Exact results in supersymmetric field theory.

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EXACT RESULTS IN SUPERSYMMETRIC FIELD THEORY

By

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Abstract

This thesis examines $\mathcal{N} = 2$ Super-Yang-Mills theory where the low-energy effective action of the theory is governed by a holomorphic function called the prepotential. The Seiberg-Witten solution of the theory determines the prepotential in terms of an complex curve and, once we compactify the theory on a circle, we will examine the identification of this complex curve with the spectral curve of the Calogero-Moser integrable system. Since the supersymmetry restricts the perturbative contributions to the prepotential, the results we gain are exact. Further, they are independent of the compactification radius. The generalization to the quiver models, with gauge group $SU(N)^k$, is introduced along with the spin generalization of the integrable system. The massive vacua of these theories have been determined previously, here we examine the case of a specific gauge group in order to determine the complete phase structure, including the massless vacua.

We then move on to determining contributions coming from instantons to the prepotential of the theory with gauge group $SU(N)$. We see how by lifting the theory onto 5 dimensions the functional integral on the instanton moduli space is realized as a quantum mechanical σ-model with the moduli space as a target. However, just such a model is shown to calculate a particular index of the manifold, in this case a particular equivariant index since the space has isometries. We account for the non-compact nature of the moduli space by removing boundary terms and then calculate explicit results in the case of $SU(2)$.

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Contents

Chapter 1

Introduction

A beautiful aspect of some supersymmetric theories is that certain quantities can be calculated exactly, i.e. the perturbative expansion terminates and semi-classical effects coming from instantons can be calculated exactly. This allows us to determine the precise vacuum structure of the theory. We will examine two situations where this is possible, both centring on the Seiberg-Witten solution of $\mathcal{N} = 2$ Super-Yang-Mills (SYM) theory [1, 2].

It is well known that the $\mathcal{N} = 2$ SYM theory can be described by a low-energy effective action. This is because the D- and F-flatness conditions cause the lowest component of the chiral supermultiplet to gain a non-zero vacuum-expectation-value (VEV), hence the massive degrees of freedom can be integrated out. The restrictions imposed by supersymmetry (holomorphicity, duality and renormalizability) allow us to determine the perturbative contributions exactly since the expansion terminates at one-loop. We are then left with determining the non-perturbative contributions which are captured by the leading order semi-classical[1] contributions of instantons.

[1]The term *semi-classical* is used since the constant g^2 appears in the partition in just the same place as Planck's constant $\hbar$, if we were to reinstate all physical units, thus the limit $g^2 \to 0$ is identical to $\hbar \to 0$.

The work of Seiberg-Witten was to show how the prepotential, $\mathcal{F}$, which contained all this information was determined by an auxiliary complex curve, which itself could be viewed as a Riemann surface.

1.1 Exact Superpotentials

The first topic we will consider is the exact determination of vacua in mass-deformed *quiver* models, where the gauge group is $\mathrm{SU}(N)^k$. The original example, with k set to 1, is the mass-deformed $\mathcal{N} = 4$ theory where the exact $\mathrm{SL}(2,\mathbb{Z})$ S-duality[2] of the pure theory is broken but remains as a symmetry relating different vacua in the softly-broken theory.[3] Upon breaking the theory to $\mathcal{N} = 2$ supersymmetry we find the low-energy effective action is solved by the Seiberg-Witten curve. However, we can further break to $\mathcal{N} = 1$ supersymmetry and doing so lifts the Coulomb branch of the theory except at singularities, where the Seiberg-Witten curve degenerates. Thus we find the curve's singularities exactly define the vacua of the $\mathcal{N} = 1^\star$ theory. It was shown by Donagi and Witten [3] that, in the $\mathcal{N} = 1^\star$ theory with gauge group $\mathrm{SU}(N)$, there were $\sum_{d|N} d$ massive vacua which lie on sublattices of the torus and so form a finite-dimensional representation of $\mathrm{SL}(2,\mathbb{Z})$.

An alternative approach is to compactify on $\mathbb{R}^3 \times S^1$ [4]. Now the Coulomb branch of the $\mathcal{N} = 2$ theory has twice the dimension of the uncompactified theory due to Wilson lines and dual photons around the compact dimension. It turns out that we then have exactly the correct degrees of freedom to identify the $\mathcal{N} = 2$ theory with

[2]The full $\mathrm{SL}(2,\mathbb{Z})$ symmetry of $\mathcal{N} = 4$ SYM contains two generators: S takes the coupling $\tau \to -\frac{1}{\tau}$ and interchanges electric and magnetic charges, while T takes $\tau \to \tau + 1$ and arises since the θ component of the coupling is periodic.

[3]Mass-deforming, or softly-breaking, a supersymmetric theory is done by the addition of a superpotential and the resulting theory is labelled by the remaining supersymmetries and a star, e.g. $\mathcal{N} = 2^\star$ or $\mathcal{N} = 1^\star$.

a completely integrable dynamical system. Even without the compactification the Seiberg-Witten curve is identical to the spectral curve of the elliptic Calogero-Moser system; now however we find the Coulomb branch, as a complex manifold, is itself simply the complexified phase space of the integrable system. The Wilson lines and dual photons describe the missing *angle* variables. The superpotential which reduces the theory to $\mathcal{N} = 1$ supersymmetry can then be calculated by identifying it with one of the Hamiltonians. What is not so apparent is that this picture captures all the quantum corrections. We thus obtain exact results. It also turns out that the Hamiltonian is independent of the radius, R, of the compact dimension and so the results are still valid in the $R \to \infty$ limit.

These results were then generalized to the quiver theories, with gauge groups $\mathrm{SU}(N)^k$, in [5], where the integrable system is now a spin generalization of the Calogero-Moser system. Again the massive vacua in the general case were determined and again found to lie on sublattices of the torus, τ. However, a classification of the massless vacua is not known. One reason for this is the following: since the vacua are identified by the critical points of the exact superpotential, which is just a Hamiltonian of the integrable system, they are also equilibrium points in the flow generated by the particular Hamiltonian. The massive vacua are special in that they are equilibrium points of all the Hamiltonian flows, while the massless vacua are only equilibrium points for a subset of the flows. Our investigation will determine all vacua, massive and massless, in the specific case of $\mathrm{SU}(2) \times \mathrm{SU}(2)$.

1.2 Index Theory on the Instanton Moduli Space

The instanton contributions to the prepotential, $\mathcal{F}$, can, in principle, be calculated either from the Seiberg-Witten curve or directly from the functional integral. Let us consider the latter approach. In the semi-classical limit the functional integral localizes around the instanton solutions and we need only integrate out the fluctuations to leading order. We are still left with a complicated integral on the instanton moduli space, $\hat{\mathfrak{M}}_k$, for each level of instanton charge, k. The second topic of this thesis is thus to determine if the instanton coefficients contain any topological information about the instanton moduli space.

The functional integral on the instanton moduli space takes the form of the partition function of a zero-dimensional supersymmetric σ-model with $\hat{\mathfrak{M}}_k$ as a target [6]. However, we know from the work of Alvarez-Gaume [7] how the partition function of a *quantum mechanical* supersymmetric σ-model calculates a topological index of the target space. Thus to make the connection we must lift the original gauge theory onto $\mathbb{R}^4 \times S^1$ allowing the σ-model to become quantum mechanical. Once this is done we can realize the prepotential in terms of the number of zero-energy states in the σ-model. We will also be able to identify these vacuum states with the BPS states of instanton *dyons* in the parent theory.

This connection is further motivated through the recent work of Nekrasov [8]. Here the prepotential is calculated by considering both *equivariant cohomology* introduced by the hyper-Kähler structure of $\hat{\mathfrak{M}}_k$, and the action of the Lorentz symmetries of the space. We will use this formula, as well as the method of Csaki et al [9], to explicitly determine the prepotential in five dimensions, and so determine explicit index numbers.

Of course, everything is not quite so simple. The target space, $\hat{\mathfrak{M}}_k$, is not compact. Firstly, the moduli space has orbifold type singularities corresponding to instantons of zero size. These can be removed by deforming the space in a way consistent with its hyper-Kähler structure. Secondly, the moduli space is non-compact as instantons become arbitrarily separated. This can be countered by setting a cut-off on the separation of clumps of the instantons. Then we can calculate the partition function with a boundary and take the boundary to infinity. Since the identification of the partition function with the Witten index only holds in the compact case, we will see how the boundary terms change the story, and so how to access the true index.

There is an alternative method, known as *geometric engineering* of gauge theories, where the $\mathcal{N} = 2$ SU(N) dynamics are found embedded in Type IIA string theory in 10 dimensions, compactified on a Calabi-Yau threefold with gravity decoupled. In such a method the prepotential is determined by contributions from *worldsheet* instantons, and the formula which we calculate when taking the boundary effects into account is in fact identical to a formula given by geometric engineering.

As well as investigating the above for pure $\mathcal{N} = 2$ we shall also examine the theory with an adjoint hypermultiplet, the $\mathcal{N} = 2^*$ theory. This will in fact allow us to interpolate between the $\mathcal{N} = 2$ theory and the $\mathcal{N} = 4$ theory for which the prepotential is VEV independent (i.e. the beta function is zero). It is then an interesting point to see how the boundary terms are unchanged, yet the whole story is consistent with previous results. This will be done for the explicit case of SU(2), with predictions made for the more general result.

1.3 Outline

The outline of the thesis is as follows: in Chapter 2 we examine the Exact Superpotentials discussed above. We begin with a brief look at the classical results in the $\mathcal{N} = 2^{\star}$ theory. Then we detail some basics regarding integrable systems and how the Seiberg-Witten solution can be identified with particular such systems. The compactification method is then described allowing the exact quantities to be found. With this structure in mind we move on to the quiver theories looking at the general results for gauge group $\mathrm{SU}(N)^k$ before detailing the calculations in the specific case of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ where we can determine the nature of all vacua. Parts of this chapter have been published in [10].

Chapter 3 sets us up on the road to calculating topological indices on the instanton moduli space. We have a brief summary of the various elliptic complexes we shall meet later, and the indices they calculate. Then the 'heat kernel' method of Alvarez-Gaume for relating these to supersymmetric quantum mechanics is detailed for the relevant cases. Parts of this chapter have been published in [11].

Chapter 4 goes on to the physics of the $\mathcal{N} = 2, 2^{\star}$ moduli space, beginning with a review of the instanton contributions. The heat kernel method becomes relevant with the lift to five dimensions, we will examine the full quantum mechanical σ-models and identify the BPS states involved. Finally, we examine the complication of non-compactness, and determine a formula for explicit determination of the indices.

Chapter 5 examines the case of gauge group SU(2) and calculates the prepotential via three different methods, all of which agree. Once the boundary contributions determined in Chapter 4 are removed we find integer values for the indices. Parts of these two chapters are to be published in [12] and [13]. We will then summarize the results and briefly examine directions for future investigation.

Chapter 2

Phase Structure of Broken Quiver Models

We begin then with the study of exact superpotentials in theories with $\mathcal{N} = 4$ or $\mathcal{N} = 2$ supersymmetry (SUSY), broken to $\mathcal{N} = 1$, and their connection with certain integrable systems. Before examining the compactification method we will introduce the types of vacua which may appear and the Seiberg-Witten solution in $\mathcal{N} = 2$. Then the model is generalized to the quiver theories, where the massive vacua have already been determined systematically. We will then examine the specific case of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ and determine all vacua.

2.1 Phases of Softly Broken $\mathcal{N} = 4$

In this section we will briefly examine the phase structure of softly-broken $\mathcal{N} = 4$ SYM theory by classical reasoning. Beginning with the 4-dimensional theory, with gauge group $\mathrm{SU}(N)$, the superspace Lagrangian has a superpotential of the form

$$\mathcal{W} = \mathrm{Tr}\left(\Phi_1[\Phi_2, \Phi_3] + \sum_{i=1}^{3} m_i \Phi_i^2\right) , \tag{2.1}$$

where we have used the notation of $\mathcal{N}=1$ superfields, so that the Φ_i are three chiral multiplets in the adjoint representation of the gauge group. To find the vacua of the theory we should solve the F-flatness equations coming from the superpotential above (2.1) as well as the D-flatness equations, coming from the kinetic terms, modulo gauge transformations. As usual, instead of imposing D-flatness, we can solve the F-flatness equations modulo complexified gauge transformations, given by $\mathrm{SL}(N,\mathbb{C})$.

After a subtle rescaling[1] we find the stationary points of $\mathcal{W}$ in (2.1) are

$$\frac{\partial \mathcal{W}}{\partial \widetilde{\Phi}_i}=0 \Longrightarrow \widetilde{\Phi}_i=\epsilon_{ijk}[\widetilde{\Phi}_j,\widetilde{\Phi}_k]\ , \tag{2.2}$$

which are simply the commutation relations of the SU(2) Lie algebra. We thus have one solution for every N-dimensional representation of SU(2). The nature of each solution depends on the decomposition of its representation, ρ, into irreducible pieces. There are broadly three possibilities:

- *Higgs*: ρ is itself irreducible, the gauge group is completely broken; the theory remains weakly coupled and has one vacuum,

- *Confining*: ρ is the sum of $d(>1)$ irreducible pieces of the same dimension; in the IR the theory flows to $\mathcal{N}=1$ SYM with gauge group SU(d), since the allowed gauge factors interchange the irreducible pieces, and has a mass gap, this is known to have d supersymmetric vacua; note this includes the trivial case when $d=N$ and the gauge group is unbroken,

- *Coulomb/Massless*: ρ is a sum of irreducible representations of different dimensions, the gauge group contains one or more Abelian factors indicating at least one massless photon.

[1] $\Phi_1=2\sqrt{m_2m_3}\widetilde{\Phi}_1$, $\Phi_2=2\sqrt{m_3m_1}\widetilde{\Phi}_2$ and $\Phi_3=-2\sqrt{m_1m_2}\widetilde{\Phi}_3$.

We thus arrive at the results of [3], quoted in the introduction, that the total number of massive vacua is the sum of the divisors of N. Massless vacua can occur for $N \geq 3$, though little is known of these in general.

2.2 Integrable Systems and the Seiberg-Witten solution

The full quantum structure of $\mathcal{N} = 2$ SUSY theories is known to be intimately connected to certain integrable systems (see [14] for a full review). In many cases these systems are simply classical mechanical systems of N-particles in one dimension with position and momenta, X_i and p_i respectively, $i = 1, \ldots, N$. The systems in question are *completely integrable*, meaning we have N independent conserved quantities H_i, each generating a *flow* on the $2N$-dimensional phase space of positions and momenta.

In such an integrable system there is an alternative description of the system, with a much simpler geometrical view of the phase space, $\mathcal{M}$, as a fibre bundle described by *action-angle* variables. The action variables, a_i, are functions of the Hamiltonians and are constant along the flows; meanwhile the angle variables, θ_i, lie on the fibre at each point in $\mathcal{M}$. Table 2.1 shows how the Poisson brackets and derivatives compare with the original description.

Canonical	Action-angle
$\{p_i, X_j\} = \delta_{ij}$	$\{a_i, \theta_j\} = \delta_{ij}$
$\dot{p}_i = -\frac{\partial H}{\partial X_i}$ and $\dot{X}_i = \frac{\partial H}{\partial p_i}$	$\dot{a}_i = 0$ and $\dot{\theta}_i = \frac{\partial H}{\partial a_i}$

Table 2.1: Canonical and Action-angle variable descriptions of Hamiltonian systems.

In these systems there is a hidden Riemann surface Σ_{int}, of genus N, which plays a central role. In particular, the angle variables (considered to be complex) take values in the Jacobian torus of Σ_{int}, which itself depends on the position in the moduli space $\{a_i\}$. The Jacobian torus is defined as follows. The *period matrix*, $\tau_{ij}(a_k)$, of Σ_{int} is given by integrating holomorphic one-forms[2], ω_i, around a canonical basis of cycles $\{A_i, B_i\}$ on Σ_{int} (see Figure 2.1)

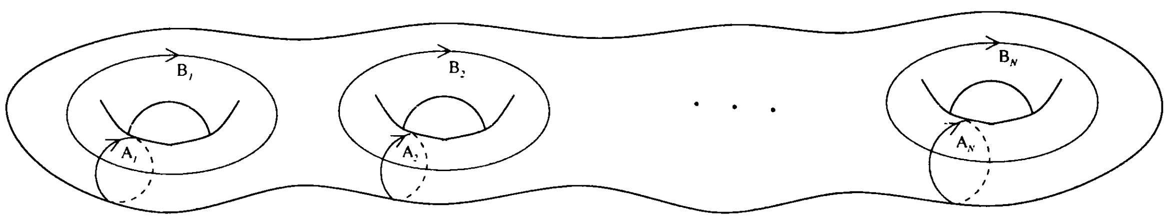


Figure 2.1: 1-cycles on a Riemann Surface, Σ_{int}.

$$\oint_{A_i} \omega_j = \delta_{ij}, \qquad \oint_{B_i} \omega_j = \tau_{ij}, \qquad i, j = 1, \ldots, N \ . \tag{2.3}$$

The angle variables are then identified with points in the Jacobian torus, $T^N(a_i)$, of Σ_{int}. The torus arises because although the angle variables live on the complex plane, thus $\{\theta_i\} \in \mathbb{C}^N$, there is a periodic identification in two directions

$$\theta_i = \theta_i + m_i + \sum_{j=1}^{N} \tau_{ij}(a_k) n_j, \qquad m_i, n_i \in \mathbb{Z} \ , \tag{2.4}$$

where τ_{ij} is just the period matrix of the Riemann surface.

The spectral curve of Σ_{int} is defined by the equation

$$F(v, z) = \det_{N\times N}(L(z) + v.\mathbf{1}) = 0 \ , \tag{2.5}$$

[2]Alternatively called abelian differentials of the first kind.

where z and v parameterize the surface and $L(z)$ is the $N \times N$ *Lax* matrix whose components, in the elliptic Calogero-Moser system which will be relevant to our physical theory, are given by

$$L_{ij}(z) = p_i\delta_{ij} + m(1-\delta_{ij})\frac{\sigma(z - X_i + X_j)}{\sigma(z)\sigma(X_i - X_j)}e^{\zeta(z)(X_i - X_j)} \ . \tag{2.6}$$

Here, m will turn out to be the mass of the adjoint hypermultiplet of the $\mathcal{N} = 2^*$ theory. $F(v, z)$ is a polynomial of degree N in v whose coefficients involve the conserved quantities, i.e. the action variables, of the system. Thus the characteristic equation (2.5) of the Lax operator generates a basis in the space of Hamiltonians.

Now we can turn to the physical theory, in particular the Seiberg-Witten solution of $\mathcal{N} = 2^*$ SYM. As outlined in the introduction, after solving the F-flatness conditions we find the low-energy effective action has the following form

$$\mathcal{S}_{\text{eff}} = \frac{1}{4\pi}\int d^4x \ \operatorname{Im}\left\{\tau_{\mu\nu}(\Phi)W_\mu^{\dot\alpha}W_{\nu\dot\alpha}\big|_{\theta\theta} + \Phi_{Du}\Phi_u^\dagger\big|_{\theta\theta\theta\theta}\right\} + \text{h.o.t} \ , \tag{2.7}$$

where the coupling constant τ is an $N \times N$ matrix, and h.o.t refers to terms at higher order in derivatives which can be ignored at low energy. Both the coupling and the field Φ_{Du} are related to the *prepotential* $\mathcal{F}(\Phi)$

$$\tau_{\mu\nu}(\Phi) = \frac{\partial^2\mathcal{F}}{\partial\Phi_\mu\partial\Phi_\nu}, \qquad\qquad \Phi_{Du} = \frac{\partial\mathcal{F}}{\partial\Phi_\mu} \ . \tag{2.8}$$

(2.7) is written in $\mathcal{N} = 1$ superspace notation [15] with one chiral superfield Φ and one gauge multiplet contained in the spinoral field W. It is important that $\mathcal{F}$ is a holomorphic function[3], as required by supersymmetry – this fact played a crucial role in Seiberg and Witten's theory which determines $\mathcal{F}$ exactly. The S-duality between magnetic and electric charges predicted to be exact in the original $\mathcal{N} = 4$ theory is

[3] $\mathcal{F}$ depends only on Φ and not its conjugate $\Phi^\dagger$.

depicted here by the 'dual' magnetic field Φ_D compared to the original electric field Φ. To 'solve' the theory we expand (2.7) in component fields and note that the F-flatness condition allows a VEV for the lowest component of the chiral fields

$$\mathrm{tr}[\phi, \phi^\dagger]^2 = 0 \implies \phi = \mathrm{diag}(a_1, \ldots, a_N) \ . \tag{2.9}$$

At generic points $\{a_i\}$, i.e. $a_i \neq a_j$, this breaks the gauge group to its maximal abelian subgroup, $\mathrm{U}(1)^N$, and so the theory is described as being on the *Coulomb* branch. The N-dimensional moduli space of the Coulomb branch can be described locally either by the a's or the dual a_D's. For a global parameterization we must determine gauge invariant operators such as the condensates $\{u_n\}$, $n = 2, \ldots, N$

$$u_n = \left\langle \mathrm{tr}_N \phi^n \right\rangle \ . \tag{2.10}$$

We are now ready to identify certain parts of the integrable system with the properties of the moduli space. Firstly, the VEVs a_i are exactly the action variables of the integrable system. They are given by integrating the 'Seiberg-Witten' meromorphic 1-form[4], λ, around the cycles A_i

$$a_i = \frac{1}{2\pi i} \oint_{A_i} \lambda, \qquad a_{D_i} = \frac{1}{2\pi i} \oint_{B_i} \lambda = \frac{\partial \mathcal{F}}{\partial a} \ , \tag{2.11}$$

where it turns out λ is simply

$$\lambda = v\, dz \ . \tag{2.12}$$

Secondly, the condensates are just the coefficients given in the spectral curve (2.5). Finally, the matrix of coupling constants is of course the period matrix of Σ_{int}. However, we have no degrees of freedom which correspond to the angle variables.

[4]An abelian differential of the third kind.

2.3 Compactification and Exact Results

Clearly the moduli space of the integrable system is not identical to the Coulomb branch of the gauge theory since we have no candidates for angle variables in the gauge theory. Thus we are motivated to compactify the gauge theory on a circle of radius R. The component of the gauge fields wrapping the compact dimension becomes a scalar quantity called the Wilson line, while the remaining three components are dual to a scalar called the dual photon. Thus, since we have an effective $\mathrm{U}(1)^N$ gauge group there are N dual photons and N Wilson lines, which together amass into N complex scalar fields that live on a complex N-torus defined by the period matrix of Σ_{int} as in (2.4). This is exactly what we want since they can now be identified with the missing angle variables which live on the Jacobian of Σ_{int}. The complete moduli spaces of both the integrable system and the Coulomb branch of the gauge theory in 3-dimensions thus have $2N$-complex dimensions. Hence the moduli space of the gauge theory on $\mathbb{R}^3 \times S^1$ is identified with the whole phase space of the integrable system.

This becomes even more productive once we consider further soft-breaking to $\mathcal{N} = 1$ SUSY. This is done by adding a tree-level superpotential[5]

$$\frac{1}{g}\mathrm{Tr}\mathcal{W}(\Phi) , \tag{2.13}$$

and can be viewed as a perturbation lifting most of the Coulomb branch to leave just the vacua of the $\mathcal{N} = 1$ theory, given by critical points of the superpotential. However $\mathcal{W}(\Phi)$ reduces to a polynomial function of the VEVs, $\tilde{\mathcal{W}}(a_i)$ and in the integrable system such a function is a conserved quantity and hence is a particular Hamiltonian.

[5]for example, the mass terms in (2.1).

Thus to determine the vacua we must simply find the stationary points of the relevant Hamiltonian with respect to the moduli, e.g.

$$\frac{\partial H}{\partial p_i} = \frac{\partial H}{\partial X_i} = 0 \ . \tag{2.14}$$

This is clearly an equilibrium point of the system by Hamilton's equations. Finally, we will find that the Hamiltonians do not depend on the compactification radius, R, and so the results are valid in the $R \to \infty$ limit. Let us examine an example of how this works for the case determined in [4]. Here Dorey determined the exact form of the $\mathcal{N} = 1^\star$ superpotential in (2.1) in terms of new chiral superfields made from combining the dual photons, σ^i, and the Wilson lines, ϕ^i

$$X^i = -i(\sigma^i + \tau\phi^i) \ , \tag{2.15}$$

as well as the original fermionic degrees of freedom given by p_i. Arguments either by semi-classical reasoning on the basis of three- and four-dimensional instanton contributions, or by the properties of the Seiberg-Witten curve[6] led to the result

$$\mathcal{W}(p, X) = \sum_i \frac{p_i^2}{2} - m_1 m_2 m_3 \sum_{i>j} \wp(X_i - X_j) \ , \tag{2.16}$$

where $\wp(z)$ is the Weierstrass function (see Appendix A for details of elliptic functions used throughout). (2.16) is just the first Hamiltonian of the elliptic Calogero-Moser system introduced above. It then becomes a simple matter to determine the stationary points of this function

$$\frac{\partial \mathcal{W}}{\partial p_i} = p_i = 0, \qquad \frac{\partial \mathcal{W}}{\partial X_i} = m_1 m_2 m_3 \sum_{j \neq i} \wp'(X_i - X_j) = 0 \ . \tag{2.17}$$

The properties of odd elliptic functions can be used to show that the massive vacua live on sublattice points of the fundamental parallelogram of the defining torus, τ.

[6]That is an even, elliptic, non-constant, holomorphic function with a 2nd order pole

An interesting point is the $\mathrm{SL}(2,\mathbb{Z})$-duality, which would have been exact in the parent $\mathcal{N}=4$ theory, now permutes the vacua; this is because it is simply the modular transformations of the torus and so permutes the sublattice points. The single massless vacua in SU(3) was also determined in [4], however the systematic classification of the massless vacua in the general case, $N>3$, has not been found, though there have been some partial results in [16].

2.4 Quiver Theories and Massive Vacua

The quiver theories are in some sense a generalization of the $\mathcal{N}=4$ theory, with gauge group $\mathrm{SU}(N)^k$. They are also finite theories but have $\mathcal{N}=2$ SUSY rather than $\mathcal{N}=4$. In fact these theories are intimately related to stacked D3-branes in Type IIB string theory, however I will not discuss this aspect. The mass-deformed $\mathcal{N}=4$ case described above is simply the $k=1$ case. We can again make a semi-classical analysis similar to the $\mathcal{N}=4$ case. The details are in [5]. Here we will concentrate on the compactification method described in the previous section.

The field content consists of (i) for each $\mathrm{SU}(N)$ factor an $\mathcal{N}=1$ vector multiplet and adjoint-valued chiral multiplet Φ_i, $i=1,\ldots,k$ and (ii) chiral multiplets $Q_i,\tilde{Q}_i$ in the $(\mathbf{N},\bar{\mathbf{N}})$ and $(\bar{\mathbf{N}},\mathbf{N})$ representations of $\mathrm{SU}(N)_i\times\mathrm{SU}(N)_{i+1}$, respectively. The tree-level superpotential has the form

$$\mathcal{W}=\frac{1}{g^2}\mathrm{Tr}\left\{\Phi_iQ_i\tilde{Q}_i-Q_i\Phi_{i+1}\tilde{Q}_i+m_iQ_i\tilde{Q}_i+\mu_i\Phi_i^2\right\}\ , \tag{2.18}$$

where we assume that the labels are defined modulo N. Here, m_i are the $\mathcal{N}=2$ supersymmetry preserving masses of the hypermultiplets and μ_i are the $\mathcal{N}=2\to\mathcal{N}=1$ breaking masses of the adjoint chiral multiplets.

The integrable system related to this theory is now a spin generalization of the elliptic Calogero-Moser system. It describes the motion of N particles in one dimension with positions X_a, momenta p_a and a $k \times k$ 'spin' matrix $\mathcal{J}^a$. The Hamiltonian of the system is[7]

$$\begin{aligned} H_0 = \sum_a p_a^2 + \sum_{a\neq b}\sum_{ij} \mathcal{J}^a_{ji}\mathcal{J}^b_{ij}\frac{\sigma(X_{ab}+z_{ji})}{\sigma(X_{ab})\sigma(z_{ji})}\big(\zeta(X_{ab}+z_{ji})-\zeta(X_{ab})\big) \\ -\tfrac{1}{2}\sum_{i\neq j}\Big[\sum_a \mathcal{J}^a_{ij}\mathcal{J}^a_{ji} - Nm_im_j\Big]\Big(\wp(z_{ij})-\zeta(z_{ij})^2\Big)\ . \end{aligned} \tag{2.19}$$

Here, $\wp(z)$, $\sigma(z)$ and $\zeta(z)$ are the (quasi-)elliptic functions defined on torus of half-periods $\omega_1 = i\pi$ and $\omega_2 = i\pi\tau$ (so of complex structure τ). In the above, the separation between the particles is given by $X_{ab} \equiv X_a - X_b$ while $z_{ij} \equiv z_i - z_j$ are "inhomogeneities", $k-1$ external parameters (since only the differences matter). In our application, the k independent complex coupling constants τ_i of each of the $\mathrm{SU}(N)$ factors of the gauge group are associated to the k independent parameters $\{\tau, z_i\}$ in the following way. Firstly we order the z_i so that $0 \leq \mathrm{Re}\, z_i \leq \mathrm{Re}\, z_{i+1} \leq 2\pi \mathrm{Im}\,\tau$. Then

$$\tau_i = i\frac{z_{i+1}-z_i}{2\pi}, \qquad i=1,\ldots,k-1\ , \qquad \tau_k = i\frac{z_1-z_k}{2\pi}+\tau\ . \tag{2.20}$$

To define the dynamical system the dynamical variables have the non-vanishing Poisson brackets [17]

$$\{X_a,p_b\} = \delta_{ab}\ , \qquad \{\mathcal{J}^a_{ij},\mathcal{J}^b_{kl}\} = \delta_{ab}\big(\delta_{jk}\mathcal{J}^a_{il} - \delta_{il}\mathcal{J}^a_{kj}\big)\ . \tag{2.21}$$

In fact, in the application to gauge theory, the spins are not arbitrary $k \times k$ matrices, rather they have rank one and so we can define them in terms of new variables Q_{ai} and $\tilde{Q}_{ia}$

$$\mathcal{J}^a_{ij} = \tilde{Q}_{ia}Q_{aj}\ . \tag{2.22}$$

[7] We have written the following in terms of spins $\mathcal{J}^a$ rather than $\mathcal{S}^i$ in [5]. The relation between the two representation can be determined from (2.22).

If we take all the inhomogeneities z_i equal, then (2.19) simplifies to

$$H_0 = \sum_a p_a^2 - \sum_{a \neq b} \mathrm{Tr}\big(\mathcal{J}^a \mathcal{J}^b\big) \wp(X_{ab}) \ , \tag{2.23}$$

the dynamical system analyzed in [17]. The system is completely integrable, even when the z_i are arbitrary, so there exists a basis of action-angle variables for which the Hamiltonian (2.19) is but one of a set of Hamiltonians.

For the application to gauge theory, we have to impose additional conditions on the spins. The reduction can be defined as a symplectic quotient by the abelian symmetries

$$Q_{ai} \to e^{\phi_a} Q_{ai} e^{\psi_i} \ , \quad \tilde{Q}_{ia} \to e^{-\psi_i} \tilde{Q}_{ia} e^{-\phi_a} \ . \tag{2.24}$$

In all there are $N + k - 1$ independent symmetries. Taking the symplectic quotient involves imposing the momentum map constraints

$$\sum_a Q_{ai} \tilde{Q}_{ia} = N m_i \ , \qquad \sum_i Q_{ai} \tilde{Q}_{ia} = \sum_i m_i \ , \tag{2.25}$$

along with an ordinary quotient by the symmetries (2.24). Notice that hypermultiplet masses m_i enter via (2.25). In (2.19) we note that the centre-of-mass motion is completely decoupled and so we set $\sum_a p_a = \sum_a X_a = 0$. Once this has been done the phase space (after the symplectic quotient on the spins) has the dimension $2k(N-1)$: precisely the complex dimension of the Coulomb branch of the compactified $\mathrm{SU}(N)^k$ theory.

The remaining conserved quantities can be extracted from the Lax operator described in [5]. Of particular importance for us is the basic Hamiltonian (2.19) along with the following k others

$$
\begin{aligned}
H_i &= 2\sum_a p_a \mathcal{J}^a_{ii} - 2\sum_{a\neq b}\sum_{j(\neq i)} \mathcal{J}^a_{ji}\mathcal{J}^b_{ij}\frac{\sigma(X_{ab}+z_{ji})}{\sigma(X_{ab})\sigma(z_{ji})} \\
&\quad +2\sum_{j(\neq i)}\Big[\sum_a \mathcal{J}^a_{ij}\mathcal{J}^a_{ji} - Nm_i m_j\Big]\zeta(z_{ij}) \ . \qquad (2.26)
\end{aligned}
$$

of which only $k-1$ are independent since $\sum_{i=1}^k H_i = 0$.

