Some generators of Lp-sub-Markovian semigroups in the half-space R(n+1)/0+.

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Some generators of L_p-sub-Markovian semigroups in the half-space $\mathbb{R}^{n+1}_{0+}$

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Summary

In this thesis we study pseudo-differential operators of the form

$$-A_{\pm} = -\psi(D_{x'}) \pm \frac{\partial}{\partial x_{n+1}}, \quad (x', x_{n+1}) \in \mathbb{R}^{n+1}_{0+},$$

where $\psi(D_{x'})$ is an operator with real continuous negative definite symbol $\psi\colon \mathbb{R}^n \to \mathbb{R}$, acting on functions depending on $x' \in \mathbb{R}^n$. Further we consider the fractional powers $(-A_{\pm})^{\alpha}$, $0 < \alpha < 1$, of $-A_{\pm}$. After determining the domains in $L_p(\mathbb{R}^{n+1}_{0+})$ of these operators in terms of Bessel-type potential spaces and studying some properties of these function spaces, we prove that with these domains $-(-A_{\pm})^{\alpha}$ are generators of L_p-sub-Markovian semigroups. Then we extend this result and show that the operators

$$-(-A_{\pm})^{\alpha} - p(x', D_{x'})$$

also generate L_p-sub-Markovian semigroups, if the pseudo-differential operator $p(x', D_{x'})$ is $(-A_{\pm})^{\alpha}$-bounded and the symbol $p(x', \xi')$ of $p(x', D_{x'})$ is with respect to ξ' a continuous negative definite function. In the end we proved the continuity of the pseudo-differential operator with continuous negative definite symbol (with certain condition on the growth of the Lévy measure) between the Besov spaces.

Contents

Chapter 0

Introduction

In this thesis we study pseudo-differential operators of some special type, and give conditions under which these operators are generators of L_p-sub-Markovian semigroups, i.e. strongly continuous contractions semigroups on L_p, which have the sub-Markovian property.

More precisely, we consider operators of the form

$$-A_\pm = -\psi(D_{x'}) \pm \frac{\partial}{\partial x_{n+1}}, \quad (x', x_{n+1}) \in \mathbb{R}^{n+1}_{0+},$$

where $\psi(D_{x'})$ is an operator with real continuous negative definite symbol $\psi\colon \mathbb{R}^n \to \mathbb{R}$, acting on functions depending on $x' \in \mathbb{R}^n$. Further we will consider the fractional powers $(-A_\pm)^\alpha$, $0 < \alpha < 1$, of $-A_\pm$. We determine the domains in $L_p(\mathbb{R}^{n+1}_{0+})$ of these operators, and prove that with these domains $-(-A_\pm)^\alpha$ are generators of L_p-sub-Markovian semigroups. Then we extend this result by proving that the operators

$$-(-A_\pm)^\alpha - p(x', D_{x'})$$

also generate L_p-sub-Markovian semigroups, if the pseudo-differential operator $p(x', D_{x'})$ is $(-A_\pm)^\alpha$-bounded and the symbol $p(x', \xi')$ of $p(x', D_{x'})$ is with respect to ξ' a continuous negative definite function.

We are interested in ψ-Bessel potential spaces $H_p^{\psi,s}$ on the half-space $\mathbb{R}^{n+1}_{0+}$, $1 \le p < \infty$, $-\infty < s < \infty$, ψ is a real-valued continuous negative definite function. Such spaces were defined in Farkas, Jacob, Schilling [FJS1] on $\mathbb{R}^n$, and they are some generalization of Triebel-Lizorkin spaces $F^s_{p2} (= H^s_p)$, which are called Bessel potential spaces of order s. Our main references on function spaces are Triebel ([T1] and [T2]), see also the books of Adams [A], and Besov, Il'in, Nikol'skii [BIN]. Such function spaces can be constructed as spaces of functions which can be approximated by sequences of smooth functions with finite support (see Triebel [T1]) or as sets of functions which together with their (fractional) derivatives belong to some Banach space (see Sobolev [So], Lizorkin [Liz2], [Liz3]); see also Nikol'skii [N2], Besov [Be1], [Be2]).

However Bessel potential spaces can also be defined as in Bagby [Bag] or Aronszajn and Smith [AS]. Namely, in this paper the spaces H_p^s are defined as the images of the spaces L_p under certain operators, i.e.

$$J_s f = F^{-1}\big((1+|x|^2+it)^{-s/2}\hat{f}\big), \quad x \in \mathbb{R}^n, t \in \mathbb{R},$$

(see[Bag]), or

$$G_s f = F^{-1}\big((1+|x|^2)^{-s/2}\hat{f}\big), \quad x \in \mathbb{R}^n,$$

(see [AS]). For the extensive references on Bessel potential spaces as domains of Riesz and Bessel potentials we refer to the survey of fractional derivatives and integrals of Samko et al. [S], see also Rubin [R] for a lot of results on mapping properties, restrictions and extensions of fractional integrals and derivatives, etc.

We want to define the $\Re$-Bessel potential spaces on the half-space $\mathbb{R}^{n+1}_{0+}$, where $\Re = Re(\chi_\pm(\xi))^\alpha$, $\chi_\pm = \psi(\xi')\pm i\xi_{n+1}$, and ψ is a continuous negative definite function which satisfies certain assumptions. The assumptions we need (see **A1** and **A2**, Assumption 2.1.17) are such that the function $e^{i\alpha arg\, \chi}$ is an L_p-Fourier multiplier (for Fourier multipliers see for example Triebel [T1], and also the paper of Lizorkin [Liz1]). This enables us to show, that the $(\chi_\pm(\xi))^\alpha$-Bessel potential space (i.e. the space constructed with the complex-valued continuous negative definite function $(\chi_\pm(\xi))^\alpha$) is equivalent to the $\Re$-Bessel potential space (i.e. the space constructed with the real continuous negative definite function $\Re$). Since the set $\mathbb{R}^{n+1}_{0+}$ has a boundary, we have (see for the case of Bessel potentials Triebel ([T1] and [T2]), also Lions, Magenes [LM] in the special case $p=2$) two possibilities how to define the $\Re$-Bessel potential space on $\mathbb{R}^{n+1}_{0+}$:

i) as $H^{\Re,s}_{p+}$ –the space of restrictions to $\mathbb{R}^{n+1}_{0+}$of functions in $H^{\Re,s}_p(\mathbb{R}^{n+1})$;

ii) as $\tilde{H}^{\Re,s}_{p+}$ –the space of functions from $H^{\Re,s}_p(\mathbb{R}^{n+1})$ which have support in $\mathbb{R}^{n+1}_{0+}$.

As soon as we have defined these spaces, many questions arise, i.e. the dense sets in $H^{\Re,s}_{p+}$ and in $\tilde{H}^{\Re,s}_{p+}$, the problems of extensions to $H^{\Re,s}_p(\mathbb{R}^{n+1})$ and the restriction to the initial spaces, the embedding and interpolation results, isomorhic mappings, etc. Such problems for Sobolev spaces, Triebel-Lizorkin spaces, Besov spaces, and many others, were studied a lot. See Triebel [T1] and [T2], the books of Adams [A], Besov, Il'in, Nikol'skii [BIN], Nikol'skii [N2] for the L_p-theory of Sobolev spaces. In particular for methods of restriction and extensions across the boundary and embedding theorems we refer to Lions and Magenes [LM] for the L_2 theory of Sobolev spaces. Bennett and Sharpley [BS] is a good reference on interpolation theory, see also the references given there. As far as our task is concerned, we solve these problems for $H^{\Re,s}_{p+}$ and in $\tilde{H}^{\Re,s}_{p+}$.

We also consider the Lipschitz spaces in order to find conditions under which a pseudo-differential operator (which satisfies additional assumptions) is continuous between some Sobolev spaces ($W_p^s = B^s_{pp}$, s is not integer, see [T2]). For the Lipschitz spaces we refer to Krein [Kre], Stein [S2], see also Kufner, John and Fucik [KJF], and Triebel [T1].

The results on function spaces enable us to solve some boundary-value problems related to the equation

$$(\lambda+(-A_\pm)^\alpha)f(x) = g(x), \quad x = (x', x_{n+1}) \in \mathbb{R}_+^{n+1},$$
$$f \in D((-A_\pm)^\alpha), \quad g \in L_p(\mathbb{R}_+^{n+1}), \quad 0 < \alpha \leq 1. \tag{1}$$

As the domains of $(-A_\pm)^\alpha$ we take $H_{p+}^{\Re,s}$ and $\tilde{H}_{p+}^{\Re,s}$. For the solutions of boundary-value problems for elliptic differential operators in H_p^s we refer to Taira [Ta] and Edmunds, Triebel [ET], see also Lions and Magenes [LM], Hörmander [H] for the case $p = 2$.

Now we turn to the probabilistic part, namely, to applying these results on function spaces to construct an L_p-sub-Markovian semigroup. However, it is not trivial to associate a Markov provess to an L_p-sub-Markovian semigroup. It was done in the case $p = 2$, see [FOT].

To construct a random process $(X_t)_{t\geq 0}$ means to give a triple $(\Omega, \mathcal{F}, P(dx))$, where Ω is a set, $\mathcal{F}$ is a σ-algebra on Ω, and $P(dx)$ is a probability measure. To construct a Lévy process, i.e. a process with stationary and independent increments, continuous in probability, this complicated problem becomes easier, since the characteristic function $E[e^{i\xi X_t}]$ of a Lévy process X_t has a very special form: it can be expressed as (see for example Jacob [J1], or Bertoin [Ber])

$$E[e^{i\xi X_t}] = e^{-t\psi(\xi)}, \quad \xi \in \mathbb{R}^n, \quad t > 0,$$

where $\psi(\xi)$ is a continuous negative definite function. It is known that any continuous negative definite function $\psi(\xi)$ has a Lévy-Khinchin representation:

$$\psi(\xi) = c + i(\xi, b) + \sum_{i,j=1}^n q_{ij}\xi_i\xi_j + \int_{\mathbb{R}^n\setminus\{0\}} (1 - e^{-i\xi x} - \frac{i\xi x}{1+|x|^2})\, \nu(dx),$$

where $c \geq 0$, $b \in \mathbb{R}^n$, the quadratic form $\sum_{i,j=1}^n q_{ij}\xi_i\xi_j$ is positive semidefinite, and $\nu(dx)$ is a Lévy measure, i.e. it is a measure such that

$$\int_{\mathbb{R}^n\setminus\{0\}} (1 \wedge |x|^2)\, \nu(dx) < \infty.$$

For the processes with independent increments we refer to Skorokhod [Sk], see also the classical treatise of Gikhman and Skorokhod [GS2].

To construct a random process one may use probabilistic approaches, i.e. knowing its sample paths, for example solving stochastic differential equations, see Itô [I], or Gikhman, Skorokhod [GS3], or analytical approaches, solving Kolmogorov's equation, see Feller [Fe] and Dynkin [Dy]. The approach we will use is the second one, namely we will construct a semigroup of operators $(T_t)_{t\geq 0}$ having a generator $(A, D(A))$:

$$Af = \lim_{t\to 0} \frac{T_t f - f}{t}, \quad f \in D(A).$$

Certainly, if we obtain a sub-Markovian semigroup, i.e. a strongly continuous contraction semigroup in L_p such that $0 \leq u \leq 1$ a.e. implies $0 \leq T_t u \leq 1$ a.e., then we may associate a Markov process $((X_t)_{t\geq 0}, P^x)_{x\in\mathbb{R}^n}$ with $(T_t)_{t\geq 0}$. This process and the semigroup are linked by

$$T_t u(x) = E^x(u(X_t))$$

or with $u = \chi_A$ we find

$$p_t(x, A) = (T_t \chi_A)(x)$$

for the transition function. However in both relations one has to take into account certain exceptional sets.

Not all operators with continuous negative definite symbols are generators of strongly continuous contraction semigroups. To be a generator of a strongly continuous contraction semigroup an operator A must satisfy the conditions of the Hille-Yosida Theorem. We will follow the formulation of this theorem given in Jacob [J1], or Pazy [Pa], for a different formulation of it see [GS1]. In addition, for an operator to be a generator of an L_p-sub-Markovian semigroup, it is necessary to be a Dirichlet operator: it must satisfy the inequality:

$$\int_{\mathbb{R}^n} A\,u\big((u-1)^+\big)^{p-1}\,dx \leq 0.$$

For the Dirichlet operators we refer in the case $p = 2$ to Ma and Röckner [MR], Bouleau, Hirsch [BH], and [J1] in the general case. Moreover, having a generator A of an L_p-sub-Markovian semigroup we may perturb it by an A-bounded Dirichlet operator B, and the operator $(A + B, D(A))$ is still a generator of some sub-Markovian semigroup. For such perturbation results we refer to Pazy [Pa], Jacob [J1], [J2].

In the first chapter we provide definitions and results from the theory of one-parameter operator semigroups. We give the definitions of L_p-sub-Markovian semigroups, Dirichlet operators, and quote the Hille-Yosida theorem, which gives us the necessary and sufficient conditions under which a closed operator is a generator of a strongly continuous contraction semigroup. The theory of subordination in the sense of Bochner gives us a way of constructing new semigroups starting with a strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ and a convolution semigroup of measures $(\eta_t)_{t\geq 0}$ with supports in $[0, \infty)$. Such convolution semigroups of measures correspond to so-called Bernstein functions, which will play a great rôle in the second chapter. We also provide some examples of Bernstein functions, one of them, namely, $f(x) = x^\alpha$, $0 < \alpha < 1$, corresponds to the one-sided stable semigroup of order α, $(\sigma_\alpha(x, t)\,dx)_{t\geq 0}$, which are the the probability measures (which have "Lévy stable" densities $\sigma_\alpha(x, t)$) related to an α-stable process (see Ibragimov, Linnik [IL] for more about such processes). We only list some properties of the stable density functions $(\sigma_\alpha(x, t))_{t\geq 0}$, which are important for us, and discuss related Mittag-Leffler-type functions, see Podlubny [Pod]. In the end of Section 1.1 the definition of Bessel functions is given in order to provide our work with examples in Section 3.2.

In Section 1.2 after some preparations we prove for two generators $(A, D(A))$ and $(B, D(B))$ of strongly continuous contraction semigroups in $L_p(\mathbb{R}^n)$ and $L_p(\mathbb{R}^m)$, respectively, that the closure of the operator $A \oplus B$, defined on the product space $D(A) \otimes D(B)$, is a generator of a strongly continuous contraction semigroup, which is sub-Markovian, if $(A, D(A))$ and $(B, D(B))$ generate sub-Markovian semigroups. A similar result for Feller semigroups was proved in Krägeloh [Kr].

In Section 1.3 we give the definition of a pseudo-differential operator, and of A-bounded operators. Actually the restriction in Definition 1.3.1 that $0 < \varepsilon < 1$ is not necessary for the A-boundedness of an operator Q (see [KK] or Pazy [Pa] for a more general definition of A-boundedness). It is essential when we want to prove that having a generator $(A, D(A))$ of a sub-Markovian semigroup in L_p, the perturbed operator $(A + Q, D(A))$, where Q is A-bounded with $0 < \varepsilon < 1$, is a generator of an L_p-sub-Markovian semigroup too.

Section 2.1 is devoted to the studying of $\Re$-Bessel potential spaces on $\mathbb{R}^{n+1}$ and on the half-space $\mathbb{R}^{n+1}_{0\pm}$, where $\Re = Re(\chi_\pm(\xi))^\alpha$, $\chi_\pm(\xi) = \psi(\xi') \pm i\xi_{n+1}$, and ψ is a continuous negative definite function. First, we show, that under the conditions stated in Assumption 2.1.17 the functions $e^{z\, arg \chi}$, $0 < Im\, z < 1$, are L_p-Fourier multipliers, which enables us to find the domains of operator $(-A_\pm)^\alpha$, $symb(-A_\pm) = \chi_\pm$ in $L_p(\mathbb{R}^{n+1})$, and show, that the operator $(-A_\pm)^\alpha$ is an isomorphism between $H_p^{\Re,s}$ and $H_p^{\Re,s-2}$ (or between $H_p^{|\chi|^{\alpha,s}}$ and $H_p^{|\chi|^{\alpha,s-2}}$, since the result that $e^{z\, arg \chi}$, $0 < Im\, z < 1$, is an L_p-Fourier multiplier gives us also the equivalence of the spaces $H_p^{\Re,s}$ and $H_p^{|\chi|^{\alpha},2}$, see Remark 2.1.23). Next we obtain an interpolation theorem, which says that we can obtain the space $H_p^{\Re,2}$ by complex interpolation between the spaces L_p and $H_p^{\chi,2,1}$, see Theorem 2.1.27. Further, we define the spaces $H_{p\pm}^{\Re,s}$ and $\tilde{H}_{p\pm}^{\Re,s}$ on $\mathbb{R}^{n+1}_{0+}$ (see Definition 2.21), and prove that the operators $(-A_\pm)^\alpha$ are isomorphisms between $\tilde{H}_{p\pm}^{\Re,s}$ and $\tilde{H}_{p\pm}^{\Re,s-2}$ as well as between $H_{p\mp}^{\Re,s}$ and $H_{p\mp}^{\Re,s}$. In the proof of the first statement we use the same method as given in the proof of Theorem 2.10.3 [T2], and the proof of the second statement is based on the fact that the spaces $H_{p\mp}^{\Re,s}$ are the factor spaces of $H_p^{\Re,s}$ with respect to $\tilde{H}_{p\pm}^{\Re,s}$.

The proofs of the existence of the retraction and the corresponding coretraction are different to those in Triebel [T2], because the method given there is not applicable in our situation. But we can construct the isomorphisms from the spaces $H_p^{\Re,s}$, $\tilde{H}_{p+}^{\Re,s}$ and $H_{p+}^{\Re,s}$ to the classical Bessel potential spaces (on the whole space $\mathbb{R}^{n+1}$ and on the half-space $\mathbb{R}^{n+1}_{0+}$), and in such a way we can reduce our problem to the problem already solved.

We also prove embedding and interpolation theorems for $\tilde{H}_{p+}^{\Re,s}$ and $H_{p+}^{\Re,s}$, (see Theorem 2.2.9, Theorem 2.2.15 and Remark 2.2.16), and find dense sets in these spaces (Theorem 2.2.10).

Chapter 3 is devoted to constructing L_p-sub-Markovian semigroups.

While checking the third condition of the Hille-Yosida theorem, i.e. the solvability of equation (1) for any $g \in L_p(\mathbb{R}^{n+1}_{0+})$ with some boundary conditions, we see that different boundary conditions lead to different semigroups. We consider two types of boundary condition, namely, Dirichlet and Neumann boundary conditions. We find a representation of the L_p-sub-Markovian semigroups, generated by $-(-A_\pm)^\alpha$, $0 < \alpha \leq 1$, with some domain in $L_p(\mathbb{R}^{n+1}_{0+})$, with these boundary conditions, as well as the correponding resolvent operators.

First we consider the case $\alpha = 1$, in which the operators $-A_\pm = -\psi(D_{x'}) \pm \frac{\partial}{\partial x_{n+1}}$, $(x', x_{n+1}) \in \mathbb{R}^{n+1}_{0+}$, are defined on the product spaces $H_p^{\psi,2}(\mathbb{R}^n) \otimes H^1_{p,+}(\mathbb{R}_{0+})$ and $H_p^{\psi,2}(\mathbb{R}^n) \otimes \tilde{H}^1_{p,+}(\mathbb{R}_{0+})$. Theorem 1.2.3 gives that the closure of $(-A_+, H_p^{\psi,2} \otimes H^1_{p,+})$ and $(-A_+, H_p^{\psi,2} \otimes \tilde{H}^1_{p,+})$, respectively, with respect to the graph norm of A_+, generates L_p-sub-Markovian semigroups. In both cases we find the corresponding semigroups and the resolvent operators. In the first case we consider non-zero Dirichlet boundary conditions, and zero Neumann boundary conditions in the second.

The results obtained in Chapter 2 give the necessary background to extend the case $\alpha = 1$ to $0 < \alpha < 1$, i.e. to prove that the operators $-(-A_+)^\alpha$ on some domains are generators of L_p-sub-Markovian semigroups, and we find these domains. More precisely, we find in Chapter 2 the domains $D((-A_\pm)^\alpha)$ of $-(-A_\pm)^\alpha$, and Theorems 2.2.5 and 2.2.6 give that $(-A_\pm)^\alpha$ is an isomorphism between these domains and $L_p(\mathbb{R}^{n+1}_{0+})$, which is essential when we solve the boundary value problem (1). Therefore, we have all necessary tools to prove that the operators $(-(-A_\pm)^\alpha, D((-A_\pm)^\alpha))$ are the generators of L_p-sub-Markovian semigroups, for different $D((-A_\pm)^\alpha) \subset L_p(\mathbb{R}^{n+1}_{0+})$.

Since we know the semigroups generated by $(-A_\pm, D(A_\pm))$, subordination in sense of Bochner gives us the candidates for semigroups generated by $(-(-A_\pm)^\alpha, D((-A_\pm)^\alpha))$ (with some boundary conditions). As in the case $\alpha = 1$, different boundary conditions lead to different semigroups, see Theorems 3.1.4, Remark 3.1.2 and Theorem 3.1.5. In Chapter 3.2 in order to illustrate our work we gave a few examples of such generators and semigroups.

In Chapter 3.3 we proved that an operator $(-(-A_\pm)^\alpha, D((-A_\pm)^\alpha))$ perturbed by $(-A_\pm)^\alpha$-bounded pseudo-differential operator still generates an L_p-sub-Markovian semigroup. We also give a few examples of such pseudo-differential operators.

Finally, in Chapter 3.4 we proved that a pseudo-differential operator with negative definite symbol is continuous between the Lipschitz spaces of order λ, $0 < \lambda < 1$, if we pose a condition on the growth of the density of the corresponding Lévy measure. This result leads to the continuity of such a pseudo-differential operator between the Besov spaces W_p^s. For the case of elliptic pseudo-differential operators see Edmunds, Triebel [ET].

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Chapter 1

Preliminaries

1.1 Sub-Markovian semigroups. Basic definitions

We refer to the books of Jacob [J1] and Yosida [Y] in presenting some definitions and theorems from the theory of one-parameter semigroups (see also the monograph of Hille and Phillips [HP]).

Let $(X, \|\cdot\|)_X$ be a real or complex Banach space.

Definition 1.1.1. A. A one-parameter family $(T_t)_{t\geq 0}$ of bounded linear operators $T_t : X \to X$ is called a (one parameter) **semigroup** of operators, if $T_0 = I$ and $T_{t+s} = T_t \circ T_s$ hold for all $s, t \geq 0$.

B. We call $(T_t)_{t\geq 0}$ **strongly continuous** if

$$\lim_{t\to 0} \|T_t u - u\|_X = 0$$

for all $u \in X$.

C. The semigroup $(T_t)_{t\geq 0}$ is called a **contraction semigroup**, if for all $t \geq 0$

$$\|T_t\| \leq 1$$

holds, and we denote by $\|T_t\|$ the operator norm $\|T_t\|_{X,X}$.

D. We call a strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ on $L_p(\mathbb{R}^n, \mathbb{R})$, $1 < p < \infty$, **sub-Markovian** if for all $u \in L^p(\mathbb{R}^n, \mathbb{R})$ such that $0 \leq u \leq 1$ almost everywhere (a.e.) it follows that $0 \leq T_t u \leq 1$ a.e.

Definition 1.1.2. Let $(T_t)_{t\geq 0}$ be a strongly continuous semigroup of operators on a Banach space $(X, \|\cdot\|_X)$. The **generator** A of $(T_t)_{t\geq 0}$ is defined by

$$A\,u := \lim_{t\to 0} \frac{T_t u - u}{t} \quad \text{strongly,}$$

with domain

$$D(A) := \{u \in X| \quad \lim_{t\to 0} \frac{T_t u - u}{t} \quad \text{exists as a strong limit.}\}$$

Proposition 1.1.3. A. For each strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ on X there exists a closed operator A with domain $D(A)$, dense in $(X, \|\cdot\|_X)$, which is the generator of $(T_t)_{t\geq 0}$.
B. For each $t \geq 0$ the operator T_t maps $D(A)$ into itself.
Conversely, if $(A, D(A))$ is the generator of strongly continuous semigroup on X, then A is a closed operator and $D(A)$ is dense in X. (See [J1] or [Y]).

We are especially interested in sub-Markovian semigroups and their generators. For the semigroup to be sub-Markovian it is necessary and sufficient that its generator is a Dirichlet operator, see Definition 4.6.7, [J1].

Definition 1.1.4. A closed, densely defined linear operator $A : D(A) \to L_p(\mathbb{R}^n, \mathbb{R})$, $1 < p < \infty$, $D(A) \subset L_p(\mathbb{R}^n, \mathbb{R})$, is called a **Dirichlet operator** if for all $u \in D(A)$ the relation

$$\int_{\mathbb{R}^n} (A\,u)\big((u-1)^+\big)^{p-1}\, dx \leq 0 \tag{1.1}$$

holds.

Proposition 1.1.5. Suppose that a Dirichlet operator $(A, D(A))$ on $L_p(\mathbb{R}^n, \mathbb{R})$, $1 < p < \infty$, generates a strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ on $L_p(\mathbb{R}^n, \mathbb{R})$. Then $(T_t)_{t\geq 0}$ is sub-Markovian.

The next proposition is the consequence of Theorems 4.6.11 and 4.6.12, [J1].

Proposition 1.1.6. Let $(A, D(A))$ be a Dirichlet operator on $L_p(\mathbb{R}^n, \mathbb{R})$. Then it is dissipative, i.e.

$$\|\lambda u - A\,u\|_p \geq \lambda \|u\|_p$$

for all $u \in D(A)$ and $\lambda > 0$.

Now we can formulate the necessary and sufficient conditions for the operator $(A, D(A))$ to be the generator of a strongly continuous contraction semigroup, see for example [J1], Theorem 4.1.33. Denote by $R(A)$ the range of operator A.

Theorem 1.1.7 (Hille-Yosida theorem). A closed linear operator $(A, D(A))$ on a Banach space $(X, \|\cdot\|_X)$ is the generator of a strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ if and only if the following conditions hold:

1. $D(A) \subset X$ is dense;

2. A is a dissipative operator;

3. $R(\lambda - A) = X$ for some $\lambda > 0$.

In Chapter 2 we will deal with the subordinated semigroups. As a preparation we present already now some definitions, see [J1].

Define by $\delta_y(dx)$ the **Dirac measure** at y

$$\delta_y(A) = \begin{cases} 1, & y \in A \\ 0, & y \notin A \end{cases} \tag{1.2}$$

which has the property that

$$\int_G f(x)\delta_y(dx) = f(y)$$

for all functions which are locally integrable (with respect to Lebesgue measure) on some domain G, $y \in G$.

Definition 1.1.8. A family $(\mu_t)_{t\geq 0}$ of Borel measures on $\mathbb{R}^n$ is called a **convolution semigroup** on $\mathbb{R}^n$ if i) $\mu_t(\mathbb{R}^n) \leq 1$ for all $t \geq 0$; ii) $\mu_s * \mu_t = \mu_{t+s}$, $s, t \geq 0$ and $\mu_0 = \delta_0$; iii) $\mu_t \to \delta_0$ vaguely as $t \to 0$.

The convolution semigroups are closely related with **negative definite functions**, i.e. with functions $\psi : \mathbb{R}^n \to \mathbb{C}$ such that $\psi(0) \geq 0$ and the function $\xi \mapsto (2\pi)^{-n/2}e^{-t\psi(\xi)}$ is **positive definite** for $t \geq 0$. Recall that a function $u : \mathbb{R}^n \to \mathbb{C}$ is positive definite if for every $k \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$, and for any vectors $\xi_1, \ldots, \xi_k \in \mathbb{R}^n$ we have

$$\sum_{i,j=1}^{k} u(\xi_i - \xi_j)\lambda_i\overline{\lambda_j} \geq 0.$$

Denote by $\hat{g}$ the Fourier transform of a function $g \in L_1(\mathbb{R}^n)$,

$$\hat{g}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\xi x} g(x)\, dx, \tag{1.3}$$

and by $\hat{\mu}_t$ the Fourier transform of the measure μ_t:

$$\hat{\mu}_t(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\xi x} \mu_t(dx). \tag{1.4}$$

For Fourier transforms we refer to [RS2].

Theorem 1.1.9. For any convolution semigroup $(\mu_t)_{t\geq 0}$ on $\mathbb{R}^n$ there exists a uniquely determined continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{C}$ such that

$$\hat{\mu}_t(\xi) = (2\pi)^{-n/2}e^{-t\psi(\xi)}, \quad t \geq 0 \quad \text{and} \quad \xi \in \mathbb{R}^n \tag{1.5}$$

holds and the converse is also true, i.e. for every continuous negative definite function ψ there exists uniquely determined convolution semigroup, such that (1.5) holds. (See for the proof [J1]).

Taking a convolution semigroup, $(\eta_t)_{t\geq 0}$ supported by $[0,\infty)$ and a strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ we may define the **subordinated semigroup**

$$T_t^\eta f(x) = \int_0^\infty T_s f(x)\eta_t(ds) \quad \text{as a Bochner integral.} \tag{1.6}$$

By Theorem 4.3.1. [J1] the semigroup $(T_t^\eta)_{t\geq 0}$ is again a strongly continuous contraction semigroup. There is a one-to-one correspondence between $(\eta_t)_{t\geq 0}$ and some special class of functions, called Bernstein functions.

Definition 1.1.10. A real-valued function $f \in C^\infty((0,\infty))$ is called a **Bernstein function** if

$$f \geq 0 \quad \text{and} \quad (-1)^k \frac{d^k f(x)}{dx^k} \leq 0$$

holds for all $k \in \mathbb{N}$.

Similar to Theorem 1.1.9 we have (see [J1])

Theorem 1.1.11. Let $f : (0,\infty) \to \mathbb{R}$ be a Bernstein function. Then there exists a unique convolution semigroup $(\eta_t)_{t\geq 0}$ supported by $[0,\infty)$ such that

$$L(\eta_t)(x) = e^{-tf(x)}, \quad x > 0, \quad t > 0$$

holds, and the converse in also true.

Here $L(f)(x)$ and $L(\eta)(x)$ denote the Laplace transform (see [Pod]) of the function f and of the measure η respectively:

$$L(f)(z) = \int_0^\infty e^{-zt} f(t)\, dt, \quad L(\eta)(z) = \int_0^\infty e^{-zt}\eta(dt), \tag{1.7}$$

where the measure η is such that $\operatorname{supp}\eta \subset [0,\infty)$ and $\int_0^\infty e^{-xs}\eta(ds) < \infty$ for all $x > 0$.

Therefore if ψ is a continuous negative definite function with convolution semigroup $(\mu_t)_{t\geq 0}$, and f is a Bernstein function with convolution semigroup $(\nu_t)_{t\geq 0}$, then the function $f(\psi)$ is again continuous negative definite, and the convolution measure which corresponds to $f(\psi)$ is

$$\mu_t^f(dx) = \int_0^\infty \mu_s(dx)\nu_t(ds). \tag{1.8}$$

We will give some examples of Bernstein functions f which we will use later, and the corresponding families of measures $(\nu_t)_{t\geq 0}$.

I. $f(x) = x^\alpha$, $x \geq 0$, $0 < \alpha < 1$, which corresponds to the convolution semigroup $(\eta_t^{(\alpha)})_{t\geq 0}$. The family of measures $\eta_t^{(\alpha)}(dx) = \sigma_\alpha(x,t)dx$, is called **one-sided stable semigroup of order** α, and the functions $\sigma_\alpha(x,t)$, $t > 0$, are called the **Lévy stable density functions**. For the properties of these functions see [Y], section IX.11, or [Pod]. We only note some properties of $\sigma_\alpha(x,t)$ which are necessary for us.

1. The Laplace transform of $\sigma_\alpha(x,t)$, $t > 0$ with respect to x is

$$\int_0^\infty e^{-xz}\sigma_\alpha(x,t)\,dx = e^{-tz^\alpha}, \quad Re\, z > 0; \tag{1.9}$$

2. The Laplace transform of $\sigma_\alpha(x,t)$, $x > 0$ with respect to t is

$$\int_0^\infty e^{-t\mu}\sigma_\alpha(x,t)\,dt = \frac{e'_\alpha(x,\mu)}{-\mu}, \quad \mu > 0, \tag{1.10}$$

where

$$\frac{e'_\alpha(x,\mu)}{-\mu} = -\mu x^{\alpha-1}E_{\alpha,\alpha}(-\mu y^\alpha) = \sum_{k=1}^\infty \frac{\alpha k(-\mu)^k y^{\alpha k-1}}{\Gamma(\alpha k+1)} \tag{1.11}$$

is the derivative of the **Mittag-Leffler type function** $e_\alpha(x,\mu)$, $\mu > 0$:

$$e_\alpha(x,\mu) := E_{\alpha,1}(-\mu y^\alpha) = \sum_{k=0}^\infty \frac{(-\mu)^k y^{\alpha k}}{\Gamma(\alpha k+1)}, \quad x > 0. \tag{1.12}$$

Here

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad \alpha,\beta > 0 \tag{1.13}$$

is the **Mittag-Leffler function**. For the properties of such functions see [BE], vol.3, §18.1.

