On the geometry related to jump processes: Investigating transition functions of Levy and Levy-type processes.

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On the Geometry Related to Jump Processes –

Investigating Transition Functions of Lévy and Lévy-type Processes

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DECLARATION

This work has not previously been accepted in substance for any degree and is not being concurrently submitted in candidature for any degree.

Signed

Date ..29 March 2010........

STATEMENT 1

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STATEMENT 2

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Abstract

In this thesis, we study some geometrical aspects of metric measure spaces $(\mathbb{R}^n, \psi^{1/2}, \mu)$, where μ is a locally finite regular Borel measure and $\psi^{1/2}$ a metric on $\mathbb{R}^n$ which arises from a continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{R}$ which satisfies $\psi(\xi) \geqslant 0$ with $\psi(\xi) = 0$ if, and only if, $\xi = 0$. This study is motivated by the investigation of a transition density estimate for pure jump processes on a general metric measure space. To gain a better insight into the behaviour of transition functions of symmetric Lévy processes in this general setting, it seems desirable to understand geometrical properties of their underlying state spaces. More precisely, we show completeness of the metric spaces $(\mathbb{R}^n, \psi^{1/2})$ and study under which circumstances open balls $B^\psi(x,r)$, $x \in \mathbb{R}^n$, $r > 0$, with respect to this metric are convex. Moreover, we focus on conditions of the metric measure spaces $(\mathbb{R}^n, \psi^{1/2}, \mu)$ for the balls to satisfy the volume growth property

$$\mu(B^\psi(x,R)) \leqslant C_\psi(r,R)\, \mu(B^\psi(x,r))$$

for μ-almost all $x \in \mathbb{R}^n$, $0 < r < R$ and a constant $C_\psi(r,R) \geqslant 1$. Finally, we show that the homogeneity property of a metric measure space can be applied to our case and provide some results associated with the construction of a Hajłasz-Sobolev space over $(\mathbb{R}^n, \psi^{1/2}, \lambda^{(n)})$, where $\lambda^{(n)}$ denotes the n-dimensional Lebesgue measure.

Contents

Introduction

In this thesis, we study some geometrical aspects of metric measure spaces $(\mathbb{R}^n, \psi^{1/2}, \mu)$, where μ is a locally finite regular Borel measure and $\psi^{1/2}$ a metric on $\mathbb{R}^n$ which arises from a real-valued continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{R}$ satisfying $\psi(\xi) = 0$ if, and only if, $\xi = 0$. Due to a theorem by I.J. Schoenberg, the function $\psi^{1/2}$ indeed gives an appropriate metric in order to study the behaviour of Lévy and Lévy-type processes in terms of estimates of their transition functions. In particular, the volume of metric balls is an essential part in the study of pointwise transition function estimates.

It has been a well-studied topic in the past years to obtain upper and lower bounds for transition functions of certain stochastic processes. The interest in estimates has not been restricted to $\mathbb{R}^n$ only, but has included Riemannian manifolds and general metric measure spaces. To give a brief overview about the contributions to this field we start with a diffusion process $(X_t)_{t\geqslant 0}$, i.e. a Markov process with continuous trajectories, on $\mathbb{R}^n$. A prototype of a diffusion process is the Brownian motion. Starting at some point $x \in \mathbb{R}^n$ its transition probability is given by

$$\mathbb{P}^x(X_t \in A) = \int_A p_t(x, y)\, dy \tag{1}$$

where $A \subset \mathbb{R}^n$ is a Borel set and

$$p_t(x, y) := p_t(x - y) = (4\pi t)^{-n/2} \exp\left(-\frac{|x - y|^2}{4t}\right). \tag{2}$$

Equation (1) can be read as the probability that the process $(X_t)_{t\geqslant 0}$ hits the set A at a specified time t. From an analytic point of view, transition functions of diffusion processes are governed by convection-diffusion equations. In fact, (2) is the fundamental solution to the heat equation $\partial_t u = \Delta u$. More precisely, the integral

$$u(t, x) = \int_{\mathbb{R}^n} p_t(x, y)\, v(y)\, dy$$

for a continuous bounded function v on $\mathbb{R}^n$ is the solution of the Cauchy problem $\partial_t u = \Delta u$ with the initial condition $u(0, x) = v(x)$. Adopting the viewpoint of functional calculus, p_t can be understood as the integral kernel of the operator $e^{-t\Delta}$ and the solution can be written as $u(t, \cdot) = \exp(-\Delta t)\, v$. Due to its connection to the heat equation $p_t(x, y)$ is sometimes also called the *heat kernel*. In particular, (2) is often named the *Gaussian heat kernel* and the Laplacian the *generator* of the underlying stochastic process, which is the Wiener process.

In this work we will focus on Lévy processes of jump-type, which are generated by certain pseudo-differential operators. Lévy processes are a special case of Feller processes which themselves form a sub-class of Markov processes. A Lévy process is a stochastically continuous process $(X_t)_{t\geqslant 0}$ which has independent and stationary increments. Its generator $-\psi(D)$ can be represented as a pseudodifferential operator of the form

$$-\psi(D)u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\, \psi(\xi)\, \hat{u}(\xi)\, d\xi \tag{3}$$

for $u \in \mathcal{S}(\mathbb{R}^n)$, by which we denote the Schwartz space over $\mathbb{R}^n$. A special role is played by the symbol $\psi : \mathbb{R}^n \to \mathbb{C}$ of such a pseudodifferential operator. Every Lévy process $(X_t)_{t\geqslant 0}$ with starting point $x \in \mathbb{R}^n$ is determined by its characteristic function

$$\mathbb{E}^x[e^{i(X_t - x)\cdot\xi}] = e^{-t\overline{\psi(\xi)}}, \quad t \geqslant 0,$$

where $\psi : \mathbb{R}^n \to \mathbb{C}$ is a continuous negative definite function and the symbol of the associated generator. The function $\overline{\psi}$ is also called the *characteristic exponent* of the Lévy process, and ψ has the Lévy-Khinchin representation

$$\psi(\xi) = c + i(d\cdot\xi) + q(\xi) + \int_{\mathbb{R}^n\setminus\{0\}} \left(1 - e^{-ix\cdot\xi} - \frac{ix\cdot\xi}{1+|\xi|^2}\right) \frac{1+|x|^2}{|x|^2}\, \mu(dx),$$

where $c \geqslant 0$, $d \in \mathbb{R}^n$, $q \geqslant 0$ a positive semidefinite quadratic form on $\mathbb{R}^n$, and μ a finite measure on $\mathbb{R}^n \setminus \{0\}$. For more details and a proof of this formula see [45]. In this work we will restrict ourselves to symmetric Lévy processes in which case $\psi(\xi)$ is real-valued. Moreover, we will only consider Lévy processes $(X_t)_{t\geqslant 0}$ which have no diffusion part, i.e. they are of pure jump. Then the Lévy-Khinchin representation of $\psi : \mathbb{R}^n \to \mathbb{R}$ reduces to

$$\psi(\xi) = \int_{\mathbb{R}^n\setminus\{0\}} (1 - \cos(y\cdot\xi))\, \nu(dy)$$

with the Lévy measure ν associated with the continuous negative definite function ψ, which satisfies the integrability condition $\int_{\mathbb{R}^n\setminus\{0\}} (|x|^2 \wedge 1)\, \nu(dx) < \infty$.

We assume that the Lévy process $(X_t)_{t\geqslant 0}$ possesses a density, which is of the form

$$p_t(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\, e^{-t\psi(\xi)}\, d\xi$$

with a continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{R}$ such that the characteristic function $e^{-t\psi(\cdot)} \in L^1(\mathbb{R}^n)$. For some continuous negative definite functions $\psi(\xi)$, the density $p_t(x)$ can be given in a concrete form as in the diffusion case $\psi(\xi) = |\xi|^2$, where the density is the Gaussian (2). For instance, the negative definite function $\psi_C(\xi) = |\xi|$ is related to the Cauchy process with the corresponding density

$$p_t^C(x) = \Gamma\Big(\frac{n+1}{2}\Big) \frac{t}{(\pi(t^2 + |x|^2))^{\frac{n+1}{2}}}, \tag{4}$$

and $\psi_H(\xi) = \sqrt{|\xi|^2 + m^2} - m$ for $m > 0$ yields a relativistic stable process with density

$$p_t^H(x) = 2(2\pi)^{-\frac{n+1}{2}} m^{\frac{n+1}{2}}\, e^{mt}\, t\, (|x|^2 + t^2)^{-\frac{n+1}{2}}\, K_{\frac{n+1}{2}}(m\sqrt{|x|^2 + t^2}). \tag{5}$$

We will refer to these density functions in § 3. These densities are related to continuous negative definite functions which satisfy

$$\frac{\psi_H(\xi)}{\psi_C(\xi)} \to 1 \quad \text{as } |\xi| \to \infty,$$

hence they are in particular asymptotically comparable. The transition densities themselves, however, show a rather different qualitative behaviour. For example, the heavy tails of the Cauchy density do not allow $x\,p_t^C(x)$ to be in $L^1(\mathbb{R}^n)$, whereas we have $x\,p_t^H(x) \in L^p(\mathbb{R}^n)$ for any $p \geqslant 1$, i.e. all absolute moments of the underlying process exist. This already is an indication that it is desirable to gain a better understanding of the behaviour of transition functions of Lévy processes.

In the Hilbert space setting $L^2(\mathbb{R}^n)$ transition density estimation has a close connection to symmetric Dirichlet forms. For analytic and probabilistic aspects of the extensive theory of Dirichlet forms see [52] or [26, 27]. Starting with a symmetric Markov process with generator $-\psi(D)$ as given in (3), which is a symmetric operator, we can introduce a bilinear form, the associated *Dirichlet form*, by

$$\mathcal{E}(u,v) := (\psi(D)^{1/2}u, \psi(D)^{1/2}v)_{L^2} = \int_{\mathbb{R}^n} \psi(D)\, u(x)\, \overline{v(x)}\, dx$$

for $v \in \mathrm{D}(\psi(D)^{1/2})$ and $u \in \mathrm{D}(\psi(D)^{1/2}) \cap \mathrm{D}(\psi(D))$. A characterisation of the domains of $\psi(D)$ in terms of function spaces was introduced by Farkas, Jacob and Schilling [25] and will be discussed in § 1.5. If $-\psi(D)$ is a differential operator $\mathcal{E}(u,v)$ is called a *local Dirichlet form*, whereas in the case of $-\psi(D)$ being an integral operator involving difference terms, which is the case for pure jump processes, the corresponding form $\mathcal{E}(u,v)$ is called a *non-local Dirichlet form*. Dirichlet forms of either type have been employed frequently in the study of heat kernel estimates in the L^2-setting as they provide a powerful technique to handle symmetric Markov semigroups and their corresponding processes.

Estimation of heat kernels has its origins in the works of Nash [54] and Aronson [2], both in the context of elliptic and parabolic equations given in divergence form. From the probabilistic point of view, it has become of interest to estimate the transition density of a process on a general metric measure space (X,d,μ), where (X,d) is a metric space and μ a Radon measure on X. In this setting, Sturm [58] has presented a method for constructing a diffusion process on X for quasi-every starting point using Dirichlet form techniques. In the case of a diffusion process on a metric measure space (X,d,μ) the aim is to get a Gaussian estimate of the form

$$p_t(x,y) \approx \frac{C_1}{\mu(B^d(x,\sqrt{t}))} \exp\left(-\frac{d^2(x,y)}{C_2\, t}\right), \tag{6}$$

where $B^d(x,r)$ denotes an open ball with respect to the metric $d(x,y)$ of radius $r > 0$ centred at the point $x \in X$. Here, $\approx$ stands for the existence of an upper and lower bound. For example, in $(\mathbb{R}^n, |\cdot|, \lambda^{(n)})$ this reads as

$$p_t(x,y) \approx C_1\, t^{-n/2} \exp\left(-\frac{|x-y|^2}{C_2\, t}\right).$$

In a series of papers [3, 30, 31, 32, 36], Grigor'yan et al. discuss heat kernel estimates of self-similar type, as they appear in, e.g., fractals. They study bounds of the form

$$p_t(x,y) \approx t^{-\alpha/\beta}\, \Phi\left(\frac{d(x,y)}{t^{1/\beta}}\right) \tag{7}$$

with parameters α and β and a function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ on general metric measure spaces (X,d,μ). Under certain conditions on the space (X,d,μ), and under the assumption that a heat kernel p_t satisfies (7) for μ-almost all $x,y \in X$, it is shown in [31] that the following scenarios occur. In the case of diffusions on (X,d,μ), which corresponds to a local operator as generator of the process, the function Φ and the parameter α and β satisfy the exponential estimate

$$\Phi(s) \approx c_1 \exp\left(-c_2\, s^{\frac{\beta}{\beta-1}}\right), \quad 2 \leqslant \beta \leqslant \alpha+1,$$

whereas Φ is of the polynomial nature

$$\Phi(s) \approx (1+s)^{-(\alpha+\beta)}, \quad 0 < \beta \leqslant \alpha+1$$

in the non-local case. Recently, Grigor'yan, Hu and Lau [35] studied the estimate (7) under the assumption on the volume of metric balls in (X,d,μ) to be of doubling nature. For an overview of various equivalent characterisations of heat kernel upper estimates for regular local Dirichlet forms on volume doubling metric measure spaces see [33].

Besides metric measure spaces see e.g. [31] for a survey about heat kernel estimates on Riemannian manifolds for the Laplace-Beltrami operator Δ. Here, we name in particular the works of Li and Yau [51], who, under certain assumptions on certain manifolds, derived an estimate of the form (6), where $d(x, y)$ is understood as the geodesic distance of x and y on the manifold. Results for more general manifolds have also been achieved by Davies [19] and Grigor'yan [29]. Their contributions technically work under a certain gradient assumption on the geodesic distance d on the manifold, which is a property that typically fails in general metric measure spaces.

In the context of symmetric Markov processes a different approach has been taken by Davies [17, 18], Varopoulos, Saloff-Coste and Coulhon [62], and Carlen, Kusuoka and Stroock [10]. In our language, on a complete separable metric measure space (X, d, μ) they used Dirichlet form techniques provided in $L^2(X, \mu)$ to derive the uniform upper bound

$$p_t(x, y) \leqslant C\, t^{-n/2} \tag{8}$$

for its transition density from Sobolev- and Nash-type inequalities. For the transition density of the Brownian motion (8) is fulfilled in particular if $x = y$. Therefore, it has become customary to call this upper bound an *on-diagonal estimate.* We will discuss these results in some more detail in § 3.2. Moreover, Davies has given a method in [17] how to obtain a decay estimate for p_t if $x \neq y$, which is actually equivalent to (8).

In recent years research has been extended towards heat kernel estimates of symmetric jump processes corresponding to non-local Dirichlet forms. In [48] Kolokoltsov obtained heat kernel estimates for stable-like processes in the Euclidean space $\mathbb{R}^n$. More recently, in [4, 5, 6] estimates of transition density functions of processes have been derived which are related to non-local operators of fixed order. In the case of non-local operators of variable order contributions have been made by Chen, Kim and Kumagai in [11, 12, 13, 14]. In [13] Chen and Kumagai, see also [14, 12], introduced an estimate for the transition function of a symmetric jump process corresponding to fractional Laplacian-like operators of variable order. An example for such a process is the relativistic stable process, whose density function is given by (5). Under the assumption of a uniform volume doubling property of the measure μ in (X, d, μ) they obtain the estimate

$$p_t(x) \approx c_1 \left(\frac{1}{\mu(B^d(0, \phi^{-1}(t)))} \wedge \frac{t}{\mu(B^d(0, d(x, 0)))\, \phi(c_2\, d(x, 0))} \right) \tag{9}$$

by employing a strictly increasing continuous function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\phi(0) = 0$ and $\phi(1) = 1$, which can in fact be obtained from the jump intensity of the process under consideration. For two examples the performance of the upper bound provided by (9) are studied in § 3. It turns out that on the diagonal $x = 0$ for all $t > 0$ and also asymptotically for large t and x, the estimate (9) approximates the actual density very well. However, for moderate values of t and x it does not yield a good model of the shape of the transition function. This gives again an indication that it is desirable to gain a better understanding of the behaviour of p_t in geometric terms.

To sum up, in the diffusion processes case on (X, d, μ), the aim is to get estimates of the Gaussian form (6) as noted above. For the case of pure jump processes obtaining heat kernel estimates for p_t involving geometric terms first requires a link between the metric d and and the process under consideration. Here, a result by Schoenberg [55, 56], see also [8], comes into play. It ensures that a metric space (X, d) isometrically embeds into Hilbert space $(H, \langle \cdot, \cdot \rangle_H)$ if, and only if, for $x, y \in M$ the metric satisfies $d^2(x, y) = \psi(x - y)$ with a continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{R}$. We show in § 3 that $\sqrt{\psi}$ indeed gives a metric provided $\psi : \mathbb{R}^n \to \mathbb{R}$ is continuous and negative definite with $\psi(\xi) = 0$ if, and only if, $\xi = 0$. As a consequence, if we equip the set X with

$d(x,y) = \psi^{1/2}(x-y)$ we can embed the metric space $(X, \psi^{1/2})$ into a Hilbert space while keeping the distances. Eventually, this ensures that we can measure distances with the help of the inner product $\langle \cdot, \cdot \rangle$ on the Hilbert space. More precisely, for $x, y \in X$, paths $\gamma : [0,1] \to X$, $\gamma(0) = x$, $\gamma(1) = y$, on the metric space $(X, \psi^{1/2})$ have arclengths

$$\ell(\gamma) = \int_0^1 \langle \dot\gamma(t), \dot\gamma(t) \rangle_H^{1/2} \, dt = \int_0^1 \psi^{1/2}(\dot\gamma(t)) \, dt \, .$$

The aim of this work is to study the geometry in the metric spaces $(X, \psi^{1/2})$ involving a real-valued continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{R}$ giving rise to a metric due to its properties. This should give a basis for the analysis of transition functions p_t of jump processes in geometric terms. Very recently, V. Knopova and R. Schilling [49] obtained an on-diagonal upper bound for the heat kernel in the symmetric jump processes case involving the measure $\lambda^{(n)}(B^\psi(0, \sqrt{t}))$ of a ball with respect to the metric $\psi^{1/2}$, where ψ is continuous and negative definite.

The outline of this work is as follows. In § 1 we provide the analytic and probabilistic background, introducing the concepts of continuous negative definite functions, strongly continuous contraction semigroups and their relation to Lévy processes. We also introduce translation invariant symmetric Dirichlet forms which appear in the context of estimation of transition density functions of symmetric diffusion and jump processes in Hilbert space settings. The first paragraph is mainly based on [44, 45, 25, 18, 62]. As a new result, we connect the notion of a weighted function space by Triebel [59] to the definition of a symbol class by Hoh [42, 43], see also [46]. We show in § 1.7 that the latter is a class of admissible weight functions in the sense of Triebel.

In § 2 we collect some analytic results in general metric spaces as provided in the monograph [40]. It covers, amongst other topics, the Vitali Covering Theorem and function spaces over metric spaces, in particular the introduction of Sobolev spaces on metric spaces. The latter needs a formulation that is suitable for spaces on which a weak derivative is a priori not defined. In particular, we will concentrate on the definition of Sobolev spaces according to Hajłasz [37].