The Hamiltonians parameterize the Coulomb branch of the four-dimensional theory prior to compactification. In particular, the k independent Hamiltonians H_0 along with H_i are identified with the subspace of quadratic condensates $\mathrm{Tr}\,\Phi_i^2$. In [5], the unique combination of Hamiltonians corresponding to the diagonal combination was identified

$$
\sum_{i=1}^k \mathrm{Tr}\,\Phi_i^2 = k\,H^* \ , \qquad (2.27)
$$

where

$$
H^* = H_0 - \tfrac{1}{k}\sum_{i\neq l}\zeta(z_{il})H_i \ . \qquad (2.28)
$$

The fact that there is a non-trivial function multiplying the H_i is necessary in order that H^* has the appropriate modular properties. The superpotential in the three-dimensional compactification corresponding to an arbitrary $\mathcal{N}=1$ deformation of the theory is then simply a combination

$$
\mathcal{W} = -\frac{1}{g^2}\Big(\lambda_0 H^* + \sum_{i=1}^k \lambda_i H_i\Big) \ . \qquad (2.29)
$$

In [5] the stationary points of the diagonal contribution were determined. This restricts us to the case of all the μ_i being equal, or in (2.29) $\lambda_0 = k\mu$ and $\lambda_i = 0$ for $i > 0$. The massive vacua then have a similar structure to those found in Section 2.3, remembering that that case represents the $k=1$ situation here. The important point in their determination is that the massive vacua correspond in the dynamical system to equilibrium configurations of *all* the Hamiltonians, whereas the massless vacua correspond to equilibrium configurations of just the particular Hamiltonian corresponding to $\mathcal{W}$. Thus the massive vacua are given by

$$p_a = 0\,, \qquad X_a = \frac{2r_a}{q}\omega_1 + \frac{2s_a}{p}\omega_2\,, \qquad 0 \le r_a < q\,, \quad 0 \le s_a < p\,, \quad pq = N\,, \tag{2.30}$$

where the pair (r_a, s_a) picks out a sublattice point, and modular transformations thereof. This means that the derivative of the second term in each Hamiltonian, (2.19) and (2.26), is zero by the properties of odd elliptic functions[8]. It just remains to determine the spin matrices. The ansatz for this was given, and proved, in [5]

$$\mathcal{J}^a_{ij} = \sqrt{m_i m_j}\left(\frac{\rho_j}{\rho_i}\right)^{r_a}\left(\frac{\lambda_j}{\lambda_i}\right)^{s_a} e^{2z_{ji}\left[\frac{r_a}{q}\zeta(\omega_1)+\frac{s_a}{p}\zeta(\omega_2)\right]}\,. \tag{2.31}$$

where ρ_i and λ_i are arbitrary q^{th} and p^{th} roots of unity, respectively. It suffices to note that this results in the identity

$$\mathcal{J}^a_{ij}\mathcal{J}^a_{ji} = m_i m_j \qquad\qquad \forall a\,, \tag{2.32}$$

thus the final term in each Hamiltonian will be zero.

Note in particular that, by the modular properties of the torus and since the N^{k-1} massive vacua lie on sublattices of the torus, the massive vacua are permuted by the $\mathrm{SL}(2,\mathbb{Z})$-duality symmetry.

[8]Either terms cancel in pairs or give no contribution since the difference $X_{ab} = X_a - X_b$ is a half-period.

2.5 The Vacua of the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ Broken Quiver Theory

In order to get a concrete picture of the massless vacua it is useful to consider a particular example. Here we shall examine the case with gauge group $\mathrm{SU}(2)\times\mathrm{SU}(2)$. Not only will we find all the vacua but we will also determine the condensates u_2 exactly.

2.5.1 The Phase Structure via Semi-classical Reasoning

We begin by investigating the vacuum structure by solving the F-flatness conditions modulo complex gauge transformations in the conventional way. The analysis has been done in the more general setting of the $\mathrm{SU}(N)^k$ theory, in [5], where the F-flatness conditions derived from (2.18) are

$$\begin{aligned}
\tilde{Q}_{i-1}Q_{i-1} - Q_i\tilde{Q}_i &= 2\mu_i\Phi_i + \lambda_i.\mathbf{1} \ , \\
\Phi_{i+1}\tilde{Q}_i - \tilde{Q}_i\Phi_i &= m_i\tilde{Q}_i \ , \\
Q_i\Phi_{i+1} - \Phi_i Q_i &= m_i Q_i \ ,
\end{aligned} \tag{2.33}$$

where the λ_i are Lagrange multipliers coming from requirement that the Φ_i are traceless. We can then use complex gauge transformations to diagonalize Φ_i. For the moment we assume that the masses are all generic.

First of all there are confining vacua for which $\Phi_i = Q_i = \tilde{Q}_i = 0$ and the gauge symmetry is completely unbroken. We expect that the theory at low energy is precisely pure $\mathcal{N}=1$ Yang-Mills with gauge group $\mathrm{SU}(2)\times\mathrm{SU}(2)$. Since each $\mathrm{SU}(2)$ factor is independent and each on its own yields two independent vacua, in all we expect four confining vacua.

There are two Higgs vacua in which the gauge group is completely broken. For the first

$$\Phi_1 = \tfrac{1}{2}(m_1 + m_2)\mathrm{diag}\big(1,-1\big)\ , \qquad \Phi_2 = \tfrac{1}{2}(m_1 - m_2)\mathrm{diag}\big(1,-1\big)\ . \tag{2.34}$$

and

$$\begin{aligned} Q_1 &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\ , \quad \tilde{Q}_1 = \begin{pmatrix} 0 & m_1(\mu_1+\mu_2) + m_2(\mu_1-\mu_2) \\ 0 & 0 \end{pmatrix}\ , \\ Q_2 &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\ , \quad \tilde{Q}_2 = \begin{pmatrix} m_1(\mu_1-\mu_2) + m_2(\mu_1+\mu_2) & 0 \\ 0 & 0 \end{pmatrix}\ . \end{aligned} \tag{2.35}$$

The other Higgs vacuum is obtained by sending $\Phi_1 \leftrightarrow \Phi_2$ along with

$$\begin{aligned} Q_1 &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\ , \quad \tilde{Q}_1 = \begin{pmatrix} 0 & m_1(\mu_1-\mu_2) + m_2(\mu_1+\mu_2) \\ 0 & 0 \end{pmatrix}\ , \\ Q_2 &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\ , \quad \tilde{Q}_2 = \begin{pmatrix} 0 & 0 \\ 0 & m_1(-\mu_1+\mu_2) + m_2(\mu_1+\mu_2) \end{pmatrix}\ . \end{aligned} \tag{2.36}$$

All-in-all there are 6 vacua with a mass gap: 4 confining and 2 Higgs.

There are two massless, or Coulomb, vacua each with an unbroken U(1) factor. For the first

$$\begin{aligned} &\Phi_1 = \tfrac{\mu_2 m_2}{\mu_1+\mu_2}\mathrm{diag}\big(1,-1\big)\ , \qquad \Phi_2 = \tfrac{\mu_1 m_2}{\mu_1+\mu_2}\mathrm{diag}\big(-1,1\big)\ , \\ &Q_1 = \tilde{Q}_1 = 0, \quad Q_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\ , \quad \tilde{Q}_2 = \frac{4m_2\mu_1\mu_2}{\mu_1+\mu_2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\ , \end{aligned} \tag{2.37}$$

whilst for the second

$$\begin{aligned} &\Phi_1 = \tfrac{\mu_2 m_1}{\mu_1+\mu_2}\mathrm{diag}\big(1,-1\big)\ , \qquad \Phi_2 = \tfrac{\mu_1 m_1}{\mu_1+\mu_2}\mathrm{diag}\big(1,-1\big)\ , \\ &Q_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\ , \quad \tilde{Q}_1 = \frac{4m_1\mu_1\mu_2}{\mu_1+\mu_2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\ , \quad Q_2 = \tilde{Q}_2 = 0\ . \end{aligned} \tag{2.38}$$

The analysis above holds for generic values of the masses. However, for particular values of the masses flat directions emerge and different vacua can be related. Of course at this stage we emphasize that we are not taking into account any of the quantum effects. To start with, if m_1 or m_2 vanish then the two Higgs vacua are related by a flat direction. For instance with $m_2 = 0$ we have

$$\begin{aligned} &\Phi_1 = \Phi_2 = \tfrac{1}{2} m_1 \mathrm{diag}\big(1,-1\big) \ , \qquad Q_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ , \\ &\tilde{Q}_1 = \begin{pmatrix} 0 & m_1(\mu_1+\mu_2) \\ 0 & 0 \end{pmatrix} \ , \quad Q_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ , \quad \tilde{Q}_2 = \begin{pmatrix} x & 0 \\ 0 & x - m_1(\mu_1 - \mu_2) \end{pmatrix} \ . \end{aligned} \tag{2.39}$$

Here, x parameterizes the flat direction. In an analogous way, one of the massless vacua is related to the confining vacuum by a flat direction.

2.5.2 The Exact Superpotential

We can now use (2.19) to determine the full quantum structure of the theory. In the case of an $\mathrm{SU}(2) \times \mathrm{SU}(2)$ quiver, we can be more explicit about the symplectic reduction on the spins. Solving the moment map conditions (2.25) and fixing the symmetries (2.24) can be achieved, for instance, by parameterizing them with two variables $\{x, y\}$ such that

$$Q_{ai} = \begin{pmatrix} 1 & m_2 - y \\ e^{-x\frac{m_1 - y}{y - m_2}} & m_2 + y \end{pmatrix} \ , \qquad \tilde{Q}_{ia} = \begin{pmatrix} y + m_1 & e^{x}(y - m_2) \\ 1 & 1 \end{pmatrix} \ . \tag{2.40}$$

The Poisson bracket that one derives from (2.21) is then simply $\{x, y\} = 1$.

Once this has been done, the dynamical system in this case has a four-dimensional phase space parameterized by $X \equiv X_1 - X_2$, $p \equiv \frac{1}{2}(p_1 - p_2)$, x and y with Poisson brackets

$$\{X, p\} = 1 \ , \qquad \{x, y\} = 1 \ , \tag{2.41}$$

$$\{X, x\} = \{X, y\} = \{p, x\} = \{p, y\} = 0 \ . \tag{2.42}$$

The two Hamiltonians (2.19) and (2.26) are

$$H_0 = 2p^2 + 2(2y^2 - m_1^2 - m_2^2)\wp(X) - 2e^x(y^2 - m_2^2)\frac{\sigma(X-z)}{\sigma(X)\sigma(z)}\big(\zeta(X-z) - \zeta(X)\big)$$
$$+ 2e^{-x}(y^2 - m_1^2)\frac{\sigma(X+z)}{\sigma(X)\sigma(z)}\big(\zeta(X+z) - \zeta(X)\big) + 2y^2(\wp(z) - \zeta(z)^2) \ , \qquad (2.43a)$$
$$H_1 = 4py - 4y^2\zeta(z) + 2e^x(y^2 - m_2^2)\frac{\sigma(X-z)}{\sigma(X)\sigma(z)} + 2e^{-x}(y^2 - m_1^2)\frac{\sigma(X+z)}{\sigma(X)\sigma(z)} \ , \qquad (2.43b)$$

where $z \equiv z_{12}$. It is straightforward to check that H_0 and H_1 Poisson-commute.

Now we turn to the rôle of the dynamical system in our gauge theory. First of all, we have to complexify the coordinates so that $\{X, p, x, y\}$ are all treated as complex quantities and the phase space now becomes a hyper-Kähler space with a distinguished complex structure. This is clear in the formulation of the integrable system as a Hitchin system [5, 18, 19] which has the form of a hyper-Kähler quotient [20]. The symplectic form of the dynamical system is identified with the closed $(2,0)$ form with respect to the distinguished complex structure. From a field theory point-of-view, the action variables form a basis of coordinates on the Coulomb branch of the theory before compactification on the circle. The question of how to relate H_0 and H_1 to the gauge invariants operators $\mathrm{Tr}\,\Phi_i^2$, $i = 1, 2$, was addressed in [5]. There is a unique combination which has the required properties[9] to be identified with the average combination

$$\tfrac{1}{2}\mathrm{Tr}\,\big(\Phi_1^2 + \Phi_2^2\big) \equiv H^* = H_0 - \zeta(z)H_1 \ . \qquad (2.44)$$

whilst the quantity H_1 is identified with the difference

$$\mathrm{Tr}\,\big(\Phi_2^2 - \Phi_1^2\big) \equiv H_1 \ , \qquad (2.45)$$

[9]These are having dimension 2, modular weight 2 as well as being invariant under both gauge group permutations and the S-duality.

since this must hold under the interchange of subscripts 1 and 2

$$\mathrm{Tr}\left(\Phi_1^2 - \Phi_2^2\right) \equiv H_2 = -H_1 \ . \tag{2.46}$$

It follows that

$$\mathrm{Tr}\,\Phi_1^2 = H^* - \tfrac{1}{2}H_1 \ , \qquad \mathrm{Tr}\,\Phi_2^2 = H^* + \tfrac{1}{2}H_1 \ . \tag{2.47}$$

We can now identify the general $\mathcal{N} = 1^*$ deformation of the superpotential with the following linear combination of the action variables

$$W = -\frac{1}{g^2}\left(\mu_1 \mathrm{Tr}\,\Phi_1^2 + \mu_2 \mathrm{Tr}_2\,\Phi_2^2\right) = -\frac{1}{g^2}(\mu_1 + \mu_2)\tilde{H} \ , \tag{2.48}$$

where

$$\begin{aligned}\tilde{H} = H^* + \tfrac{1}{2}\beta H_1 &= 2p^2 + 4\alpha p y + 2e^{x}(y^2 - m_2^2)\tilde{\phi}(X) + 2e^{-x}(y^2 - m_1^2)\phi(X) \\ &+ 2(2y^2 - m_1^2 - m_2^2)\wp(X) + 2y^2(\wp(z) - \zeta(z)^2 - 2\alpha\zeta(z)) \ ,\end{aligned} \tag{2.49}$$

where we have defined the constants

$$\alpha = -\zeta(z) + \tfrac{1}{2}\beta \ , \qquad \beta = \frac{\mu_2 - \mu_1}{\mu_1 + \mu_2} \tag{2.50}$$

along with the functions

$$\begin{aligned}\phi(X) &= \frac{\sigma(X+z)}{\sigma(X)\sigma(z)}\left(\zeta(X+z) - \zeta(X) + \alpha\right) \ , \\ \tilde{\phi}(X) &= \frac{\sigma(X-z)}{\sigma(X)\sigma(-z)}\left(\zeta(X-z) - \zeta(X) - \alpha\right) \ .\end{aligned} \tag{2.51}$$

2.5.3 The Quantum Phase Structure

Supersymmetric vacua are obtained by extremizing the superpotential. In the SU(2)×SU(2) theory this implies $\tilde{H}$ in (2.49). The relevant equations are thus

$$\frac{\partial \tilde{H}}{\partial p} = 4p + 4\alpha y = 0 \ , \tag{2.52a}$$

$$\frac{\partial \tilde{H}}{\partial y} = 4y\{2\wp(X) + \wp(z) - \tfrac{1}{4}\beta^2 + e^x\tilde{\phi}(X) + e^{-x}\phi(X)\} = 0 \ , \tag{2.52b}$$

$$\frac{\partial \tilde{H}}{\partial x} = 2e^x(y^2 - m_2^2)\tilde{\phi}(X) - 2e^{-x}(y^2 - m_1^2)\phi(X) = 0 \ , \tag{2.52c}$$

$$\frac{\partial \tilde{H}}{\partial X} = 2(2y^2 - m_1^2 - m_2^2)\wp'(X) + 2e^x(y^2 - m_2^2)\tilde{\phi}'(X) + 2e^{-x}(y^2 - m_1^2)\phi'(X) = 0 \ . \tag{2.52d}$$

In simplifying (2.52b) with have already solved (2.52a) for p

$$p = -\alpha y \ . \tag{2.53}$$

One branch of solutions is then obtained by solving (2.52b) with $y = 0$. It then follows that there are two solutions of (2.52c), which we label by $n_1 = 1, 2$, for which

$$e^x = (-1)^{n_1}\frac{m_1}{m_2}\sqrt{\frac{\phi(X)}{\tilde{\phi}(X)}} \ . \tag{2.54}$$

Using standard elliptic function identities, along with (2.53) and (2.54), the final equation (2.52d) can be recast in the form

$$\wp'(X)\Big(m_1^2 + (-1)^{n_1}m_1m_2\gamma\big(\phi\tilde{\phi}\big)^{-1/2} + m_2^2\Big) = 0 \ , \tag{2.55}$$

where we have defined the quantity

$$\gamma = 2\wp(X) + \wp(z) - \frac{\beta^2}{4} \ . \tag{2.56}$$

For later use, one can show, again using standard elliptic function identities, that

$$\phi(X)\tilde{\phi}(X) = \wp^2(X) + \tfrac{1}{4}\beta^2\wp(z) + \wp^2(z) + \wp(X)\wp(z) - \tfrac{1}{4}\beta^2\wp(X) + \tfrac{1}{2}\beta\wp'(z) - \tfrac{1}{4}g_2 \ , \tag{2.57}$$

from which one deduces

$$\gamma^2 - 4\phi\tilde{\phi} = g_2 - 3\wp^2(z) - 2\beta\wp'(z) - \tfrac{3}{2}\beta^2\wp(z) + \tfrac{1}{16}\beta^4 \ . \tag{2.58}$$

As a consequence the left-hand side is independent of X.

For generic masses the solution to (2.55) is $\wp'(X) = 0$, *i.e.* X is a half-period

$$X \in \{\omega_1,\ \omega_2,\ \omega_1 + \omega_2\} \ , \tag{2.59}$$

which we label $X_c = i\pi, i\pi\tau, i\pi(\tau+1)$, $c = 1, 2, 3$.

In order to assess whether these six vacua are massive or massless, we compute the determinant of the Hessian matrix of $\tilde{H}$

$$\begin{aligned}\mathrm{Det}\left[\frac{\partial^2 \tilde{H}}{\partial x_i \partial x_j}\right] &= \frac{1}{m_2^2\phi\tilde{\phi}}\Big((-1)^{n_1} m_1 m_2 \gamma + (m_1^2 + m_2^2)(\phi\tilde{\phi})^{1/2}\Big) \\ &\times \Big(2\phi\tilde{\phi}\wp''(X)\Big[(-1)^{n_1} m_1 m_2 \gamma + (m_1^2 + m_2^2)(\phi\tilde{\phi})^{1/2}\Big] - (-1)^{n_1} m_1 m_2 (\gamma^2 - 4\phi\tilde{\phi})\wp'(X)\Big) \ .\end{aligned} \tag{2.60}$$

It can be shown that the above is generically non-zero, and hence there are no zero eigenvalues, thus all six vacua are massive. The values of the condensates in these six massive vacua are

$$\mathrm{Tr}\ \Phi_i^2 = -2(m_1^2 + m_2^2)\wp(X) - 4(-1)^{n_1}\frac{m_1 m_2}{(\phi\tilde{\phi})^{1/2}}\Big(\phi\tilde{\phi} + \tfrac{1}{2}(\beta - (-1)^i)\big(\beta(\wp(z) - \wp(X)) + \wp'(z)\big)\Big) \ . \tag{2.61}$$

These six vacua are precisely the vacua found in [5] for general k and N. It is tempting to identify them with the six massive vacua, two Higgs and four confining, that we found in Section 2.5.1 and this turns out to be correct. In order to pin down the relation, consider the semi-classical expansion of the condensates in each of the vacua as described in [21]. The expansions we need can be deduced from the following expansions of the (quasi-)elliptic functions

$$\begin{aligned}
\wp(X) &= \frac{1}{12} + \frac{e^{-X}}{(1-e^{-X})^2} + \sum_{n=1}^{\infty}\left\{\frac{e^{-X}q^n}{(1-e^{-X}q^n)^2} + \frac{e^{X}q^n}{(1-e^{X}q^n)^2} - \frac{2q^n}{(1-q^n)^2}\right\}, \\
\sigma(X) &= e^{\zeta(i\pi)X^2/(2i\pi)}(e^{X/2} - e^{-X/2})\prod_{n=1}^{\infty}\frac{1-q^n(e^X - e^{-X}) + q^{2n}}{(1-q^n)^2}, \\
\zeta(X) &= X\frac{\zeta(i\pi)}{i\pi} + \frac{1}{2}\coth(X/2) - \sum_{n=1}^{\infty}\frac{q^n(e^X + e^{-X})}{1-q^n(e^X+e^{-X})+q^{2n}}.
\end{aligned} \tag{2.62}$$

The condensates can be written in terms of the complex couplings of each gauge group factor

$$q_1 = e^{2\pi i\tau_1} = e^{-z}, \qquad q_2 = e^{2\pi i\tau_2} = qe^{z}, \tag{2.63}$$

where $q = e^{2\pi i\tau}$. It is easy to see that the condensates have an expansion in terms of the quantities

$$e^{-X}q^n, \quad e^{X}q^{n+1}, \quad q^n, \quad e^{-z}q^n, \quad e^{z}q^{n+1}, \tag{2.64}$$

with $n = 0, 1, 2, \ldots$. Given the values for X in (2.59), it is clear that the vacua with $X = i\pi$ have an expansion in integer powers of q_1 and q_2. Hence, the two vacua with $X = i\pi$ are identified with the Higgs vacua in which the condensates have a conventional semi-classical instanton expansion in integer powers of q_1 and q_2. The vacua with $X = i\pi\tau$ or $i\pi(\tau+1)$ have an expansion which includes powers of the fractional instanton factor $q^{1/2}$. This is characteristic of a confining vacuum. Hence, we identify the four vacua with these values of X, and $n_1 = 1, 2$, with the four confining vacua identified in Section 2.5.1.

Now we return to the equations for the vacua (2.52a)-(2.52d) and choose a different branch of solutions obtained by solving (2.52b) with

$$e^{x} = \frac{-\gamma + (-1)^{n_2}\sqrt{\gamma^2 - 4\phi\tilde{\phi}}}{2\tilde{\phi}}, \tag{2.65}$$

rather than $y = 0$. There are two solutions of this type labelled by $n_2 = 1, 2$. Then (2.52c) is solved for y giving

$$y = \sqrt{\frac{m_2^2 e^x \tilde{\phi} - m_1^2 e^{-x} \phi}{e^x \tilde{\phi} - e^{-x} \phi}} \; . \tag{2.66}$$

Choosing the opposite sign for y can be shown to lead to an equivalent solution due to the presence of discrete symmetries which we have hitherto ignored[10]. In particular the values of the condensates will not depend on it.

The final equation (2.52d) becomes

$$(m_1^2 - m_2^2) \frac{\partial \sqrt{\gamma^2 - 4\phi\tilde{\phi}}}{\partial X} = 0 \, , \tag{2.67}$$

which is identically zero for all values of X since the combination (2.58) is independent of X.

The two solutions are obviously massless vacua since each corresponds to a line of critical points parameterized by X. This is further confirmed by the determinant of the Hessian, which has the form

$$\begin{aligned} \operatorname{Det}\left[\frac{\partial^2 \tilde{H}}{\partial x_i \partial x_j}\right] =& 16(m_1^2 - m_2^2)\left(m_2^2 - \frac{4 m_1^2 \phi\tilde{\phi}}{\left(\gamma \mp \sqrt{\gamma^2 - 4\phi\tilde{\phi}}\right)^2}\right) \frac{\tilde{\phi}^2}{\left(\gamma \mp \sqrt{\gamma^2 - 4\phi\tilde{\phi}}\right)} \\ & \cdot \left(\frac{\partial^2(\gamma^2 - 4\phi\tilde{\phi})}{\partial X^2} - 2\left(\frac{\partial\sqrt{\gamma^2 - 4\phi\tilde{\phi}}}{\partial X}\right)^2\right) , \end{aligned} \tag{2.68}$$

which is clearly zero for all values of the parameters. The values of the condensates

[10]This comes from a physically irrelevant freedom in the definition (2.40) which takes $y \to -y$.

in these two massless vacua are

$$\begin{aligned}\mathrm{Tr}\ \Phi_i^2 =& (m_1^2+m_2^2)\big(\wp(z)-\tfrac{1}{2}(-1)^i\beta+\tfrac{1}{4}\beta^2\big) \\ &+(-1)^{n_2}\frac{m_1^2-m_2^2}{\sqrt{\gamma^2-4\phi\tilde{\phi}}}\Big(\gamma^2-4\phi\tilde{\phi}+\tfrac{1}{2}\beta(\beta-(-1)^i)\big(3\wp(z)+2\wp'(z)-\tfrac{1}{4}\beta^2\big)\Big)\ .\end{aligned} \tag{2.69}$$

The discussion of the vacuum structure above has been established in the case where the masses $\{m_i\}$ and $\{\mu_i\}$ are generic. For special values the vacua can merge. First of all, if X equals a half period and y in (2.66) equals 0, which requires

$$m_2^2 e^x \tilde{\phi} - m_1^2 e^{-x}\phi = 0\ , \tag{2.70}$$

where x is given by (2.65), then a massless vacuum meets what was once one of the massive vacua. Solving these equations leads to a condition on the ratio of the hypermultiplet masses m_1/m_2. In this way either of the massless vacua can meet any of the 6 massive vacua at 12 special values for m_1/m_2

$$\frac{m_1}{m_2} = (-1)^{n_1}\frac{-\gamma+(-1)^{n_2}\sqrt{\gamma^2-4\phi\tilde{\phi}}}{2\sqrt{\phi\tilde{\phi}}}\Bigg|_{X=X_c}\ . \tag{2.71}$$

Finally the two massless vacua merge together when

$$\gamma^2-4\phi\tilde{\phi} = g_2 - 3\wp^2(z) - 2\beta\wp'(z) - 3\beta^2\wp(z)/2 + \beta^4/16 = 0\ . \tag{2.72}$$

A final point concerns the SL(2, $\mathbb{Z}$) S-duality. Since the only effect of this is to take $\tau \to \tau + 1$ we see that it simply permutes the confining vacua, while leaving the Higgs vacua untouched. The massless vacua are obviously unaffected since they can take any value for X.

2.5.4 Discussion

We have calculated the exact phase structure, i.e. the number and nature of the vacua at generic and specific values of the parameters, and the condensates of the two adjoint-valued scalar fields in the mass deformed $\mathrm{SU}(2) \times \mathrm{SU}(2)$ finite quiver theory. The strategy involved compactifying the theory on a circle of finite radius so that the low-energy degrees-of-freedom are all scalar. However, the values calculated remain valid in the decompactification limit. In this way, we have been able to show how the exact structure of vacua matches the one deduced from an analysis of the tree-level superpotential in the four-dimensional theory. Our analysis acts as an enumerative example of the work of [5] where the massive vacua were found for the general gauge group $\mathrm{SU}(N)^k$. We have seen in this case that the massive vacua are identifiable and conform to the simple lattice structure shown in [5]. However, it has also been possible to determine the massless vacua which could not be found in the general case; the exact solutions for these vacua do not have any obvious generalization given their complicated form, (2.65) and (2.66).

Chapter 3

Index Theory

We now move on to the second topic of the thesis: the determination of the prepotential in $\mathcal{N} = 2, 2^*$ SYM theory in terms of a SUSY quantum mechanical σ-model with the instanton moduli space as a target. Alvarez-Gaume [7, 22] has shown how SUSY quantum mechanical systems with a particular target space calculate topological indices on that space. In this chapter we will cover some preliminaries of index theory and then detail the heat kernel method used in [7, 22] for the cases we shall need.

A key property in the study of SUSY theories is the Witten index, $\mathrm{Tr}(-1)^F$, where F is the fermion number of each state. This calculates the difference between the number of bosonic and fermionic zero-energy states

$$\mathrm{Tr}(-1)^F = n_B(E=0) - n_F(E=0) \ . \tag{3.1}$$

The fact that only zero energy states contribute can be seen be examining the supersymmetry algebra

$$\{Q, Q\} = H, \qquad [Q, H] = 0 \ . \tag{3.2}$$

Then any fermionic state $|f\rangle$ with positive energy E has a related bosonic state $|b\rangle = \alpha Q\,|f\rangle$, where $\alpha = \sqrt{E/2}$ as $2Q^2 = H$. Since H and Q commute we find that $|f\rangle$ and $|b\rangle$ are degenerate and so do not contribute to the index. Thus only the zero-energy states, which are singlets and invariant under supersymmetry contribute to the index. The aim of this chapter is to introduce some mathematical aspects of Index theory, and the heat kernel method for calculating the Witten index of quantum mechanical systems, such that we can tackle the physical theory with confidence in the following chapter.

3.1 The Atiyah-Singer Index Theorem and Elliptic Complexes

The subject of index theory was opened up by the beautiful theory of Atiyah-Singer [23–26] where the index of an elliptic complex (a fibre bundle E and a differential operator D), was given in terms of characteristic classes integrated over the base space. Thus the index is viewed as a topological quantity of the base space[1] $\mathcal{M}$. However, the characteristic classes are themselves defined in terms of local quantities: differential forms. Thus it appears we can calculate topological indices from local properties of $\mathcal{M}$. The Atiyah-Singer theorem is explicitly stated by

$$\mathrm{Ind}(E,D) = (-1)^{n(n+1)/2}\int_{\mathcal{M}} ch\left(\bigoplus_r (-1)^r E_r\right) \left.\frac{Td(T_p\mathcal{M}^{\mathbb{C}})}{e(T_p\mathcal{M})}\right|_{\text{vol}} . \tag{3.3}$$

Here n is the dimension of the base manifold, $\mathcal{M}$, the *vol* indicates we take the top form while the functions $ch(E)$, $Td(\mathcal{M})$ and $e(\mathcal{M})$ are characteristic classes. Full details of how this works for specific cases, as well as the details of elliptic complexes, index theory and underlying differential geometry can of course be found in e.g. [27, 28]. We will instead just examine a few of the relevant cases.

[1] An assumption which we will have to relax later on is that $\mathcal{M}$ is compact.

3.1.1 de Rham complex

The typical example is that of the de Rham complex which describes the space of p-forms, $\Lambda^{(p)}$, on a manifold $\mathcal{M}$, of dimension n. There is an exterior derivative taking p-forms to $(p+1)$-forms

$$d_p : \Lambda^{(p)} \to \Lambda^{(p+1)} , \tag{3.4}$$

and its conjugate

$$d_p^\dagger = - \star d_p \star : \Lambda^{(p)} \to \Lambda^{(p-1)} . \tag{3.5}$$

From these we can construct an elliptic operator, which is in fact the *Laplacian* operating on p-forms

$$\Delta\omega_p = d_{p-1} d_p^\dagger \omega_p + d_{p+1}^\dagger d_p \omega_p . \tag{3.6}$$

The index of this elliptic complex is in this case the *Euler character*

$$\mathrm{Ind}(\Lambda, d) = \sum_{p=0}^{n} (-1)^p \dim H^p(\mathcal{M}, \mathbb{R}) = \sum_{p=0}^{n} (-1)^p b_p = \chi(\mathcal{M}) , \tag{3.7}$$

where $H^p(\mathcal{M}, G)$ is the p'th cohomology class of the gauge group, G, on $\mathcal{M}$ and b_p are the *betti* numbers. We can represent the Euler character in terms of eigenvalues, x_i, of the 2-form curvature of the tangent bundle[2] of $\mathcal{M}$,

$$\chi(\mathcal{M}) = \int_{\mathcal{M}} e(T_p\mathcal{M}) = \int_{\mathcal{M}} \prod_{i=1}^{n/2} x_i . \tag{3.8}$$

3.1.2 Dolbeault complex

Let us briefly examine another example, the *Dolbeault complex.* Here we examine complex manifolds which admit holomorphic co-ordinates, $z^i = x^i + iy^i$, and anti-holomorphic ones, $\bar{z}^i = x^i - iy^i$. The algebra of forms can now be split in the

[2]The tangent bundle of $\mathcal{M}$ is denoted $T\mathcal{M}$, and $T_p\mathcal{M}$ refers to the tangent bundle of $\mathcal{M}$ at the point $p \in \mathcal{M}$.

following way

$$\Lambda = \oplus_{p,q}\Lambda^{(p,q)} , \tag{3.9}$$

where the first index refers to holomorphic co-ordinates and the second to anti-holomorphic ones. Thus an element of $\Lambda^{(p,q)}$ is of the form

$$f_{i_1\cdots i_p j_1\cdots j_q} dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q} . \tag{3.10}$$

We now have exterior derivatives, ∂ w.r.t. z^i, and $\bar{\partial}$ w.r.t. $\bar{z}^i$. The Dolbeault complex[3] is then given by

$$\xrightarrow{\bar{\partial}} \Lambda^{(0,q)} \xrightarrow{\bar{\partial}} \Lambda^{(0,q+1)} \xrightarrow{\bar{\partial}} , \tag{3.11}$$

with the Laplacian operator again given by the combination

$$\bar{\Delta} = \bar{\partial}\bar{\partial}^{\dagger} + \bar{\partial}^{\dagger}\bar{\partial} , \tag{3.12}$$

and the index in analogy with (3.7) is called the *arithmetic* or *Todd genus*

$$\mathrm{Ind}(\Lambda^{(0,q)}, \bar{\partial}) = \sum_{p=0}^{n}(-1)^p h_{(0,p)} = \int_{\mathcal{M}} Td(T_p\mathcal{M}) , \tag{3.13}$$

where $h_{(p,q)}$ are the *hodge* numbers and the Todd class, $Td(T_p\mathcal{M})$ is

$$Td(T_p\mathcal{M}) = \prod_{i=1}^{n/2} \frac{x_i}{1 - e^{-x_i}} . \tag{3.14}$$

3.1.3 Spin complex

Finally, let us examine the *spin complex*, which can be defined if $\mathcal{M}$ is a $2n$-dim *spin manifold.* Our fibre, $\mathcal{S}$, is a 2^n-dim space defining spinors, $\psi \in \mathcal{S}$, and may be separated into positive and negative chirality parts

$$\begin{aligned} \gamma^{2n+1}\psi^{\pm} &= \pm\psi^{\pm} , \\ \psi^{\pm} \in \mathcal{S}^{\pm} , &\qquad \mathcal{S} = \mathcal{S}^{+} \oplus \mathcal{S}^{-} . \end{aligned} \tag{3.15}$$

[3]The complex can alternatively be composed of holomorphic forms and operators, $(\Lambda^{(p,0)}, \partial)$.