3. The Laplace transform of $\frac{e'_\alpha(x,\mu)}{-\mu}$ in x is

$$L_{x\to z}\Big[\frac{e'_\alpha(x,\mu)}{-\mu}\Big] = \frac{1}{\mu+z^\alpha}, \quad Re\, z > 0; \tag{1.14}$$

4. The Laplace transform of $e_\alpha(x,\mu)$ in x is

$$L_{x\to z}[e_\alpha(x,\mu)] = \frac{z^\alpha}{\mu+z^\alpha}, \quad Re z > 0. \tag{1.15}$$

We also note, that only for $\alpha = \frac{1}{2}$ we know an explicit expression for $\sigma_\alpha(x,t)$:

$$\sigma_{1/2}(x,t) = \frac{1}{2\sqrt{\pi}}x^{-3/2}te^{-\frac{t^2}{4x}}, \quad t,x > 0. \tag{1.16}$$

For the following two examples see [J1], Chapter 3.9.

II. The Bernstein function $f(x) = \ln(1+x)$ corresponds to the convolution semigroup of measures

$$\nu_t(dx) = \lambda^{(1)}(dx)\chi_{(0,\infty)}(x)\frac{1}{\Gamma(t)}x^{t-1}e^{-x}, \tag{1.17}$$

where $\lambda^{(1)}(dx)$ is one-dimensional Lebesgue measure.

III. The Bernstein function $f(x) = 1 - e^{-xs}$ corresponds to discrete measures, namely the Poisson semigroup with jumps of size s, i.e.

$$\nu_t(x) = \sum_{k=0}^{\infty} e^{-t} \frac{t^k}{k!} \delta_{ks}. \tag{1.18}$$

For later purposes we will need a subclass of Bernstein functions.

Definition 1.1.12. A function $f : (0, +\infty) \to \mathbb{R}$ is called a **complete Bernstein function** is there is a Bernstein function g such that

$$f(x) = x^2 L(g)(x)$$

holds for all $x > 0$, where $L(g)$ is the Laplace transform of function g.

For the following statement see [J1], Theorem 3.9.29.

Lemma 1.1.13. A function f is a complete Bernstein function if and only if it is a Bernstein function having the representation

$$f(x) = a + bx + \int_{0+}^{\infty} (1 - e^{-sx})\, \mu(ds), \quad x > 0,$$

where $a, b \geq 0$, the measure μ is given by $\mu(ds) = m(s)\, ds$, and the density $m(s)$ satisfies

$$m(s) = \int_{0+}^{\infty} e^{-ts} \tau(dt), \quad s > 0$$

and $\tau(dt)$ is the measure on $(0, +\infty)$ which satisfies

$$\int_{0+}^{1} \frac{\tau(dt)}{t} + \int_{1}^{\infty} \frac{\tau(dt)}{t^2} < \infty.$$

While solving the equations of the form

$$(\lambda I - A)u = v, \quad v \in X, \quad u \in D(A),$$

we need to know what is the **resolvent set** of the operator $(A, D(A))$, defined on the Banach space X (see for example [B]).

Definition 1.1.14. The resolvent set $\rho(A)$ of A consists of all $\lambda \in \mathbb{C}$ such that $\lambda I - A$ is surjective and one has a continuous inverse $(\lambda I - A)^{-1}$ defined on $R(\lambda I - A) = X$.

Knowing the strongly continuous contraction semigroup generated by the operator $(A, D(A))$, we can construct the **resolvent** $R_\lambda := (\lambda - A)^{-1}$, $\lambda \in \rho(A)$ (see [Y], IX.4, or Lemma 4.1.18, [J1].

Lemma 1.1.15. Let $(T_t)_{t\geq 0}$ be a strongly continuous contraction semigroup on the Banach space $(X, \|\cdot\|_X)$ with generator $(A, D(A))$. Then $\{\lambda \in \mathbb{C}| \; Re\,\lambda > 0\} \subset \rho(A)$ and we have

$$R_\lambda g = (\lambda - A)^{-1} g = \int_0^\infty e^{-\lambda t} T_t g \, dt \tag{1.19}$$

for all $g \in X$ and $Re\,\lambda > 0$.

In Chapter 2 we will find with help of this lemma the solution to the equation

$$(\lambda + (-A)^\alpha) f = g, \quad g \in L_p(\mathbb{R}^{n+1}_{0+})$$

for different extensions of the operator $-(-A)^\alpha$, $0 < \alpha \leq 1$.

In the end of this paragraph we want to give the definition and some properties of the function which will occur later in Chapter 2 while constructing some examples of strongly continuous semigroups. For the definitions below see [BE], vol.2, §7.2.

First we suppose that ν is not an integer.

The function

$$J_\nu(z) = \sum_{m=0}^\infty \frac{(-1)^m \left(\frac{z}{2}\right)^{2m+\nu}}{m!\Gamma(m+\nu+1)} \tag{1.20}$$

is called the **Bessel function of order ν of first kind**, and the function

$$I_\nu(z) = e^{-\frac{\nu\pi}{2}} J_\nu(z e^{\frac{i\pi}{2}}) \tag{1.21}$$

is called the **modified Bessel function of first kind**. Further, the function

$$K_\nu(z) = \frac{\pi}{2\sin(\nu\pi)}[I_{-\nu}(z) - I_\nu(z)] \tag{1.22}$$

is called the **modified Bessel function of third kind**. For real ν and positive z the functions $I_\nu(z)$ are real, so for such ν and z $K_\nu(z)$ is real too.

Note that

$$K_{(-\nu)}(z) = K_\nu(z).$$

For ν being an integer, $\nu = n$, the function K_n is defined as

$$K_n(z) = \lim_{\nu\to n} K_\nu(z) = \frac{(-1)^n}{2}\Big[\frac{\partial I_{-\nu}}{\partial \nu} - \frac{\partial I_\nu}{\partial \nu}\Big]_{\nu=n}.$$

For many other representations of Bessel functions we refer to [BE], vol.2.

1.2 Semigroups on product spaces

Consider the measure spaces (X, μ_1) and (Y, μ_2) such that $L_p(X, d\mu_1)$ and $L_p(Y, d\mu_2)$, $1 < p < \infty$, are separable.

In this section we will study an operator of the type $C = A + B$, where the domains of A and B, $D(A)$ and $D(B)$, are in $L_p(X, d\mu_1)$ and in $L_p(Y, d\mu_2)$ respectively. The theorem we will prove states that the operator $(C, D(C))$, where $D(C)$ is the closure of $D(A) \otimes D(B)$ with respect to the graph norm of C, i.e. $D(C) = \overline{D(A) \otimes D(B)}$, is a generator of strongly continuous contraction semigroup, or even sub-Markovian semigroup, if $(A, D(A))$ and $(B, D(B))$ are. To prove this theorem we need some notions of topological tensor products, see for the definitions the book of Trèves [Tre].

Definition 1.2.1. A. Let $E(X)$ and $E(Y)$ be two spaces of complex-valued functions defined on X and Y respectively. We shall denote by $f \otimes g$ the function on $X \times Y$:

$$(f \otimes g)(x, y) = f(x)g(y).$$

B. We denote by $E(X) \otimes E(Y)$ the linear subspace of the space of all complex functions defined on $X \times Y$ spanned by elements of type $f \otimes g$, i.e. a function $h : X \times Y \to \mathbb{C}$ belongs to $E(X) \otimes E(Y)$ if and only if there exist finitely many functions $f_1, f_2, \ldots, f_n \in E(X)$, $g_1, g_2, \ldots, g_n \in E(Y)$, $n \in \mathbb{N}$, such that

$$h = \sum_{i=1}^{n} f_i \otimes g_i.$$

C. For the linear mappings $A : E(X) \to E(X)$ and $B : E(Y) \to E(Y)$ respectively the tensor product $A \otimes B$ is a linear operator on $E(X) \otimes E(Y)$, defined as

$$(A \otimes B)(f \otimes g) = (A\,f) \otimes (B\,g)$$

(see [Tre], Proposition 29.2).

For our purposes we will consider $E(X) = L_p(X, d\mu_1)$ and $E(Y) = L_p(Y, d\mu_2)$, $1 < p < \infty$. In this case for $f \in L_p(X, d\mu_1)$, $g \in L_p(Y, \mu_2)$

$$\|f \otimes g\|_{p, X \times Y} = \|f\|_{p,X} \|g\|_{p,Y}.$$

Lemma 1.2.2. Let E_1 and E_2 be the dense subsets in $L_p(X, d\mu_1)$ and in $L_p(Y, d\mu_2)$. Then, the set $E_1 \otimes E_2$ is dense in $L_p(X \times Y, d\mu_1 \otimes d\mu_2)$.

Proof. From Corollary 39.3 [Tre] we can deduce that $L_p(X, d\mu_1) \otimes L_p(Y, d\mu_2)$ is dense in $L_p(X \times Y, d\mu_1 \otimes d\mu_2)$. Therefore, for $z \in L_p(X \times Y, d\mu_1 \otimes d\mu_2)$ and $\varepsilon > 0$ there exists $h \in L_p(X, d\mu_1) \otimes L_p(Y, d\mu_2)$ such that

$$\|z - h\|_{p, X \times Y} < \frac{\varepsilon}{2}.$$

Since $h \in L_p(X, d\mu_1) \otimes L_p(Y, d\mu_2)$, it has the representation

$$h = \sum_{i=1}^{n} f_i \otimes g_i, \quad f_i \in L_p(X, d\mu_1), \quad g_i \in L_p(Y, d\mu_2), \quad i = 1, \ldots, n$$

for some n. Therefore in order to prove the lemma we may approximate each term $f_i \otimes g_i$, $i = 1, \ldots, n$ by a function from $E_1 \otimes E_2$. Since E_1 is dense in $L_p(X, d\mu_1)$ and E_2 is dense in $L_p(Y, d\mu_2)$, for $f_i \in L_p(X, d\mu_1)$, $g_i \in L_p(Y, d\mu_2)$ and an arbitrary $\varepsilon > 0$ there exists $\varphi_i \in E_1$, $\psi_i \in E_2$, $i = 1, \ldots, n$ such that

$$\|f_i - \varphi_i\|_{p,X} < \frac{\varepsilon}{4Kn}, \quad \|g_i - \psi_i\|_{p,Y} \leq \frac{\varepsilon}{4Kn}$$

where K is a constant which we will define later.

Consider $f_i \otimes g_i$, $i = 1, \ldots, n$.

$$\|f_i \otimes g_i - \varphi_i \otimes \psi_i\|_{p,X\times Y} = \|f_i g_i - \varphi_i g_i + \psi_i g_i - \varphi_i \psi_i\|_{p,X\times Y} \leq \|g_i\|_{p,Y}\|f_i - \varphi_i\|_{p,X} + \\ + \|\varphi_i\|_{p,X}\|g_i - \psi_i\|_{p,Y} \leq \max_{i=1,\ldots,n}(\|g_i\|_{p,Y}, \|\varphi_i\|_{p,X})\frac{\varepsilon}{2Kn}.$$

Setting $K = \max_{i=1,\ldots,n}(\|g_i\|_{p,Y}, \|\varphi_i\|_{p,X})$ we obtain $\|f_i g_i - \varphi_i \psi_i\|_{p,X\times Y} < \frac{\varepsilon}{2n}$ for each $i = 1, \ldots, n$. Let $v = \sum_{i=1}^{n} \varphi_i \otimes \psi_i$. Then

$$\|z - v\|_{p;X\times Y} \leq \|z - h\|_{p;X\times Y} + \|h - v\|_{p;X\times Y} \leq \\ \leq \Big\| \sum_{i=1}^{n} f_i \otimes g_i - \sum_{i=1}^{n} \varphi_i \otimes \psi_i \Big\|_{p,X\times Y} = \Big\| \sum_{i=1}^{n} (f_i \otimes g_i - \varphi_i \otimes \psi_i) \Big\|_{p,X\times Y} \\ \leq \sum_{i=1}^{n} \|f_i \otimes g_i - \varphi_i \otimes \psi_i\|_{p,X\times Y} < \varepsilon$$

proving that $E_1 \otimes E_2$ is dense in $L_p(X \times Y, d\mu_1 \otimes d\mu_2)$. □

Using Lemma 1.2.2 we can construct sub-Markovian semigroups on product spaces.

Theorem 1.2.3. Consider the operators $(A, D(A))$ and $(B, D(B))$, $D(A) \subset L_p(X, d\mu_1)$, $D(B) \subset L_p(Y, d\mu_2)$, such that they can be extended to generators of strongly continuous contraction semigroups $(T_1(t))_{t\geq 0}$ and $(T_2(t))_{t\geq 0}$ on $L_p(X, d\mu_1)$ and $L_p(Y, d\mu_2)$ respectively. Then, the closure $(C, \overline{D(A) \otimes D(B)}^{\|\cdot\|_C})$ of the operator $C_0 = A \oplus B = A \otimes I_X + I_Y \otimes B$ with domain $D(C_0) = D(A) \otimes D(B)$, generates a strongly continuous contraction semigroup $(T(t))_{t\geq 0}$ on $L_p(X \times Y, d\mu_1 \otimes d\mu_2)$

Here $\|f\|_C = \|Cf\|_{p,X\times Y} + \|f\|_{p,X\times Y}$ is the graph norm of operator C.

Proof. First we show that $(T_1(t) \otimes T_2(t))_{t\geq 0}$ is a semigroup of contractions on $L_p(X \times Y, d\mu_1 \otimes d\mu_2)$.

That $(T_1(t) \otimes T_2(t))_{t\geq 0}$ is a semigroup is clear if we rewrite $T_1(t) \otimes T_2(t)$ as

$$T_1(t) \otimes T_2(t) = (T_1(t) \otimes I_Y) \circ (I_X \otimes T_2(t)) = (T_1(t) \otimes I) \circ (I \otimes T_2(t)).$$

Consider $h \in Lin(f_i, g_i) := \{h : h = \sum_{i=1}^n f_i \otimes g_i,\ f_i \in L_p(X, d\mu_1);\ g_i \in L_p(Y, d\mu_2)\}$. Since for $y \in Y$ we know that $\sum_{i=1}^n f_i(\cdot)T_2(t)g_i(y) \in L_p(X, d\mu_1)$ a.e., and $T_1(t)$ is a contraction then

$$\begin{aligned}
\|(T_1(t) \otimes T_2(t))h\|_{p,X\times Y}^p &= \|(T_1(t) \otimes T_2(t)) \sum_{i=1}^n f_i \otimes g_i\|_{p,X\times Y}^p = \\
&= \big\| \|T_1(t)(\sum_{i=1}^n f_i \otimes T_2(t) g_i)\|_{p,X}^p \big\|_{p,Y}^p \\
&\leq \big\| \| \sum_{i=1}^n f_i \otimes T_2(t) g_i\|_{p,X}^p \big\|_{p,Y}^p
\end{aligned}$$

and further, since for each $x \in X$ it holds that $\sum_{i=1}^n f_i(x) g_i(\cdot) \in L_p(X, d\mu_2)$ a.e., and T_2 is a contraction, then, by Fubini's theorem

$$\begin{aligned}
\big\| \|T_2(t) \sum_{i=1}^n f_i \otimes g_i\|_{p,X}^p \big\|_{p,Y}^p &= \big\| \|T_2(t) \sum_{i=1}^n f_i \otimes g_i\|_{p,Y}^p \big\|_{p,X}^p \\
&\leq \big\| \| \sum_{i=1}^n f_i \otimes g_i\|_{p,Y}^p \big\|_{p,X}^p = \|h\|_{p,X\times Y}^p
\end{aligned}$$

Since $Lin(f_i, g_i)$ is dense in $L_p(X \times Y, d\mu_1 \otimes d\mu_2)$, the operators $(T_1(t) \otimes T_2(t))_{t\geq 0}$ can be extended by continuity to a contracting operator $T(t)$, $t \geq 0$, on $L_p(X \times Y, d\mu_1 \otimes d\mu_2)$.

Now we will prove the strong continuity of $(T_1 \otimes T_2)_{t\geq 0}$. For $h \in Lin(f_i, g_i)$ and $t > 0$ we have

$$\begin{aligned}
\|(T_1(t) \otimes T_2(t))h - h\|_p^p &\leq \sum_{i=1}^n \|T_1(t) f_i \otimes T_2(t) g_i - f_i g_i\|_p^p \\
&\leq \sum_{i=1}^n \|T_1(t) f_i \otimes T_2(t) g_i - f_i T_2(t) g_i + f_i T_2(t) g_i - f_i g_i\|_p^p \\
&\leq \sum_{i=1}^n (\|T_1(t) f_i - f_i\|_{p,X}^p \|T_2(t) g_i\|_{p,Y}^p + \|f_i\|_{p,X}^p \|T_2(t) g_i - g_i\|_{p,Y}^p).
\end{aligned}$$

Since $(T_1(t))_{t\geq 0}$ and $(T_2(t))_{t\geq 0}$ are strongly continuous contraction semigroups on $L_p(X, d\mu_1)$ and on $L_p(Y, d\mu_2)$ respectively, we can choose such $\delta > 0$, the same for all $i = 1, \ldots, n$ such that for $0 < t < \delta$

$$\|T_1(t) f_i - f_i\|_{p,X}^p \|g_i\|_{p,Y}^p < \frac{\varepsilon}{6n}$$

$$\|T_2(t) g_i - g_i\|_{p,Y}^p \|f_i\|_{p,X}^p < \frac{\varepsilon}{6n}$$

and thus

$$\|(T_1(t)\otimes T_2(t))h-h\|_p^p<\frac{\varepsilon}{3}\quad\text{for}\quad 0<t<\delta,$$

which gives is that $(T_1(t)\otimes T_2(t))_{t\geq 0}$ is strongly continuous on $Lin(f_i,g_i)$. Because of the density of this set in $L_p(X\times Y,d\mu_1\otimes d\mu_2)$, for $f\in L_p(X\times Y,d\mu_1\otimes d\mu_2)$ there exists h (let us take the same h as before) such that

$$\|f-h\|_{p,X\times Y}^p<\frac{\varepsilon}{3}.$$

Then for $0<t<\delta$, since $T(t)=T_1(t)\otimes T_2(t)$ on $Lin(f_i,g_i)$

$$\begin{aligned}\|T(t)f-f\|_p^p&=\|(T(t)f-T(t)h)+(T(t)-I)h-f+h\|_p^p\\&\leq\|T(t)(f-h)\|_p^p+\|(T(t)-I)h\|_p^p+\|h-f\|_p^p<\varepsilon.\end{aligned}$$

Thus, $(T(t))_{t\geq 0}$ is strongly continuous on $L_p(X\times Y,d\mu_1\otimes d\mu_2)$. By Proposition 1.1.3 there exists a closed operator $(\tilde{C},D(\tilde{C}))$ which generates this semigroup. We will show that $D_1=D(A)\otimes D(B)\subset D(\tilde{C})$ and $\tilde{C}\big|_{D_1}=C$. Take $h=\sum_{i=1}^n f_i\otimes g_i$, $f_i\in D(A)$, $g_i\in D(B)$, $i=1,\ldots,n$. For $t>0$:

$$\begin{aligned}\|\frac{T(t)h-h}{t}-(A+B)h\|_{p,X\times Y}^p\leq\sum_{i=1}^n(\|(\frac{T_1(t)f_i-f_i}{t}-Af_i)g_i\|_{p,X\times Y}^p+\\+\|T_1(t)f_i\otimes(\frac{T_2(t)g_i-g_i}{t}-Bg_i)\|_{p,X\times Y}^p+\|T_1(t)f_i-f_i\|_{p,X}^p\|Bg_i\|_{p,Y}^p)\end{aligned}$$

and choosing for $\varepsilon>0$ a $\delta>0$ such that for $0<t<\delta$, $i=1,\ldots,n$,

$$\|(\frac{T_1(t)f_i-f_i}{t}-Af_i)\|_{p,X}^p\|g_i\|_{p,Y}^p<\frac{\varepsilon}{3n},\quad\|(\frac{T_2(t)g_i-g_i}{t}-Bg_i)\|_{p,Y}^p\|f_i\|_{p,X}^p<\frac{\varepsilon}{3n}$$

and

$$\|T_1(t)f_i-f_i\|_{p,X}^p\|Bg_i\|_{p,Y}^p<\frac{\varepsilon}{3n}$$

we get

$$\|\frac{T(t)h-h}{t}-Ch\|_{p,X\times Y}^p<\varepsilon\quad\text{for}\quad 0<t<\delta,$$

and therefore $D_1\subset D(\tilde{C})$. Since the closure of $(A,D(A))$ and $(B,D(B))$ generates a strongly continuous contraction semigroup on $L_p(X,d\mu_1)$ and on $L_p(X,d\mu_2)$, then, by Proposition 1.1.3 $T_1(t):D(A)\to D(A)$, $T_2(t):D(B)\to D(B)$ and thus $T(t):D_1\to D_1$. By Lemma 1.2.2 D_1 is dense in $L_p(X\times Y,d\mu_1\otimes d\mu_2)$ and therefore by Proposition 4.3.6 [J1] D_1 is a core for $\tilde{C}$, i.e. $D_1\subset D(\tilde{C})$ and $\tilde{C}\big|_{D_1}=C$. Then, by Theorem 4.1.40 [J1] C and $\tilde{C}$ generate the same semigroup, $(T(t))_{t\geq 0}$, and $C=\tilde{C}$. □

Remark 1.2.4. Under the conditions of Theorem 1.2.3 the closure of the operator $(C,D(C))$ is a generator of an L_p-sub-Markovian semigroup if $(A,D(A))$ and $(B,D(B))$ are.

Proof. Consider $h \in L_p(X \times Y, d\mu_1 \otimes d\mu_2)$, and a sequence $(h_n)_{n\geq 0} \subset L_p(X, d\mu_1) \otimes L_p(Y, d\mu_2)$, such that $h_n \to h$ as $n \to \infty$, $h_n = \sum_{i=1}^{m(n)} f_{i,n} \otimes g_{i,n}$. Since

$$(T_1(t) \otimes T_2(t))h_n(x,y) = \sum_{i=1}^{m(n)} T_1(t)f_{i,n}(x) \otimes T_2(t)g_{i,n}(y)$$

and for all i, n

$$0 \leq T_1(t)f_{i,n} \leq 1, \quad 0 \leq T_2(t)g_{i,n} \leq 1 \quad \text{a.e.} \quad \text{if} \quad 0 \leq f_{i,n}, g_{i,n} \leq 1 \quad \text{a.e.},$$

then

$$0 \leq (T_1(t) \otimes T_2(t))h_n \leq 1 \quad \text{a.e.} \tag{1.23}$$

By the Riesz theorem (see for example [B], Theorem 4.3) there exists a subsequence $(h_{n_k})_{k\geq 0}$ of $(h_n)_{n\geq 0}$ such that

$$(T_1(t) \otimes T_2(t))h_{n_k} \to (T_1(t) \otimes T_2(t))h \quad \text{a.e.},$$

and therefore passing to the limit as $k \to \infty$ in

$$0 \leq (T_1(t) \otimes T_2(t))h_{n_k} \leq 1,$$

we have, extending the operators $(T_1(t) \otimes T_2(t))_{t\geq 0}$ to $(T(t))_{t\geq 0}$

$$0 \leq T(t)h \leq 1 \quad \text{a.e.};$$

i.e. the semigroup $(T(t))_{t\geq 0}$ is sub-Markovian. □

Remark 1.2.5. The statements similar to Lemma 1.2.2 and Theorem 1.2.3 were proved in the work of A.Krägeloh, [Kr], for the Feller and strong Feller semigroups.

1.3 Some notions about pseudo-differential operators

An operator $p(x, D)$ defined on $C_0^\infty(\mathbb{R}^n)$ is said to be a pseudo-differential operator with symbol $p(x, \xi)$ if it allows the representation

$$p(x,D)f(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\xi} p(x,\xi)\hat{f}(\xi)\, d\xi, \quad x \in \mathbb{R}^n. \tag{1.24}$$

In general it is not possible to find the domain of such operators in L_p in terms of function spaces. Later for our purposes we will consider such operators as perturbations of operators $(A, D(A))$, which are the generators of L_p-sub-Markovian semigroups. To do this we need the operator $p(x, D)$ to be A-bounded.

Definition 1.3.1. Let $(A, D(A))$ and $(Q, D(Q))$ be two linear operators on the Banach space $(X, \|\cdot\|_X)$, such that $D(A) \subset D(Q)$ and for some $\varepsilon \in [0,1)$ and $\beta = \beta(\varepsilon) \geq 0$

$$\|Qf\|_X \leq \varepsilon\|Af\|_X + \beta\|f\|_X \tag{1.25}$$

for all $f \in D(A)$. Then the operator Q is called A**-bounded** and ε is called an A bound for Q.

Remark 1.3.2. Let $(-A, D(A))$ be a generator of a strongly continuous contraction semigroup on a Banach space $(X, \|\cdot\|_X)$. For $0 < \alpha < 1$ the operator $(-A)^\alpha$ is A-bounded and (1.25) holds for all $0 < \varepsilon < 1$.

(For the proof see [J1], Proposition 4.3.25).

Adding an A-bounded operator to a generator $(-A, D(A))$ of an L_p-sub-Markovian semigroup we again obtain a generator of an L_p-sub-Markovian semigroup, see [J2], Theorem 2.8.1:

Theorem 1.3.3. Let $(-A, D(A))$ be a pseudo-differential operator which generates a sub-Markovian semigroup in L_p, $1 < p < \infty$. If an operator $-p(x, D)$ is L_p-dissipative, A-bounded, and if in addition $(-A - p(x, D), D(A))$ is an L_p-Dirichlet operator, then $(-A - p(x, D), D(A))$ is a generator of an L_p-sub-Markovian semigroup.

In Chapter 2 we will give examples of such A-bounded operators, where the symbol of it is a continuous negative definite function satisfying some additional conditions, and so $(-A - p(x, D), D(A))$ generates an L_p-sub-Markovian semigroup if $(-A, D(A))$ does.

Chapter 2

Bessel-type potential spaces

2.1 Bessel-type potential spaces on $\mathbb{R}^n$

We will start with ψ-Bessel potential spaces on $\mathbb{R}^n$. For the following definitions see [J2] and [FJS1].

Definition 2.1.1. Let $(T_t^{(p)})_{t\geq 0}$, $1 \leq p < \infty$, be an L_p-sub-Markovian semigroup on $L_p(\mathbb{R}^n;\mathbb{R})$. We define the **gamma transform** $(V_r^{(p)})_{r\geq 0}$ by

$$V_r^{(p)} u := \frac{1}{\Gamma(\frac{r}{2})} \int_0^\infty t^{r/2-1} e^{-t} T_t^{(p)} u \, dt, \quad u \in L_p(\mathbb{R}^n;\mathbb{R}).$$

The following theorem shows a connection between the gamma transform of the strongly continuous contraction semigroup and its generator.

Theorem 2.1.2. Let $(T_t^{(p)})_{t\geq 0}$ be an L_p-sub-Markovian semigroup on $L_p(\mathbb{R}^n;\mathbb{R})$ with generator $(A^{(p)}, D(A^{(p)}))$. For all $r > 0$ and $u \in L_p(\mathbb{R}^n;\mathbb{R})$ we have

$$V_r^{(p)} u = (I - A^{(p)})^{-r/2} u.$$

In particular, each $V_r^{(p)}$ is injective.

(See for the proof Theorem 3.1.9 [J2]).

Since the operators $V_r^{(p)}$, $r > 0$, are injective, we can give

Definition 2.1.3. The **Bessel-type potential spaces** associated with $(T_t^{(p)})_{t\geq 0}$ are defined by

$$\mathcal{F}_{r,p}(\mathbb{R}^n;\mathbb{R}) := V_r^{(p)}(L_p(\mathbb{R}^n;\mathbb{R})) \tag{2.1}$$

with the norm

$$\|u\|_{F_{r,p}} := \|v\|_{L_p} \quad \text{for} \quad u = V_r^{(p)} v. \tag{2.2}$$

We are interested in the case when the L_p-sub-Markovian semigroup $(T_t^{(p)})_{t\geq 0}$ is associated with a continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{R}$. Denote by $(A^{(p)}, D(A^{(p)}))$ the generator of such a semigroup, and we know (see [J1]) that $S(\mathbb{R}^n) \subset D(A^{(p)})$. On $S(\mathbb{R}^n)$ the generator has the representation (see Example 4.1.13, [J1])

$$A^{(p)}u(x) = -\psi(D)u(x) = -(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\psi(\xi)\hat{u}(\xi)\, d\xi. \tag{2.3}$$

Further, since the representations of $A^{(2)}$ and $A^{(p)}$ coincide on $S(\mathbb{R}^n)$, for $u \in S(\mathbb{R}^n)$ we find

$$V_r^{(p)}u = (I - A^{(p)})^{-r/2}u = (I - A^{(2)})^{-r/2}u = V_r^{(2)}u,$$

and

$$V_r^{(2)}u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi}(1 + \psi(\xi))^{-r/2}\hat{u}(\xi)\, d\xi,$$

which implies on $S(\mathbb{R}^n)$ that

$$\|u\|_{\mathcal{F}_{r,p}} = \|(1 + \psi(D))^{r/2}u\|_{L_p} = \|F^{-1}((1 + \psi(\cdot))^{r/2}\hat{u})\|_{L_p}. \tag{2.4}$$

Consider the continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{R}$ with the representation

$$\psi(\xi) = \int_{\mathbb{R}^n\setminus\{0\}} (1 - cos(y\cdot\xi))\, \nu(dy) \tag{2.5}$$

where the Lévy measure $\nu(dy)$ is such that

$$\int_{\mathbb{R}^n\setminus\{0\}} (|y|^2 \wedge 1)\, \nu(dy) < \infty.$$

We call a continuous negative definite function with such a representation a **continuous negative definite function of type 1**.

We will also consider continuous negative definite functions

$$\chi(\xi) = \psi(\xi') \pm i\xi_{n+1} \tag{2.6}$$

where $\psi(\xi')$ is of type 1, $\xi' \in \mathbb{R}^n$, $\xi_{n+1} \in \mathbb{R}$. A continuous negative definite function with the representation (2.6) we will call a **continuous negative definite function of type 2.**

Definition 2.1.4. Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a continuous negative definite function having the representation (2.5). For $s \geq 0$ define the **ψ-Bessel potential space of order s** as

$$H_p^{\psi,s} = H_p^{\psi,s}(\mathbb{R}^n) := \mathcal{F}_{s,p}(\mathbb{R}^n) = (I - A^{(p)})^{-s/2}(L_p(\mathbb{R}^n))$$

(or $D((1 - A)^{s/2}) = H_p^{\psi,s}$, A is an operator with the symbol $\psi(\xi)$, see [J2], p.279-281).

We will understand $\psi(D)u$ in the following sense: if $\psi(\xi)$, $\xi \in \mathbb{R}^n$, is a real continuous negative definite function with the representation (2.5), then the function

$$\psi_R(\xi) = \int_{\mathbb{R}^n \setminus \{0\}} (1 - \cos(y\xi)) 1_{B_R(0)}(y)\, \nu(dy),$$

is from $C^\infty(\mathbb{R}^n)$. Therefore $F^{-1}(\psi_R(\xi)\hat{f}(\xi))$ is well defined for $f \in S(\mathbb{R})$, and

$$\psi(D)f = F^{-1}(\psi_R(\xi)\hat{f}(\xi)) + \int_{B_R^c(0)} (u(x) - u(x-y))\, \nu(dy).$$

Thus $F^{-1}((1+\psi(\xi))^s \hat{f}(\xi))$ and $F^{-1}(\psi^s(\xi)\hat{f}(\xi))$ are understood as $(1+\psi(D))^s f = F^{-1}((1+\psi(\xi))^s \hat{f}(\xi))$ and $(\psi(D))^s f = F^{-1}(\psi^s(\xi)\hat{f}(\xi))$, and it was proved that

$$\|u\|_{H_p^{\psi,s}} = \|F^{-1}((1+\psi(\cdot))^{s/2}\hat{u}(\cdot)\|_{L_p}, \tag{2.7}$$

see [J2] or [FJS2].

Definition 2.1.5. Let ψ be of type 1, $1 < p < \infty$, and $s < 0$. The space $H_p^{\psi,s}(\mathbb{R}^n)$ is defined as the closure of $S(\mathbb{R}^n)$ with respect to the norm

$$\|u\|_{H_p^{\psi,s}} = \|F^{-1}((1+\psi(\cdot))^{s/2}\hat{u}(\cdot)\|_{L_p}, \quad s < 0. \tag{2.8}$$

Lemma 2.1.6. The Schwartz space $S(\mathbb{R}^n)$ is dense in $H_p^{\psi,s}$ for every $1 < p < \infty$, $-\infty < s < \infty$ (see Proposition 3.3.14, [J2]).

For the spaces $H_p^{\psi,s}$, ψ of type 1, the following interpolation theorem was proved (see [J2], p.295, Theorem 3.3.38).

Theorem 2.1.7. Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a continuous negative definite function, $0 < p_0, p_1 < \infty$, $s_0, s_1 \in \mathbb{R}$, $0 < \theta < 1$. For $s = (1-\theta)s_0 + \theta s_1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ it follows that

$$[H_{p_0}^{\psi,s_0}, H_{p_1}^{\psi,s_1}]_\theta = H_p^{\psi,s}$$

holds.

