2) that

$$\begin{aligned}
&(\mathbb{E}\|\bar{X}(t) - \bar{X}(\lfloor t \rfloor)\|_H^2)^{1/2} \\
&\le \|e^{\lfloor t \rfloor A}\{e^{(t-\lfloor t \rfloor)A} - 1\}x\|_H \\
&\quad + \int_0^{\lfloor t \rfloor} (\mathbb{E}\|e^{(\lfloor t \rfloor - s)A}\{e^{(t-\lfloor t \rfloor)A} - \mathbf{1}\} b(X(s))\|_H^2)^{1/2} \mathrm{d}s \\
&\quad + \Big(\int_0^{\lfloor t \rfloor} \mathbb{E}\|e^{(\lfloor t \rfloor - s)A}\{e^{(t-\lfloor t \rfloor)A} - \mathbf{1}\}\sigma^1(X(s))\|_{HS}^2 \mathrm{d}s\Big)^{1/2} \\
&\quad + \int_{\lfloor t \rfloor}^t (\mathbb{E}\|b(X(s))\|_H^2)^{1/2} \mathrm{d}s + \Big(\int_{\lfloor t \rfloor}^t \mathbb{E}\|\sigma^1(X(s))\|_{HS}^2 \mathrm{d}s\Big)^{1/2} \\
&=: F_1(t) + F_2(t) + F_3(t) + F_4(t) + F_5(t).
\end{aligned} \tag{6.4.11}$$

Let $\rho := (\theta_1 \wedge \theta_2)/2$. In view of (6.4.5), (6.4.6), (H1) and the boundedness of $(-A)^{-(1-\rho/2)}$, one has

$$\begin{aligned} F_1(t) &= \|(-A)^{-(1-\rho/2)} e^{\lfloor t\rfloor A}(-A)^{-\rho/2}\{e^{(t-\lfloor t\rfloor)A} - \mathbf{1}\}(-A)x\|_H^2 \\ &\le \|(-A)^{-(1-\rho/2)} e^{\lfloor t\rfloor A}\|^2 \cdot \|(-A)^{-\rho/2}\{e^{(t-\lfloor t\rfloor)A} - \mathbf{1}\}(-A)x\|_H^2 \\ &\le C\|(-A)^{-(1-\rho/2)}\|^2 \cdot \|Ax\|_H^2 \triangle^\rho. \end{aligned}$$

Also, by (6.4.5) and (6.4.6), we obtain from (6.4.10) that for $\tilde\theta \in (0,1)$

$$\begin{aligned} &\sum_{k=2}^{5} F_k(t) \\ &\le C\triangle^{1/2} + C\int_0^{\lfloor t\rfloor} \|(-A)^\rho e^{\tilde\theta(\lfloor t\rfloor - s)A}\| \cdot \|e^{(1-\tilde\theta)(\lfloor t\rfloor - s)A}\| \\ &\quad \times \|(-A)^{-\rho}\{e^{(t-\lfloor t\rfloor)A} - 1\}\|\mathrm{d}s \\ &\quad + C\Big(\int_0^{\lfloor t\rfloor} \|(-A)^\rho e^{\tilde\theta(\lfloor t\rfloor - s)A}\|^2 \cdot \|e^{(1-\tilde\theta)(\lfloor t\rfloor - s)A}\|^2 \\ &\quad \times \|(-A)^{-\rho}\{e^{(t-\lfloor t\rfloor)A} - 1\}\|^2\mathrm{d}s\Big)^{1/2} \\ &\le C\triangle^{1/2} + C\triangle^\rho \int_0^{\lfloor t\rfloor} (\tilde\theta s)^{-\rho} e^{-\alpha(1-\tilde\theta)s}\mathrm{d}s \\ &\quad + C\triangle^\rho\Big(\int_0^{\lfloor t\rfloor} (\tilde\theta s)^{-2\rho} e^{-2\alpha(1-\tilde\theta)s}\mathrm{d}s\Big)^{1/2}. \end{aligned} \tag{6.4.12}$$