Now our operator is the Dirac operator

$$\begin{aligned} D &= \gamma^\alpha(\partial_\alpha + \omega_\alpha) = i \not{\nabla}\mathcal{D}^+ , \\ D^\dagger &= i \not{\nabla}\mathcal{D}^- , \end{aligned} \tag{3.16}$$

where ω_α is the *spin connection*, $\mathcal{D}^\pm$ are projection operators onto $\mathcal{S}^\pm$ and $\mathcal{D}^+\mathcal{D}^- = 0$. The spin complex is then

$$\mathcal{S}^+ \underset{D^\dagger}{\overset{D}{\rightleftarrows}} \mathcal{S}^- . \tag{3.17}$$

The index is thus the *number of +ve chirality zero-energy spinors* minus the *number of -ve chirality zero-energy spinors.* This splitting can in fact be produced in any complex of forms by considering *even* and *odd* forms.

$$\mathrm{Ind}(\mathcal{S}^\pm, \mathcal{D}) = \dim\ker D - \dim\ker D^\dagger = \nu_+ - \nu_- , \tag{3.18}$$

The index calculates the $\hat{A}$ *genus*

$$\mathrm{Ind}(\mathcal{S}^\pm, \mathcal{D}) = \int_{\mathcal{M}} \hat{A}(T_p\mathcal{M}) = \int_{\mathcal{M}} \prod_{i=1}^{n/2} \frac{x_i/2}{\sinh(x_i/2)} , \tag{3.19}$$

where we note that this appears similar to the arithmetic genus. In fact the two classes are equal when the base space is Ricci flat[4].

3.1.4 Hirzebruch complex

An interesting possibility is to find a complex which calculates an index which interpolates between the Euler character and the arithmetic class. Such a complex was determined by [29, 30] and achieves this by taking a deformation of the de Rham complex. Remembering that on a complex manifold we can split the p-forms into

[4]This implies $\sum_i x_i = 0$. A relevant is example is when $\mathcal{M}$ is hyper-Kähler.

(anti)-holomorphic forms, with $d = \partial + \bar{\partial}$, we see that the Euler character can similarly be split

$$\chi(\mathcal{M}) = \sum_{p=0}^{n} (-1)^p b_p = \sum_{p,q}^{n} (-1)^{p+q} h_{(p,q)} \ . \tag{3.20}$$

If we deform this by a parameter y we obtain the χ_y-genus

$$\chi_y(\mathcal{M}) = \sum_{p,q} (-1)^p y^q h_{(q,p)} = \int_{\mathcal{M}} \prod_{i=1}^{n/2} \frac{x_i(1 + ye^{-x_i})}{1 - e^{-x_i}} \ , \tag{3.21}$$

which clearly interpolates between the Euler character at $y = -1$ and the Todd genus at $y = 0$. Although the integral form for the χ_y genus has the correct asymptotic properties it does not follow from the Atiyah-Singer formula; we will prove the identity below.

3.2 Heat Kernel Method

As stated in the introduction, the work of Alvarez-Gaume [7, 22] (see also [31]) developed a method of calculating the Witten index of supersymmetric quantum-mechanical σ-models that meshes precisely with the mathematical results above. Here we will detail the method for the case when the Hilbert space of states can be realized in terms of forms, and so is related to the de Rham complex[5], before outlining how other cases differ.

We remember that the Witten index gains contributions only from the zero-energy states and so we can instead calculate the regularized trace

$$\mathrm{Tr}(-1)^F e^{-\beta\mathcal{H}} \ , \tag{3.22}$$

since only the relevant states have zero eigenvalues of the Hamiltonian, $\mathcal{H}$. Note further that we can take the limit $\beta \to 0$ and the calculation will still hold. This

[5] The details of this realization are detailed in Chapter 4.

density can also be interpreted as the partition function of an ensemble at temperature β^{-1}, hence the name adopted by the calculation. However, the factor of $(-1)^F$ implies *periodic* boundary conditions on the fermions

$$\mathrm{Tr}(-1)^F e^{-\beta\mathcal{H}} = \int_{\mathrm{PBC}} [d^n X(t)]\,[d^n\psi(t)]\, e^{-\mathcal{S}_E[X,\psi]} \ , \tag{3.23}$$

where $\mathcal{S}_E$ is the Euclidean action. The most general Minkowski space Lagrangian[6], as in [7] though with slight alterations in notation, is given entirely by requiring supersymmetry

$$\mathcal{L}_M = \frac{1}{2}g_{\mu\nu}(X)\dot{X}^\mu\dot{X}^\nu + \frac{i}{2}g_{\mu\nu}(X)\psi^\mu_\alpha D_t\psi^\nu_\alpha + \frac{1}{4}R_{\mu\nu\sigma\rho}\psi^\mu_1\psi^\nu_1\psi^\sigma_2\psi^\rho_2 \ , \tag{3.24}$$

where

$$D_t\psi^\nu_\alpha = \dot{\psi}^\nu_\alpha + \Gamma^\nu_{\sigma\rho}\dot{X}^\sigma\psi^\rho_\alpha \ , \tag{3.25}$$

and $R_{\mu\nu\sigma\rho}$ is the 4-form curvature. The Lagrangian (3.24) admits 2 supersymmetry transformations (as in [7] we shall call this the $\mathcal{N}=1$ theory)

$$\begin{aligned} \delta X^\mu &= -i\epsilon_\alpha\psi^\mu_\alpha \ , \\ \delta\psi^\mu_\alpha &= \dot{X}^\mu\epsilon_\alpha + i\Gamma^\mu_{\nu\rho}\epsilon_\beta\psi^\nu_\beta\psi^\rho_\alpha \ . \end{aligned} \tag{3.26}$$

3.2.1 Calculating the Euler Character

The first step towards calculating (3.23) is to expand small fluctuations around constant configurations

$$X^\mu \to x^\mu + \delta X^\mu(t) \ , \qquad \psi^\mu_\alpha \to \eta^\mu_\alpha + \delta\psi^\mu_\alpha(t) \ . \tag{3.27}$$

We can then integrate out the fluctuations separately, hence at each point x^μ we can use normal co-ordinates such that

$$g_{\mu\nu}(x) = \delta_{\mu\nu} \ , \tag{3.28}$$

[6] We will of course have to Wick rotate with $t \to -it$.

along with the identity

$$\psi^\mu_\alpha \Gamma_{\mu\sigma\rho}(X)\dot{X}^\sigma \psi^\rho_\alpha = \tfrac{1}{2}\delta X^\mu R_{\mu\nu\sigma\rho}\eta^\sigma_\alpha \eta^\rho_\alpha \delta \dot{X}^\rho + \dots \ . \tag{3.29}$$

The action in (3.23) then splits in two

$$\mathcal{S}_E = \int dt\, \mathcal{L} = \beta \mathcal{L}_{(0)} + \int_0^\beta dt\, \mathcal{L}_{(2)} + \dots \ , \tag{3.30}$$

since the fluctuations are orthogonal to the constant co-ordinates, and we will be taking the $\beta \to 0$ limit, the quadratic terms will give us the lowest order terms. The constant part is simply

$$\mathcal{L}_{(0)} = -\tfrac{1}{4} R_{\mu\nu\sigma\rho}\eta^\mu_1 \eta^\nu_1 \eta^\sigma_2 \eta^\rho_2 \ , \tag{3.31}$$

while the quadratic part itself splits into bosonic and fermionic contributions

$$\mathcal{L}_{(2)} = \tfrac{1}{2}\delta X^\mu \Delta^B_{\mu\nu}\delta X^\nu + \tfrac{1}{2}\delta\psi^\mu \Delta^F_{\mu\nu}\delta\psi^\nu \ . \tag{3.32}$$

The bosonic operator is, up to a total derivative and to lowest order in β,

$$\Delta^B_{\mu\nu} = -\delta_{\mu\nu}\partial_t^2 + \tfrac{1}{2}R_{\mu\nu\sigma\rho}\eta^\sigma_\alpha \eta^\rho_\alpha \partial_t + \dots \ . \tag{3.33}$$

The fermionic operator is, of course, matrix-valued

$$\Delta^F_{\mu\nu} = \begin{pmatrix} \delta_{\mu\nu}\partial_t - \frac{1}{2}R_{\mu\nu\sigma\rho}\eta^\sigma_2\eta^\rho_2 & -\frac{1}{2}R_{\mu\nu\sigma\rho}\eta^\sigma_1\eta^\rho_2 \\ \frac{1}{2}R_{\mu\nu\sigma\rho}\eta^\sigma_1\eta^\rho_2 & \delta_{\mu\nu}\partial_t - \frac{1}{2}R_{\mu\nu\sigma\rho}\eta^\sigma_1\eta^\rho_1 \end{pmatrix} + \dots \ . \tag{3.34}$$

At this point we can perform the functional integral over the fluctuations before calculating the resulting determinants over all momentum modes around the compact dimension (parameterized by t). However, as an explanatory case, we will reverse the order of these integrals and explicitly expand the fluctuations[7] into Fourier modes

[7] Remember we will be taking the limit $\beta \to 0$ and so need only determine the lowest order terms.

periodic in β

$$
\begin{aligned}
\delta X^\mu &\rightarrow \sum_{n=1}^{\infty} \phi_n^\mu \cos(\frac{2\pi n t}{\beta}) + \varphi_n^\mu \sin(\frac{2\pi n t}{\beta}) \ , \\
\delta \psi_\alpha^\mu &\rightarrow \sum_{n=1}^{\infty} \xi_n^{\mu,\alpha} \cos(\frac{2\pi n t}{\beta}) + \chi_n^{\mu,\alpha} \sin(\frac{2\pi n t}{\beta}) \ ,
\end{aligned} \tag{3.35}
$$

where we note that the time derivatives will give factors of $\pm(\frac{2\pi n}{\beta})$.

The integral dt then picks out terms of the same mode along with a factor of $\beta/2$. At this point we should rescale both the fermionic and bosonic degrees of freedom to ensure that the lowest order contributions in both sectors are independent of β, and thus do not go to zero or infinity in the $\beta \rightarrow 0$ limit. From (3.31) we note that to pick up the constant contributions we must scale some of the fermions with β. We choose to scale the η_1's like $\beta^{-1/2}$ while the η_2's do not scale with β. Thus terms containing η_2 will come in at higher orders of β and so can be discarded. We also need to scale the δX by $\sqrt{\beta/2}$.

To reformulate the functional integral over the constant bosonic and fermionic modes in terms of an index we note that the bosonic degrees of freedom parametrise the target space of the σ-model, i.e. they *are* the coordinates of $\mathcal{M}$. Meanwhile the fermionic degrees of freedom (η_1 in particular) behave just like one-forms on $\mathcal{M}$ since they are Grassmann valued. Thus integrating a p-form over a p dimensional manifold gives

$$
\begin{aligned}
\int_{\mathcal{M}} \omega &= \int \omega_{\mu_1 \cdots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \ , \\
&= p! \int \omega_{1\cdots p} \, dx^1 \cdots dx^p = p! \int d^p x \, \omega_{1\cdots p} \ .
\end{aligned} \tag{3.36}
$$

However, from (3.23), we will at some point reach

$$
\int d^p x \, d^p \eta_1 \left(\omega_{\mu_1 \cdots \mu_p} \eta_1^{\mu_1} \cdots \eta_1^{\mu_p}\right) = p! \int d^p x \, \omega_{1\cdots p} \ . \tag{3.37}
$$

The η_2 integrals, meanwhile, will be done explicitly.

Thus, scaling the η_1's by $\beta^{-1/2}$, we can define the 2-form curvature function, with the usual forms dx^σ replaced by the fermions η_1^σ

$$R_{\mu\nu} = \tfrac{1}{2} R_{\mu\nu\sigma\rho}\eta_1^\sigma\eta_1^\rho \, . \tag{3.38}$$

We can then rewrite (3.30) as

$$\begin{aligned} &\int d\eta_2 \quad \exp\left[-\eta_2^\mu \; R_{\mu\nu}/2 \; \eta_2^\nu\right] \\ &\cdot\int d\phi\, d\varphi \quad \exp\left[-\frac{1}{2}\sum_{n=1}^{\infty}\begin{pmatrix}\phi & \varphi\end{pmatrix}_n^\mu \mathcal{B}_{\mu\nu}\begin{pmatrix}\phi \\ \varphi\end{pmatrix}_n^\nu\right] \\ &\cdot\int d\xi\, d\chi \quad \exp\left[-\frac{1}{2}\sum_{n=1}^{\infty}\begin{pmatrix}\xi^1 & \chi^1 & \xi^2 & \chi^2\end{pmatrix}_n^\mu \mathcal{F}_{\mu\nu}\begin{pmatrix}\xi^1 \\ \chi^1 \\ \xi^2 \\ \chi^2\end{pmatrix}_n^\nu\right] \; . \end{aligned} \tag{3.39}$$

Here the bosonic matrix is

$$\mathcal{B}_{\mu\nu} = \begin{pmatrix} (\pi n)^2\delta_{\mu\nu} & \pi n\frac{R_{\mu\nu}}{2} \\ -\pi n\frac{R_{\mu\nu}}{2} & (\pi n)^2\delta_{\mu\nu} \end{pmatrix} , \tag{3.40}$$

while the fermionic matrix is

$$\mathcal{F}_{\mu\nu} = \begin{pmatrix} 0 & \pi n\delta_{\mu\nu} & 0 & 0 \\ -\pi n\delta_{\mu\nu} & 0 & 0 & 0 \\ 0 & 0 & \frac{R_{\mu\nu}}{2} & \pi n\delta_{\mu\nu} \\ 0 & 0 & -\pi n\delta_{\mu\nu} & \frac{R_{\mu\nu}}{2} \end{pmatrix} . \tag{3.41}$$

We can now do these Gaussian integrals explicitly resulting in determinant factors ($\det^{-1/2}$ for bosons and $\det^{1/2}$ for fermions). After cancelling factors of πn we reach

$$\det{}^{1/2}(R_{\mu\nu})\prod_{n=1}^{\infty}\frac{\det^{1/2}\left(1+\frac{1}{(\pi n)^2}(R_{\mu\nu}/2)^2\right)}{\det^{1/2}\left(1+\frac{1}{(\pi n)^2}(R_{\mu\nu}/2)^2\right)} , \tag{3.42}$$

and so clearly the bosonic and fermionic determinants cancel exactly[8]. Now remembering that we are identifying the η_1's with forms on the manifold, while the x's parameterize the manifold, we see that the remaining integrals imply that we integrate the remaining top-form over the manifold

$$\int dx\, d\eta_1 \det{}^{1/2}(R_{\mu\nu}) = \int_{\mathcal{M}} \det{}^{1/2}(R_{\mu\nu})\Big|_{vol} = \int_{\mathcal{M}} \prod_{i=1}^{n/2} x_i \;, \tag{3.43}$$

where n is the dimension of $\mathcal{M}$ and x_i are the eigenvalues of $R_{\mu\nu}/4\pi$. Thus we see the heat kernel method gives the correct index, i.e. the Euler character for the de Rham complex.

3.2.2 $\hat{A}$ genus

To calculate the Dirac genus as in [22] we must modify the quantum theory, removing half of the fermions. This can be achieved by setting $\psi_1 = \psi_2 = \psi/\sqrt{2}$, which means the curvature term in (3.24) disappears[9]. As we shall see in Section 4.1.1 the Hilbert space can now be realized in terms of spinors and so the system will be identified with the spin complex. There are now no terms from the constant configurations but the fluctuation determinants do not cancel. The bosonic determinant is unchanged, however the fermionic part has no curvature term, (neither is it matrix-valued since there is only one variety of fermion, ψ). Thus we are left with

$$\prod_{n=1}^{\infty} \det{}^{-\frac{1}{2}} \left(1 + \frac{1}{(\pi n)^2}(R_{\mu\nu}/2)^2\right) \; . \tag{3.44}$$

However this is simply the standard product expansion of the sinh function

$$\sinh(x) = x . \prod_{k=1}^{\infty} \left(1 + \frac{x^2}{\pi^2 k^2}\right) \; , \tag{3.45}$$

[8]Note the determinants in (3.42) do not include constant zero ($n = 0$) modes, since these are included in the final integration over the manifold.

[9]This also implies we have only 1 supersymmetry charge and this situation is sometimes known as $\mathcal{N} = \frac{1}{2}$ SUSY.

We have finally then

$$\begin{aligned} \mathrm{Tr}(-1)^F e^{-\beta\mathcal{H}} &= \int_{\mathcal{M}} \det{}^{\frac{1}{2}} \left(\frac{R_{\mu\nu}/2}{\sinh(R_{\mu\nu}/2)} \right) , \\ &= \int_{\mathcal{M}} \prod_{i=1}^{n/2} \frac{x_i/2}{\sinh(x_i/2)} . \end{aligned} \tag{3.46}$$

We thus get the $\hat{A}$-genus as expected in the spin complex.

3.2.3 χ_y genus

In general there is no closed σ-model which allows us to determine the arithmetic genus. However we should be able to deform the $\mathcal{N}=1$ model in just the same way as the χ_y genus deformed the Euler character. In [32] the χ_y genus was related to SUSY quantum mechanics via a twisting of the boundary conditions on the fermions. Instead, we would like to calculate it by deforming the Lagrangian (3.24) by a soft-breaking[10] term

$$\mathcal{L}_m = -\tfrac{1}{2} m \psi_2^\mu \omega_{\mu\nu} \psi_2^\nu + c , \tag{3.47}$$

where m is the mass parameter, $\omega_{\mu\nu}$ is the *fundamental 2-form* and c is a constant arising via a normal ordering prescription. In [11] it was shown how it can be fixed by ensuring the Hamiltonian operator annihilates the vacuum. Here we will leave it undefined for now.

We should also note that the mass term breaks half the supersymmetry (those related to the ϵ_2), hence we have one unbroken supersymmetry and we should find the limit $m \to \infty$ calculates the arithmetic genus, whereas we return to two unbroken supersymmetries as $m \to 0$. The heat kernel method works as above with the constant

[10] Meaning this term will break part of the supersymmetry of the theory.

part of the Lagrangian now

$$\mathcal{L}_{(0)} = -\tfrac{1}{4}R_{\mu\nu\sigma\rho}\eta_1^\mu\eta_1^\nu\eta_2^\sigma\eta_2^\rho + \tfrac{1}{2}m\eta_2^\mu\omega_{\mu\nu}\eta_2^\nu + c\ , \tag{3.48}$$

while the fermionic determinant becomes

$$\Delta^F_{\mu\nu} = \begin{pmatrix} \partial_t\delta_{\mu\nu} & 0 \\ 0 & \partial_t\delta_{\mu\nu} - \tfrac{1}{2}R_{\mu\nu\sigma\rho}\eta_1^\sigma\eta_1^\rho - m\omega_{\mu\nu} \end{pmatrix} + \dots\ , \tag{3.49}$$

where we have already discarded terms containing η_2. This time we have contributions from the constant configurations as well as non-cancelling fluctuation determinants

$$\det{}^{1/2}(R_{\mu\nu}/2 - \beta m\omega_{\mu\nu}/2)\prod_{n=1}^{\infty}\frac{\det^{1/2}\left(1+\frac{1}{(\pi n)^2}(R_{\mu\nu}/2-\beta m\omega_{\mu\nu}/2)^2\right)}{\det^{1/2}\left(1+\frac{1}{(\pi n)^2}(R_{\mu\nu}/2)^2\right)}\ , \tag{3.50}$$

then using the identity (3.45) we find

$$e^{-\beta c}\int_{\mathcal{M}}\det{}^{1/2}\left(R_{\mu\nu}\frac{\sinh(R_{\mu\nu}/2-\beta m\omega_{\mu\nu}/2)}{\sinh(R_{\mu\nu}/2)}\right) = \int_{\mathcal{M}}\prod_{i=1}^{n/2}\frac{x_i(1+ye^{-x_i})}{1-e^{-x_i}}\ , \tag{3.51}$$

where $y = -e^{-\beta m}$. In the process the constant c is set to $mn/2$. Thus we see the deformation (3.47) calculates the χ_y genus with suitable limits at asymptotic values of the parameter.

3.3 Killing Vectors and Equivariant Indices

We will now consider the case when our quantum-mechanical system contains coupling to a *Killing* vector field, $\phi(X)$. A Killing vector field generates an isometry of the manifold preserving the metric

$$\mathfrak{L}_\phi\, g = 0\ , \tag{3.52}$$

where $\mathfrak{L}_\phi$ is the Lie derivative, and so represents the direction of symmetries of the manifold. Systems with such fields were considered in [33–39] in the context of the semi-classical quantization of monopoles. For now we will need only the Euclidean Lagrangian related to the $\hat{A}$ genus

$$\mathcal{L}_E = \tfrac{1}{2} g_{\mu\nu} \left(\dot{X}^\mu \dot{X}^\nu + \phi^\mu \phi^\nu + \psi^\mu D_t \psi^\nu - \psi^\sigma \psi^\nu \nabla_\sigma \phi^\mu \right) . \tag{3.53}$$

The key point in deriving the integral expression now is that the functional integral (3.23) may depend on β via the combination $\beta\vec{\phi}$. So we can take a limit in which $\beta \to 0$ keeping $\beta\vec{\phi}$ fixed. In this limit, only constant paths $X^\mu(t)$ which are fixed by the isometry ϕ contribute. The functional integral consequently only receives contributions from the neighbourhood of the fixed-point set $\mathcal{M}^\phi \subset \mathcal{M}$ under ϕ. Using the Riemannian metric on $\mathcal{M}$, we can decompose the tangent bundle along the $\mathcal{M}^\phi$ as

$$T\mathcal{M}\Big|_{\mathcal{M}^\phi} \simeq T\mathcal{M}^\phi \oplus \mathcal{N} , \tag{3.54}$$

where $\mathcal{N}$ is the normal bundle along $\mathcal{M}^\phi$. The curvature tensor splits as follows

$$R\Big|_{\mathcal{M}^\phi} = R^\phi \oplus R^\mathcal{N} , \tag{3.55}$$

where R^ϕ is the Riemann curvature of $\mathcal{M}^\phi$ and $R^\mathcal{N}$ is a connection in the normal bundle.

Evaluating the functional integral according to the heat kernel method described above, we find that we are now calculating the *equivariant index* of the particular complex. Remembering that η^μ are identified with 1-forms in $T^*\mathcal{M}^\phi$, so

$$\Omega_{\mu\nu}\eta^\nu = 0 , \tag{3.56}$$

where the skew-symmetric matrix $\Omega_{\mu\nu}$, the Riemannian moment, gives the action of the Lie derivative, $\mathfrak{L}_\phi$, in the normal bundle $\mathcal{N}$

$$\Omega_{\mu\nu} = i\nabla_\mu \phi_\nu \Big|_{\mathcal{M}^\phi} . \tag{3.57}$$

So the fluctuations $\delta X^\mu(t)$ include the constant modes and fluctuations orthogonal to $\mathcal{M}^\phi$ as well as the fluctuations along $\mathcal{M}^\phi$, while to match this, $\delta\psi^\mu(t)$ includes all the fluctuations in the dual normal bundle as well as non-constant fluctuations in $T^*\mathcal{M}^\phi$. We now expand the Euclidean action to quadratic order in the fluctuations, using the identity

$$\tfrac{1}{2}(\nabla_\mu\phi_\nu - \nabla_\nu\phi_\mu) = \Omega_{\mu\nu} + \tfrac{1}{2}\Omega^\rho_\sigma R_{\rho\lambda\mu\nu}\delta X^\sigma \delta X^\lambda + \cdots \ . \tag{3.58}$$

As before, when calculating the Dirac index we find there are no constant contributions while the fluctuations are now given by

$$\begin{aligned} \Delta^B_{\mu\nu} &= (-\partial_t\delta^\lambda_\mu + \Omega^\lambda_\mu)(-\partial_t\delta_{\lambda\nu} + \Omega_{\lambda\nu} + \tfrac{1}{2}R_{\lambda\nu\sigma\rho}\eta^\sigma\eta^\rho) + \cdots \ , \\ \Delta^F_{\mu\nu} &= (-\partial_t\delta_{\mu\nu} + \Omega_{\mu\nu}) + \cdots \ . \end{aligned} \tag{3.59}$$

Performing the functional integral over the fluctuations allows partial cancellation of the brackets in (3.59). In contrast to the case with no Killing vector, where the constant modes were included in the final integration over the manifold rather than in the fluctuations, we will only be integrating over the fixed point set $\mathcal{M}^\phi$. Thus, as stated above, we should include the zero mode fluctuations in the normal bundle. Splitting the integral according to $T\mathcal{M}^\phi \oplus \mathcal{N}$ the fluctuations gives

$$\det{}^{1/2}\Big(\frac{R^\phi_{\mu\nu}/2}{\sinh R^\phi_{\mu\nu}/2}\Big)\det{}^{1/2}\Big(\frac{1}{2\sinh(\beta\Omega_{\mu\nu} + R^{\mathcal{N}}_{\mu\nu})/2}\Big) \ , \tag{3.60}$$

where $R^\phi_{\mu\nu} = \frac{1}{2}R^\phi_{\mu\nu\sigma\rho}\eta^\sigma\eta^\rho$ and $R^{\mathcal{N}}_{\mu\nu} = \frac{1}{2}R^{\mathcal{N}}_{\mu\nu\sigma\rho}\eta^\sigma\eta^\rho$ and we have again scaled η^μ by $\beta^{-1/2}$. Now replacing Grassmann coordinates η^μ with 1-forms in $T^*\mathcal{M}^\phi$, and being careful with the overall normalization[11], we have shown that the functional integral (3.23) of the Lagrangian (3.53) is equal to

$$\sum_\alpha \Big(\frac{\beta}{2\pi i}\Big)^{n_\alpha/2}\int_{\mathcal{M}^\phi(\alpha)} \det{}^{1/2}\Big(\frac{R^\phi_{\mu\nu}/2}{\sinh R^\phi_{\mu\nu}/2}\Big)\det{}^{1/2}\Big(\frac{1}{2\sinh(\beta\Omega_{\mu\nu} + R^{\mathcal{N}}_{\mu\nu})/2}\Big) \ , \tag{3.61}$$

[11] see [40].

where we have allowed for the possibility that the fixed-point set has more than one component labelled by α. In the above n_α is the dimension of the particular component $\mathcal{M}^\phi(\alpha)$. The combination $\beta\Omega_{\mu\nu} + R^{\mathcal{N}}_{\mu\nu}$ is the *equivariant curvature* of the normal bundle.

We are now going to examine an equivariant generalization of the χ_y genus, combining all the various complications we have encoutered above[12]. The heat kernel has not been determined before for this system. We begin with the Euclideanized Lagrangian

$$\begin{aligned}\mathcal{L}_E &= \tfrac{1}{2}g_{\mu\nu}\Big(\dot{X}^\mu\dot{X}^\nu + \phi^\mu\phi^\nu + \psi^\mu_\alpha D_t\psi^\nu_\alpha + (-1)^\alpha\psi^\sigma_\alpha\psi^\nu_\alpha\nabla_\sigma\phi^\mu\Big)\\ &\quad -\tfrac{1}{4}R_{\mu\nu\sigma\rho}\psi^\mu_1\psi^\nu_1\psi^\sigma_2\psi^\rho_2 + \tfrac{1}{2}m\psi^\mu_2\omega_{\mu\nu}\psi^\nu_2 + c\ . \end{aligned}\tag{3.62}$$

we have additional constant pieces

$$\mathcal{L}_{(0)} = -\tfrac{1}{4}R_{\mu\nu\sigma\rho}\eta^\mu_1\eta^\nu_1\eta^\sigma_2\eta^\rho_2 + \tfrac{1}{2}m\eta^\mu_2\omega_{\mu\nu}\eta^\nu_2 + c\ , \tag{3.63}$$

and while the bosonic fluctuations are as in (3.59), the fermionic matrix is

$$\Delta^F = \begin{pmatrix} -\partial_t\delta_{\mu\nu} + \Omega_{\mu\nu} & 0 \\ 0 & \partial_t\delta_{\mu\nu} + \Omega_{\mu\nu} + \tfrac{1}{2}R_{\mu\nu\sigma\rho}\eta^\sigma_1\eta^\rho_1 - m\omega_{\mu\nu}\end{pmatrix} + \cdots\ , \tag{3.64}$$

to leading order, where, since we will be scaling η_1 by $\beta^{-1/2}$, terms with any η_2's will be at next order and so will again be suppressed in the $\beta \to 0$ limit. Combining all the contributions we obtain

$$\begin{aligned}&\det{}^{1/2}\Big(R^\phi_{\mu\nu} - \beta m\omega^\phi_{\mu\nu}\Big)\det{}^{1/2}\Big(\frac{R^\phi_{\mu\nu}/2}{\sinh R^\phi_{\mu\nu}/2}\Big)\det{}^{1/2}\Big(\frac{1}{2\sinh(\beta\Omega_{\mu\nu} + R^{\mathcal{N}}_{\mu\nu})/2}\Big)\\ &.\det{}^{1/2}\Big(\frac{\sinh(R^\phi_{\mu\nu} - \beta m\omega^\phi_{\mu\nu})/2}{(R^\phi_{\mu\nu} - \beta m\omega^\phi_{\mu\nu})/2}\Big)\det{}^{1/2}\Big(2\sinh(\beta\Omega_{\mu\nu} + R^{\mathcal{N}}_{\mu\nu} - \beta m\omega^{\mathcal{N}}_{\mu\nu})/2\Big),\end{aligned}\tag{3.65}$$

[12]Note that in comparison to [33–39] we have taken ϕ_4 to be the only non-zero Killing vector, in accordance with our needs in the next chapter.

which means our functional integral is

$$e^{-\beta mn/2}\sum_{\alpha}\Big(\frac{\beta}{2\pi i}\Big)^{n_\alpha}\int_{\mathcal{M}^\phi(\alpha)} \det{}^{1/2}\Big(R^\phi_{\mu\nu}\frac{\sinh(R^\phi_{\mu\nu}-\beta m\omega^\phi_{\mu\nu})/2}{\sinh R^\phi/2}\Big)$$
$$\times\det{}^{1/2}\Big(\frac{\sinh(\beta\Omega_{\mu\nu}+R^{\mathcal{N}}_{\mu\nu}-\beta m\omega^{\mathcal{N}}_{\mu\nu})/2}{\sinh(\beta\Omega_{\mu\nu}+R^{\mathcal{N}}_{\mu\nu})/2}\Big), \tag{3.66}$$

3.4 Hyper-Kähler Manifolds

In the next chapter we shall see how the quantum mechanical σ-models arise in the calculation of instanton coefficients in $\mathcal{N}=2$ SYM theories. An important property of these theories is that the manifold concerned, the instanton moduli space, is hyper-Kähler. Here we shall briefly examine some of the consequences of this for the indices we have calculated.

Complex manifolds are even dimensional manifolds which admit a complex structure $\mathbf{I}: T\mathcal{M}\to T\mathcal{M}$, satisfying $\mathbf{I}^2=-1$ as well as an integrability condition given by the vanishing of the Neijenhuis tensor for any two vectors X and Y

$$[\mathbf{I}X,\mathbf{I}Y]-[X,Y]-\mathbf{I}[X,\mathbf{I}Y]-\mathbf{I}[\mathbf{I}X,Y]=0\ . \tag{3.67}$$

The complex structure defines the *fundamental 2-form*

$$\omega(X,Y)=g(\mathbf{I}X,Y)\ , \tag{3.68}$$

for any two vectors X and Y. A Kähler manifold is one for which $\omega_{\mu\nu}$, now called the *Kähler form*, is closed, $d\omega=0$. A hyper-Kähler manifold then has 3 independent complex structures $\mathbf{I}^{(c)}$ (and so 3 Kähler forms) which satisfy the algebra

$$\mathbf{I}^{(c)}\mathbf{I}^{(d)}=-\delta^{cd}+\epsilon^{cde}\mathbf{I}^{(e)}\ . \tag{3.69}$$

These three complex structures quadruple the number of supersymmetries compared with a generic manifold and we shall see this explicitly in the next chapter. In terms of notation the $\mathcal{N} = 1$ theory, which had 2 supersymmetries, now has 8, while the $\mathcal{N} = \frac{1}{2}$ theory now has 4. Further a Killing vector, ϕ, is "tri-holomorphic" if it preserves each of the 3 complex structures

$$\mathfrak{L}_\phi \mathbf{I}^{(c)} = 0, \qquad c = 1, 2, 3 \; . \tag{3.70}$$

An important property of hyper-Kähler manifolds is that they are Ricci flat, i.e. their curvature vanishes

$$\sum_i x_i = 0 \; . \tag{3.71}$$

As noted previously, this allows us to identify the arithmetic and the $\hat{A}$ indices, since then

$$\prod_i (1 - e^{-x_i}) = \prod_i 2 \sinh(x_i/2) \; . \tag{3.72}$$

Chapter 4

$\mathcal{N} = 2, 2^\star$ Prepotentials

In the previous chapter we saw how the Witten index of a supersymmetric quantum mechanical system could be determined through the functional integral (3.23). We will now move on to see how such a structure arises in a quantum field theory. To this end, we return to the low-energy effective action of a pure $\mathcal{N} = 2$ SYM gauge theory (2.7). Remember that the coupling and the dual fields were given in terms of a holomorphic prepotential, $\mathcal{F}$. By the work of Seiberg-Witten [1,2] it was shown how the prepotential had a simple classical piece, while the perturbative contributions were exact at one-loop. What remains is to determine the instanton contributions

$$\mathcal{F} = \overbrace{\tfrac{1}{2}\tau_0 \sum_i a_i^2}^{\text{class.}} - \overbrace{\frac{1}{8\pi i}\sum_\alpha (a\cdot\alpha)^2 \log(\frac{(a\cdot\alpha)^2}{\Lambda^2})}^{\text{pert.}} + \overbrace{\frac{1}{4\pi i}\sum_{k=1}^{\infty} \mathcal{F}_k \Lambda^{2kN}}^{\text{inst.}}, \tag{4.1}$$

where α is a positive root of the gauge group SU(N). The instanton co-efficients are determined implicitly by the Seiberg-Witten curve (2.11), we will use this as a method of direct calculation in Section 5.2. However, the $\mathcal{F}_k$ can in fact be calculated directly from the functional integral by localization around the instanton solution and integrating out fluctuations to leading order (as was done in the heat kernel method).