We will also be interested in some other spaces related to continuous negative definite functions.

Definition 2.1.8. Denote by $B_{\psi,p}^s(\mathbb{R}^n)$, $1 \le p \le \infty$, $s \in \mathbb{R}$, the space

$$B_{\psi,p}^s = B_{\psi,p}^s(\mathbb{R}^n) = \{u | u \in S', \quad \|(1+|\psi(\xi)|)^{s/2}\hat{u}(\xi)\|_{L_p} < \infty\} \tag{2.9}$$

Here $\psi : \mathbb{R}^n \to \mathbb{C}$ is a continuous negative definite function.

Similar to Theorem 2.1.7 we can prove

Theorem 2.1.9. Let $\psi : \mathbb{R}^n \to \mathbb{C}$ be a continuous negative definite function, $\theta \in [0,1]$, $s = (1-\theta)s_1 + \theta s_2$, $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$, $1 \le p_1, p_2 \le \infty$. Then

$$[B^{s_1}_{\psi,p_1}, B^{s_2}_{\psi,p_2}]_\theta = B^s_{\psi,p}. \tag{2.10}$$

Before we give the proof of Theorem 2.1.9, we recall some definitions taken from [J2], p.293-294, or [T2], §1.9.1.

Let $G = \{z \in \mathbb{C};\ 0 < Re\, z < 1\}$. For two complex Banach spaces $(X_0, \|\cdot\|_{X_0})$ and $(X_1, \|\cdot\|_{X_1})$ both embedded into some Hausdorff space $\mathcal{X}$, set $X := X_0 + X_1$, equipped with the norm $\|\cdot\|_X := \max(\|\cdot\|_{X_0}, \|\cdot\|_{X_1})$ which is equivalent to the norm $\|\cdot\|_{X_0} + \|\cdot\|_{X_1}$, and which turns X into a Banach space. Denote by $W(G,X)$ the space of all continuous functions $\omega : \overline{G} \to X$ with the following properties:

1) $\omega\big|_G$ is analytic and $\sup_{z\in\overline{G}} \|\omega(z)\|_X < \infty$;

2) $\omega(iy) \in X_0$ and $\omega(1+iy) \in X_1$, for $y \in \mathbb{R}$ with continuous maps $y \mapsto \omega(iy)$ and $y \mapsto \omega(1+iy)$.

3) $\|\omega\|_{W(G,X)} := \max(\sup \|\omega(iy)\|_{X_0}, \sup \|\omega(1+iy)\|_{X_1}) < \infty$.

By the maximum principle $(W(G,X), \|\cdot\|_{W(G,X)})$ is a Banach space. We call $\{X_0, X_1\}$ an interpolation couple, and for any interpolation couple define its **complex interpolation space**

$$[X_0; X_1]_\theta := \{u \in X;\ \text{there exists } \omega \in W(G,X) \text{ such that } \omega(\theta) = u\} \tag{2.11}$$

and on $[X_0, X_1]_\theta$ we introduce the norm

$$\|u\|_{[X_0,X_1]_\theta} := \inf\{\|\omega\|_{W(G,X)},\quad \omega \in W(G,X) \text{ and } \omega(\theta) = u\} \tag{2.12}$$

With this norm $([X_0, X_1]_\theta, \|\cdot\|_{[X_0,X_1]_\theta})$ is a Banach space too.

Lemma 2.1.10. Let $\{X_0, X_1\}$ be an interpolation couple and let $0 < \theta < 1$. Then we have

$$\|u\|_{[X_0,X_1]_\theta} = \inf\{(\sup_{y\in\mathbb{R}} \|\omega(iy)\|_{X_0})^{1-\theta} (\sup_{y\in\mathbb{R}} \|\omega(1+iy)\|_{X_1})^\theta\} \tag{2.13}$$

where the inf ranges over all $\omega \in W(G,X)$ such that $\omega(\theta) = u$. $\square$

Proof of Theorem 2.1.9. Let $X := B^{s_1}_{\psi,p_1} + B^{s_2}_{\psi,p_2} \hookrightarrow S'(\mathbb{R}^n)$, and $G = \{z \in \mathbb{C},\ 0 < Re\, z < 1\}$. Let $u \in B_\theta$, $B_\theta = [B^{s_1}_{\psi,p_1}, B^{s_2}_{\psi,p_2}]_\theta$, and choose any $\omega \in W(G,X)$ with $\omega(\theta) = u$. We define on G

$$g_\omega(z) = e^{(z-\theta)^2}(1 + |\psi(\xi)|)^{\frac{(1-z)s_1 + zs_2}{2}} F_\xi(\omega(z)).$$

It is clear that $g_\omega(z)$ is analytic in G, continuous in $\overline{G}$, and since $\sup_{z\in\overline{G}} \|\omega(z)\|_X < \infty$ then $g_\omega(z)$ is also bounded in $\overline{G}$. For $y \in \mathbb{R}$ we have

$$g_\omega(iy) = e^{(iy-\theta)^2}(1+|\psi(\xi)|)^{\frac{iy(s_2-s_1)}{2}}(1+|\psi(\xi)|)^{s_1/2}F_\xi(\omega(iy))$$

and

$$g_\omega(1+iy) = e^{(iy+1-\theta)^2}(1+|\psi(\xi)|)^{\frac{iy(s_1-s_2)}{2}}(1+|\psi(\xi)|)^{s_2/2}F_\xi(\omega(1+iy)).$$

Since $|(1+|\psi|)^{i\mu}| = |\exp\{i\mu\ln(1+|\psi|)\}| \le 1$, $\mu \in \mathbb{R}$, then

$$\|g_\omega(iy)\|_{p_1} \le M_1\|\omega(iy)\|_{B^{s_1}_{\psi,p_1}}$$

and

$$\|g_\omega(1+iy)\|_{p_2} \le M_2\|\omega(1+iy)\|_{B^{s_2}_{\psi,p_2}}.$$

Let $v(\xi) = (1+|\psi(\xi)|)^{s/2}F_\xi(\omega(\theta))$. Since $L_p = [L_{p_1}, L_{p_2}]_\theta$ for $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$, $1 \le p_1, p_2 \le \infty$, $0 < \theta < 1$, we have, using Lemma 2.1.10 for the spaces L_{p_1} and L_{p_2}, that

$$\begin{aligned}\|u\|_{B^s_{\psi,p}} =& \|(1+|\psi|)^{s/2}F(\omega(\theta))\|_p = \|v\|_p = \\ =& \inf_{\substack{g\in W(G,S'(\mathbb{R}^n))\\ g(\theta)=v}}\{(\sup_{y\in\mathbb{R}}\|g(iy)\|_{p_1})^{1-\theta}(\sup_{y\in\mathbb{R}}\|g(1+iy)\|_{p_2})^\theta\} \\ \le& (\sup_{y\in\mathbb{R}}\|g_\omega(iy)\|_{p_1})^{1-\theta}(\sup_{y\in\mathbb{R}}\|g_\omega(1+iy)\|_{p_2})^\theta \\ \le& M_1^{1-\theta}M_2^\theta(\sup_{y\in\mathbb{R}}\|\omega(iy)\|_{B^{s_1}_{\psi,p_1}})^{1-\theta}(\sup_{y\in\mathbb{R}}\|\omega(1+iy)\|_{B^{s_2}_{\psi,p_2}})^\theta\end{aligned}$$

and applying again Lemma 2.1.10 now to the spaces $B^{s_1}_{\psi,p_1}$ and $B^{s_2}_{\psi,p_2}$, we obtain

$$\|u\|_{B^s_{\psi,p}} \le M_1^{1-\theta}M_2^\theta\|u\|_{B_\theta}.$$

Now we prove the converse imbedding. Let g be an arbitrary function from $W(G, S'(\mathbb{R}^n))$, $g(\theta) = v$, $g(iy) \in L_{p_1}(\mathbb{R}^n)$, $g(1+iy) \in L_{p_2}(\mathbb{R}^n)$. Define

$$\omega_g(z) = e^{(z-\theta)^2}(1+|\psi(\xi)|)^{\frac{(1-z)s_1-zs_2}{2}}F_\xi(g(z)).$$

We have for $y \in \mathbb{R}$

$$\omega_g(iy) = e^{(iy-\theta)^2}(1+|\psi(\xi)|)^{\frac{iy(s_1-s_2)}{2}}(1+|\psi(\xi)|)^{s_1/2}F_\xi(g(iy))$$

and

$$\omega_g(1+iy) = e^{(iy+1-\theta)^2}(1+|\psi(\xi)|)^{\frac{iy(s_1-s_2)}{2}}(1+|\psi(\xi)|)^{s_2/2}F_\xi(g(1+iy)),$$

and then, again applying Lemma 2.1.10 we have

$$\begin{aligned}\|u\|_{B_\theta} &= \inf_{\substack{\omega\in W(G,X)\\ \omega(\theta)=u}} \{(\sup_{y\in\mathbb{R}} \|\omega(iy)\|_{B^{s_1}_{\psi,p_1}})^{1-\theta}(\sup_{y\in\mathbb{R}} \|\omega(1+iy)\|_{B^{s_2}_{\psi,p_2}})^{\theta}\}\\ &\le (\sup_{y\in\mathbb{R}} \|\omega_g(iy)\|_{B^{s_1}_{\psi,p_1}})^{1-\theta}(\sup_{y\in\mathbb{R}} \|\omega_g(1+iy)\|_{B^{s_2}_{\psi,p_2}})^{\theta}\\ &\le M_3^{1-\theta}M_4^{\theta}(\sup_{y\in\mathbb{R}} \|g(iy)\|_{p_1})^{1-\theta}(\sup_{y\in\mathbb{R}} \|g(1+iy)\|_{p_2})^{\theta},\end{aligned}$$

and then

$$\|u\|_{B_\theta} \le M_3^{1-\theta}M_4^{\theta}\|v\|_p = M\|u\|_{B^s_{\psi,p}}.$$

Thus $B^s_{\psi,p} \subset B_\theta$, which completes the proof. □

Definition 2.1.11. Let $\chi = \chi_+(\xi) = \psi(\xi') + i\xi_{n+1}$ or $\chi = \chi_-(\xi) = \psi(\xi') - i\xi_{n+1}$ be a continuous negative definite function of type 2. We define

$$H_p^{\chi,s,1} = H_p^{\chi,s,1}(\mathbb{R}^n \times \mathbb{R}) := \overline{H_p^{\psi,s}(\mathbb{R}^n) \otimes H_p^1(\mathbb{R})}^{\|\cdot\|_A}, \quad s \ge 0,$$

where $H_p^1(\mathbb{R}) = \{f \in L_p(\mathbb{R}),\ f' \in L_p(\mathbb{R})\}$ is a classical Sobolev space of order 1, and $\|\cdot\|_A$ is the graph norm $\|Af\|_p + \|f\|_p$ of the operator A with symbol $symb(-A) = \psi(\xi') \pm i\xi_{n+1}$.

To proceed further we quote some results from [J2] and [FJS1] to show that if ψ is of type 1, then $(-\psi(D), H_p^{\psi,2})$ generates an L_p-sub-Markovian semigroup. We can also state this for a function of type 2, namely that $(-A, H_p^{\chi,2,1})$, $symb(A) = \chi$, generates an L_p-sub-Markovian semigroup.

Consider first the case when ψ is of type 1.

In Theorem 2.1.15 [FJS1] or Theorem 3.3.11.[J2] it was proved that for $1 < p < \infty$ $D(A) = H_p^{\psi,2}$, and $S(\mathbb{R}^n)$ is an operator core for $(-A, H_p^{\psi,2})$, $symb(-A) = \psi(\xi)$.

Since for $\psi : \mathbb{R}^n \to \mathbb{C}$-a continuous negative definite function, the operator A, $symb(-A) = \psi$, satisfies on $C_0^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} (-A\,u)(u - 1^+)^{p-1}\,dx \le 0 \tag{2.14}$$

and extends from $C_0^\infty(\mathbb{R}^n)$ to $D(A)$ with (2.14) (see Example 4.6.29, [J1]), it is a Dirichlet operator on $D(A) = H_p^{\psi,2}$, and therefore it is dissipative on $H_p^{\psi,2}$ (see Propositions 4.6.4-4.6.12, [J1]). In Corollary 3.3.13 [J2] it was proved that for $t, s \ge 0$ and $1 < p < \infty$

$$(I + A)^{s/2} : H_p^{\psi,t+s} \to H_p^{\psi,t}$$

is bijective, continuous with continuous inverse. Then, the equation

$$(I + A)u = f$$

$u \in H_p^{\psi,2}$, $f \in L_p$, has a unique solution.

Summarizing the statements we listed, we deduce

Theorem 2.1.12. Let $1 < p < \infty$, and consider the operator A, $symb(-A) = \psi$, ψ is of type 1. The operator $(-A, H_p^{\psi,2})$ is a generator of a strongly continuous contraction semigroup on $L_p(\mathbb{R}^n)$ which is sub-Markovian.

The next theorem which we need later on we have taken from [J2] (Theorem 3.3.18).

Theorem 2.1.13. For $u \in S(\mathbb{R}^n)$ and $1 < p < \infty$ the estimates

$$\gamma_0(\|F^{-1}(\psi^{s/2}\hat{u})\|_p + \|u\|_p) \leq \|u\|_{H_p^{\psi,s}} \leq \gamma_1(\|F^{-1}(\psi^{s/2}\hat{u})\|_p + \|u\|_p) \qquad (2.15)$$

hold and by Lemma 2.2 (2.15) extends to all $u \in H_p^{\psi,s}$.

Consider now the operators of type 2. For example, let $symb(-A) = \psi(\xi') + i\xi_{n+1}$. Since $\psi(\xi')$ is of type 1, then $(-\psi(D_{x'}), H_p^{\psi,2})$ is a generator of an L_p-sub-Markovian semigroup.

Consider the operator $-\frac{d}{dx}$ on $S(\mathbb{R})$. We can see, that $-\frac{d}{dx}$ is the conjugate operator to $\frac{d}{dx}$: for $u, v \in S(\mathbb{R}^n)$

$$\int_{\mathbb{R}} u\, v'\, dx = -\int_{\mathbb{R}} u'\, v\, dx$$

and then it is closable (see [G], Theorem II.2.6). Taking the closure of $S(\mathbb{R})$ with respect to the graph norm of $-\frac{d}{dx}$ when considered as a closable operator in L_p we deduce that the domain of $-\frac{d}{dx}$ is H_p^1. Since the symbol $i\xi$ of $-\frac{d}{dx}$ is a continuous negative definite function, $-\frac{d}{dx}$ is a Dirichlet operator on $C_0^\infty(\mathbb{R})$, and therefore it is a Dirichlet operator on H_p^1, and we conclude, that it is dissipative on H_p^1. Further, we can see, that

$$f(x) = \int_0^\infty e^{-\lambda t} g(x-t)\, dt$$

is the solution to the equation

$$\lambda f + f' = g, \quad g \in L_p(\mathbb{R})$$

where the equality is understood to hold a.e., and $f \in H_p^1$. Thus, all conditions of Hille-Yosida theorem are satisfied, and therefore we have proved

Lemma 2.1.14. The operator $(-\frac{d}{dx}, H_p^1)$ generates a strongly continuous contraction semigroup which is sub-Markovian.

Note that the main point in this lemma is that we have a precise knowledge of the domain of the generator.

Take now the tensor product of spaces $H_p^{\psi,2}$ and H_p^1, $H = H_p^{\psi,2} \otimes H_p^1$, and consider the operator $-A$ on the closure of H with respect to the graph norm. Applying Theorem 1.2.3 to the operator $(-A, H_p^{\chi,2,1})$ we obtain

Theorem 2.1.15. The operator $(-A, H_p^{\chi,2,1})$ is the generator of an L_p-sub-Markovian semigroup.

The equivalence of the graph norm of $-A$ and the norm

$$\begin{aligned}|||f||| &= \|(I+A)f\|_p = \|F^{-1}(1+\psi(\xi'))\hat{f}) - f'_{x_{n+1}}\|_p = \|F^{-1}(1+\psi(\xi')+i\xi_{n+1})\hat{f})\|_p \\ &= \|F^{-1}Re\,\chi(\xi)\hat{f} + iF^{-1}Im\,\chi(\xi)\hat{f}\|_p =: \|f\|_{\chi,2,1}\end{aligned}$$

follows from Theorem 3.1.25 [J2] for $r = 1$:

Theorem 2.1.16. Let $(-A, D(A))$ be the generator of an L_p-sub-Markovian semigroup, and $0 < r \leq 1$. Then for all $u \in D(A)$ we have

$$\frac{1}{3}(\|(-A)^r u\|_p + \|u\|_p) \leq \|(I-A)^r u\|_p \leq \|(-A)^r u\|_p + \|u\|_p.$$

Now we want to extend Definition 2.1.4 to the case when ψ is not only real.

Let $(A, D(A))$ be a generator of a strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ on the Banach space X. Then we can define its fractional powers:

$$(-A)^\alpha \varphi = \frac{1}{\Gamma(-\alpha)}\int_0^\infty t^{-\alpha-1}(T_t\varphi - \varphi)\,dt, \quad \varphi \in D(A), \tag{2.16}$$

and

$$(-A)^{-\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}T_t\varphi\,dt, \quad \varphi \in X, \tag{2.17}$$

where $0 < \alpha < 1$ (see [S], equations (5.84), (5.86) and [Y], equation IX.11.5).

Formulas (2.16) and (2.17) are called Balakrishnan's formulas.

Consider the fractional power of $-A_\pm$, $symb\,(-A_\pm) = \psi(\xi') \pm i\xi_{n+1} = \chi_\pm(\xi)$, where ψ is of type 1. On $S(\mathbb{R}^n)$ the operators $(-A_\pm)^\alpha$ have the representation

$$(-A_\pm)^\alpha f(x) = (2\pi)^{-(n+1)/2}\int_{\mathbb{R}^{n+1}} e^{i\xi x}(\psi(\xi') \pm i\xi_{n+1})^\alpha \hat{f}(\xi)\,d\xi, \tag{2.18}$$

or

$$\widehat{(-A_\pm)^\alpha f}(\xi) = (\psi(\xi') \pm i\xi_{n+1})^\alpha \hat{f}(\xi),$$

where $\xi = (\xi', \xi_{n+1})$ and $x = (x', x_{n+1})$.

We need an additional assumption on ψ.

Assumption 2.1.17. Let ψ be a continuous negative definite function such that
A1. $\psi(\xi) = f(\phi(\xi))$, where f is a Bernstein function, and ϕ, $\phi(0) > 0$, is a continuous negative definite function such that for all i, $1 \leq i \leq n$, ϕ'_i exists and does not depend on ξ_j, $i \neq j$ (we will denote by g'_i the derivative of function $g(\xi_1, \dots, \xi_n)$ with respect to ξ_i);
A2.

$$\sup_{\xi\in\mathbb{R}^n}\left|\frac{\xi_1\dots\xi_k\phi'_1\dots\phi'_k}{\phi^k}\right| < \infty, \quad \forall k, \quad k = 1,\dots,n.$$

Such functions ϕ exist, for example, Assumption 2.1.17 is satisfied for $\psi = f(1+|\xi|^2)$, where f is a Bernstein function.

Example 2.1.18. The continuous negative definite functions

$$\psi(\xi) = (1+|\xi|^2)^{\beta/2}, \quad 0<\beta<2 \tag{2.19}$$

$$\psi(\xi) = \ln(1+(1+|\xi|^2)^{\beta/2}), \quad 0<\beta<2 \tag{2.20}$$

$$\psi(\xi) = (1+|\xi|^2)^{\beta/2}\ln(1+(1+|\xi|^2)^{\beta/2}), \quad 0<\beta<1 \tag{2.21}$$

satisfy Assumption 2.1.17 A2.

For the last example we refer to [FL]. For a continuous negative definite functions $(\chi_\pm(\xi))^\alpha$, where $\chi_\pm(\xi) = \psi(\xi') \pm i\xi_{n+1}$, we will consider $(\chi_\pm)^\alpha$ -Bessel potential spaces, $H_p^{|\chi_\pm|^\alpha,t}$ and $H_p^{\Re,t}$, where $\Re = Re(\chi_\pm(\xi))^\alpha$, $1<p<\infty$, $t\in\mathbb{R}$.

We claim that the operators $(-A_\pm)^\alpha$, $symb(-A_\pm) = \chi_\pm$, are isomorphisms between $H_p^{|\chi_\pm|^\alpha,t}$ and $H_p^{|\chi_\pm|^\alpha,t-2}$.

For the proof we need the notion of a Fourier multiplier (see [J2], Definition 3.3.34, or [T1], §5.1).

Definition 2.1.19. Let $1\le p,q\le\infty$. We call a distribution $m\in S'(\mathbb{R}^n)$ a Fourier multiplier of type (p,q) if

$$\|m\|_{M_{p,q}} := \sup\{\frac{\|F^{-1}(m\hat{\varphi})\|_q}{\|\varphi\|_p}, \quad 0\neq\varphi\in S(\mathbb{R}^n)\} < \infty \tag{2.22}$$

The set of all Fourier multipliers of type (p,q) is denoted by $M_{p,q}$.

In order to prove that a function is an L_p-Fourier multiplier, we will check whether the conditions of Lizorkin's Fourier multiplier theorem (see [Liz1]) are satisfied:

Theorem 2.1.20. Let $m\in L_\infty(\mathbb{R}^n)$ be a function such that

$$\sup_{\xi\in\mathbb{R}^n} |(\xi\cdot\partial)^\alpha m(\xi)| \le c \tag{2.23}$$

for all $\alpha\in\mathbb{N}_0^n$, $\alpha_j\in\{0,1\}$, $j=1,\dots,n$, and $(\xi\cdot\partial)^\alpha = (\xi_1\partial_1)^{\alpha_1}\cdot\ldots\cdot(\xi_n\partial_n)^{\alpha_n}$. Then m is an L_p-Fourier multiplier for $1<p<\infty$.

Now we are ready to prove our result.

Theorem 2.1.21. Let $\chi_\pm(\xi) = \psi(\xi')\pm i\xi_{n+1}$, $\xi=(\xi',\xi_{n+1})\in\mathbb{R}^{n+1}$, and $\psi:\mathbb{R}^n\to\mathbb{R}$ is a continuous negative definite function, which satisfies **A1** and **A2**. Then the functions $e^{z\arg\chi_\pm}$ are L_p-Fourier multipliers for $z\in\mathbb{C}$, $-1<Im\,z<1$.

Proof. We will consider $\chi_+ = \chi$, the proof for χ_- is similar. First, we prove that $e^{i\arg\chi}$ is an L_p-Fourier multiplier, $1 < p < \infty$.

Since

$$e^{i\arg\chi} = \frac{\psi}{|\chi|} + i\frac{\xi_{n+1}}{|\chi|} = \frac{\psi}{(\psi^2+\xi_{n+1}^2)^{1/2}} + \frac{i\xi_{n+1}}{(\psi^2+\xi_{n+1}^2)^{1/2}} = I_0 + iJ_0$$

we have to prove that I_0 and J_0 are L_p-Fourier multipliers. To do this let us check the conditions of Theorem 2.1.20.

First consider I_0. Differentiating I_0 with respect to ξ_{n+1}, we obtain

$$\frac{\partial I_0}{\partial \xi_{n+1}} = -\frac{\xi_{n+1}\psi}{(\psi^2+\xi_{n+1}^2)^{3/2}} = I_1$$

with

$$\sup_{\xi\in\mathbb{R}^{n+1}} |\xi_{n+1}I_1| < \infty,$$

and

$$\frac{\partial I_1}{\partial \xi_1} = \xi_{n+1}\big(-\frac{\psi_1'}{(\psi^2+\xi_{n+1}^2)^{3/2}} + \frac{3\psi^2\psi_1'}{(\psi^2+\xi_{n+1}^2)^{5/2}}\big) = I_2.$$

Analogously,

$$\frac{\partial J_0}{\partial \xi_{n+1}} = \frac{1}{(\psi^2+\xi_{n+1}^2)^{1/2}} - \frac{\xi_{n+1}^2}{(\psi^2+\xi_{n+1}^2)^{3/2}} = J_1,$$

and we see, that

$$\sup_{\xi\in\mathbb{R}^{n+1}} |\xi_{n+1}J_1| < \infty.$$

Further,

$$\frac{\partial J_1}{\partial \xi_1} = -\frac{\psi\psi_1'}{(\psi^2+\xi_{n+1}^2)^{3/2}} + \frac{3\xi_{n+1}^2\psi\psi_1'}{(\psi^2+\xi_{n+1}^2)^{5/2}} = J_2.$$

Note, that since ψ satisfies **A1** and **A2**, and since for a Bernstein function $f(s)$ we have (see [J1], Theorem 3.9.34)

$$|f^{(k)}(s)| \le \frac{k!}{s^k} f(s), \quad s > 0, \quad k \ge 0,$$

we obtain the estimates:

$$\begin{aligned} |\psi_1'| &= |f'(\phi)\phi_1'| \le b_1\Big|\frac{\psi\phi_1'}{\phi}\Big|; \\ |\psi_{2,1}''| &= |f''(\phi)\phi_1'\phi_2'| \le b_2\Big|\frac{\psi\phi_1'\phi_2'}{\phi^2}\Big|; \\ &\cdots\cdots\cdots \\ |\psi_{k,\dots,1}^{(k)}| &= |f^k(\phi)\phi_1'\phi_2'\dots\phi_k'| \le b_k\Big|\frac{\psi\phi_1'\phi_2'\dots\phi_k'}{\phi^k}\Big|, \end{aligned} \tag{2.24}$$

where b_i, $i \geq 1$, are some constants.

Therefore, since $\psi(\xi) > 0$, $\xi \in \mathbb{R}^{n+1}$, we obtain:

$$\sup_{\xi \in \mathbb{R}^{n+1}} |\xi_1 \xi_{n+1} J_2| \leq a_1 \sup_{\xi \in \mathbb{R}^{n+1}} \Big\{ \Big|\frac{\xi_1 \phi_1'}{\phi}\Big| + 3\Big|\frac{\xi_1 \phi_1'}{\phi}\Big| \Big\} < \infty,$$

and

$$\sup_{\xi \in \mathbb{R}^{n+1}} |\xi_1 \xi_{n+1} I_2| \leq a_1 \sup_{\xi \in \mathbb{R}^{n+1}} \Big\{ \Big|\frac{\xi_1 \phi_1'}{\phi}\Big| + 3\Big|\frac{\xi_1 \phi_1'}{\phi}\Big| \Big\} < \infty,$$

where a_1 is some constant.

To see the general rule, let us find the third derivatives of I_0 and J_0. We obtain, that

$$\frac{\partial I_2}{\partial \xi_2} = \xi_{n+1} \Big\{ -\frac{\psi_{2,1}''}{(\psi^2 + \xi_{n+1}^2)^{3/2}} + \frac{3\psi\psi_2'\psi_1'}{(\psi^2 + \xi_{n+1}^2)^{5/2}} + \frac{6\psi\psi_2'\psi_1' + \psi^2\psi_{2,1}''}{(\psi^2 + \xi_{n+1}^2)^{5/2}} - \frac{15\psi^3\psi_2'\psi_1'}{(\psi^2 + \xi_{n+1}^2)^{7/2}} \Big\} = I_3.$$

and

$$\frac{\partial J_2}{\partial \xi_2} = -\frac{\psi_2'\psi_1' + \psi_{2,1}''}{(\psi^2 + \xi_{n+1}^2)^{3/2}} + \frac{3\psi^2\psi_2'\psi_1'}{(\psi^2 + \xi_{n+1}^2)^{5/2}} + \frac{3\xi_{n+1}^2(\psi_2'\psi_1' + \psi_{2,1}'')}{(\psi^2 + \xi_{n+1}^2)^{5/2}} - \frac{15\xi_{n+1}^2\psi^2\psi_2'\psi_1'}{(\psi^2 + \xi_{n+1}^2)^{7/2}} = J_3.$$

By **A2** we have

$$\sup_{\xi \in \mathbb{R}^{n+1}} |\xi_2 \xi_1 \xi_{n+1} I_3| \leq a_2 \sup_{\xi \in \mathbb{R}^{n+1}} \Big|\frac{\xi_1 \xi_2 \phi_1' \phi_2'}{\phi^2}\Big| < \infty$$

and

$$\sup_{\xi \in \mathbb{R}^{n+1}} |\xi_2 \xi_1 \xi_{n+1} J_3| \leq a_2 \sup_{\xi \in \mathbb{R}^{n+1}} \Big|\frac{\xi_1 \xi_2 \phi_1' \phi_2'}{\phi^2}\Big| < \infty,$$

where a_2 is again some constant.

We see, that the derivatives $\frac{\partial I_1}{\partial \xi_1}$ and $\frac{\partial J_1}{\partial \xi_1}$ consist of the terms with the representation (up to multipliers which may depend on ξ_{n+1}):

$$\frac{\psi^l \psi_1'}{(\psi^2 + \xi_{n+1}^2)^{(1+2r)/2}}, \quad r = 1, 2 \quad \text{and} \quad l < 2r + 1,$$

and the representation of the derivatives $\frac{\partial I_2}{\partial \xi_2}$ and $\frac{\partial J_2}{\partial \xi_2}$ contains the terms

$$\frac{\psi^l P(\psi_1', \psi_2', \psi_{21}'')}{(\psi^2 + \xi_{n+1}^2)^{(1+2r)/2}}, \quad r = 1, 2, 3 \quad \text{and} \quad l < 2r + 1.$$

Here $P(\psi_1', \psi_2', \psi_{21}'')$ is a polynomial expression of the first degree by each variable: it consists of the terms $\psi_1'\psi_2'$ and ψ_{21}'' (with some coefficients, maybe dependent on ξ_{n+1}). This leads to a proposition, that all terms of k-th derivatives $\frac{\partial I_k}{\partial \xi_k}$ and $\frac{\partial J_k}{\partial \xi_k}$ are of the form

$$\frac{\psi^l P_k(\psi_1', \dots, \psi_k', \dots, \psi_{k\dots 1}^{(k)})}{(\psi^2 + \xi_{n+1}^2)^{(1+2r)/2}}, \quad r = 1, 2, \dots, k \quad \text{and} \quad l < 2r + 1, \tag{2.25}$$

where $P_k(\psi_1', \dots, \psi_k', \dots, \psi_{k\dots 1}^{(k)})$ is a polynomial expression of the form

$$P_k(\psi_1', \dots, \psi_k', \dots, \psi_{k\dots 1}^{(k)}) = \sum_{l_1 + \dots + l_s = k} c(l_1, \dots, l_s)\ \psi^{(l_1)} \dots \psi^{(l_s)}, \tag{2.26}$$

where all the derivatives in $\psi^{(l_1)}, \dots, \psi^{(l_s)}$ are with respect to different variables, and in $\psi^{(l_i)}$, $i = 1, \dots, s$ the derivatives in appear exactly once.
We will prove by induction that (2.25) holds for all k, $1 \le k \le n$. For $m = 1$ we already showed, that (2.25) holds. Suppose that (2.25) holds for $m = k$, and let us consider the case $m = k + 1$. Differentiating (2.25) with respect to ξ_{k+1}, we obtain

$$\frac{l\psi^{l-1}\psi_{k+1}' P_k + \psi^l (P_k)_{k+1}'}{(\psi^2 + \xi_{n+1}^2)^{(1+2r)/2}} + (1 + 2r)\frac{P_k \psi^{l+1} \psi_{k+1}'}{(\psi^2 + \xi_{n+1}^2)^{(1+2r)/2+1}} \tag{2.27}$$

(of course we skip differentiation of ψ^l if $l = 0$). Since $\psi_{k+1}' P_k$ and $(P_k)_{k+1}'$ again consists the terms of form (2.26), we see, that for $m = k + 1$ our proposition is also true. Therefore by induction we obtain that $\frac{\partial I_k}{\partial \xi_k}$ and $\frac{\partial J_k}{\partial \xi_k}$ are of form (2.25).
Having (2.25) with $l < 2r + 1$ and with P_k calculated as (2.26), we see, that since

$$|P_k| \le c\frac{\phi_1' \dots \phi_k'}{\phi^k},$$

we come to the estimates (we denote by $I_{k+1} = \frac{\partial I_k}{\partial \xi_k}$ and by $J_{k+1} = \frac{\partial J_k}{\partial \xi_k}$) for $1 \le k \le n$:

$$\sup_{\xi \in \mathbb{R}^{n+1}} |\xi_k \dots \xi_1 \xi_{n+1} I_{k+1}| < \infty \tag{2.28}$$

and

$$\sup_{\xi \in \mathbb{R}^{n+1}} |\xi_k \dots \xi_1 \xi_{n+1} J_{k+1}| < \infty \tag{2.29}$$

and thus

$$\sup_{\xi \in \mathbb{R}^{n+1}} |\xi_k \dots \xi_1 \xi_{n+1} \partial_{k,\dots,1,n+1}^{(k+1)} e^{i \arg \chi}| < \infty, \quad k = 1, \dots, n. \tag{2.30}$$

We may write $\partial_{k,\dots,1,n+1}^{(k+1)} e^{i \arg \chi}$ as

$$\partial_{k,\dots,1,n+1}^{(k+1)} e^{i \arg \chi} = e^{i \arg \chi} Q((\arg \chi)', (\arg \chi)'', \dots, (\arg \chi)^{(k)}),$$

where $Q(\cdot, \ldots, \cdot)$ is a polynomial of arguments $\arg\chi^{(i)}$, $i = 1, \ldots, k$, with some complex coefficients. Similarly,

$$\partial^{(k+1)}_{k,\ldots,1,n+1} e^{i\alpha \arg\chi} = e^{\alpha i \arg\chi} Q_1((\arg\chi)', (\arg\chi)'', \ldots, (\arg\chi)^{(k)}, \alpha), \quad -1 < \alpha < 1,$$

and

$$\partial^{(k+1)}_{k,\ldots,1,n+1} e^{\theta \arg\chi} = e^{\theta \arg\chi} Q_2((\arg\chi)', (\arg\chi)'', \ldots, (\arg\chi)^{(k)}, \theta), \quad \theta \in \mathbb{R},$$

where the polynomials Q_1 and Q_2 are different from Q only in coefficients, which now depend on α and θ respectively. Note, that since $-\frac{\pi}{2} \leq \arg\chi \leq \frac{\pi}{2}$, then the function $|e^{\theta \arg\chi(\xi)}|$ is bounded for all $\xi \in \mathbb{R}^n$.