Observe that

$$\int_0^{\lfloor t\rfloor} s^{-\rho} e^{-\alpha(1-\tilde\theta)s}\mathrm{d}s \le (\alpha(1-\tilde\theta))^{\rho-1}\int_0^\infty s^{-\rho}e^{-s}\mathrm{d}s = (\alpha(1-\tilde\theta))^{\rho-1}\Gamma(1-\rho),$$

and similarly

$$\int_0^{\lfloor t\rfloor} s^{-2\rho} e^{-2\alpha(1-\tilde\theta)s}\mathrm{d}s \le (2\alpha(1-\tilde\theta))^{2\rho-1}\Gamma(1-2\rho),$$

where $\Gamma(\cdot)$ is the Gamma function. Hence

$$\sum_{k=2}^{4} F_k(t) \le C\triangle^{(\theta_1\wedge\theta_2)/2}.$$

This, together with the estimate of $F_1(t)$, gives that

$$\sup_{t\ge 0} \mathbb{E}\|\bar X(t) - \bar X(\lfloor t\rfloor)\|_H^2 \le C\triangle^{\theta_1\wedge\theta_2}.$$

Noting that $\bar{X}(t) = X(t) - \bar{Z}(t)$ and utilizing (6.4.4), one has

$$\sup_{t\geq 0} \mathbb{E}\|X(t) - X(\lfloor t \rfloor)\|_H^2 \leq C\triangle^{\theta_1\wedge\theta_2}. \tag{6.4.13}$$

Since

$$\begin{aligned}\|(1-\pi_n)(-A)^{-\theta_1}x\|_H^2 &= \Big\| \sum_{k=n+1}^{\infty} \lambda_k^{-\theta_1}\langle x, e_k\rangle_H e_k \Big\|_H^2 \\ &\leq \lambda_n^{-2\theta_1}\|x\|_H^2, \quad x \in H,\end{aligned}$$

we arrive at

$$\|(1-\pi_n)(-A)^{-\theta_1}\|^2 \leq \lambda_n^{-2\theta_1}. \tag{6.4.14}$$

By virtue of the Itô isometry, (H2), (6.4.14), (6.4.5) and (6.4.6), it follows that

$$\begin{aligned}&\mathbb{E}\|\bar{Z}(t) - \tilde{Y}^n(t)\|_H^2 \\ &\leq 2\int_0^t \|e^{sA}(1-\pi_n)\sigma^0\|_{HS}^2 \mathrm{d}s \\ &\quad + 2\int_0^t \|(-A)^{-\theta_1}(1 - e^{(s-\lfloor s\rfloor)}A)(-A)^{\theta_1}e^{(t-s)A}\sigma_n^0\|_{HS}^2 \mathrm{d}s \\ &\leq 2\|(1-\pi_n)(-A)^{-\theta_1}\|^2 \int_0^t \|(-A)^{\theta_1}e^{sA}\sigma^0\|_{HS}^2 \mathrm{d}s \\ &\quad + C\triangle^{2\theta_1}\int_0^t \|(-A)^{\theta_1}e^{sA}\sigma_n^0\|_{HS}^2 \mathrm{d}s \\ &\leq C(\|(1-\pi_n)(-A)^{-\theta_1}\|^2 + \triangle^{2\theta_1})\int_0^t \|(-A)^{\theta_1}e^{sA}\sigma^0\|_{HS}^2 \mathrm{d}s \\ &\leq C(\lambda_n^{-2\theta_1} + \triangle^{2\theta_1}).\end{aligned} \tag{6.4.15}$$

Following the argument of (6.4.11), we have

$$\begin{aligned}&(\mathbb{E}\|\bar{X}(t) - Z^n(t)\|_H^2)^{1/2} \\ &\leq \|e^{tA}(1-\pi_n)x\|_H \\ &\quad + \int_0^t \|e^{(t-s)A}(1-\pi_n)\|(\mathbb{E}\|b(X(s))\|_H^2)^{1/2}\mathrm{d}s \\ &\quad + \Big(\int_0^t \|e^{(t-s)A}(1-\pi_n)\|^2\mathbb{E}\|\sigma^1(X(s))\|_{HS}^2\mathrm{d}s\Big)^{1/2} \\ &\quad + \int_0^t \|e^{(t-s)A}\|(\mathbb{E}\|b_n(X(s)) - b_n(X(\lfloor s\rfloor))\|_H^2)^{1/2}\mathrm{d}s \\ &\quad + \Big(\int_0^t \|e^{(t-s)A}\|^2\mathbb{E}\|\sigma_n^1(X(s)) - \sigma_n^1(X(\lfloor s\rfloor))\|_{HS}^2\mathrm{d}s\Big)^{1/2} \\ &\quad + \int_0^t \|e^{(t-s)A}\|(\mathbb{E}\|b_n(X(\lfloor s\rfloor)) - b_n(Y^n(\lfloor s\rfloor))\|_H^2)^{1/2}\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&+\Big(\int_0^t \|e^{(t-s)A}\|^2\mathbb{E}\|\sigma_n^1(X(\lfloor s\rfloor))-\sigma_n^1(Y^n(\lfloor s\rfloor))\|_{HS}^2\mathrm{d}s\Big)^{1/2}\\
&+\int_0^t \|e^{(t-s)A}\{1-e^{(s-\lfloor s\rfloor)A}\}\|(\mathbb{E}\|b(Y^n(\lfloor s\rfloor))\|_H^2)^{1/2}\mathrm{d}s\\
&+\Big(\int_0^t \|e^{(t-s)A}\{1-e^{(s-\lfloor s\rfloor)A}\}\|^2\mathbb{E}\|\sigma^1(Y^n(\lfloor s\rfloor))\|_{HS}^2\mathrm{d}s\Big)^{1/2}\\
&=:\sum_{i=1}^{9}G_i(t).
\end{aligned}
\tag{6.4.16}
$$