In classifying the instanton solutions we are led to the concept of a *moduli space* of inequivalent[1] solutions. Since the instanton solutions are finite action classical solutions of gauge theories in a four-dimensional Euclidean space, they can initially be classified by a topological charge, k. Thus the moduli space splits into distinct manifolds $\mathfrak{M}_k$, although it should be noted they are not strictly speaking manifolds since they have conical-type singularities. Each individual moduli space itself contains a lot of structure, they are all in fact hyper-Kähler manifolds, of dimension $4kN$. We will also remove a factor giving the overall position of the instanton configuration, leaving us with the *centred moduli space*, $\hat{\mathfrak{M}}_k$: $\mathfrak{M}_k \approx \hat{\mathfrak{M}}_k \times \mathbb{R}^4$.

The coordinates on the moduli space, X^μ, describe the collective properties of the gauge field, and so are called *collective coordinates.* Thus the functional integral at the instanton solution can be converted into a collective coordinate integral where the metric of the moduli space is restricted by the ADHM constraints, developed by Atiyah, Drinfeld, Hitchin and Manin for the construction of instantons.

4.1 Instantons in Four and Five dimensions

We begin with the instanton contributions to the prepotential in four dimensions. The coefficient of the k-instanton contribution $\mathcal{F}_k$ can be determined from the instanton-induced long-distance behaviour of certain correlators or from the instanton contributions to the condensate u_2 [6, 41, 42]. The first explicit calculations done were at the one instanton level in [43] for gauge group SU(2). This was extended to the two instanton level using the ADHM construction of the instanton moduli space in [41,42], thereby providing the first non-trivial checks of Seiberg-Witten theory for SU(2). Explicit calculations for SU(N) at the one instanton level were then performed in [44]. At

[1]where 'equivalence' is thought of as up to *local* gauge transformations. See [6] for more information regarding all aspects of instantons.

this time it was discovered how the multi-instanton contribution could be formulated even though the ADHM constraints could not be solved [44].

It was then found that the k-instanton contribution could be formulated as the matrix partition function for a supersymmetric σ-model, see Section VIII of [6]. The ADHM construction is then naturally interpreted as a gauged linear σ-model version of this matrix integral. The argument is as follows: the four-point anti-chiral fermion correlator $\langle\bar{\lambda}\bar{\lambda}\bar{\psi}\bar{\psi}\rangle$ contains the fourth derivative (with respect to the VEVs) of the prepotential as well as an integral over the location of the instanton. However, by inserting the long-distance behaviour of the fermions into the collective coordinate integral we obtain the same integral over the location of the instanton as well as the fourth derivative of the integral of a σ-model over $\hat{\mathfrak{M}}_k$.

Before proceeding, we need to address the issue of singularities. As it stands, the instanton moduli space has orbifold type singularities which correspond physically to situations where an instanton shrinks to zero size[2]. It is important to stress that there is nothing inherently pathological about these singularities, however, they do add certain complications. There are two ways to deal with them:

- Work in the covering space of the hyper-Kähler quotient construction, which means working with a gauged linear σ-model.
- Deform the space in order to remove the singularities. It turns out that there is a natural way to do this, consistent with the hyper-Kähler structure of the moduli space. This deformed instanton moduli space naturally arises when the parent field theory is formulated in a non-commutative spacetime (see [6] and

[2]It would appear from the instanton solutions, explicit at one instanton, that the field strength goes to zero as the instanton shrinks, yet there must still be a topological charge, hence the integration measure has a singularity.

references therein). It has been argued that the deformation does not alter the prepotential itself, except by physically unimportant VEV-independent terms.

It is more convenient to use the deformation approach since then $\widehat{\mathfrak{M}}_k$ is a smooth space and we can use the index theory on smooth manifolds as developed in the previous chapter. We can then write the instanton coefficient as an integral over the smooth manifold $\widehat{\mathfrak{M}}_k$ as

$$\mathcal{F}_k = \int d^nX\, d^n\psi\, e^{-\mathcal{S}_E}\ , \tag{4.2}$$

where the "instanton action" is

$$\mathcal{S}_E = \tfrac{1}{2}g_{\mu\nu}(X)\big(\phi^{\dagger\mu}(X)\phi^\nu(X) - \psi^\sigma\psi^\nu\nabla_\sigma\phi^{\dagger\mu}(X)\big)\ . \tag{4.3}$$

Here, $g_{\mu\nu}(X)$ is the metric on $\widehat{\mathfrak{M}}_k$, in some local coordinate system X^μ, $\mu=1,\ldots,n$, where $n=4(kN-1)$, ∇ is the Levi-Civita connection and ψ^μ are the Grassmann collective coordinates of the instanton. Notice that the measure is simple because the factors of $\sqrt{g}$ cancel between the fermions and bosons. The fields $\phi^\mu(X)$ come from the non-zero VEVs in the parent theory. They in fact form components of a Killing vector in the complexified tangent space to the hyper-Kähler space $\widehat{\mathfrak{M}}_k$ which arises in the following way. The space $\widehat{\mathfrak{M}}_k$ admits a group of SU(N) isometries which correspond to the action of global gauge transformations on the instanton solution. The Killing vector ϕ corresponds to the particular element of the associated (complexified) Lie algebra of SU(N) picked out by the VEV $\vec{v}$ of the scalar field.[3] As noted at the end of the previous chapter the Killing vector ϕ is tri-holomorphic Killing meaning it preserves both the metric—therefore Killing—and the 3 independent complex structures—therefore tri-holomorphic—of the hyper-Kähler space $\widehat{\mathfrak{M}}_k$.

[3] The VEV picks out an element of the Lie algebra of SU(N) which we can choose in the Cartan subalgebra to be $\vec{v}\cdot\vec{H}$, where $\vec{H}$ denotes the Cartan generators.

At this stage, we should caution that in practice we do not actually have the explicit form of the metric $g_{\mu\nu}(X)$, beyond $k = 1$, due to the infamous ADHM constraints which must be resolved in order to get an explicit parameterization of the moduli space of instantons $\hat{\mathfrak{M}}_k$. No solutions of the ADHM constraints are known beyond $k = 2$, but in these cases, as we have already mentioned, we can formulate the integral (4.2) as a zero-dimensional gauged linear σ-model. In this formulation the ADHM constraints are imposed via Lagrange multipliers and there is an auxiliary $\mathrm{U}(k)$ gauge field. These details are not required in what follows but can be found in the review [6].

What is not apparent about the integral in (4.2) is whether it computes anything topological associated to the instanton moduli space. In fact, for general gauge groups, the coefficients $\mathcal{F}_k$ are complicated functions of the VEVs involving rational coefficients. For instance, in the simplest case of gauge group $\mathrm{SU}(2)$ the first few coefficients extracted from Seiberg-Witten theory are [1]

$$\mathcal{F}_1 = \frac{1}{2a^2}\ , \qquad \mathcal{F}_2 = \frac{5}{64a^6}\ , \qquad \mathcal{F}_3 = \frac{3}{64a^{10}}\ . \tag{4.4}$$

Having said this, the integral (4.2) looks superficially like the integral formula for a topological index (3.23), with the key difference being (4.3) describes a zero-dimensional σ-model, whereas in Chapter 3 we examined quantum mechanical, or 1-d, σ-models.

As explained in the introduction, the connection to index theory is achieved by lifting the $\mathcal{N} = 2$ theory to five dimensions where the extra dimension is compact a with period that we take to be β. To start with let us suppose $\beta = \infty$. In five dimensional Minkowski spacetime there is only a simple one-loop perturbative contribution to the prepotential [45]; in particular, there are no non-perturbative contributions. The reason is that there are no genuine instantons having finite action in five dimensions. All this changes when the (Euclidean) time becomes compact. In $\mathbb{R}^4 \times S^1$, there are instantons because conventional four-dimensional instantons can be embedded in the four non-compact dimensions and have finite five-dimensional action; namely $8\pi^2 k\beta/g_5^2$. Next one can evaluate the semi-classical contributions from these finite action configurations to the five-dimensional prepotential, which we denote $\mathcal{F}^\beta$. It is perhaps not surprising these contributions generalize (4.1) in a fairly obvious way. The instanton contributions to the prepotential in five dimensions are

$$\mathcal{F}^\beta = \mathcal{F}^\beta_{\text{pert}} + \frac{1}{4\pi i\beta^2}\sum_{k=1}^{\infty}\mathcal{F}^\beta_k e^{-8\pi^2 k\beta/g_5^2} \ , \tag{4.5}$$

where the partition function of the zero-dimensional supersymmetric σ-model is replaced by the partition function of the supersymmetric quantum mechanical σ-model on the same target space realized by a Euclidean functional integral in periodic time with periodic boundary conditions

$$\mathcal{F}^\beta_k = \int_{\text{PBC}} [d^n X(t)]\,[d^n\psi(t)]\, e^{-S_E[X,\psi]} \ . \tag{4.6}$$

The instanton action $S[X,\psi]$ is now a functional of the instanton collective coordinates which are now functions of t. The structure of the Euclidean action $S_E[X,\psi]$ is almost completely determined by supersymmetry. The point is that the parent field theory has 8 real supersymmetries—$\mathcal{N}=2$ in four dimensions—and since instanton solutions break half the supersymmetries the quantum mechanical σ-model must admit 4 real supersymmetries. Notice that the number of bosonic and fermionic fields are equal and there is a unique quantum mechanical system of this type: the $\mathcal{N}=\frac{1}{2}$ quantum mechanical σ-model coupled to a Killing vector described in Section 3.3, with a (Minkowski space) Lagrangian (to compare with (3.53))

$$\mathcal{L}=\tfrac{1}{2}g_{\mu\nu}\Big(\dot{X}^\mu\dot{X}^\nu-\phi^\mu\phi^\nu+i\psi^\mu D_t\psi^\nu-i\psi^\sigma\psi^\nu\nabla_\sigma\phi^\mu\Big)\ . \tag{4.7}$$

In the generic situation this system has one real supersymmetry

$$\delta X^\mu=-i\epsilon\psi^\mu\ ,\qquad \delta\psi^\mu=(\dot{X}^\mu-\phi^\mu)\epsilon\ . \tag{4.8}$$

However, since the target space $\hat{\mathfrak{M}}_k$ is hyper-Kähler there are 3 additional supersymmetries

$$\delta X^\mu=i\epsilon^c(\mathbf{I}^{(c)}\cdot\psi)^\mu\ ,\qquad \delta\psi^\mu=(\mathbf{I}^{(c)}\cdot(\dot{X}-\phi))^\mu\epsilon^c-i\epsilon^c(\mathbf{I}^{(c)}\cdot\psi)^\nu\psi^\sigma\Gamma^\mu_{\nu\sigma}\ , \tag{4.9}$$

which involve the three independent complex structures $\mathbf{I}^{(c)}$, $c=1,2,3$. These supersymmetries are also preserved by the coupling to ϕ since the latter is tri-holomorphic (3.70).

Now we describe how dimensional reduction gives the four-dimensional contributions to the prepotential in (4.2). The first point to make is that in the five-dimensional theory there is only a *real* scalar field and its real VEV determines the Killing vector ϕ on the instanton moduli space. Note in five dimensions this Killing vector lives in the tangent space of $\hat{\mathfrak{M}}_k$ and *not* its complexification. Now we dimensionally reduce. In the parent theory the component of the gauge field around the circle becomes a scalar field and can have a VEV, *i.e.* a Wilson line, and along with the real scalar field, ϕ^μ, gives the complex scalar field in the four-dimensional parent theory. At the collective coordinate level the extra VEV arises from the possibility of performing a non-trivial Scherk-Schwarz dimensional reduction. This kind of dimensional reduction allows the real Killing vector to become complex. For the most part, and without loss of generality, we shall take the Wilson line to be zero, and so

ϕ will always be a real Killing vector, with the understanding that it may easily be re-instated by complexifying the VEV as well as the associated Killing vector.

The fact that the instanton action of the compactified theory reduces to the four-dimensional one in the $\beta \to 0$ limit, means that the instanton corrections to the prepotential also reduce

$$\mathcal{F}_k^\beta \xrightarrow{\beta\to 0} \beta^{2(1-kN)}\mathcal{F}_k \ . \tag{4.10}$$

The factor of β here goes along with the five-dimensional instanton action term to give the appropriate power of the Λ-parameter of the four-dimensional theory

$$\Lambda^{2N} = \beta^{-2N} e^{-8\pi^2\beta/g_5^2} \ . \tag{4.11}$$

4.1.1 Quantum Mechanics on the Instanton Moduli Space

We have seen that the instanton contribution to the prepotential in five dimensions involves the partition function of a supersymmetric quantum mechanical σ-model with $\widehat{\mathfrak{M}}_k$ as the target space. In this section, we consider this quantum mechanical system in more detail in order to explicitly determine the vacuum states and identify the differential operators we saw in Chapter 3. Again we refer the reader to [33–37] for more details.

Let us describe the canonical quantization of the system and the Hilbert space of states. The canonical (anti-)commutators are

$$[X^\mu, p^\nu] = ig^{\mu\nu} \ , \qquad \{\psi^\mu, \psi^\nu\} = g^{\mu\nu} \ . \tag{4.12}$$

The Hilbert space can be realized as the space of sections of the spin bundle over $\widehat{\mathfrak{M}}_k$ with $\psi^\mu = \gamma^\mu/\sqrt{2}$ where the γ^μ are the usual γ-matrices

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \ , \tag{4.13}$$

We choose an off-diagonal basis of γ-matrices, with the analogue of "γ^5"

$$\gamma^{2n+1} \equiv \prod_{\mu=1}^{2n} \gamma^{\mu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{4.14}$$

Equivalently, since $\hat{\mathfrak{M}}_k$ is hyper-Kähler, one can realize the Hilbert space in terms of holomorphic forms but we prefer to use spinors such that we can compare with the σ-model associated to the Spin complex in Section 3.1.3. The Hilbert space has the usual split into bosonic and fermionic states

$$\mathcal{H} = \mathcal{H}^{+} \oplus \mathcal{H}^{-} , \qquad (-1)^F : \ \mathcal{H}^{\pm} \to \pm\mathcal{H}^{\pm} , \tag{4.15}$$

realized as chiral spinors and anti-chiral spinors, respectively, so the fermion parity is

$$(-1)^F \equiv \gamma^{2n+1} . \tag{4.16}$$

The supercovariant momentum operator, defined by

$$\pi_{\mu} = p_{\mu} - \tfrac{i}{4}\Gamma^{\nu}_{\mu\rho}[\psi_{\nu}, \psi^{\rho}] , \tag{4.17}$$

is then represented by the covariant derivative $\pi_{\mu} = -i\nabla_{\mu}$ on spinors derived from the Riemannian spin connection. The supersymmetry charge which generates the supersymmetry (4.8) is

$$Q = \psi^{\mu}(\pi_{\mu} - \phi_{\mu}) \equiv \tfrac{1}{\sqrt{2}}\big(\not{D} - \gamma^{\mu}\phi_{\mu}\big) . \tag{4.18}$$

There are corresponding generators $Q^{(c)}$ for the additional supersymmetries (4.9). The quantity on the right-hand side of (4.18) is proportional to the *equivariant* Dirac operator

$$\not{D}_{\phi} = \gamma^{\mu}\big(\nabla_{\mu} - \phi_{\mu}\big) . \tag{4.19}$$

In the off-diagonal basis of γ matrices

$$\not{D}_\phi = \begin{pmatrix} 0 & \not{D}_\phi^- \\ \not{D}_\phi^+ & 0 \end{pmatrix} , \tag{4.20}$$

the supersymmetry algebra is

$$\{Q,Q\} = 2(\mathcal{H}-\mathcal{Z}) , \qquad \{Q^{(c)},Q^{(d)}\} = 2\delta_{cd}(\mathcal{H}-\mathcal{Z}) , \qquad \{Q,Q^{(c)}\} = 0 . \tag{4.21}$$

Here, $\mathcal{H}$ is the Hamiltonian and $\mathcal{Z}$ is a *central charge* which commutes with all the supersymmetry charges and the Hamiltonian. Explicitly we have

$$\begin{aligned} \mathcal{H} &= \tfrac{1}{2}\Big(g^{-1/2}\pi_\mu g^{1/2} g^{\mu\nu}\pi_\nu + \phi_\mu\phi^\mu + i(\nabla_\mu\phi_\nu)\psi^\mu\psi^\nu\Big) , \\ \mathcal{Z} &= \phi^\mu\pi_\mu - \tfrac{i}{2}(\nabla_\mu\phi_\nu)\psi^\mu\psi^\nu , \end{aligned} \tag{4.22}$$

and one can easily show that $i\mathcal{Z} = \mathfrak{L}_\phi$, the Lie derivative acting on spinors. The central charge is related to the electric charge of the correspond state in the parent theory in a simple way. Recall that the Killing vector ϕ is associated to an element of the Lie algebra of the SU(N) gauge group on the instanton moduli space. In fact it is the element $\vec{\phi}\cdot\vec{H}$, where $\vec{\phi}$ is the vector of VEVs and $\vec{H}$ are the Cartan generators. Hence

$$\mathcal{Z} = \vec{\phi}\cdot\vec{q} , \tag{4.23}$$

where $\vec{q}$ is the vector of electric charges associated to the Cartan subgroup $\mathrm{U}(1)^{N-1} \subset \mathrm{SU}(N)$. In particular, $\vec{q}$ must be quantized to lie in the root lattice of SU(N).[4]

States in the quantum mechanical system describe states of instanton dyons in the parent field theory in five dimensions. In particular, single particle states correspond to normalizable states. There is an important class of states in the field theory known as BPS states and these come in small representations of supersymmetry. In a theory

[4]Eigenvalues in the weight lattice do not occur since all fields are adjoint valued.

with 8 supercharges these small multiplets consist of 4 states since they are annihilated by $\frac{1}{2}$ the supersymmetry charges of the parent theory. In the quantum mechanical system single particle BPS states of instanton dyons[5] correspond to normalizable states which are annihilated by the four supersymmetry charges Q and $Q^{(c)}$ since these are the four charges which preserve the original bosonic instanton configuration

$$Q\,|\Psi\rangle = Q^{(c)}\,|\Psi\rangle = 0\ , \qquad \langle\Psi|\Psi\rangle < \infty\ . \tag{4.24}$$

Thus at the BPS states the supersymmetry generators will commute, thus (4.21) defines the BPS equation as

$$\mathcal{H} = \mathcal{Z}\ . \tag{4.25}$$

4.1.2 The $\mathcal{N}=2^\star$ Theory

We now turn our attention to the story with an adjoint hypermultiplet, the $\mathcal{N}=2^\star$ theory in four dimensions. We will describe this theory as softly-broken pure $\mathcal{N}=4$, which, at $m=0$, has a zero beta function and so the prepotential is exact at the classical level. There are non-physical terms (VEV-independent) which have the form

$$\mathcal{F}_k^{\mathcal{N}=4} = N\sum_{n|k}\frac{1}{n}\ , \tag{4.26}$$

and we shall see later how these are relevant to our story. In physical terms however, we no longer have a renormalization scale, Λ. Instead we have just the dimensionless coupling constant

$$\tau(a) = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}\ , \tag{4.27}$$

and we will use $q=e^{2\pi i\tau}$. Once we give half of the fermions a mass the theory acquires a richer structure. Firstly, we get non-trivial corrections to the effective action, and

[5]The states are called dyons since they carry two charges, the topological instanton charge and the VEV-induced 'electric' charge.

secondly the theory should 'interpolate' between the pure theories at $\mathcal{N} = 4$ and $\mathcal{N} = 2$. Meanwhile the non-trivial corrections are contained within the prepotential in the usual way

$$\begin{aligned}\mathcal{F}^m &= \frac{1}{2}\tau_0 \sum_i a_i^2 - \frac{1}{8\pi i} \sum_\alpha \left((a \cdot \alpha)^2 \log(a \cdot \alpha)^2 - \tfrac{1}{2}(a \cdot \alpha + m)^2 \log(a \cdot \alpha + m)^2 \right. \\ &\qquad \left. -\tfrac{1}{2}(a \cdot \alpha - m)^2 \log(a \cdot \alpha - m)^2 \right) + \frac{1}{4\pi i} \sum_{k=1}^{\infty} \mathcal{F}_k^m q^k \ . \end{aligned} \tag{4.28}$$

as determined in [46]. Here also the first few instanton contributions in SU(2) were calculated. In [47] the first 8 instanton coefficients were calculated from Seiberg-Witten theory with the gauge group SU(2). The first three are

$$\begin{aligned}\mathcal{F}_1^m &= \frac{1}{2}\frac{m^4}{a^2} \ , \\ \mathcal{F}_2^m &= \frac{3}{2}\frac{m^4}{a^2} - \frac{3}{4}\frac{m^6}{a^4} + \frac{5}{64}\frac{m^8}{a^6} \ , \\ \mathcal{F}_3^m &= 2\frac{m^4}{a^2} - 4\frac{m^6}{a^4} + \frac{5}{2}\frac{m^8}{a^6} - \frac{7}{12}\frac{m^{10}}{a^8} + \frac{3}{64}\frac{m^{12}}{a^{10}} \ . \end{aligned} \tag{4.29}$$

Again, the instanton contributions can alternatively be calculated from first principles. The prepotential can be reformulated as the matrix partition function of a σ-model through the four-point correlator.

Now however we must be careful with the contribution from the centre-of-mass piece, $\mathbb{R}^4$, that we removed from the moduli space. The prepotential now has a factor from the partition function on $\mathbb{R}^4$ with the mass term. In four dimensions this simply pulls down a factor of im for each pair of fermions. Thus we can rewrite the instanton coefficient as an integral of a zero-dimensional σ-model over the smooth manifold $\widehat{\mathfrak{M}}_k$ with an additional factor

$$\mathcal{F}_k^m = -m^2 \int d^n X \, d^{2n}\psi \, e^{-S_E} \ , \tag{4.30}$$

where the "instanton action", including the mass deformation, is

$$\begin{aligned}\mathcal{S}_E &= \tfrac{1}{2}g_{\mu\nu}(X)\big(\phi^{\dagger\mu}(X)\phi^\nu(X) - \psi^\sigma_\alpha\psi^\nu_\alpha\nabla_\sigma\phi^{\dagger\mu}(X)\big) - \tfrac{1}{4}R_{\mu\nu\sigma\rho}\psi^\mu_1\psi^\nu_1\psi^\sigma_2\psi^\rho_2 \\ &\quad +\tfrac{1}{2}m\psi^\mu_2\omega_{\mu\nu}\psi^\nu_2 - c. \end{aligned} \tag{4.31}$$

Just as in the $\mathcal{N}=2$ case the $\phi^\mu(X)$ is a tri-holomorphic complex Killing vector[6].

Clearly the mass term allows us to interpolate between the two pure theories, firstly with the $\mathcal{N}=4$ theory being accessible in the simple $m\to 0$ limit. Alternatively the second set of fermions, ψ_2, now acts as a massive adjoint hypermultiplet for the $\mathcal{N}=2$ theory, and the supersymmetries related to ψ_2 are broken. To reduce to the pure $\mathcal{N}=2$ theory care must be taken with the limits. In the four dimensional case we can take $m\to\infty$ so long as we also take $\tau\to i\infty$ such that $qm^{2N}=\Lambda^{2N}$ is kept fixed[7]. At the classical superfield level, the superpotential is such that two of the three chiral multiplets are required to be zero by the critical point equations. This leaves the vector multiplet and one chiral multiplet forming the $\mathcal{N}=2$ supermultiplet.

Just as in Section 4.1 we can make the connection with index theory by lifting to five dimensions on a circle. The four-dimensional instantons can continue to be embedded in the four non-compact dimensions, with unchanged action $8\pi^2k\beta/g_5^2$. The instanton contributions to the prepotential in five dimensions are thus

$$\mathcal{F}^{(\beta,m)} = \mathcal{F}^{(\beta,m)}_{\text{pert}} + \frac{1}{4\pi i\beta^2}\sum_{k=1}^{\infty}\mathcal{F}^{(\beta,m)}_k e^{-8\pi^2k\beta/g_5^2}\,, \tag{4.32}$$

where the partition function of the zero-dimensional supersymmetric linear σ-model is replaced by the partition function of the supersymmetric quantum mechanical

[6]The pure $\mathcal{N}=4$ theory contains three complex Killing vectors, but the addition of the mass deformation requires a superpotential in the parent theory which allows only one non-zero VEV. The particular vector which we allow to be non-zero can be determined by requiring the SUSY algebra to reduce to the $\mathcal{N}=2$ algebra.

[7]We can see that this combination picks out the highest powers of m in each line of (4.29) and thus agrees with (4.4).

linear σ-model on the same target space realized by a Euclidean functional integral in periodic time with periodic boundary conditions

$$\mathcal{F}_k^{(\beta,m)} = -4\sinh^2(\beta m/2)\int_{\mathrm{PBC}} [d^n X(t)]\,[d^{2n}\psi(t)]\,e^{-S_E[X,\psi]}\,. \qquad (4.33)$$

Note we have now taken into account the integration over $\mathbb{R}^4$ in the parent theory of the hypermultiplet fermions with mass and kinetic terms (around momentum modes around the compact dimension, including the zero modes) and the kinetic terms (without the zero modes) of the original fermions and bosons.[8]

Just as in the 4 dimensional case we should find our σ-model as a deformation of pure $\mathcal{N}=4$ SYM. Here there are 16 supersymmetries in the parent field theory, broken to 8 supersymmetries on the instanton solutions. Allowing a non-zero mass will break a further 4 supersymmetries. Since we have twice as many fermions as in the pure $\mathcal{N}=2$ case we are led to the deformed $\mathcal{N}=1$ Lagrangian coupled to a Killing vector, examined in Section 3.3. The (Minkowski space) Lagrangian is, c.f. (3.62)[9]

$$\begin{aligned}\mathcal{L} &= \tfrac{1}{2}g_{\mu\nu}\Big(\dot{X}^\mu\dot{X}^\nu - \phi^\mu\phi^\nu + i\psi^\mu_\alpha D_t\psi^\nu_\alpha + i(-1)^\alpha\psi^\sigma_\alpha\psi^\nu_\alpha\nabla_\sigma\phi^\mu\Big)\\ &\quad +\tfrac{1}{4}R_{\mu\nu\sigma\rho}\psi^\mu_1\psi^\nu_1\psi^\sigma_2\psi^\rho_2 - \tfrac{1}{2}m\psi^\mu_2\omega_{\mu\nu}\psi^\nu_2\,. \end{aligned} \qquad (4.34)$$

When $m=0$ this system has two real supersymmetries

$$\delta X^\mu = -i\epsilon_\alpha\psi^\mu_\alpha\,, \qquad \delta\psi^\mu_\alpha = \dot{X}^\mu\epsilon_\alpha + (-1)^\alpha\phi^\mu\epsilon_\alpha + i\Gamma^\mu_{\nu\rho}\epsilon_\beta\psi^\nu_\beta\psi^\rho_\alpha\,. \qquad (4.35)$$

[8] Without the hypermultiplets, the contributions from fermions and bosons would cancel, hence we did not have to allow for a multiplicative factor in the pure $\mathcal{N}=2$ theory.

[9] In [33–37] there were five real Killing vectors. By the argument given in the footnote after (4.31) we have set all but ϕ_4 to 0. Breaking a different linear combination of the supersymmetry, via a different mass deformation, would result in a different central charge appearing in the correct place in the SUSY algebra, and so we would allow a different Killing vector to be non-zero. Also, we have no need to include the constant, c in (3.47), on the understanding that we are calculating the χ_y index multiplied by some power of $e^{-\beta m}$.

As explained in Section 3.4, the hyper-Kähler nature of the moduli space allows 6 additional supersymmetries giving us the 8 required by the instanton solution

$$\delta X^\mu = i\epsilon^c_\alpha(\mathbf{I}^{(c)}\cdot\psi)^\mu_\alpha\,, \qquad \delta\psi^\mu_\alpha = (\mathbf{I}^{(c)}\cdot(\dot{X}+(-1)^\alpha\phi))^\mu\epsilon^c_\alpha - i(\mathbf{I}^{(c)}\cdot\Gamma)^\mu_{\nu\rho}\epsilon^c_\beta(\mathbf{I}^{(c)}\cdot\psi)^\nu_\beta(\mathbf{I}^{(c)}\cdot\psi)^\rho_\alpha\,, \tag{4.36}$$

which involve the three independent complex structures $\mathbf{I}^{(c)}$, $c = 1, 2, 3$.

The reduction to four dimensions is just the same as in the pure $\mathcal{N} = 2$ case with the real Killing field ϕ combining with the Wilson line around the compact dimension to form a complex scalar field. Again the instanton corrections themselves reduce

$$\mathcal{F}^{(\beta,m)}_k \xrightarrow{\beta\to 0} \beta^{2(1-kN)}\mathcal{F}^m_k\,. \tag{4.37}$$

The factor of β here goes along with the five-dimensional instanton action term to give the appropriate power of the q-parameter of the four-dimensional theory

$$q = \beta^{-2N}e^{-8\pi^2\beta/g_5^2}\,. \tag{4.38}$$

This process does not commute with the reduction to the pure $\mathcal{N} = 2$ theory ($m \to \infty$). In the compactified five dimensions such a limit is no longer smooth. This is because the mass term becomes a periodic variable due to the Kaluza-Klein modes and so the limit $m \to \infty$ is nonsensical. Further, since we have Kaluza-Klein modes around the compact dimension as well as mass modes it would be non-trivial to separate these out. We will see later further examples of this discontinuity in the five dimensional theory. However, we will see that we can pick out the correct modes by hand once we calculate explicit results. To summarize we have a 'flow' between theories as shown in Figure 4.1.

Let us describe the canonical quantization of the $\mathcal{N} = 4$ system [37, 39] before

$$
\begin{tikzcd}
\text{4d } \mathcal{N}=2 & & \text{5d } \mathcal{N}=2 \arrow[ll, "\beta \to 0"', "\Lambda^{2N} = \beta^{-2N} e^{-8\pi^2\beta/g_5^2} \text{ fixed}"] \\
\text{4d } \mathcal{N}=2^{\star} \arrow[u, "{m \to \infty,\ \Lambda^{2N} = m^{2N} q \text{ fixed}}"'] \arrow[d, "m \to 0"'] & & \text{5d } \mathcal{N}=2^{\star} \arrow[ll, "\beta \to 0"', "q = \beta^{-2N} e^{-8\pi^2\beta/g_5^2} \text{ fixed}"] \arrow[u, dashed, "m \to \infty"] \arrow[d, "m \to 0"] \\
\text{4d } \mathcal{N}=4 & & \text{5d } \mathcal{N}=4 \arrow[ll, "\beta \to 0"']
\end{tikzcd}
$$

Figure 4.1: Flows between theories of different dimension and mass content.

adding the mass deformation. The canonical (anti-)commutators are

$$[X^{\mu}, p^{\nu}] = i g^{\mu\nu} , \qquad \{\psi^{\mu}_{\alpha}, \psi^{\nu}_{\beta}\} = g^{\mu\nu}\delta_{\alpha\beta} . \tag{4.39}$$

The supercovariant momentum operator is the obvious generalization of (4.17)

$$\pi_{\mu} = p_{\mu} - \tfrac{1}{2}\Gamma^{\nu}_{\ \mu\rho}\psi_{\alpha\nu}\psi^{\rho}_{\alpha} , \tag{4.40}$$

while the supersymmetry charge which generates the supersymmetry of the theory is

$$Q_{\alpha} = \psi^{\mu}_{\alpha}\pi_{\mu} + (-1)^{\alpha}\psi^{\mu}_{\alpha}\phi_{\mu} , \tag{4.41}$$

with corresponding generators, $Q^{(c)}$ for the additional supersymmetries. The supersymmetry algebra is

$$\begin{aligned}\{Q_\alpha, Q_\beta\} &= \{Q^{(c)}_\alpha, Q^{(c)}_\beta\} = 2\delta_{\alpha\beta}(\mathcal{H} + (-1)^\alpha \mathcal{Z}) \ , \\ \{Q^{(c)}_\alpha, Q^{(d)}_\beta\} &= \{Q_\alpha, Q^{(c)}_\beta\} = 0 \ .\end{aligned} \tag{4.42}$$

Now

$$\begin{aligned}&\mathcal{H} = \tfrac{1}{2}\Big(g^{-1/2}\pi_\mu g^{1/2} g^{\mu\nu}\pi_\nu + g^{\mu\nu}\phi_\mu\phi_\nu - i(-1)^\alpha D_\mu\phi_\nu\psi^\mu_\alpha\psi^\nu_\alpha - \tfrac{1}{2}R_{\mu\nu\sigma\rho}\psi^\mu_1\psi^\nu_1\psi^\sigma_2\psi^\rho_2\Big) \ , \\ &\mathcal{Z} = \phi^\mu\pi_\mu - \tfrac{i}{2}D_\mu\phi_\nu\psi^\mu_\alpha\psi^\nu_\alpha \ .\end{aligned} \tag{4.43}$$

Again the Killing vector is associated to an element in the Lie algebra of the gauge group

$$\mathcal{Z} = \vec{\phi}\cdot\vec{q} \ . \tag{4.44}$$

Whereas in the pure $\mathcal{N}=2$ case we realized the Hilbert space in terms of spinors it is easier in $\mathcal{N}=4$ to describe the system in terms of (anti-)holomorphic coordinates, this should allow us to identify the Hilbert space of the σ-model with the de Rham and Dolbeault complexes. To begin with we can combine ψ_1 and ψ_2 to form creation and annihilation operators, respectively

$$b^{\mu\dagger} = \psi^\mu_1 + i\psi^\mu_2 \ , \quad b^\mu = \psi^\mu_1 - i\psi^\mu_2 \ . \tag{4.45}$$

These obey the standard operator algebra

$$\begin{aligned}\{b^\mu, b^{\nu\dagger}\} &= g^{\mu\nu} \ , \\ \{b^\mu, b^\nu\} &= \{b^{\mu\dagger}, b^{\nu\dagger}\} = 0 \ .\end{aligned} \tag{4.46}$$

Since the b's are Grassmann values we can identify the Hilbert space of states with the de Rham complex

$$b^{\mu_1\dagger}\cdots b^{\mu_p\dagger}\,|0\rangle \longleftrightarrow dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p} \ . \tag{4.47}$$

Further, we can combine the supercharges[10]

$$\begin{aligned} Q &= Q_1 - iQ_2 = b^\mu \pi_\mu + \cdots \longrightarrow dx\frac{\partial}{\partial x} + \cdots = d + \cdots \ , \\ Q^\dagger &= Q_1 + iQ_2 = b^{\mu\dagger}\pi_\mu + \cdots \longrightarrow d^\dagger + \cdots \ . \end{aligned} \tag{4.48}$$

Thus without the mass term the Hilbert space is realized in terms of forms, and so the system is identified with the de Rham complex.