Therefore, since (2.30) holds, then

$$\sup_{\xi \in \mathbb{R}^{n+1}} |\xi_k \ldots \xi_1 \xi_{n+1} \partial^{(k+1)}_{k,\ldots,1,n+1} e^{i\alpha \arg\chi}| < \infty \tag{2.31}$$

and

$$\sup_{\xi \in \mathbb{R}^{n+1}} |\xi_k \ldots \xi_1 \xi_{n+1} \partial^{(k+1)}_{k,\ldots,1,n+1} e^{\theta \arg\chi}| < \infty, \tag{2.32}$$

hold for all $k = 1, \ldots, n$. It can be seen from the calculations that the order in which we take the derivatives does not matter. We considered the situation when the derivative in ξ_{n+1} appear in the begining, but it can be proved that the estimates will be the same if it appears in between or in the end; the order of the derivatives with respect to ξ_i, $i = 1, \ldots, n$ does not matter.

Therefore we have proved, that $e^{z \arg\chi}$, $0 < Im\, z < 1$, $Re\, z \in \mathbb{R}$, is an L_p-Fourier multiplier. □

Recall that $\Re = Re(\chi_\pm(\xi))^\alpha$, $0 < \alpha < 1$, is again a continuous negative definite function.

Theorem 2.1.22. Under the conditions of Theorem 2.1.21 the operator $(-A_\pm)^\alpha : H_p^{\Re,t} \to H_p^{\Re,t-2}$, $symb(-A_\pm) = \chi_\pm$, is an isomorphism.

Proof. Consider $\chi_+ = \chi$, and denote by $\theta = \theta(\xi) = \alpha \arg\chi(\xi)$. From the proof of Theorem 2.1.21 (see (2.28)) we can see that $\sin\theta$ is an L_p-Fourier multiplier. We will prove that $\frac{1}{\cos\theta}$ is an L_p-Fourier multiplier too, which gives us that $\tan\theta$ is an L_p-Fourier multiplier, because for $u \in L_p$

$$\left\|F^{-1}\left(\frac{\sin\theta(\xi)}{\cos\theta(\xi)} \hat{f}(\xi)\right)\right\|_p \leq c_1 \|F^{-1}(\sin\theta(\xi)\hat{f}(\xi))\|_p \leq c_2 \|f\|_p.$$

Note that for $0 < \alpha < 1$ we have $|\theta(\xi)| = |\alpha \arg\chi(\xi)| < \frac{\alpha\pi}{2}$, and $\cos\theta(\xi) > 0$ for all $\xi \in \mathbb{R}^{n+1}$.

We find that

$$\begin{aligned}(\frac{1}{\cos\theta})^{(k)}_{k,\dots,1} &= -(\frac{\sin\theta}{\cos^2\theta}\theta'_1)^{(k-1)}_{k,\dots,2}\\ &= (\frac{\theta'_1\theta'_2(\cos^3\theta - 2\sin\theta\cos\theta}{\cos^4\theta})^{(k-2)}_{k,\dots,3}\\ &\dots\dots\dots\\ &= \frac{P_k(\theta'_1,\theta'_2,\dots,\theta'_k,\dots,\theta^{(k)}_{k,\dots,1},\sin\theta,\cos\theta)}{\cos^{2k}\theta},\end{aligned}$$

where P_k is a polynomial (compare (2.26))

$$P_k(\theta'_1,\theta'_2,\dots,\theta'_k,\dots,\theta^{(k)}_{k,\dots,1},\sin\theta,\cos\theta) = \sum_{l_1+\dots+l_s=k} c(l_1,\dots,l_s)\theta^{(l_1)}\dots\theta^{(l_s)},$$

such that the coefficients $c(l_1,\dots,l_s)$ depend on θ only as $\sin^i\theta$ and $\cos^j\theta$, $1 \le i,j \le k$, and all the derivatives in $\theta^{(l_1)}\dots\theta^{(l_s)}$ are with respect to different variables.
Further, since $\theta = \theta(\xi) = \alpha\arg\chi(\xi) = \alpha\, arctan\, \frac{\xi_{n+1}}{\psi(\xi')}$, $\chi(\xi) = \psi(\xi') + i\xi_{n+1}$, for $i = 1,\dots,n$, ($\psi(\xi') > 0$, $\xi' \in \mathbb{R}^n$ in view of **A1**)

$$\theta'_i = -\alpha\frac{\psi}{(\psi^2+\xi^2_{n+1})^{1/2}}\frac{\xi_{n+1}\psi'_i}{\psi^2} = -\frac{\alpha\xi_{n+1}\psi'_i}{(\psi^2+\xi^2_{n+1})^{1/2}\psi},$$

and for $i = n+1$ we have

$$\theta'_{n+1} = -\frac{\alpha}{(\psi^2+\xi^2_{n+1})^{1/2}},$$

which gives that $\xi_i\theta'_i(\xi)$, $i = 1,\dots,n+1$ is bounded for all $\xi \in \mathbb{R}^{n+1}$.
Therefore we conclude (see the proof of Theorem 2.1.21) that

$$\begin{aligned}&\sup_{\xi\in\mathbb{R}^{n+1}}\left|\xi_1\dots\xi_n\Big(\frac{1}{\cos\theta(\xi)}\Big)^{(k)}_{k,\dots,1}\right| =\\ &= \sup_{\xi\in\mathbb{R}^{n+1}}\left|\xi_1\dots\xi_n\frac{P_k(\theta'_1,\theta'_2,\dots,\theta'_k,\dots,\theta^{(k)}_{k,\dots,1},\sin\theta,\cos\theta)}{\cos^{2k}\theta}\right|\\ &\le c(\alpha)\sum_{l_1+\dots+l_s=k}\sup_{\xi\in\mathbb{R}^{n+1}}|\xi_1\dots\xi_n\theta^{(l_1)}\dots\theta^{(l_s)}| < \infty,\end{aligned}$$

i.e. $\frac{1}{\cos\theta}$ is an L_p -Fourier multiplier.
Further, since

$$\chi^\alpha(\xi) = \Re(\xi)(1 + i\tan(\theta(\xi))),$$

and $\Re(0) \neq 0$, then, by Theorem 2.1.13

$$\begin{aligned}\|F^{-1}(\chi^\alpha(\xi)\hat f(\xi))\|_{H^{\Re,t-2}_p} &\le c_1\|F^{-1}(\Re^{(t-2)/2}\Re(1+i\tan\theta)\hat f)\|_p\\ &\le c_2\|F^{-1}(\Re^{t/2}\hat f)\|_p \le c_3\|f\|_{H^{\Re,t}_p}\end{aligned}$$

and, since $\chi^{-\alpha} = \Re^{-1}(1 - i\tan\theta)\cos^2\theta$

$$\|F^{-1}(\chi^{-\alpha}(\xi)\hat{f}(\xi))\|_{H_p^{\Re,t}} \leq c_4\|F^{-1}(\Re^{t/2}\Re^{-1}(1 - i\tan\theta)\cos^2\theta\hat{f})\|_p \leq c_5\|F^{-1}(\Re^{(t-2)/2}\hat{f})\|_p \leq c_6\|f\|_{H_p^{\Re,t}},$$

which proves our theorem. □

Remark 2.1.23. Since $\cos(\alpha arg\chi)$ is a Fourier multiplier by Theorem 2.1.21, we can see, that the spaces $H_p^{\Re,s}$ and $H_p^{|\chi|^\alpha,s}$ coincide, and thus $(-A_\pm)^\alpha : H_p^{|\chi|^\alpha,t} \to H_p^{|\chi|^\alpha,t-2}$, is a continuous, bijective mapping with continuous inverse.

Theorem 2.1.24. The operator $(-(-A_\pm)^\alpha, H_p^{\Re,2})$ is a generator of an L_p-sub-Markovian semigroup, $symb(-A_\pm) = \chi_\pm$.

Proof. From Theorems 2.1.22 and 2.1.16 we deduce, that the graph norm of $(-A_\pm)^\alpha$ is equivalent to the norm $\|\cdot\|_{\Re,2}$ and $H_p^{\Re,2}$ is the domain of $(-A_\pm)^\alpha$, and $-(-A_\pm)^\alpha$ is a Dirichlet operator (and so it is dissipative) on $H_p^{\Re,2}$.
We are going to use the following theorem, see [B], Theorem VIII.3.3:

Let E_1, E_2 be two Banach spaces. If $A \in L(E_1, E_2)$ is invertible, and $B \in L(E_1, E_2)$ is such that $\|B\| \leq \|A^{-1}\|^{-1}$, then $A + B$ is invertible (here $L(E_1, E_2)$ is the space of all continuous linear operators from E_1 to E_2, and $\|A\|$ means the norm of an operator A, i.e. $\|A\|_{E_1\to E_2} = \sup_{\|x\|=1}\|Ax\|_{E_2}$).
This theorem and Theorem 2.1.22 give us that since $(-A)^\alpha$ is bijective from $H_p^{\Re,2}$ to L_p, then for $\lambda : |\lambda| < \|(-A)^{-\alpha}\|^{-1}$ the operator $\lambda + (-A)^\alpha$ is invertible and therefore the equation

$$\lambda f + (-A)^\alpha f = g$$

is uniquely solvable for any $g \in L_p$, $|\lambda| < \|(-A)^{-\alpha}\|^{-1}$
So, we have that $\lambda_0 + (-A)^\alpha$ is invertible on $H_p^{\Re,2}$, $|\lambda_0| < \|(-A)^{-\alpha}\|^{-1}$ and then $\lambda_0 \in \rho(-(-A)^\alpha)$-the resolvent set of $-(-A)^\alpha$. But then by Lemma 4.1.27 [J1] $(0, \infty) \subset \rho(-(-A)^\alpha)$, and then the equation

$$\lambda f + (-A)^\alpha f = g$$

is uniquely solvable for all $\lambda > 0$, $g \in L_p$.

Thus, all conditions of Hille-Yosida theorem are satisfied, and the theorem is proved. □

Remark 2.1.25. We can also see from Theorem 2.1.21 that if $\psi(\xi')$ satisfies **A1**,**A2**, and $\psi(\xi') \geq (1 + |\xi'|^2)^{\beta/2}$, then the function $\frac{(1+|\xi'|^2)^{\beta/2}}{\psi(\xi')}$ is an L_p-Fourier multiplier.

From Theorem 2.1.21 we see that the spaces $H_p^{|\chi|^\alpha,t}$ and $H_p^{\Re,t}$ are equivalent, where $\chi(\xi)=\psi(\xi')+i\xi_{n+1}$, ψ satisfies **A1** and **A2**. In order to make our notation less awkward, we will work with $H_p^{\Re,t}$-spaces, having in mind that all the statements that are valid for $H_p^{\Re,t}$, are also true for $H_p^{|\chi|^\alpha,t}$.

In the next step we will give a link between the spaces $H_p^{\chi,2,1}$ and $H_p^{\Re,2}$. It is essential that these spaces are the domains of generators of strongly continuous contraction semigroups.

For $-A_\pm$, $symb\,(-A_\pm)=\chi_\pm$, being a generator of a strongly continuous contraction semigroup, we can define (at least on $S(\mathbb{R}^{n+1})$ the complex powers $(-A_\pm)^{i\theta}$, $\theta\in\mathbb{R}$, of $-A_\pm$ as a continuation of (2.16) to the the complex plane.

Considering the operator $(-A_\pm)^{i\theta}$ on $S(\mathbb{R}^{n+1})$, we may calculate that

$$-\widehat{(-A_\pm)^{i\theta}f}(\xi)=(\chi_\pm(\xi))^{i\theta}\hat{f}(\xi),\quad \xi\in\mathbb{R}^{n+1},$$

so in this case $(\chi_\pm(\xi))^{i\theta}$ is the symbol of $(-A_\pm)^{i\theta}$.

Our next step will be the following lemma.

Lemma 2.1.26. Let ψ be a continuous negative definite function which satisfies **A1** and **A2**. Then the operators $(-A_\pm)^{i\theta}$, $symb(-A_\pm)=\chi_\pm=\psi(\xi')\pm i\xi_{n+1}$, are bounded in $L_p(\mathbb{R}^{n+1})$ for all $\theta\in\mathbb{R}$.

Proof.. Again we will show that the conditions of Theorem 2.1.20 are satisfied for $\chi=\chi_+$, the proof for χ_- is similar.
Let $I_0=(\psi(\xi')+i\xi_{n+1})^{i\theta}$. We see, that

$$\begin{aligned}
&\frac{\partial I_0}{\partial\xi_1}=i\theta(\psi+i\xi_{n+1})^{i\theta-1}\psi_1'=I_1,\\
&\frac{\partial I_1}{\partial\xi_2}=i\theta(i\theta-1)(\psi+i\xi_{n+1})^{i\theta-2}\psi_1'\psi_2'+i\theta(\psi+i\xi_{n+1})^{i\theta-1}\psi_{21}''=I_2,\\
&\frac{\partial I_2}{\partial\xi_3}=i\theta(i\theta-1)(i\theta-2)(\psi+i\xi_{n+1})^{i\theta-3}\psi_1'\psi_2'\psi_3'+i\theta(\psi+i\xi_{n+1})^{i\theta-2}\\
&\quad(\{(i\theta-1)\psi_{21}''\psi_3'+\psi_2'\psi_{31}''\}+\psi_3'\psi_{21}'')+i\theta(\psi+i\xi_{n+1})^{i\theta-1}\psi_{321}'''=I_2,\qquad(2.33)\\
&\cdots\cdots\cdots\\
&\frac{\partial I_{k-1}}{\partial\xi_k}=(\psi+i\xi_{n+1})^{i\theta}\sum c(l,m_1,\ldots,m_l)\psi^{(m_1)}_{j_{m_1+1},\ldots,j_1}\cdots\\
&\frac{\cdots\psi^{(m_i)}_{j_{m_i+m_{i-1}+\ldots+m_1+1},\ldots,j_{m_{i-1}+\ldots+m_1+1}}\cdots\psi^{(m_l)}_{j_{m_l+m_{l-1}+\ldots+m_1+1},\ldots,j_{m_{l-1}+\ldots+m_1+1}}}{(\psi+i\xi_{n+1})^l},
\end{aligned}$$

where the sum is taken over all l, $1\le l\le k$, $m_1+\ldots+m_l=k-1$, and the indexes $j_{\ldots}$ such that in every term $\psi^{(m_i)}_{j_{m_i+m_{i-1}+\ldots+m_1+1},\ldots,j_{m_{i-1}+\ldots+m_1+1}}$ we have $j_{m_i+m_{i-1}+\ldots+m_1+1},>$ $j_{m_i+m_{i-1}+\ldots+m_1}>\ldots>j_{m_{i-1}+\ldots+m_1+1}$, in the product $\psi^{(m_1)}_{j_{m_1+1},\ldots,j_1}\cdots$

$\dots\psi^{(m_i)}_{j_{m_i}+m_{i-1}+\dots+m_1+1},\dots,j_{m_{i-1}+\dots+m_1+1}\dots\ \psi^{(m_l)}_{j_{m_l}+m_{l-1}+\dots+m_1+1},\dots,j_{m_{l-1}+\dots+m_1+1}$ all the indexes are different, and one of the indexes $j_{m_1+1}, j_{m_1+m_2+1},\dots,j_{m_l+\dots+m_1+1}$ equals k. Let us prove (2.33) by induction. For $k=1$ it holds. Suppose (2.33) holds for k, and let us see what we obtain in the case $k+1$. Differentiating $I_k = \frac{\partial I_{k-1}}{\partial \xi_k}$ we obtain

$$\frac{\partial I_k}{\partial \xi_{k+1}} = \frac{i\theta\psi'_{k+1}I_k}{\psi+i\xi_{n+1}} + (\psi+i\xi_{n+1})^{i\theta}\sum lc(l,m_1,\dots,m_l)\psi^{(m_1)}_{j_{m_1+1},\dots,j_1}\dots$$
$$\frac{\dots\psi^{(m_i)}_{j_{m_i}+m_{i-1}+\dots+m_1+1,\dots,j_{m_{i-1}+\dots+m_1+1}}\dots\psi^{(m_l)}_{j_{m_l}+m_{l-1}+\dots+m_1+1,\dots,j_{m_{l-1}+\dots+m_1+1}}\psi'_{k+1}}{(\psi+i\xi_{n+1})^{l+1}}$$
$$+ (\psi+i\xi_{n+1})^{i\theta}\sum c(l,m_1,\dots,m_l)(\psi^{(m_1)}_{j_{m_1+1},\dots,j_1}\dots$$
$$\frac{\dots\psi^{(m_i)}_{j_{m_i}+m_{i-1}+\dots+m_1+1,\dots,j_{m_{i-1}+\dots+m_1+1}}\dots\psi^{(m_l)}_{j_{m_l}+m_{l-1}+\dots+m_1+1,\dots,j_{m_{l-1}+\dots+m_1+1}})'_{k+1}}{(\psi+i\xi_{n+1})^{l}}.$$

One may notice, that the right-hand side is of the same form as (3.11). Therefore (3.11) is true.

Further,

$$\frac{\partial I_k}{\partial \xi_{n+1}} = -\theta\frac{(\psi+i\xi_{n+1})^{i\theta}I_k}{\psi+i\xi_{n+1}} + (\psi+i\xi_{n+1})^{i\theta}\sum c(l,m_1,\dots,m_l)(-il)\psi^{(m_1)}_{j_{m_1+1},\dots,j_1}\dots$$
$$\frac{\dots\psi^{(m_i)}_{j_{m_i}+m_{i-1}+\dots+m_1+1,\dots,j_{m_{i-1}+\dots+m_1+1}}\dots\psi^{(m_l 0}_{j_{m_l}+m_{l-1}+\dots+m_1+1,\dots,j_{m_{l-1}+\dots+m_1+1}}}{(\psi+i\xi_{n+1})^{l+1}}.$$

Thus, in view of (2.24) and **A2**, we have that

$$\sup_{\xi\in\mathbb{R}^{n+1}}\left|\xi_1\dots\xi_k\xi_{n+1}\frac{\partial I_k}{\partial\xi_{n+1}}\right| \le c\sup_{\xi\in\mathbb{R}^{n+1}}\left|\frac{\xi_1\dots\xi_k\phi'_1\dots\phi'_k}{\phi^k}\right| < \infty$$

for all k, $1\le k\le n$, and thus $(\chi(\xi))^{i\theta}$ is an L_p-Fourier multiplier for all $\theta\in\mathbb{R}$, which leads to the estimate

$$\|(-A)^{i\theta}f\|_p = \|F^{-1}((\chi(\xi))^{i\theta}\hat{f}(\xi))\|_p \le c\|f\|_p. \quad \square$$

Note that for an operator $-A$ being a generator of a strongly continuous contraction semigroup the operator $(-A)^z$, $0 < Re\, z < 1$ is an analytic function of z. To show this, consider again the representation of $(-A)^\alpha$ via Balakrishnan's formula (2.16). From Lemma 2.1.26 $(-A)^{i\theta}$ is well defined and bounded in $L_p(\mathbb{R}^{n+1})$, then we may extend

$(-A)^\alpha$ to the strip $G = \{z : 0 < Re\, z < 1\}$ in the following way:

$$(-A)^z f(x) = \frac{1}{\Gamma(-z)} \int_0^\infty \frac{(T_\lambda - I)f(x)}{\lambda^{1+z}}\, d\lambda =$$
$$= \frac{1}{\Gamma(-z)} \int_0^\infty \lambda^{-z} \frac{(T_\lambda - I)f(x)}{\lambda}\, d\lambda =$$
$$= \frac{1}{\Gamma(-z)} \int_0^\infty e^{-z \ln \lambda} \frac{(T_\lambda - I)f(x)}{\lambda}\, d\lambda =$$
$$= \frac{1}{\Gamma(-z)} \int_{\mathbb{R}} e^{-z\, u}(T_{e^u} - I)f(x)\, du,$$

and we can see that $(-A)^z$ is a bilateral Laplace transform (see [DS], VIII.2.1, p.642) in u of $(T_{e^u} - I)f(x)$. Since $(T_t)_{t\geq 0}$ is a strongly continuous contraction semigroup, then for $f \in D(A)$ the latter function belongs to $L_1^{loc}(\mathbb{R})$:

$$\int_a^b |(T_{e^u} - I)f(x)|\, du = \int_{e^a}^{e^b} |\frac{T_\lambda f(x) - f(x)}{\lambda}|\, d\lambda \quad \text{for all } -\infty < a, b < \infty.$$

Therefore, $(-A)^z$ is an analytic function in G as the Laplace transform of an L_1^{loc}-function (see [DS]).
Later we will need the analyticity of $(-A)^z$, $-1 < Re\, z < 0$, which follows, since the operator-valued function $(-A)^{z-1}$, $0 < Re\, z < 1$, is analytic, and $-1 < Re\,(1-z) < 0$.
Let χ satisfy the conditions of Lemma 2.1.26, and suppose we have a theorem such as:

Theorem 2.1.27. Let $D(A)$ be the domain of the operator $-A$, $symb(-A) = \chi(\xi) = \psi(\xi') + i\xi_{n+1}$, ψ satisfies **A1** and **A2**, and $0 < \alpha < 1$. Then

$$[D((-A)^0), D((-A)^1)]_\alpha = [L_p, H_p^{\chi,2,1}]_\alpha = D((-A)^\alpha).$$

(This theorem is a modification of Theorem 1.15.3 [T2], see also [See]).

Then, since $D((-A)^\alpha) = H_p^{\Re,2}$, $\Re(\xi) = Re(\chi(\xi))^\alpha$, we obtain the following relation between $H_p^{\chi,2,1}$ and $H_p^{\Re,2}$:

$$[L_p, H_p^{\chi,2,1}]_\alpha = H_p^{\Re,2}.$$

Proof of Theorem 2.1.27 Let $u \in H_0 = [L_p, H_p^{\chi,2,1}]_\alpha$, $X = L_p + H_p^{\chi,2,1}$ and the space $W(G,X)$ as we defined before, in particular we have $G = \{z : 0 < Re\, z < 1\}$. Define the function $g(z) = e^{(z-\alpha)^2}(-A)^{\alpha - z} f$ for $f \in D(A) = H_p^{\chi,2,1}$. Note $D(A)$ is dense in $D((-A)^\alpha)$. First we check that $g(z)$ belongs to $W(G,X)$.

1) $g(z)$ is analytic (since $(-A)^z$ is) and $\sup_{z\in\overline{G}} \|g(z)\|_X < \infty$ (because $f \in D(A)$);

2) For $f \in D(A)$ the inequalities

$$\|g(iy)\|_p = \|e^{(iy-\alpha)^2}(-A)^{-iy}(-A)^\alpha f\|_p \leq c\|(-A)^\alpha f\|_p < \infty;$$

$$\|g(1+iy)\|_{H^{\chi,2,1}} = \|e^{(1+iy-\alpha)}(-A)^{-iy}(-A)^{\alpha-1}f\|_{H^{\chi,2,1}}$$
$$\leq c_1\|(-A)^{\alpha-1}f\|_{H^{\chi,2,1}} \leq c_2\|(-A)^{\alpha}f\|_p < \infty$$

hold, and $g(iy) \in L_p$, $g(1+iy) \in H^{\chi,2,1}$.
Then by the definition of the norm in $W(G,X)$

$$\|g\|_{W(G,X)} = \max\{\sup_{y\in\mathbb{R}}\|g(iy)\|_p, \sup_{y\in\mathbb{R}}\|g(1+iy)\|_{H^{\chi,2,1}}\}$$
$$\leq c_3\|(-A)^{\alpha}f\|_p.$$

Therefore, by the definition of the norm in the interpolation space (see (2.12))

$$\|f\|_{H_0} \leq \|g\|_{W(G,X)}$$

and arrive at

$$\|f\|_{H_0} \leq c_5\|(-A)^{\alpha}f\|_p.$$

For the proof of the inverse inequality choose $g_{\alpha}(z) = e^{(z-\alpha)^2}(-A)^z\omega(z)$, $\omega(\alpha) = u$. Clearly, $g_{\alpha}(\alpha) = (-A)^{\alpha}u \in L_p$, and applying Lemma 2.1.10 to the space L_p (see also [J2], equation (3.218)), we obtain

$$\|(-A)^{\alpha}u\|_p = \inf\{(\sup_y\|g(iy)\|_p)^{1-\alpha}(\sup_y\|g(1+iy)\|_p)^{\alpha}\}$$
$$\leq (\sup_y\|g_{\omega}(iy)\|_p)^{1-\alpha}(\sup_y\|g_{\omega}(1+iy)\|_p)^{\alpha}\}$$
$$\leq c(\sup_y\|\omega(iy)\|_p)^{1-\alpha}(\sup_y\|\omega(1+iy)\|_{H_p^{\chi,2,1}})^{\alpha}$$

and taking inf over all ω such that $\omega(\alpha) = u$ we obtain

$$\|(-A)^{\alpha}u\|_p \leq c\|u\|_{H_0}$$

which proves our theorem. □

2.2 Bessel-type potential spaces on $\mathbb{R}^n_{0+}$

Now we turn to the half-spaces.

Definition 2.2.1. Let ψ be of type 1 or 2. We define

$$H^{\psi,s}_{p,+} = \{f : \exists g \in H^{\psi,s}_p, \quad f = g|_{\mathbb{R}^n_{0+}}\}$$

with the norm

$$\|f\|_{\psi,s,+} = \|f\|_{\psi,s,p,+} = \inf_{g\in H^{\psi,s}_p,\, f=g|_{\mathbb{R}^n_{0+}}}\|g\|_{\psi,s,p},$$

and

$$\tilde{H}^{\psi,s}_{p,+} = \{f : f \in H^{\psi,s}_p, \quad \operatorname{supp} f \subset \mathbb{R}^n_{0+}\}$$

with the same norm as on $H^{\psi,s}_p$.
Similarly we can define the spaces $H^{\psi,s}_{p,-}$ and $\tilde{H}^{\psi,s}_{p,-}$.

Remark 2.2.2. The space $\tilde{H}^{\psi,s}_{p,+}$ is a closed subspace of $H^{\psi,s}_{p,+}$. Indeed, consider a Cauchy sequence $(f_n)_{n\geq 1}$, $f_n \in \tilde{H}^{\psi,s}_{p,+}$. For such a sequence there exists a function $f \in \tilde{H}^{\psi,s}_{p,+}$ such that $f_n \to f$ as $n \to \infty$. Therefore, by Riesz' lemma, there exists a subsequence $(f_{n(k)})_{k\geq 1}$ of $(f_n)_{n\geq 1}$, such that $f_{n(k)} \to f$ a.e. as $k \to \infty$. Suppose that there exists $D \subset \mathbb{R}^n_{0-}$, $\lambda^{(n)}(D) > 0$ (where $\lambda^{(n)}$ is the n-dimensional Lebesgue measure) such that $f \neq 0$ a.e. in D. Then there exists at least one $f_{n(k_0)}$ such that $f_{n(k_0)} \neq 0$ a.e. on $D' \subset D$, $\lambda^{(n)}(D') > 0$, which is a contradiction to that $f_{n(k_0)} \in \tilde{H}^{\psi,s}_{p,+}$. Therefore, *supp* $f \subset \mathbb{R}^n_{0+}$ too, and thus $\tilde{H}^{\psi,s}_{p,+}$ is closed.

For the next definition we refer to [DS], I.11, p.38, and II.4.21, p.72, or [Zhu], 1.1.8.

Definition 2.2.3. Let X be a Banach space, and B its closed subspace. The factor space X/B is the set of all sets of the form $x + B$, $x \in X$. It is a Banach space with the norm

$$|||x + B||| = \inf_{z \in B} \|x + z\|_X.$$

Remark 2.2.4. Since $\tilde{H}^{\psi,s}_{p,-}$ is closed, we can take the factor-space of $H^{\psi,s}_p$ with respect to $\tilde{H}^{\psi,s}_{p,-}$, and this factor-space consists of sets $f + \tilde{H}^{\psi,s}_{p,-}$, and

$$|||f + \tilde{H}^{\psi,s}_{p,-}||| = \inf_{h \in \tilde{H}^{\psi,s}_{p,-}} \|f + h\|_{H_{\psi,s,p}},$$

and we can see, that the norm $|||\cdot|||$ is equivalent to the norm of the function f in $H^{\psi,s}_{p,+}$, and therefore

$$H^{\psi,s}_{p,+} = H^{\psi,s}_p / \tilde{H}^{\psi,s}_{p,-}, \quad s \in \mathbb{R}. \tag{2.34}$$

Our next aim is to prove the lifting property of the operator $(-A_+)^\alpha$, $symb(-A_+) = \chi_+$, in $\tilde{H}^{\Re,t}_{p,+}$.

Theorem 2.2.5. Let $-\infty < t < \infty$, $1 < p < \infty$. Then

$$(-A_+)^\alpha : \tilde{H}^{\Re,t}_{p,+} \to \tilde{H}^{\Re,t-2}_{p,+}$$

isomorphically, where $-A_+$ is an operator with the symbol χ_+.

Proof. We will do similarly to Theorem 2.10.3 [T2]. Thanks to Theorem 2.1.22 we know that $(-A_+)^\alpha$: $H^{\Re,t}_p \to H^{\Re,t-2}_p$. What we need to know is that if $f \in C_0^\infty(\mathbb{R}^{n+1})$ with

$\operatorname{supp} f \subset \mathbb{R}^{n+1}_{0+}$, then $\operatorname{supp}(-A_+)^\alpha f \subset \mathbb{R}^{n+1}_{0+}$. For this we use the Paley-Wiener theorem (see [Y], p.226-229).

Let $g \in C_0^\infty(\mathbb{R})$ be such that $\operatorname{supp} g \in (-\infty, \varepsilon)$ for some $\varepsilon > 0$. Then we derive the estimate for the Fourier-Laplace transform $\hat{g}(z)$ of g, where $z = \xi + i\eta$:

$$\begin{aligned} |(1+|z|)^N \hat{g}(z)| &= \Big| \int_{-\infty}^{\varepsilon} \frac{e^{-izx}}{(2\pi)^{1/2}} (1-(-\triangle)^{1/2})^N g(x)\,dx \Big| \\ &= \Big| \int_{-\infty}^{\varepsilon} \frac{e^{x\eta - ix\xi}}{(2\pi)^{1/2}} (1-(-\triangle)^{1/2})^N g(x)\,dx \Big| \\ &\le \int_{-\infty}^{\varepsilon} \Big| \frac{e^{x\eta}}{(2\pi)^{1/2}} (1-(-\triangle)^{1/2})^N g(x) \Big|\,dx \\ &\le C_{g,N} e^{\eta\varepsilon} \end{aligned} \tag{2.35}$$

for all $N \in \mathbb{N}$ and some constant $C_{g,N}$.

Consider

$$F^{-1}((i\xi_{n+1} + \psi(\xi'))^\alpha \hat{f}(\xi)) = \int_{\mathbb{R}^n} \int_{-\infty}^{\infty} e^{i(x',\xi') + ix_{n+1}\xi_{n+1}} (i\xi_{n+1} + \psi(\xi'))^\alpha \hat{f}(\xi)\,d\xi_{n+1}\,d\xi'.$$

Since the function $iz + \psi(\xi')$, $z \in \mathbb{C}$, has a root $z_0 : Re\, z_0 = 0$, $Im\, z_0 = \psi(\xi') > 0$, we extend (see [T], §3.1) the function $(iz + \psi(\xi'))^\alpha \hat{f}(\xi', z)$ to the lower half-plane of $\mathbb{C}$. Consider the rectangle $\{-k \le Re\, z \le k,\ -N \le Im\, z \le 0\}$, where $k, N \le 0$. Since in view of (2.35) for $f \in C_0^\infty(\mathbb{R}^{n+1})$, $\operatorname{supp} f \subset \mathbb{R}^n \times (-\infty, \varepsilon)$,

$$|(1 + \psi(\xi') + |z|)^N \hat{f}(\xi', z)| \le C_{f,N,\varepsilon} e^{\eta\varepsilon}$$

or

$$|\hat{f}(\xi, z)| \le \frac{C_{f,N,\varepsilon} e^{\eta\varepsilon}}{|(1 + \psi(\xi') + |z|)^N|} \tag{2.36}$$

holds for some constant $C_{f,N,\varepsilon}$ (uniformly in ξ', because we can make N large, and the growth in ξ' in the denominator will "kill" the grouth in ξ' in the nominator).