In § 3 we study transition function estimates in various settings. We show how Dirichlet forms play a role in the setting of symmetric Markov processes on $L^2(\mathbb{R}^n; \mathbb{R})$ and discuss the approach taken by Davies [17, 18], Varopoulos, Saloff-Coste and Coulhon [62], and Carlen, Kusuoka and Stroock [10] to derive heat kernel estimates from Sobolev- and Nash-type inequalities. We then turn to the topic of transition function estimation on more general spaces and give a brief survey about results obtained by Sturm [58] and Grigor'yan et al., where we mainly rely on the surveys [30, 31]. Finally, we focus on the heat kernel estimate for jump processes proposed by Chen and Kumagai [13]. We use MATHEMATICA as a tool to compute the upper bound in (9) for p_t^C and p_t^H given by (4) and (5), respectively, and to visualise the behaviour of the upper heat kernel for $t > 0$. The results provide a motivation to study geometrical aspects related to Lévy processes of jump type more closely.

As motivated by the result of Schoenberg which we give in § 3.4, we equip $\mathbb{R}^n$ with the metric $\psi^{1/2}$, where $\psi : \mathbb{R}^n \to \mathbb{R}$ is continuous and negative definite with $\psi(\xi) = 0$ if, and only if, $\xi = 0$, and we study properties of the metric measure spaces $(\mathbb{R}^n, \psi^{1/2}, \mu)$ in § 4. In particular, we show the completeness of the metric spaces $(\mathbb{R}^n, \psi^{1/2})$ and study under which circumstances open balls $B^\psi(x, \rho)$, $x \in \mathbb{R}^n$, $\rho > 0$, with respect to the metric $\psi^{1/2}$ are convex. We then focus on the volume growth condition in the metric measure space. In particular, we fix a metric $d_0 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and a measure μ_0 on the σ-algebra $\mathcal{B}(\mathbb{R}^n, d_0)$ of Borel sets on $\mathbb{R}^n$ with respect to d_0, such that open balls in $(\mathbb{R}^n, d_0, \mu_0)$ satisfy the growth condition

$$\mu_0(B^{d_0}(x, R)) \leqslant C_{d_0}(r, R) \, \mu_0(B^{d_0}(x, r))$$

for two radii $0 < r < R$, a constant $C_{d_0}(r, R) \geqslant 1$ and μ_0-almost all $x \in \mathbb{R}^n$. We study how a metric $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is related to d_0, and a measure μ is connected to μ_0 such that the spaces $(\mathbb{R}^n, d, \mu_0)$ and $(\mathbb{R}^n, d_0, \mu)$ also satisfy the volume growth condition, respectively. The results obtained can be applied to metric measure spaces equipped with a metric that arises from a continuous negative definite function. Moreover, based on the homogeneity property of a metric measure space in Chater III of [15] we note in § 4.3 that this concept applies to our settings. Finally, we provide some results on the construction of a Hajłasz-Sobolev space $M^{1,p}(\mathbb{R}^n, \psi^{1/2}, \lambda^{(n)})$ in § 4.4. We prove, analogously to the general case in [40], that $M^{1,p}(\mathbb{R}^n, \psi^{1/2}, \lambda^{(n)})$ is a Banach space and show that for $u \in M^{1,p}(\mathbb{R}^n, \psi^{1/2}, \lambda^{(n)})$ a Poincaré-type inequality holds, which is an adaptation of a result in [39].

Index of Notation

$\mathbb{N}$ set of natural numbers
$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$
$\mathbb{N}_0^n$ set of all multi-indices
$\mathbb{R}$ set of all real numbers
$\mathbb{R}_+ = \{x \in \mathbb{R} : x \geqslant 0\}$
$\mathbb{R}^n$ Euclidean vector space
$\mathbb{C}$ set of all complex numbers
$B^d(x, r)$ open ball of radius r with respect to metric d
$D(z_0, r) := \{z \in \mathbb{C} : |z - z_0| \leqslant r\}$ closed disk in the complex plane about z_0
$a \wedge b = \min\{a, b\}$
$a \vee b = \max\{a, b\}$
$A \setminus B$ set theoretical difference of two sets
A^c complement of the set A
$\mathrm{conv}(A)$ convex hull of a set A
$\mathrm{diam}(A)$ diameter of a set A
$\bar{A}$ closure of a set A
$\mathring{A}$ interior of a set A
$\mathcal{A}$ σ-algebra
$(\Omega, \mathcal{A})$ measurable space
$(\Omega, \mathcal{A}, \mu)$ measure space
$\mathcal{B}(X, d)$ set of Borel sets in metric space (X, d)
$\sigma(\mathcal{O})$ σ-algebra generated by $\mathcal{O}$
$\mathcal{B}^{(n)}$, $\mathcal{B}(\mathbb{R}^n, |\cdot|)$ set of Borel sets in $(\mathbb{R}^n, |\cdot|)$

$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$ characteristic function of A
$u_{|A}$ restriction of u to A
$u^+ = u \vee 0$ positive part of u
$u^- = -(u \vee 0)$ negative part of u
$\mathrm{sgn}\, u = \begin{cases} \frac{u}{|u|}, & u \neq 0 \\ 0, & u = 0 \end{cases}$ sign of u
$\mathrm{Re}\, u$ real part of complex-valued u
$\mathrm{Im}\, u$ imaginary part of complex-valued u
$(u_\nu)_{\nu \in \mathbb{N}}$ sequence of functions u_ν
$u \circ v$ composition of functions
$u * v$ convolution of functions
$D^\alpha u = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$
K_τ Bessel function of second kind and order τ

$\hat{u}$, $\mathcal{F}u$ Fourier transform of u
$\mathcal{F}^{-1}u$ inverse Fourier transform of u
$\mathcal{L}u$ Laplace transform of u
$\operatorname{supp}(u)$ support of function u
$\mathsf{R}(u)$ range of a function u
$\ker(\phi)$ kernel of a mapping ϕ

$\lambda^{(n)}$ Lebesgue measure in $\mathbb{R}^n$
μ regular Borel or Radon measure
ε_a Dirac measure at $a \in \mathbb{R}^n$
$\operatorname{cap}(A)$ capacity of set A
$\operatorname{supp}(\mu)$ support of a measure μ
$(\mu_t)_{t\geqslant 0}$ convolution semigroup of sub-probability measures
$\int_{a+}^{b} = \int_{(a,b]}$
$⨍_A [\ldots]\, d\mu = \frac{1}{\mu(A)} \int_A [\ldots]\, d\mu$

(X,d) metric space with metric $d(\cdot,\cdot)$ on $X \times X$
(X,d,μ) metric measure space with metric d and measure μ
$B_b(\Omega)$ space of bounded Borel measurable functions
$C(G)$ space of continuous functions on G
$C_0(G)$ space of continuous functions on G with compact support
$C_\infty(G)$ space of continuous functions on G vanishing at infinity
$C_b(G)$ space of bounded continuous functions on G
$C^k(G)$ space of k-times continously differentiable functions on G
$C_0^k(G) = C^k(G) \cap C_0(G)$
$C^\infty(G)$ space of smooth functions on G
$C^\infty(\mathbb{R}^n;\mathbb{K})$ space of smooth functions on $\mathbb{R}^n$ with values in $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$
$C_0^\infty(G) = C^\infty(G) \cap C_0(G)$
$L^p(\Omega,\mu)$ Lebesgue space of μ-measurable functions on Ω
$L^p(G) = L^p(G,\lambda^{(n)})$
$L^p(\mathbb{R}^n;\mathbb{K})$ Lebesgue space of functions on $\mathbb{R}^n$ with values in $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$
$L^p_{\mathrm{loc}}(G) = \{u \in B(G) : u_{|K} \in L^p(G) \text{ for all } K \subset G \text{ compact}\}$
$W^{m,p}(G)$ classical Sobolev space
$H^m(G) = W^{m,2}(G)$
$H_2^s(G)$ classical Bessel potential space over G
$H_p^{\psi,s}(G)$ ψ-Bessel potential space over G of order s with respect to L^p
$H_p^{\psi,s}(\mathbb{R}^n;\mathbb{K})$ ψ-Bessel potential space of $\mathbb{K}$-valued functions on $\mathbb{R}^n$ with $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$
$\mathcal{S}(\mathbb{R}^n;\mathbb{K})$ Schwartz space of $\mathbb{K}$-valued functions on $\mathbb{R}^n$ with $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$

$(X,\|\cdot\|_X)$ Banach space with norm $\|\cdot\|_X$
X^* dual space of a topological vector space
$X \hookrightarrow Y$ continuous embedding of X into Y
$\mathsf{D}(A)$ domain of operator A
$\mathsf{R}(A)$ range of operator A
$(A,\mathsf{D}(A))$ linear operator with domain $\mathsf{D}(A)$
$\operatorname{gr}(A)$ graph of operator A
$R_\lambda = (\lambda - A)^{-1}$ resolvent of operator A

$\rho(A)$ resolvent set of operator A
$\sigma(A)$ spectrum of operator A
$\psi(D)$ pseudodifferential operator with symbol $\psi(\xi)$
$(T_t)_{t\geqslant 0}$ (one parameter) semigroup of operators
$(T_t^{(p)})_{t\geqslant 0}$ (one parameter) semigroup of operators on $L^p(\mathbb{R}^n)$, $1 < p < \infty$
$(T_t^{(\infty)})_{t\geqslant 0}$ (one parameter) semigroup of operators on $C_\infty(\mathbb{R}^n)$
$A^{(p)}$ generator of $(T_t^{(p)})_{t\geqslant 0}$
$A^{(\infty)}$ generator of $(T_t^{\infty})_{t\geqslant 0}$

$\ell(\gamma)$ length of a curve γ
$d(\cdot,\cdot)$ distance in arbitrary metric space
$|\cdot|$ Euclidean distance on $\mathbb{R}^n$
$\|u\|_X$ norm of u in the space X
$\|u\|_A = \|u\|_X + \|Au\|_X$ graph norm of A
$\|A\| = \|A\|_{X-Y}$ operator norm of $A : X \to Y$
$\mathcal{E}(\cdot,\cdot)$ Dirichlet form
$(\cdot,\cdot)_H, \langle\cdot,\cdot\rangle_H$ inner product in the Hilbert space H

Introductory Concepts

§1 Preliminaries

This first paragraph has an introductory character and provides the necessary analytic and probabilistic background. We start with some basic concepts formulated on general metric spaces (X, d). We further collect definitions and well-known results on continuous positive and negative definite functions on $\mathbb{R}^n$ that are related to certain one-parameter semigroups of operators which themselves can be associated to particular stochastic processes. In particular we focus on semigroups of Feller and sub-Markovian operators and their generators which are pseudodifferential operators. At the end of this paragraph we show that the symbol class studied by Hoh in [42, 43] is a class of admissible weight functions in the sense of Triebel [59].

1.1 Basic concepts

We will start with summarising some basic notation from Analysis and Measure Theory.

Definition 1.1 (Metric space). A *metric space* is a pair (X, d) that consists of a set $X \neq \emptyset$ and a distance function $d(x, y)$, which is a real-valued function on $X \times X$ such that $d(x, y) \geqslant 0$ for all $x, y \in X$, $d(x, y) = 0$ if, and only if, $x = y$, the symmetry condition $d(x, y) = d(y, x)$ for all $x, y \in X$ as well as the triangle inequality

$$d(x, y) \leqslant d(x, z) + d(z, y) \quad \text{for all } x, y, z \in X \tag{1.1}$$

hold. The mapping $d : X \times X \to \mathbb{R}$ is called a *metric*. For a subset $U \subset X$ we call the metric $d_{|U \times U}$ the *induced metric* on U. Sometimes it is convenient to allow a weaker version of (1.1), namely

$$d(x, y) \leqslant C(d(x, z) + d(z, y)) \quad \text{for all } x, y, z \in X \text{ and } C \geqslant 1,$$

in which case (X, d) is said to be a *quasi-metric space* and d a *quasi-metric*.

Note that in a metric space the condition $d(x, y) \geqslant 0$ follows from the symmetry condition, the triangle inequality and the definiteness condition:

$$2\, d(x, y) = d(x, y) + d(x, y) = d(x, y) + d(y, x) \geqslant d(x, x) = 0\,.$$

However, giving this condition is essential when dealing with *semi-metric spaces* (X, d), in which case we allow $d(x, y) = 0$ to hold without having $x = y$. A *semi-quasimetric space* is defined accordingly. Recall the notion of a norm on $X = \mathbb{R}^n$.

Definition 1.2 (Norm on $\mathbb{R}^n$). A *norm* on $\mathbb{R}^n$ is a real-valued function $N(x)$ such that $N(x) \geqslant 0$ for all $x \in \mathbb{R}^n$, the definiteness condition $N(x) = 0 \Leftrightarrow x = 0$, the homogeneity condition $N(\lambda x) = |\lambda|\, N(x)$, $x \in \mathbb{R}^n$, $c \in \mathbb{R}$, and the triangle inequality

$$N(x+y) \leqslant N(x) + N(y) \quad \text{for all } x, y \in \mathbb{R}^n \tag{1.2}$$

hold. Again, a weaker version of (1.2) is given by

$$N(x+y) \leqslant C(N(x) + N(y)) \quad \text{for all } x, y \in \mathbb{R}^n,\ C \geqslant 1,$$

in which case N is said to be a *quasi-norm*. If the definiteness condition is weakened so that $x = 0$ implies $N(x) = 0$, we say that N is a *semi-norm*.

If N is a norm on $\mathbb{R}^n$ then $d(x,y) = N(x-y)$ defines a metric on $\mathbb{R}^n$.

We consider a metric space (X, d) and fix $y \in X$ and $\mathbb{R} \ni r > 0$. Then we write

$$B^d(y,r) = \{x \in X : d(x,y) < r\}$$

for the open ball with centre y and radius r. In our considerations it is often helpful to indicate the dependence on the metric d. Whenever there is no need to emphasise which metric is used we may sometimes write $B(y,r)$ or $B_r(y)$ instead of $B^d(y,r)$.

Definition 1.3. **A.** For a non-empty set $A \subset X$ of a metric space (X,d) we define its *diameter* with respect to the metric d as

$$\operatorname{diam}(A) := \operatorname{diam}_d(A) := \sup\{d(x,y) : x, y \in A\}.$$

Then A is said to be *bounded* with respect to d if $\operatorname{diam}(A) < \infty$.
B. A subset $A \subset \mathbb{R}^n$ is said to be *convex*, if for any elements $x, y \in A$ and $0 \leqslant t \leqslant 1$ we have that the convex combination $tx + (1-t)y$ also belongs to A.

We will first give the definition of a general measurable space. We quote the definition from [47].

Definition 1.4 (Measurable space). Let $\Omega \neq \emptyset$ be a set. A *σ-algebra* $\mathcal{A}$ over Ω is a collection of subsets of Ω satisfying $\Omega \in \mathcal{A}$, $A \in \mathcal{A}$ implies $A^c \in \mathcal{A}$, and $A_k \in \mathcal{A}$ for $k \in \mathbb{N}$ implies $\bigcup_{k\in\mathbb{N}} A_k \in \mathcal{A}$. The elements in $\mathcal{A}$ are called *$\mathcal{A}$-measurable sets* and the tuple $(\Omega, \mathcal{A})$ is called a *measurable space*.

Using these concepts we define a measure on $(\Omega, \mathcal{A})$.

Definition 1.5 (Measure). Let $(\Omega, \mathcal{A})$ be a measurable space. A function $\mu : \mathcal{A} \mapsto \mathbb{R} \cup \{\infty\}$ is called a *measure* if

(i) $\mu(A) \geqslant 0$ for all $A \in \mathcal{A}$,

(ii) $\mu(\emptyset) = 0$, and

(iii) for all sequences $(A_j)_{j\in\mathbb{N}}$ of pairwise disjoint sets in $\mathcal{A}$ we have the σ-additivity property

$$\mu\Big(\bigcup_{j=1}^{\infty} A_j\Big) = \sum_{j=1}^{\infty} \mu(A_j).$$

If $\mu(\Omega) = 1$ the measure μ is a *probability measure.*

Definition 1.6 (Measure space). Let $(\Omega, \mathcal{A})$ be a measurable space, μ a measure on $\mathcal{A}$. Then the triple $(\Omega, \mathcal{A}, \mu)$ is called a *measure space.* If $\mu(\Omega) = 1$, it is called a *probability space.*

In the following we will work with a special σ-algebra on a metric space (X, d) which is generated by open sets.

Definition 1.7 (σ-algebra of Borel sets). Let (X, d) be a metric space. We consider the system of open subsets $\mathcal{O} = \{B \subset X : B \text{ open}\}$. The system $\sigma(\mathcal{O}) =: \mathcal{B}(X, d)$ generated by $\mathcal{O}$ is the *σ-algebra of Borel sets* in X. The elements in $\mathcal{B}(X, d)$ are called *Borel sets* in X with respect to d.

Remark 1.8. In the metric space $(\mathbb{R}^n, |\cdot|)$ we will sometimes denote the σ-algebra of Borel sets by $\mathcal{B}^{(n)}$ for short, i.e. $\mathcal{B}^{(n)} := \mathcal{B}(\mathbb{R}^n, |\cdot|)$.

We further need the following concepts.

Definition 1.9. Let $(X, \mathcal{B}(X, d))$ be a measurable space.
A. An outer measure μ is a *Borel measure* if all Borel sets B are measurable with respect to μ, i.e. if $\mu(A) = \mu(A \cap B) + \mu(A \setminus B)$ for all subsets $A \subset X$.
B. μ is *locally finite* if for every $x \in X$ there exists an $r > 0$ such that $\mu(B^d(x, r)) < \infty$.
C. An outer measure μ is *Borel regular* if it is a Borel measure with the property that for every set $A \subset X$ there exists a Borel set $B \in \mathcal{B}(X, d)$ such that $A \subset B$ and $\mu(A) = \mu(B)$.
D. μ is a *Radon measure* if it is a Borel measure and

(a) $\mu(K) < \infty$ for compact $K \subset X$,

(b) μ is inner regular, i.e. for a Borel set $B \subset X$ we have $\mu(B) = \sup\{\mu(K) : K \subset B \text{ compact}\}$,

(c) μ is outer regular, i.e. for a Borel set $B \subset X$ we have $\mu(B) = \inf\{\mu(U) : U \supset B \text{ open}\}$.

An immediate consequence of Definition 1.9 is the fact that Radon measures are always Borel regular. Examples for Radon measures are the n-dimensional Lebesgue measure $\lambda^{(n)}$ on $\mathbb{R}^n$, and also the Dirac measure

$$\delta_a(A) = \begin{cases} 1, & a \in A \\ 0, & a \notin A \end{cases} = \chi_A(a)$$

is a Radon measure on any metric space X. Another example is the standard Gaussian measure $\gamma^{(n)}$ on $\mathbb{R}^n$ given by

$$\gamma^{(n)}(A) = \frac{1}{(2\pi)^{n/2}} \int_A \exp\Big(-\frac{1}{2}|x|^2\Big)\, \lambda^{(n)}(dx).$$

In later chapters we will deal with metric measure spaces, which consist of a metric space (X, d) and a measure μ which is assumed to be Borel regular. We denote these spaces by the triple (X, d, μ). Further, we assume that balls in X have a positive but finite measure, i.e. μ is locally finite in the sense of Definition 1.9.

For us, a central definition is the following.

Definition 1.10 (Volume doubling measure). Let (X, d, μ) be a metric measure space with regular Borel measure μ. Let $r > 0$ and $B^d(x, r)$, $x \in X$, be a ball of radius r centred at x with respect to the metric d. The measure μ is said to be a *volume doubling measure* if there exists a constant $C_d \geqslant 1$, such that

$$\mu(B(x, 2r)) \leqslant C_d\, \mu(B(x, r)) \tag{1.3}$$

for μ-almost all $x \in X$. Then the metric measure space is called a *volume doubling space*. We will call μ a *locally doubling measure* if there exists $\rho > 0$ such that (1.3) holds for almost all x and $r < \rho$. Accordingly, we will call (X, d, μ) a *locally doubling metric measure space*.