Since the instanton moduli space is a Kähler manifold we can also find a basis of (anti-)holomorphic co-ordinates z^i, $\bar{z}^j$, where the metric is then off-diagonal[11] and $j = 1, ..., n/2$. The Hilbert space is then given by $\bar{\psi}_1^j$ and ψ_2^j as creation operators while ψ_1^j and $\bar{\psi}_2^j$ as annihilation operators, with each set separately satisfying the standard algebra. The states of the Hilbert space can be identified with the elements of the Dolbeault complex via

$$\psi_2^{j_1} \cdots \psi_2^{j_p} \bar{\psi}_1^{k_1} \cdots \bar{\psi}_1^{k_q} |0\rangle \longleftrightarrow dz^{j_1} \wedge \cdots \wedge dz^{j_p} \wedge d\bar{z}^{k_1} \wedge \cdots d\bar{z}^{k_q} \ . \tag{4.49}$$

The supercharges are now[12]

$$\begin{aligned} Q_1 = \psi_1^\mu \pi_\mu + \cdots = \psi_1^j \partial_{z^j} + \bar{\psi}_1^j \partial_{\bar{z}^j} + \cdots &= \star\partial\star + \bar{\partial} + \cdots = -\bar{\partial}^\dagger + \bar{\partial} + \cdots \ , \\ Q_2 = \psi_2^\mu \pi_\mu + \cdots = \psi_2^j \partial_{z^j} + \bar{\psi}_2^j \partial_{\bar{z}^j} + \cdots &= \partial + \star\bar{\partial}\star + \cdots = \partial - \partial^\dagger + \cdots \ . \end{aligned} \tag{4.50}$$

The important point here is that the holomorphic and anti-holomorphic differentials are split in just the right way so that when we break half the supersymmetries (by adding the mass term) the Hilbert space of the σ-model is identified with the Dolbeault complex with massive holomorphic degrees of freedom.

[10] We will ignore terms dependent on the Killing vectors since these will turn the differential operator into an equivariant operator and for now we just want to determine which differential operator the supercharges correspond to.

[11] Meaning non-zero components for terms mixing holomorphic components with anti-holomorphic components.

[12] Note the adjoints of ∂ and $\bar{\partial}$ are defined as in [27].

Now we deform the algebra by including the mass term in the Lagrangian, equivalently the Hamiltonian changes by $\mathcal{H} \to \mathcal{H}' = \mathcal{H} + \mathcal{H}_m$. This breaks the supersymmetries generated by Q_2 and $Q_2^{(c)}$ and so the algebra becomes simply

$$\{Q_1, Q_1\} = \{Q_1^{(c)}, Q_1^{(c)}\} = 2(\mathcal{H}' - \mathcal{H}_m - \mathcal{Z}) \ . \tag{4.51}$$

Note that in the language of the Dolbeault complex the new term is just the number operator counting the holomorphic degree of a state.

We saw above how the normalizable, or BPS, states of the $\mathcal{N} = 2$ quantum mechanical system came in multiplets of 4. These are called $\frac{1}{2}$-BPS states since they are annihilated by half of the original SUSY charges. In the pure $\mathcal{N} = 4$ theory there are now two possibilities since there are generically two independent central charges:

- $\frac{1}{4}$-BPS states from $\mathcal{H} = \mathcal{Z}_1$ or $\mathcal{Z}_2$, these are annihilated by 4 SUSY charges,
- $\frac{1}{2}$-BPS states from $\mathcal{H} = \mathcal{Z}_1 = \mathcal{Z}_2$, which annihilate all the 8 remaining charges.

However, it was noted in [39] among others, that with only one non-zero Killing vector we can have only $\frac{1}{2}$-BPS states whose solutions are annihilated, in this case, by the 4 remaining SUSY charges

$$Q_1 \left|\Psi\right\rangle = Q_1^{(c)} \left|\Psi\right\rangle = 0 \ , \qquad \langle\Psi|\Psi\rangle < \infty \ . \tag{4.52}$$

Thus at the BPS states, set by the right-hand side of (4.51) being zero,

$$\mathcal{H}' = \vec{\phi} \cdot \vec{q} + mj \ . \tag{4.53}$$

4.2 Boundary Effects

We have seen that if the target space of σ-model, $\hat{\mathfrak{M}}_k$, were compact, the partition functions (4.6) and (4.33) would compute the Witten index

$$\int_{\text{PBC}} [d^n X(t)]\,[d^n\psi(t)]\,e^{-S_E[X,\psi]} \overset{\text{compact}}{=} \operatorname{Tr}(-1)^F e^{-\beta\mathcal{H}}\,, \tag{4.54}$$

as shown in Chapter 3.

However, in our case the target space is not compact. This fact has already been mentioned in Section 4.1. The non-compactness due to the unchecked growth of an instanton turns out not to be a problem in the presence of the scalar VEVs which acts as a cut-off on an instanton's size. The problematic non-compactness involves instantons becoming arbitrarily separated in $\mathbb{R}^4$. This is correlated with the fact that single particle BPS states have to be normalizable. In the non-compact case a natural definition of the Witten index involves imposing a normalizability, or L^2, condition. The Witten index with L^2 condition, which we now denote as Ind_k, to make the instanton number explicit, receives contributions only from the single particle BPS states (4.24)

$$\operatorname{Tr}_{L^2}(-1)^F e^{-\beta\mathcal{H}} = \text{Ind}_k\,. \tag{4.55}$$

Since the SUSY charges are identified with differential operators we can see that, in terms of the geometry of $\hat{\mathfrak{M}}_k$, the Witten index is the *equivariant* index of the Dirac and Dolbeault operators with L^2 condition

$$\begin{aligned}\text{Ind}_k &= \operatorname{Tr}_{L^2,\ker \not{D}^+}\left(e^{i\beta\mathfrak{L}_\phi}\right) - \operatorname{Tr}_{L^2,\ker \not{D}^-}\left(e^{i\beta\mathfrak{L}_\phi}\right)\,,\\ \text{Ind}_k^m &= \operatorname{Tr}_{L^2,\ker\bar\partial}\left(e^{i\beta\mathfrak{L}_\phi+i\beta\mathfrak{L}_m}\right) - \operatorname{Tr}_{L^2,\ker\bar\partial^\dagger}\left(e^{i\beta\mathfrak{L}_\phi+i\beta\mathfrak{L}_m}\right)\,.\end{aligned} \tag{4.56}$$

for the $\mathcal{N}=2,2^*$ theories respectively.

The problem is that the L^2 index is no longer equal to the partition function of the theory because the latter receives contributions from the non-normalizable states of the continuum. The reason is that although supersymmetry pairs bosonic and fermionic states it does not guarantee that the spectral density of bosonic and fermionic states is equal. Just this kind of situation was described by Sethi and Stern in [48]. In order to make progress we have to consider in more detail the "dangerous" non-compact asymptotic regions of the moduli space. These regions consist of configurations of well-separated clumps of instantons of lower charge

$$k \to (k_1)^{n_1} + (k_2)^{n_2} + \cdots + (k_p)^{n_p} \ , \qquad \sum_{j=1}^{p} n_j k_j = k \ , \tag{4.57}$$

where n_j gives the number of clumps with instanton number k_j. The moduli space in this regime is approximately a direct product

$$\hat{\mathfrak{M}}_k \ \overset{\text{cluster}}{\longrightarrow} \ \frac{\mathrm{Sym}^{n_1}(\hat{\mathfrak{M}}_{k_1} \times \mathbb{R}^4) \times \cdots \times \mathrm{Sym}^{n_p}(\hat{\mathfrak{M}}_{k_p} \times \mathbb{R}^4)}{\mathbb{R}^4} \ , \tag{4.58}$$

where the quotient is by the overall position. Note that the factors of the moduli space for clumps of the same instanton number involve the symmetric product

$$\mathrm{Sym}^n(\mathfrak{M}) = \frac{\mathfrak{M} \times \cdots \times \mathfrak{M}}{S_n} \ , \tag{4.59}$$

where S_n is the symmetric group on n objects which acts by permutation.

The way to define the L^2 index is to introduce an explicit cut-off on the moduli space by restricting the separations of the clumps of instantons to be some scale R. This defines a manifold with a boundary, $\hat{\mathfrak{M}}_k(R)$. This has the effect of lifting the continuum modes. The L^2 index can then be defined as the partition function of the quantum mechanical system on $\hat{\mathfrak{M}}_k(R)$ evaluated in the limit $\beta \to \infty$ followed by[13]

[13]The order of limits here is important.

$R \to \infty$

$$\mathrm{Ind}_k = \lim_{R\to\infty} \lim_{\beta\to\infty} \mathrm{Tr}\,(-1)^F e^{-\beta\mathcal{H}} \,. \tag{4.60}$$

Sethi and Stern then showed that the L^2 index can be written as a sum of a principal, or bulk, and a defect, or boundary, term

$$\mathrm{Ind}_k = P_k + D_k \,. \tag{4.61}$$

The principal or bulk term, as its name suggests, is the partition function of the quantum mechanical system on $\hat{\mathfrak{M}}_k(R)$ evaluated in the limit $\beta \to 0$ followed by $R \to \infty$

$$P_k = \lim_{R\to\infty} \lim_{\beta\to 0} \mathrm{Tr}\,(-1)^F e^{-\beta\mathcal{H}} \,. \tag{4.62}$$

The defect term D_k can be expressed as an integral over the boundary of $\hat{\mathfrak{M}}_k(R)$, however, we will not need the explicit form which may be found in [48]. Notice that the principal term is precisely the functional integral in periodic time

$$P_k = \mathcal{F}_k^\beta \,, \tag{4.63}$$

in the present context. So in order to relate $\mathcal{F}_k^\beta$ to the L^2 index Ind_k we have to find the defect contribution D_k.

In order to evaluate the defect term we generalize and expand upon a heuristic argument originally due to Yi [49], developed by Green and Gutperle [50] and used in [51] in the context of monopole bound states. The basic argument is tantamount to the usual assumption in quantum mechanics and quantum field theory that we can define asymptotic scattering states. Essentially, this means that we can split the Hamiltonian of the system as: $\mathcal{H} = \mathcal{H}^{(0)} + \mathcal{H}_{\mathrm{int}}$. Here $\mathcal{H}^{(0)}$ is an appropriate free Hamiltonian. The asymptotic particle states are eigenstates of $\mathcal{H}^{(0)}$ and $\mathcal{H}_{\mathrm{int}}$ accounts for the interactions between these states which are assumed to have finite range.

The idea is that the defect contribution to the index comes from the asymptotic region where the interactions between the asymptotic states can be neglected. It can therefore be calculated with respect to the free Hamiltonian, $\mathcal{H}^{(0)}$, giving

$$D_k = D_k^{(0)} \,. \tag{4.64}$$

As the free Hamiltonian certainly has no bound-states we have,

$$\mathrm{Ind}_k^{(0)} = P_k^{(0)} + D_k^{(0)} = 0 \qquad \Longrightarrow \qquad D_k = D_k^{(0)} = -P_k^{(0)} \,, \tag{4.65}$$

where

$$P_k^{(0)} = \lim_{R\to\infty} \lim_{\beta\to 0} \mathrm{Tr}\,(-1)^F e^{-\beta \mathcal{H}^{(0)}} \,. \tag{4.66}$$

In the above $\mathcal{H}^{(0)}$ describes the quantum mechanical system with a target space given by the right-hand side of (4.58) with a cut-off R imposed on the separations of the clumps. In other words there are no interactions between the clumps of instantons. We can now calculate $P_k^{(0)}$ as the partition function (4.66). The asymptotic space (4.58) describes $n = \sum n_i$ free particles moving in $\mathbb{R}^4$, where for each i, there are n_i particles carrying an internal space $\hat{\mathfrak{M}}_{k_i}$. The Hamiltonian will reflect the decomposition of the "position" and "internal" spaces

$$\mathcal{H}^{(0)} = \mathcal{H}_{\text{pos}} + \mathcal{H}_{\text{in}} \,. \tag{4.67}$$

Here, $\mathcal{H}_{pos}$ is the free Hamiltonian for n supersymmetric particles on $\mathbb{R}^4$ modulo the centre-of-mass motion (with a restriction that particles cannot be separated by more than a distance R).

At this point, we must take account of the symmetric products in (4.58). This can be done by working in the covering space and then by explicitly inserting projectors onto the states invariant under the various products of the symmetric groups. The projectors can be written as

$$\mathbb{P}=\frac{1}{n_1!\cdots n_p!}\sum_\pi M_\pi^{\rm pos}\times M_\pi^{\rm in}\ , \tag{4.68}$$

where the sum is over the elements of the product of symmetric groups $S_{n_1}\times\cdots\times S_{n_p}$ and $M_\pi^{\rm in,pos}$ is the operator which represents π on the position and internal Hilbert spaces. The partition function we want can then be written as

$$\mathrm{Tr}\big((-1)^Fe^{-\beta\mathcal{H}^{(0)}}\mathbb{P}\big)=\frac{1}{n_1!\cdots n_p!}\sum_\pi\mathrm{Tr}\big((-1)^Fe^{-\beta\mathcal{H}_{\rm pos}}M_\pi\big)\,\mathrm{Tr}\big((-1)^Fe^{-\beta\mathcal{H}_{\rm in}}M_\pi\big)\ , \tag{4.69}$$

evaluated in the limit $\beta\to0$ (with fixed $\beta\vec{\phi}$ and βm) and then $R\to\infty$.

In (4.69), let us consider the piece coming from the motion in $\mathbb{R}^4$ with a particular element $\pi\in S_{n_1}\times\cdots\times S_{n_p}$. To evaluate this partition function

$$\lim_{R\to\infty}\lim_{\beta\to0}\mathrm{Tr}\big((-1)^Fe^{-\beta\mathcal{H}_{\rm pos}}M_\pi\big)\ , \tag{4.70}$$

essentially one can follow the steps in [50, 51]. Remember R is a cut-off giving the maximum separation of any two particles. Concentrating briefly on the $N=2$ theory, the system is a set of n free particles with positions X^i_μ, $\mu=1,\ldots,4$, and fermionic partners ψ^i_μ, with the centre-of-mass motion frozen out. In (4.70), M_π gives the action of an element of the symmetric group S_n

$$\pi:\qquad X^i_\mu\to(M_\pi)^i_jX^j_\mu\ ,\qquad \psi^i_\mu\to(M_\pi)^i_j\psi^j_\mu\ . \tag{4.71}$$

In the limit $\beta\to0$, we can use the standard heat kernel representation of the propagator to write (4.70) as

$$\mathrm{Tr}_\psi\big((-1)^FM_\pi\big)\int_{|X^i-X^j|\le R}d^{4(n-1)}X\,\frac{e^{-(X-M_\pi\cdot X)^2/2\beta}}{(2\pi\beta)^{2(n-1)}}\ . \tag{4.72}$$

We now evaluate (4.72). It is not difficult to show that the fermionic trace yields $(\det'(1 - M_\pi))^2$, where the prime means excluding the zero eigenvalue which arises from the centre-of-mass degrees-of-freedom $\sum_i \psi^i_m$. However, $1 - M_\pi$ has additional zero eigenvectors and vanishes identically unless π is a cyclic permutation of the n particles. The bosonic integrals give a factor $(\det'(1-M_\pi))^{-4}$ and it might be thought that a non-cyclic permutation would lead to a divergence. However, as argued in [50], we should take account of the cut-off R. A zero eigenvector in the bosonic determinant should actually be interpreted as $\sim R^{-1}$. However, the fermion determinant vanishes identically and so only cyclic permutations can lead to a non-zero result and for these we can then safely take $R \to \infty$.

When we consider the theory with matter we might expect to obtain another fermionic factor in (4.72). However, it turns out this is not so. Remember that we must take $\beta \to 0$ as our first limit but keep βm fixed which implies we are sending $m \to \infty$. Thus the mass scale has already been 'integrated out' relative to R and so does not affect the calculation in the $R \to \infty$ limit.

It is not difficult to show that for a cyclic permutation $\det'(1 - M_\pi) = n$. Finally, with and without matter, we have the result

$$\mathrm{Tr}\big((-1)^F e^{-\beta\mathcal{H}_{\mathrm{pos}}} M_\pi\big) = \begin{cases} n^{-2} & \pi \text{ is a cyclic permutation} \\ 0 & \text{otherwise}\ . \end{cases} \tag{4.73}$$

From this result, we see immediately that, since the result is only non-vanishing for a cyclic permutation of the n particles, non-zero contributions to the partition function can only come from asymptotic regions (4.58) where all the instanton clumps have the same instanton number

$$\hat{\mathfrak{M}}_k \xrightarrow{\text{cluster}} \frac{\mathrm{Sym}^n(\hat{\mathfrak{M}}_{k/n} \times \mathbb{R}^4)}{\mathbb{R}^4}\ . \tag{4.74}$$

Clearly n must be an integer divisor of k.

With this result in mind we now turn to the contribution from the internal part in (4.69). We need to evaluate

$$\mathrm{Tr}\big((-1)^F e^{-\beta\mathcal{H}_{\rm in}} M_\pi\big) , \qquad (4.75)$$

where π is a cyclic permutation of the n particles. The n internal spaces are completely decoupled, $\mathcal{H}_{\rm in} = \sum_i \mathcal{H}_i$, so the trace is easily expressed in terms of a sum over the eigenstates of each separate Hamiltonian $\mathcal{H}_i$. Remember that each particle has the same instanton number k/n and so each Hamiltonian $\mathcal{H}_i$ is identical. Let us denote the orthonormal eigenvectors as $|\psi^i_\alpha\rangle$, where i is the particle label. Let f_α be the fermion number and $\vec{\lambda}_\alpha$ be a vector of the quantum numbers of the eigenstates[14], so, with constants $\vec{c}$, we have

$$\mathcal{H}_i\left|\psi^i_\alpha\right\rangle = \vec{c}\cdot\vec{\lambda}_\alpha\left|\psi^i_\alpha\right\rangle , \qquad F\left|\psi^i_\alpha\right\rangle = f_\alpha\left|\psi^i_\alpha\right\rangle . \qquad (4.76)$$

The symmetric group M_π acts as a simple permutation of the particle label $i \to \pi(i)$. Hence, (4.75) is

$$\sum_{\{\alpha_i\}}(-1)^{\sum_i f_{\alpha_i}}\, e^{-\beta\sum_i \vec{c}\cdot\vec{\lambda}_{\alpha_i}}\, \langle\psi^1_{\alpha_1}\cdots\psi^n_{\alpha_n}|\psi^{\pi(1)}_{\alpha_1}\cdots\psi^{\pi(n)}_{\alpha_n}\rangle . \qquad (4.77)$$

Now we use the fact that π is a cyclic permutation. The inner product in the above is zero unless all the particles are in the same state $\alpha_1=\alpha_2=\cdots=\alpha_n$. Hence, for a cyclic permutation, (4.77) is

$$\sum_\alpha(-1)^{nf_\alpha}\,(-1)^{\ell(\pi)f_\alpha}\,e^{-n\beta\vec{c}\cdot\vec{\lambda}_\alpha} , \qquad (4.78)$$

[14]In the $\mathcal{N}=2$ theory $\vec{\lambda}_\alpha=\vec{q}_\alpha$, the 'electric' charge, while the states in the $\mathcal{N}=2^*$ theory carry $\vec{\lambda}_\alpha=(\vec{q}_\alpha,j_\alpha)$, including the holomorphic degree.

where $\ell(\pi)$ is the length of the element $\pi \in S_n$. For a cyclic permutation $\ell(\pi)$ is even/odd according to whether n is odd/even. Therefore (4.78) can be written in terms of the index, refined in terms of the quantum numbers, on $\hat{\mathfrak{M}}_{k/n}$

$$\sum_{\alpha}(-1)^{f_\alpha}\, e^{-n\beta\vec{c}\cdot\vec{\lambda}_\alpha} = \sum_{\vec{\lambda}} \mathrm{Ind}_{k/n,\vec{\lambda}}\, e^{-n\beta\vec{c}\cdot\vec{\lambda}} = \mathrm{Ind}_{k/n} \; . \tag{4.79}$$

Putting together the internal part (4.79) with the position part (4.73) in (4.69) and using the fact that there are $(n-1)!$ cyclic permutations gives the contribution to $P_k^{(0)}$ from the clustering region (4.74)

$$(n-1)! \times \frac{1}{n!} \times \frac{1}{n^2} \times \sum_{\vec{\lambda}} \mathrm{Ind}_{k/n,\vec{\lambda}}\, e^{-n\beta\vec{c}\cdot\vec{\lambda}} \; . \tag{4.80}$$

Summing over n, the integer divisors of k excluding 1, we have

$$P_k^{(0)} = \sum_{n|k\neq 1} \sum_{\vec{\lambda}} \frac{\mathrm{Ind}_{k/n,\vec{\lambda}}}{n^3}\, e^{-n\beta\vec{c}\cdot\vec{\lambda}} \; . \tag{4.81}$$

Then using the fact that $D_k^{(0)} = -P_k^{(0)}$ and (4.64), the defect contribution with interactions is

$$D_k = -\sum_{n|k\neq 1} \sum_{\vec{\lambda}} \frac{\mathrm{Ind}_{k/n,\vec{\lambda}}}{n^3}\, e^{-n\beta\vec{c}\cdot\vec{\lambda}} \; . \tag{4.82}$$

Finally, from (4.61) and (4.63), we have our goal, the relation between the instanton coefficients of the prepotential and index theory. We note from (4.25) that the BPS states in the $\mathcal{N} = 2$ theory have quantum numbers of electric charge, thus

$$\mathcal{F}_k^{\beta} = \sum_{n|k} \sum_{\vec{q}} \frac{\mathrm{Ind}_{k/n,\vec{q}}}{n^3}\, e^{-n\beta\vec{\phi}\cdot\vec{q}} \; . \tag{4.83}$$

Notice that in contrast to (4.81), the sum now includes $n = 1$.

By summing over instanton number k with the factor $\exp(-8\pi^2 k\beta/g_5^2)$, we can write an expression for the prepotential in terms of the L^2 index as

$$\mathcal{F}^{\beta} = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{\mathrm{Ind}_{k,\vec{q}}}{n^3} \left(e^{-\beta\vec{\phi}\cdot\vec{q} - 8\pi^2 k/g_5^2}\right)^n \; . \tag{4.84}$$

Notice that the perturbative piece can be included as above, through the $k = 0$ terms with the definition

$$\mathrm{Ind}_{0,\vec{q}} = \begin{cases} 1 & \vec{q} \text{ is a positive root} \\ 0 & \text{otherwise .} \end{cases} \tag{4.85}$$

The BPS states of the $\mathcal{N} = 2^\star$ theory also carry quantum numbers, j, of the number operator on ψ_2. Thus, extracting a factor of $(-1)^j$ from the index for convenience, we have

$$\mathcal{F}_k^{(\beta,m)} = \sum_{n|k} \sum_{\vec{q},j} \frac{\mathrm{Ind}_{k/n,\vec{q},j}}{n^3} (-1)^j e^{-n\beta\vec{\phi}\cdot\vec{q} - n\beta m j} \ . \tag{4.86}$$

Note that if we take the $m \to 0$ limit of this there is no obvious n^{-1} dependence as required by (4.26). We will have to see in the explicit index results how this can be consistent.

4.3 Discussion

We have seen how by lifting $\mathcal{N} = 2$ and $\mathcal{N} = 2^\star$ SYM theories onto a fifth compact dimension the prepotential governing the low-energy effective action can be viewed as the partition function of a particular supersymmetric quantum mechanical σ-model. Thus, as shown in Chapter 3, such a function should be calculating an index on the target space of the σ-model, here the instanton moduli space, $\hat{\mathfrak{M}}_k$. It is possible to identify the BPS states of the quantum mechanical system with instanton dyons in the physical theory. The complicating factor is that the instanton moduli space is non-compact, however, once this is taken into account, (4.83) and (4.86) give new formulae for realizing the prepotentials in terms of the index. The next chapter is devoted to doing just this for the case of gauge group SU(2).

Chapter 5

Explicit Index Results

In this chapter we shall derive explicit integers for the equivariant index and so confirm the index structure determined in the previous chapter. We will in fact use three different methods to calculate the prepotential in the case of SU(2), both with and without adjoint matter. The first uses the functional integral calculation resulting from the heat kernel method described in Chapter 3. This requires knowledge of the metric on the moduli space of instantons which, due to the ADHM constraints, is unknown above $k = 1$. Thus to gather results at higher instanton levels we will use the method of Csaki et al [9] where the five dimensional coupling, τ_5, is lifted from the four dimensional coupling, τ_4, by using Nekrasov's generalization of Seiberg-Witten theory to 5 dimensions [52]. We then integrate twice to find the prepotential. This method is very efficient even up to seven or eight instantons as we will see. However, it does not pick up any VEV-independent contributions which, although they are physically irrelevant, will enable us to confirm our results are consistent. Furthermore this method is very difficult to generalize to other gauge groups. Hence we move on to the third and most powerful method based on the equivariant cohomology of the instanton moduli space and localization. This uses a powerful formula derived by

Nekrasov [8] in terms of sums over partitions. We will use this formula to explicitly determine the prepotential for general gauge groups up to three instantons, as well as confirming the results for gauge group SU(2) up to four instantons.

5.1 Explicit One Instanton Integral

First, then, let us discuss the one instanton case in full detail leading to an evaluation of the formulas (3.61) and (3.66). The moduli space $\hat{\mathfrak{M}}_1$ for gauge group SU(2) is the Eguchi-Hanson space. The metric may be written

$$ds^2 = \frac{1}{\sqrt{1+4\zeta^2/r^4}}\big(dr^2 + \tfrac{1}{4}r^2(d\psi + \cos\theta\, d\varphi)^2\big) + \tfrac{1}{4}r^2\sqrt{1+4\zeta^2/r^4}\big(d\theta^2 + \sin^2\theta\, d\varphi^2\big)\ . \tag{5.1}$$

Here, ζ is the parameter that regulates the singularity of the instanton moduli space, as $\zeta \to 0$ the space becomes the orbifold $\mathbb{R}^4/\mathbb{Z}_2$. The range of the coordinates is

$$0 \le r \le \infty\ , \qquad 0 \le \theta \le \pi\ , \qquad 0 \le \varphi \le 2\pi\ , \qquad 0 \le \psi \le 2\pi\ . \tag{5.2}$$

The size of the instanton ρ is related to the radial coordinate via

$$\rho^4 = \tfrac{1}{4}r^4 + \zeta^2\ . \tag{5.3}$$

A global gauge transformation in the unbroken $U(1) \subset SU(2)$ acts on the moduli space as a shift in the angle φ and so the Killing vector ϕ is

$$\phi = v\frac{\partial}{\partial\varphi}\ , \tag{5.4}$$

where v is the magnitude of the VEV. It is clear that fixed-point set of ϕ consists of two discrete points at $r = 0$ and $\theta = 0, \pi$. In the vicinity of the fixed-point at $\theta = 0$, the metric is

$$ds^2 = \frac{1}{2\zeta}\big(du^2 + u^2 d\sigma^2\big) + \frac{\zeta}{2}\big(d\theta^2 + \theta^2 d\varphi^2\big) + \cdots\ , \tag{5.5}$$

where $u = r^2/2$ and $\sigma = \psi + \varphi$. Hence, there are 2 copies of $\mathbb{R}^2$ with polar coordinates (u, σ) and (θ, φ). The Killing vector generates a simple rotation in σ and φ. In this case the Riemannian moment $\Omega_{\mu\nu}$ is an imaginary 4×4-dimensional skew-symmetric matrix with eigenvalues $(v, -v, v, -v)$. We can see this by converting to cartesian coordinates

$$x_1 = u \cos\sigma \ , \qquad x_2 = \theta \cos\varphi \ ,$$
$$y_1 = u \sin\sigma \ , \qquad y_2 = \theta \sin\varphi \ , \tag{5.6}$$

under which the Killing vector becomes

$$\phi = v\frac{\partial}{\partial\varphi} = v(-y_1\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial y_1} - y_2\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial y_2}) \ . \tag{5.7}$$

Since in the neighbourhood of the fixed point the space is Euclidean the connections, and so the curvature $R_{\mu\nu}$, are zero, thus

$$\Omega_{\mu\nu} = i\nabla_\mu\phi_\nu|_{\hat{\mathfrak{M}}_1^\phi} = i\partial_\mu\phi_\nu = iv \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \ . \tag{5.8}$$

The second fixed-point yields an identical contribution to the index. Applying the formula (3.61) gives the first instanton contribution as

$$\mathcal{F}_1^\beta = 2 \times \frac{1}{(2\sinh(\beta v/2))^2} = \frac{1}{2\sinh^2(\beta v/2)} \ . \tag{5.9}$$

The theory with an adjoint hypermultiplet inhabits the same space $\hat{\mathfrak{M}}_1$, we simply have to determine the Kähler form, $\omega^{\mathcal{N}}_{\mu\nu}$, and use (3.66), noting that since the fixed-point set consists simply of points the contributions come entirely from the normal bundle. The three Kähler forms are given in equation (32) in [53]. The form we need is

$$\omega = \frac{1}{2}\frac{1}{\sqrt{1+4\zeta^2/r^4}} r dr \wedge (d\psi + \cos\theta d\varphi) + \frac{r^2}{4}\sqrt{1+4\zeta^2/r^4}(-\sin\theta d\theta \wedge d\varphi) \ . \quad (5.10)$$

Again we reduce to the fixed point set of ϕ and convert to cartesian coordinates to obtain

$$\omega^{\mathcal{N}} = \frac{1}{2\zeta} dx_1 \wedge dy_1 - \frac{\zeta}{2} dx_2 \wedge dy_2 \ . \quad (5.11)$$

Thus the two-form $\omega^{\mathcal{N}}_{\mu\nu}$ turns out to be an anti-symmetric matrix with eigenvalues $(1, -1, -1, 1)$. Thus from (3.66), and remembering the extra factor in (4.33) coming from integrating the mass terms over $\mathbb{R}^4$, we find

$$\begin{aligned} \mathcal{F}_1^{(\beta,m)} &= -4\sinh^2(\beta m/2) \times 2 \times \frac{\sinh(\beta(m+v)/2)\sinh(\beta(m-v)/2)}{(\sinh(\beta v/2))^2} \ , \\ &= -8\sinh^2(\beta m/2)\left(1 - \frac{\sinh(\beta m/2)^2}{\sinh(\beta v/2)^2}\right) \ . \end{aligned} \quad (5.12)$$

5.2 Seiberg-Witten Solution

Since the metric on the moduli space is unknown above $k = 1$ we must find an alternative method of calculating the prepotential. A particularly useful prescription is given in [9] since it allows computation up to high instanton charges ($k = 7, 8$ limited only by computation time). We begin with the Seiberg-Witten curve for 4-dimensional SU(2) pure gauge theory[1]

$$u = \tfrac{1}{2}p^2 + \Lambda^2 \cosh(X) \ . \quad (5.13)$$

Here (p, X) are the 'momentum' and 'position' parameters respectively, while Λ is the dynamically generated scale. Of course u is the first gauge invariant parameter

[1]Here we begin with the reparameterization of Nekrasov [54], rather than the original Seiberg-Witten curves, where the relation to the periodic Toda chain is explicit.

k	1	2	3	4	5	6	7	8
$G(k)$	$\frac{1}{2^2}$	$\frac{5}{2^6}$	$\frac{9}{2^7}$	$\frac{1469}{2^{14}}$	$\frac{4471}{2^{15}}$	$\frac{121191}{2^{19}}$	$\frac{441325}{2^{20}}$	$\frac{866589165}{2^{30}}$

Table 5.1: Values of G_k.

and clearly has the form of a Hamiltonian. However, just as described in Section 2.2, the adjoint scalar field ϕ has gained a VEV, a. The gauge invariant modulus in the weak coupling regime can then be given by

$$u = \left\langle \mathrm{Tr}\phi^2 \right\rangle = \tfrac{1}{2}a^2 + \sum_{k=1}^{\infty} G_k \frac{\Lambda^{4k}}{a^{4k-2}} , \tag{5.14}$$

where the coefficients G_k (obtained from [55]) are given in Table 5.1. The coupling constant τ also has a weak coupling expansion

$$\tau(a) = \frac{4\pi i}{g^2(a)} + \frac{\theta(a)}{2\pi} = \frac{i}{\pi} \log \frac{a^2}{\Lambda^2} + \sum_{k=1}^{\infty} \tau_k \frac{\Lambda^{4k}}{a^{4k}} , \tag{5.15}$$

where the coefficients τ_k are related to the G_k via the Matone relation [56, 57]

$$\tau_k = -\frac{i}{\pi} \frac{(4k-2)(4k-1)}{2k} G_k . \tag{5.16}$$

The meromorphic differential (see (2.12)) is just $\lambda = p\, dX$, with the A and B cycles chosen to ensure the correct asymptotic behaviour of a as $u \to \infty$. Thus

$$a(u) = \frac{1}{2\pi i} \int_{-i\pi}^{i\pi} dX\, p \to \sqrt{2u} . \tag{5.17}$$

Introducing two new parameters

$$\omega \equiv \sqrt{2u} , \qquad \nu_4 \equiv \frac{\Lambda^2}{u} , \tag{5.18}$$

we can rewrite (5.17) as

$$\frac{\partial a(u)}{\partial \omega} = \frac{1}{2\pi i} \int_{-i\pi}^{i\pi} \frac{dX}{\sqrt{1 - \nu_4 \cosh(X)}} . \tag{5.19}$$

The crux of the method of [9] is to determine a similar equation in the 5-dimensional theory. Since this theory is on the compactified manifold $\mathbb{R}^4 \times S^1$ the scalar field ϕ combines with the component of the gauge field in the compact dimension; the resulting complex scalar field, χ, then develops the VEV, now denoted A. The gauge invariant modulus is now

$$U = \langle \mathrm{Tr} \frac{\cosh(2\beta\chi)}{\beta^2} \rangle = \frac{\cosh(\beta A)}{\beta^2} + \text{instantons.} \tag{5.20}$$

The spectral curve describing the low-energy dynamics is related to the relativistic Toda chain, only, as in [9], we have an additional *a priori* unknown function $f(\beta\Lambda)$

$$U = \frac{1}{\beta^2} \cosh(\beta p) \sqrt{1 + 2\beta^2 \Lambda^2 f(\beta\Lambda) \cosh(X)} \ , \tag{5.21}$$

where

$$f(\beta\Lambda) = 1 + \sum_{k=1}^{\infty} f_k \beta^{4k} \Lambda^{4k} \ , \tag{5.22}$$

and the coefficients f_k are to be determined in the following calculation. The VEV $A(U)$ can again be shown to have the correct asymptotic behaviour

$$A(U) = \frac{1}{2\pi i} \int_{-i\pi}^{i\pi} dX \, p \overset{u\to\infty}{\longrightarrow} \beta^{-1} \cosh^{-1}(\beta^2 U) \ , \tag{5.23}$$

and so we again define two new parameters

$$\cosh(\alpha) \equiv \beta^2 U \ , \qquad \nu_5 \equiv \frac{2 f(\beta\Lambda) \beta^2 \Lambda^2}{\sinh^2(\alpha)} \ . \tag{5.24}$$

Thus in analogy to (5.19) we find

$$\frac{\partial A(U)}{\partial \alpha} = \frac{\beta^{-1}}{2\pi i} \int_{-i\pi}^{i\pi} \frac{dX}{\sqrt{1 - \nu_5 \cosh(X)}} \ . \tag{5.25}$$

The same equations determine the dual VEVs A_D and a_D except with the integration running from 0 to $\cosh^{-1}(u/\Lambda^2)$. Now it is clear that equations (5.19) and (5.25) are identified when the parameters satisfy $\nu_4 = \nu_5$

$$\frac{\partial A}{\partial \alpha}\beta = \left.\frac{\partial a}{\partial \omega}\right|_{\nu_4=\nu_5} , \quad \frac{\partial A_D}{\partial \alpha}\beta = \left.\frac{\partial a_D}{\partial \omega}\right|_{\nu_4=\nu_5} . \tag{5.26}$$

Also, by definition, $\nu_4 = \nu_5$ implies

$$u = \tilde{U} \equiv \frac{\beta^4 U^2 - 1}{2\beta^2 f(\beta\Lambda)} . \tag{5.27}$$

Now we note from the dependence of the coupling on the scalars in (2.8) that

$$\tau_{5d}(U) = \frac{\partial}{\partial A}\left(\frac{\partial \mathcal{F}^\beta}{\partial A}\right) = \frac{\partial A_D}{\partial A} , \tag{5.28}$$

and

$$\tau_{4d}(u) = \frac{\partial}{\partial a}\left(\frac{\partial \mathcal{F}}{\partial a}\right) = \frac{\partial a_D}{\partial a} , \tag{5.29}$$

but by (5.26) the two differentials on the right-hand side can be identified. Thus to calculate the five dimensional coupling we simply need to determine τ_{4d} at the specific value of $u = \tilde{U}$.