The integrals along $\{Re\, z = -k,\ Im\, z$ from $-N$ to $0\}$ and $\{Re\, z = k,\ Im\, z$ from 0 to $-N\}$ tend to 0 as $k \to \infty$. Indeed, integrating along $\{Re\, z = -k,\ Im\, z$ from $-N$ to $0\}$ we obtain

$$\begin{aligned} &\Big| \int_{-N}^{0} e^{i(x',\xi') - ikx_{n+1} - x\tau} (-ik - \tau + \psi(\xi'))^\alpha \hat{f}(\xi', -k + i\tau)\,d\tau \Big| \\ &\qquad \le C_{f,N,\varepsilon} \int_{-N}^{0} e^{(\varepsilon - x_{n+1})\tau} \frac{(k^2 + (\psi(\xi') - \tau)^2)^{\alpha/2}}{1 + \psi(\xi') + (\tau^2 + k^2)^{1/2})^N}\,d\tau, \end{aligned}$$

and the right-hand side tends to 0 as $k \to \infty$ by the Lebesgue's dominated convergence theorem.

For the integral along $\{Re\, z = k,\ Im\, z$ from 0 to $-N\}$ the estimate is similar.

Therefore in view of the Cauchy theorem, we obtain

$$\Big|\int_{-\infty}^{\infty} e^{i(x',\xi')+ix_{n+1}z}(iz+\psi(\xi'))^{\alpha}\hat{f}(\xi)\,dz$$

$$= \lim_{k\to\infty}\int_{-iN-k}^{-iN+k} e^{i(x',\xi')+ix_{n+1}z}(iz+\psi(\xi'))^{\alpha}\hat{f}(\xi',z)\,dz$$

$$= \int_{-iN-\infty}^{-iN+\infty} e^{i(x',\xi')+ix_{n+1}z}(iz+\psi(\xi'))^{\alpha}\hat{f}(\xi',z)\,dz$$

$$= \int_{-\infty}^{\infty} e^{i(x',\xi')+ix_{n+1}\tau-Nx_{n+1}}(i\tau+N+\psi(\xi'))^{\alpha}\hat{f}(\xi',\tau-iN)\,d\tau.$$

In view of (2.36) we have for some large N and a constant $C_{f,N,\varepsilon}$

$$\left|e^{i(x',\xi')+ix_{n+1}\tau-Nx_{n+1}}(i\tau+N+\psi(\xi'))^{\alpha}\hat{f}(\xi',\tau-iN)\right| \le \frac{C_{f,N,\varepsilon}e^{-(\varepsilon-x_{n+1})N}(\tau^2+(N+\psi)^2)^{\alpha/2}}{|(1+\psi(\xi')+(\tau^2+N^2)^{1/2})^N|},$$

and thus by the Lebesgue's donimated convergence theorem we get

$$\int_{-\infty}^{\infty} e^{i(x',\xi')+ix_{n+1}z}(iz+\psi(xi'))^{\alpha}\hat{f}(\xi',z)\,dz \to 0 \quad \text{as} \quad N\to\infty.$$

Thus, if $f\in C_0^{\infty}(\mathbb{R}^{n+1})$ and $\operatorname{supp} f\subset\mathbb{R}^n\times(-\infty, varepsilon)$ then

$$F^{-1}((i\xi_{n+1}+\psi(\xi'))^{\alpha}\hat{f}(\xi))(x',x_{n+1})=0, \tag{2.37}$$

and letting $\varepsilon\to 0$ we obtain (2.37) for $f\in C_0^{\infty}(\mathbb{R}^{n+1})$, $\operatorname{supp} f\in\mathbb{R}^n\times(\infty,0]$. By the density agruments

$$\operatorname{supp} F^{-1}((i\xi_{n+1}+\psi(\xi'))^{\alpha}\hat{f}(\xi))(x',x_{n+1})\subset\mathbb{R}^n\times[0,\infty)$$

for all $f\in\tilde{H}^{\Re,2}_{p,+}$. $\square$

From Theorem 2.2.5 we immediately obtain

Theorem 2.2.6. Let $-\infty<t<+\infty$, $1<p<\infty$. Then

$$(-A_-)^{\alpha}: H^{\Re,t}_{p+}\to H^{\Re,t-2}_{p+} \tag{2.38}$$

isomorphically, where $-A_-$ is an operator with the symbol χ_-.

Proof. First, notice that since $arg\,\chi_+ = -arg\,\chi_-$, we have

$$H_p^{Re(\chi_+)^{\alpha},t}=H_p^{Re(\chi_-)^{\alpha},t}=H_p^{\Re,t}.$$

Similarly to the proof of Theorem 2.2.5 we can prove that

$$(-A_-)^\alpha : \tilde{H}^{\Re,t}_{p-} \longrightarrow \tilde{H}^{\Re,t-2}_{p-}.$$

In this case we must go to the upper half-plane of $\mathbb{C}$, and take functions $f \in C_0^\infty(\mathbb{R}^{n+1})$ with supports in $\mathbb{R}^n \times (-\varepsilon, +\infty)$. Since by Remark 2.2.2 it holds that $H^{\Re,t}_p = H^{\Re,t}_{p+} \oplus \tilde{H}^{\Re,t}_{p-}$, it follows that we have (2.38). □

Next we prove some density results and embedding theorems. In the rest of this Chapter we assume that all the spaces are defined on $\mathbb{R}^n$ or on the half-spaces $\mathbb{R}^n_{0+}$ or $\mathbb{R}^n_{0-}$.

Let ψ be of type 1 and

$$1 + \psi(\xi) \geq c_0(1 + |\xi|^2)^{r_0}, \quad 0 \leq r_0 \leq 1 \tag{2.39}$$

In Lemma 3.3.31 and the proof of Corollary 3.3.34 [J2] it was stated that

$$\frac{(1 + |\cdot|^2)^{t/2}}{(1 + \psi_R(\cdot))^{s/2}} \in M_{p,q} \quad \text{if } s > \frac{t+n}{r_0},\ t > 0,$$

$$\frac{(1 + \psi_R(\cdot))^{s/2}}{(1 + |\cdot|^2)^{t/2}} \in M_{p,q} \quad \text{if } t > s + n.$$

where $\psi_R(\cdot)$ is

$$\psi_R(\xi) = \int_{\mathbb{R}^n \setminus \{0\}} (1 - \cos(y\xi))\chi_{B(0)}(y)\nu(dy) \tag{2.40}$$

and the norms $\|u\|_{\psi,s}$ and $\|u\|_{\psi,R,s} = \|(id + \psi_R(D))^{s/2}u\|_p$ are equivalent (see [J2], p.281-282). For the next theorem see [J2], Theorem 3.3.28.

Theorem 2.2.7. Let ψ_1, ψ_2 be of type 1. Further let $s, r \in \mathbb{R}$ and $1 < p, q < \infty$. Then the continuous embedding

$$H^{\psi_1,s}_p \hookrightarrow H^{\psi_2,r}_q$$

holds if and only if $m = (1 + \psi_2)^{r/2}(1 + \psi_1)^{-s/2} \in M_{pq}$.

Using this theorem we can prove (see also Lemma 3.3.31 and Corollary 3.3.34 [J2])

Theorem 2.2.8. Let $t, s > 0$ and ψ be of type 1. Then

$$\tilde{H}^{\psi,s}_{p,+} \hookrightarrow \tilde{H}^{\psi,t}_{p+}, \quad \text{and} \quad H^{\psi,s}_{p,+} \hookrightarrow H^t_{p+}$$

if $s > \frac{t+n}{r_0}$.

Proof. The first embedding follows from the definition of the space $\tilde{H}^{\psi,s}_{p,+}$ and Theorem 2.2.7, since the operator of the embedding (id) does not change the support of a function.

From Theorem 2.2.7 we have

$$\|u\|_{|\cdot|^2,t} \leq c\|u\|_{\psi,s}, \quad s > \frac{t+n}{r_0}, \tag{2.41}$$

Consider u as an extension of an element of $H^{\psi,s}_{p,+}$ (or $H^t_{p,+}$) to $\mathbb{R}^n$. Taking in (2.41) inf over all such extensions, we obtain the statement for $H^{\psi,s}_{p,+}$ and $H^t_{p,+}$. □

Note that if ψ satisfies assumptions **A1** and **A2**, then the statement of Theorem 2.2.8 holds if

$$(1+\psi(\xi))^{s/2} \geq (1+|\xi|^2)^{t/2},$$

see the proof of Theorem 2.1.21.

For $f \in H$, H is a subspace of H^t_p, $t > \frac{1}{p}$, define by $rest_t f$ the operator

$$rest_t f = \{f(x',0), \frac{\partial f(x',0)}{\partial x_n}, \dots, \frac{\partial^{[t-\frac{1}{p}]^-} f(x',0)}{\partial x_n^{[t-\frac{1}{p}]^-}}\},$$

where $[x]^-$ denotes the largest integer, less then x. From Theorem 2.2.8 we can derive another embedding theorem

Theorem 2.2.9. Let $1 < p < \infty$, ψ is of type 1 and satisfies (2.39), $s > \frac{(k+n)p+n}{pr_0}$, $k \in \mathbb{N}$. Then

$$H^{\psi,s}_{p,+} \hookrightarrow C^k_\infty(\mathbb{R}^n_{0+}) \quad \text{and} \quad \tilde{H}^{\psi,s}_{p,+} \hookrightarrow \dot{C}^k_\infty(\mathbb{R}^n_{0+})$$

where $\dot{C}^k_\infty(\mathbb{R}^n_{0+}) = \{f : f \in C^k_\infty(\mathbb{R}^n_{0+}),\ rest_t f = 0\}$, $C^k_\infty(\mathbb{R}^n_{0+}) = C^k_\infty(\mathbb{R}^n)\big|_{\mathbb{R}^n_{0+}}$, $k + \frac{n}{p} < t < sr_0 - n$.

Proof. From Theorem 2.2.8 we know that $H^{\psi,s}_{p,+} \hookrightarrow H^t_{p,+}$ and $\tilde{H}^{\psi,s}_{p,+} \hookrightarrow \tilde{H}^t_{p,+}$ for $s > \frac{t+n}{r_0}$. Since $H^t_{p,+} \hookrightarrow C^k_\infty(\mathbb{R}^n_{0+})$, $t > k + \frac{n}{p}$, which follows from Theorem 4.6.2 [T2], we have $H^{\psi,t}_{p,+} \hookrightarrow C^k_\infty(\mathbb{R}^n_{0+})$ for $s > \frac{(k+n)p+n}{pr_0}$.

Consider now $\tilde{H}^{\psi,t}_{p,+}$. Let t not be equal to $\frac{1}{p} + k$, where k is an integer. Then by Theorems 2.9.3.a and 2.10.3.a [T2]

$$\tilde{H}^t_{p,+} = \overset{\circ}{H}{}^t_{p,+} = \{f : f \in H^t_{p,+}, \quad rest_t f = 0\}$$

which gives us the second statement of the Theorem. □

Using Theorem 2.2.8 it is easy to find dense subsets in $H^{\psi,t}_{p,+}$ and $\tilde{H}^{\psi,t}_{p,+}$.

Theorem 2.2.10. a) The set $C_0^\infty(\mathbb{R}^n_{0+}) = C_0^\infty(\mathbb{R}^n)\big|_{\mathbb{R}^n_{0+}}$ is dense in $H^{\psi,s}_{p,+}$ for all $s > 0$.

b) The set $C_0^\infty(\mathbb{R}^n_+) = \{f : f \in C_0^\infty(\mathbb{R}^n), \operatorname{supp} f \subset \mathbb{R}^n_{0+}\}$ is dense in $\tilde{H}^{\psi,s}_{p,+}$ for all $s > 0$.

Proof. a) From Lemma 2.1.6 we have that $C_0^\infty(\mathbb{R}^n)$ is dense in $H^{\psi,s}_p$. Let $f \in H^{\psi,s}_p$. Then given $\varepsilon > 0$ there exists $\varphi_0 \in C_0^\infty(\mathbb{R}^n)$ such that $\|f - \varphi_0\|_{\psi,s} < \varepsilon$. Define $g = f\big|_{\mathbb{R}^n_{0+}}$, $\varphi_1 = \varphi_0\big|_{\mathbb{R}^n_{0+}}$. Then $\|g - \varphi_1\|_{\psi,s,+} \le \|f - \varphi_0\|_{\psi,s} < \varepsilon$, which shows the density of $C_0^\infty(\mathbb{R}^n_{0+})$ in $H^{\psi,s}_{p,+}$.

b) The proof follows from Theorem 2.2.8 and Corollary 2.10.3/1 [T2], in which it was proved that $C_0^\infty(\mathbb{R}^n_+)$ is dense in $\tilde{H}^t_{p,+}$. □

Denote by $L(A, B)$ the space of continuous linear operators from A to B, where A and B are normed vector spaces.

For the next definition see [T2], §1.2.4.

Definition 2.2.11. Let A and B be two complex Banach spaces. The operator $R \in L(A, B)$ is called a retraction, if there exists an operator $S \in L(B, A)$ such that

$$RS = I. \tag{2.42}$$

An operator S such that (2.42) holds is called a coretraction which corresponds to R.

Now we want to prove the existence of a retraction and a coretraction in the spaces $\tilde{H}^{\Re,s}_{p,+}$.

Theorem 2.2.12. Let $1 < p < \infty$, $s \in \mathbb{R}$. Then for all s there exists a coretraction from $\tilde{H}^{\Re,s}_{p,+}$ to $H^{\Re,s}_p$, and for all s, $|s| < 2N$ there exists a retraction from $H^{\Re,s}_p$ to $\tilde{H}^{\Re,s}_{p,+}$.

Before we prove this theorem we recall Theorem 2.10.4/2 proved in [T2].

Theorem 2.2.13. Let $1 < p < \infty$, $-\infty < s < \infty$. Then the mapping

$$Sf = \begin{cases} f, & x_n \ge 0 \\ 0 & x_n < 0 \end{cases}$$

is the coretraction from $\tilde{H}^s_{p,+}$ to H^s_p which corresponds to the retraction R, which is the extension of an operator $\tilde{R}$ to a continuous operator from H^s_p to $\tilde{H}^s_{p,+}$, $|s| < N$, where

$$\tilde{R}\varphi(x) = \chi_+(x)\big(\varphi(x) - \sum_{j=1}^{N+1} a_j \varphi(x'; -\lambda_j x_n)\big).$$

Here $\varphi \in C_0^\infty(\mathbb{R}^n)$, χ_+ is the characteristic function of $\mathbb{R}^n_{0+}$, $0 < \lambda_1 < \cdots < \lambda_{N+1} < \infty$ and the coefficients a_j are such that

$$\frac{\partial^k}{\partial x_n^k}\varphi(x', x_n)\Big|_{x_n=0} = \sum_{j=1}^{N+1} a_j \frac{\partial^k}{\partial x_n^k}\varphi(x', -\lambda_j x_n)\Big|_{x_n=0}.$$

We also refer to Theorem 2.10.3.a [T2] in which it was proved that

$$J_s f = F^{-1}(ix_n + (1+|x'|^{1/2})^s \hat{f})$$

is the isomorphic mapping from $\tilde{H}^{\sigma}_{p,+}$ to $\tilde{H}^{\sigma-s}_{p,+}$.

Proof of Theorem 2.2.12. From Theorem 2.1.22 we can derive, applying $(-A_+)^{\pm\alpha}$ N times, that $(-A_+)^{\alpha N}$: $H^{\Re,2N}_p \to L_p$ isomorphically, and then, from Theorem 2.2.5 we have, that $(-A_+)^{\alpha N}$: $\tilde{H}^{\Re,2N}_{p,+} \to L_p$ isomorphically. Then, using Theorem 2.2.13 and Theorem 2.10.3.a [T2] we can construct the diagrams

$$\begin{array}{ccccc} H^{\psi,2N}_p & \xrightarrow{(-A_+)^{\alpha N}} & L_p & \xrightarrow{J_s^{-1}} & H^s_p \\ \big\downarrow R_0 & & & & \big\downarrow R \\ \tilde{H}^{\psi,2N}_{p,+} & \xleftarrow{(-A_+)^{-\alpha N}} & L_{p,+} & \xleftarrow{J_s} & \tilde{H}^s_{p,+} \end{array}$$

and

$$\begin{array}{ccccc} H^{\psi,2N}_p & \xleftarrow{(-A_+)^{-\alpha N}} & L_p & \xleftarrow{J_s} & H^s_p \\ \big\uparrow S_0 & & & & \big\uparrow S \\ \tilde{H}^{\psi,2N}_{p,+} & \xrightarrow{(-A_+)^{\alpha N}} & L_{p,+} & \xrightarrow{J_s^{-1}} & \tilde{H}^s_{p,+} \end{array}$$

and without loss of generality we can put $s = 2N$ in the definition of J_s.

Since all the operators are isomorphisms, $S_0 = (-A_+)^{-\alpha N} J_{2N} S \, J_{2N}^{-1} (-A_+)^{\alpha N}$ is the coretraction from $\tilde{H}^{\Re,2N}_{p,+}$ to $H^{\Re,2N}_p$ which corresponds to the retraction $R_0 = (-A_+)^{-\alpha N} J_{2N} R \, J_{2N}^{-1} (-A_+)^{\alpha N}$. The same is true for the spaces $\tilde{H}^{\Re,-2N}_{p,+}$ and $H^{\Re,-2N}_p$. Then, after applying Theorem 1.2.4 [T2] we obtain that $R_0 S_0 = I$, or that S_0 and R_0 are the coretraction and the retraction for the spaces $\tilde{H}^{\Re,s}_{p,+}$ and $H^{\Re,s}_p$, $|s| < 2N$. □

The interpolation theorem for the spaces $\tilde{H}^{\Re,s}_{p,+}$ follows after applying Theorem 1.17.1/1 [T2]:

Theorem 2.2.14. Let $\{A_0, A_1\}$ be an interpolation couple, and B is a complement subspace of $A_0 + A_1$, and the projection to B is from $L(\{A_0, A_1\}, \{A_0, A_1\})$. Let F be an arbitrary interpolation functor. Then $(\{A_0 \cap B, A_1 \cap B\})$ is also an interpolation couple and

$$F(\{A_0 \cap B, A_1 \cap B\}) = F(\{A_0, A_1\}) \cap B.$$

(See [T2] for the definition of the complement subspace).

Then, if we take $A_0 = H^{\Re,s_0}_{p_0}$, $A_1 = H^{\Re,s_1}_{p_1}$, $B = \{f \in A_0 + A_1, \operatorname{supp} f \subset \mathbb{R}^{n+1}_+\}$ and since for finite s the restriction to B is a retraction, we have

Theorem 2.2.15. Let $1 < p_0, p_1 < \infty$, $-\infty < s_1, s_0 < \infty$, $0 < \theta < 1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $s = (1-\theta)s_0 + \theta s_1$. Then

$$[\tilde{H}^{\Re,s_0}_{p_0,+}, \tilde{H}^{\Re,s_1}_{p_1,+}]_\theta = \tilde{H}^{\Re,s}_{p,+}.$$

Remark 2.2.16. Since the space $H^{\Re,2}_{p,+}$ is the factor space of $H^{\Re,s}_p$ with respect to $\tilde{H}^{\Re,2}_{p,-}$, then for $1 < p_0, p_1 < \infty$, $-\infty < s_1, s_0 < \infty$, $0 < \theta < 1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $s = (1-\theta)s_0 + \theta s_1$, we have

$$[H^{\Re,s_0}_{p_0,+}, H^{\Re,s_1}_{p_1,+}]_\theta = H^{\Re,s}_{p,+}.$$

The proof follows from the theorem below, taken from [T2] (see Theorem 1.17.2):

Theorem 2.2.17. Let $\{A_0, A_1\}$ be an interpolation couple, and C be complemented subspace of $A_1 + A_2$, and Q is a projection on C such that $Q \in L(\{A_0, A_1\}, \{A_0, A_1\})$. Further, let F be an arbitrary interpolating functor. Then $\{A_0/A_0 \cap C, A_1/A_1 \cap C\}$ is n interpolation couple as well, and

$$F(\{A_0/A_0 \cap C, A_1/A_1 C\}) = F(\{A_0, A_1\})/(F(\{A_0, A_1\}) \cap C).$$

We only need to put $A_0 = H^{\Re,s_0}_{p_0}$, $A_1 = H^{\Re,s_1}_{p_1}$, $C = \{f \in A_0 + A_1,\ \operatorname{supp} f \subset \mathbb{R}^{n+1}_+\}$.

Theorem 2.2.18. For all s, $-\infty < s < +\infty$, $1 < p < \infty$, ψ is of type 1, the restriction from $H^{\Re,s}_p$ to $H^{\Re,s}_{p,+}$ is a retraction and for all N there exists a coretraction which does not depend on p and s, $|s| < N$.

Proof. The proof of this theorem is a modification of the proof of Theorem 2.2.12 by taking the coretraction S_1

$$S_1 f = \begin{cases} f, & x_n \geq 0 \\ \sum_{j=1}^{N+1} a_j f(x'; -\lambda_j x_n), & x_n < 0 \end{cases}$$

instead of S, S_1 corresponds to the retraction R_1 -the restriction to the half-space $\mathbb{R}^n_{0+}$, and by applying the operators $I_s f = F^{-1}((1+|\xi'|^2)^{1/2} - i\xi_n)^s \hat{f})$ and $(-A_-)^{\pm\alpha}$ to construct the retraction and coretraction between spaces $H^{\Re,s}_{p,+}$ and $H^{\Re,s}_p$. $\square$

In the end of this section we give another consequence of Theorem 2.1.21.

Let ψ be a continuous negative definite function. By the same argument as in the proof of Theorem 2.1.21 we can prove (if **A1** and **A2** hold for ψ), that

$$M(\xi) = \frac{\psi(\xi) + a}{\psi(\xi) + b} \tag{2.43}$$

is a Fourier multiplier, since

$$\sup_{\xi \in \mathbb{R}^n} \left| \xi_1 \ldots \xi_k \partial^k_{1\ldots k} M(\xi) \right| < \infty. \tag{2.44}$$

Moreover, for $a < b$ the function $\xi \mapsto M(\xi)$ is again a continuous negative definite function (see [J1], Corollary 3.6.14).

Similarly, the function

$$M_1(\xi) = \frac{\psi^*(\xi) + a}{\psi(\xi) + b} \tag{2.45}$$

where ψ^* is another continuous negative definite function which also satisfies **A1** and **A2** as well as

$$c_1\psi \le \psi^* \le c_2\psi, \tag{2.46}$$

is an L_p -multiplier. Indeed, we can calculate (as we did in the proof of Theorem 2.1.21) that $(M_1)^{(k)}_{k,\ldots,1,n+1}$ consists of the terms of the form:

$$\frac{P_l(\psi_1^{*(1)}, \ldots, \psi_l^{*(1)}, \ldots, \psi_{l,\ldots,1}^{*(l)}, \psi_1', \ldots, \psi_l', \ldots, \psi_{l,\ldots,1}^{(l)})}{(\psi + b)^r}, \quad 0 \le l \le r \le n,$$

where

$$\begin{aligned} &P_l(\psi_1^{*(1)}, \ldots, \psi_l^{*(1)}, \ldots, \psi_{l,\ldots,1}^{*(l)}, \psi_1', \ldots, \psi_l', \ldots, \psi_{l,\ldots,1}^{(l)}) \\ &\qquad = \sum_{l_1+\cdots+l_s=l} c(l_1, \ldots, l_s)\psi^{(l_1)} \ldots \psi^{(l_i)}\psi^{*(l_{i+1})} \ldots \psi^{*(l_s)}. \end{aligned}$$

Since both ψ and ψ^* satisfy **A1** and **A2**, and (2.46) holds, we find

$$\sup_{\xi \in \mathbb{R}^n} \left| \xi_1 \ldots \xi_l \xi_{n+1} \frac{P_l}{(b + \psi)^r} \right| < \infty$$

for all $0 < l < r \le n + 1$, i.e. M_1 is an L_p -Fourier multiplier. We obtained

Corollary 2.2.19. Let ψ_1 and ψ_2 be two continuous negative definite functions satisfying **A1** and **A2**, and such that

$$0 < C \le \frac{1 + \psi_1}{1 + \psi_2} \le D < \infty.$$

Then $H_p^{\psi_1,s} = H_p^{\psi_2,s}$ for $-\infty < s < \infty$, $1 < p < \infty$.

Proof. Clearly, from **A1** and **A2** the function

$$\frac{1 + \psi_1}{1 + \psi_2},$$

is n times differentiable, and the Corollary follows from the results above. □

Chapter 3

Some generators of L_p-sub-Markovian semigroups

3.1 Generators of L_p-sub-Markovian semigroups

Now we can formulate results similar to Theorem 2.1.24 for the half-plane. Consider the operator $-A_+$ with the symbol $\chi_+(\xi) = \psi(\xi') + i\xi_{n+1}$; where ψ is of type 1 and satisfies **A1** and **A2**. By Theorem 2.1.12 we have that $(-\psi(D_{x'}), H_p^{\psi,2})$ is the generator of an L_p-sub-Markovian semigroup. Further, $(-\frac{d}{dx}, H^1_{p,+})$ is closed as well as $(-\frac{d}{dx}, \tilde{H}^1_{p,+})$, where $\tilde{H}^1_{p,+} = \overset{\circ}{H}{}^1_{p,+} = \overline{C_0^\infty}^{\|\cdot\|_{H^1_p}}(\mathbb{R}_+)$ (see Theorem 2.9.8 [T2] for the definition of $\overset{\circ}{H}{}^s_{p,+}$ and Theorem 2.10.3.b [T2] for the proof that if $s \neq n + \frac{1}{p}$ then $\tilde{H}^s_{p,+}(\mathbb{R}^n) = \overset{\circ}{H}{}^s_{p,+}(\mathbb{R}^n)$). The spaces $\tilde{H}^1_{p,+}$ and $H^1_{p,+}$ are dense in $L_p(\mathbb{R}_+)$ (see the proof of Theorem 2.2.10) and since $-\frac{d}{dx}$ is a Dirichlet operator on H^1_p (see the proof of Lemma 2.1.14), it is also a Dirichlet operator on $H^1_{p,+}$ and on $\tilde{H}^1_{p,+}$.

We can solve the equation

$$\lambda u + u' = f$$

uniquely for any $\lambda > 0$ and $f \in L_p(\mathbb{R}_{0+})$ with Neumann or Dirichlet boundary conditions (we may take Neumann or Dirichlet conditions when $u \in H^1_{p,+}$ and zero Dirichlet conditions when $u \in \tilde{H}^1_{p,+}$, because for any $u \in \tilde{H}^1_{p,+}$ $u(0) = 0$).

Therefore, by the Hille-Yosida theorem, $(-\frac{d}{dx}, H^1_{p,+})$ and $(-\frac{d}{dx}, \tilde{H}^1_{p,+})$ are the generators of L_p-sub-Markovian semigroups.

In the following assume that

$$\psi(\xi') \geq (1 + |\xi'|^2)^\delta$$

for some $\delta > 0$.

Applying Remark 1.2.4 we have

Theorem 3.1.1. The operators $(-A_+, H^{\chi,2,1}_{p,+})$ and $(-A_+, \tilde{H}^{\chi,2,1}_{p,+})$ are generators of L_p-sub-Markovian semigroups $(T^{(1)}_t)_{t\geq 0}$ and $(T^{(2)}_t)_{t\geq 0}$ respectively. In addition we find

$$\begin{aligned} T^{(1)}_t f(x) = &\int_{\mathbb{R}^n} \frac{f(x'-y', x_{n+1}-t)\mathcal{W}_t(y')\,dy'}{(2\pi)^{\frac{n}{2}}} 1_{\{x_{n+1}\geq t\}}(x) \\ &+ \int_{\mathbb{R}^n} \frac{f(x'-y', 0)\mathcal{W}_{x_{n+1}}(y')\,dy'}{(2\pi)^{\frac{n}{2}}} 1_{\{x_{n+1}< t\}}(x) \end{aligned} \tag{3.1}$$

and

$$T^{(2)}_t f(x) = \int_{\mathbb{R}^n} \frac{f(x'-y', x_{n+1}-t)\mathcal{W}_t(y')\,dy'}{(2\pi)^{\frac{n}{2}}} 1_{\{x_{n+1}\geq t\}}(x) \tag{3.2}$$

where $\mathcal{W}_t = F^{-1}(e^{-\psi(\xi')t})$ exists as the function ψ satisfies **A1** and **A2**, and $f(x', x_{n+1}) = 0$ for $x_{n+1} \leq 0$ in (3.2).

Proof. First we show that $H^{\chi,2,1}_{p,+}$ and $\tilde{H}^{\chi,2,1}_{p,+}$ are the domains of A_+, i.e. that $\overline{H^{\psi,2}_p \otimes H^1_{p,+}}^{\|\cdot\|_A} = H^{\chi,2,1}_{p,+}$ and $\overline{H^{\psi,2}_p \otimes \tilde{H}^1_{p,+}}^{\|\cdot\|_A} = \tilde{H}^{\chi,2,1}_{p,+}$.

Since

$$H^{\psi,2}_p \otimes \tilde{H}^1_{p,+} = \{f : f \in H^{\psi,2}_p \otimes H^1_p, \quad \operatorname{supp} f \subset \mathbb{R}^{n+1}_{0+}\},$$

then

$$\overline{H^{\psi,2}_p \otimes \tilde{H}^1_{p,+}}^{\|\cdot\|_A} = \{f : f \in \overline{H^{\psi,2}_p \otimes H^1_p}^{\|\cdot\|_A}, \quad \operatorname{supp} f \subset \mathbb{R}^{n+1}_{0+}\} = \tilde{H}^{\chi,2,1}_{p,+}$$

and analogously since

$$H^{\psi,2}_p \otimes H^1_{p,+} = \{f : \quad \text{there exists} \quad g \in H^{\psi,2}_p \otimes H^1_p, \quad g\big|_{\mathbb{R}^{n+1}_{0+}} = f\},$$

then

$$\overline{H^{\psi,2}_p \otimes H^1_{p,+}}^{\|\cdot\|_A} = \{f : \quad \text{there exists} \quad g \in \overline{H^{\psi,2}_p \otimes H^1_p}^{\|\cdot\|_A}, \quad g\big|_{\mathbb{R}^{n+1}_{0+}} = f\} = H^{\chi,2,1}_{p,+}.$$

By Remark 1.2.4 we obtain that $(-A_+, H^{\chi,2,1}_{p,+})$ and $(-A_+, \tilde{H}^{\chi,2,1}_{p,+})$ are generators of L_p-sub-Markovian semigroups. To find these semigroups, we will do the following. Consider the equation

$$(\lambda + A_+)f(x) = g(x), \quad x \in \mathbb{R}^{n+1}_+. \tag{3.3}$$

In the following calculations denote by $\hat{g}(\xi, \eta)$ the function $L_{x_{n+1}\to\eta}F_{x'\to\xi'}(g(x', x_{n+1}))$, where $L_{x_{n+1}\to\eta}$ is the Laplace transform, $F_{x'\to\xi'}$ the Fourier transform, and denote $\hat{g}(\xi', 0) = F_{x'\to\xi'}(g(x', 0))$, $\hat{g}(\xi', x_{n+1}) = F_{x'\to\xi'}(g(x', x_{n+1}))$.

Taking the Fourier transform $F_{x'\to\xi'}$ of the left-hand part of (3.3),

$$\begin{aligned}
&F_{x'\to\xi'}\big((\lambda+A_+)f\big)(\xi',x_{n+1})\\
&\quad=(2\pi)^{-n/2}\int_{\mathbb{R}^n}e^{-i(\xi',x')}\Big[(\lambda+\psi(D_{x'}))+\frac{\partial}{\partial x_{n+1}}f(x',x_{n+1})\Big]\,dx'\\
&\quad=\lambda\hat f(\xi',x_{n+1})+\psi(\xi')\hat f(\xi',x_{n+1})+\frac{\partial}{\partial x_{n+1}}\hat f(\xi',x_{n+1}),
\end{aligned}$$

and then the Laplace transform $L_{x_{n+1}\to\eta}$,

$$\begin{aligned}
&L_{x_{n+1}\to\eta}F_{x'\to\xi'}\big((\lambda+A_+)f\big)(\xi',x_{n+1})\\
&\quad=\int_0^\infty e^{-\eta x_{n+1}}F_{x'\to\xi'}\big((\lambda+A_+)f\big)(\xi',x_{n+1})\,dx_{n+1}\\
&\quad=(\lambda+\psi(\xi'))\int_0^\infty e^{-\eta x_{n+1}}\hat f(\xi',x_{n+1})dx_{n+1}\\
&\qquad+\int_0^\infty e^{-\eta x_{n+1}}\frac{\partial}{\partial x_{n+1}}\hat f(\xi',x_{n+1})dx_{n+1}\\
&\quad=(\lambda+\psi(\xi')+\eta)\hat f(\xi,\eta)-\hat f(\xi',0),
\end{aligned}$$

we finally derive that

$$(\lambda+\psi(\xi')+\eta)\hat f(\xi,\eta)-\hat f(\xi',0)=\hat g(\xi,\eta),$$

or

$$\hat f(\xi,\eta)=\frac{\hat g(\xi,\eta)+\hat f(\xi',0)}{(\lambda+\psi(\xi')+\eta)} \tag{3.4}$$

is the $L_{x_{n+1}\to\eta}F_{x'\to\xi'}$-transform of the solution to (3.3) with some boundary conditions. Consider the operator

$$\begin{aligned}
T_t^{(1)}g(x)&=\int_{\mathbb{R}^n}\frac{g(x'-y',x_{n+1}-t)\mathcal{W}_t(y')\,dy'}{(2\pi)^{\frac{n}{2}}}1_{\{x_{n+1}\ge t\}}(x)\\
&\quad+\int_{\mathbb{R}^n}\frac{g(x'-y',0)\mathcal{W}_{x_{n+1}}(y')\,dy'}{(2\pi)^{\frac{n}{2}}}1_{\{x_{n+1}<t\}}(x)
\end{aligned}$$

where $g\in L_p(\mathbb{R}^{n+1}_{0+})$.