Remark 1.11. Let us note that (1.3) implies that for any number $\lambda \geqslant 1$ we get $\mu(B(x, \lambda r)) \leqslant C_d(\lambda)\, \mu(B(x, r))$ and therefore

$$\mu(B(x, R)) \leqslant C_d(R, r)\, \mu(B(x, r))$$

for all positive radii $R > r > 0$.

Definition 1.12. A set $X \neq \emptyset$ equipped with a (quasi-) metric d is said to be a *space of homogeneous type* if for each $x \in X$ the balls $B^d(x, r)$ of some radius r form a basis for the open neighbourhoods at x, and if the *homogeneity condition* holds: there exists an $N > 0$ such that for all $x \in X$ and all radii $r > 0$ the ball $B^d(x, r)$ contains at most N points x_i, $i = 1, \ldots, n$, such that $d(x_i, x_j) > \frac{r}{2}$ for $i \neq j$.

A fundamental reference for analysis on homogeneous spaces is [15]. When studying analysis on doubling metric measure spaces, many results that depend on covering arguments can be generalised to spaces of homogeneous type. These include e.g. the study of certain function spaces on spaces of homogeneous type, which turn out to have properties comparable to the classical analogues, see e.g. [50] and references therein.

1.2 Continuous negative definite functions

Continuous negative definite functions will play a central role in this work due to their property of giving rise to a metric if we restrict ourselves to real-valued functions. Therefore, this paragraph serves to introduce these functions and, closely connected to them, positive definite as well as (complete) Bernstein functions.

Definition 1.13 (Positive definite function). The function $g : \mathbb{R}^n \to \mathbb{C}$ is a *positive definite function* if for any $k \in \mathbb{N}$ and $\xi^1, \ldots, \xi^k \in \mathbb{R}^n$ the matrix $\left(g(\xi^j - \xi^l)\right)_{j,l=1,\ldots,k}$ is positive Hermitian, i.e. $\sum_{j,l=1}^k g(\xi^j - \xi^l)\lambda_j\, \bar{\lambda}_l \geqslant 0$ for any $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$.

The Fourier transform of a bounded Borel measure μ on $\mathbb{R}^n$ is given by

$$\hat{\mu}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \mu(dx)\,. \tag{1.4}$$

Since as a bounded Borel measure we have $\mu \in \mathcal{S}'(\mathbb{R}^n)$, its Fourier transform as defined in (1.4) coincides with that obtained when considering μ as a distribution, i.e.

$$\begin{aligned} \langle \mu, \hat{\phi} \rangle &= \int_{\mathbb{R}^n} \hat{\phi}(\xi)\, \mu(d\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} e^{-ix\cdot\xi} \phi(x)\, dx \right] \mu(d\xi) \\ &= \int_{\mathbb{R}^n} \left[(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \mu(d\xi) \right] \phi(x)\, dx = \langle \hat{\mu}, \phi \rangle \end{aligned}$$

for all $\phi \in \mathcal{S}(\mathbb{R}^n)$, where $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space of all rapidly decreasing functions on $\mathbb{R}^n$ and $\mathcal{S}'(\mathbb{R}^n)$ the space of tempered distributions, i.e. a distribution with continuous extension to $\mathcal{S}(\mathbb{R}^n)$. The following lemma holds (see Lemma 3.5.4 in [45]).

Lemma 1.14. *Let μ be a bounded Borel measure on $\mathbb{R}^n$. Then $\hat{\mu}$ is a uniformly continuous positive definite function.*

The following theorem is fundamental in the theory of positive definite functions and relates these functions to bounded and positive measures on $\mathbb{R}^n$. Its proof is given in [45], Theorem 3.5.7.

Theorem 1.15 (Bochner). *A function $g : \mathbb{R}^n \to \mathbb{C}$ is the Fourier transform of a bounded Borel measure μ on $\mathbb{R}^n$ with total mass $|\mu|$ if, and only if, g is continuous, positive definite and $g(0) = \hat{\mu}(0) = (2\pi)^{-n/2}|\mu|$.*

Note that Bochner's Theorem states the converse to Lemma 1.14, i.e. given a continuous positive definite function g on $\mathbb{R}^n$, then there exists a bounded Borel measure μ on $\mathbb{R}^n$ such that $\hat{\mu} = g$.

We define negative definite functions as follows.

Definition 1.16 (Negative definite function). A function $\psi : \mathbb{R}^n \to \mathbb{C}$ is called *negative definite* if the matrix $\left(\psi(\xi^j)+\overline{\psi(\xi^l)}-\psi(\xi^j-\xi^l)\right)_{j,l=1,\dots,k}$ is positive Hermitian for any $k \in \mathbb{N}$ and $\xi^1, \dots, \xi^k \in \mathbb{R}^n$, i.e.

$$\sum_{j,l=1}^{k} (\psi(\xi^j) + \overline{\psi(\xi^l)} - \psi(\xi^j - \xi^l))\lambda_j \bar{\lambda}_l \geqslant 0 .$$

Note that Definition 1.16 is related to Definion 3.23 as for $k \in \mathbb{N}$, $\xi_1, \dots, \xi_k \in \mathbb{R}^n$ and $\lambda_1, \dots, \lambda_k \in \mathbb{C}$ with $\sum_{j=1}^k \lambda_j = 0$,

$$\begin{aligned} 0 &\leqslant \sum_{j,l=1}^{k} (\psi(\xi^j) + \overline{\psi(\xi^l)} - \psi(\xi^j - \xi^l))\lambda_j \bar{\lambda}_l \\ &= \sum_{j=1}^{k} \lambda_j \Big(\sum_{l=1}^{k} \overline{\lambda_l \psi(\xi^l)} \Big) + \sum_{l=1}^{k} \bar{\lambda}_l \Big(\sum_{j=1}^{k} \lambda_j \psi(\xi^j) \Big) - \sum_{j,l=1}^{k} \psi(\xi^j - \xi^l)\, \lambda_j \bar{\lambda}_l \\ &= - \sum_{j,l=1}^{k} \psi(\xi^j - \xi^l)\, \lambda_j \bar{\lambda}_l , \end{aligned}$$

i.e. $\sum_{j,l=1}^{k} \psi(\xi^j - \xi^l)\, \lambda_j \bar{\lambda}_l \leqslant 0$ and therefore some authors have called these functions (conditionally) negative definite, having in mind Definition 3.23.

Remark 1.17 (First properties of negative definite functions). A negative definite function always satisfies $\psi(0) \geqslant 0$ which follows immediately from the definition. Moreover, consider the 2×2-matrix

$$\begin{pmatrix} \psi(\xi) + \overline{\psi(\xi)} - \psi(0) & \psi(\xi) + \overline{\psi(0)} - \psi(\xi) \\ \psi(0) + \overline{\psi(\xi)} - \psi(-\xi) & \psi(0) + \overline{\psi(0)} - \psi(0) \end{pmatrix} \tag{1.5}$$

which is positive Hermitian. Since for a positive Hermitian matrix $A = (a_{ij})$ we have $a_{ij} = \overline{a_{ji}}$ it follows that

$$\psi(\xi) = \overline{\psi(-\xi)} ,$$

which gives the symmetry $\psi(\xi) = \psi(-\xi)$ if ψ is real-valued. Considering the determinant of (1.5), which is real, then we find

$$\overline{\psi(0)}\left[\psi(\xi) + \overline{\psi(\xi)} - 2\psi(0)\right] \geqslant 0$$

if $\psi(\xi) + \overline{\psi(\xi)} = 2\operatorname{Re}\psi(\xi) \geqslant 2\psi(0)$ or

$$\operatorname{Re}\psi(\xi) \geqslant \psi(0)\,.$$

Further, the function $\xi \mapsto \psi(\xi) - \psi(0)$ is negative definite whenever ψ is negative definite, and $\xi \mapsto \psi(\xi) := g(0) - g(\xi)$ is negative definite whenever g is positive definite.

The following theorem is fundamental in the theory of negative definite functions, cf. [45], Chapter 3.6.

Theorem 1.18 (Schoenberg). *A function $\psi : \mathbb{R}^n \to \mathbb{C}$ is negative definite if, and only if, $\psi(0) \geqslant 0$ and the function $\xi \mapsto e^{-t\psi(\xi)}$ is positive definite for all $t > 0$.*

Proof. Let $\psi : \mathbb{R}^n \to \mathbb{C}$ be a negative definite function. For any $k \in \mathbb{N}$ and $\xi^1, \ldots, \xi^k \in \mathbb{R}^n$ the matrix

$$\left(a_{j,l}\right)_{j,l} := \left(\psi(\xi^j) + \overline{\psi(\xi^l)} - \psi(\xi^j - \xi^l)\right)_{j,l=1,\ldots,k}$$

is positive Hermitian. By $a_{j,l}^m$ we denote the m-th power of each component in the matrix. Then, by Lemma 3.5.9 in [45] the matrices with the entries $a_{j,l}^m$ for some $m > 0$ and $\frac{a_{j,l}^m}{m!}$ are also positive Hermitian. As the exponential function is holomorphic on the whole complex plane the power series $e^{a_{j,l}} = \sum_{m=1}^{\infty} \frac{a_{j,l}^m}{m!}$ has infinite radius of convergence. Therefore, we may interchange the sums to get $\sum_{j,l=1}^{k} e^{a_{j,l}}\, \lambda_j\, \bar{\lambda}_l \geqslant 0$, and hence

$$\left(e^{a_{j,l}}\right)_{j,l} = \left(e^{\psi(\xi^j) + \overline{\psi(\xi^l)} - \psi(\xi^j - \xi^l)}\right)_{j,l=1,\ldots,k}$$

is positive Hermitian. It follows

$$\sum_{j,l=1}^{k} e^{-\psi(\xi^j - \xi^l)} \eta_j\, \bar{\eta}_l \;=\; \sum_{j,l=1}^{k} e^{\psi(\xi^j) + \overline{\psi(\xi^l)} - \psi(\xi^j - \xi^l)} \lambda_j\, \bar{\lambda}_l \;\geqslant\; 0$$

with $\eta_i = \lambda_1\, e^{-\psi(\xi^i)} \in \mathbb{C}$. Hence $\xi \mapsto e^{-\psi(\xi)}$ is a positive definite function. Further, for a negative definite function $\xi \mapsto \psi(\xi)$ and $t > 0$ we find that $\xi \mapsto t\psi(\xi)$ is negative definite and the arguments above yield $\xi \mapsto (2\pi)^{-n/2} e^{-t\psi}$ is positive definite.

Conversely, let $\xi \mapsto e^{-t\psi(\xi)}$ be a positive definite function and $\psi(\xi)$ be such that $\psi(0) \geqslant 0$. Then $e^{-t\psi(0)} \leqslant 1$ for all $t > 0$ and by the remarks above the function $\xi \mapsto \frac{1}{t}\,(1 - e^{-t\psi(\xi)})$ is negative definite. As the set of negative definite functions is closed under pointwise convergence, see Lemma 3.6.7 in [45], we find that

$$\lim_{t \to 0} \frac{1}{t}\,(1 - e^{-t\psi(\xi)}) = \psi(\xi)$$

is negative definite. □

Without proof we quote the corollary below from [45].

Corollary 1.19. *A negative definite function $\psi : \mathbb{R}^n \to \mathbb{C}$ that is continuous at the origin $0 \in \mathbb{R}^n$ is continuous on the whole of $\mathbb{R}^n$.*

The following lemma provides useful properties of continuous negative definite functions.

Lemma 1.20 (Properties of continuous negative definite functions). *Let $\psi : \mathbb{R}^n \to \mathbb{C}$ be a continuous negative definite function. Then, for all $\xi, \eta \in \mathbb{R}^n$ we have*

(i) $|\psi(\xi+\eta)|^{1/2} \leqslant |\psi(\xi)|^{1/2} + |\psi(\eta)|^{1/2}$,

(ii) $\big||\psi(\xi)|^{1/2} - |\psi(\eta)|^{1/2}\big| \leqslant |\psi(\xi-\eta)|^{1/2}$,

(iii) $|\psi(\xi) + \psi(\eta) - \psi(\xi \pm \eta)| \leqslant 2\,(\operatorname{Re}\psi(\xi)\operatorname{Re}\psi(\eta))^{1/2}$,

(iv) $|\psi(\xi)| \leqslant c_\psi(1+|\xi|^2)$, *where* $c_\psi = 2\sup_{|\eta|\leqslant 1}|\psi(\eta)|$.

(v) $\left(\frac{1+|\psi(\xi)|}{1+|\psi(\eta)|}\right)^s \leqslant 2^{|s|}(1+|\psi(\xi-\eta)|)^{|s|}$, $s > 0$,

(vi) $1 + |\psi(\xi \pm \eta)| \leqslant (1+|\psi(\xi)|)(1+|\psi(\eta)|^{1/2})^2$,

(vii) $\psi(\xi) \geqslant c_0\,|\xi|^{2r_0}$, *for* $c_0, r_0 > 0$ *and* $|\xi|$ *large.*

The proofs of these properties are given in [45]. Since for the negative definite function $\psi(\xi) = |\xi|^2$ (see Example 1.27 below) the inequality in Lemma 1.20 (v) is called *Peetre's inequality*, we call it *generalised Peetre's* or *Peetre-type inequality* for general negative definite functions. Corollary 1.19 can be proved with the help of Lemma 1.20 (vi). Moreover, Lemma 1.20 (i) ensures that a triangle inequality holds for square roots of real-valued negative definite functions, which is due to the fact that $\psi : \mathbb{R}^n \to \mathbb{R}$ is symmetric, see Remark 1.17. Later, we will restrict ourselves to work with negative definite functions which only take values in $\mathbb{R}$ and are continuous at $0 \in \mathbb{R}^n$.

An important property of a continuous negative definite function is its representation given in the following theorem.

Theorem 1.21 (Lévy-Khinchin). *For a continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{C}$ there exist a constant $c > 0$, a vector $d \in \mathbb{R}^n$, a symmetric positive semidefinite quadratic form q on $\mathbb{R}^n$ and a finite measure μ on $\mathbb{R}^n \setminus \{0\}$ such that*

$$\psi(\xi) = c + i(d\cdot\xi) + q(\xi) + \int_{\mathbb{R}^n\setminus\{0\}} \left(1 - e^{-ix\cdot\xi} - \frac{ix\cdot\xi}{1+|x|^2}\right)\frac{1+|x|^2}{|x|^2}\,\mu(dx), \tag{1.6}$$

where c, d, q and μ are uniquely determined by ψ. Conversely, given c, d, q and μ as above, the function ψ as given in (1.6) is continuous negative definite.

Definition 1.22 (Lévy measure). Let μ be the measure in the Lévy-Khinchin representation of the continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{C}$. The measure

$$\nu(dx) := \frac{1+|x|^2}{|x|^2}\,\mu(dx)$$

defined on the σ-field $\mathcal{B}(\mathbb{R}^n\setminus\{0\})$ of all Borel sets on $\mathbb{R}^n\setminus\{0\}$ is called the *Lévy measure* associated with ψ.

Corollary 1.23. *Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a real-valued continuous negative definite function. Then there exist $c > 0$, q and ν as given in Theorem 1.21 and Definition 1.22 such that*

$$\psi(\xi) = c + q(\xi) + \int_{\mathbb{R}^n\setminus\{0\}} (1 - \cos(x\cdot\xi))\,\nu(dx). \tag{1.7}$$

The formulae (1.6) and (1.7) are commonly referred to as the *Lévy-Khinchin representation* of ψ, depending on the context, i.e. ψ being either complex- or real-valued.

The measure ν is a Radon measure on $\mathbb{R}^n \setminus \{0\}$ and satisfies

$$\int_{\mathbb{R}^n\setminus\{0\}} (|x|^2 \wedge 1)\, \nu(dx) < \infty\,.$$

Later, we will mainly restrict ourselves to cases in which $\psi : \mathbb{R}^n \to \mathbb{R}$ has the pure integral representation

$$\psi(\xi) = \int_{\mathbb{R}^n\setminus\{0\}} (1 - \cos(x\cdot\xi))\, \nu(dx)\,. \tag{1.8}$$

Below we will give some examples of continuous negative definite functions and their relation to certain convolution semigroups of measures. First, however, recall the following notions of convergence of measures.

Definition 1.24 (Convergence of measures). Let $\Omega \subset \mathbb{R}^n$ be a locally compact set, $(\mu_t)_{t\geqslant 0}$ a family of bounded Borel measures on Ω. Considering the limit

$$\lim_{t\to\infty} \int_\Omega u(x)\, \mu_t(dx) = \int_\Omega u(x)\, \mu_0(dx)\,, \tag{1.9}$$

we say that $(\mu_t)_{t\in\mathbb{N}}$ *converges weakly* to μ_0 if (1.9) holds for all $u \in C_b(\Omega)$, where $C_b(\Omega)$ is the set of bounded continuous functions on Ω. The sequence $(\mu_t)_{t\in\mathbb{N}}$ is said to *converge with respect to* C_∞ to μ_0 if (1.9) is satisfied for all $u \in C_\infty(\Omega)$, i.e. the set of all continuous functions vanishing at infinity. Furthermore, we say that $(\mu_t)_{t\in\mathbb{N}}$ *converges vaguely* to μ_0 if (1.9) holds for all $u \in C_0(\Omega)$, i.e. the set which contains functions of bounded support on Ω.

Remark 1.25. Due to the inclusions $C_0(\Omega) \subset C_\infty(\Omega) \subset C_b(\Omega)$ we have the following implications:

weak convergence $\Rightarrow$ convergence with respect to C_∞ $\Rightarrow$ vague convergence.

Definition 1.26 (Convolution semigroup of measures). A family $(\mu_t)_{t\geqslant 0}$ of sub-probability measures on $\mathbb{R}^n$, i.e. $\mu_t(\mathbb{R}^n) \leqslant 1$ for all $t \geqslant 0$, is called a *convolution semigroup* if $\mu_t * \mu_s = \mu_{t+s}$ for any $s, t \geqslant 0$, $\mu_0 = \varepsilon_0$ (Dirac measure in 0) and $\mu_t \to \varepsilon_0$ vaguely for $t \to 0$.