A straightforward computation shows that

$$\|e^{tA}(1-\pi_n)x\|_H^2=\sum_{i=n+1}^{\infty}e^{-2\lambda_i t}\langle x,e_i\rangle_H^2.$$

This further gives that

$$\|e^{tA}(1-\pi_n)\|^2\le e^{-2\lambda_n t} \tag{6.4.17}$$

and that

$$G_1(t)\le\Big(\sum_{i=n+1}^{\infty}\frac{e^{-2\lambda_i t}}{\lambda_i^2}\lambda_i^2\langle x,e_i\rangle_H^2\Big)^{1/2}\le\lambda_n^{-1}\|Ax\|_H \tag{6.4.18}$$

by recalling that $\{\lambda_i\}_{i\ge1}$ is a nondecreasing sequence. By (6.4.10) and (6.4.17), one has

$$
\begin{aligned}
&G_2(t)+G_3(t)\\
&\le C\int_0^t\|e^{(t-s)A}(1-\pi_n)\|\mathrm{d}s+C\Big(\int_0^t\|e^{(t-s)A}(1-\pi_n)\|^2\mathrm{d}s\Big)^{1/2}\\
&\le C\int_0^t e^{-\lambda_n(t-s)}\mathrm{d}s+C\Big(\int_0^t e^{-2\lambda_n(t-s)}\mathrm{d}s\Big)^{1/2}\\
&\le C(\lambda_n^{-1}+\lambda_n^{-1/2}).
\end{aligned}
\tag{6.4.19}
$$

Taking (H1), (H3) and (6.4.13) into account gives that

$$
\begin{aligned}
&G_4(t)+G_5(t)\\
&\le C\triangle^{(\theta_1\wedge\theta_2)/2}\Big\{\int_0^t\|e^{(t-s)A}\|\mathrm{d}s+\Big(\int_0^t\|e^{(t-s)A}\|^2\mathrm{d}s\Big)^{1/2}\Big\}\\
&\le C\triangle^{(\theta_1\wedge\theta_2)/2}.
\end{aligned}
\tag{6.4.20}
$$

Next, note from (H1) and (H3) that

$$
\begin{aligned}
&G_6(t)+G_7(t)\\
&\le \int_0^t \|e^{(t-s)A}\|(\mathbb{E}\|b(X(\lfloor s\rfloor))-b(Y^n(\lfloor s\rfloor))\|_H^2)^{1/2}\mathrm{d}s\\
&\quad+\Big(\int_0^t \|e^{(t-s)A}\|^2\mathbb{E}\|\sigma^1(X(\lfloor s\rfloor))-\sigma^1(Y^n(\lfloor s\rfloor))\|_{HS}^2\mathrm{d}s\Big)^{1/2}\\
&\le \sup_{0\le s\le t}(\mathbb{E}\|b(X(\lfloor s\rfloor))-b(Y^n(\lfloor s\rfloor))\|_H^2)^{1/2}\int_0^t\|e^{(t-s)A}\|\mathrm{d}s\\
&\quad+\sup_{0\le s\le t}(\mathbb{E}\|\sigma^1(X(\lfloor s\rfloor))-\sigma^1(Y^n(\lfloor s\rfloor))\|_H^2)^{1/2}\Big(\int_0^t\|e^{(t-s)A}\|^2\mathrm{d}s\Big)^{1/2} \qquad (6.4.21)\\
&\le \alpha^{-1}\sup_{0\le s\le t}(\mathbb{E}\|b(X(\lfloor s\rfloor))-b(Y^n(\lfloor s\rfloor))\|_H^2)^{1/2}\\
&\quad+(2\alpha)^{-1/2}\sup_{0\le s\le t}(\mathbb{E}\|\sigma^1(X(\lfloor s\rfloor))-\sigma^1(Y^n(\lfloor s\rfloor))\|_H^2)^{1/2}\\
&\le \tau\sup_{0\le s\le t}(\mathbb{E}\|X(s)-Y^n(s)\|_H^2)^{1/2}\\
&\le \tau\sup_{0\le s\le t}(\mathbb{E}\|\bar X(s)-Z^n(s)\|_H^2)^{1/2}+\tau\sup_{0\le s\le t}(\mathbb{E}\|\bar Z(s)-\tilde Y^n(s)\|_H^2)^{1/2},
\end{aligned}
$$