It is bounded in $L_p(\mathbb{R}^{n+1}_{0+})$. Indeed, since $\mathcal{W}_t(y')=F^{-1}(e^{-t\psi(\xi')})$ is an L_p-multiplier, then $T_t^{(1)}g(\cdot,x_{n+1})\in L_p(\mathbb{R}^n)$ for $g\in L_p(\mathbb{R}^{n+1}_{0+})$. Further, the first term in the representation of $T_t^{(1)}g(x',\cdot)$ belongs to $L_p(\mathbb{R}_{0+})$ since $g(x',\cdot)$ does, and the second is bounded and with finite support with respect to x_{n+1}.

Let us do the following. Let $S(\mathbb{R}^{n+1}_{0+})=S(\mathbb{R}^{n+1})\Big|_{\mathbb{R}^{n+1}_{0+}}$. If we show that $g\in S(\mathbb{R}^{n+1}_{0+})$ the resolvent

$$R_\lambda g=\int_0^\infty e^{-\lambda t}T_t^{(1)}g(x)\,dt$$

leads to (3.4), then by density of $S(\mathbb{R}^{n+1}_{0+})$ in $L_p(\mathbb{R}^{n+1}_{0+})$ and since A_+ is an isomorptism between $H_p^{\chi,2,1}$ and $L_p(\mathbb{R}^{n+1})$, we obtain that $(T_t^{(1)})_{t\geq 0}$ defined on $L_p(\mathbb{R}^{n+1}_{0+})$ is a semigroup generated by $-A_+$.

We may rewrite $T_t^{(1)}g(x)$ as

$$T_t^{(1)}g(x) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{i(x',\xi')-t\psi(\xi')}\{\hat{g}(\xi',x_{n+1}-t)1_{[0,x_{n+1})}(t) + \hat{g}(\xi',0)1_{[x_{n+1},\infty)}(t)\}\,d\xi'.$$

Since

$$e^{-\lambda t+i(x',\xi')-t\psi(\xi')}\{\hat{g}(\xi',x_{n+1}-t)1_{[0,x_{n+1})}(t)+\hat{g}(\xi',0)1_{[x_{n+1},\infty)}(t)\}\,d\xi'.$$

belongs to $L_1(\mathbb{R}^n)$ w.r.t. ξ', and to $L_1(0,\infty)$ w.r.t. t, then we can apply Fubini theorem and obtain

$$\begin{aligned}L_{t\to\lambda}T_t^{(1)}g &= (2\pi)^{-n/2}\int_{\mathbb{R}^n}\int_0^\infty e^{i(x',\xi')-t\psi(\xi')-\lambda t}\{\hat{g}(\xi',x_{n+1}-t)1_{[0,x_{n+1})}(t)\\ &\quad + \hat{g}(\xi',0)1_{[x_{n+1},\infty)}(t)\}\,dtd\xi'\\ &= F^{-1}_{\xi'\to x'}[L_{t\to\lambda}\{e^{-t\psi(\xi')}\{\hat{g}(\xi',x_{n+1}-t)1_{[0,x_{n+1})}(t)+\hat{g}(\xi',0)1_{[x_{n+1},\infty)}(t)\}\}].\end{aligned}$$

But $F_{x'\to\xi'}F^{-1}_{\xi'\to x'} = I$, and we come to

$$\begin{aligned}&L_{x_{n+1}\to\eta}F_{x'\to\xi'}L_{t\to\lambda}(T_t^{(1)})\\ &\quad= L_{x_{n+1}\to\eta}L_{t\to\lambda}\{e^{-t\psi(\xi')}\{\hat{g}(\xi',x_{n+1}-t)1_{[0,x_{n+1})}(t)+\hat{g}(\xi',0)1_{[x_{n+1},\infty)}(t)\}\}.\end{aligned}$$

Actually from now the order of integration does not matter, but we can change

$$L_{x_{n+1}\to\eta}L_{t\to\lambda} = L_{t\to\lambda}L_{x_{n+1}\to\eta},$$

since the function

$$e^{-\lambda t-x_{n+1}\eta-t\psi(\xi')}\{\hat{g}(\xi',x_{n+1}-t)1_{[0,x_{n+1})}(t)+\hat{g}(\xi',0)1_{[x_{n+1},\infty)}(t)\}$$

belongs to $L_1(\mathbb{R}^n)$ w.r.t. ξ', and to $L_1(0,\infty)$ w.r.t. t and x_{n+1}.

We want to check if the Laplace transform of semigroup $(T_t^{(1)})_{t\geq 0}$ gives us the solution to (3.3) with some boundary conditions. Therefore we will calculate

$$L_{x_{n+1}\to\eta}F_{x'\to\xi'}L_{t\to\lambda}(T_t^{(1)}g). \tag{3.5}$$

Applying the Fubini theorem to (3.5) we get:

$$\begin{aligned}
&L_{t\to\lambda}L_{x_{n+1}\to\eta}F_{x'\to\xi'}(T_t^{(1)}g)\\
&= L_{t\to\lambda}L_{x_{n+1}\to\eta}\big[1_{[t,\infty)}(x_{n+1})\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{e^{-i(\xi',x')}g(x'-y',x_{n+1}-t)\mathcal{W}_t(y')}{(2\pi)^n}\,dy'\,dx'\big]\\
&+\quad L_{t\to\lambda}L_{x_{n+1}\to\eta}\big[1_{[0,t)}(x_{n+1})\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{e^{-i(\xi',x')}g(x'-y',0)\mathcal{W}_{x_{n+1}}(y')}{(2\pi)^n}\,dy'\,dx'\big]\\
&= L_{t\to\lambda}L_{x_{n+1}\to\eta}\big[\hat{g}(\xi',x_{n+1}-t)e^{-t\psi(\xi')}1_{[t,\infty)}(x_{n+1})+1_{[0,t)}(x_{n+1})\hat{g}(\xi',0)e^{-x_{n+1}\psi(\xi')}\big]\\
&= L_{t\to\lambda}\big[e^{-t\psi(\xi')}\int_0^\infty e^{-\eta x_{n+1}}\hat{g}(\xi',x_{n+1}-t)1_{[t,\infty)}(x_{n+1})\,dx_{n+1}\\
&\quad+\hat{g}(\xi',0)\int_0^\infty e^{-\eta x_{n+1}}e^{-x_{n+1}\psi(\xi')}1_{[0,t)}(x_{n+1})\,dx_{n+1}\big]\\
&= L_{t\to\lambda}\big[e^{-t\psi(\xi')}\int_0^\infty e^{-\eta\tau}\hat{g}(\xi',\tau)\,d\tau+\frac{\hat{g}(\xi',0)}{\psi(\xi')+\eta}(1-e^{-t(\psi(\xi')+\eta)})\big]\\
&= L_{t\to\lambda}\big[e^{-t\psi(\xi')}\hat{g}(\xi',\eta)+\frac{\hat{g}(\xi',0)}{\psi(\xi')+\eta}(1-e^{-t(\psi(\xi')+\eta)})\big]\\
&= \frac{\hat{g}(\xi',\eta)}{\lambda+\psi(\xi')+\eta}+\frac{\hat{g}(\xi',0)}{\psi(\xi')+\eta}\big[\frac{1}{\lambda}-\frac{1}{\lambda+\psi(\xi')+\eta}\big]\\
&= \frac{\hat{g}(\xi',\eta)}{\lambda+\psi(\xi')+\eta}+\frac{\hat{g}(\xi',0)}{\lambda(\lambda+\psi(\xi')+\eta)}.
\end{aligned}$$

From (3.3) we see, that if $A_+f(x',0)=0$, then

$$\lambda f(x',0)=g(x',0).$$

Therefore

$$L_{t\to\lambda}L_{x_{n+1}\to\eta}F_{x'\to\xi'}(T_t^{(1)}g)=\frac{\hat{g}(\xi',\eta)}{\lambda+\psi(\xi')+\eta}+\frac{\hat{f}(\xi',0)}{\lambda+\psi(\xi')+\eta},$$

which equals to (3.4).

Therefore, the Laplace transform of $T_t^{(1)}g$ indeed gives the solution to (3.3) with the boundary condition $A_+f(x',0)=0$. Since there is one-to-one correspondence between the images and pre-images of Fourier-Laplace transform, we conclude that the operators $(T_t^{(1)})_{t\geq 0}$ form a strongly continuous contraction semigroup with the generator $(-A_+,H_p^{\chi,2.1})$, which proves (3.1).
To prove (3.2) we only need to put $g(x',0)=0$, $x'\in\mathbb{R}^n$. ☐

Remark 3.1.2. The semigroup generated by $(-A_+,H_p^{\chi,2.1})$ can be different, if we pose different boundary conditions for (3.3). If we take

$$\frac{\partial}{\partial x_{n+1}}f(x',0)=0,$$

then the semigroup generated by $(-A_+, H_p^{\chi,2.1})$ is the following:

$$T_t^{(1')}g(x) = \int_{\mathbb{R}^n} \frac{f(x'-y', x_{n+1}-t)\mathcal{W}_t(y')\,dy'}{(2\pi)^{\frac{n}{2}}} 1_{[t,\infty)}(x_{n+1}) + \int_{\mathbb{R}^n} \frac{f(x'-y',0)\mathcal{W}_t(y')\,dy'}{(2\pi)^{\frac{n}{2}}} 1_{[0,t)}(x_{n+1}). \tag{3.6}$$

Proof. Indeed,

$$\begin{aligned}
&L_{t\to\lambda}L_{x_{n+1}\to\eta}F_{x'\to\xi'}(T_t^{(1')}g)\\
&= L_{t\to\lambda}L_{x_{n+1}\to\eta}\big[1_{\{x_{n+1}\geq t\}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{e^{-i(\xi',x')}g(x'-y',x_{n+1}-t)\mathcal{W}_t(y')}{(2\pi)^n}\,dy'\,dx'\big]\\
&+\quad L_{t\to\lambda}L_{x_{n+1}\to\eta}\big[1_{\{0\leq x_{n+1}<t\}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{e^{-i(\xi',x')}g(x'-y',0)\mathcal{W}_t(y')}{(2\pi)^n}\,dy'\,dx'\big]\\
&= L_{t\to\lambda}\big[e^{-t\psi(\xi')}\hat{g}(\xi',\eta)+\hat{g}(\xi',0)e^{-t\psi(\xi')}\int_0^t e^{-\eta x_{n+1}}\,dx_{n+1}\big]\\
&= L_{t\to\lambda}\big[e^{-t(\psi(\xi')+\eta)}\hat{g}(\xi',\eta)+\frac{\hat{g}(\xi',0)e^{-t\psi(\xi')}}{\eta}(1-e^{-\eta t})\big]\\
&= \hat{g}(\xi',\eta)\int_0^\infty e^{-t(\lambda+\psi(\xi')+\eta)}\,dt+\frac{\hat{g}(\xi',0)}{\eta}\int_0^\infty\big(e^{-t(\lambda+\psi(\xi'))}-e^{-t(\lambda+\psi(\xi')+\eta)}\big)\,dt\\
&= \frac{\hat{g}(\xi',\eta)}{\lambda+\psi(\xi')+\eta}+\frac{\hat{g}(\xi',0)}{(\lambda+\psi(\xi'))(\lambda+\psi(\xi')+\eta)},
\end{aligned}$$

and since in view of the zero Neumann boundary condition

$$\lambda f(x',0)+\psi(D_{x'})f(x',0)=g(x',0),$$

we obtain

$$L_{s\to\lambda}L_{x_{n+1}\to\eta}F_{x'\to\xi'}(T_t^{(1')}g)=\frac{\hat{g}(\xi',\eta)}{\lambda+\psi(\xi')+\eta}+\frac{\hat{f}(\xi',0)}{\lambda+\psi(\xi')+\eta}.$$

Therefore, the Laplace transform of $T_t^{(1')}g$, $g \in L_p(\mathbb{R}_{0+}^{n+1})$ is the solution to the equation (3.3) with zero Neumann boundary conditions, and thus the operator $(-A_+, H_p^{\chi,2.1})$ with this boundary condition is the generator of the L_p-sub-Markovian semigroup $(T_t^{(1')})_{t\geq 0}$. □

Theorem 3.1.3. The operator $(-A_-, H_{p,+}^{\chi,2,1})$ is the generator of L_p-sub-Markovian semigroup $(T_t^{(3)})_{t\geq 0}$:

$$T_t^{(3)}f(x)=\int_{\mathbb{R}^n}\frac{f(x'-y',x_{n+1}+t)\mathcal{W}_t(y')\,dy'}{(2\pi)^{\frac{n}{2}}}. \tag{3.7}$$

Proof. By the same considerations as we gave in the proof of Theorem 3.1.1 we may obtain that $(-A_-, H^{\chi,2,1}_{p,+})$ is a generator of some strongly continuous contraction semigroup. We show now, that (3.7) is one of (depends of the boundary condition) the possible semigroups. Proceeding as in the proof of Theorem 3.1.1,

$$\begin{aligned}
&L_{t\to\lambda}L_{x_{n+1}\to\eta}F_{x'\to\xi'}(T^{(3)}_t g) = L_{s\to\lambda}L_{x_{n+1}\to\eta}\big(\hat{g}(\xi', x_{n+1}+t)e^{-t\psi(\xi')}\big)\\
&= L_{t\to\lambda}\big(e^{-t(\psi(\xi')-\eta)}\int_0^\infty e^{-\eta(x_{n+1}+t)}\hat{g}(\xi', x_{n+1}+t)\,dx_{n+1}\big)\\
&= L_{t\to\lambda}\big(e^{-t(\psi(\xi')-\eta)}\int_{-t}^\infty e^{-\eta(x_{n+1}+t)}\hat{g}(\xi', x_{n+1}+t)\,dx_{n+1}\\
&\quad - e^{-t(\psi(\xi')-\eta)}\int_{-t}^0 e^{-\eta(x_{n+1}+t)}\hat{g}(\xi', x_{n+1}+t)\,dx_{n+1}\big)\\
&= L_{t\to\lambda}\big(\hat{g}(\xi',\eta)e^{-t(\psi(\xi')-\eta)}\big) + \int_0^\infty\int_0^t e^{-\lambda t - t\psi(\xi')+\eta x_{n+1}}\hat{g}(\xi', -x_{n+1}+t)\,dx_{n+1}\,dt\\
&= \frac{\hat{g}(\xi',\eta)}{\lambda+\psi(\xi')-\eta} + \int_0^\infty e^{\eta x_{n+1}}\int_{x_{n+1}}^\infty e^{-t(\psi(\xi')+\lambda)}\hat{g}(\xi', -x_{n+1}+t)\,dt\,dx_{n+1}\\
&= \frac{\hat{g}(\xi',\eta)}{\lambda+\psi(\xi')-\eta} + \int_0^\infty e^{-x_{n+1}(\lambda+\psi(\xi')-\eta)}\int_0^\infty e^{-\tau(\lambda+\psi(\xi))}\hat{g}(\xi',\tau)\,d\tau\,dx_{n+1}\\
&= \frac{\hat{g}(\xi',\eta)+\hat{g}(\xi',\lambda+\psi(\xi'))}{\lambda+\psi(\xi')-\eta}.
\end{aligned}$$

Thus, $L_{t\to\lambda}T^{(3)}_t g$ is the solution to $(\lambda + A_-)f = g$ with the boundary condition

$$f(x', 0) = F^{-1}_{\xi'\to x'}(\hat{g}(\xi', \lambda + \psi(\xi'))),$$

and therefore the semigroup $(T^{(3)}_t)_{t\geq 0}$ is generated by $(-A_-, H^{\chi,2,1}_{p+})$. □

Now let us consider the fractional power of $-A_+$, $(-A_+)^\alpha$, $0 < \alpha < 1$. First consider functions from $D_1 = \tilde{H}^{\chi,2,1}_{p,+}$ and $D_2 = H^{\chi,2,1}_{p,+}$. From Theorem 4.3.7 [J1] the domain of the generator A_+ of strongly continuous contraction semigroup is dense in $D((-A_+)^\alpha)$, and $D(A_+)$ is a core for $(-A_+)^\alpha$, $0 < \alpha < 1$. Then

$$\overline{D_1}^{\|\cdot\|_{(-A_+)^\alpha}} = \overline{D_1}^{\|\cdot\|_{\Re,2}} = \{f : \|f\|_{\Re,2} < \infty, \quad \operatorname{supp} f \subset \mathbb{R}^{n+1}_{0+}\} = \tilde{H}^{\Re,2}_{p,+}$$

and analogously

$$\begin{aligned}
\overline{D_2}^{\|\cdot\|_{(-A_+)^\alpha}} &= \overline{D_2}^{\|\cdot\|_{\Re,2}} = \overline{\{f : \exists g : \ \|g\|_{\chi,2,1} < \infty,\ g|_{\mathbb{R}^{n+1}_{0+}} = f\}}^{\|\cdot\|_{\Re,2}}\\
&= \{f : \exists g : \quad \|g\|_{\Re,2} < \infty,\ g|_{\mathbb{R}^{n+1}_{0+}} = f\} = H^{\Re,2}_{p,+}.
\end{aligned}$$

For the operator $-(-A_-)^\alpha$, $symb(-A_-) = \chi_-$, the situation is similar.

To solve the boundary problem for the operator $(-A_\pm)^\alpha$, we need the existence of the trace $f(\cdot, x_{n+1})$ if $f \in D((-A_\pm)^\alpha)$. Since

$$(\psi^2(\xi') + \xi_{n+1}^2)^{\alpha/2} \geq \frac{\psi^\alpha(\xi') + |\xi_{n+1}|^\alpha}{2},$$

then by Theorem 2.1.21 and Lizorkin multiplier theorem we obtain that

$$H_{p,+}^{\Re,2} \hookrightarrow \overline{H_p^{\psi^\alpha,2}(\mathbb{R}^n) \otimes H_p^\alpha(\mathbb{R}_{0+})}^{\|\cdot\|}$$

where the closure is taken with respect to the graph norm of the operator $\psi(D_{x'})^\alpha + (-\triangle_{x_{n+1}})^{\alpha/2}$. Thus, since the trace in the space $H_p^\alpha(\mathbb{R}_{0+})$ exists for $\frac{1}{p} < \alpha < 1$, for such α the trace will exist in the space $H_{p,+}^{\Re,2}$. Analogously, for $\frac{1}{p} < \alpha < 1$ the trace exists in the space $\tilde{H}_{p,+}^{\Re,2}$ and equals to zero.

Now we are ready to prove that $(-(-A_+)^\alpha, \tilde{H}_{p,+}^{\Re,2})$ and $(-(-A_-)^\alpha, H_{p,+}^{\Re,2})$ are the generators of L_p-sub-Markovian semigroups.

Theorem 3.1.4. For $1 < p < \infty$ and $\frac{1}{p} < \alpha < 1$ the operators $(-(-A_+)^\alpha, \tilde{H}_{p,+}^{\Re,2})$ and $(-(-A_-)^\alpha, H_{p,+}^{\Re,2})$, *symb* $(-A_\pm) = \chi_\pm(\xi') = \psi(\xi') \pm i\xi_{n+1}$, ψ satisfies **A1** and **A2**, are the generators of L_p-sub-Markovian semigroups $(T_t^{(4)})_{t\geq 0}$ and $(T_t^{(5)})_{t\geq 0}$ given by

$$T_t^{(4)} f(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \int_0^{x_{n+1}} f(x' - y', x_{n+1} - s) \mathcal{W}_s(y') \sigma_\alpha(s,t)\, ds\, dy' \tag{3.8}$$

and

$$T_t^{(5)} f(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \int_0^{\infty} f(x' - y', x_{n+1} + s) \mathcal{W}_s(y') \sigma_\alpha(s,t)\, ds\, dy' \tag{3.9}$$

where $\sigma_\alpha(s,t)$ is the measure which corresponds to the Bernstein function x^α, $0 < \alpha < 1$, by the formula

$$e^{-t\,z^\alpha} = \int_0^\infty e^{-z\,s} \sigma_\alpha(s,t)\, ds, \quad t > 0 \quad Re\, z > 0.$$

Proof. To prove this theorem we apply again the Hille-Yosida theorem.

1)From Theorem 2.2.10 it follows that the spaces $\tilde{H}_{p,+}^{\Re,2}$ and $H_{p,+}^{\Re,2}$are dense in $L_p(\mathbb{R}_{0+}^{n+1})$.

2) Since $-(-A_+)^\alpha$ and $-(-A_-)^\alpha$ are Dirichlet operators on $H_p^{\Re,2}$ (by Theorem 2.1.24), and by Theorem 2.2.5 and Theorem 2.2.6 $(-A_+)^\alpha$ and $(-A_-)^\alpha$ respectively are isomorphisms between $\tilde{H}_{p,+}^{\Re,2}$ and $H_{p,+}^{\Re,2}$, then these operators are Dirichlet operators on $\tilde{H}_{p,+}^{\Re,2}$ and $H_{p,+}^{\psi,2}$, and therefore they are dissipative on these spaces.

3) In Theorem 2.2.5 and 2.2.6 we have proved that

$$(-A_+)^\alpha : \tilde{H}_{p,+}^{\Re,2} \to L_p(\mathbb{R}_{0+}^{n+1})$$

and

$$(-A_-)^\alpha : H_{p,+}^{\Re,2} \to L_p(\mathbb{R}_{0+}^{n+1})$$

are isomorphisms. Therefore the equations

$$\begin{matrix} (-A_+)^\alpha f(x) = g(x), \quad x \in \mathbb{R}^{n+1}_+ \\ f(x',0) = 0 \end{matrix} \qquad \text{and} \qquad \begin{matrix} (-A_-)^\alpha f(x) = g(x), \quad x \in \mathbb{R}^{n+1}_+ \\ f(x',0) = h(x') \end{matrix} \tag{3.10}$$

are uniquely solvable in $\tilde{H}^{\Re,2}_{p,+}$ and in $H^{\Re,2}_{p,+}$, respectively, $g \in L_p(\mathbb{R}^{n+1}_{0+})$, $h \in L_p(\mathbb{R}^n)$ and it can be expressed in terms of g, since the operator $(-A_-)^\alpha$ is an isomorphism from $H^{\Re,2}_{p,+}$ to $L_p(\mathbb{R}^{n+1}_{0+})$.
(Note that if we can solve equations (3.10), we can also solve for $g \in L_p(\mathbb{R}^{n+1}_{0+})$, $h \in L_p(\mathbb{R}^n)$ the equations

$$(\lambda + (-A_-)^\alpha) f = g, \quad x \in \mathbb{R}^{n+1}_+ \tag{3.11}$$

subject to the boundary conditions

$$f(x',0) = h(x'),$$

where h again depends on g, and

$$(\lambda + (-A_+)^\alpha) f = g, \quad x \in \mathbb{R}^{n+1}_+ \tag{3.12}$$

with $f(x',0) = 0$, for all $\lambda > 0$).
Note also, that $(-(-A_+)^\alpha, \tilde{H}^{\Re,2}_{p,+})$ and $(-(-A_-)^\alpha, H^{\Re,2}_{p,+})$ are Dirichlet operators as the restrictions to $\mathbb{R}^{n+1}_{0+}$ of operators $(-(-A_+)^\alpha, \tilde{H}^{\Re,2}_p)$ and $(-(-A_-)^\alpha, H^{\Re,2}_p)$, which are Dirichlet operators, see Theorem 2.1.24.

Thus, all conditions of Hille-Yosida theorem are satisfied, and the operators $(-(-A_+)^\alpha, \tilde{H}^{\Re,2}_{p,+})$ and $(-(-A_-)^\alpha, H^{\Re,2}_{p,+})$ are the generators of some L_p- sub-Markovian semigroups.

Let the semigroups $(T^{(4)}_t)_{t\geq 0}$ and $(T^{(5)}_t)_{t\geq 0}$ be obtained by subordination of $(T^{(2)}_t)_{t\geq 0}$ and $(T^{(3)}_t)_{t\geq 0}$ respectively with the convolution semigroup $\eta^{(\alpha)}_t(ds) = \sigma_\alpha(s,t)ds$, see (1.6). To see that they are generated by $(-(-A_+)^\alpha, \tilde{H}^{\Re,2}_{p,+})$ and $(-(-A_-)^\alpha, H^{\Re,2}_{p,+})$, we will do the following:

We know from Balakrishnan's formula (2.17) that for $g \in H^{\Re,2}_{p,+}$ we have as representation for $(-A_-)^{-\alpha}$:

$$\begin{aligned} (-A_-)^{-\alpha} g &= \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{T^{(5)}_t g}{t^{1-\alpha}}\, dt \\ &= \frac{1}{\Gamma(\alpha)} \int_0^\infty \int_{\mathbb{R}^n} \frac{g(x'-y', x_{n+1}+t)\mathcal{W}_t(y')}{(2\pi)^{\frac{n}{2}} t^{1-\alpha}}\, dy'\, dt. \end{aligned}$$

Let $g \in S(\mathbb{R}^{n+1}_{0+})$. We can rewrite

$$T^{(5)}_t g = F^{-1}_{\xi\to x}(e^{-t(\psi(\xi')-i\xi_{n+1})^\alpha}\hat{g}).$$

The function

$$e^{-t(\lambda+(\psi(\xi')-i\xi_{n+1})^{\alpha})}\hat{g}) \in L_1(\mathbb{R}^{n+1}\times(0,\infty)),$$

therefore

$$L_{t\to\eta}F^{-1}_{\xi\to x}(\ldots.) = F^{-1}_{\xi\to x}L_{t\to\lambda}(\ldots..).$$

Thus, with the help of (1.10), (1.12) and Lemma 1.1.15, we find by Lebesgue dominated convergence theorem that for $g \in S(\mathbb{R}^{n+1}_{0+})$

$$\begin{aligned}(-A_-)^{-\alpha}g &= \lim_{\lambda\to 0} R_\lambda g = \lim_{\lambda\to 0}\int_0^\infty e^{-\lambda t}T_t^{(5)}g\,dt = \\ &= \lim_{\lambda\to 0}\int_0^\infty\int_{\mathbb{R}^n}\frac{g(x'-y',x_{n+1}+t)\mathcal{W}_t(y')}{(2\pi)^{\frac{n}{2}}}\frac{e'_\alpha(t,\lambda)}{-\lambda}\,dy'\,dt \\ &= \frac{1}{\Gamma(\alpha)}\int_0^\infty\int_{\mathbb{R}^n}\frac{g(x'-y',x_{n+1}+t)\mathcal{W}_t(y')}{(2\pi)^{\frac{n}{2}}t^{1-\alpha}}\,dy'\,dt,\end{aligned}$$

and by the density arguments $(-A_-)^{-\alpha}$ has the same representation for $g \in L_p(\mathbb{R}^{n+1}_{0+})$. Thus, the operator $(-(-A_-)^\alpha, H^{\Re,2}_{p,+})$ with the boundary condition

$$f(x',0) = \frac{1}{\Gamma(\alpha)}\int_0^\infty\int_{\mathbb{R}^n}\frac{g(x'-y',t)\mathcal{W}_t(y')}{(2\pi)^{\frac{n}{2}}t^{1-\alpha}}\,dy'\,dt$$

generates the semigroup $(T_t^{(5)})_{t\geq 0}$. By the same consideration the operator $(-(-A_+)^\alpha, \tilde{H}^{\Re,2}_{p,+})$ with zero Dirichlet boundary conditions generates the semigroup $(T_t^{(4)})_{t\geq 0}$. □

We also can show that $(-(-A_+)^\alpha, H^{\Re,2}_{p,+})$ is a generator of an L_p-sub-Markovian semigroup. The difference from the previous theorem is that the operator $(-A_+)^\alpha$ is not an isomorphism between $H^{\Re,2}_{p,+}$ and $L_p(\mathbb{R}^{n+1}_{0+})$, and we need to solve the boundary-value problem

$$(\lambda + (-A_+)^\alpha)f(x) = g(x), \quad x \in \mathbb{R}^{n+1}_+, g \in L_p(\mathbb{R}^{n+1}_{0+}), \tag{3.13}$$

for some boundary conditions, and show, that the solution belongs to $H^{\Re,2}_{p,+}$.

We obtained in Theorem 3.1.1 and Remark 3.1.2 that $(T_t^{(1)})_{t\geq 0}$ and $(T_t^{(1')})_{t\geq 0}$ are strongly continuous contraction semigroups generated by $(-A_+, H^{\chi,2,1}_{p,+})$ with different boundary conditions. By (1.6) the candidates for the semigroups generated by $(-(-A_+)^\alpha, H^{\Re,2}_{p,+})$ are the semigroups obtained by subordination with the Bernstein function $f(x) = x^\alpha$, $x > 0$, $0 < \alpha < 1$:

$$\begin{aligned}T_t^{(6)}g(x) &= (2\pi)^{-n/2}\int_0^{x_{n+1}}\int_{\mathbb{R}^n} g(x'-y',x_{n+1}-s)\mathcal{W}_s(y')\sigma_\alpha(s,t)\,dy'\,ds \\ &\quad + (2\pi)^{-n/2}\int_{x_{n+1}}^\infty\int_{\mathbb{R}^n} g(x'-y',0)\mathcal{W}_{x_{n+1}}(y')\sigma_\alpha(s,t)\,dy'\,ds,\end{aligned} \tag{3.14}$$

which is obtained by subordination with $f(x)$ from $(T_t^{(1)})_{t\geq 0}$, and

$$T_t^{(6')}g(x) = (2\pi)^{-n/2}\int_0^{x_{n+1}}\int_{\mathbb{R}^n} g(x'-y', x_{n+1}-s)\mathcal{W}_s(y')\sigma_\alpha(s,t)\,dy'\,ds + (2\pi)^{-n/2}\int_{x_{n+1}}^{\infty}\int_{\mathbb{R}^n} g(x'-y',0)\mathcal{W}_s(y')\sigma_\alpha(s,t)\,dy'\,ds, \tag{3.15}$$

which is obtained by subordination with $f(x)$ from $(T_t^{(1')})_{t\geq 0}$. These semigroups are again (by Theorem 4.3.1, [J1]) strongly continuous and contracting.

Theorem 3.1.5. For $1 < p < \infty$, $\frac{1}{p} < \alpha < 1$, the operator $(-(-A_+)^\alpha, H_{p,+}^{\Re,2})$, *symb*$(-A_+)$ $= \chi_+(\xi) = \psi(\xi') + i\xi_{n+1}$, where ψ satisfies **A1** and **A2**, is the generator of the L_p-sub-Markovian semigroups (3.14) and (3.15), depending on the boundary conditions.

Proof. As we proved in Theorem 3.1.4, the space $H_{p,+}^{\Re,2}$ is dense in $L_p(\mathbb{R}_{0+}^{n+1})$, and by Theorem 2.2.10 and $-(-A_+)^\alpha$ is a Dirichlet operator on $H_{p,+}^{\Re,2}$ as a restriction of the Dirichlet operator $(-(-A_+)^\alpha, H_p^{\Re,2})$ to the half-space.
We need to check whether problem (3.13) is solvable for all $g \in L_p(\mathbb{R}_{0+}^{n+1})$ (with some boundary conditions) and that the solution belongs to $H_{p,+}^{\Re,2}$.
We may decompose the solution to (3.13) into two parts: $f = f_1 + f_2$, where f_1 is the solution to

$$(\lambda + (-A_+)^\alpha)f_1(x) = g(x), \quad x \in \mathbb{R}_+^{n+1}, \qquad f_1(x',0) = 0 \tag{3.16}$$

and f_2 is the solution to

$$(\lambda + (-A_+)^\alpha)f_2(x) = 0, \quad x \in \mathbb{R}_+^{n+1} \tag{3.17}$$

for some boundary condition.

We already know (from Theorem 3.1.1) that the solution to (3.16) exists for all $g \in L_p(\mathbb{R}_{0+}^{n+1})$ and belongs to $\tilde{H}_{p,+}^{\Re,2}$.