Example 1.27. Consider the family $(\mu_t)_{t\geqslant 0}$ of measures on $\mathbb{R}^n$ having the functions $(g_t)_{t\geqslant 0}$ defined by

$$g_t(x) = (4\pi t)^{-n/2} e^{-|x|^2/(4t)} \qquad \text{for } x \in \mathbb{R}^n$$

as densities with respect to the Lebesgue measure on $\mathbb{R}^n$. For the Fourier transform $\hat{g}_t$ of g_t, we need to compute

$$(4\pi t)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\, e^{-|x|^2}\, dx = \prod_{j=1}^{n} (4\pi t)^{-1/2} \int_{\mathbb{R}} e^{-ix_j\xi_j}\, e^{-x_j^2}\, dx_j\,.$$

Now consider only the integral over $\mathbb{R}$,

$$\int_{\mathbb{R}} e^{-ity-y^2}\, dy = \int_{\mathbb{R}} e^{-(y^2+ity)}\, dy = e^{-t^2/4} \int_{\mathbb{R}} e^{-(y+it/2)^2}\, dy\,.$$

Use the change of variable $z := y + it/2$, then $dz = dy$ and therefore

$$\int_{\mathbb{R}} e^{-ity-y^2}\,dt = e^{-t^2/4}\int_{\mathbb{R}} e^{-z^2}\,dz = e^{-t^2/4}\sqrt{\pi}\,.$$

This yields

$$(4\pi t)^{-n/2}\int_{\mathbb{R}^n} e^{-ix\cdot\xi-|x|^2}\,dx = (4\pi t)^{-n/2}\prod_{j=1}^{n}\sqrt{\pi}e^{-\xi_j^2/4} = (4t)^{-n/2}\,e^{-|\xi|^2/4}\,.$$

Now, for $x \in \mathbb{R}^n$ and $a \in \mathbb{R}$, $a \neq 0$, the following formula holds:

$$\widehat{f(ax)}(\xi) = |a|^{-n}\,\hat{f}\Big(\frac{\xi}{a}\Big), \qquad \xi \in \mathbb{R}^n\,. \tag{1.10}$$

In this case, $f(x) = e^{-|x|^2}$ and $a = (4t)^{-1/2}$. Together, it follows from (1.10) and the above calculation that

$$\begin{aligned}\hat{g}_t(\xi) &= (2\pi)^{-n/2}\,(4\pi t)^{-n/2}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}\,e^{-|x|^2/(4t)}\,dx\\ &= (2\pi)^{-n/2}\,(4t)^{-n/2}\,(4t)^{n/2}\,e^{-t|\xi|^2} = (2\pi)^{-n/2}\,e^{-t|\xi|^2}\,.\end{aligned}$$

This family $(\mu_t)_{t\geqslant 0}$ of measures forms a convolution semigroup on $\mathbb{R}^n$, the *Brownian* (or *Gaussian*) *semigroup* on $\mathbb{R}^n$. The associated continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{R}$ is the function $\xi \mapsto |\xi|^2$.

Given a convolution semigroup $(\mu_t)_{t\geqslant 0}$ on $\mathbb{R}^n$ then its Fourier transform can be characterised by a continuous negative definite function as we have seen in Example 1.27. In fact, the following theorem holds; see Theorem 3.6.4 in [45].

Theorem 1.28. *For any convolution semigroup $(\mu_t)_{t\geqslant 0}$ there exists a continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{C}$ such that*

$$\hat{\mu}_t(\xi) = (2\pi)^{-n/2}\,e^{-t\psi(\xi)}\,.$$

By Theorem 1.15, each $\hat{\mu}_t$ is a continuous positive definite function.

Examples 1.29. A. For general $0 < \alpha \leqslant 2$ the function $\xi \mapsto |\xi|^\alpha$ is continuous negative definite. The associated convolution semigroup $(\mu_t^\alpha)_{t\geqslant 0}$ of measures on $\mathbb{R}^n$ is called the *symmetric stable semigroup of order* α and is characterised by

$$\hat{\mu}_t^\alpha(\xi) = (2\pi)^{-n/2}\,e^{-t|\xi|^\alpha} \qquad \text{for } t > 0,\ \xi \in \mathbb{R}^n\,.$$

Since the function $\hat{\mu}_t^\alpha$ is integrable we get $\mu_t^\alpha = g_t^\alpha\,\lambda^{(n)}$ where g_t^α is the inverse Fourier transform of $\hat{\mu}_t^\alpha$, i.e.

$$\mu_t^\alpha(x) = \mathcal{F}^{-1}(\hat{\mu}_t^\alpha)(x) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\,\hat{\mu}_t^\alpha(\xi)\,d\xi = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\,e^{-t\,|\xi|^\alpha}\,d\xi\,,$$

which is now the density of the semigroup corresponding to $\xi \mapsto |\xi|^\alpha$ for $0 < \alpha \leqslant 2$. Here as in the following we denote the inverse Fourier transform by $\mathcal{F}^{-1}$. For $\alpha = 1$ we have an explicit form of g_t as the semigroup $(\mu_t)_{t\geqslant 0}$ becomes the Cauchy semigroup with μ_t having the density

$$g_t(x) = \Gamma\Big(\frac{n+1}{2}\Big)\,\frac{t}{(\pi(t^2+|x|^2))^{\frac{n+1}{2}}} \qquad \text{for } t \geqslant 0,\ x \in \mathbb{R}\,,$$

for $t \geqslant 0$ and $x \in \mathbb{R}^n$. The symmetric stable semigroup of order 2 is the Brownian semigroup with $g_t(x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$ as density, see Example 1.27.

B. For $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ it follows from Lemma 3.6.7 in [45] that $\psi(\xi,\eta) := \psi_1(\xi) + \psi_2(\eta) : \mathbb{R}^n \to \mathbb{C}$ is negative definite whenever $\psi_1 : \mathbb{R}^{n_1} \to \mathbb{C}$ and $\psi_2 : \mathbb{R}^{n_2} \to \mathbb{C}$ are negative definite. Hence, in particular, the function $\psi(\xi,\eta) = |\xi|^\alpha + |\eta|^\beta$ with $\alpha, \beta \in (0,2]$, $\xi \in \mathbb{R}^{n_1}$, $\eta \in \mathbb{R}^{n_2}$, is a continuous negative definite function, see Proposition 1.34 below.

We now define a further class of functions which, as it turns out, operate on the class of negative definite functions.

Definition 1.30 (Bernstein function). A function $f \in C^\infty((0,\infty))$ is called a *Bernstein function* if

$$f \geqslant 0 \quad \text{and} \quad (-1)^k f^{(k)} \leqslant 0$$

for all $k \in \mathbb{N}$.

Note that a Bernstein function is positive, increasing and concave. The set of Bernstein functions forms a convex cone which contains the positive constant functions. Similar to continuous negative definite functions, Bernstein functions also have an integral representation.

Theorem 1.31. *Let f be a Bernstein function. Then f has the representation*

$$f(x) = a + bx + \int_{0+}^\infty (1 - e^{-xs})\,\mu(ds)\,, \quad x > 0\,,$$

where $a, b \geqslant 0$ are constants and μ is a measure on $(0,\infty)$ satisfying

$$\int_{0+}^\infty \frac{s}{s+1}\,\mu(ds) < \infty\,.$$

Conversely, any f having this representation is a Bernstein function.

Below we consider an important subclass of Bernstein functions.

Definition 1.32. A function $f : \mathbb{R}_+ \to \mathbb{R}$ is called a *complete Bernstein function* if there exists a Bernstein function g such that $f(x) = x^2\,\mathcal{L}g(x)$, where $\mathcal{L}g(x)$ is the Laplace transform of the function g defined by

$$\mathcal{L}g(x) := \int_0^\infty e^{-xs} g(s)\,ds\,.$$

Theorem 1.33. *A function $f : \mathbb{R}_+ \to \mathbb{R}$ is a complete Bernstein function if, and only if, it has the integral representation*

$$f(x) = a + bx + \int_{0+}^\infty \frac{x}{s+x}\,\rho(ds)$$

with a measure ρ on $(0,\infty)$ which satisfies the integrability condition $\int_{0+}^\infty (1+s)^{-1}\,\rho(ds) < \infty$.

Examples of (complete) Bernstein functions and their respective representation formulae are given in section 1.2 of [25]. An extensive list of complete Bernstein functions can be found in the Appendix of [47]; see also the forthcoming monograph by Schilling, Song and Vondraček [57].

Bernstein functions operate on negative definite functions in the following way:

Proposition 1.34. *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a Bernstein function and $\psi : \mathbb{R}^n \to \mathbb{R}$ a continuous negative definite function. Then $f \circ \psi : \mathbb{R}^n \to \mathbb{R}$ is also a continuous negative definite function.*

Later, we will use this result to obtain more examples of continuous negative definite functions.

Definition 1.35. In the situation of Proposition 1.34 the continuous negative definite function $(f \circ \psi)(\xi)$ is called *subordinate* to $\xi \mapsto \psi(\xi)$ with respect to f.

In the following paragraphs we will often work with radially symmetric continuous negative definite functions.

Definition 1.36 (Radially symmetric function). A function $\phi : \mathbb{R}^n \to \mathbb{R}$ is called *radially symmetric in n dimensions* if it is a function of the Euclidean norm of its argument.

As we have seen in Example 1.27 the function $\xi \to |\xi|^2$ is continuous and negative definite for $\xi \in \mathbb{R}^n$ and any $n \in \mathbb{N}$. Then by Definition 1.36 and Proposition 1.34 the function $\xi \to f(|\xi|^2)$ is a radially symmetric continuous negative definite function for any Bernstein function $f : \mathbb{R}_+ \to \mathbb{R}$. In fact, the converse also holds true, i.e. given that $\xi \to f(|\xi|^2)$ is continuous and negative definite then f is a Bernstein function; see Theorem 3.9.25 in [45]. The following theorem, taken from [45], characterises those continuous negative definite functions that are subordinate to $\xi \to |\xi|^2$ and, hence, are radially symmetric.

Theorem 1.37. *A continuous negative definite function ψ is subordinate to the function $\xi \mapsto |\xi|^2$ if, and only if, its Lévy-Khinchin representation is of the form*

$$\psi(\xi) = \psi(0) + b\,|\xi|^2 + \int_{\mathbb{R}^n \setminus \{0\}} (1 - \cos(y \cdot \xi))\, m(|y|^2)\, dy\,,$$

where m is the Laplace transform of a measure ν on $(0, \infty)$ satisfying

$$\int_{0+}^{1} s^{-\frac{n}{2}}\, \nu(ds) + \int_{1}^{\infty} s^{-\frac{n}{2}-1}\, \nu(ds) < \infty\,.$$

The Bernstein function f such that $\psi(\xi) = f(|\xi|^2)$ holds is given by

$$f(x) = \psi(0) + bx + \int_{0+}^{\infty} (1 - e^{-xs})(4\pi s)^{\frac{n}{2}}\, \Phi(\nu)(ds)\,,$$

where $\Phi(\nu)$ is the image measure of ν with respect to the mapping $s \mapsto \Phi(s) = (4s)^{-1}$.

Remark 1.38. We will later consider continuous negative definite functions with $\psi(\xi) = 0$ if, and only if, $\xi = 0$. Now, in order for the composition $f \circ \psi$ to satisfy $(f \circ \psi)(0) = 0$ whenever $\psi(0) = 0$, it is a necessary and sufficient condition for f to be a Bernstein function with $f(0) = 0$; see Remark 3.9.24 in [45].

In general, a continuous negative definite function with Lévy-Khinchin representation (1.6) need not be differentiable. However, a result due to W. Hoh [42], see also [45], ensures that a certain integrability property of the Lévy measure of $\psi : \mathbb{R}^n \to \mathbb{R}$ yields not only classical differentiability up to a certain order, but also upper bounds for the derivatives for all multi-indices $\alpha \in \mathbb{N}_0^n$.

Theorem 1.39. *Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a continuous negative definite function with representation (1.7). Assume that*

$$M_j := \int_{\mathbb{R}^n\setminus\{0\}} |y|^j \, \nu(dy) < \infty$$

for $2 \leqslant j \leqslant m$, $m \in \mathbb{N}$. Then ψ is m-times continuously differentiable and for all multi-indices $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leqslant m$ we find

$$|\partial^\alpha \psi(\xi)| \leqslant \begin{cases} \psi(\xi) & \alpha = 0 \\ c_1\, \psi^{1/2}(\xi) & |\alpha| = 1 \\ c_{|\alpha|} & |\alpha| \geqslant 2 \end{cases}$$

with $c_1 = (2\,M_2)^{1/2} + 2\lambda^{1/2}$, $c_2 = M_2 + 2\lambda$ and $c_{|\alpha|} = M_{|\alpha|}$ for $3 \leqslant |\alpha| \leqslant m$. The constant λ denotes the maximal eigenvalue of the quadratic form $q(\xi)$ in the Lévy-Khinchin representation of ψ.

In § 1.7 we will use Theorem 1.39 to show that continuous negative definite functions $\psi : \mathbb{R}^n \to \mathbb{R}$ are admissible weights in the sense of Triebel [59] for weighted function spaces.

We will need the following concept in the context of properties of metric measure spaces.

Definition 1.40 (Comparable negative definite functions). We call two continuous negative definite functions $\psi_1, \psi_2 : \mathbb{R}^n \to \mathbb{R}$ *comparable* if there exist constants $0 < \lambda_0 \leqslant \lambda_1 < \infty$ such that

$$\lambda_0 \leqslant \frac{\psi_1(\xi)}{\psi_2(\xi)} \leqslant \lambda_1 \, .$$

Example 1.41. Let $\psi_1(\xi) = \sqrt{|\xi|^2 + m^2} - m$, $m > 0$, and $\psi_2(\xi) = |\xi|$. Then we find $0 \leqslant \frac{\psi_1(\xi)}{\psi_2(\xi)} \leqslant 1$ for any $m > 0$ and

$$\frac{\psi_1(\xi)}{\psi_2(\xi)} = \frac{\sqrt{|\xi|^2 + m^2} - m}{|\xi|} \to 1 \quad \text{as } |\xi| \to \infty \, .$$

The functions ψ_1 and ψ_2, however, are not comparable in the sense of Definition 1.40.

If continuous negative definite funtions have the Lévy-Khinchin representation

$$\psi_i(\xi) = \int_{\mathbb{R}^n\setminus\{0\}} (1 - \cos(y \cdot \xi))\, \nu_i(dy) \, , \qquad i = 1, 2,$$

and if their Lévy measures $\nu_i(dy) = g_i(y)\, \lambda^{(n)}(dy)$ are such that $0 < \lambda_0\, g_1(y) \leqslant g_2(y) \leqslant \lambda_1\, g_1(y)$ then it follows that ψ_1 and ψ_2 are comparable in the sense of Definition 1.40.

1.3 On one-parameter operator semigroups

In this paragraph we give a brief survey of some general results for one-parameter semigroups of operators and their generators on Banach spaces. An essential one in this context is the famous result by Hille and Yosida, whose theorem gives a characterisation of operators that generate strongly continuous contraction semigroups. Studying operators of the form $T_t u = \mu_t * u$ with a convolution semigroup μ_t and a function u on an appropriate function space leads to special classes of operator semigroups. For some of these there exist similar characterisations of their generators. We start with the following essential definition, taken from [45].

Definition 1.42 (Strongly continuous contraction semigroup). A family of bounded linear operators $(T_t)_{t\geqslant 0}$ on a Banach space $(X, \|\cdot\|_X)$ forms a *one-parameter operator semigroup* if the T_t satisfy

(i) $T_0 = I$,

(ii) $T_{s+t} = T_s \circ T_t$ for all $s, t \geqslant 0$ (semigroup property).

We call the semigroup *strongly continuous* if

(iii) $\lim_{t\to 0+} \|(T_t - I)u\|_X = 0$ for all $u \in X$.

A semigroup is called a *contraction semigroup* if

(iv) $\|T_t\|_X \leqslant 1$ for all $t \geqslant 0$.

Example 1.43. For $t \geqslant 0$ let us consider the operator

$$T_t u(x) = \mu_t * u(x) = \int_{\mathbb{R}^n} u(x-y)\,\mu_t(dy) \tag{1.11}$$

with a convolution semigroup $(\mu_t)_{t\geqslant 0}$. The integral is well-defined for all $u \in C_\infty(\mathbb{R}^n;\mathbb{R})$ and on $\mathcal{S}(\mathbb{R}^n;\mathbb{R})$ we have

$$\mathcal{F}(T_t u)(\xi) = \widehat{\mu_t * u}(\xi) = (2\pi)^{n/2}\hat{\mu}_t(\xi)\,\hat{u}(\xi) = \hat{u}(\xi)\,e^{-t\psi(\xi)}$$

for $t \geqslant 0$, where $\psi : \mathbb{R}^n \to \mathbb{R}$ is the continuous negative definite function associated with the convolution semigroup. Applying the inverse Fourier transform we see that T_t has the form of a *pseudodifferential operator* acting on $\mathcal{S}(\mathbb{R}^n;\mathbb{R})$, i.e.

$$T_t u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-t\psi(\xi)} \hat{u}(\xi)\,d\xi\,.$$

The fact that $\mathcal{F}(T_t u) = \hat{u}(\xi)\,e^{-t\psi(\xi)}$ shows that for $u \in \mathcal{S}(\mathbb{R}^n;\mathbb{R})$ we have $\hat{u} \in \mathcal{S}(\mathbb{R}^n;\mathbb{R})$ and $\mathcal{F}(T_t u) \in L^1(\mathbb{R}^n;\mathbb{R})$. Now the lemma of Riemann-Lebesgue in the theory of Fourier analysis ensures that the Fourier transform is a continuous linear operator from $L^1(\mathbb{R}^n;\mathbb{R})$ to $C_\infty(\mathbb{R}^n;\mathbb{R})$, which implies that $T_t u \in C_\infty(\mathbb{R}^n;\mathbb{R})$. Moreover, for $u \in C_\infty(\mathbb{R}^n;\mathbb{R})$ we get

$$|T_t u(x)| \leqslant \int_{\mathbb{R}^n} |u(x-y)|\,\mu_t(dy) \leqslant \|u\|_\infty \mu(\mathbb{R}^n) \leqslant \|u\|_\infty$$

as $\mu_t(\mathbb{R}^n) \leqslant 1$ by definition. Since the right-hand side is independent of t and x, this inequality implies that T_t is a contraction on $C_\infty(\mathbb{R}^n;\mathbb{R})$.

The operators T_t also have the semigroup property $T_{t+s}u(x) = (T_t \circ T_s)u(x)$. To verify this, we need to recall the definition of the convolution semigroup, i.e.

$$\mu_{t+s}(y) = (\mu_t * \mu_s)(y) = \int_{\mathbb{R}^n} \mu_t(y-z)\,\mu_s(dz).$$

This yields

$$\begin{aligned}
T_{t+s}u(x) &= \int_{\mathbb{R}^n} u(x-y)\,\mu_{t+s}(dy) = \int_{\mathbb{R}^n} u(x-y)\Big(\int_{\mathbb{R}^n} \mu_t(y-z)\,\mu_s(dz)\Big)(dy)\\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(x-y)\,\mu_t(y-z)\,\mu_s(dz)\,(dy) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(x-y-z)\,\mu_s(dz)\,\mu_t(dy)\\
&= \int_{\mathbb{R}^n} T_s u(x-y)\,\mu_t(dy) = T_t(T_s u(x))\,.
\end{aligned}$$

Further, we have

$$T_0u(x) = \int_{\mathbb{R}^n} u(x-y)\,\mu_0(dy) = \int_{\mathbb{R}^n} u(x-y)\,\varepsilon_0(dy) = u(x)$$

and thus $T_0 = I$. Further, it is shown in Example 4.1.3 in [45] that $(T_t)_{t\geqslant 0}$ is strongly continuous for $t \to 0+$, hence a strongly continuous contraction semigroup on $C_\infty(\mathbb{R}^n;\mathbb{R})$.

The above example motivates the definition of the following special classes of operator semigroups.

Definition 1.44. A. A strongly continuous contraction semigroup $(T_t)_{t\geqslant 0}$ of real-valued operators on $C_\infty(\mathbb{R}^n;\mathbb{R})$, equipped with the usual supremum norm $\|\cdot\|_\infty$, is called a *Feller semigroup* if T_t is positivity preserving for any $t \geqslant 0$, i.e. $u \geqslant 0$ implies $T_tu \geqslant 0$.