where $\tau\in(0,1)$ is defined by (6.4.9). Following the argument of (6.4.12) leads to

$$G_8(t)+G_9(t)\le C\triangle^{(\theta_1\wedge\theta_2)/2}. \qquad (6.4.22)$$

Substituting (6.4.18)-(6.4.22) into (6.4.16) yields that

$$\sup_{t\ge0}(\mathbb{E}\|\bar X(t)-Z^n(t)\|_H^2)^{1/2}\le C(\lambda_n^{-1/2}+\triangle^{(\theta_1\wedge\theta_2)/2})$$

due to $\tau\in(0,1)$. Consequently the desired assertion follows from (6.4.15). □

Theorem 6.4.4. Let (H1)-(H4) and (6.4.3) hold. Then, for any $\varepsilon>0$, there exist $n>0$ and $\varepsilon>0$ such that

$$\mathrm{d}_{\mathbb{L}}(\pi^{n,\triangle}(\cdot),\pi(\cdot))\le\varepsilon.$$

Proof. Fix $x\in H$ and let $\epsilon>0$ be arbitrary. By Lemma 6.4.3 there exist $\triangle^*\in(0,1)$ and sufficiently large $n>0$ such that

$$\mathrm{d}_{\mathbb{L}}(\mathbb{P}_{k\triangle}(x,\cdot),\mathbb{P}_k^{n,\triangle}(x_n,\cdot))\le\epsilon/3,\quad \triangle\in(0,\triangle^*).$$

For the previous $n > 0$, by Theorem 6.2.2, there exist $\triangle_0 \in (0,1)$ and $T_1 > 0$ such that

$$\mathrm{d}_{\mathbb{L}}(\mathbb{P}_k^{n,\triangle}(x_n,\cdot),\pi^{n,\triangle}(\cdot)) \le \epsilon/3$$

whenever $\triangle \in (0,\triangle_0)$ and $k\triangle \ge T_1$. Furthermore, due to Lemma 6.4.1 there exists $T_2 > 0$ such that

$$\mathrm{d}_{\mathbb{L}}(\mathbb{P}_t(x,\cdot),\pi(\cdot)) \le \epsilon, \quad t \ge T_2.$$

Let $T := T_1 \vee T_2$ and $k = [T/\triangle] + 1$ for any $\triangle < \triangle^* \wedge \triangle_0$. Then the desired assertion follows from the triangle inequality

$$\begin{aligned}\mathrm{d}_{\mathbb{L}}(\pi^{n,\triangle}(\cdot),\pi(\cdot)) \le \mathrm{d}_{\mathbb{L}}(\mathbb{P}_{k\triangle}(x,\cdot),\pi(\cdot)) + d_{\mathbb{L}}(\mathbb{P}_{k\triangle}(x,\cdot),\mathbb{P}_k^{n,\triangle}(x_n,\cdot))\\ + \mathrm{d}_{\mathbb{L}}(\mathbb{P}_k^{n,\triangle}(x_n,\cdot),\pi^{n,\triangle}(\cdot)).\end{aligned}$$

□

Remark 6.4.2. For the finite-dimensional case, finite-time convergence of numerical scheme is enough to discuss the limit of stationary distribution of numerical solution [56, Theorem 6.23, p266]. While for the infinite-dimensional case, we need the *uniform convergence* of EM scheme (6.2.8) to reveal the limit behavior of $\pi^{n,\triangle}$, which is quite different from the finite-dimensional cases, and therefore (6.4.9) is imposed. On the other hand, for the finite-time convergence of EM scheme (6.2.8), condition (6.4.9) can be deleted by checking the argument of Lemma 6.4.3 and combining with the Gronwall inequality.

Remark 6.4.3. By following the procedure of this chapter, numerical approximation of stationary distribution of SPDEs with jumps can also be discussed, which will be reported in forthcoming paper.

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