To find the solution to (3.13), we will follow the proof of Theorem 3.1.1, namely, first take the Fourier transform $F_{x'\to\xi'}$ of the left-hand part of (3.13), and then the Laplace transform $L_{x_{n+1}\to\eta}$. Since $F^{-1}symb(-(-A_+)^{\alpha-1}) = F^{-1}((\chi_+(\xi))^{\alpha-1})$ exists, then

$$\begin{aligned}
&L_{x_{n+1}\to\eta}F_{x'\to\xi'}\big((\lambda + (-A_+)^\alpha)f\big) = \lambda\hat{f}(\xi',\eta) + L_{x_{n+1}\to\eta}F_{x'\to\xi'}\big((-A_+)^{\alpha-1}(-A_+)f\big)\\
&= \lambda\hat{f}(\xi',\eta) + L_{x_{n+1}\to\eta}F_{x'\to\xi'}(-(-A_+)^{\alpha-1})\big[L_{x_{n+1}\to\eta}F_{x'\to\xi'}((\psi(D_{x'}) + \frac{\partial}{\partial x_{n+1}})f)\big]\\
&= \lambda\hat{f}(\xi',\eta) + (\psi(\xi')+\eta)^{\alpha-1}(\psi(\xi')\hat{f}(\xi',\eta) + \eta\hat{f}(\xi',\eta) - \hat{f}(\xi',0))\\
&= \big(1 + (\psi(\xi')+\eta)^\alpha\big)\hat{f}(\xi',\eta) - (\psi(\xi')+\eta)^{\alpha-1}\hat{f}(\xi',0).
\end{aligned}$$

Therefore the $L_{x_{n+1}\to\eta}F_{x'\to\xi'}$ -transform of the solution to (3.13) is

$$\hat{f}(\xi',\eta) = \frac{\hat{g}(\xi',\eta) + (\psi(\xi')+\eta)^{\alpha-1}\hat{f}(\xi',0)}{\lambda + (\psi(\xi')+\eta)^{\alpha}} \tag{3.18}$$

Note, that since $\alpha > 1/p$ then for $f \in H_{p,+}^{\mathrm{R},2}$ the trace exists, and $f(x',\cdot) \in C_\infty([0,\infty))$. Then $f(x',x_{n+1}) \to f(x',0)$ as $x_{n+1}\to 0$, x' is fixed. Let $x_{n+1}\in[0,\varepsilon]$, $\varepsilon>0$. For such x_{n+1} the integral

$$\int_0^\infty \frac{T_t^{(1)} f(x',x_{n+1}-t) - f(x)}{t^{1+\alpha}}\,dt$$

converges uniformly, i.e.

$$\sup_{x_{n=1}\in[0,\varepsilon]}\left|\int_A^\infty \frac{T_t^{(1)} f(x',x_{n+1}-t) - f(x)}{t^{1+\alpha}}\,dt\right| \to 0 \quad \text{as} \quad A\to\infty.$$

Therefore we construct the fractional power of $-A_+$ with help of the semigroup $(T_t^{(1)})_{t\geq 0}$, see Balakrishnan's formula (2.16), and get, passing to the limit at $x_{n+1}\to 0$,

$$\begin{aligned}(-A_+)^\alpha f(x',0) &= \frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \frac{T_t^{(1)} f(x',0) - f(x',0)}{t^{1+\alpha}}\,dt \\ &= \frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \frac{f(x',0) - f(x',0)}{t^{1+\alpha}}\,dt = 0,\end{aligned}$$

with $\lambda f(x',0) = g(x',0)$. From (3.16) and (3.17) we see that if $\lambda f(x',0) = g(x',0)$, then $\lambda f_2(x',0) = g(x',0)$.

We need to check whether f_2, $\hat{f}_2(\xi',\eta) = \dfrac{(\psi(\xi')+\eta)^{\alpha-1}\hat{g}(\xi',0)}{\lambda(\lambda+(\psi(\xi')+\eta)^\alpha)} \in H_{p,+}^{\Re,2}$, where $g\in L_p(\mathbb{R}_{0+}^{n+1})$-the right-hand side of (3.16). But it follows from (3.17) that it is enough to check whether $f_2\in L_p(\mathbb{R}_{0+}^{n+1})$.

Suppose that in (3.17) $f_2 = f_2^* \cdot 1_{\{x_{n+1}\geq 0\}}$, for some f_2^*. Then, we obtain, taking the Fourier transform of both sides of (3.17), that $F(f_2)(\xi) = \dfrac{(\psi(\xi')+i\xi_{n+1})^{\alpha-1}F(f_2)(\xi',0)}{\lambda(\lambda+(\psi(\xi')+i\xi_{n+1})^\alpha)}$, and we can apply the theorems about Fourier multipliers we got before.
Let $\delta>0$ be such that

$$c((1+\xi_{n+1}^2)(1+|\xi'|^2))^\delta \leq \lambda + |\chi_+(\xi)|^\alpha.$$

(Clearly, δ should be less then $\frac{\alpha}{2}$ and it depends on α). From Remark 2.1.25 we can derive that

$$\frac{c((1+\xi_{n+1}^2)(1+|\xi'|^2))^\delta}{\lambda+(\chi_+(\xi))^\alpha}$$

is an L_p-Fourier multiplier. In such a way we reduced the estimate of $F(f_2)(\xi)$ to an estimate for

$$\frac{\hat{g}(\xi',0)}{((1+\xi_{n+1}^2)(1+|\xi'|^2))^\delta}.$$

The inverse Fourier transform of $\frac{1}{(|\xi|^2+1)^\delta}$ is (see [BE1], vol.1)

$$F^{-1}\Big(\frac{1}{(|\xi|^2+1)^\delta}\Big) = \Big(\frac{|x|}{2}\Big)^{\delta-1/2}\frac{1}{\Gamma(\delta)\pi}K_{(\delta-1/2)}(|x|),$$

where $K_\nu(x)$ is he Bessel function of third kind. Since $K_\nu(x) \sim c_1 e^{-x}$ as $x \to \infty$, and $K_\nu(x) \sim c_2 x^{-\nu}$ as $x \to 0$ (see [BE]), we see that

$$F^{-1}\Big(\frac{1}{(|\xi|^2+1)^\delta}\Big) \in L_p(\mathbb{R}).$$

Further for $g(\cdot, 0) \in L_p(\mathbb{R}^n)$

$$F^{-1}\Big(\frac{\hat{g}(\xi',0)}{(|\xi'|^2+1)^\delta}\Big) \in L_p(\mathbb{R}^n),$$

and therefore

$$\Big\|F^{-1}_{\xi\to x}\Big(\frac{(\chi_+(\xi))^{\alpha-1}\hat{g}(\xi',0)}{\lambda+(\chi_+(\xi))^\alpha}\Big)\Big\|_p \le \Big\|F^{-1}_{\xi\to x}\Big(\frac{\hat{g}(\xi',0)}{\lambda+(\chi_+(\xi))^\alpha}\Big)\Big\|_p < \infty.$$

Thus, by Hille-Yosida theorem, $(-(-A_+)^\alpha, H^{\Re,2}_{p,+})$ with the boundary condition $(-A_+)^\alpha f(x',0) = 0$ is the generator of an L_p-sub-Markovian semigroup. It remains to prove that this semigroup is (3.14).

We will follow the same onsiderations as in the proof of Theorem 3.1.1. Let $g \in S(\mathbb{R}^{n+1}_{0+})$. Rewrite $T^{(6')}_t$ as

$$T^{(6')}_t g = (2\pi)^{-n/2}\int_0^{x_{n+1}}\int_{\mathbb{R}^n} e^{i(x\,\xi')-s\psi(\xi')}\hat{g}(\xi', x_{n+1}-s)\sigma_\alpha(s,t)\,d\xi'\,ds$$
$$+ (2\pi)^{-n/2}\int_{x_{n+1}}^\infty e^{i(x\,\xi')-s\psi(\xi')}\hat{g}(\xi',0)\sigma_\alpha(s,t)\,d\xi'\,ds.$$

Then since

$$e^{-s\psi(\xi')+i(x',\xi')-\lambda t}[\hat{h}(\xi', x_{n+1}-s)\sigma_\alpha(s,t) - \hat{g}(\xi',0)\sigma_\alpha(s,t)]$$

belongs to $L_1(\mathbb{R}^n)$ w.r.t. ξ', and to $L_1(0,\infty)$ w.r.t. s and t, we can applying Fubini theorem and using (1.19), that for the solution of (3.13)

$$\begin{aligned}
f(x) = R_\lambda g(x) &= \int_0^\infty e^{-\lambda t}T^{(6)}_t g(x)\,dt\\
&= \int_0^{x_{n+1}}\int_{\mathbb{R}^n}\frac{g(x'-y', x_{n+1}-s)\mathcal{W}_s(y')e'_\alpha(s,\lambda)}{-\lambda(2\pi)^{n/2}}\,dy'\,ds\\
&\quad + \int_{x_{n+1}}^\infty\int_{\mathbb{R}^n}\frac{g(x'-y',0)\mathcal{W}_{x_{n+1}}(y')}{(2\pi)^{n/2}}\frac{e'_\alpha(s,\lambda)}{-\lambda}\,dy'\,ds\\
&= \int_0^{x_{n+1}}\int_{\mathbb{R}^n}\frac{g(x'-y', x_{n+1}-s)\mathcal{W}_s(y')e'_\alpha(s,\lambda)}{-\lambda(2\pi)^{n/2}}\,dy'\,ds\\
&\quad + \int_{\mathbb{R}^n}\frac{g(x'-y',0)\mathcal{W}_{x_{n+1}}(y')e_\alpha(x_{n+1},\lambda)}{\lambda(2\pi)^{n/2}}\,dy'
\end{aligned}$$

holds. Taking the Fourier transform $F_{x'\to\xi'}$ and then the Laplace transform $L_{x_{n+1}\to\eta}$ of the right and the left side of this equality, we obtain, using (1.14) and (1.15) that

$$\begin{aligned}\hat{f}(\xi,\eta) &= L_{x_{n+1}\to\eta}\Big[\int_0^{x_{n+1}} \hat{g}(\xi', x_{n+1}-s)e^{-s\psi(\xi')}\frac{e'_\alpha(s,\lambda)}{-\lambda} \\ &\quad + \hat{g}(\xi',0)e^{-x_{n+1}\psi(\xi')}\frac{e_\alpha(s,\lambda)}{\lambda}\,ds\Big] \\ &= \frac{\hat{g}(\xi',\eta)+\lambda^{-1}(\psi(\xi')+\eta)^{\alpha-1}\hat{g}(\xi',0)}{\lambda+(\psi(\xi')+\eta)^\alpha},\end{aligned}$$

which is equal to (3.18), regarding that $\lambda\hat{f}(\xi',0)=\hat{g}(\xi',0)$.

Thus, the strongly continuous contraction semigroup $(T_t^{(6)})_{t\geq 0}$ is indeed generated by $(-(-A_+)^\alpha, H_{p,+}^{\Re,2})$ with the boundary condition $(-A_+)^\alpha f(x',0)=0$.

Now we want to prove that the operator $(-(-A_+)^\alpha, H_{p,+}^{\Re,2})$ with zero Neumann boundary condition $\frac{\partial}{\partial x_{n+1}} f(x',0)=0$ generates the semigroup $(T_t^{(6')})_{t\geq 0}$.

To do this we will follow the proof of Theorem 3.1.4, i.e. we calculate $(-A_+)^{-\alpha}$ for $g\in S(\mathbb{R}^{n+1}_{0+})$ as

$$\begin{aligned}(-A_+)^{-\alpha}g &= \lim_{\lambda\to 0} R_\lambda g = \lim_{\lambda\to 0}\int_0^\infty e^{-\lambda t}T_t^{(6')}g\,dt \\ &= \lim_{\lambda\to 0}\Big(\int_0^{x_{n+1}}\int_{\mathbb{R}^n}\frac{g(x'-y',x_{n+1}-s)\mathcal{W}_s(y')e'_\alpha(s,\lambda)}{-\lambda(2\pi)^{n/2}}\,dy'\,ds \\ &\quad + \int_{x_{n+1}}^\infty\int_{\mathbb{R}^n}\frac{g(x'-y',0)\mathcal{W}_s(y')}{(2\pi)^{n/2}}\frac{e'_\alpha(s,\lambda)}{-\lambda}\,dy'\,ds\Big) \\ &= \frac{1}{\Gamma(\alpha)}\int_0^{x_{n+1}}\int_{\mathbb{R}^n}\frac{g(x'-y',x_{n+1}-s)\mathcal{W}_s(y')}{s^{1-\alpha}(2\pi)^{n/2}}\,dy'\,ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_{x_{n+1}}^\infty\int_{\mathbb{R}^n}\frac{g(x'-y',0)\mathcal{W}_s(y')}{s^{1-\alpha}(2\pi)^{n/2}}\,dy'\,ds.\end{aligned}$$

But this is exacly the operator obtained with help of Balakrishnan's formula:

$$\begin{aligned}(-A_+)^{-\alpha}g(x) &= \frac{1}{\Gamma(\alpha)}\int_0^\infty\frac{T_t^{(1')}g(x)-g(x)}{s^{1-\alpha}}\,ds \\ &= \frac{1}{\Gamma(\alpha)}\int_0^{x_{n+1}}\int_{\mathbb{R}^n}\frac{g(x'-y',x_{n+1}-s)\mathcal{W}_s(y')}{s^{1-\alpha}(2\pi)^{n/2}}\,dy'\,ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_{x_{n+1}}^\infty\int_{\mathbb{R}^n}\frac{g(x'-y',0)\mathcal{W}_s(y')}{s^{1-\alpha}(2\pi)^{n/2}}\,dy'\,ds.\end{aligned}$$

(Since $g\in S(\mathbb{R}^{n+1}_{0+})$ the trace exists, and by the density argument the same representation for $(-A_+)^\alpha$ holds for $g\in L_p(\mathbb{R}^{n+1}_{0+})$).

It remains for us to show, that $f(x)=L_{t\to\lambda}T_t^{(6')}g(x)$ is the solution to (3.13) with zero Neumann boundary condition.

We have:

$$\begin{aligned} f(x) =& L_{t\to\lambda} T_t^{(6')} g(x) \\ =& \int_0^{x_{n+1}} \int_{\mathbb{R}^n} \frac{g(x'-y', x_{n+1}-s)\mathcal{W}_s(y')e'_\alpha(s,\lambda)}{-\lambda(2\pi)^{n/2}} dy'\, ds \\ &+ \int_{x_{n+1}}^{\infty} \int_{\mathbb{R}^n} \frac{g(x'-y',0)\mathcal{W}_s(y')e'_\alpha(x_{n+1},\lambda)}{-\lambda(2\pi)^{n/2}} dy'\, ds \end{aligned}$$

Differentiating with respect to x_{n+1}, we get

$$\begin{aligned} \frac{\partial}{\partial x_{n+1}} f(x) =& \int_0^{x_{n+1}} \int_{\mathbb{R}^n} \frac{\frac{\partial}{\partial x_{n+1}} g(x'-y', x_{n+1}-s)\mathcal{W}_s(y')e'_\alpha(s,\lambda)}{-\lambda(2\pi)^{n/2}} dy'\, ds \\ &+ \int_{\mathbb{R}^n} \frac{g(x'-y',0)\mathcal{W}_{x_{n+1}}(y')e'_\alpha(x_{n+1},\lambda)}{-\lambda(2\pi)^{n/2}} dy' \\ &- \int_{\mathbb{R}^n} \frac{g(x'-y',0)\mathcal{W}_{x_{n+1}}(y')e'_\alpha(x_{n+1},\lambda)}{-\lambda(2\pi)^{n/2}} dy' \\ =& \int_0^{x_{n+1}} \int_{\mathbb{R}^n} \frac{\frac{\partial}{\partial x_{n+1}} g(x'-y', x_{n+1}-s)\mathcal{W}_s(y')e'_\alpha(s,\lambda)}{-\lambda(2\pi)^{n/2}} dy'\, ds, \end{aligned}$$

which tends to zero in a.e. sense if $x_{n+1} \to 0$:

$$\frac{\partial}{\partial x_{n+1}} f(x',0) = 0 \quad \text{a.e.}$$

Thus, the operator $(-(-A_+)^\alpha, H_{p,+}^{\Re,2})$ with the zero Neumann boundary condition $\frac{\partial}{\partial x_{n+1}} f(x',0) = 0$ generates the semigroup $(T_t^{(6')})_{t\geq 0}$. □

3.2 Examples

In this section we want to give several examples. We will give the representations of some semigroups generated by $(-A_\pm, H_{p,+}^{\chi,2,1})$ and $(-A_+, \tilde{H}_{p,+}^{\chi,2,1})$, where $symb(-A_\pm) = \chi_\pm$, because the properties of the fractional powers of these operators follow via subordination.

First consider the two-dimensional case.

I. Consider a real-valued continuous negative definite function $p(\xi)$ of the form

$$p(\xi) = 2\delta \ln cosh(\frac{a\xi}{2}). \tag{3.19}$$

This function is a particular case of the characteristic exponent of the **Meixner process**:

$$\psi_{m,\delta,a,b}(\xi) := -im\xi + 2\delta\big(\ln cosh(\frac{a\xi - ib}{2}) - ln \cos(\frac{b}{2})\big), \tag{3.20}$$

where $m \in \mathbb{R}$, $\delta > 0$, $a > 0$ and $-\pi < b < \pi$. The densities of the measures μ_t which correspond to $\psi_{m,\delta,a,b}$ are given by

$$p_t^{m,\delta,a,b}(x) = \frac{\left(2\cos\left(\frac{b}{2}\right)\right)^{2\delta t}}{2\pi a \Gamma(2\delta t)} e^{\frac{b}{a}(x-mt)} \big|\Gamma(\delta t + i\frac{x-mt}{a})\big|^2.$$

(For more references on the Meixner processes see [Gr], and also [BB]).

Since we are interested in real-valued symbols ψ, we posed $m = b = 0$. In this case the density of the convolution measure is

$$p_t^{\delta,a}(x) = p_t^{0,\delta,a,0}(x) = \frac{2^{\delta t}}{2\pi a \Gamma(2\delta t)} \big|\Gamma(\delta t + \frac{ix}{a})\big|^2. \tag{3.21}$$

For a continuous negative definite function $p(\xi)$ defined in (3.20) we define $\chi_\pm(\xi,\eta) = 1 + p(\xi) \pm i\eta$. We added 1 in order to make $\psi(\xi) = 1 + p(\xi) > 0$, since as we saw this was essential in Theorem 2.1.21 and Theorem 2.2.5.

Since

$$p'(\xi) = \delta a \, tanh(\frac{a\xi}{2})$$

is a bounded function, we find (see the proof of Theorem 2.1.21)

$$\sup_{(\xi,\eta)\in\mathbb{R}^2} \big|\xi\eta\partial^2_{\xi,\eta} e^{i\alpha arg \chi_\pm}\big| < \infty$$

as well as

$$\sup_{(\xi,\eta)\in\mathbb{R}^2} \big|\xi\partial_\xi e^{i\alpha arg \chi_\pm}\big| < \infty, \qquad \sup_{(\xi,\eta)\in\mathbb{R}^2} \big|\eta\partial_\eta e^{i\alpha arg \chi_\pm}\big| < \infty.$$

Thus the function $e^{i\alpha arg \chi_\pm}$ is an L_p-Fourier multiplier, and therefore we can apply the results obtained in the previous section to construct the semigroups generated by $(-(A_\pm)^\alpha, H^{\Re,2}_{p,+})$ and $(-(-A_+)^\alpha, \tilde{H}^{\Re,2}_{p,+})$, where $\Re = Re(\chi_\pm(\xi,\eta))^\alpha$, $symb(-A_\pm) = \chi_\pm(\xi,\eta)$.

Knowing (3.21), we can explicitly construct the semigroups generated by $(-A_\pm, H^{\chi,2,1}_{p,+})$ and $(-A_+, \tilde{H}^{\chi,2,1}_{p,+})$.

Note that the density (with respect to the Dirac measure) of the convolution measure which corresponds to a constant continuous negative definite function $\psi(\xi) \equiv \lambda$ is

$$\mu_{t,\lambda}(x) = e^{-\lambda t}. \tag{3.22}$$

Therefore the density of the convolution measures $\mu_t(dx)$ associated with the continuous negative definite function $\xi \mapsto 1 + p(\xi)$ is

$$\begin{aligned}\mu_t(x) &= F^{(-1)}\big(e^{-t(1+p(\xi))}\big)(x) = \mu_{t,1} * p_t^{\delta,a}(x) \\ &= e^{-t} p_t^{\delta,a}(x).\end{aligned}$$

Thus, substituting $\mu_t(x)$ in (3.1), (3.2) and (3.7) we obtain that the strongly continuous contraction semigroups

$$
\begin{aligned}
T_t^{(1)} f(x) = & \int_{\mathbb{R}} \frac{f(x_1 - y_1, x_2 - t) e^{-t} p_t^{\delta,a}(y_1)\, dy_1}{2\pi} 1_{[t,\infty)}(x_2) \\
& + \int_{\mathbb{R}} \frac{f(x_1 - y_1, 0) e^{-t} p_t^{\delta,a}(y_1)\, dy_1}{2\pi} 1_{[0,t)}(x_2)
\end{aligned}
\tag{3.23}
$$

and

$$
T_t^{(2)} f(x) = \int_{\mathbb{R}} \frac{f(x_1 - y_1, x_2 - t) e^{-t} p_t^{\delta,a}(y_1)\, dy_1}{2\pi} 1_{[t,\infty)}(x_2) \tag{3.24}
$$

and

$$
T_t^{(3)} f(x) = \int_{\mathbb{R}} \frac{f(x_1 - y_1, x_2 + t) e^{-t} p_t^{\delta,a}(y_1)\, dy_1}{2\pi} \tag{3.25}
$$

are generated by $(-A_+, H_{p,+}^{\chi,2,1})$, $(-A_+, \tilde{H}_{p,+}^{\chi,2,1})$ and $(-A_-, H_{p,+}^{\chi,2,1})$ respectively (with the boundary conditions as in Theorems 3.1.1 and 3.1.3), $symb(-A_\pm) = \chi(\xi, \eta)$.

II. Consider the Bernstein function $f(x) = \ln(1 + x)$, $x > 0$. The corresponding convolution semigroup is (see (1.17))

$$
\nu_t(dx) = \lambda^{(1)}(dx) 1_{(0,\infty)}(x) \frac{1}{\Gamma(t)} x^{t-1} e^{-x}, \tag{3.26}
$$

where $\lambda^{(1)}(dx)$ is one-dimensional Lebesgue measure.

Denote by $\psi(\xi) = f(1 + |\xi|^2) = \ln(2 + |\xi|^2)$, $\xi \in \mathbb{R}^n$. Such function ψ satisfies **A1** and **A2**. To find $\mathcal{W}_t(y) = F^{(-1)}(e^{-t\psi(\cdot)})(y)$, i.e. the density of the convolution measure corresponding to ψ, we use (1.8).

We know, that the convolution measure

$$
\mu_t^{(1)}(dx) = \lambda^{(n)}(dx) \frac{e^{-\frac{|x|^2}{4t}}}{(4\pi t)^{n/2}}
$$

(where $\lambda^{(n)}(dx)$ is the n-dimensional Lebesgue measure) is associated with the continuous negative definite function $|\xi|^2$. Then, with help of (3.22) we obtain the convolution measure of function $\phi(\xi) = 1 + |\xi|^2$:

$$
\mu_t(dx) = \lambda^{(n)}(dx) \frac{e^{-\frac{|x|^2}{4t} - t}}{(4\pi t)^{n/2}} \tag{3.27}
$$

Further, with the help of (1.8) we obtain the convolution measure corresponding to $\psi = f(\phi)$:

$$
\begin{aligned}
\mu_t^f(dx) &= \int_0^\infty \frac{e^{-2s} e^{-\frac{|x|^2}{4s}} s^{t-1}}{\Gamma(t)(4\pi s)^{n/2}}\, ds \lambda^{(n)}(dx) \\
&= \frac{2}{(4\pi)^{n/2}\Gamma(t)} \left(\frac{1}{|x|^2}\right)^{t-\frac{n}{2}-1} K_{t-\frac{n}{2}}(\sqrt{2}|x|) \lambda^{(n)}(dx).
\end{aligned}
\tag{3.28}
$$

We used that

$$L_{s\to p}\Big(e^{-\frac{a}{4s}}s^{\nu-1}\Big) = \int_0^\infty e^{-ps}s^{\nu-1}e^{-\frac{a}{4s}}\,ds = 2\Big(\frac{2}{ap}\Big)^{\nu-1}K_\nu(\sqrt{ap}),$$

where $K_\nu(x)$ is the modified Bessel function of the third kind.

Substituting (3.28) into (3.1), (3.2) and (3.7) we obtain the following strongly continuous contraction semigroups

$$\begin{aligned} T_t^{(1)}f(x) &= \int_{\mathbb{R}^n}\frac{2f(x'-y',x_{n+1}-t)K_{t-\frac{n}{2}}(\sqrt{2}|y'|)}{(4\pi)^{n/2}\Gamma(t)(2\pi)^{\frac{n}{2}}}\Big(\frac{1}{|y'|^2}\Big)^{t-\frac{n}{2}-1}dy'1_{[t,\infty)}(x_{n+1}) \\ &+\int_{\mathbb{R}^n}\frac{2f(x'-y',0)K_{t-\frac{n}{2}}(\sqrt{2}|y'|)}{(4\pi)^{n/2}\Gamma(t)(2\pi)^{\frac{n}{2}}}\Big(\frac{1}{|y'|^2}\Big)^{t-\frac{n}{2}-1}dy'1_{[0,t)}(x_{n+1}) \end{aligned} \tag{3.29}$$

$$T_t^{(2)}f(x) = \int_{\mathbb{R}^n}\frac{2f(x'-y',x_{n+1}-t)K_{t-\frac{n}{2}}(\sqrt{2}|y'|)}{(4\pi)^{n/2}\Gamma(t)(2\pi)^{\frac{n}{2}}}\Big(\frac{1}{|y'|^2}\Big)^{t-\frac{n}{2}-1}dy'\,1_{[t,\infty)}(x_{n+1}) \tag{3.30}$$

and

$$T_t^{(3)} = \int_{\mathbb{R}^n}\frac{2f(x'-y',x_{n+1}+t)K_{t-\frac{n}{2}}(\sqrt{2}|y'|)}{(4\pi)^{n/2}\Gamma(t)(2\pi)^{\frac{n}{2}}}\Big(\frac{1}{|y'|^2}\Big)^{t-\frac{n}{2}-1}dy' \tag{3.31}$$

which are generated by $(-A_+, H_{p,+}^{\chi,2,1})$, $(-A_+, \tilde{H}_{p,+}^{\chi,2,1})$ and $(-A_-, H_{p,+}^{\chi,2,1})$ respectively, $symb(-A_\pm) = \chi_\pm(\xi) = \ln(2+|\xi'|^2)\pm i\xi_{n+1}$, $\xi\in\mathbb{R}^{n+1}$.

3.3 Perturbation of the generator of an L_p-sub-Markovian semigroup

In the previous section we discussed the class of operators of the form $A = (\psi(D) \pm \frac{\partial}{\partial x_{n+1}})^\alpha$, $0<\alpha<1$, that are the generators of L_p-sub-Markovian semigroups. Now we want to extend this class by adding a perturbing operator $p(x', D_{x'})$, $x'\in\mathbb{R}^n$.
Suppose that $p(x', D_{x'})$ satisfies the estimate

$$\|p(x',D_{x'})f\|_p \le c\|f\|_{W_p^{\mu_0}(\mathbb{R}^n)} \tag{3.32}$$

with some $0<\mu_0<1$, where W_p^s is a Sobolev space,

$$W_p^s = \begin{cases} H_p^s, & s \quad \text{is an integer,} \\ B_{pp}^s, & s \quad \text{s in not an integer.} \end{cases}$$

Since for $p\ge 2$ the embedding $H_p^s(\mathbb{R}^n)\subset B_{pp}^s(\mathbb{R}^n)$ holds for $s\ge 0$, see [T2], Remark 2.3.3/4, then

$$\|p(x',D_{x'})f\|_p \le c\|f\|_{H_p^{\mu_0}(\mathbb{R}^n)}. \tag{3.33}$$

Let $(A, D(A))$ and $(B, D(B))$ be some operators, $D(A) \subset L_p(\mathbb{R}^n)$, $D(B) \subset L_p(\mathbb{R}^m)$, and the Schwartz spaces on $\mathbb{R}^n$ and $\mathbb{R}^m$ are dense in $D(A)$ and $D(B)$ respectively.

Denote by $D(A \otimes B)$ the closure of the space $S(\mathbb{R}^{n+m})$ with respect to the graph norm of $A \otimes B$. Also, we assume that for $f \in D(B)$

$$\|f\|_p \leq c\|Bf\|_p$$

holds.

Lemma 3.3.1. Let $(A, D(A))$ and $(B, D(B))$ be as before, and let an operator Q be defined on $D(A)$ and A-bounded. Then $(Q \otimes I, D(A \otimes B))$ is $A \otimes B$-bounded.

Proof. From Lemma 1.2.2 we know, that the space $C_0^\infty(\mathbb{R}^n) \otimes C_0^\infty(\mathbb{R}^m)$ is dense in $L_p(\mathbb{R}^{n+m})$. From the conditions of our lemma we see, that $C_0^\infty(\mathbb{R}^n) \otimes C_0^\infty(\mathbb{R}^m)$ is dense in $D(A) \otimes D(B)$, which is in turn is dense in $D(A \otimes B)$. Then for $f \in D(A \otimes B)$ there exists a sequence $(f_n)_{n \geq 0} = (f_{1n} \otimes f_{2n})_{n \geq 0}$ which converges to f as $n \to \infty$. For this sequence we have, since Q is A-bounded,

$$\begin{aligned} \|(Q \otimes I)(f_{1n} \otimes f_{2n})\|_{L_p(\mathbb{R}^{n+m})} &= \|Qf_{1n}\|_{L_p(\mathbb{R}^n)}\|f_{2n}\|_{L_p(\mathbb{R}^m)} \\ &\leq \left(\varepsilon\|Af_{1n}\|_{L_p(\mathbb{R}^n)} + \delta\|f_{1n}\|_{L_p(\mathbb{R}^n)}\right)\|f_{2n}\|_{L_p(\mathbb{R}^m)} \\ &\leq \varepsilon\|Af_{1n} \otimes Bf_{2n}\|_{L_p(\mathbb{R}^{n+m})} + \delta\|f_{1n} \otimes f_{2n}\|_{L_p(\mathbb{R}^{n+m})} \\ &= \varepsilon\|A \otimes Bf_n\|_{L_p(\mathbb{R}^{n+m})} + \delta\|f_n\|_{L_p(\mathbb{R}^{n+m})} \end{aligned} \tag{3.34}$$

for some $0 < \varepsilon < 1$ and $\delta > 0$. Passing in (3.34) to the limit as $n \to \infty$, we obtain

$$\|(Q \otimes I)f\|_{L_p(\mathbb{R}^{n+m})} \leq \varepsilon\|A \otimes Bf\|_{L_p(\mathbb{R}^{n+m})} + \delta\|f\|_{L_p(\mathbb{R}^{n+m})},$$

proving the lemma. □

With help of Remark 1.3.2 we see from (3.33) that $p(x', D_{x'})$ is $(1 - \triangle_{x'})^{\mu/2}$-bounded with an arbitrary ε-bound, if $\mu_0 = \kappa\mu$, $0 < \kappa, \mu < 1$. Applying Lemma 3.3.1 with $A = (1 - \triangle_{x'})^{\mu/2}$, $B = (1 - \triangle_{x_{n+1}})^{\mu/2}$, we obtain that $p(x', D_{x'})$ is $\tilde{C}$-bounded, where $\tilde{C} = \left((1 - \triangle_{x'})(1 - \triangle_{x_{n+1}})\right)^{\mu/2}$.

But for $symb\,\tilde{C} = p(\xi) = ((1 + |\xi'|^2)(1 + \xi_{n+1}^2))^{\mu/2}$ we have

$$p(\xi) \leq (1 + |\xi|^2)^\mu, \quad 0 < \mu < 1,$$

and

$$\partial^{(k+1)}_{\xi_k, \dots, \xi_1, \xi_{n+1}} p(\xi) = \frac{(\frac{\mu}{2})^2(\frac{\mu}{2} - 1)\dots(\frac{\mu}{2} - k + 1)\xi_1 \dots \xi_k \xi_{n+1}}{(1 + |\xi'|^2)^{k - \frac{\mu}{2}}(1 + |\xi_{n+1}|^2)^{1 - \frac{\mu}{2}}}.$$

Then

by the same way as in the proof of Theorem 2.1.21 we derive that $\frac{p(\xi)}{(1+|\xi|^2)^\mu}$ is an L_p-Fourier multiplier, which leads to the estimate

$$\|\tilde{C}f\|_p \le c\|(1-\triangle)^\mu f\|_p.$$

Thus, the operator $p(x', D_{x'})$ which satisfied (3.32) with $\mu_0 = \mu\kappa$, where $0 < \mu < 1$, $0 < \kappa < 1$, is C-bounded, $C = (1-\triangle)^\mu$.