At this point we divert from the method of [9] and directly determine $a(A)$. A simple rearrangement of (5.26) gives

$$\frac{\partial A}{\partial a} = \beta^{-1}\left.\frac{\partial \alpha}{\partial \omega}\right|_{\nu_4=\nu_5} , \tag{5.30}$$

while the rhs can be determined from (5.27). Finally we arrive at

$$A = \int \frac{\sqrt{f(\beta\Lambda)}}{\sqrt{1 + 2\beta^2 f(\beta\Lambda) u(a)}}\, da . \tag{5.31}$$

The rest of the calculation must be done on *Mathematica* along the following lines. First insert (5.14) and (5.22) into (5.31) and determine the integral order by order in Λ. At this point the coefficients of the function $f(\beta\Lambda)$ are determined by requiring integrability of the coupling, τ, to give the prepotential. They are satisfyingly simple

$$f_k = (-1)^k . \tag{5.32}$$

Inverting the result of the integration gives an explicit function $a(A)$, whose leading order term is simply

$$a = \sinh(\beta A) + \mathcal{O}(\Lambda^4) \ . \tag{5.33}$$

Inserting the full expansion into the 4-d coupling (5.15) and re-expanding in instanton charge gives the 5-d coupling τ_5. All that is left is to integrate twice to determine the prepotential $\mathcal{F}^\beta(A)$.

We will now determine a similar method for the $\mathcal{N} = 2^\star$ theory. The Seiberg-Witten curve was generalized in [54] and is in fact the Hamiltonian of the Calogero-Moser system, where the position coordinate has become elliptic

$$u_m = \tfrac{1}{2}p^2 + \tfrac{m^2}{2}\wp(X) \ . \tag{5.34}$$

We now determine the weak-coupling expansion of the invariant modulus. This can be calculated from the prepotential of the 4-d theory, as derived in [47], by using the Matone relation. We must allow for VEV-independent contributions, as in [58], here depicted by the constant c. The resulting expansion has the form

$$u_m = \tfrac{1}{2}a^2 + \tfrac{1}{2}(a+m)^2 + \tfrac{1}{2}(a-m)^2 + cm^2 + \sum_{k=1}^{\infty} G_{m,k}(a,m)q^k \ . \tag{5.35}$$

This shows that the asymptotic behaviour of the VEV, a, is again

$$a(u) = \frac{1}{2\pi i}\int_{-i\pi}^{i\pi} dX\, p \to \sqrt{2u_m} \ , \tag{5.36}$$

and the new parameters are

$$\omega_m \equiv \sqrt{2u_m} \ , \qquad \mu_4 \equiv \frac{m^2}{2u_m} \ , \tag{5.37}$$

allowing us to rewrite (5.36) as

$$\frac{\partial a(u_m)}{\partial \omega_m} = \frac{1}{2\pi i}\int_{-i\pi}^{i\pi} \frac{dX}{\sqrt{1-\mu_4\wp(X)}} \ . \tag{5.38}$$

The spectral curve of the 5-d $\mathcal{N} = 2^*$ theory was given in [54] and describes the relativistic Calogero-Moser system. However, just as in the pure gauge theory, we should allow for an unknown function to account for ambiguities in the curve of the massive theory, these get determined later in the calculation, just as above. Thus

$$U_m = \cosh(\beta p)\sqrt{1 + \frac{\wp(X)}{\wp(\beta m) + f(q)}} , \tag{5.39}$$

where from the first few terms of the calculation it is suggested that the function f is the first Eisenstein series

$$f(q) = -\tfrac{1}{6} + E_2(q) . \tag{5.40}$$

Then

$$A(U_m) = \frac{1}{2\pi i}\int_{-i\pi}^{i\pi} dX\, p \to \cosh^{-1}(U_m) , \tag{5.41}$$

and the new parameters are

$$\cosh(\alpha_m) \equiv U_m , \qquad \mu_5 \equiv \frac{1}{\sinh^2(\alpha_m)\,(\wp(\beta m) + f(q))} , \tag{5.42}$$

allowing us to rewrite (5.41) as

$$\frac{\partial A(U_m)}{\partial \alpha_m} = \frac{\beta^{-1}}{2\pi i}\int_{-i\pi}^{i\pi} \frac{dX}{\sqrt{1 - \mu_5\wp(X)}} . \tag{5.43}$$

The identification of the couplings follows just as in the massless case. We now find the integral analogous to (5.31) to be

$$\beta A = \int \frac{1}{\sqrt{2u(a) + m^2(\wp(\beta m) + f(q))}}\, da . \tag{5.44}$$

The calculation of the prepotential follows as above.

5.3 Equivariant Cohomology

We will now use the method of calculating the prepotential given by Nekrasov [8]. Later we will examine the significance of Nekrasov's result for our calculations. However here we will just use the formula to confirm our results for SU(2). In fact we will use the similar formula of Bruzzo et al [59], since it will enable us to switch between pure and massive theories, and between four and five dimensions.

The contributions to the prepotential are derived in terms of a 'partition function' $\mathcal{Z}$, which is itself just a sum over coloured partitions

$$\mathcal{Z}(\mathbf{a};\epsilon_1,\epsilon_2,\Lambda) = \mathcal{Z}^{pert}(\mathbf{a};\epsilon_1,\epsilon_2,\Lambda)\sum_{\vec{k}} q^{|\vec{k}|}\mathcal{Z}_{\vec{k}}(\mathbf{a};\epsilon_1,\epsilon_2)\ , \tag{5.45}$$

where

$$\mathcal{Z}_k = \sum_{\{Y_\lambda\}}\prod_{\lambda,\tilde{\lambda}=1}^{N}\prod_{s\in Y_\lambda} f(E(s))\ . \tag{5.46}$$

At each instanton level, k, we have k boxes to distribute between N Young's tableaux. Each configuration of boxes counts once. We then pick one of the Young's tableaux and label it Y_λ and cycle over all other tableaux with the index $Y_{\tilde{\lambda}}$. Now we can define ν_{i_λ} to be the length of the i_λ'th row in Y_λ, while $\tilde{\nu}'_{j_\lambda}$ is the height of the j_λ'th column in $Y_{\tilde{\lambda}}$, where (i_λ, j_λ) is the position in Y_λ of the box s which again we cycle over. This is all numerated by the functions

$$\begin{aligned} E(s) &= a_{\lambda\tilde{\lambda}} - \epsilon_1 h(s) + \epsilon_2(v(s)+1)\ , \\ h(s) &= \nu_{i_\lambda} - j_\lambda\ ,\ \ v(s) = \tilde{\nu}'_{j_\lambda} - i_\lambda\ . \end{aligned} \tag{5.47}$$

The combinatorics of how all this works is fully spelt out in Appendix B. The function $f(x)$ allows us to calculate each $\mathcal{Z}_k$ without specifying the nature of the theory we are examining. In general f depends on $\epsilon = \epsilon_1+\epsilon_2$, however to obtain the prepotential we will set $\epsilon_1 = -\epsilon_2 = \hbar$, then $f(x)$ is as given in Table 5.2. Note in particular that f is

$f(x)$	$4-\text{dim}$	$5-\text{dim}$
$\mathcal{N}=2$	$\frac{1}{x^2}$	$\frac{1}{\sinh^2(\beta x)}$
$\mathcal{N}=2^\star$	$\frac{(x-m)(x+m)}{x^2}$	$\frac{\sinh(\beta(x+m))\sinh(\beta(x-m))}{\sinh^2(\beta x)}$

Table 5.2: The function $f(x)$ in theories of various dimensions and matter.

now an even function in all cases. Also, in the limit of $m \to 0$ we see that $f(x) = 1, \forall x$ in both 4 and 5 dimensions. Thus using the identity proved by Nekrasov [8]

$$\mathcal{F}(\mathrm{a}, \Lambda) = \lim_{\hbar \to 0} \left(\hbar^2 \log(\mathcal{Z})\right) \ . \tag{5.48}$$

we can show, as in [59]

$$\mathcal{F}^{\mathcal{N}=4} = N \sum_{k=1}^{\infty} q^k \sum_{n|k} \frac{1}{n} \ . \tag{5.49}$$

Hence, any results derived from this calculation will give the correct $m = 0$ limit. With the notation

$$T_i(x) = \prod_{j \neq i} f(a_{ij} + x) \ , \quad T_i = T_i(0) \ , \tag{5.50}$$

we find the one instanton term is

$$\mathcal{Z}_1 = \sum_i f(\hbar) T_i \ . \tag{5.51}$$

For two instantons there is a little more work, from which we get

$$\begin{aligned} \mathcal{Z}_2 \ &= \ \sum_i f(2\hbar) f(\hbar) T_i \left[T_i(\hbar) + T_i(-\hbar)\right] \\ &+ \frac{1}{2} \sum_{i \neq j} f^2(\hbar) f(a_{ij} - \hbar) f(a_{ji} - \hbar) \frac{T_i}{f(a_{ij})} \frac{T_j}{f(a_{ji})} \ . \end{aligned} \tag{5.52}$$

On to the $k = 3$ contribution

$$\begin{aligned}
\mathcal{Z}_3 &= \sum_i f(3\hbar)f(2\hbar)f(\hbar)T_i\left[T_i(\hbar)T_i(2\hbar) + T_i(-\hbar)T_i(-2\hbar)\right] \\
&\quad + \sum_i f(3\hbar)f^2(\hbar)T_i(\hbar)T_iT_i(-\hbar) \\
&\quad + \sum_{i\neq j} f(2\hbar)f^2(\hbar)f(a_{ij})\frac{T_i}{f(a_{ij})}\frac{T_j}{f(a_{ji})} \\
&\qquad \cdot \left[f(a_{ij}-2\hbar)f(a_{ji}-\hbar)\frac{T_i(-\hbar)}{f(a_{ij}-\hbar)} + f(a_{ij}-\hbar)f(a_{ji}-2\hbar)\frac{T_i(\hbar)}{f(a_{ij}+\hbar)}\right] \\
&\quad + \frac{1}{3!}\sum_{i\neq j\neq k} f^3(\hbar)f(a_{ij}-\hbar)f(a_{ik}-\hbar)f(a_{ji}-\hbar)f(a_{jk}-\hbar)f(a_{ki}-\hbar)f(a_{kj}-\hbar) \\
&\qquad \cdot \frac{T_i}{f(a_{ij})f(a_{ik})}\frac{T_j}{f(a_{ji})f(a_{jk})}\frac{T_k}{f(a_{ki})f(a_{kj})} \ . \qquad (5.53)
\end{aligned}$$

We can then use the identity (5.48) along with some identities given in Appendix B to calculate the prepotential in either 4 or 5 dimensions and with or without adjoint matter. The 4 dimensional results are given in [8], here we will give the results for 5 dimensions. With the notation

$$T_i = T_i(0)\ , \quad T_i^{(n)} = \frac{\partial^n}{\partial x^n}T_i(x)\bigg|_{x\to 0}\ , \qquad (5.54)$$

and considering the pure $\mathcal{N} = 2$ theory first we have

$$T_i(x) = \prod_{j\neq i}\frac{1}{\sinh^2\left(\beta(a_{ij}+x)\right)}\ , \qquad (5.55)$$

the first three terms in the prepotential are[2]

$$
\begin{aligned}
\mathcal{F}_1^\beta &= -\sum_i T_i \,, \\
\mathcal{F}_2^\beta &= \frac{1}{2}\sum_i T_i^2 - \frac{1}{4\beta^2}\sum_i T_i T_i^{(2)} - \sum_{i\neq j}\frac{T_i T_j}{\sinh^2(\beta a_{ij})} \,, \\
\mathcal{F}_3^\beta &= -\frac{16}{27}\sum_i T_i^3 + \frac{1}{9\beta^2}\sum_i T_i T_i^{(1)} T_i^{(1)} + \frac{5}{9\beta^2}\sum_i T_i^2 T_i^{(2)} - \frac{1}{12\beta^4}\sum_i T_i T_i^{(2)} T_i^{(2)} \\
&\quad -\frac{1}{18\beta^4}\sum_i T_i T_i^{(1)} T_i^{(3)} - \frac{1}{36\beta^4}\sum_i T_i^2 T_i^{(4)} - 5\sum_{i\neq j}\frac{T_i^2 T_j}{\sinh^4(\beta a_{ij})} \\
&\quad -\frac{1}{\beta^2}\sum_{i\neq j}\frac{T_i T_i^{(2)} T_j}{\sinh^2(\beta a_{ij})} - \frac{1}{\beta}\sum_{i\neq j} T_i T_i^{(1)} T_j \sinh(2\beta a_{ij}) \\
&\quad -2\sum_{i\neq j\neq k}\frac{T_i T_j T_k}{\sinh^2(\beta a_{ij})\sinh^2(\beta a_{ik})} \,. \qquad (5.56)
\end{aligned}
$$

Meanwhile the $\mathcal{N} = 2^\star$ case, where

$$
T_i(x) = \prod_{j\neq i}\frac{\sinh\left(\beta(a_{ij}+x-m)\right)\sinh\left(\beta(a_{ij}+x+m)\right)}{\sinh\left(\beta(a_{ij}+x)\right)^2} \,, \qquad (5.57)
$$

and

$$
\begin{aligned}
\Pi_{ij} &= \coth^2(\beta a_{ij}) - \frac{1}{2}\coth^2(\beta(a_{ij}+m)) - \frac{1}{2}\coth^2(\beta(a_{ij}-m)) \,, \\
\Xi_{ij} &= \coth^4(\beta a_{ij}) - \frac{1}{2}\coth^4(\beta(a_{ij}+m)) - \frac{1}{2}\coth^4(\beta(a_{ij}-m)) \,, \\
\Sigma_{ij} &= \sinh(2\beta a_{ij}) - \frac{1}{2}\sinh(2\beta(a_{ij}+m)) - \frac{1}{2}\sinh(2\beta(a_{ij}-m)) \,, \qquad (5.58)
\end{aligned}
$$

[2]We have removed a factor of β^{-2} which appears explicitly in (4.5). The remaining factors of β cancel factors coming from the derivatives of T_i.

and $\mathcal{S}_m = \sinh(\beta m)$, gives

$$
\begin{aligned}
\mathcal{F}_1^{(\beta,m)} &= \mathcal{S}_m^2 \sum_i T_i \,, \\
\mathcal{F}_2^{(\beta,m)} &= \frac{1}{2}\mathcal{S}_m^2(3+\mathcal{S}_m^2)\sum_i T_i^2 - \frac{\mathcal{S}_m^4}{4\beta^2}\sum_i T_i T_i^{(2)} - \mathcal{S}_m^4 \sum_{i\neq j} T_i T_j \Pi_{ij} \,, \\
\mathcal{F}_3^{(\beta,m)} &= \frac{4}{27}\mathcal{S}_m^2(9+6\mathcal{S}_m^2+4\mathcal{S}_m^4)\sum_i T_i^3 - \frac{\mathcal{S}_m^4(3+\mathcal{S}_m^2)}{9\beta^2}\sum_i T_i T_i^{(1)} T_i^{(1)} \\
&\quad - \frac{5\mathcal{S}_m^4(3+\mathcal{S}_m^2)}{9\beta^2}\sum_i T_i^2 T_i^{(2)} + \frac{\mathcal{S}_m^6}{12\beta^4}\sum_i T_i T_i^{(2)} T_i^{(2)} + \frac{\mathcal{S}_m^6}{18\beta^4}\sum_i T_i T_i^{(1)} T_i^{(3)} \\
&\quad + \frac{\mathcal{S}_m^6}{36\beta^4}\sum_i T_i^2 T_i^{(4)} - 6(\mathcal{S}_m^4+\mathcal{S}_m^6)\sum_{i\neq j} T_i^2 T_j \Pi_{ij} + \frac{\mathcal{S}_m^6}{\beta^2}\sum_{i\neq j} T_i T_i^{(2)} T_j \Pi_{ij} \\
&\quad + \mathcal{S}_m^6 \sum_{i\neq j} T_i^2 T_j \left(2\left(\Pi_{ij}\right)^2 + 3\Xi_{ij} - 4\coth^2(\beta(a_{ij}+m))\coth^2(\beta(a_{ij}-m))\right) \\
&\quad - \frac{\mathcal{S}_m^6}{\beta}\sum_{i\neq j} T_i T_i^{(1)} T_j \Sigma_{ij} + 2\mathcal{S}_m^6 \sum_{i\neq j\neq k} T_i T_j T_k \Pi_{ij}\Pi_{ik} \,.
\end{aligned} \tag{5.59}
$$

5.4 Full Results

5.4.1 SU(2) Prepotential and Indices

For the pure $\mathcal{N} = 2$ theory the methods of Csaki et al and Nekrasov give identical results, after removing the explicit factor of $(4\pi i\beta^2)^{-1}$ in (4.5), we find

$$\begin{aligned}
\mathcal{F}_1^\beta &= \frac{1}{2\sinh^2(\beta A)}\,,\\
\mathcal{F}_2^\beta &= \frac{1}{64\sinh^4(\beta A)}+\frac{5}{64\sinh^6(\beta A)}\,,\\
\mathcal{F}_3^\beta &= \frac{1}{846\sinh^6(\beta A)}+\frac{17}{576\sinh^8(\beta A)}+\frac{3}{64\sinh^{10}(\beta A)}\,,\\
\mathcal{F}_4^\beta &= \frac{1}{8192\sinh^8(\beta A)}+\frac{39}{4096\sinh^{10}(\beta A)}+\frac{1521}{32768\sinh^{12}(\beta A)}\\
&\quad+\frac{1469}{32768\sinh^{14}(\beta A)}\,,\\
\mathcal{F}_5^\beta &= \frac{1}{64000\sinh^{10}(\beta A)}+\frac{37}{12800\sinh^{12}(\beta A)}+\frac{6263}{204800\sinh^{14}(\beta A)}\\
&\quad+\frac{6413}{81920\sinh^{16}(\beta A)}+\frac{4471}{81920\sinh^{18}(\beta A)}\,,\\
\mathcal{F}_6^\beta &= \frac{1}{442368\sinh^{12}(\beta A)}+\frac{125}{147456\sinh^{14}(\beta A)}+\frac{6515}{393216\sinh^{16}(\beta A)}\\
&\quad+\frac{290293}{3538944\sinh^{18}(\beta A)}+\frac{221839}{1572864\sinh^{20}(\beta A)}+\frac{40397}{524288\sinh^{22}(\beta A)}\,,\\
\mathcal{F}_7^\beta &= \frac{1}{2809856\sinh^{14}(\beta A)}+\frac{195}{802816\sinh^{16}(\beta A)}+\frac{51075}{6422528\sinh^{18}(\beta A)}\\
&\quad+\frac{850485}{12845056\sinh^{20}(\beta A)}+\frac{11887}{57344\sinh^{22}(\beta A)}+\frac{982043}{3670016\sinh^{24}(\beta A)}\\
&\quad+\frac{441325}{3670016\sinh^{26}(\beta A)}\,,\\
\mathcal{F}_8^\beta &= \frac{1}{16777216\sinh^{16}(\beta A)}+\frac{287}{4194304\sinh^{18}(\beta A)}+\frac{117691}{33554432\sinh^{20}(\beta A)}\\
&\quad+\frac{1517109}{33554432\sinh^{22}(\beta A)}+\frac{242756049}{1073741824\sinh^{24}(\beta A)}+\frac{274866637}{536870912\sinh^{26}(\beta A)}\\
&\quad+\frac{2268722897}{4294967296\sinh^{28}(\beta A)}+\frac{866589165}{4294967296\sinh^{30}(\beta A)}\,.
\end{aligned} \tag{5.60}$$

The explicit calculation given in Section 5.1 then agrees with the one instanton results here once we identify $v = 2A$. We are finally ready to determine the equivariant index in the pure SU(2) theory. Using (4.83), where since the vector of VEVs $\vec{\phi}$ is simply proportional to v we get

$$\mathcal{F}_k^\beta = \sum_{n|k}^{\infty} \sum_{l=1}^{\infty} \frac{\mathrm{Ind}_{k/n,l}}{n^3} e^{-2nl\beta A} . \tag{5.61}$$

It is then a simple matter to remove the lower index contributions at each level of instanton charge k and thus extract the indices as in Table 5.3 at the end of this chapter.

5.4.2 Again with Matter

The results for the $\mathcal{N} = 2^\star$ theory are again in full agreement with each other (although the results from Nekrasov's formula need a rescaling $m \to m/2$). With the notation $\mathcal{S}_A = \sinh(\beta A)$ and $\mathcal{S}_m = \sinh(\beta m/2)$ we have

$$\begin{aligned}
\mathcal{F}_1^{(\beta,m)} &= -8\mathcal{S}_m^2 + \frac{8}{\mathcal{S}_A^2}\mathcal{S}_m^4 , \\
\mathcal{F}_2^{(\beta,m)} &= -4(3\mathcal{S}_m^2 + \mathcal{S}_m^4) + \frac{4}{\mathcal{S}_A^2}(6\mathcal{S}_m^4 - 4\mathcal{S}_m^6) + \frac{4}{\mathcal{S}_A^4}(-12\mathcal{S}_m^6 + \mathcal{S}_m^8) + \frac{20}{\mathcal{S}_A^6}\mathcal{S}_m^8 , \\
\mathcal{F}_3^{(\beta,m)} &= -\frac{32}{27}(9\mathcal{S}_m^2 + 6\mathcal{S}_m^4 + 4\mathcal{S}_m^6) + \frac{32}{\mathcal{S}_A^2}(\mathcal{S}_m^4 - 4\mathcal{S}_m^6) + \frac{128}{\mathcal{S}_A^4}(-2\mathcal{S}_m^6 + 3\mathcal{S}_m^8) \\
&\quad + \frac{128}{27\mathcal{S}_A^6}(135\mathcal{S}_m^8 - 78\mathcal{S}_m^{10} + \mathcal{S}_m^{12}) + \frac{64}{9\mathcal{S}_A^8}(-84\mathcal{S}_m^{10} + 17\mathcal{S}_m^{12}) + \frac{48}{\mathcal{S}_A^{10}}\mathcal{S}_m^{12} , \\
\mathcal{F}_4^{(\beta,m)} &= -2(7\mathcal{S}_m^2 + 7\mathcal{S}_m^4 + 8\mathcal{S}_m^6 + 4\mathcal{S}_m^8) + \frac{8}{\mathcal{S}_A^8}(-1554\mathcal{S}_m^{10} + 2031\mathcal{S}_m^{12} - 252\mathcal{S}_m^{14} + \mathcal{S}_m^{16}) \\
&\quad + \frac{8}{\mathcal{S}_A^2}(7\mathcal{S}_m^4 - 46\mathcal{S}_m^6 + 16\mathcal{S}_m^8) + \frac{8}{\mathcal{S}_A^{10}}(2151\mathcal{S}_m^{12} - 1426\mathcal{S}_m^{14} + 78\mathcal{S}_m^{16}) \\
&\quad + \frac{4}{\mathcal{S}_A^4}(-180\mathcal{S}_m^6 + 795\mathcal{S}_m^8 - 280\mathcal{S}_m^{10} - 8\mathcal{S}_m^{12}) + \frac{26}{\mathcal{S}_A^{12}}(-440\mathcal{S}_m^{14} + 117\mathcal{S}_m^{16}) \\
&\quad + \frac{4}{\mathcal{S}_A^6}(1095\mathcal{S}_m^8 - 2684\mathcal{S}_m^{10} + 584\mathcal{S}_m^{12}) + \frac{2938}{\mathcal{S}_A^{14}}\mathcal{S}_m^{16} ,
\end{aligned}$$

$$
\begin{aligned}
\mathcal{F}_5^{(\beta,m)} \;=\;& -\frac{16}{125}(75\mathcal{S}_m^2 + 100\mathcal{S}_m^4 + 280\mathcal{S}_m^6 + 320\mathcal{S}_m^8 + 128\mathcal{S}_m^{10}) \\
& + \frac{16}{\mathcal{S}_A^2}(3\mathcal{S}_m^4 - 56\mathcal{S}_m^6 + 48\mathcal{S}_m^8) \\
& + \frac{256}{\mathcal{S}_A^4}(-6\mathcal{S}_m^6 + 53\mathcal{S}_m^8 - 56\mathcal{S}_m^{10} + 4\mathcal{S}_m^{12}) \\
& + \frac{256}{\mathcal{S}_A^6}(67\mathcal{S}_m^8 - 370\mathcal{S}_m^{10} + 299\mathcal{S}_m^{12} - 24\mathcal{S}_m^{14}) \\
& + \frac{128}{\mathcal{S}_A^8}(-756\mathcal{S}_m^{10} + 2637\mathcal{S}_m^{12} - 1432\mathcal{S}_m^{14} + 96\mathcal{S}_m^{16}) \\
& + \frac{128}{125\mathcal{S}_A^{10}}(292275\mathcal{S}_m^{12} - 633300\mathcal{S}_m^{14} + 217420\mathcal{S}_m^{16} - 9920\mathcal{S}_m^{18} + 16\mathcal{S}_m^{20}) \\
& + \frac{256}{25\mathcal{S}_A^{12}}(-51150\mathcal{S}_m^{14} + 66695\mathcal{S}_m^{16} - 13120\mathcal{S}_m^{18} + 296\mathcal{S}_m^{20}) \\
& + \frac{128}{25\mathcal{S}_A^{14}}(101010\mathcal{S}_m^{16} - 72660\mathcal{S}_m^{18} + 6263\mathcal{S}_m^{20}) \\
& + \frac{64}{5\mathcal{S}_A^{16}}(-21000\mathcal{S}_m^{18} + 6413\mathcal{S}_m^{20}) + \frac{286144}{5\mathcal{S}_A^{18}}\mathcal{S}_m^{20} \;,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{F}_6^{(\beta,m)} \;=\; & -\frac{16}{27}(27\mathcal{S}_m^2 + 45\mathcal{S}_m^4 + 120\mathcal{S}_m^6 + 216\mathcal{S}_m^8 + 192\mathcal{S}_m^{10} + 64\mathcal{S}_m^{12}) \\
& + \frac{32}{\mathcal{S}_A^2}(3\mathcal{S}_m^4 - 50\mathcal{S}_m^6 + 92\mathcal{S}_m^8 - 8\mathcal{S}_m^{10}) \\
& + \frac{16}{\mathcal{S}_A^4}(-180\mathcal{S}_m^6 + 2565\mathcal{S}_m^8 - 5480\mathcal{S}_m^{10} + 1648\mathcal{S}_m^{12}) \\
& + \frac{16}{27\mathcal{S}_A^6}(83835\mathcal{S}_m^8 - 820728\mathcal{S}_m^{10} + 1439232\mathcal{S}_m^{12} - 467424\mathcal{S}_m^{14} + 9216\mathcal{S}_m^{16} \\
& \qquad -256\mathcal{S}_m^{18}) \\
& + \frac{256}{9\mathcal{S}_A^8}(-16317\mathcal{S}_m^{10} + 109122\mathcal{S}_m^{12} - 143253\mathcal{S}_m^{14} + 38619\mathcal{S}_m^{16} - 1144\mathcal{S}_m^{18}) \\
& + \frac{256}{\mathcal{S}_A^{10}}(9813\mathcal{S}_m^{12} - 44944\mathcal{S}_m^{14} + 42024\mathcal{S}_m^{16} - 8500\mathcal{S}_m^{18} + 240\mathcal{S}_m^{20}) \\
& + \frac{64}{27\mathcal{S}_A^{12}}(-3486780\mathcal{S}_m^{14} + 10843065\mathcal{S}_m^{16} - 6953376\mathcal{S}_m^{18} + 974520\mathcal{S}_m^{20} \\
& \qquad -20640\mathcal{S}_m^{22} + 16\mathcal{S}_m^{24}) \\
& + \frac{64}{9\mathcal{S}_A^{14}}(2379195\mathcal{S}_m^{16} - 4921980\mathcal{S}_m^{18} + 2053674\mathcal{S}_m^{20} - 177180\mathcal{S}_m^{22} \\
& \qquad +2000\mathcal{S}_m^{24}) \\
& + \frac{64}{3\mathcal{S}_A^{16}}(-1011780\mathcal{S}_m^{18} + 1331619\mathcal{S}_m^{20} - 325782\mathcal{S}_m^{22} + 13030\mathcal{S}_m^{24}) \\
& + \frac{64}{27\mathcal{S}_A^{18}}(7030197\mathcal{S}_m^{20} - 5326962\mathcal{S}_m^{22} + 580586\mathcal{S}_m^{24}) \\
& + \frac{32}{3\mathcal{S}_A^{20}}(-667584\mathcal{S}_m^{22} + 221839\mathcal{S}_m^{24}) + \frac{1292704}{\mathcal{S}_A^{22}}\mathcal{S}_m^{24}\,,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{F}_7^{(\beta,m)} \;=\;& -\frac{64}{343}(49\mathcal{S}_m^2 + 98\mathcal{S}_m^4 + 588\mathcal{S}_m^6 + 1680\mathcal{S}_m^8 + 2464\mathcal{S}_m^{10} + 1792\mathcal{S}_m^{12} + 512\mathcal{S}_m^{14}) \\
& + \frac{64}{\mathcal{S}_A^2}(\mathcal{S}_m^4 - 44\mathcal{S}_m^6 + 128\mathcal{S}_m^8 - 32\mathcal{S}_m^{10}) \\
& + \frac{256}{\mathcal{S}_A^4}(-18\mathcal{S}_m^6 + 399\mathcal{S}_m^8 - 1408\mathcal{S}_m^{10} + 944\mathcal{S}_m^{12} - 64\mathcal{S}_m^{14}) \\
& + \frac{256}{\mathcal{S}_A^6}(465\mathcal{S}_m^8 - 7082\mathcal{S}_m^{10} + 21631\mathcal{S}_m^{12} - 15608\mathcal{S}_m^{14} + 1968\mathcal{S}_m^{16}) \\
& + \frac{128}{\mathcal{S}_A^8}(-12948\mathcal{S}_m^{10} + 141141\mathcal{S}_m^{12} - 341400\mathcal{S}_m^{14} + 216080\mathcal{S}_m^{16} - 28928\mathcal{S}_m^{18} \\
& \qquad + 256\mathcal{S}_m^{20}) \\
& + \frac{128}{\mathcal{S}_A^{10}}(107991\mathcal{S}_m^{12} - 850744\mathcal{S}_m^{14} + 1569984\mathcal{S}_m^{16} - 797504\mathcal{S}_m^{18} + 94688\mathcal{S}_m^{20} \\
& \qquad - 1280\mathcal{S}_m^{22}) \\
& + \frac{1024}{\mathcal{S}_A^{12}}(-71170\mathcal{S}_m^{14} + 406071\mathcal{S}_m^{16} - 557408\mathcal{S}_m^{18} + 215726\mathcal{S}_m^{20} - 20576\mathcal{S}_m^{22} \\
& \qquad + 288\mathcal{S}_m^{24}) \\
& + \frac{512}{343\mathcal{S}_A^{14}}(168207494\mathcal{S}_m^{16} - 971432532\mathcal{S}_m^{18} + 689156041\mathcal{S}_m^{20} - 194051732\mathcal{S}_m^{22} \\
& \qquad + 13603408\mathcal{S}_m^{24} - 153216\mathcal{S}_m^{26} + 64\mathcal{S}_m^{28}) \\
& + \frac{256}{49\mathcal{S}_A^{16}}(-110126520\mathcal{S}_m^{18} + 321731109\mathcal{S}_m^{20} - 225408260\mathcal{S}_m^{22} + 43395744\mathcal{S}_m^{24} \\
& \qquad - 1966720\mathcal{S}_m^{26} + 12480\mathcal{S}_m^{28}) \\
& + \frac{256}{49\mathcal{S}_A^{18}}(168425103\mathcal{S}_m^{20} - 340459756\mathcal{S}_m^{22} + 158751264\mathcal{S}_m^{24} - 18672752\mathcal{S}_m^{26} \\
& \qquad + 408600\mathcal{S}_m^{28}) \\
& + \frac{1024}{49\mathcal{S}_A^{20}}(-42376992\mathcal{S}_m^{22} + 56298277\mathcal{S}_m^{24} - 15684074\mathcal{S}_m^{26} + 850485\mathcal{S}_m^{28}) \\
& + \frac{1024}{7\mathcal{S}_A^{22}}(3844491\mathcal{S}_m^{24} - 3019198\mathcal{S}_m^{26} + 380384\mathcal{S}_m^{28}) \\
& + \frac{512}{7\mathcal{S}_A^{24}}(-2791096\mathcal{S}_m^{26} + 982043\mathcal{S}_m^{28}) + \frac{225958400}{7\mathcal{S}_A^{26}}\mathcal{S}_m^{28}\,. \qquad (5.62)
\end{aligned}
$$

The index formula (4.86) is now

$$\mathcal{F}_k^\beta = \sum_{n|k}^{\infty} \sum_{l,j} \frac{(-1)^j \mathrm{Ind}_{k/n,l,j}}{n^3} e^{-2nl\beta A} e^{-nj\beta m} \ . \tag{5.63}$$

Then we can extract the indices as in Tables 5.4-5.10, to be found at the end of this chapter, noting that they are symmetric in $j \to -j$ we only quote the results for positive j.