B. A *sub-Markovian semigroup on L^p*, $1 \leqslant p \leqslant \infty$, is a strongly continuous contraction semigroup $(T_t)_{t\geqslant 0}$ of real-valued operators on $L^p(\mathbb{R}^n;\mathbb{R})$ with the property $0 \leqslant u \leqslant 1$ a.e. implies $0 \leqslant T_tu \leqslant 1$ a.e.

C. A strongly continuous contraction semigroup $(T_t)_{t\geqslant 0}$ is called *symmetric* if $(T_tu, v) = (u, T_tv)$ for all $u, v \in L^p(\mathbb{R}^n;\mathbb{R}) \cap L^2(\mathbb{R}^n;\mathbb{R})$ in the case of L^p-sub-Markovian semigroups and for $u, v \in C_\infty(\mathbb{R}^n;\mathbb{R}) \cap L^2(\mathbb{R}^n;\mathbb{R})$ in the case of Feller semigroups.

D. We call the semigroup $(T_t)_{t\geqslant 0}$ *analytic* if the operators T_tu admit an analytic extension T_zu to a sector $\Sigma_{\theta,d_0} = \{z \in \mathbb{C} : \arg(z - d_0) < \theta\}$.

In the case of symmetric L^2-sub-Markovian semigroups we have the following result, which can be found in [16].

Theorem 1.45. *Let $(T_t^{(2)})_{t\geqslant 0}$ be a symmetric sub-Markovian semigroup on $L^2(\mathbb{R}^n)$. Then it extends from $L^2(\mathbb{R}^n)\cap L^p(\mathbb{R}^n)$ to a sub-Markovian semigroup $(T_t^{(p)})_{t\geqslant 0}$ on $L^p(\mathbb{R}^n)$ for $1 \leqslant p < \infty$. Further, it holds that every symmetric sub-Markovian semigroup on $L^2(\mathbb{R}^n)$ and its extensions to $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ are analytic.*

A central notion in the theory of semigroups is that of an infinitesimal generator.

Definition 1.46 (Infinitesimal generator). Let $(T_t)_{t\geqslant 0}$ be a strongly continuous contraction semigroup on a Banach space $(X, \|\cdot\|_X)$. Its *infinitesimal generator* is given by the operator

$$Au := \lim_{t\to 0+} \frac{T_tu - u}{t} \qquad \text{(in the strong sense)}$$

for $u \in X$ on the domain

$$\mathsf{D}(A) := \left\{ u \in X : \lim_{t\to 0+} \frac{T_tu - u}{t} \text{ exists in } X \right\}.$$

In § 1.6 we will introduce ψ-Bessel potential spaces and show that for certain generators $(A^{(p)}, \mathsf{D}(A^{(p)}))$ their domains coincide with these function spaces; see also [25] and [46].

Example 1.43 (rev.) By the discussion above it is clear that the family $(T_t)_{t\geqslant 0}$ consisting of operators (1.11) can be extended to a Feller semigroup, which we will denote by $(T_t^{(\infty)})_{t\geqslant 0}$. If fact, it can also be extended to a semigroup acting on $L^2(\mathbb{R}^n;\mathbb{R})$. To see this we note that

$\mathcal{S}(\mathbb{R}^n;\mathbb{R}) \subset L^2(\mathbb{R}^n;\mathbb{R})$ dense and use Plancherel's Theorem, which says that the Fourier transform is an isometric isomorphism from L^2 to L^2. This yields

$$\|T_t u\|_{L^2} = \|\mathcal{F}(T_t u)\|_{L^2} \leqslant \|u\|_{L^2},$$

hence the extension is a contraction, too.

It is also of interest to obtain the actual form of the generator of $(T_t)_{t\geqslant 0}$. Using the convolution theorem and the equality $\mathcal{F}(T_t u)(\xi) = \hat{u}(\xi)e^{-t\psi(\xi)}$ for $t \geqslant 0$, $u \in \mathcal{S}(\mathbb{R}^n;\mathbb{R})$, we obtain the representation

$$\begin{aligned} Au(x) &= \lim_{t\to 0+} \frac{T_t - I}{t}\, u(x) = \lim_{t\to 0+} \mathcal{F}^{-1}\Big(\frac{e^{-t\psi}-1}{t}\,\hat{u}\Big)(x) \\ &= -\mathcal{F}^{-1}(\psi\,\hat{u})(x) = -(2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\psi(\xi)\,\hat{u}(\xi)\,d\xi\,. \end{aligned}$$

Hence, not only T_t, but also its generator is a pseudodifferential operator, but with the symbol $-\psi(\xi)$.

The notion of an operator resolvent is essential in the study of generators of one-parameter semigroups. Its definition is as follows, taken from [45].

Definition 1.47 (Operator resolvent). Let A be a closed linear operator on a Banach space $(X, \|\cdot\|_X)$ with domain $\mathsf{D}(A) \subset X$. The family $(R_\lambda)_{\lambda\in\rho(A)}$ of operators R_λ defined by

$$R_\lambda u := (\lambda - A)^{-1}u\,,$$

where $\lambda - A$ is short for $\lambda I - A$, is called the *resolvent of A*, and the set

$$\begin{aligned} \rho(A) := \{&\lambda \in \mathbb{C} : \lambda - A \text{ is surjective and has a continuous} \\ &\text{inverse defined on } \mathsf{R}(\lambda - A) = X\} \end{aligned}$$

is called the *resolvent set of A*.

The resolvent of a generator $(A, \mathsf{D}(A))$ of a strongly continuous contraction semigroup $(T_t)_{t\geqslant 0}$ on a Banach space $(X, \|\cdot\|_X)$ has a representation as a Laplace integral

$$R_\lambda u = \int_0^\infty e^{-\lambda t} T_t u\, dt$$

giving a way to express R_λ with the help of the semigroup operators. For the resolvent set we have

$$\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\} \subset \rho(A)\,.$$

The theorem of Hille and Yosida, cf. e.g. [16, 21], gives conditions for operators to generate a general strongly continuous semigroup of contraction operators.

Theorem 1.48 (Hille, Yosida; 1948). *Let A be a linear (and in general unbounded) operator on a Banach space $(X, \|\cdot\|_X)$. Then A generates a strongly continuous contraction semigroup if, and only if, the following conditions hold:*

(i) A is closed and $\mathsf{D}(A) \subset X$ dense,

(ii) for all $\lambda > 0$ we have $\lambda \in \rho(A)$ with $\|\lambda(\lambda - A)^{-1}\| \leqslant 1$.

To prove this result, Yosida used the *Yosida approximation* in order to reduce the problem of working with unbounded operators A to the case of bounded ones which we will denote by A_λ. The essential idea is to use the operator λR_λ, which satisfies the condition of Theorem 1.48, is a bounded operator and approximates the identity as $\lambda \to \infty$, since for $u \in \mathsf{D}(A)$,

$$\begin{aligned} \|\lambda R_\lambda u - u\|_X &= \|(\lambda - A + A)(\lambda - A)^{-1}u - u\|_X = \|A(\lambda - A)^{-1}u\|_X \\ &= \|(\lambda - A)^{-1}Au\|_X \leq \frac{\|Au\|_X}{\lambda} \to 0 \text{ as } \lambda \to \infty, \end{aligned}$$

where we used condition (ii) of Theorem 1.48. Hence $\lim_{\lambda\to\infty} \lambda R_\lambda u = u$ for all $u \in X$, since $\mathsf{D}(A) \subset X$ dense. The Yosida approximation is then defined by

$$A_\lambda := A\lambda R_\lambda = \lambda A(\lambda - A)^{-1} = \lambda^2(\lambda - A)^{-1} - \lambda I$$

for $\lambda > 0$. Then the bounded operators A_λ approximate A, since for $u \in \mathsf{D}(A)$ it follows

$$A_\lambda u = \lambda A(\lambda - A)^{-1}u = \lambda(\lambda - A)^{-1}(Au) \to Au \text{ as } \lambda \to \infty.$$

Remark 1.49. Feller, Miyadera and Phillips (1952; see e.g. [21], Theorem 3.8) extended Theorem 1.48 to hold for strongly continuous semigroups $(T_t)_{t\geqslant 0}$ that are not necessarily contractive. They showed that a linear operator A generates a semigroup of operators that satisfy $\|T_t\|_X \leqslant M\, e^{\omega t}$ with $M \geqslant 1$ and growth bound $\omega \in \mathbb{R}$ if A is closed, densely defined, $(\omega, \infty) \in \rho(A)$ and $\|(\lambda - A)^{-n}\|_X \leqslant M(\lambda - \omega)^{-n}$ for all $\lambda > \omega$ and $n \in \mathbb{N}$. A variant of the theorem by Hille and Yosida was found by Lumer and Phillips, which does not need estimates for all natural exponents of the resolvent, but instead uses the notion of *dissipativity* of the operator A which is an easily verifiable condition.

Definition 1.50 (Dissipativity). A linear operator $A : \mathsf{D}(A) \to X$, defined on a subset of the Banach space $(X, \|\cdot\|_X)$, is called *dissipative* if $\|(\lambda - A)u\|_X \geqslant \lambda\, \|u\|_X$ for all $\lambda > 0$ and $u \in \mathsf{D}(A)$.

Theorem 1.51 (Lumer, Phillips; 1961). *Let A be a linear operator on $(X, \|\cdot\|_X)$, where X is a Banach space. Then A generates a strongly continuous contraction semigroup if, and only if, the following conditions are satisfied:*

(i) A is closed and $\mathsf{D}(A) \subset X$ dense,

(ii) A is dissipative and

(iii) the range $\mathsf{R}(\lambda - A)$ is dense in X for some $\lambda > 0$.

One major concern in the theory of semigroups related to Markov processes is to classify generators of Feller and L^p-sub-Markovian semigroups. It turns out that for the classification of generators of Feller semigroups we need the following concept; see [44, 45].

Definition 1.52 (Positive maximum principle). Let $A : \mathsf{D}(A) \to C_\infty(\mathbb{R}^n)$ be a linear operator on $\mathsf{D}(A) \subset C_\infty(\mathbb{R}^n)$. Then A satisfies the *positive maximum principle* if

$$u \in \mathsf{D}(A) \text{ and } u(x_0) = \sup_{x\in\mathbb{R}^n} u(x) \geqslant 0 \text{ imply } Au(x_0) \leqslant 0. \tag{1.12}$$

For these operators we have the following result; see Lemma 4.5.2 in [45].

Lemma 1.53. *Let $A : \mathsf{D}(A) \to C_\infty(\mathbb{R}^n)$ with $\mathsf{D}(A) \subset C_\infty(\mathbb{R}^n)$ satisfy* (1.12). *Then A is dissipative.*

In [45], Chapter 4.5, it is also shown that the generator $(A^{(\infty)}, \mathsf{D}(A^{(\infty)}))$ of a Feller semigroup satisfies the positive maximum principle (1.12). The following theorem by Ph. Courrège provides a characterisation of operators satisfying (1.12), hence it gives a characterisation of the generator of a Feller semigroup $(T_t^{(\infty)})_{t\geqslant 0}$. If the Feller semigroup arises from operators of the form (1.11), see Example 1.43, we have already seen that its generator has the form of a pseudodifferential operator. The theorem by Courrège gives this result for operators whose symbols are of a more general form.

Theorem 1.54 (Courrège). *Suppose A is a linear operator on $C_0^\infty(\mathbb{R}^n)$ and satisfies* (1.12). *Then $A^{(\infty)}$ has the representation*

$$A^{(\infty)}u(x) = -q(x, D)\, u(x) = -(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\, q(x,\xi)\, \hat{u}(\xi)\, d\xi$$

as a pseudodifferential operator, where $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ is a measurable locally bounded function such that $\xi \mapsto q(x,\xi)$ is continuous negative definite and therefore admits a Lévy-Khinchin representation.

Now the following theorem gives a classification of those operators that generate a Feller semigroup. It is a variant of the Theorems 1.48 and 1.51 taking the positive maximum principle into account.

Theorem 1.55 (Hille-Yosida-Ray). *A linear operator $(A^{(\infty)}, \mathsf{D}(A^{(\infty)}))$ on the Banach space $C_\infty(\mathbb{R}^n) \supset \mathsf{D}(A^{(\infty)})$ generates a Feller semigroup if, and only if,*

(i) $A^{(\infty)}$ is closed and $\mathsf{D}(A^{(\infty)}) \subset C_\infty(\mathbb{R}^n)$ dense,

(ii) $A^{(\infty)}$ satisfies the positive maximum principle (1.12),

(iii) the range $\mathsf{R}(\lambda - A^{(\infty)})$ is dense in $C_\infty(\mathbb{R}^n)$ for some $\lambda > 0$.

For an operator A to be the generator of a sub-Markovian semigroup on $L^p(\mathbb{R}^n)$ it is necessary to be a Dirichlet operator, which is a concept we will define in § 1.5. Before that, we will give a brief introduction to subordination in the sense of Bochner of operator semigroups. It is a technique used to obtain new semigroups from given ones.

1.4 Subordinating operator semigroups

This section gives only a short overview about obtaining new semigroups from given ones by subordination. We include this section in order to apply the subordination technique in the following paragraphs, where we have to deal with operators of fractional powers. In particular, we will encounter the Γ-transform of an L^p-sub-Markovian semigroup $(T_t^{(p)})_{t\geqslant 0}$ in § 1.6, which gives an example of a subordinated semigroup. We will show that complete Bernstein functions, as introduced in § 1.2 play a central role. For a detailed discussion of this topic, see [45, 24, 25].

Let $(T_t)_{t\geqslant 0}$ denote a strongly continuous contraction semigroup on $L^p(\mathbb{R}^n)$ or $C_\infty(\mathbb{R}^n)$ with generator $(A, D(A))$. We consider a convolution semigroup of measures $(\eta_t)_{t\geqslant 0}$ supported in $[0,\infty)$. Then we have the correspondence

$$\mathcal{L}\eta_t(x) = \int_0^\infty e^{-sx}\,\eta_t(ds) = e^{-tf(x)} \quad \text{for } x, t > 0$$

between a convolution semigroup of sub-probability measures on $[0,\infty)$ and a Bernstein function f. The convolution semigroup $(\eta_t)_{t\geqslant 0}$ is uniquely determined in this case, so we have a one-to-one correspondence between f and $(\eta_t)_{t\geqslant 0}$, see Chapter 3.9 in [45]. Without proof we will quote the following fundamental result.

Theorem 1.56 ([45], Theorem 4.3.1). *Let $(T_t)_{t\geqslant 0}$ be a strongly continuous semigroup of contractions on a Banach space $(X, \|\cdot\|_X)$, the semigroup $(\eta_t)_{t\geqslant 0}$ with corresponding Bernstein function f be defined as above, and for $u \in X$ let*

$$T_t^f u := \int_0^\infty T_s u\,\eta_t(ds)\,.$$

Then the integral is well-defined and $(T_t^f)_{t\geqslant 0}$ is again a strongly continuous contraction semigroup.

Definition 1.57. In the situation of Theorem 1.56 we call the semigroup $(T_t^f)_{t\geqslant 0}$ *subordinate (in the sense of Bochner) to $(T_t)_{t\geqslant 0}$ with respect to f.*

The property of $(T_t)_{t\geqslant 0}$ being a Feller- or an L^p-sub-Markovian semigroup carries over to the subordinated semigroup $(T_t^f)_{t\geqslant 0}$, see Corollary 4.3.4. in [45].

Our next goal is to determine the generator $(A^f, D(A^f))$ of the semigroup $(T_t^f)_{t\geqslant 0}$. For a representation of A^f we need the notion of a (complete) Bernstein function, see Definitions 1.30 and 1.32. It is shown in Theorem 1.4.3. in [25] that the generator of a subordinated semigroups with respect to a complete Bernstein function f has the representation

$$A^f u = -au + bAu + \int_0^\infty A(s-A)^{-1}u\,\rho(ds)\,, \quad u \in D(A)$$

with the domain $D(A^f) = D(A)$ if $b \neq 0$ and

$$D(A^f) = \left\{u \in X : \lim_{k\to\infty} \int_0^k A(s-A)^{-1}u\,\rho(ds) \text{ exists weakly in } X\right\},$$

if $b = 0$, where X is either the space $L^p(\mathbb{R}^n)$ or $C_\infty(\mathbb{R}^n)$. As mentioned in [24] subordination of operator semigroups brings forth a functional calculus which is in accordance with that of fractional powers of operators, which we get in the special case $f(s) = s^\alpha$ for $0 < \alpha < 1$. More precisely, for a complete Bernstein function f we can represent the generator A^f of a subordinated semigroup as $A^f = -f(-A)$, and $f(A)$ is given by the Dunford integral

$$f(A) = \frac{1}{2\pi i}\int_\gamma f(s)\,(s-A)^{-1}\,ds\,,$$

where γ is a closed path around the spectrum of the operator A in the complex plane.

For the remainder of this paragraph we want to give an example of a subordinated semigroup.

Example 1.58 (Γ-transform of an L^p-sub-Markovian semigroup). The Γ-transform $(V_r^{(p)})_{r\geqslant 0}$ of an L^p-sub-Markovian semigroup $(T_t^{(p)})_{t\geqslant 0}$ is given by

$$V_r^{(p)}u := \frac{1}{\Gamma(r/2)}\int_0^\infty t^{\frac{r}{2}-1}\, e^{-t}\, T_t^{(p)}u\,dt\,, \qquad u\in L^p(\mathbb{R}^n)\,.$$

Its corresponding convolution semigroup is the Γ-semigroup

$$\eta_t(ds) = \frac{1}{\Gamma(r/2)}\, s^{\frac{r}{2}-1}\, e^{-s}\, \chi_{[0,\infty)}(s)\, ds$$

and the corresponding Bernstein function $f(s) = \frac{1}{2}\,\log(1+s)$.

The Γ-transform of an L^p-sub-Markovian semigroup is again an L^p-sub-Markovian semigroup obtained by subordination as we have

$V_{r_1}^{(p)}\,V_{r_2}^{(p)} = V_{r_1+r_2}^{(p)}$	(semigroup property)
$\Vert V_r^{(p)}u\Vert_{L^p} \leqslant \Vert u\Vert_{L^p}$	(contraction property)
$\lim_{r\to 0+} V_r^{(p)}u = u$ in $L^p(\mathbb{R}^n)$	(strong continuity in 0)
$0\leqslant u\leqslant 1$ a.e. implies $0\leqslant V_r^{(p)}u\leqslant 1$ a.e.	(sub-Markov property)

A connection between the Γ-transform $(V_r^{(p)})_{r\geqslant 0}$ and the generator $(A^{(p)}, D(A^{(p)}))$ of the underlying semigroup $(T_t^{(p)})_{t\geqslant 0}$ is given in the following theorem.

Theorem 1.59. *Let $(T_t^{(p)})_{t\geqslant 0}$ be an L^p-sub-Markovian semigroup and $(A^{(p)}, D(A^{(p)}))$ its generator. For all $r>0$ and $u\in L^p(\mathbb{R}^n)$ we have*

$$V_r^{(p)}u = (I - A^{(p)})^{-r/2}u\,.$$

We will see in § 1.6 that the operators $V_r^{(p)}$ appear in the construction of Bessel-type potential spaces.

1.5 Translation-invariant symmetric Dirichlet forms

In contrast to Feller semigroups, where Courrège's theorem classifies their generators, we cannot make use of similar pointwise statements such as the positive maximum principle in any L^p-setting. It turns out that the following concept is essential to obtain a characterisation of generators of sub-Markovian semigroups, see e.g. Chapter 4.6 in [45].