We want $p(x', D_{x'})$ to be controlled by the operator $(\chi_\pm(D))^\alpha$.

Lemma 3.3.2. Let ψ satisfy **A1** and **A2**, and

$$(1+|\xi|^2)^\mu \le |\chi_\pm(\xi)|^\alpha, \quad 0 < \mu < 1.$$

Then

$$H_p^{\Re,2} \subset H_p^{2\mu}. \tag{3.35}$$

Proof. Denote $\chi = \chi_+$, for the case $\chi = \chi_-$ the proof is similar. Differentiating with respect to ξ_{n+1} we obtain

$$\frac{\partial}{\partial \xi_{n+1}}\Big(\frac{1}{\chi^\alpha}\Big) = -\frac{i\alpha}{\chi^{1+\alpha}}.$$

Let now ξ_{n+1} be fixed. For fixed ξ_{n+1} the functions χ and χ^α satisfy **A1** and **A2** if ψ does. We can calculate that

$$\Big|\Big(\frac{1}{\chi^\alpha}\Big)^{(k)}\Big| \le \Big|\frac{c_1}{\chi^\alpha}\frac{\phi_1'\dots\phi_k'}{\phi^k}\Big|$$

and

$$\Big|\Big(\frac{1}{\chi}\Big)^{(k)}\Big| \le \Big|\frac{c_2}{\chi}\frac{\phi_1'\dots\phi_k'}{\phi^k}\Big|,$$

from where

$$\Big|\Big(\frac{1}{\chi^{\alpha+1}}\Big)^{(k)}\Big| \le \Big|\frac{c_3}{\chi^{1+\alpha}}\Big(\frac{\phi_1'\dots\phi_k'}{\phi^k}\Big)^2\Big|$$

follows. Further (see again the proof of Theorem 2.1.21),

$$\sup_{\xi\in\mathbb{R}^{n+1}}\Big|\xi_1\dots\xi_k\xi_{n+1}\Big(\frac{(1+|\xi|^2)^\mu}{\chi^\alpha(\xi)}\Big)^{(k+1)}_{\xi_k,\dots,\xi_1,\xi_{n+1}}\Big| < \infty.$$

In such a way $\frac{(1+|\xi|^2)^\mu}{\chi^\alpha(\xi)}$ is a L_p-Fourier multiplier and $H_p^{\Re,2} \subset H_p^{2\mu}$. $\square$

Corollary 3.3.3. Suppose that χ and μ are as in Lemma 3.3.2. Then

$$H_{p,+}^{\Re,2} \subset H_{p,+}^{2\mu}, \quad \tilde{H}_{p,+}^{\Re,2} \subset \tilde{H}_{p,+}^{2\mu}.$$

Proof. From (3.35) we have that

$$\|f\|_{H_p^{2\mu}} \leq c\|f\|_{H_p^{\Re,2}} \tag{3.36}$$

for $f \in H_p^{\Re,2}$. To prove the first embedding we may consider the function f in (3.36) as extensions to $\mathbb{R}_{0-}^{n+1}$. Taking infimum over all such extensions, we obtain the first embedding.
To prove the second, we only need to take the functions with support in $\mathbb{R}_{0+}^{n+1}$ in (3.36). □

Theorem 3.3.4. Let $(-A)^\alpha$ be an operator with a continuous negative definite symbol $(\chi(\xi))^\alpha = (\psi(\xi') + i\xi_{n+1})^\alpha$, $0 < \alpha < 1$, such that for some $0 < \mu < 1$

$$(1 + |\xi|^2)^\mu \leq c|\chi(\xi)|^\alpha,$$

the function ψ satisfies **A1** and **A2**, and further let a pseudo-differential operator $p(x', D_{x'})$, satisfy (3.32) with $\mu_0 = \kappa\mu$, $0 < \kappa < 1$, and be an L_p- Dirichlet operator. Then the operator $(-(-A)^\alpha - p(x', D_{x'}), H_p^{\Re,2})$ is a generator of an L_p-sub-Markovian semigroup.

Proof. We obtained before that $p(x', D_{x'})$ is $(1 - \triangle_x)^\mu$-bounded (with an ε-bound $0 < \varepsilon < 1$), $x = (x', x_{n+1}) \in \mathbb{R}^{n+1}$. From Lemma 3.3.2 we see that $p(x', D_{x'})$ is also $(-A)^\alpha$-bounded. The statement of our theorem follows after applying Theorems 2.1.24 and 1.3.3. □

As an example of a pseudo-differential operator $p(x, D)$ for which (3.32) holds we take $p(x, D)$ of the form

$$-p(x, D)f = \int_{\mathbb{R}^n \setminus \{0\}} (f(x - y) - f(x))\nu(x, dy),$$

with the Lévy kernel $\nu(x, dy) = g(x, y)dy$, and $g(x, y) : \mathbb{R}^n \times \mathbb{R}^n \to \overline{\mathbb{R}}_+$ is a measurable function satisfying with $0 < \delta < \mu_0 < 1$ the estimate $g(x, y) \leq \frac{M}{|y|^{n+\delta}}$.

We want to give a few examples of $(-A)^\alpha$ -bounded pseudo-differential operators with symbols of special kind.

Consider a pseudo-differential operator the symbol of which is of the form:

$$p(x, \xi) = \int_E a(x, y)q(y, \xi)\, dy \tag{3.37}$$

where $E \subset \mathbb{R}^d$ for some d, the function $q(x, \xi)$ is a symbol of another pseudo-differential operator $q(x, D)$ which satisfy with some real continuous negative definite function ψ_0 the inequality

$$\|q(y, D)f\|_{L_p(dx)} \leq c(y)\|f\|_{H_p^{\psi_0,2}} \quad \text{for} \quad f \in H_p^{\psi_0,2}, \quad y \in E, \tag{3.38}$$

$$|a(x,y)| \leq a(y) \quad \forall x \in \mathbb{R}^n, \quad y \in E, \tag{3.39}$$

and for $a(y)$ it holds

$$\int_E |a(y)c(y)|\, dy < c_0. \tag{3.40}$$

Then

$$\|p(\cdot, D)f\|_p \leq \|f\|_{H_p^{\psi_0,2}} \int_E |a(y)c(y)|\, dy \leq c_0 \|\psi_0(D)f\|_p + \delta \|f\|_p \tag{3.41}$$

for some c_0 and δ. Therefore, for $0 < c_0 < 1$ the operator $p(x, D)$ is $\psi(D)$ -bounded. For example, $q(\sigma, \xi)$ may be a continuous negative definite function which depends on parameter σ, for example, $q(\sigma, \xi) = (\sigma^2 + |\xi|^2)^\alpha$, $0 < \alpha < 1$.

Note that when the domain of integration E is bounded, the situation becomes simple because of mean value theorem (see, for example, [Fik], Vol.II, §644): there exists $\sigma_0 \in E$ such that for the symbol $p(x, \xi)$ it holds:

$$p(x,\xi) = \int_E a(x,\sigma) q(\sigma, \xi)\, d\sigma = V(E) a(x, \sigma_0) q(\sigma_0, \xi),$$

where $V(E)$ is the volume of E, and then

$$\begin{aligned} -p(x,D)f(x) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\xi x} p(x,\xi) \hat{f}(\xi)\, d\xi \\ &= V(E)(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\xi x} a(x,\sigma_0) q(\sigma_0, \xi) \hat{f}(\xi)\, d\xi \\ &= -V(E) a(x, \sigma_0) q(\sigma_0, D) f(x). \end{aligned}$$

As another example consider the case when in (3.45) the operator $q(\sigma, \xi) = 1 - e^{-\sigma\psi(\xi)}$, which is for fixed $\sigma > 0$ is a continuous negative definite function, and $a(x, \sigma)$ is a density of some convolution measure with support on $[0, \infty)$:

$$a(x,\sigma) d\sigma = \mu_x(d\sigma).$$

Corollary 3.3.5. Let $p(x', D_{x'})$ be a pseudo-differential operator with symbol

$$p(x', \xi') = \int_E a(x', \sigma) q(\sigma, \xi')\, d\sigma$$

which satisfies (3.38), (3.39) and (3.40) with $0 < c_0 < 1$ and $\psi_0 = \psi^\alpha$, and suppose that $-p(x', D_{x'})$ is a Dirichlet operator on $C_0^\infty(\mathbb{R}^n)$. Then the operator $(-p(x, D) - (-A_\pm)^\alpha, H_{p,+}^{\Re,2})$ and $(-p(x, D) - (-A_+)^\alpha, \tilde{H}_{p,+}^{\Re,2})$, with $symb(-A_\pm) = \psi(\xi') \pm i\xi_{n+1}$, ψ satisfies **A1** and **A2**, are generators of L_p-sub-Markovian semigroups.

The proofs follow from Lemma 3.3.1, Theorem 1.3.3, Theorem 3.1.4 and Theorem 3.1.5.

3.4 Continuity of certain pseudo-differential operators in Besov spaces

Our next purpose is to prove some mapping property of a pseudo- differential operator of special kind. For this we need the **Lipschitz spaces of order** α.

Definition 3.4.1. For $0 < \alpha < 1$ define the **Lipschitz space of order** α by

$$\Lambda_\alpha = \{f \in C_\infty(\mathbb{R}^n), \quad \|f(\cdot - t) - f(\cdot)\|_\infty \leq A|t|^\alpha\}, \tag{3.42}$$

with the norm

$$\|f\|_{\Lambda_\alpha} = \|f\|_\infty + \sup_{|t|>0} \frac{\|f(\cdot - t) - f(\cdot)\|_\infty}{|t|^\alpha} \tag{3.43}$$

on it.

For $\alpha = 1$ define the **Lipschitz space of order 1**, namely, Λ_1 as the closure of the space $C_0^\infty(\mathbb{R}^n)$ with respect to the norm (3.43) with $\alpha = 1$.

Note that in Λ_α, $0 < \alpha < 1$, the space C_0^∞ is NOT dense. We need to work with the subspace Λ_α^0 of Λ_α.

Definition 3.4.2. We will say that $f \in \Lambda_\alpha^0$, $0 < \alpha < 1$, if $f \in \Lambda_\alpha$ and

$$\lim_{|t|\to 0} \frac{\|f(x - t) - f(x)\|_\infty}{|t|^\alpha} = 0. \tag{3.44}$$

For the spaces Λ_α^0 we have

Theorem 3.4.3. The closure of the space Λ_1 with respect to the norm $\|\cdot\|_{\Lambda_\alpha}$ is equal to Λ_α^0.

Proof. Since $C_0^\infty \subset \Lambda_1 \subset \Lambda_\alpha$ and C_0^∞ is dense in Λ_1, we see, that the space $\overline{\Lambda_1}^{\|\cdot\|_{\Lambda_\alpha}}$ is equivalent to $\overline{C_0^\infty}^{\|\cdot\|_{\Lambda_\alpha}}$.
We will follow the proof given in [Kre] for $[0, 1]$, Chapter III, §3.2.
Clearly, for all functions from Λ_1 (3.44) holds. Let $f_m \in \Lambda_1$ and $f_m \to f$ in Λ_α, i.e. $f \in \Lambda_\alpha$. Fix $\varepsilon > 0$. There exists N_0, such that

$$\|f_m\|_{\Lambda_1} \leq c, \quad \|f_m - f\|_{\Lambda_\alpha} < \frac{\varepsilon}{2}$$

for all $m \geq N_0$ and let $\delta > 0$ be such that for $|t| < \delta$

$$|t|^{1-\alpha} < \frac{\varepsilon}{2c}.$$

Then

$$\begin{aligned}\frac{|f(x-t)-f(x)|}{|t|^\alpha} &\le \frac{|f(x-t)-f_m(x-t)-(f(x)-f_m(x))|}{|t|^\alpha} \\ &+ \frac{|f_m(x-t)-f_m(x)|}{|t|^\alpha} \\ &\le \|f_m-f\|_{\Lambda_\alpha}+\|f_m\|_{\Lambda_1}|t|^{1-\alpha} \\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon,\end{aligned}$$

and f satisfies (3.44).

Let now $f \in \Lambda_\alpha$ and satisfy (3.44), i.e. $f \in \Lambda_\alpha^0$. We will show that there exists a sequence $(f_m)_{m\ge 0}$ from Λ_1, that converges to f in the $\|\cdot\|_{\Lambda_\alpha}$-norm.

Consider

$$f_m(x) = m^n \int_x^{x+1/m} f(\tau)\, d\tau - m^n \int_0^{1/m} f(\tau)\, d\tau.$$

Here and further we write

$$\int_x^{x+1/m} \dots d\tau = \int_{x_1}^{x_1+1/m} \dots \int_{x_n}^{x_n+1/m} \dots d\tau_1 \dots d\tau_n,$$

and $\int_0^{1/m} \dots d\tau$ is defined analogously.

The functions f_m, $m \ge 0$, are once continuously differentiable (with respect to each x_i), and therefore belong to Λ_1. Making a change of variables, we obtain

$$f_m(x) = m^n \int_0^{1/m} [f(x+\theta)-f(\theta)]\, d\theta,$$

which gives

$$f_m(x) - f(x) = m^n \int_0^{1/m} [f(x+\theta)-f(\theta)-f(x)]\, d\theta.$$

Define

$$[f_m(x-t)-f(x+t)] - [f_m(x)-f(x)] = \Psi(f_m-f,t);$$

in this notation

$$\|f_m-f\|_{\Lambda_\alpha} = \|f_m-f\|_\infty + \sup_{|t|>0} \frac{|\Psi(f_m-f,t)|}{|t|^\alpha}.$$

Since $f \in \Lambda_\alpha$ and (3.44) holds, then for $\varepsilon > 0$ there exists $\delta > 0$ such that for $|t| < \delta$ the function f satisfies

$$\frac{|f(x+t)-f(x)|}{|t|^\alpha} < \frac{\varepsilon}{2} \qquad \forall x \in \mathbb{R}^n.$$

Therefore

$$\frac{|\Psi(f_m - f, t)|}{|t|^\alpha} = \frac{m^n}{|t|^\alpha}\Big|\int_0^{1/m} [f(x+\theta+t) - f(x+t) - f(x+\theta) + f(x)]\,d\theta\Big|$$
$$\le m^n \int_0^{1/m} \Big(\frac{|f(x+\theta+t) - f(x+\theta)|}{|t|^\alpha} + \frac{|f(x+t) - f(x)|}{|t|^\alpha}\Big)\,d\theta$$
$$< m^n \cdot \frac{1}{m^n}\Big(\frac{\varepsilon}{2} + \frac{\varepsilon}{2}\Big) = \varepsilon.$$

Let $|t| \ge \delta$. Then

$$\frac{|\Psi(f_m - f, t)|}{|t|^\alpha} \le m^n \int_0^{1/m} \frac{|\theta|^\alpha}{|t|^\alpha}\Big(\frac{|f(x+\theta+t) - f(x+t)|}{|\theta|^\alpha} + \frac{|f(x+\theta) - f(x)|}{|\theta|^\alpha}\Big)\,d\theta$$
$$\le 2m^n \|f\|_{\Lambda_\alpha} \int_0^{1/m} \frac{|\theta|^\alpha}{|t|^\alpha}\,d\theta$$
$$\le 2\delta^{-\alpha}\|f\|_{\Lambda_\alpha} m^n \int_0^{1/m} |\theta|^\alpha\,d\theta,$$

and we again may chose N_0 such large that for all $m \ge N_0$

$$m^n \int_0^{1/m} |\theta|^\alpha\,d\theta < \frac{\varepsilon}{2\delta^{-\alpha}\|f\|_{\Lambda_\alpha}}.$$

Therefore $\|f_m - f\|_{\Lambda_\alpha} < \varepsilon$ for $m \ge N_0$. □

Consider now the integro-differential operator of the form:

$$p(x,D)f(x) = \int_{\mathbb{R}^n \setminus \{0\}} (f(x-y) - f(x))\nu(x,dy) \tag{3.45}$$

Theorem 3.4.4. Let $p(x,D)$ be as in (3.45), $\nu(x,dy) = g(x,y)dy$, $0 < \alpha, \sigma, \delta < 1$, $\delta < \alpha < \sigma$, the function $g(x,y)$ is differentiable in x,

$$|g(x,y)| \le \frac{c}{|y|^{n+\delta}}, \tag{3.46}$$

and

$$\int_{|y|<1} |y|^\sigma g'_x(x,y)\,dy + \int_{|y|\ge 1} g'_x(x,y)\,dy < \infty \tag{3.47}$$

uniformly in x.
Then $p(x,D) : \Lambda^0_\sigma \to \Lambda^0_{\sigma-\alpha}$ continuously.

Proof. Let $f \in \Lambda_\sigma^0$. For such f we have

$$\sup_{|y|>0} \frac{\|f(\cdot - y) - f(\cdot)\|_\infty}{|y|^\sigma} < \infty. \tag{3.48}$$

Since

$$\begin{aligned}
\|p(\cdot, D)f\|_\infty &= \sup_{x\in\mathbb{R}^n} |p(x,D)f(x)| = \sup_{x\in\mathbb{R}^n} \Big| \int_{\mathbb{R}^n} (f(x-y) - f(x))\nu(x,dy)\Big| \\
&\le \sup_{x\in\mathbb{R}^n} \Big| \int_{|y|\le 1} \frac{f(x-y)-f(x)}{|y|^\sigma} |y|^\sigma \nu(x,dy)\Big| \\
&\quad + 2\|f\|_\infty \sup_{x\in\mathbb{R}^n} \int_{|y|>1} \nu(x,dy) \\
&\le \sup_{y\in\mathbb{R}^n} \frac{\|f(\cdot - y) - f(\cdot)\|_\infty}{|y|^\sigma} \sup_{x\in\mathbb{R}^n} \int_{|y|\le 1} |y|^\sigma \nu(x,dy) \\
&\quad + 2\|f\|_\infty \sup_{x\in\mathbb{R}^n} \int_{|y|>1} \nu(x,dy) \le c\|f\|_{\Lambda_\sigma},
\end{aligned} \tag{3.49}$$

then it remains to check that

$$\begin{aligned}
\sup_{|h|>0} \frac{1}{|h|^{\sigma-\alpha}} \Big\| \int_{\mathbb{R}^n} & (f(\cdot - h - y) - f(\cdot - h))\nu(\cdot - h, dy) \\
& - \int_{\mathbb{R}^n} (f(\cdot - y) - f(\cdot))\nu(\cdot, dy)\Big\|_\infty < c\|f\|_{\Lambda_\sigma}.
\end{aligned}$$

Note, that if $|h| > 1$ we obtain

$$\begin{aligned}
\frac{1}{|h|^{\sigma-\alpha}} \Big| \int_{\mathbb{R}^n} & (f(x-h-y) - f(x-h))\nu(x-h, dy) \\
& - \int_{\mathbb{R}^n} (f(x-y) - f(x))\nu(x,dy)\Big| \\
& \le \Big| \int_{\mathbb{R}^n} (f(x-h-y) - f(x-h))\nu(x-h,dy)\Big| \\
& \quad + \Big| \int_{\mathbb{R}^n} (f(x-y) - f(x))\nu(x,dy)\Big|,
\end{aligned}$$

and therefore for $f \in \Lambda_\sigma^0$

$$\sup_{|h|\ge 1} \frac{\|\tau_h p(\cdot, D)f(\cdot) - p(\cdot, D)f(\cdot)\|_\infty}{|h|^{\sigma-\alpha}} < \infty, \tag{3.50}$$

where $\tau_h g = g(\cdot - h)$, and thus $p(x, D)$ maps Λ_σ^0 continuously into $C_\infty(\mathbb{R}^n)$.

Next consider the case $|h| < 1$. If we rewrite

$$p(x-h, D)f(x-h) - p(x,D)f(x)$$

as

$$
\begin{aligned}
&p(x-h,D)f(x-h)-p(x,D)f(x)=\\
&\quad=\int_{\mathbb{R}^n}\{(f(x-h-y)-f(x-h))-(f(x-y)-f(x))\}g(x,y)\,dy\\
&\quad\quad+\int_{\mathbb{R}^n}(f(x-h-y)-f(x-h))(g(x-h,y)-g(x,y))\,dy\\
&\quad=I_1(x)+I_2(x).
\end{aligned}
$$

Since from the mean-value theorem

$$|g(x-h,y)-g(x,y)|=|h|g'_x(\theta,y),$$

for some θ, in view of condition (3.47) we obtain

$$
\begin{aligned}
I_2(x)&\le|h|\sup_{|y|>0}\frac{\|f(\cdot-h-y)-f(\cdot-h)\|_\infty}{|y|^\sigma}\int_{|y|<1}|y|^\sigma g'_x(\theta,y)\,dy\\
&\quad+2|h|\|f\|_\infty\int_{|y|\ge1}g'_x(\theta,y)\,dy\le|h|\|f\|_{\Lambda_\sigma},
\end{aligned}
$$

and therefore

$$\frac{\|I_2\|_\infty}{|h|^{\sigma-\alpha}}\le|h|^{1+\alpha-\sigma}\|f\|_{\Lambda_\sigma}. \tag{3.51}$$

To estimate $\frac{\|I_1\|_\infty}{|h|^{\sigma-\alpha}}$ we divide it into two parts

$$
\begin{aligned}
\frac{|I_1(x)|}{|h|^{\sigma-\alpha}}&=|h|^{\alpha-\sigma}\Big(\int_{|y|<|h|}\{(f(x-h-y)-f(x-h))-(f(x-y)-f(x))\}g(x,y)\,dy\\
&\quad+\int_{|y|\ge|h|}\{(f(x-h-y)-f(x-h))-(f(x-y)-f(x))\}g(x,y)\,dy\Big)\\
&\le\sup_{|y|>0}\frac{\|(f(\cdot-h-y)-f(\cdot-h))-(f(\cdot-y)-f(\cdot))\|_\infty}{|y|^\sigma}|h|^{\alpha-\sigma}\int_0^h\frac{c\,dy}{|y|^{n+\delta-\sigma}}\\
&\quad+\sup_{0<|h|<1}\sup_{|y|>|h|}\frac{\|(f(\cdot-h-y)-f(\cdot-y))-(f(\cdot-h)-f(\cdot))\|_\infty}{|h|^\sigma}|h|^\alpha\int_h^\infty\frac{c\,dy}{|y|^{n+\delta}}\\
&\le C\|f\|_{\Lambda_\sigma}|h|^{\alpha-\delta},
\end{aligned}
$$

and thus for all $|h|\le1$ and some constant c_0

$$\sup_{|h|\le1}\frac{\|p(\cdot-h,D)f(\cdot-h)-p(\cdot,D)f(\cdot)\|_\infty}{|h|^{\sigma-\alpha}}\le c_0\|f\|_{\Lambda_\sigma}. \tag{3.52}$$

Combining (3.52) with (3.50) we arrive at

$$\sup_{|h|>0}\frac{\|p(\cdot-h,D)f(\cdot-h)-p(\cdot,D)f(\cdot)\|_\infty}{|h|^{\sigma-\alpha}}\le c\|f\|_{\Lambda_\sigma}, \tag{3.53}$$

and together with (3.49) we obtain that

$$p(x, D) : \Lambda_\sigma^0 \to \Lambda_{\sigma-\alpha}.$$

Moreover, we can see from (3.52) that

$$\lim_{|h|\to 0} \frac{\|p(\cdot - h, D)f(\cdot - h) - p(\cdot, D)f(\cdot)\|_\infty}{|h|^{\sigma-\alpha}} = 0, \tag{3.54}$$

which gives us the statement of the theorem. □

Let $(X_0, \|\cdot\|_{X_0})$ be a Banach space and $X_1 \subset X_0$ a dense subspace such that there is a norm $\|\cdot\|_{X_1}$ turning $(X_1, \|\cdot\|_{X_1})$ into a Banach space and

$$\|f\|_{X_0} \le c\|f\|_{X_1} \quad \text{for all} \quad f \in X_1.$$

For the following theorem is taken from [Kre] (see also [J1], Theorem 2.8.7).

Theorem 3.4.5. Let $(X_0, \|\cdot\|_{X_0})$ and $(X_1, \|\cdot\|_{X_1})$ be two Banach spaces as above and further let $(Y_0, \|\cdot\|_{Y_0})$ and $(Y_1, \|\cdot\|_{Y_1})$ be two Banach spaces satisfying the same conditions as X_0 and X_1. Suppose that $T : X_0 \to X_1$ is a bounded linear operator such that $Af \in Y_k$ for $f \in X_k$, and

$$\|Af\|_{Y_k} \le M_k\|f\|_{X_k}, \quad k = 0, 1.$$

Then A maps continuously $X_\theta = [X_0, X_1]_\theta$ into $Y_\theta = [Y_0, Y_1]_\theta$, where X_θ and Y_θ are complex interpolation spaces, and we have the estimate:

$$\|Af\|_{Y_\theta} \le M_0^{1-\theta}M_1^\theta\|f\|_{X_\theta}, \quad \theta \in [0, 1]. \tag{3.55}$$

Applying this theorem, we derive

Theorem 3.4.6. Let $p(x, D)$ be as in Theorem 3.4.4, and assume in addition that for $t > \alpha + 1 + \frac{n}{2}$ the operator $p(x, D)$ is such that

$$p(x, D) : \quad H^t \to H^{t-\alpha}. \tag{3.56}$$

continuously, where $H^t = H_2^t$ is the Sobolev space of order t. Then for $p = \frac{2}{\theta}$, $s = (1-\theta)\sigma + \theta t$, and $0 < \theta < 1$ we obtain

$$p(x, D) : \quad W_p^s \to W_p^{s-\alpha}, \tag{3.57}$$

where $0 < \alpha < \sigma < 1$.

Proof. We know from Theorem 3.4.4 that $p(x,D)\colon \Lambda^0_\sigma \to \Lambda^0_{\sigma-\alpha}$, and for $t > \alpha + 1 + \frac{n}{2}$ the spaces H^t and $H^{t-\alpha}$ are dense in C^1_∞. Then, since the last space is dense in Λ^0_σ and $\Lambda^0_{\sigma-\alpha}$, the space H^t is dense in Λ^0_σ and the space $H^{t-\alpha}$ is dense in $\Lambda^0_{\sigma-\alpha}$. Therefore, applying Theorem 3.4.5 we obtain that $p(x,D)$ is continuous from $[\Lambda^0_\sigma, H^t]_\theta$ to $[\Lambda^0_{\sigma-\alpha}, H^{t-\alpha}]_\theta$ for some $0 < \theta < 1$. But by the definition of the space Λ^0_σ the norm on it is $\|\cdot\|_{\Lambda_\sigma}$, therefore the norm on the space $[\Lambda^0_\sigma, H^t]_\theta$ is equivalent to the norm on the space $[\Lambda_\sigma, H^t]_\theta$. For $0 < \sigma < 1$ we have $\Lambda_\sigma = B^\sigma_{\infty,\infty}$ (see [T1], §2.3.5), and since $H^t = B^t_{2,2}$, then

$$[\Lambda_\sigma, H^t]_\theta = [B^\sigma_{\infty,\infty}, B^t_{2,2}]_\theta = B^{(1-\theta)\sigma+\theta t}_{\frac{2}{\theta},\frac{2}{\theta}} = B^s_{p,p} = W^s_p. \tag{3.58}$$

Analogously

$$[\Lambda_{\sigma-\alpha}, H^{t-\alpha}]_\theta = W^{s-\alpha}_p. \tag{3.59}$$

Equations (3.58) and (3.59) gives us that for $f \in C^\infty_0$ it holds

$$\|p(x,D)f\|_{W^{s-\alpha}_p} \leq c\|f\|_{W^s_p}, \tag{3.60}$$

and since the set C^∞_0 is dense in both $W^{s-\alpha}_p$ and W^s_p, then (3.60) holds for $f \in W^s_p$. □

Notation

General Notation

x^+	x, if $x > 0$, and 0, when $x \leq 0$
$a \wedge b$	$= min(a, b)$
$\mathbb{R}$	real numbers
$\mathbb{R}^n$	Euclidean vector space of dimention n
$\mathbb{R}^{n+1}_{0+}$	$= \mathbb{R}^{n+1} \times [0, \infty)$
$\mathbb{R}^{n+1}_{+}$	$= \mathbb{R}^{n+1} \times (0, \infty)$
$\mathbb{C}$	complex numbers
$Im\, f$	imaginary part of f
$Re\, f$	real part of f
$\operatorname{supp} f$	support of a function
$\overline{D}$	closure of a set D
1_D	characteristic function of a set D
$A^{(p)}$	an operator A defined on a subspace of L_p
$D(A)$	domain of an operator A
$R(A)$	range of an operator A
$\rho(A)$	resolvent set of an operator A, Definition 1.1.14
R_λ	resolvent of an operator A at λ
$(T_t)_{t\geq 0}$	one-parameter semigroup of operators, Definition 1.1.1

Functions and Distributions

$\hat{g}, Fg$	Fourier transform of a function, (1.3)
$\hat{\mu}_t$	Fourier transform of a measure, (1.4)
$L(f)$	Laplace transform, (1.7)
$f * g$	convolution of functions
$f \circ g$	composition of functions
$f \otimes g$	tensor product of functions, Definition 2.1.4, A
$A \otimes B$	tensor product of operators, Definition 2.1.4, C
$E(X) \otimes E(Y)$	tensor product of spaces, Definition 2.1.4, B
T_t^η	subordinated semigroup, (1.6)
μ_t^f	subordinate convolution semigroup, (1.8)
$\Gamma(t)$	Gamma function
$\sigma_\alpha(x,t)$	density of one-side stable semigroup of measures of order α, (1.9)
$e_\alpha(x,\mu)$	Mittag-Leffler type function, (1.12)
$E_{\alpha,\beta}(z)$	two-parameter Mittag-Leffler function, (1.13)
$J_\nu(z)$	(1.20)
$I_\nu(z)$	(1.21)
$K_\nu(z)$	(1.22)
$\lambda^{(n)}(dx)$	n-dimensional Lebesgue measure
$\delta_x(dx)$	Dirac measure, (1.2)
$\Re$	$= Re\,(\chi_\pm(\xi))^\alpha$
$\mathcal{W}_t$	Theorem 3.1.1
$\psi(D)$	pseudo-differential operator with symbol $\psi(\xi)$
$\psi_R(\xi)$	(2.40)

$p(x, D)$	pseudo-differential operator
$p(x, \xi)$	symbol of a pseudo-differential operator $p(x, D)$
$V_r^{(p)}$	Gamma-transform, Definition 2.1.1
$T_t^{(1)}$	(3.1)
$T_t^{(1')}$	(3.6)
$T_t^{(2)}$	(3.2)
$T_t^{(3)}$	(3.7)
$T_t^{(4)}$	(3.8)
$T_t^{(5)}$	(3.9)
$T_t^{(6)}$	(3.14)
$T_t^{(6')}$	(3.15)
$p_t^{\delta,a}(x)$	density of the Meixner process, (3.21)

Function Spaces and Norms

$L_p(G)$	Lebesgue space over a set $G \subset \mathbb{R}^n$ with respect to the Lebesgue measure $\lambda(dx)$
$L(A, B)$	space of continuous linear operators from A to B
$\mathcal{F}_{r,p}(\mathbb{R}^n, \mathbb{R})$	Bessel-type potential space, Definition 2.1.3
$\|u\|_{\mathcal{F}_{r,p}}$	norm in the space $\mathcal{F}_{r,p}(\mathbb{R}^n, \mathbb{R})$, (2.2)
$S(\mathbb{R}^n)$	Schwartz space
$S'(\mathbb{R}^n)$	dual to the Schwartz space
$H_p^{\psi,s}$	ψ-Bessel potential space, Definition 2.1.4
$B_{\psi,p}^s$	Definition 2.1.8
$H_p^{\chi,s,1}$	Definition 2.1.11
$H_{p,+}^{\psi,s}$	Definition 2.2.1

$\tilde{H}^{\psi,s}_{p,+}$	Definition 2.2.1
B^s_{pq}	Besov space
W^s_p	Sobolev space in L_p of integer order s
$H^s = H^s_2$	L_2-Sobolev space of fractional order s
Λ_α	Lipschitz space of order α, Definition 3.4.1
$\|f\|_{\Lambda_\alpha}$	norm in the Lipschitz space of order α, see (3.43)
Λ^0_α	Definition 3.4.2
$W(G,X)$	p.16
$[X_0, X_1]_\theta$	complex interpolation space, (2.11)
$C(\mathbb{R}^n)$	space of continuous functions on $\mathbb{R}^n$
$C_\infty(\mathbb{R}^n)$	space of continuous functions on $\mathbb{R}^n$ vanishing at infinity
$C^k_\infty(\mathbb{R}^n)$	$=\{f \in C_\infty(\mathbb{R}^n),\ \partial^{(l)} f \in C_\infty(\mathbb{R}^n),\ \|l\| \leq k\}$
$C^\infty_0(\mathbb{R}^n)$	space of infinitely many times differentiable functions on $\mathbb{R}^n$ with compact support
$C^k_\infty(\mathbb{R}^n_{0+})$	Definition 2.2.9
$\dot{C}^k_\infty(\mathbb{R}^n_{0+})$	Definition 2.2.9
$C^\infty_0(\mathbb{R}^n_{0+})$	Definition 2.2.10, a
$C^\infty_0(\mathbb{R}^n_+)$	Definition 2.2.10, b

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