5.4.3 Consistent Limits

There are a number of consistency checks to be made on the results, in effect to ensure that the 'flows' in Figure 4.1 are correct. Firstly, we should take the $\beta \to 0$ limit and regain previous results. This limit can be taken smoothly, and allowing for the β scaling in (4.10), we find both the $\mathcal{N} = 2$ and $\mathcal{N} = 2^\star$ results produce the correct 4d expansions with which we started in Section 5.2.

Secondly, as noted many times previously, our $\mathcal{N} = 2^\star$ results should interpolate between the $\mathcal{N} = 4$ theory, where there are only VEV independent contributions, and the $\mathcal{N} = 2$ results we found above. As previously noted, the simple limit $m \to \infty$ does not give the correct results since the mass parameter is a periodic variable. Explicitly, however, we can hand pick the terms of the highest power of $\mathcal{S}_m$ in each instanton co-efficient of (5.62). Then the limit $m \to \infty$ on these terms gives the results in (5.60).

There is one question which remains to be answered. Do the results in (5.62) give the correct terms in the $m \to 0$ limit? This limit in fact only takes contributions from the lowest power of $\mathcal{S}_m$ in each instanton coefficient, although the limit $m \to 0$ is smooth. Importantly, such a term is always in the VEV independent part (i.e. $\mathcal{O}(\mathcal{S}_A^0)$) which could only be obtained from Nekrasov's formula and which we have determined up to 4 instantons for gauge group SU(2). Indeed, the coefficients are correct

$$\begin{aligned}
\mathcal{F}_1^{(\beta,m)} &= -8\mathcal{S}_m^2 &&\xrightarrow{m\to 0} -2 = -2(\tfrac{1}{1}) \, , \\
\mathcal{F}_2^{(\beta,m)} &= -12\mathcal{S}_m^2 - 4\mathcal{S}_m^4 &&\xrightarrow{m\to 0} -3 = -2(\tfrac{1}{1}+\tfrac{1}{2}) \, , \\
\mathcal{F}_3^{(\beta,m)} &= -\tfrac{32}{3}\mathcal{S}_m^2 - \tfrac{64}{9}\mathcal{S}_m^4 - \tfrac{144}{27}\mathcal{S}_m^6 &&\xrightarrow{m\to 0} -\tfrac{8}{3} = -2(\tfrac{1}{1}+\tfrac{1}{3}) \, , \\
\mathcal{F}_4^{(\beta,m)} &= -14\mathcal{S}_m^2 - 14\mathcal{S}_m^4 - 16\mathcal{S}_m^6 - 8\mathcal{S}_m^8 &&\xrightarrow{m\to 0} -\tfrac{7}{2} = -2(\tfrac{1}{1}+\tfrac{1}{2}+\tfrac{1}{4}) \, ,
\end{aligned} \tag{5.64}$$

however, this was built in to the Nekrasov formula, as we saw in (5.49). So this is no surprise. Instead, we need to understand how the n^{-1} dependence comes out of (4.86). Rather than taking $m \to 0$ we should initially pick out the VEV independent terms, i.e. states with zero 'electric' charge

$$\sum_{n|k}\sum_{j} \frac{\mathrm{Ind}_{k/n,0,j}}{n^3}(-1)^j e^{-nj\beta m} \, . \tag{5.65}$$

But from our results up to 4 instantons we find

$$\sum_j \mathrm{Ind}_{k,0,j}(-1)^j e^{-j\beta m} = -8\sinh^2(\beta m/2) \, , \quad \forall k \, , \tag{5.66}$$

which means our expression for the $\mathcal{N}=4$ limit is

$$\mathcal{F}_k^{(\beta,m\to 0)} = -\sum_{n|k} 4N\frac{\sinh^2(n\beta m/2)}{n^3}, \tag{5.67}$$

where the n^{-1} dependence in the $m \to 0$ limit is explicit. Further the first two coefficients for SU(3) can be calculated and follow exactly the same pattern. Thus it has been possible to predict the VEV independent contributions to $\mathcal{F}^{(\beta,m)}$ above 4 instantons as given in (5.62).

5.5 Discussion

In this chapter three methods of calculation have been used to determine the prepotential of the theories with gauge group SU(2). An important result is that these agree, in particular the Seiberg-Witten calculation, lifted to 5-d via the method of Csaki et al, agrees with geometric method of Nekrasov. The prepotentials have then been expanded according to (4.83) and (4.86) to show the underlying index structure. Thus it is shown that the Seiberg-Witten solution of $\mathcal{N} = 2, 2^{\star}$ SYM theories carries topological information about the instanton moduli space. Also the method of using equivariant cohomology in the derivation of Nekrasov's formula is shown to be valid.

l	$\text{Ind}_{1,l}$	$\text{Ind}_{2,l}$	$\text{Ind}_{3,l}$	$\text{Ind}_{4,l}$	$\text{Ind}_{5,l}$	$\text{Ind}_{6,l}$	$\text{Ind}_{7,l}$	$\text{Ind}_{8,l}$
1	2	0	0	0	0	0	0	0
2	4	0	0	0	0	0	0	0
3	6	6	0	0	0	0	0	0
4	8	32	8	0	0	0	0	0
5	10	110	110	10	0	0	0	0
6	12	288	756	288	12	0	0	0
7	14	644	3556	3556	644	14	0	0
8	16	1280	13072	27264	13072	1280	16	0
9	18	2340	40338	153324	153324	40338	2340	18
10	20	4000	109120	690400	1252040	690400	109120	4000
11	22	6490	266266	2627482	7877210	7877210	2627482	266266
12	24	10080	597888	8757888	40635264	67008672	40635264	8757888
13	26	15106	1253538	26216372	179141716	455426686	455426686	179141716
14	28	21952	2481024	71783040	694898120	2589961248	3986927140	2589961248
15	30	31080	4675050	182298480	2423174610	12732898500	28667463630	28667463630

Table 5.3: The values of the index $\text{Ind}_{k,m}$.

l	$\mathrm{Ind}_{1,l,0}$	$\mathrm{Ind}_{1,l,1}$	$\mathrm{Ind}_{1,l,2}$
1	12	8	2
2	24	16	4
3	36	24	6
4	48	32	8
5	60	40	10
6	72	48	12
7	84	56	14
8	96	64	16
9	108	72	18
10	120	80	20

Table 5.4: The values of the index $\mathrm{Ind}_{1,l,j}$.

l	$\mathrm{Ind}_{2,l,0}$	$\mathrm{Ind}_{2,l,1}$	$\mathrm{Ind}_{2,l,3}$	$\mathrm{Ind}_{2,l,3}$	$\mathrm{Ind}_{2,l,4}$
1	56	39	12	1	0
2	368	272	104	16	0
3	1548	1173	492	99	6
4	4896	3776	1680	384	32
5	12780	9955	4580	1125	110
6	28944	22704	10680	2736	288
7	58912	46417	22148	5831	644
8	110272	87168	42016	11264	1280
9	193104	152991	74268	20169	2340
10	320240	254160	124040	34000	4000

Table 5.5: The values of the index $\mathrm{Ind}_{2,l,j}$.

l	$\mathrm{Ind}_{3,l,0}$	$\mathrm{Ind}_{3,l,1}$	$\mathrm{Ind}_{3,l,3}$	$\mathrm{Ind}_{3,l,3}$	$\mathrm{Ind}_{3,l,4}$	$\mathrm{Ind}_{3,l,5}$	$\mathrm{Ind}_{3,l,6}$
1	208	152	56	8	0	0	0
2	3376	2608	1168	272	24	0	0
3	29556	23544	11682	3384	492	24	0
4	177600	144224	76184	24992	4608	384	8
5	820440	674920	371530	131840	27900	3000	110
6	3111624	2582928	1462896	547200	126408	15744	756
7	10113824	8450120	4884068	1895432	464520	63504	3556
8	29037440	24377536	14301232	5699648	1456128	211968	13072
9	75299220	63444600	37638324	15299280	4028616	613008	40338
10	179376880	151558640	90684560	37417680	10078200	1584000	109120

Table 5.6: The values of the index $\mathrm{Ind}_{3,l,j}$.

l	$\mathrm{Ind}_{4,l,0}$	$\mathrm{Ind}_{4,l,1}$	$\mathrm{Ind}_{4,l,2}$	$\mathrm{Ind}_{4,l,3}$	$\mathrm{Ind}_{4,l,4}$
1	684	513	208	39	2
2	23168	18432	9104	2608	368
3	368748	302301	164604	57222	11682
4	3727616	3114848	1802128	702304	174560
5	27496320	23279710	14035460	5899725	1660030
6	160036992	136786944	84943392	37658448	11530416
7	772761948	665189140	422231824	194633586	63310646
8	3205450368	2774211584	1790561728	849981184	289153280
9	11718721764	10185027714	6659378640	3233503476	1138106394
10	38505396480	33577886720	22179988480	10962266800	3961757200

l	$\mathrm{Ind}_{4,l,5}$	$\mathrm{Ind}_{4,l,6}$	$\mathrm{Ind}_{4,l,7}$	$\mathrm{Ind}_{4,l,8}$
1	0	0	0	0
2	16	0	0	0
3	1173	36	0	0
4	24992	1680	32	0
5	291000	27900	1125	10
6	2303472	272016	15744	288
7	13842255	1872192	134211	3556
8	67475712	10027648	826624	27264
9	278795979	44356896	4028967	153324
10	1007178000	168629200	16432000	690400

Table 5.7: The values of the index $\mathrm{Ind}_{4,l,j}$.

l	$\mathrm{Ind}_{5,l,0}$	$\mathrm{Ind}_{5,l,1}$	$\mathrm{Ind}_{5,l,3}$	$\mathrm{Ind}_{5,l,3}$	$\mathrm{Ind}_{5,l,4}$	
1	2032	1560	684	152	12	
2	131248	106576	56276	18432	3376	
3	3494304	2921136	1692216	660816	164604	
4	55192896	46994656	28838680	12515264	3704640	
5	608353820	534646800	334995060	156124000	51717860	
6	5121040320	4458033360	2930263380	1437646464	515520240	
7	34851795440	30557061976	20531301812	10469087160	3980407396	
8	199408762432	175813034304	120164606032	63124490752	25096414208	
9	986934423600	874017580104	605523654684	325647058536	134063859132	
10	4317795441000	3837499618800	2687823524380	1472960451200	623465558480	
l	$\mathrm{Ind}_{5,l,5}$	$\mathrm{Ind}_{5,l,6}$	$\mathrm{Ind}_{5,l,7}$	$\mathrm{Ind}_{5,l,8}$	$\mathrm{Ind}_{5,l,9}$	$\mathrm{Ind}_{5,l,10}$
1	0	0	0	0	0	0
2	272	4	0	0	0	0
3	23544	1548	24	0	0	0
4	702304	76184	3776	48	0	0
5	11651760	1660030	131840	4580	40	0
6	130587696	22143528	2303472	126408	2736	12
7	1097143152	209920508	26044480	1872192	63504	644
8	7369325120	1539872256	216187648	18577664	826624	13072
9	41333307120	9231320196	1419210720	138332880	7375536	153324
10	199740522240	46971805440	7736611120	827488040	50145520	125040

Table 5.8: The values of the index $\mathrm{Ind}_{5,l,j}$.

l	$\mathrm{Ind}_{6,l,0}$	$\mathrm{Ind}_{6,l,1}$	$\mathrm{Ind}_{6,l,3}$	$\mathrm{Ind}_{6,l,3}$	$\mathrm{Ind}_{6,l,4}$
1	5616	4382	2032	513	56
2	645568	532480	295608	106576	23168
3	27181116	23053218	13974168	5929812	1692216
4	637896512	550611008	352523696	165058944	55050496
5	10036106780	8769781430	5831716400	2920156835	1080124920
6	117165258000	103316991264	70651462296	37154322144	14801507136
7	1081487624952	960361631572	670961822384	366256581761	154282770698
8	8241278604032	7358871601536	5228972034400	2939010095744	1292867200768

l	$\mathrm{Ind}_{6,l,5}$	$\mathrm{Ind}_{6,l,6}$	$\mathrm{Ind}_{6,l,7}$	$\mathrm{Ind}_{6,l,8}$	$\mathrm{Ind}_{6,l,9}$
1	1	0	0	0	0
2	2608	104	0	0	0
3	302301	29556	1173	6	0
4	12515264	1802128	144224	4896	32
5	286136215	51717860	5899725	371530	9955
6	4362629184	917938008	130587696	11530416	547200
7	49200012022	11545442328	1912159487	209920508	13842255
8	437930088960	111542753952	20643112192	2639977984	216187648

l	$\mathrm{Ind}_{6,l,10}$	$\mathrm{Ind}_{6,l,11}$	$\mathrm{Ind}_{6,l,12}$
1	0	0	0
2	0	0	0
3	0	0	0
4	0	0	0
5	60	0	0
6	10680	48	0
7	464520	5831	14
8	10027648	211968	1280

Table 5.9: The values of the index $\mathrm{Ind}_{6,l,j}$.

l	$\mathrm{Ind}_{7,l,0}$	$\mathrm{Ind}_{7,l,1}$	$\mathrm{Ind}_{7,l,3}$	$\mathrm{Ind}_{7,l,3}$	$\mathrm{Ind}_{7,l,4}$
1	14592	11552	5616	1560	208
2	2845536	2376304	1372048	532480	131248
3	181793880	155935848	97907364	44267184	13974168
4	6109989376	5329787104	3525186080	1748372608	637140224
5	132672257960	117100468440	80305330700	42438404880	17028387960
6	2082169931328	1853954006256	1305980398608	723122690400	311118469320
7	25323135217884	22701204282736	16326420269754	9369710171584	4250862025788
8	250179192526592	225492761233344	164867352869488	97338445015552	46039555065856

l	$\mathrm{Ind}_{7,l,5}$	$\mathrm{Ind}_{7,l,6}$	$\mathrm{Ind}_{7,l,7}$	$\mathrm{Ind}_{7,l,8}$	$\mathrm{Ind}_{7,l,9}$
1	8	0	0	0	0
2	18432	1168	16	0	0
3	2921136	368748	23544	492	0
4	165058944	28838680	3114848	177600	3776
5	5068348520	1080124920	156124000	14035460	674920
6	102180053232	24956243424	4362629184	515520240	37658448
7	1502930603104	405657123410	81171706952	11545442328	1097143152
8	17237444964608	5022319450624	1111957462336	180954297344	20643112192

l	$\mathrm{Ind}_{7,l,10}$	$\mathrm{Ind}_{7,l,11}$	$\mathrm{Ind}_{7,l,12}$	$\mathrm{Ind}_{7,l,13}$	$\mathrm{Ind}_{7,l,14}$
1	0	0	0	0	0
2	0	0	0	0	0
3	0	0	0	0	0
4	8	0	0	0	0
5	12780	40	0	0	0
6	1462896	22704	72	0	0
7	63310646	1895432	22148	56	0
8	1539872256	67475712	1456128	11264	16

Table 5.10: The values of the index $\mathrm{Ind}_{7,l,j}$.

Chapter 6

Summary

In this thesis we have examined two situations where we are able to calculate exact results in supersymmetric gauge theories supersymmetry. These centre around the low-energy effective action of $\mathcal{N} = 2$ gauge theory in 4 dimensions (2.7) and the superpotential of $\mathcal{N} = 1$ theories.

In the first case we were able to identify the Seiberg-Witten curve of the theory compactified on a circle with the spectral curve of a particular integrable system. We saw initially how such an identification occurs in the simple case of mass deformed $\mathcal{N} = 4$ theory with gauge group $\mathrm{SU}(N)$, studied in [4]. In this case the superpotential of the field theory, broken to $\mathcal{N} = 1$, is naturally identified with the governing Hamiltonian of the Calogero-Moser integrable system. We were thus able to determine, in principle, the vacuum states of the theory simply by determining the stationary points of the Hamiltonian, which is independent of the compactification radius. The results are thus valid in the 4-dimensional limit. The superpotential is expressed in terms of the Weierstrass elliptic function, $\wp$, and the massive vacua are given by configurations of 'particles' in the integrable system where they have zero 'momentum' and are positioned at sublattice points of the fundamental parallelogram of $\wp(X)$.

We then examined the more general setup involved in the quiver models in [5]. Here the gauge group in $\mathrm{SU}(N)^k$ and the integrable system is a spin generalization of the Calogero-Moser system. The phase structure for any N and k had been determined, but only for the massive vacua of the theory. This is partly because such vacua are given by equilibrium values for *all* the hamiltonians of the integrable system. With the $k = 1$ results generalizing naturally, it was just left to find a parametrization of the spin matrices which solved the critical point equations for all the hamiltonians of the theory.

The story for the massless vacua is not so easy to pin down. The point is that they are given by equilibrium values of just a subset of the hamiltonians and their generic parameterization in unknown. We thus examined the specific case with gauge group $\mathrm{SU}(2) \times \mathrm{SU}(2)$ in the hope of determining the massless vacua in this case. It turns out that the equations are solved entirely by the spin matrices and the 'position' variable X is a free parameter, i.e. there are flat directions in the parameter space. This is confirmed by the fact that the determinant of the Hessian matrix is non-zero for the massive vacua and zero for the massless vacua. In this specific case we are also able to exactly determine the values of the condensates, $\mathrm{Tr}\Phi_i^2$. It is hoped that examining the massless vacua in this simple case will lead us to determine such vacua for the general gauge group.

Recent work by Dijkgraaf and Vafa [60–62] has developed a new method for calculating the superpotential in terms of the planar diagrams of a matrix model. The results of [5] have been confirmed with this method, and it may be possible that new light may be shed on the massless vacua of the gauge theory.

Our second topic was to go back to the low-energy effective action and determine the prepotential $\mathcal{F}$ for theories with $\mathcal{N} = 2$ supersymmetry, with and without a massive adjoint hypermultiplet, in terms of a topological index. The fact that, due to supersymmetry, the perturbative expansion terminates means we are left needing only to determine the instanton contributions. For the general case this involves integrating over the *centred instanton moduli space*, $\hat{\mathfrak{M}}_k$. When we lifted the theory to 5 dimensions we saw how the functional integral calculated exactly the Witten index of a quantum mechanical σ-model, as we had met in Chapter 3, with $\hat{\mathfrak{M}}_k$ as a target. Thus the instanton expansion is given as a topological quantity on the moduli space and is realized as counting the ground states of the σ-model. Further, the supersymmetry charges of the σ-model are indentified with the Dirac and Dolbeault operators in $\mathcal{N} = 2, 2^*$ theories respectively. Thus the Hilbert space is realized as the space of spinors/forms. The $\frac{1}{2}$-BPS states of the model, found in Chapter 4, describe the instanton dyon states of the parent field theory in 5 dimensions and carry a U(1) electric charge.

An important qualification for the definition of the index theory is that the manifold in question is compact and does not include any singularities. However, the instanton moduli space breaks both these properties. Fortunately we can deform the space to remove the singularities, related to instantons of zero size, without changing the general properties of the space. The non-compact nature of the space, related to instantons becoming arbitrarily separated, requires careful consideration. First we must identify the normalizable states of the theory by imposing the L^2 condition on the Witten index. This can then be split into principal and defect terms, where the principal contribution is exactly the prepotential we would like to calculate. Thus to explicitly determine the index contributions we must remove the defect terms from the prepotential at each instanton level. We thus arrived at the identity (4.84).

As noted in the introduction, there is a motivation for finding such a formula directly in the field theory from *geometric engineering* techniques. Here Type IIA string theory in 10 dimensions is examined on a 'non-compact' Calabi-Yau threefold. Topologically, for the SU(2) theory, the CY space involves an S^2 fibration over an S^2 base. Then, once gravity has been decoupled, the remaining dynamics describes the $\mathcal{N} = 2$ gauge theory whose prepotential can be found from a tree-level string computation. In equation (1.2) of [63] the prepotential is expressed as a sum of *worldsheet instantons*

$$\mathcal{F}_{\text{insta.}} = \sum_{n=0}^{\infty} \sum_{m,k=1}^{\infty} \frac{d_{m,n}}{k^3} q_b^{nk} q_f^{mk} , \tag{6.1}$$

where b and f refer to the *base* and the *fibre* respectively. We note in particular the power of k^{-3} in analogy to (4.84). The $d_{m,n}$ are the number of primitive worldsheet instantons wrapping n times round the base and m times round the fibre.

Clearly we should obtain integer values for the index since in the parent theory we are calculating the number of vacuum/BPS states. In Chapter 5 we determined these integers for gauge group SU(2), by three different methods. The first used the explicit parametrization of the metric on the one-instanton space, given by the ADHM constaints, realized in terms of the Eguchi-Hanson space. Second, we lifted the five-dimensional prepotential from the four-dimensional Seiberg-Witten curve according to the method of Csaki et al. Finally we used the explicit formula determined by Nekrasov. This last method demands a little more explanation. First however we can note that the pure $\mathcal{N} = 2$ results in Table 5.3 are in fact just those invariants, $d_{m,n}$, found in the $\mathbf{F}_0$ geometric engineering scenario in [63]. In [64] it has been shown how the gauge theory with adjoint matter may be engineered.

As well as considering the gauge symmetries of the field theory, Nekrasov's formula also considers the action of the Lorentz group on the manifold and this solves the non-compactness problem. In particular, the fixed point set under both symmetries becomes discrete and so the system is in essence a counting problem. The result is a sum over (N)-coloured partitions of k. The method uses non-commutative geometry and involves a parameter, $\hbar$, the scale of the background deformation. The prepotential is then identified by taking the deformation parameter to zero. However, the partition function $\mathcal{Z}$ contains a lot more information. In fact it can be viewed as

$$\mathcal{Z} = exp\left(\sum_{g=0}^{\infty} \mathcal{F}_g \hbar^{2g-2}\right) , \tag{6.2}$$

where the $\mathcal{F}_g$ should not be confused with the instanton coefficients $\mathcal{F}_k$ but are instead 'higher genus' contributions. We see that (6.2) implies the formula (5.48) where $\mathcal{F}_0$, i.e. $g = 0$, is identified as the prepotential. However, in [52] it was shown how the theory with a non-zero deformation parameter ($\hbar \neq 0$) is actually seen as the partition function of topological string theory compactified on a Calabi-Yau threefold. As such it is also the partition function of the geometrically engineered theory, where the parameter $\hbar$ is identified with the string coupling g_s. Thus the fact that the results from all three methods agree in the $\mathcal{N} = 2^\star$ theory effectively means we have checked the Seiberg-Witten curve calculation with the results from geometric engineering. It would be an interesting point to investigate if the gauge theory can be deformed in such a way as to pick up the $\hbar$ expansion.

Finally, we had to ensure the $\mathcal{N} = 2^\star$ prepotential had the correct asymptotic behaviour as the mass scaled to 0, i.e. to agree with the $\mathcal{N} = 4$ result contributions (4.26) it must have an n^{-1} dependence. We noted that the terms surviving in this limit would come from VEV-independent contributions to the prepotential in $\mathcal{N} = 2^\star$

and so we could only determine this limit from the results of Nekrasov's formula which we had already seen would give the correct limit. However, it was still possible to determine how (4.86) could give the correct limit since it was found that the contributions from instantons of zero 'electric' charge (thus VEV-independent) are in fact equal at each instanton order. This property should hold for all gauge groups.

Appendix A

Elliptic Functions

In this appendix we briefly review some of the properties of elliptic functions which are relevant to this thesis. An elliptic function is a doubly-periodic function which is analytic except at poles and has no other singularities in the finite part of the plane. There are no such functions of order 1 and for order 2 we have two classes: i) Weierstrassian - with a single irreducible double pole, and ii) Jacobian - with 2 simple poles; we will concentrate only on the first. The Weierstrassian function can be written as

$$\wp(z) = \frac{1}{z^2} + \sum_{m,n\neq 0} \left[\frac{1}{(z - 2mw_1 - 2nw_2)^2} - \frac{1}{(2mw_1 + 2nw_2)^2} \right], \tag{A.1}$$

where the sum is done without $m = n = 0$. $2w_1$ and $2w_2$ are the two periods of the function which is even

$$\begin{aligned} \wp(z + 2w_1) = \wp(z + 2w_2) &= \wp(z + 2w_1 + 2w_2) = \wp(z), \\ \wp(z) &= \wp(-z), \\ \wp'(z)^2 &= 4\wp^3(z) - g_2\wp(z) - g_3, \\ \wp'(z) &= -\wp'(-z), \end{aligned} \tag{A.2}$$

where g_2 and g_3 are the invariants of the Weierstrassian. Clearly the first differential of $\wp$ is odd and elliptic. In fact any even elliptic function can be expressed as a rational function of $\wp$ and any odd elliptic function can be expressed as $\wp'$ times a rational function of $\wp$. Note that any odd elliptic function has roots ($F(z) = 0$) at half-lattice points, e.g.

$$\begin{aligned} \wp'(w_1) = -\wp'(-w_1) &= -\wp'(-w_1 + 2w_1) = -\wp'(w_1), \\ \wp'(w_1) = \wp'(w_2) &= \wp'(w_1 + w_2) = 0, \end{aligned} \tag{A.3}$$

and similarly for any other odd elliptic function.

Related to the Weierstrassian are two quasi-elliptic functions which obey the following relations

$$\begin{aligned} \sigma(z) &= -\sigma(-z), \\ \sigma(z + 2w_i) &= -\sigma(z)e^{2(z+w_i)\zeta(w_i)}, \\ \wp(z) - \wp(x) &= -\frac{\sigma(z-x)\sigma(z+x)}{\sigma^2(z)\sigma^2(x)}, \end{aligned} \tag{A.4}$$

and

$$\begin{aligned} \zeta(z) &= -\zeta(-z), \\ \zeta(z + 2w_i) &= \zeta(z) + 2\zeta(w_i), \\ \zeta(z+x) - \zeta(z) - \zeta(x) &= \frac{1}{2}\frac{\wp'(z) - \wp'(x)}{\wp(z) - \wp(x)}, \\ (\zeta(z - z_1) - \zeta(z - z_2) + \zeta(z_1 - z_2))^2 &= \wp(z - z_1) + \wp(z - z_2) + \wp(z_1 - z_2). \end{aligned} \tag{A.5}$$

We will be using a particular function of these, for some parameter β

$$\begin{aligned} \phi(X) &= \frac{\sigma(X+z)}{\sigma(X)\sigma(z)}\left(\zeta(X+z) - \zeta(X) - \zeta(z) + \tfrac{1}{2}\beta\right), \\ \tilde{\phi}(X) &= \frac{\sigma(X-z)}{\sigma(X)\sigma(-z)}\left(\zeta(X-z) - \zeta(X) - \zeta(-z) - \tfrac{1}{2}\beta\right), \end{aligned} \tag{A.6}$$

which leads to the following identity

$$\begin{aligned}
\phi(X)\tilde{\phi}(X) &= -\frac{\sigma(X+z)\sigma(X-z)}{\sigma^2(X)\sigma^2(z)}\left(\zeta(X+z)-\zeta(X)-\zeta(z)+\tfrac{1}{2}\beta\right)\\
&\qquad\cdot\left(\zeta(X-z)-\zeta(X)-\zeta(-z)-\tfrac{1}{2}\beta\right)\\
&= (\wp(X)-\wp(z))\cdot\frac{1}{2}\left(\frac{\wp'(X)-\wp'(z)}{\wp(X)-\wp(z)}+\beta\right)\cdot\frac{1}{2}\left(\frac{\wp'(X)+\wp'(z)}{\wp(X)-\wp(z)}-\beta\right),\\
&= \frac{1}{4}\left(\frac{\wp'^2(X)-\wp'^2(z)}{\wp(X)-\wp(z)}+2\beta\wp'(z)-\beta^2(\wp(X)-\wp(z))\right),\\
&= \frac{1}{4}\frac{4\wp^3(X)-g_2\wp(X)-g_3-4\wp^3(z)+g_2\wp(z)+g_3}{\wp(X)-\wp(z)}\\
&\qquad+\frac{1}{2}\beta\wp'(z)-\frac{1}{4}\beta^2(\wp(X)-\wp(z)),\\
&= \frac{\wp^3(X)-\wp^3(z)}{\wp(X)-\wp(z)}-\frac{g_2}{4}+\frac{1}{2}\beta\wp'(z)-\frac{1}{4}\beta^2(\wp(X)-\wp(z)), \qquad\qquad (A.7)\\
&= \wp^2(X)+\wp(X)\wp(z)+\wp^2(z)-\frac{g_2}{4}+\frac{1}{2}\beta\wp'(z)-\frac{1}{4}\beta^2(\wp(X)-\wp(z)).
\end{aligned}$$

We also need the combination $\gamma^2 - 4\phi\tilde{\phi}$ where

$$\gamma = 2\wp(X) + \wp(z) - \beta^2/4, \tag{A.8}$$

thus

$$\gamma^2 - 4\phi\tilde{\phi} = g_2 - 3\wp^2(z) - 2\beta\wp'(z) - \frac{3}{2}\beta^2\wp(z) + \frac{1}{16}\beta^4, \tag{A.9}$$

which importantly is independent of X.

Appendix B

Combinatorics of Young's tableaux

In this appendix we enumerate the contributions to the partition function $\mathcal{Z}_k$ for Nekrasov's (Bruzzo's) formula (5.46). All possible Young's tableaux, and their contributions are given in Tables B.1-B.4. In order to calculate the prepotential for general gauge groups we have used the following identities

$$\left(\sum_i T_i\right)^2 = \sum_i T_i^2 + \sum_{i\neq j} T_i T_j, \tag{B.1}$$

$$\begin{aligned}
\left(\sum_i T_i\right)^3 &= \sum_i T_i^3 + 3\sum_{i\neq j} T_i^2 T_j + \sum_{i\neq j\neq k} T_i T_j T_k, \\
\left(\sum_i T_i\right)\left(\sum_j T_j^2\right) &= \sum_i T_i^3 + \sum_{i\neq j} T_i^2 T_j, \\
\left(\sum_i T_i\right)\left(\sum_{j\neq k} T_j T_k\right) &= 2\sum_{i\neq j} T_i^2 T_j + \sum_{i\neq j\neq k} T_i T_j T_k.
\end{aligned} \tag{B.2}$$

$\{Y_\lambda\}$	λ	$\tilde{\lambda}$	s	ν_{i_λ}	$\tilde{\nu}'_{j_\lambda}$	$h(s)$	$v(s)$	$a_{\lambda\tilde{\lambda}}$	$f(E(s))$
$\sum_\alpha \square$									
	α	α	(1,1)	1	1	0	0	0	$f(\epsilon_2)$
	α	β	(1,1)	1	0	0	-1	$a_{\alpha\beta}$	$T_\alpha(0)$

Table B.1: Young's tableau at the one instanton level.