Definition 1.60 (Dirichlet operator). A closed, densely defined linear operator $A:\mathsf{D}(A)\to L^p(\mathbb{R}^n)$, $1<p<\infty$, with domain $\mathsf{D}(A)\subset L^p(\mathbb{R}^n)$, is called a *Dirichlet operator* if

$$\int_{\mathbb{R}^n}(Au)\big((u-1)^+\big)^{p-1}\,dx \leqslant 0$$

for all $u\in\mathsf{D}(A)$.

For any Dirichlet operator $(A,\mathsf{D}(A))$ on $L^p(\mathbb{R}^n)$, $1<p<\infty$, it holds that

$$\int_{\mathbb{R}^n}(Au)(\operatorname{sgn}u)|u|^{p-1}\,dx = \int_{\mathbb{R}^n}(Au)\,|u^+|^{p-1}\,dx - \int_{\mathbb{R}^n}(Au)\,|u^-|^{p-1}\,dx \leqslant 0\,, \tag{1.13}$$

see Proposition 4.6.11 in [45].

Lemma 1.61. *A Dirichlet operator A on $L^p(\mathbb{R}^n)$ is dissipative, i.e.* $\|(\lambda - A)u\|_{L^p} \geqslant \lambda \|u\|_{L^p}$.

Proof. Using (1.13) and Hölder's inequality we obtain

$$\begin{aligned} \lambda \|u\|_{L^p}^p &\leqslant \lambda \int_{\mathbb{R}^n} |u|^p \, dx - \int_{\mathbb{R}^n} (Au)(\operatorname{sgn} u)|u|^{p-1} \, dx = \int_{\mathbb{R}^n} ((\lambda - A)u)(\operatorname{sgn} u)|u|^{p-1} \, dx \\ &\leqslant \|(\lambda - A)u\|_{L^p} \|(\operatorname{sgn} u)|u|^{p-1}\|_{L^q} = \|(\lambda - A)u\|_{L^p} \|u\|_{L^p}^{p/q}, \end{aligned}$$

where q is the conjugate exponent to p. Dividing both sides by $\|u\|_{L^p}^{p/q}$ gives the result. □

Note that due to Minkowksi's inequality and the dissipativity of A with $\lambda = 1$ we have the continuous embedding $\mathsf{D}(A) \hookrightarrow L^p(\mathbb{R}^n)$, since

$$\|u\|_{L^p} \leqslant \|(I - A)u\|_{L^p} \leqslant \|u\|_{L^p} + \|(-A)u\|_{L^p} = \|u\|_A ,$$

where $\|\cdot\|_A$ denotes the graph norm on $\mathsf{D}(A)$. The Dirichlet operator A is by definition closed and densely defined. If we further assume that the range $\mathsf{R}(\lambda - A)$ is dense in $L^p(\mathbb{R}^n)$ for some $\lambda > 0$, it follows from Theorem 1.51 that A generates an L^p-sub-Markovian semigroup. Conversely, it can be shown that every L^p-sub-Markovian generator is necessarily a Dirichlet operator on $L^p(\mathbb{R}^n)$, see the remarks in [45] leading to Lemma 4.6.6.

For the rest of this paragraph we consider the case $p = 2$ and work with symmetric sub-Markovian semigroups $(T_t^{(2)})_{t \geqslant 0}$ on $L^2(\mathbb{R}^n; \mathbb{R})$. In this setting the concept of a Dirichlet operator was introduced by Bouleau and Hirsch [9]. In general, however, neither the definition of a sub-Markovian semigroup nor that of a Dirichlet operator requires symmetry. For general p we can find the definition of the latter in [45]. The generator $(A, \mathsf{D}(A))$ of a symmetric L^2-sub-Markovian semigroup is a self-adjoint Dirichlet operator. In fact, we can state the following theorem (see [44], Theorem 3.2) giving also the converse.

Theorem 1.62. *A self-adjoint operator Let $A : \mathsf{D}(A) \to L^2(\mathbb{R}^n; \mathbb{R})$, $\mathsf{D}(A) \subset L^2(\mathbb{R}^n; \mathbb{R})$ dense, be a self-adjoint linear operator. Then it is a Dirichlet operator if, and only if, it generates a symmetric sub-Markovian semigroup on $L^2(\mathbb{R}^n; \mathbb{R})$.*

The fact that A is a Dirichlet operator implies

$$\int_{\mathbb{R}^n} (-Au)u \, dx = (-Au, u)_{L^2} \geqslant 0$$

by using (1.13). Hence $-A$ is a non-negative and self-adjoint operator. Then the following theorem ensures the existence of $(-A)^{1/2}$ and of a positive semidefinite bilinear form $(\mathcal{E}, \mathsf{D}(\mathcal{E}))$ on $L^2(\mathbb{R}^n; \mathbb{R})$ defined by

$$\mathcal{E}(u, v) := (-Au, v)_{L^2} = ((-A)^{1/2}u, (-A)^{1/2}v)_{L^2} \tag{1.14}$$

for $v \in \mathsf{D}(\mathcal{E}) := \mathsf{D}((-A)^{1/2})$ and $u \in \mathsf{D}(A) \cap \mathsf{D}(\mathcal{E})$. We quote this result from [45], Theorem 4.7.5.

Theorem 1.63. *Let $(A, \mathsf{D}(A))$ be a closed and densely defined self-adjoint operator on $L^2(\mathbb{R}^n; \mathbb{R})$, which satisfies* (1.13)*. Then there exists a closed, symmetric and positive semidefinite bilinear form $(\mathcal{E}, \mathsf{D}(\mathcal{E}))$ on $L^2(\mathbb{R}^n; \mathbb{R})$ defined by* (1.14)*, which satisfies the Cauchy-Schwarz inequality*

$$|(-Au, v)_{L^2}| \leqslant \|(-A)^{1/2}u\|_{L^2} \|(-A)^{1/2}v\|_{L^2}, \quad u \in \mathsf{D}(A),\ v \in \mathsf{D}(A^{1/2}).$$

Further, we have $\mathsf{D}(A) \hookrightarrow \mathsf{D}(\mathcal{E}) \hookrightarrow L^2(\mathbb{R}^n; \mathbb{R})$, where these continuous embeddings are dense. In particular, $(\mathsf{D}(A), \|\cdot\|_A)$ and $(\mathsf{D}(\mathcal{E}), \|\cdot\|_{\mathcal{E}})$ are Hilbert spaces equipped with the respective graph norms $\|u\|_A = \|u\|_{L^2} + \|(-A)u\|_{L^2}$ and $\|u\|_{\mathcal{E}} = \|u\|_{L^2} + \sqrt{\mathcal{E}(u, u)}$.

In this setting we can give an example of a *translation invariant symmetric Dirichlet form* involving a real-valued continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{R}$. For the general definition and a detailed introduction into the theory of not necessarily symmetric Dirichlet forms, we refer to the book of Z.-M. Ma and M. Röckner [52], which includes analytic as well as probabilistic aspects of the theory up to the construction of the related Markov processes. Another reference is the monograph by M. Fukushima et al. [27], which covers the purely symmetric case.

Definition 1.64 (Symmetric/Local Dirichlet form). **A.** Let $(\mathcal{E}, \mathsf{D}(\mathcal{E}))$ be a closed bilinear form on $L^2(\mathbb{R}^n)$. If it is symmetric and

$$\mathcal{E}(u^+ \wedge 1, u^+ \wedge 1) \leqslant \mathcal{E}(u,u) \quad \text{for all } u \in \mathsf{D}(\mathcal{E})),$$

then $(\mathcal{E}, \mathsf{D}(\mathcal{E}))$ is said to be a *symmetric Dirichlet form.*

B. A Dirichlet form $(\mathcal{E}, \mathsf{D}(\mathcal{E}))$ having the property that two functions $u, v \in C_0^\infty(\mathbb{R}^n)$ with disjoint support yield $\mathcal{E}(u,v) = 0$, is called a *local Dirichlet form.*

Example 1.65. In this example we show how in the symmetric $L^2(\mathbb{R}^n; \mathbb{R})$ setting real-valued continuous negative definite functions come into play as we can express the bilinear form (1.14) with the help of a fixed $\psi : \mathbb{R}^n \to \mathbb{R}$. Let $(\mu_t)_{t\geqslant 0}$ be the associated convolution semigroup of measures and $(T_t)_{t\geqslant 0}$ be defined as in (1.11). As already noted in §1.3 each operator has an extension from $\mathcal{S}(\mathbb{R}^n; \mathbb{R})$ to $L^2(\mathbb{R}^n; \mathbb{R})$ where it is – due to Plancherel's Theorem – a contraction. As we restrict ourselves to real-valued functions ψ, we can extend $(T_t)_{t\geqslant 0}$ from $\mathcal{S}(\mathbb{R}^n; \mathbb{R})$ to a symmetric semigroup on $L^2(\mathbb{R}^n; \mathbb{R})$ by

$$(T_t u, v)_{L^2} = \int_{\mathbb{R}^n} e^{-t\psi(\xi)} \hat{u}(\xi)\, \overline{\hat{v}(\xi)}\, d\xi = \int_{\mathbb{R}^n} \hat{u}(\xi)\, \overline{e^{-t\psi(\xi)}\, \hat{v}(\xi)}\, d\xi = (u, T_t v)_{L^2}$$

making use of Plancherel's Theorem. For $u \in \mathcal{S}(\mathbb{R}^n; \mathbb{R})$ the generator of the semigroup is given by the pseudodifferential operator

$$Au(x) = -\psi(D)u(x) = -(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \psi(\xi) \hat{u}(\xi)\, d\xi\,.$$

By the properties of the Fourier transform, the generator $-\psi(D)$ is invariant under translations, see [45], Example 4.7.28. Substituting $A = -\psi(D)$ in the definition (1.14) of $\mathcal{E}$ we find

$$\mathcal{E}(u,v) = (-Au, v)_{L^2} = \int_{\mathbb{R}^n} \psi(D)u(x)\overline{v(x)}\, dx = \int_{\mathbb{R}^n} \psi(\xi)\hat{u}(\xi)\overline{\hat{v}(\xi)}\, d\xi\,. \tag{1.15}$$

This bilinear form is an example of a *translation invariant symmetric Dirichlet form.*

Taking into account that $\psi(\xi)$ has a Lévy-Khinchin representation (1.7), we can reformulate (1.15) for $u, v \in \mathcal{S}(\mathbb{R}^n; \mathbb{R})$ as

$$\mathcal{E}(u,v) = c\int_{\mathbb{R}^n} \hat{u}(\xi)\, \overline{\hat{v}(\xi)}\, d\xi + \int_{\mathbb{R}^n} q(\xi)\, \hat{u}(\xi)\, \overline{\hat{v}(\xi)}\, d\xi + \int_{\mathbb{R}^n} \hat{u}(\xi)\, \overline{\hat{v}(\xi)} \int_{\mathbb{R}^n\setminus\{0\}} (1 - \cos(x\cdot\xi))\, \nu(dx)\, d\xi\,.$$

By the properties of the Fourier transform, Plancherel's Theorem and by expressing $\cos(y\cdot\xi)$ as

$\frac{1}{2}\left(e^{iy\cdot\xi}+e^{-iy\cdot\xi}\right)$ we arrive at

$$\begin{aligned}\mathcal{E}(u,v) &= c\int_{\mathbb{R}^n} u(x)\,v(x)\,dx + \int_{\mathbb{R}^n} q(x)\,\nabla u(x)\cdot\nabla v(x)\,dx \\ &\quad +\tfrac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\big(u(x+y)-u(x)\big)\big(v(x+y)-v(x)\big)\,\nu(dy)\,dx \\ &= c\int_{\mathbb{R}^n} u(x)\,v(x)\,dx + \int_{\mathbb{R}^n} q(x)\,\nabla u(x)\cdot\nabla v(x)\,dx \\ &\quad +\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)\,J(dx,dy)\,,\end{aligned} \tag{1.16}$$

where ν is the associated Lévy measure to ψ and J a symmetric measure without any mass on the diagonal $x=y$. The latter representation is often referred to as *Beurling-Deny representation.* Note that if u and v are $C_0^\infty(\mathbb{R}^n;\mathbb{R})$ functions with disjoint support it follows that the *local part* of $\mathcal{E}$ vanishes, i.e.

$$\int_{\mathbb{R}^n} u(x)\,v(x)\,dx + \int_{\mathbb{R}^n} q(x)\,\nabla u(x)\cdot\nabla v(x)\,dx = 0\,. \tag{1.17}$$

For this terminology, see [44].

According to the Beurling-Deny formula and (1.17), disjoint supports of u and v yield $\mathcal{E}(u,v)=0$ if, and only if, $J=0$. The generator of a local form is in general a differential operator. This is not true for non-local forms whose generators are in general pseudodifferential operators. In this work we are mainly interested in non-local Dirichlet forms having the Beurling-Deny representation

$$\mathcal{E}(u,v)=\iint_{\mathbb{R}^n}\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)\,J(x,y)\,dy\,dx \tag{1.18}$$

with a jump kernel that has a density $J(x,y)$ with respect to the Lebesgue measure.

1.6 On ψ-Bessel potential spaces

In this paragraph we study a family of Sobolev spaces related to continuous negative definite functions $\psi:\mathbb{R}^n\to\mathbb{R}$. They occur as domains $\mathsf{D}(\mathcal{E})$ of translation invariant symmetric Dirichlet forms as in (1.15) and also as domains $\mathsf{D}(A^{(p)})$ of generators of L^p-sub-Markovian semigroups. In order to characterise $\mathsf{D}(A^{(p)})$ for $1<p\leqslant\infty$ in terms of function spaces we fix a real-valued continuous negative definite function $\psi:\mathbb{R}^n\to\mathbb{R}$ having the Lévy-Khinchin representation

$$\psi(\xi)=\int_{\mathbb{R}^n\setminus\{0\}}\big(1-\cos(y\cdot\xi)\big)\,\nu(dy) \tag{1.19}$$

and assume it satisfies properties (iv) and (vii) of Lemma 1.20. Let $(T_t^{(p)})_{t\geqslant 0}$, $1\leqslant p\leqslant\infty$, be the associated L^p-sub-Markovian or Feller semigroup with generator $(A^{(p)},\mathsf{D}(A^{(p)}))$ and $\mathsf{D}(A^{(p)})\subset L^p(\mathbb{R}^n;\mathbb{R})$ or $\mathsf{D}(A^{(p)})\subset C_\infty(\mathbb{R}^n;\mathbb{R})$, respectively. In [25] we find the following result.

Proposition 1.66. *Let $\psi:\mathbb{R}^n\to\mathbb{R}$ be continuous negative definite with representation* (1.19). *Then for $1<p\leqslant\infty$ we have that $\mathcal{S}(\mathbb{R}^n;\mathbb{R})\subset\mathsf{D}(A^{(p)})$ and for $u\in\mathcal{S}(\mathbb{R}^n;\mathbb{R})$ the generator $A^{(p)}$ of the semigroup $(T_t^{(p)})_{t\geqslant 0}$ can be represented as the pseudodifferential operator $A^{(p)}u(x)=-\psi(D)u(x)$.*

Definition 1.67. Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a continuous negative definite function with Lévy-Khinchin representation (1.19). Then the space

$$H_p^{\psi,2}(\mathbb{R}^n;\mathbb{R}) := \{u \in L^p(\mathbb{R}^n;\mathbb{R}) : \|u\|_{H_p^{\psi,2}} < \infty\}$$

with the norm

$$\|u\|_{H_p^{\psi,2}} := \|(I + \psi(D))u\|_{L^p}$$

is called the *ψ-Bessel potential space of order 2* with respect to $L^p(\mathbb{R}^n;\mathbb{R})$, $1 \leqslant p < \infty$.

Given that $u \in \mathcal{S}(\mathbb{R}^n;\mathbb{R})$, assuming that $\psi \in C^\infty(\mathbb{R}^n;\mathbb{R})$ and ψ and all its derivatives are polynomially bounded we note that $\psi\,\hat{u} \in \mathcal{S}(\mathbb{R}^n;\mathbb{R})$, hence $(1+\psi)\hat{u} \in L^p(\mathbb{R}^n;\mathbb{R})$ and $\mathcal{F}^{-1}((1+\psi(\cdot))\hat{u}(\cdot))$ is defined. This gives

$$\|u\|_{H_p^{\psi,2}} = \|(I + \psi(D))u\|_{L^p} = \|\mathcal{F}^{-1}((1+\psi(\cdot))\hat{u}(\cdot))\|_{L^p}\,. \tag{1.20}$$

Therefore, to define $\|\cdot\|_{H_p^{\psi,2}}$ with the help of the Fourier transform we need that $\psi \in C^\infty(\mathbb{R}^n;\mathbb{R})$ which is in general not true. However, employing a decomposition of ψ into the sum of two appropriate continuous negative definite functions ψ_R, $\widetilde{\psi}_R$ and proving norm equivalences using these functions, we can show that (1.20) indeed makes sense. For a detailed derivation of this result, we refer to [25]. Here, we will only quote some essential ideas.

First we decompose ψ of the form (1.19) as $\psi(\xi) = \psi_R(\xi) + \widetilde{\psi}_R(\xi)$ with

$$\psi_R(\xi) = \int_{\mathbb{R}^n\setminus\{0\}} (1 - \cos(y\cdot\xi))\,\nu_R(y) := \int_{B(0,R)\setminus\{0\}} (1 - \cos(y\cdot\xi))\,\nu(dy)$$

for some $R > 0$ and

$$\widetilde{\psi}_R(\xi) = \int_{\mathbb{R}^n\setminus\{0\}} (1 - \cos(y\cdot\xi))\,\widetilde{\nu}_R(y) := \int_{B^c(0,R)} (1 - \cos(y\cdot\xi))\,\nu(dy)$$

and consider for $u \in \mathcal{S}(\mathbb{R}^n;\mathbb{R})$ the operators

$$\begin{aligned} \psi_R(D)u(x) &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\,\psi_R(\xi)\,\hat{u}(\xi)\,d\xi \qquad (1.21)\\ \widetilde{\psi}_R(D)u(x) &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\,\widetilde{\psi}_R(\xi)\,\hat{u}(\xi)\,d\xi\,, \end{aligned}$$

which gives $\psi(D)u(x) = \psi_R(D)u(x) + \widetilde{\psi}_R(D)u(x)$ for $u \in \mathcal{S}(\mathbb{R}^n;\mathbb{R})$. This makes sense due to Proposition 1.66.

We assume that $\nu(B^c(0,R)) < \infty$ and rewrite $\widetilde{\psi}_R(D)u(x)$ for $u \in \mathcal{S}(\mathbb{R}^n;\mathbb{R})$ as

$$\widetilde{\psi}_R(D)u(x) = \int_{B^c(0,R)} (u(x) - u(x-y))\,\nu(dy)$$

using the property $u(x-y) = \mathcal{F}^{-1}(e^{-iy\cdot\xi}\,\hat{u}(\xi))$. The operator $\widetilde{\psi}_R(D)$ extends from $\mathcal{S}(\mathbb{R}^n;\mathbb{R})$ to a bounded operator in $L^p(\mathbb{R}^n;\mathbb{R})$, $1 \leqslant p < \infty$, and

$$\|\widetilde{\psi}_R(D)\|_{L^p} \leqslant 2\,\|u\|_{L^p}\,\nu(B^c(0,R)) < \infty\,. \tag{1.22}$$

Turning to the operator $\psi_R(D)$ we note that $\psi_R(\xi) \in C^\infty(\mathbb{R}^n;\mathbb{R})$ as the Lévy measure ν_R supported on the bounded set $B(0,R)\setminus\{0\}$ satisfies the integrability property

$$\int_{\mathbb{R}^n\setminus\{0\}} |y|^j\,\nu_R(dy) < \infty$$

for all $j \geqslant 2$ and hence satisfies the condition of Theorem 1.39. Then, taking $u \in \mathcal{S}(\mathbb{R}^n;\mathbb{R})$ we get $\psi_R(\xi)\,\hat{u}(\xi) \in \mathcal{S}(\mathbb{R}^n;\mathbb{R})$ and $\psi_R(D)\,u(x) = \mathcal{F}^{-1}(\psi_R(\xi)\,\hat{u}(\xi))$ is extendable to $L^p(\mathbb{R}^n;\mathbb{R})$, $1 \leqslant p < \infty$, due to the density of $\mathcal{S}(\mathbb{R}^n;\mathbb{R}) \subset L^p(\mathbb{R}^n;\mathbb{R})$. We use this to give the following definition.

Definition 1.68. For $u \in L^p(\mathbb{R}^n;\mathbb{R})$ we define the norm

$$\|u\|_{\psi,R,p} := \|(I + \psi_R(D))u\|_{L^p}\,,$$

where $\psi_R(D)$ is the operator given in (1.21).

The following norm equivalences hold.

Lemma 1.69. *For $0 < R < S$ the norms $\|\cdot\|_{\psi,R,p}$ and $\|\cdot\|_{\psi,S,p}$ and moreover, $\|\cdot\|_{H_p^{\psi,2}}$ and $\|\cdot\|_{\psi,R,p}$ are equivalent for all $R > 0$.*

Lemma 1.69 justifies the definition of the norm $\|\cdot\|_{H_p^{\psi,2}}$ via the Fourier transform as in (1.20). For $1 \leqslant p < \infty$ the space $\mathcal{S}(\mathbb{R}^n)$ is dense in $H_p^{\psi,2}(\mathbb{R}^n;\mathbb{R})$. Moreover, $\mathcal{S}(\mathbb{R}^n;\mathbb{R})$ is an operator core for $(A^{(p)}, \mathsf{D}(A^{(p)}))$, i.e. $\mathcal{S}(\mathbb{R}^n;\mathbb{R}) \subset \mathsf{D}(A^{(p)})$ is dense with respect to the graph norm of $A^{(p)}$. Further, the following theorem can also be found in [25].

Theorem 1.70. *Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be continuous and negative definite with representation* (1.19). *Then $\mathsf{D}(A^{(p)}) = \mathsf{D}(-\psi(D)) = H_p^{\psi,2}(\mathbb{R}^n;\mathbb{R})$.*

Associated with the generator $A^{(p)} = -\psi(D)$ of the semigroup $(T_t^{(p)})_{t\geqslant 0}$ we can define ψ-Bessel potential spaces of higher order.

Definition 1.71. Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be continuous and negative definite. Then the space

$$H_p^{\psi,s}(\mathbb{R}^n;\mathbb{R}) := \{u \in L^p(\mathbb{R}^n;\mathbb{R}) : \|u\|_{H_p^{\psi,s}} < \infty\}$$

with the norm

$$\|u\|_{H_p^{\psi,s}} := \|(I + \psi(D))^{s/2}u\|_{L^p}$$

is called the *ψ-Bessel potential space of order s* with respect to $L^p(\mathbb{R}^n;\mathbb{R})$, $1 \leqslant p < \infty$.

In [25] it is shown that $\mathcal{S}(\mathbb{R}^n;\mathbb{R})$ is a dense subset of $H_p^{\psi,s}(\mathbb{R}^n;\mathbb{R})$ for any $s \geqslant 0$ and $1 \leqslant p < \infty$. Further, the domain of $(I + \psi(D))^{s/2}$ coincides with $H_p^{\psi,s}(\mathbb{R}^n;\mathbb{R})$, and the continuous embedding $H_p^{\psi,t}(\mathbb{R}^n;\mathbb{R}) \hookrightarrow H_p^{\psi,s}(\mathbb{R}^n;\mathbb{R})$ holds for any $1 \leqslant p < \infty$ whenever $t \geqslant s$. Similar to the case $s = 2$ it is possible to show, see [25, 46], that

$$\|u\|_{H_p^{\psi,s}} = \|\mathcal{F}^{-1}((1 + \psi(\cdot))^{s/2}\hat{u})\|_{L^p}$$

for $s \geqslant 0$ and $u \in \mathcal{S}(\mathbb{R}^n;\mathbb{R})$ by the same cut-off argument using the function $\psi_R(\xi)$, $R > 0$. Later, we will need the following norm equivalence result which is taken from [25].

Theorem 1.72. *For $u \in \mathcal{S}(\mathbb{R}^n;\mathbb{R})$ there exist constants c_0 and c_1 such that*

$$c_0\big(\|\mathcal{F}^{-1}(\psi(\cdot)^{s/2}\hat{u})\|_{L^p} + \|u\|_{L^p}\big) \leqslant \|u\|_{H_p^{\psi,s}} \leqslant c_1\big(\|\mathcal{F}^{-1}(\psi(\cdot)^{s/2}\hat{u})\|_{L^p} + \|u\|_{L^p}\big)$$

for any $1 \leqslant p < \infty$.

Note that in the L^2-case we may drop the inverse Fourier transform which reduces the estimates in Theorem 1.72 to

$$c_0\left(\|\psi(\cdot)^{s/2}\hat{u}\|_{L^2} + \|u\|_{L^2}\right) \leqslant \|u\|_{H_2^{\psi,s}} \leqslant c_1\left(\|\psi(\cdot)^{s/2}\hat{u}\|_{L^2} + \|u\|_{L^2}\right) \tag{1.23}$$

We will need the norm equivalence (1.23) when studying the domain $\mathsf{D}(\mathcal{E}) = \mathsf{D}((-A)^{1/2})$ of the Dirichlet form associated with a symmetric L^2-sub-Markovian semigroup.

For $p = 2$ the ψ-Bessel potential spaces $H^{\psi,s}(\mathbb{R}^n;\mathbb{R}) := H_2^{\psi,s}(\mathbb{R}^n;\mathbb{R})$ are Hilbert spaces where the norm is induced by the inner product

$$(u,v)_{H^{\psi,s}} := \int_{\mathbb{R}^n} (I+\psi(D))^{s/2}\,u(x)\,\overline{(I+\psi(D))^{s/2}\,v(x)}\,dx = \int_{\mathbb{R}^n} (1+\psi(\xi))^s\,\hat{u}(\xi)\,\overline{\hat{v}(\xi)}\,d\xi\,.$$

Let us make some remarks about the relation of ψ-Bessel potential spaces to classical function spaces.

Remark 1.73. For the continuous negative definite function $\psi(\xi) = |\xi|^2$ the space $H_p^{|\cdot|^2,s}(\mathbb{R}^n;\mathbb{R}) = H_p^2(\mathbb{R}^n;\mathbb{R}) = W^{2,p}(\mathbb{R}^n;\mathbb{R})$ is the classical Sobolev space of order 2 with respect to L^p. For arbitrary order s we arrive at the classical Bessel potential spaces $H_p^s(\mathbb{R}^n;\mathbb{R})$ in which the norm is given by

$$\|u\|_{H_p^s} = \|\mathcal{F}^{-1}((1+|\cdot|^2)^{s/2}\hat{u})\|_{L^p}\,.$$

Sometimes these spaces are also called *fractional Sobolev spaces.* Again, the inverse of the Fourier transform is negligible when working in the fractional Sobolev space $H_2^s(\mathbb{R}^n;\mathbb{R})$. For any continuous negative definite function ψ and all $1 \leqslant p < \infty$ we obviously have $H_p^{\psi,0}(\mathbb{R}^n;\mathbb{R}) = L^p(\mathbb{R}^n;\mathbb{R})$.

The domain $\mathsf{D}(\mathcal{E})$ of the translation invariant symmetric Dirichlet form (1.15) can also be characterised in terms of a ψ-Bessel potential space. By the fact that $\mathsf{D}(\mathcal{E}) = \mathsf{D}((-A)^{1/2})$ with A being the infinitesimal generator of a symmetric L^2-sub-Markovian semigroup we may conclude that $\mathsf{D}(\mathcal{E}) = H_2^{\psi,1}(\mathbb{R}^n;\mathbb{R})$, i.e. the set of those functions $u \in L^2(\mathbb{R}^n;\mathbb{R})$ such that $\|u\|_{H_p^{\psi,1}} = \|(1+\psi(\xi))^{1/2}\hat{u}\|_{L^2} < \infty$. Note that

$$\mathcal{E}(u,u) = \int_{\mathbb{R}^n} (1+\psi(\xi))\,|\hat{u}(\xi)|^2\,d\xi - \int_{\mathbb{R}^n} |\hat{u}(\xi)|^2\,d\xi$$

which is by Plancherel's Theorem equivalent to

$$\mathcal{E}(u,u) + \|u\|_{L^2}^2 = \int_{\mathbb{R}^n} (1+\psi(\xi))\,|\hat{u}(\xi)|^2\,d\xi = \|u\|_{H_2^{\psi,1}}^2\,.$$

Moreover, using Plancherel's Theorem again and (1.23) with $s = 1$ we have the norm equivalence of $\|\cdot\|_{H_2^{\psi,1}}$ and the graph norm $\|\cdot\|_{\mathcal{E}}$, since

$$\|u\|_{\mathcal{E}} = \|u\|_{L^2} + \|\psi(D)^{1/2}u\|_{L^2} = \|u\|_{L^2} + \|\psi(\cdot)^{1/2}\hat{u}\|_{L^2} \approx \|u\|_{H_2^{\psi,1}}\,.$$

1.7 Negative definite functions and weighted function spaces

In this paragraph we study another property of continuous negative definite functions, namely their connection to weighted function spaces in $\mathbb{R}^n$. More precisely, we assume that a real-valued continuous negative definite function $\psi(\xi)$ satisfies Theorem 1.39, i.e. $\psi(\xi)$ has classical derivatives up to a certain order with upper bounds for all multi-indices $\alpha \in \mathbb{N}_0^n$. Then it is possible to show that the class of these functions forms a class of admissible weight functions in the sense of Triebel [59], Chapter 6.

Definition 1.74. Let $n \in \mathbb{N}$. The class W^n of *admissible weight functions* is the collection of all $w \in C^\infty(\mathbb{R}^n;\mathbb{R})$, $w > 0$, which meet the following conditions:

(i) For all multi-indices $\alpha \in \mathbb{N}_0^n$ there exists $c_{|\alpha|} > 0$ such that $|\partial^\alpha w(x)| \leqslant c_{|\alpha|}\, w(x)$ for all $x \in \mathbb{R}^n$ and

(ii) there exist $c > 0$ and $\gamma \geqslant 0$ such that $w(x) \leqslant c\, w(y)\,(1+|x-y|^2)^{\gamma/2}$ for all $x, y \in \mathbb{R}^n$.

Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be continuous negative definite with the general Lévy-Khinchin representation (1.7), such that the Lévy measure ν satisfies the integrability property

$$M_j := \int_{\mathbb{R}^n\setminus\{0\}} |y|^j\, \nu_R(dy) < \infty\,, \quad 2 \leqslant j \leqslant m\,,\, m \in \mathbb{R}\,, \tag{1.24}$$

which is sufficient for ψ to be m-times continuously differentiable with the bounds

$$|\partial^\alpha \psi(\xi)| \leqslant c_{|\alpha|}\, \psi(\xi)^{\frac{2-\rho(|\alpha|)}{2}}\,, \tag{1.25}$$

where $\rho(k) := k \wedge 2$, see Theorem 1.39. We are studying smooth continuous negative definite functions $\psi : \mathbb{R}^n \to \mathbb{R}$ that satisfy (1.25) for $\xi \in \mathbb{R}^n$. This implies that ψ also fulfils

$$|\partial^\alpha(1+\psi(\xi))| \leqslant c_{|\alpha|}\, (1+\psi(\xi))^{\frac{2-\rho(|\alpha|)}{2}} \tag{1.26}$$

for $\alpha \in \mathbb{N}_0^n$, $c_{|\alpha|} > 0$. This motivates the following definition for general $m \in \mathbb{R}$.

Definition 1.75. Let $\rho(k) = k \wedge 2$. We define the class Λ_m as the class of continuous negative definite functions $\psi : \mathbb{R}^n \to \mathbb{R}$ for which

$$|\partial^\alpha(1+\psi(\xi))^{m/2}| \leqslant c_{|\alpha|}\, (1+\psi(\xi))^{\frac{m-\rho(|\alpha|)}{2}}$$

holds for all multi-indices $\alpha \in \mathbb{N}_0^n$.

For $m = 2$ the set $\Lambda := \Lambda_2$ was defined by Hoh in [42, 43], see also [46]. The following result is proved in [46], Chapter 2.4.

Lemma 1.76. *Assume that the Lévy measure ν associated with $\psi : \mathbb{R}^n \to \mathbb{R}$ satisfies $M_j < \infty$ for $2 \leqslant j \leqslant m$ and $m \in \mathbb{R}$, with M_j given in (1.24). Then $\psi \in \Lambda_m$.*

In this setting we can prove the following result.

Proposition 1.77. *Let $\rho(k) = k\wedge 2$ and define the class Λ_m as in Definition 1.75. Then $\Lambda_m \subset W^n$ for $m \in \mathbb{R}$.*

Proof. We need to check that a continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{R}$ in the class Λ_m, $m \in \mathbb{R}$, satisfies conditions (i) and (ii) of Definition 1.74. By the definition of $\rho(|\alpha|)$ we have

$$\frac{m-2}{2} \leqslant \frac{m-\rho(|\alpha|)}{2} \leqslant \frac{m}{2}\,, \qquad m \in \mathbb{R}\,. \tag{1.27}$$

Now, a continuous negative definite function $\psi \in \Lambda_m$ satisfies

$$|\partial^\alpha(1+\psi(\xi))^{m/2}| \leqslant c_{|\alpha|}\, (1+\psi(\xi))^{\frac{m-\rho(|\alpha|)}{2}} \leqslant c_{|\alpha|}\, (1+\psi(\xi))^{m/2}\,,$$

by definition and (1.27), i.e. condition (i) is fulfilled for $w(\cdot) = (1 + \psi(\cdot))^{m/2}$.

To show that (ii) holds we use the Peetre-type inequality for general powers $s \in \mathbb{R}$, see Lemma 1.20 (v), i.e.

$$\left(\frac{1+\psi(x)}{1+\psi(y)}\right)^s \leqslant 2^{|s|}\,(1+|\psi(x-y)|)^{|s|},$$

which gives together with the polynomial boundedness $|\psi(x)| \leqslant c_\psi\,(1+|x|^2)$, see Lemma 1.20 (iv),

$$\begin{aligned}
\left(\frac{1+\psi(x)}{1+\psi(y)}\right)^s &\leqslant 2^{|s|}(1+|\psi(x-y)|)^{|s|} \leqslant \left(2\,(1+c_\psi\,(1+|x-y|^2))\right)^{|s|} \\
&\leqslant \left(2\,(1+|x-y|^2)+2\,c_\psi\,(1+|x-y|^2)\right)^{|s|} \\
&= 2^{|s|}(1+c_\psi)^{|s|}\,(1+|x-y|^2)^{|s|} =: c_{|s|}\,(1+|x-y|^2)^{|s|}.
\end{aligned}$$

Hence, for $s = \frac{m}{2}$, $m \in \mathbb{R}$,

$$(1+\psi(x))^s \leqslant c_{|m|/2}\,(1+\psi(y))^s\,(1+|x-y|^2)^{|m|/2}.$$

□

Proposition 1.77 ensures that the function $w_s(\cdot) := (1+\psi(\cdot))^{s/2}$ is an admissible weight function. Hence, it shows that we can consider ψ-Bessel potential spaces as weighted L^p-spaces in the "Fourier picture", where the weights are coming from standard classes.

§2 Some Aspects of Analysis on Metric Spaces

In this chapter we provide an overview of basic concepts and general analytic results on metric spaces (X, d, μ), where μ is a locally finite regular Borel measure on X. The paragraphs included here can be applied to the metric spaces we consider in § 4. Topics include a covering theorem of Vitali type as well as Lebesgue spaces and different approaches to construct Sobolev spaces on general metric measure spaces. Most parts of this chapter are based on [40], where a wide range of results is provided in these spaces.

2.1 The Vitali covering theorem

This section is based on [40], [53] and [20]. We first provide a geometric result on collections of balls before applying it to get a Vitali-type covering for metric measure spaces. The following Lemma is sometimes called the *Vitali Covering Lemma* and is a purely geometric result with no measures involved. However, it is the main component of the Vitali Covering Theorem we give below. We quote the version of [40], Chapter 1.

Lemma 2.1 (Vitali Covering Lemma). *Let X be an arbitrary metric space, $\mathcal{B} = \{B_i : i \in I\}$ a countable collection of balls $B_i = B^d(x_i, r_i)$ in (X, d) with uniformly bounded diameter, i.e. $\rho := \sup_{i\in I} \operatorname{diam}(B_i) < \infty$. Then there exists a countable subcollection $\mathcal{G} = \{B_j : j \in J\}$, where $J \subset I$, of balls in $\mathcal{B}$ which is disjoint and*

$$\bigcup_{i\in I} B^d(x_i, r_i) \subset \bigcup_{j\in J} B^d(x_j, 5r_j)\,.$$

For the Vitali Covering Theorem, we need the following definition.

Definition 2.2 (Vitali covering). Let $A \subset X$ be covered by a family of balls $\mathcal{F} = \{B_i : i \in I\}$. Then the family $\mathcal{F}$ is said to be a *Vitali covering* if for every $x \in A$ there exist balls of arbitrarily small radii in $\mathcal{F}$ that contain x, i.e.

$$\inf\{r > 0 : B(x, r) \in \mathcal{F}\} = 0\,.$$

Using Defintion 2.2 and Lemma 2.1 we can now formulate the Vitali Covering Theorem for a doubling metric measure space (X, d, μ). This version can be found in [40].

Theorem 2.3 (Vitali Covering Theorem). *Let $A \subset X$ be a subset in a doubling metric measure space (X, d, μ) and let $\mathcal{F} = \{B(x_i, r_i) : i \in I,\, x_i \in X\}$ be a Vitali covering of A. Then there exists a countable subset $J \subset I$ such that the family $\mathcal{G} = \{B(x_j, r_j) : j \in J,\, x_j \in X\}$ consists of disjoint balls and covers μ-almost all of A, i.e.*

$$\mu\Big(A \setminus \bigcup_{j\in J} B(x_j, r_j)\Big) = 0\,.$$

Proof. Assume first that $\mu(A) < \infty$. Let U be an open set that covers A and define the family

$$\mathcal{F}_U := \{B(x_i, r_i) : i \in I,\, x_i \in X,\, B_{r_i} \subset U\}$$

of those balls that are contained in U only. Then $\mathcal{F}_U$ is still a Vitali covering of $A \subset U$, and the balls B_{r_i} have finite radii $r_i < \infty$. By Lemma 2.1 there exists a sub-collection $\mathcal{G}_U \subset \mathcal{F}_U$,

$$\mathcal{G}_U = \{B(x_j, r_j) : j \in J,\, x_j \in X,\, B_{r_j} \subset U\}$$

for a subset $J \subset I$ containing disjoint balls $B(x_j, r_j)$, $x_j \in X$. It follows for their measure that

$$\mu\Big(\bigcup_{j\in J} B(x_j, r_j)\Big) = \sum_{j\in J} \mu(B(x_j, r_j)) < \infty\,.$$

Using the doubling property of μ, we arrive at

$$\sum_{j\in J} \mu(B(x_j, 5r_j)) \leqslant C_d \sum_{j\in J} \mu(B(x_j, r_j)) < \infty\,. \tag{2.1}$$

This implies that for a given $\delta > 0$ we can find a finite index set $M \subset J$ such that

$$\sum_{j\in J\setminus M} \mu(B(x_j, r_j)) < \delta\,. \tag{2.2}$$

Thus, if we can show that

$$\mu\Big(A \setminus \bigcup_{m\in M} B(x_m, r_m)\Big) \leqslant \sum_{j\in J\setminus M} \mu(B(x_j, 5r_j)) \tag{2.3}$$

the Theorem will be shown by applying (2.1) and (2.2).