$\{Y_\lambda\}$	λ	$\tilde{\lambda}$	s	ν_{i_λ}	$\tilde{\nu}'_{j_\lambda}$	$h(s)$	$v(s)$	$a_{\lambda\tilde{\lambda}}$	$f(E(s))$
$\sum_\alpha \square\square$									
	α	α	(1,1)	2	1	1	0	0	$f(\epsilon_2-\epsilon_1)$
	α	α	(1,2)	2	1	0	0	0	$f(\epsilon_2)$
	α	β	(1,1)	2	0	1	-1	$a_{\alpha\beta}$	$T_\alpha(-\epsilon_1)$
	α	β	(1,2)	2	0	0	-1	$a_{\alpha\beta}$	$T_\alpha(0)$
$\sum_\alpha \begin{smallmatrix}\square\\\square\end{smallmatrix}$									
	α	α	(1,1)	1	2	0	1	0	$f(2\epsilon_2)$
	α	α	(2,1)	1	2	0	0	0	$f(\epsilon_2)$
	α	β	(1,1)	1	0	0	-1	$a_{\alpha\beta}$	$T_\alpha(0)$
	α	β	(2,1)	1	0	0	-2	$a_{\alpha\beta}$	$T_\alpha(-\epsilon_2)$
$\frac{1}{2}\sum_{\alpha\neq\beta} \square\oplus\square$									
	α	α	(1,1)	1	1	0	0	0	$f(\epsilon_2)$
	β	β	(1,1)	1	1	0	0	0	$f(\epsilon_2)$
	α	β	(1,1)	1	1	0	0	$a_{\alpha\beta}$	$f(a_{\alpha\beta}+\epsilon_2)$
	β	α	(1,1)	1	1	0	0	$a_{\beta\alpha}$	$f(a_{\beta\alpha}+\epsilon_2)$
	α	γ	(1,1)	1	0	0	-1	$a_{\alpha\gamma}$	$T_\alpha(0)/f(a_{\alpha\beta})$
	β	γ	(1,1)	1	0	0	-1	$a_{\beta\gamma}$	$T_\beta(0)/f(a_{\beta\alpha})$

Table B.2: Young's tableau at the two instanton level.

$\{Y_\lambda\}$	λ	$\tilde{\lambda}$	s	ν_{i_λ}	$\tilde{\nu}'_{j_{\tilde{\lambda}}}$	$h(s)$	$v(s)$	$a_{\lambda\tilde{\lambda}}$	$f(E(s))$
$\sum_{\alpha} \yng(3)$									
	α	α	(1,1)	3	1	2	0	0	$f(\epsilon_2 - 2\epsilon_1)$
	α	α	(1,2)	3	1	1	0	0	$f(\epsilon_2 - \epsilon_1)$
	α	α	(1,3)	3	1	0	0	0	$f(\epsilon_2)$
	α	β	(1,1)	3	0	2	-1	$a_{\alpha\beta}$	$T_\alpha(-2\epsilon_1)$
	α	β	(1,2)	3	0	1	-1	$a_{\alpha\beta}$	$T_\alpha(-\epsilon_1)$
	α	β	(1,3)	3	0	0	-1	$a_{\alpha\beta}$	$T_\alpha(0)$
$\sum_{\alpha} \yng(1,1,1)$									
	α	α	(1,1)	1	3	0	2	0	$f(3\epsilon_2)$
	α	α	(2,1)	1	3	0	1	0	$f(2\epsilon_2)$
	α	α	(3,1)	1	3	0	0	0	$f(\epsilon_2)$
	α	β	(1,1)	1	0	0	-1	$a_{\alpha\beta}$	$T_\alpha(0)$
	α	β	(2,1)	1	0	0	-2	$a_{\alpha\beta}$	$T_\alpha(-\epsilon_2)$
	α	β	(3,1)	1	0	0	-3	$a_{\alpha\beta}$	$T_\alpha(-2\epsilon_2)$
$\sum_{\alpha} \yng(2,1)$									
	α	α	(1,1)	2	2	1	1	0	$f(2\epsilon_2 - \epsilon_1)$
	α	α	(1,2)	2	1	0	0	0	$f(\epsilon_2)$
	α	α	(2,1)	1	2	0	0	0	$f(\epsilon_2)$
	α	β	(1,1)	2	0	1	-1	$a_{\alpha\beta}$	$T_\alpha(-\epsilon_1)$
	α	β	(1,2)	2	0	0	-1	$a_{\alpha\beta}$	$T_\alpha(0)$
	α	β	(2,1)	1	0	0	-2	$a_{\alpha\beta}$	$T_\alpha(-\epsilon_2)$

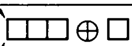
contd...

$\{Y_\lambda\}$	λ	$\tilde{\lambda}$	s	ν_{i_λ}	$\tilde{\nu}'_{j_\lambda}$	$h(s)$	$v(s)$	$a_{\lambda\tilde{\lambda}}$	$f(E(s))$
$\sum_{\alpha\neq\beta} \square\square \oplus \square$									
	α	α	(1,1)	2	1	1	0	0	$f(\epsilon_2-\epsilon_1)$
	α	α	(1,2)	2	1	0	0	0	$f(\epsilon_2)$
	β	β	(1,1)	1	1	0	0	0	$f(\epsilon_2)$
	α	β	(1,1)	2	1	1	0	$a_{\alpha\beta}$	$f(a_{\alpha\beta}+\epsilon_2-\epsilon_1)$
	α	β	(1,2)	2	0	0	-1	$a_{\alpha\beta}$	$f(a_{\alpha\beta})$
	β	α	(1,1)	1	1	0	0	$a_{\beta\alpha}$	$f(a_{\beta\alpha}+\epsilon_2)$
	α	γ	(1,1)	2	0	1	-1	$a_{\alpha\gamma}$	$T_\alpha(-\epsilon_1)/f(a_{\alpha\beta}-\epsilon_1)$
	α	γ	(1,2)	2	0	0	-1	$a_{\alpha\gamma}$	$T_\alpha(0)/f(a_{\alpha\beta})$
	β	γ	(1,1)	1	0	0	-1	$a_{\beta\gamma}$	$T_\beta(0)/f(a_{\beta\alpha})$
$\sum_{\alpha\neq\beta} \begin{smallmatrix}\square\\\square\end{smallmatrix} \oplus \square$									
	α	α	(1,1)	1	2	0	1	0	$f(2\epsilon_2)$
	α	α	(2,1)	1	2	0	0	0	$f(\epsilon_2)$
	β	β	(1,1)	1	1	0	0	0	$f(\epsilon_2)$
	α	β	(1,1)	1	1	0	0	$a_{\alpha\beta}$	$f(a_{\alpha\beta}+\epsilon_2)$
	α	β	(2,1)	1	1	0	-1	$a_{\alpha\beta}$	$f(a_{\alpha\beta})$
	β	α	(1,1)	1	2	0	1	$a_{\beta\alpha}$	$f(a_{\beta\alpha}+2\epsilon_2)$
	α	γ	(1,1)	1	0	0	-1	$a_{\alpha\gamma}$	$T_\alpha(0)/f(a_{\alpha\beta})$
	α	γ	(2,1)	1	0	0	-2	$a_{\alpha\gamma}$	$T_\alpha(-\epsilon_2)/f(a_{\alpha\beta}-\epsilon_2)$
	β	γ	(1,1)	1	0	0	-1	$a_{\beta\gamma}$	$T_\beta(0)/f(a_{\beta\alpha})$
$\frac{1}{3}\sum_{\alpha\neq\beta\neq\gamma} \square\oplus\square\oplus\square$									
	α	α	(1,1)	1	1	0	0	0	$f(\epsilon_2)$
	β	β	(1,1)	1	1	0	0	0	$f(\epsilon_2)$
	γ	γ	(1,1)	1	1	0	0	0	$f(\epsilon_2)$
	α	β	(1,1)	1	1	0	0	$a_{\alpha\beta}$	$f(a_{\alpha\beta}+\epsilon_2)$
	α	γ	(1,1)	1	1	0	0	$a_{\alpha\gamma}$	$f(a_{\alpha\gamma}+\epsilon_2)$
	β	α	(1,1)	1	1	0	0	$a_{\beta\alpha}$	$f(a_{\beta\alpha}+\epsilon_2)$
	β	γ	(1,1)	1	1	0	0	$a_{\beta\gamma}$	$f(a_{\beta\gamma}+\epsilon_2)$
	γ	γ	(1,1)	1	1	0	0	$a_{\gamma\alpha}$	$f(a_{\gamma\alpha}+\epsilon_2)$
	γ	β	(1,1)	1	1	0	0	$a_{\gamma\beta}$	$f(a_{\gamma\beta}+\epsilon_2)$
	α	δ	(1,1)	1	0	0	-1	$a_{\alpha\delta}$	$T_\alpha(0)/f(a_{\alpha\beta})f(a_{\alpha\gamma})$
	β	δ	(1,1)	1	0	0	-1	$a_{\beta\delta}$	$T_\beta(0)/f(a_{\beta\alpha})f(a_{\beta\gamma})$
	γ	δ	(1,1)	1	0	0	-1	$a_{\gamma\delta}$	$T_\gamma(0)/f(a_{\gamma\alpha})f(a_{\gamma\beta})$

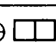
Table B.3: Young's tableau at the three instanton level.

$\{Y_\lambda\}$	λ	$\tilde{\lambda}$	s	ν_{i_λ}	$\tilde{\nu}'_{j_\lambda}$	$h(s)$	$v(s)$	$a_{\lambda\tilde{\lambda}}$	$f(E(s))$
$\sum_\alpha \ydiagram{4}$									
	α	α	(1,1)	4	1	3	0	0	$f(\epsilon_2 - 3\epsilon_1)$
	α	α	(1,2)	4	1	2	0	0	$f(\epsilon_2 - 2\epsilon_1)$
	α	α	(1,3)	4	1	1	0	0	$f(\epsilon_2 - \epsilon_1)$
	α	α	(1,4)	4	1	0	0	0	$f(\epsilon_2)$
	α	β	(1,1)	4	0	3	-1	$a_{\alpha\beta}$	$T_\alpha(-3\epsilon_1)$
	α	β	(1,2)	4	0	2	-1	$a_{\alpha\beta}$	$T_\alpha(-2\epsilon_1)$
	α	β	(1,3)	4	0	1	-1	$a_{\alpha\beta}$	$T_\alpha(-\epsilon_1)$
	α	β	(1,4)	4	0	0	-1	$a_{\alpha\beta}$	$T_\alpha(0)$
$\sum_\alpha \ydiagram{3,1}$									
	α	α	(1,1)	3	2	2	1	0	$f(2\epsilon_2 - 2\epsilon_1)$
	α	α	(1,2)	3	1	1	0	0	$f(\epsilon_2 - \epsilon_1)$
	α	α	(1,3)	3	1	0	0	0	$f(\epsilon_2)$
	α	α	(2,1)	1	2	0	0	0	$f(\epsilon_2)$
	α	β	(1,1)	3	0	2	-1	$a_{\alpha\beta}$	$T_\alpha(-2\epsilon_1)$
	α	β	(1,2)	3	0	1	-1	$a_{\alpha\beta}$	$T_\alpha(-\epsilon_1)$
	α	β	(1,3)	3	0	0	-1	$a_{\alpha\beta}$	$T_\alpha(0)$
	α	β	(2,1)	1	0	0	-2	$a_{\alpha\beta}$	$T_\alpha(-\epsilon_2)$
$\sum_\alpha \ydiagram{2,1,1}$									
	α	α	(1,1)	2	3	1	2	0	$f(3\epsilon_2 - \epsilon_1)$
	α	α	(1,2)	2	1	0	0	0	$f(\epsilon_2)$
	α	α	(2,1)	1	3	0	1	0	$f(2\epsilon_2)$
	α	α	(3,1)	1	3	0	0	0	$f(\epsilon_2)$
	α	β	(1,1)	2	0	1	-1	$a_{\alpha\beta}$	$T_\alpha(-\epsilon_1)$
	α	β	(1,2)	2	0	0	-1	$a_{\alpha\beta}$	$T_\alpha(0)$
	α	β	(2,1)	1	0	0	-2	$a_{\alpha\beta}$	$T_\alpha(-\epsilon_2)$
	α	β	(3,1)	1	0	0	-3	$a_{\alpha\beta}$	$T_\alpha(-2\epsilon_2)$

contd...

$\{Y_\lambda\}$	λ	$\tilde{\lambda}$	s	ν_{i_λ}	$\tilde{\nu}'_{j_\lambda}$	$h(s)$	$v(s)$	$a_{\lambda\tilde{\lambda}}$	$f(E(s))$
$\sum_{\alpha}$ 									
	α	α	(1,1)	2	2	1	1	0	$f(2\epsilon_2 - \epsilon_1)$
	α	α	(1,2)	2	2	0	1	0	$f(2\epsilon_2)$
	α	α	(2,1)	2	2	1	0	0	$f(\epsilon_2 - \epsilon_1)$
	α	α	(2,2)	2	2	0	0	0	$f(\epsilon_2)$
	α	β	(1,1)	2	0	1	-1	$a_{\alpha\beta}$	$T_\alpha(-\epsilon_1)$
	α	β	(1,2)	2	0	0	-1	$a_{\alpha\beta}$	$T_\alpha(0)$
	α	β	(2,1)	2	0	1	-2	$a_{\alpha\beta}$	$T_\alpha(-\epsilon_2 - \epsilon_1)$
	α	β	(2,2)	2	0	0	-2	$a_{\alpha\beta}$	$T_\alpha(-\epsilon_2)$
$\sum_{\alpha}$ 									
	α	α	(1,1)	1	4	0	3	0	$f(4\epsilon_2)$
	α	α	(2,1)	1	4	0	2	0	$f(3\epsilon_2)$
	α	α	(3,1)	1	4	0	1	0	$f(2\epsilon_2)$
	α	α	(4,1)	1	4	0	0	0	$f(\epsilon_2)$
	α	β	(1,1)	1	0	0	-1	$a_{\alpha\beta}$	$T_\alpha(0)$
	α	β	(2,1)	1	0	0	-2	$a_{\alpha\beta}$	$T_\alpha(-\epsilon_2)$
	α	β	(3,1)	1	0	0	-3	$a_{\alpha\beta}$	$T_\alpha(-2\epsilon_2)$
	α	β	(4,1)	1	0	0	-4	$a_{\alpha\beta}$	$T_\alpha(-3\epsilon_2)$
$\sum_{\alpha\neq\beta}$ 									
	α	α	(1,1)	3	1	2	0	0	$f(\epsilon_2 - 2\epsilon_1)$
	α	α	(1,2)	3	1	1	0	0	$f(\epsilon_2 - \epsilon_1)$
	α	α	(1,3)	3	1	0	0	0	$f(\epsilon_2)$
	β	β	(1,1)	1	1	0	0	0	$f(\epsilon_2)$
	α	β	(1,1)	3	1	2	0	$a_{\alpha\beta}$	$f(a_{\alpha\beta} + \epsilon_2 - 2\epsilon_1)$
	α	β	(1,2)	3	0	1	-1	$a_{\alpha\beta}$	$f(a_{\alpha\beta} - \epsilon_1)$
	α	β	(1,3)	3	0	0	-1	$a_{\alpha\beta}$	$f(a_{\alpha\beta})$
	β	α	(1,1)	1	1	0	0	$a_{\beta\alpha}$	$f(a_{\beta\alpha} + \epsilon_2)$
	α	γ	(1,1)	3	0	2	-1	$a_{\alpha\gamma}$	$T_\alpha(-2\epsilon_1)/f(a_{\alpha\beta} - 2\epsilon_1)$
	α	γ	(1,2)	3	0	1	-1	$a_{\alpha\gamma}$	$T_\alpha(-\epsilon_1)/f(a_{\alpha\beta} - \epsilon_1)$
	α	γ	(1,3)	3	0	0	-1	$a_{\alpha\gamma}$	$T_\alpha(0)/f(a_{\alpha\beta})$
	β	γ	(1,1)	1	0	0	-1	$a_{\beta\gamma}$	$T_\beta(0)/f(a_{\beta\alpha})$

contd...

$\{Y_\lambda\}$	λ	$\tilde{\lambda}$	s	ν_{i_λ}	$\tilde{\nu}'_{j_\lambda}$	$h(s)$	$v(s)$	$a_{\lambda\tilde{\lambda}}$	$f(E(s))$
$\sum_{\alpha\neq\beta}$  $\oplus \square$									
	α	α	(1,1)	2	2	1	1	0	$f(2\epsilon_2-\epsilon_1)$
	α	α	(1,2)	2	1	0	0	0	$f(\epsilon_2)$
	α	α	(2,1)	1	2	0	0	0	$f(\epsilon_2)$
	β	β	(1,1)	1	1	0	0	0	$f(\epsilon_2)$
	α	β	(1,1)	2	1	1	0	$a_{\alpha\beta}$	$f(a_{\alpha\beta}+\epsilon_2-\epsilon_1)$
	α	β	(1,2)	2	0	0	-1	$a_{\alpha\beta}$	$f(a_{\alpha\beta})$
	α	β	(2,1)	1	1	0	-1	$a_{\alpha\beta}$	$f(a_{\alpha\beta})$
	β	α	(1,1)	1	2	0	1	$a_{\beta\alpha}$	$f(a_{\beta\alpha}+2\epsilon_2)$
	α	γ	(1,1)	2	0	1	-1	$a_{\alpha\gamma}$	$T_\alpha(-\epsilon_1)/f(a_{\alpha\beta}-\epsilon_1)$
	α	γ	(1,2)	2	0	0	-1	$a_{\alpha\gamma}$	$T_\alpha(0)/f(a_{\alpha\beta})$
	α	γ	(2,1)	1	0	0	-2	$a_{\alpha\gamma}$	$T_\alpha(-\epsilon_2)/f(a_{\alpha\beta}-\epsilon_2)$
	β	γ	(1,1)	1	0	0	-1	$a_{\beta\gamma}$	$T_\beta(0)/f(a_{\beta\alpha})$
$\sum_{\alpha\neq\beta}$  $\oplus \square$									
	α	α	(1,1)	1	3	0	2	0	$f(3\epsilon_2)$
	α	α	(2,1)	1	3	0	1	0	$f(2\epsilon_2)$
	α	α	(3,1)	1	3	0	0	0	$f(\epsilon_2)$
	β	β	(1,1)	1	1	0	0	0	$f(\epsilon_2)$
	α	β	(1,1)	1	1	0	0	$a_{\alpha\beta}$	$f(a_{\alpha\beta}+\epsilon_2)$
	α	β	(2,1)	1	1	0	-1	$a_{\alpha\beta}$	$f(a_{\alpha\beta})$
	α	β	(3,1)	1	1	0	-2	$a_{\alpha\beta}$	$f(a_{\alpha\beta}-\epsilon_2)$
	β	α	(1,1)	1	3	0	2	$a_{\beta\alpha}$	$f(a_{\beta\alpha}+3\epsilon_2)$
	α	γ	(1,1)	1	0	0	-1	$a_{\alpha\gamma}$	$T_\alpha(0)/f(a_{\alpha\beta})$
	α	γ	(2,1)	1	0	0	-2	$a_{\alpha\gamma}$	$T_\alpha(-\epsilon_2)/f(a_{\alpha\beta}-\epsilon_2)$
	α	γ	(3,1)	1	0	0	-3	$a_{\alpha\gamma}$	$T_\alpha(-2\epsilon_2)/f(a_{\alpha\beta}-2\epsilon_2)$
	β	γ	(1,1)	1	0	0	-1	$a_{\beta\gamma}$	$T_\beta(0)/f(a_{\beta\alpha})$
$\frac{1}{2}\sum_{\alpha\neq\beta}$  $\oplus$ 									
	α	α	(1,1)	2	1	1	0	0	$f(\epsilon_2-\epsilon_1)$
	α	α	(1,2)	2	1	0	0	0	$f(\epsilon_2)$
	β	β	(1,1)	2	1	1	0	0	$f(\epsilon_2-\epsilon_1)$
	β	β	(1,2)	2	1	0	0	0	$f(\epsilon_2)$
	α	β	(1,1)	2	1	1	0	$a_{\alpha\beta}$	$f(a_{\alpha\beta}+\epsilon_2-\epsilon_1)$
	α	β	(1,2)	2	1	0	0	$a_{\alpha\beta}$	$f(a_{\alpha\beta}+\epsilon_2)$
	β	α	(1,1)	2	1	1	0	$a_{\beta\alpha}$	$f(a_{\beta\alpha}+\epsilon_2-\epsilon_1)$
	β	α	(1,2)	2	1	0	0	$a_{\beta\alpha}$	$f(a_{\beta\alpha}+\epsilon_2)$
	α	γ	(1,1)	2	0	1	-1	$a_{\alpha\gamma}$	$T_\alpha(-\epsilon_1)/f(a_{\alpha\beta}-\epsilon_1)$
	α	γ	(1,2)	2	0	0	-1	$a_{\alpha\gamma}$	$T_\alpha(0)/f(a_{\alpha\beta})$
	β	γ	(1,1)	2	0	1	-1	$a_{\beta\gamma}$	$T_\beta(-\epsilon_1)/f(a_{\beta\alpha}-\epsilon_1)$
	β	γ	(1,2)	2	0	0	-1	$a_{\beta\gamma}$	$T_\beta(0)/f(a_{\beta\alpha})$

contd...

$\{Y_\lambda\}$	λ	$\tilde{\lambda}$	s	ν_{i_λ}	$\tilde{\nu}'_{j_\lambda}$	$h(s)$	$v(s)$	$a_{\lambda\tilde{\lambda}}$	$f(E(s))$
$\sum_{\alpha\neq\beta} \ydiagram{1,1} \oplus \ydiagram{2}$									
	α	α	(1,1)	1	2	0	1	0	$f(2\epsilon_2)$
	α	α	(2,1)	1	2	0	0	0	$f(\epsilon_2)$
	β	β	(1,1)	2	1	1	0	0	$f(\epsilon_2 - \epsilon_1)$
	β	β	(1,2)	2	1	0	0	0	$f(\epsilon_2)$
	α	β	(1,1)	1	1	0	0	$a_{\alpha\beta}$	$f(a_{\alpha\beta} + \epsilon_2)$
	α	β	(2,1)	1	1	0	-1	$a_{\alpha\beta}$	$f(a_{\alpha\beta})$
	β	α	(1,1)	2	2	1	1	$a_{\beta\alpha}$	$f(a_{\beta\alpha} + 2\epsilon_2 - \epsilon_1)$
	β	α	(1,2)	2	0	0	-1	$a_{\beta\alpha}$	$f(a_{\beta\alpha})$
	α	γ	(1,1)	1	0	0	-1	$a_{\alpha\gamma}$	$T_\alpha(0)/f(a_{\alpha\beta})$
	α	γ	(2,1)	1	0	0	-2	$a_{\alpha\gamma}$	$T_\alpha(-\epsilon_2)/f(a_{\alpha\beta} - \epsilon_2)$
	β	γ	(1,1)	2	0	1	-1	$a_{\beta\gamma}$	$T_\beta(-\epsilon_1)/f(a_{\beta\alpha} - \epsilon_1)$
	β	γ	(1,2)	2	0	0	-1	$a_{\beta\gamma}$	$T_\beta(0)/f(a_{\beta\alpha})$
$\frac{1}{2}\sum_{\alpha\neq\beta} \ydiagram{1,1} \oplus \ydiagram{1,1}$									
	α	α	(1,1)	1	2	0	1	0	$f(2\epsilon_2)$
	α	α	(2,1)	1	2	0	0	0	$f(\epsilon_2)$
	β	β	(1,1)	1	2	0	1	0	$f(2\epsilon_2)$
	β	β	(2,1)	1	2	0	0	0	$f(\epsilon_2)$
	α	β	(1,1)	1	2	0	1	$a_{\alpha\beta}$	$f(a_{\alpha\beta} + 2\epsilon_2)$
	α	β	(2,1)	1	2	0	0	$a_{\alpha\beta}$	$f(a_{\alpha\beta} + \epsilon_2)$
	β	α	(1,1)	1	2	0	1	$a_{\beta\alpha}$	$f(a_{\beta\alpha} + 2\epsilon_2)$
	β	α	(2,1)	1	2	0	0	$a_{\beta\alpha}$	$f(a_{\beta\alpha} + \epsilon_2)$
	α	γ	(1,1)	1	0	0	-1	$a_{\alpha\gamma}$	$T_\alpha(0)/f(a_{\alpha\beta})$
	α	γ	(2,1)	1	0	0	-2	$a_{\alpha\gamma}$	$T_\alpha(-\epsilon_2)/f(a_{\alpha\beta} - \epsilon_2)$
	β	γ	(1,1)	1	0	0	-1	$a_{\beta\gamma}$	$T_\beta(0)/f(a_{\beta\alpha})$
	β	γ	(2,1)	1	0	0	-2	$a_{\beta\gamma}$	$T_\beta(-\epsilon_2)/f(a_{\beta\alpha} - \epsilon_2)$

contd...

$\{Y_\lambda\}$	λ	$\tilde{\lambda}$	s	ν_{i_λ}	$\tilde{\nu}'_{j_\lambda}$	$h(s)$	$v(s)$	$a_{\lambda\tilde{\lambda}}$	$f(E(s))$
$\frac{1}{2}\sum_{\alpha\neq\beta\neq\gamma} \square\square \oplus \square \oplus \square$									
	α	α	(1,1)	2	1	1	0	0	$f(\epsilon_2-\epsilon_1)$
	α	α	(1,2)	2	1	0	0	0	$f(\epsilon_2)$
	β	β	(1,1)	1	1	0	0	0	$f(\epsilon_2)$
	γ	γ	(1,1)	1	1	0	0	0	$f(\epsilon_2)$
	α	β	(1,1)	2	1	1	0	$a_{\alpha\beta}$	$f(a_{\alpha\beta}+\epsilon_2-\epsilon_1)$
	α	β	(1,2)	2	0	0	-1	$a_{\alpha\beta}$	$f(a_{\alpha\beta})$
	α	γ	(1,1)	2	1	1	0	$a_{\alpha\gamma}$	$f(a_{\alpha\gamma}+\epsilon_2-\epsilon_1)$
	α	γ	(1,2)	2	0	0	-1	$a_{\alpha\gamma}$	$f(a_{\alpha\gamma})$
	β	α	(1,1)	1	1	0	0	$a_{\beta\alpha}$	$f(a_{\beta\alpha}+\epsilon_2)$
	β	γ	(1,1)	1	1	0	0	$a_{\beta\gamma}$	$f(a_{\beta\gamma}+\epsilon_2)$
	γ	α	(1,1)	1	1	0	0	$a_{\gamma\alpha}$	$f(a_{\gamma\alpha}+\epsilon_2)$
	γ	β	(1,1)	1	1	0	0	$a_{\gamma\beta}$	$f(a_{\gamma\beta}+\epsilon_2)$
	α	δ	(1,1)	2	0	1	-1	$a_{\alpha\delta}$	$T_\alpha(-\epsilon_1)/f(a_{\alpha\beta}-\epsilon_1)f(a_{\alpha\gamma}-\epsilon_1)$
	α	δ	(1,2)	2	0	0	-1	$a_{\alpha\delta}$	$T_\alpha(0)/f(a_{\alpha\beta})f(a_{\alpha\gamma})$
	β	δ	(1,1)	1	0	0	-1	$a_{\beta\delta}$	$T_\beta(0)/f(a_{\beta\alpha})f(a_{\beta\gamma})$
	γ	δ	(1,1)	1	0	0	-1	$a_{\gamma\delta}$	$T_\gamma(0)/f(a_{\gamma\alpha})f(a_{\gamma\beta})$

contd...

$\{Y_\lambda\}$	λ	$\tilde{\lambda}$	s	ν_{i_λ}	$\tilde{\nu}'_{j_{\tilde\lambda}}$	$h(s)$	$v(s)$	$a_{\lambda\tilde{\lambda}}$	$f(E(s))$
$\frac{1}{2}\sum_{\alpha\neq\beta\neq\gamma} \begin{smallmatrix}\square\\\square\end{smallmatrix} \oplus \square \oplus \square$									
	α	α	(1,1)	1	2	0	1	0	$f(2\epsilon_2)$
	α	α	(2,1)	1	2	0	0	0	$f(\epsilon_2)$
	β	β	(1,1)	1	1	0	0	0	$f(\epsilon_2)$
	γ	γ	(1,1)	1	1	0	0	0	$f(\epsilon_2)$
	α	β	(1,1)	1	1	0	0	$a_{\alpha\beta}$	$f(a_{\alpha\beta}+\epsilon_2)$
	α	β	(2,1)	1	1	0	-1	$a_{\alpha\beta}$	$f(a_{\alpha\beta})$
	α	γ	(1,1)	1	1	0	0	$a_{\alpha\gamma}$	$f(a_{\alpha\gamma}+\epsilon_2)$
	α	γ	(2,1)	1	1	0	-1	$a_{\alpha\gamma}$	$f(a_{\alpha\gamma})$
	β	α	(1,1)	1	2	0	1	$a_{\beta\alpha}$	$f(a_{\beta\alpha}+2\epsilon_2)$
	β	γ	(1,1)	1	1	0	0	$a_{\beta\gamma}$	$f(a_{\beta\gamma}+\epsilon_2)$
	γ	α	(1,1)	1	2	0	1	$a_{\gamma\alpha}$	$f(a_{\gamma\alpha}+2\epsilon_2)$
	γ	β	(1,1)	1	1	0	0	$a_{\gamma\beta}$	$f(a_{\gamma\beta}+\epsilon_2)$
	α	δ	(1,1)	1	0	0	-1	$a_{\alpha\delta}$	$T_\alpha(0)/f(a_{\alpha\beta})f(a_{\alpha\gamma})$
	α	δ	(2,1)	1	0	0	-2	$a_{\alpha\delta}$	$T_\alpha(-\epsilon_2)/f(a_{\alpha\beta}-\epsilon_2)f(a_{\alpha\gamma}-\epsilon_2)$
	β	δ	(1,1)	1	0	0	-1	$a_{\beta\delta}$	$T_\beta(0)/f(a_{\beta\alpha})f(a_{\beta\gamma})$
	γ	δ	(1,1)	1	0	0	-1	$a_{\gamma\delta}$	$T_\gamma(0)/f(a_{\gamma\alpha})f(a_{\gamma\beta})$

contd...

$\{Y_\lambda\}$	λ	$\tilde{\lambda}$	s	ν_{i_λ}	$\tilde{\nu}'_{j_\lambda}$	$h(s)$	$v(s)$	$a_{\lambda\tilde{\lambda}}$	$f(E(s))$
$\frac{1}{4!}\sum_{\alpha\neq\beta\neq\gamma\neq\delta} \square\oplus\square\oplus\square\oplus\square$									
	α	α	(1,1)	1	1	0	0	0	$f(\epsilon_2)$
	β	β	(1,1)	1	1	0	0	0	$f(\epsilon_2)$
	γ	γ	(1,1)	1	1	0	0	0	$f(\epsilon_2)$
	δ	δ	(1,1)	1	1	0	0	0	$f(\epsilon_2)$
	α	β	(1,1)	1	1	0	0	$a_{\alpha\beta}$	$f(a_{\alpha\beta}+\epsilon_2)$
	α	γ	(1,1)	1	1	0	0	$a_{\alpha\gamma}$	$f(a_{\alpha\gamma}+\epsilon_2)$
	α	δ	(1,1)	1	1	0	0	$a_{\alpha\delta}$	$f(a_{\alpha\delta}+\epsilon_2)$
	β	α	(1,1)	1	1	0	0	$a_{\beta\alpha}$	$f(a_{\beta\alpha}+\epsilon_2)$
	β	γ	(1,1)	1	1	0	0	$a_{\beta\gamma}$	$f(a_{\beta\gamma}+\epsilon_2)$
	β	δ	(1,1)	1	1	0	0	$a_{\beta\delta}$	$f(a_{\beta\delta}+\epsilon_2)$
	γ	α	(1,1)	1	1	0	0	$a_{\gamma\alpha}$	$f(a_{\gamma\alpha}+\epsilon_2)$
	γ	β	(1,1)	1	1	0	0	$a_{\gamma\beta}$	$f(a_{\gamma\beta}+\epsilon_2)$
	γ	δ	(1,1)	1	1	0	0	$a_{\gamma\delta}$	$f(a_{\gamma\delta}+\epsilon_2)$
	δ	α	(1,1)	1	1	0	0	$a_{\delta\alpha}$	$f(a_{\delta\alpha}+\epsilon_2)$
	δ	β	(1,1)	1	1	0	0	$a_{\delta\beta}$	$f(a_{\delta\beta}+\epsilon_2)$
	δ	γ	(1,1)	1	1	0	0	$a_{\delta\gamma}$	$f(a_{\delta\gamma}+\epsilon_2)$
	α	ρ	(1,1)	1	0	0	-1	$a_{\alpha\rho}$	$T_\alpha(0)/f(a_{\alpha\beta})f(a_{\alpha\gamma})f(a_{\alpha\delta})$
	β	ρ	(1,1)	1	0	0	-1	$a_{\beta\rho}$	$T_\beta(0)/f(a_{\beta\alpha})f(a_{\beta\gamma})f(a_{\beta\delta})$
	γ	ρ	(1,1)	1	0	0	-1	$a_{\gamma\rho}$	$T_\gamma(0)/f(a_{\gamma\alpha})f(a_{\alpha\beta})f(a_{\gamma\delta})$
	δ	ρ	(1,1)	1	0	0	-1	$a_{\delta\rho}$	$T_\delta(0)/f(a_{\delta\alpha})f(a_{\delta\beta})f(a_{\delta\gamma})$

Table B.4: Young's tableau at the four instanton level.

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