Now, for any finite set $M \subset J$ it holds that

$$A \setminus \bigcup_{m\in M} \overline{B}(x_m, r_m) \subset \bigcup_{j\in J\setminus M} B(x_j, 5r_j)$$

holds for disjoint and closed balls $\overline{B}_{r_m}$. Indeed, as we have a Vitali covering of A, a point $x \in A \setminus \bigcup_{m\in M} \overline{B}(x_m, r_m)$ is contained in an open ball $B(x_j, r_j)$, $j \in J$, which is part of $\mathcal{G}_U$ and which does not intersect the closed and finite union $\bigcup_{m\in M} \overline{B}(x_m, r_m)$, i.e. there exists $j \in J$ such that

$$x \in B(x_j, r_j) \quad \text{and} \quad B(x_j, r_j) \cap \bigcup_{m\in M} \overline{B}(x_m, r_m) = \emptyset\,.$$

By Lemma 2.1, however, $B(x_j, r_j) \cap B(x_i, r_i) \neq \emptyset$ for some $i \in I$ and $B(x_i, r_i) \subset B(x_j, 5r_j)$ by the doubling property of μ. It follows that $x \in B(x_j, 5r_j)$ and $j \notin M$, hence we arrive at (2.3). Together this yields

$$\begin{aligned}
\mu\Big(A \setminus \bigcup_{j\in J} B(x_j, r_j)\Big) &\leqslant \mu\Big(A \setminus \bigcup_{m\in M} \overline{B}(x_m, r_m)\Big) \leqslant \mu\Big(\bigcup_{j\in J\setminus M} B(x_j, 5r_j)\Big) \\
&= \sum_{j\in J\setminus M} \mu(B(x_j, 5r_j)) \leqslant C_d \sum_{j\in J\setminus M} \mu(B(x_j, r_j)) < \delta
\end{aligned}$$

due to the disjointness of balls B_{r_j}, $j \in J$, and the doubling property of μ. As δ can be chosen arbitrarily small, the result follows in the case $\mu(A) < \infty$.

Now, let $\mu(A) = \infty$. Define the open sets

$$D_n = B(0,n) \setminus \overline{B}(0,n-1), \quad n \in \mathbb{N}.$$

Then $X = \bigcup_{n\in\mathbb{N}} D_n$. Define a family $\mathcal{F}_{D_n}$ of balls on each D_n by

$$\mathcal{F}_{D_n} := \{B(x_i, r_i) : i \in I,\, x_i \in X,\, B_{r_i} \subset D_n\}.$$

Then $\mathcal{F}_{D_n}$ is a Vitali covering of $A \cap D_n$. Similar to the argumentation above there exists a countable subset $J_n \subset I$ for each $n \in \mathbb{N}$, such that

$$\mathcal{G}_{D_n} := \{B(x_j, r_j) : j \in J_n,\, x_j \in X,\, B_{r_j} \subset D_n\}.$$

is a subfamily of $\mathcal{F}_{D_n}$ consisting of disjoint balls $B(x_j, r_j)$, $x_j \in X$, and we arrive at

$$\mu((A \cap D_n) \setminus \bigcup_{j\in J_n} B(x_j, r_j)) < \delta_n$$

for given $\delta_n > 0$ for every $n \in \mathbb{N}$, such that $\delta = \sum_{n\in\mathbb{N}} \delta_n$. Note that $\mathcal{G}_{D_n}$ is a family of disjoint sets as D_n is disjoint and $B(x_j, r_j) \subset D_n$ for all $n \in \mathbb{N}$. Hence

$$\mathcal{G} := \{B(x_j, r_j) : j \in J,\, x_j \in X\},$$

where $J := \bigcup_{n\in\mathbb{N}} J_n$, is a family of disjoint balls. Using the fact that $X = \bigcup_{n\in\mathbb{N}} D_n$, we arrive at

$$\begin{aligned}
\mu\Big(A \setminus \bigcup_{j\in J} B(x_j, r_j)\Big) &= \mu\Big(X \cap (A \setminus \bigcup_{n\in\mathbb{N}} \bigcup_{j\in J_n} B(x_j, r_j))\Big) \\
&= \mu\Big(\Big(\bigcup_{n\in\mathbb{N}} D_n\Big) \cap \Big(\bigcup_{n\in\mathbb{N}} (A \setminus \bigcup_{j\in J_n} B(x_j, r_j))\Big)\Big) \\
&= \mu\Big(\bigcup_{n\in\mathbb{N}} \Big((D_n \cap A) \setminus \bigcup_{j\in J_n} B(x_j, r_j)\Big)\Big) \\
&= \sum_{n\in\mathbb{N}} \mu\Big((D_n \cap A) \setminus \bigcup_{j\in J_n} B(x_j, r_j)\Big) < \delta
\end{aligned}$$

which yields the result. □

2.2 Lebesgue spaces on metric measure spaces

Let (X, d, μ) be a metric measure space and $1 \leqslant p \leqslant \infty$. The Lebesgue spaces $L^p(X, \mu)$ form a class of Banach spaces and are therefore of importance in many parts of analysis. In the following, let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and we consider functions $u : X \to \mathbb{K}$. More precisely, we focus on measurable functions u for which $|u|^p$ is integrable over their domain of definition and we denote

$$\|u\|_p := \Big(\int_X |u(x)|^p \mu(dx)\Big)^{1/p} < \infty. \tag{2.4}$$

The set of functions u such that (2.4) holds forms a semi-normed space, usually denoted by $\mathcal{L}^p(X, \mu)$, with $\|\cdot\|_p$ being a semi-norm, as $\|u\|_p = 0$ does not imply $u \equiv 0$ but $u = 0$ up to a set of μ-measure zero. However, if we identify two functions u and v, for which $u = v$ μ-almost everywhere, then we consider in fact the quotient space

$$L^p(X) := \mathcal{L}^p(X, \mu)/(\ker \|\cdot\|_p)$$

being now a normed space with $\|\cdot\|_p$. We will denote this norm by $\|\cdot\|_{L^p}$.

For the case $p = \infty$ we define the space $L^\infty(X, \mu)$ as the space of all measurable functions $u : X \to \mathbb{K}$ which are essentially bounded. Again, $L^\infty(X, \mu)$ is an equivalence class of functions where u and v are identified if they are equal μ-almost everywhere. The space $L^\infty(X, \mu)$ becomes a normed space with

$$\|u\|_\infty := \|u\|_{L^\infty} := \operatorname{ess\,sup} |u(x)| = \inf\{c \in \mathbb{R} : \mu(\{|u(x)| > c\}) = 0\}.$$

The L^p spaces, $1 \leqslant p \leqslant \infty$, are all Banach spaces, $L^2(X, \mu)$ is a Hilbert space on which the inner product is defined by $\int_X u(x)\,\overline{v(x)}\,\mu(dx)$.

Note that if u and v are p-times integrable functions, then so is $u + v$, as this follows from the inequality $|u + v|^p \leqslant 2^{p-1}(|u|^p + |v|^p)$. Recall the *Minkowski inequality* which is the triangle inequality in the space $L^p(X, \mu)$ and which ensures that the L^p spaces are normed vector spaces.

Minkowski's inequality. Let (X, d, μ) be a metric measure space, $1 \leqslant p \leqslant \infty$ and $u, v \in L^p(X, \mu)$. Then also $u + v \in L^p(X, \mu)$ and for its norm we get

$$\|u + v\|_{L^p} \leqslant \|u\|_{L^p} + \|v\|_{L^p}.$$

An important property of L^p spaces is that of embeddings. For $1 \leqslant p < q \leqslant \infty$ and a domain $\Omega \subset X$ with finite measure μ, the estimate

$$\|u\|_{L^p} \leqslant \mu(\Omega)^{\frac{1}{p} - \frac{1}{q}} \|u\|_{L^q} \tag{2.5}$$

for $u \in L^q(\Omega, \mu)$ means that the space $L^q(\Omega)$ is continuously embedded in $L^p(\Omega)$, which is commonly denoted by $L^q(\Omega) \hookrightarrow L^p(\Omega)$ for $q > p$. Note that the estimate above is a consequence of the Hölder inequality.

Hölder's inequality. Let (X, d, μ) be a metric measure space, $1 \leqslant p \leqslant \infty$, $u \in L^p(X, \mu)$ and $v \in L^q(X, \mu)$. Then, $u\,v \in L^1(X, \mu)$ and

$$\|u\,v\|_{L^1} \leqslant \|u\|_{L^p} \|v\|_{L^q}.$$

In the following we define locally integrable functions on a metric measure space.

Definition and Theorem 2.4. Let (X, d, μ) be a metric measure space. For a function $u : X \to \mathbb{R}$ the following are equivalent:

(i) For each compact set $K \subset X$, the function $u_{|K}$ is integrable over K.

(ii) For every point $a \in X$ there exists a neighbourhood $V \subset X$ such that $u_{|V}$ is integrable over V.

If a function u satisfies (i) or (ii) it is said to be *locally integrable* on X. The set of such functions will be denoted by $L^1_{\text{loc}}(X, \mu)$. Similarly, if $|u|^p$ is locally integrable on U we say that u belongs to the space $L^p_{\text{loc}}(X, \mu)$.

Proof. (i) $\Rightarrow$ (ii): Let $u_{|K}$ be integrable over the compact set $K \subset X$, i.e. $u_{|K} = u\chi_K$ is integrable. Setting $V := \overset{\circ}{K}$, then $V \subset K$ is open and bounded, hence χ_V is integrable and bounded. It follows that the product $(u\chi_K)\chi_V$ is integrable, but $(u\chi_K)\chi_V = u\chi_V$, so $u_{|V}$ is integrable.
(ii) $\Rightarrow$ (i): Let $K \subset X$ be compact, then there exist finitely many open sets $V_1, \dots, V_m$ such that $K \subset \bigcup_{j=1}^m V_j =: V$. By assumption $u_{|V} = u_{|\bigcup_{j=1}^m V_j}$ is integrable, hence $u_{|V_j}$ is integrable for each $j = 1, \dots, m$. Moreover we can write $K = \bigcup_{j=1}^m (K \cap V_j)$ and $K \cap V_j \subset V_j$ is a bounded subset for each $j = 1, \dots, m$. Therefore $\chi_{K\cap V_j}$ is bounded and integrable and hence $\chi_K = \chi_{\bigcup_{j=1}^m (K\cap V_j)}$ is bounded and integrable. It follows that $(u\chi_{V_j})\chi_K = u\chi_K = u_{|K}$ is integrable. □

Note that every L^p function is locally integrable whereas the converse is not true in general. Locally integrable functions occur frequently in distribution theory. In harmonic analysis, weak-L^p spaces play an essential role. We give their definition below.

Let (X, d, μ) be a metric measure space and u an L^1 function on X. Then note that for a fixed $t > 0$ we have

$$\int_X |u(x)|\, \mu(dx) \geqslant \int_{\{|u|>t\}} |u(x)|\, \mu(dx) \geqslant \int_{\{|u|>t\}} t\, \mu(dx) = t\, \mu(\{|u| > t\}),$$

or equivalently $\mu(\{|u| > t\}) \leqslant \frac{\|u\|_{L^1}}{t}$. This is an L^1 upper bound on the *distribution function* of u. Note however, for the case $X = \mathbb{R}_+$ and μ the Lebesgue measure we find that $\frac{\|u\|_{L^1}}{t} \notin L^1(\mathbb{R}_+)$, whereas $\mu(\{|u| > t\}) \in L^1(\mathbb{R}_+)$, since

$$\int_{\mathbb{R}_+} \mu(\{|u| > t\})\, dt = \|u\|_{L^1} < \infty.$$

Generally, we have for $u \in L^p(X, \mu)$ and $t > 0$

$$\|u\|_{L^p}^p = \int_X |u(x)|^p\, \mu(dx) \geqslant \int_{\{|u|>t\}} t^p\, \mu(dx) = t^p\, \mu(\{|u| > t\}) \tag{2.6}$$

for all $t > 0$. This gives rise to the following definition.

Definition 2.5. Let (X, d, μ) be a metric measure space. We call a function u to be in the *weak-$L^p(X, \mu)$ space*, denoted by $L^p_w(X, \mu)$, if there exists a constant $C > 0$ such that

$$\mu(\{x \in X : |u(x)| > t\}) \leqslant \left(\frac{C}{t}\right)^p$$

for all $t > 0$. For $u \in L^p_w(X, \mu)$ we define the quasi-norm

$$\|u\|_{L^p_w} := \sup_{t>0} \left(t^p\, \mu(\{|u| > t\})\right)^{1/p}.$$

Note that (2.6) implies the continuous embedding $L^p(X, \mu) \hookrightarrow L^p_w(X, \mu)$ and for bounded subset $\Omega \subset X$ it holds due to (2.5) that $L^q(\Omega, \mu) \hookrightarrow L^p_w(\Omega, \mu)$ for any $q \geqslant p$. Further, it is $\|u\| = 0$ if, and only if, $u = 0$ μ-almost everywhere on X, and $\|\lambda u\|_{L^p_w} = |\lambda|\, \|u\|_{L^p_w}$. Moreover, instead of the usual triangle inequality we get

$$\begin{aligned}
\|u + v\|_{L^p_w} &= \sup_{t>0} \left(t^p\, \mu(|u + v| > t)\right)^{1/p} \leqslant \sup_{t>0} \left(t^p\, 2^p\, \mu(|u| + |v| > t)\right)^{1/p} \\
&\leqslant 2\big(\sup_{t>0} \left(t^p\, \mu(|u| > t)\right)^{1/p} + \sup_{t>0} \left(t^p\, \mu(|v| > t)\right)^{1/p}\big) \\
&= 2(\|u\|_{L^p_w} + \|v\|_{L^p_w}).
\end{aligned}$$

Thus $\|\cdot\|_{L^p_w}$ is a quasi-norm. Sometimes the weak-L^p spaces are also called *Marcinkiewicz spaces.*

One application of the weak-L^p spaces is Hardy-Littlewood's maximal function theorem, which can be formulated in the context of metric measure spaces, see Heinonen [40]. We consider a function $f \in L^1(X, \mu)$ on the metric measure space (X, d, μ). Then

$$(Mf)(x) = \sup_{r>0} \frac{1}{\mu(B^d(x,r))} \int_{B^d(x,r)} |f(y)| \, \mu(dy)$$

is its *maximal function.* The definition does not exclude $Mf = \infty$.

Theorem 2.6 (Maximal Function Theorem). *The Hardy-Littlewood maximal function Mf is of weak-type $(1,1)$ on the metric measure space (X, d, μ), i.e. there exists a constant $c_1 > 0$ such that for $f \in L^1(X, \mu)$ we have*

$$\mu(\{x \in X : (Mf)(x) > t\}) \leqslant \frac{c_1}{t} \int_X |f(y)| \, \mu(dy),$$

if $\mu(B^d(x, \sigma r)) \leqslant C_d(\sigma) \, \mu(B^d(x,r))$ for a constant $C_d(\sigma)$ depending on the metric and the factor $\sigma \geqslant 1$. Further, for $1 < p \leqslant \infty$ we have the strong-type inequality

$$\|Mf\|_{L^p} \leqslant c_p \|f\|_{L^p}$$

for $f \in L^p(X, \mu)$ and a constant $c_p > 0$, i.e. the operator $f \mapsto Mf$ is bounded in $L^p(X, \mu)$.

The first part of this theorem ensures that M maps $L^1(X, \mu)$ into the weak-L^1 space, whenever μ is a doubling measure. The proof of this part needs the Vitali Covering Lemma 2.1.

Remark 2.7. Note that M is in fact a contraction on L^∞, as for $f \in L^\infty(X, \mu)$

$$|(Mf)(x)| \leqslant \sup_{r>0} \frac{1}{\mu(B^d(x,r))} \|f\|_{L^\infty} \int_{B^d(x,r)} 1 \, \mu(dy) = \|f\|_{L^\infty} \quad \text{for all } x \in X.$$

Thus $\|Mf\|_{L^\infty} \leqslant \|f\|_{L^\infty}$ and for the operator norm we have $\|M\|_{L^\infty - L^\infty} \leqslant 1$.

2.3 Sobolev spaces in a metric measure space framework

We consider the metric measure space (X, d, μ) with a regular Borel measure μ and discuss different methods how to introduce a Sobolev space on X. Let us first recall the definitions of Lipschitz and Hölder continuous functions and some properties of classical Sobolev spaces.

2.3.1 Lipschitz and Hölder continuous functions

Lipschitz continuity is a smoothness condition for a function stronger than continuity and can be defined on a general metric space. We briefly introduce the definition of Lipschitz and Hölder continuous functions in order to use these concepts later in this paragraph.

Definition 2.8. Let (X, d) be a metric space, $A \subset X$, and let $\mathbb{R}$ be equipped with the usual Euclidean distance $|\cdot|$.

A. A function $f : A \to \mathbb{R}$ is called *Lipschitz continuous* if there exists a constant $L \geqslant 0$ such that for all $x, y \in A$

$$|f(x) - f(y)| \leqslant L\, d(x, y)\,.$$

The smallest constant L, for which this inequality holds, is called the *Lipschitz constant* of f. Sometimes f is called L-Lipschitz in such cases. If $L \leqslant 1$, then f is called a *contraction.*

B. The function is *locally L-Lipschitz* if for every $x \in A$ there exists a neighbourhood $U(x) \subset A$ such that $f_{|U}$ is L-Lipschitz.

C. The function $f : A \to \mathbb{R}$ is said to be *Hölder continuous of order* α on A, $0 < \alpha \leqslant 1$, if there exists a constant $L \geqslant 1$ such that

$$|f(x) - f(y)| \leqslant L\, d^{\alpha}(x, y)$$

for all $x, y \in A$.

In particular, if f is Hölder continuous of order $\alpha \in (0, 1]$ on (X, d), then this is equivalent to f being L-Lipschitz on the metric space (X, d^{α}). Later, we will denote the class of Lipschitz continuous functions with compact support by $C_0^{0,1}(X) = C^{0,1}(X) \cap C_0(X)$ adopting the notation of Hölder spaces, where $C^{0,1}(X)$ is the space of continuous functions $u : X \to \mathbb{R}$ for which

$$\sup \left\{ \frac{|u(x) - u(y)|}{d(x, y)} : x, y \in X, x \neq y \right\} < \infty\,.$$

In Example 2.15 below we need the following result (see e.g. [22], Chapter 5.8, Theorem 6), which is valid for Lipschitz maps between Euclidean spaces.

Theorem 2.9 (Rademacher). *Consider the metric measure space* $(\mathbb{R}^n, |\cdot|, \lambda^{(n)})$ *with Lebesgue measure* $\lambda^{(n)}$ *and* $A \subset \mathbb{R}^n$. *A locally Lipschitz continuous function* $u : A \to \mathbb{R}$ *is differentiable* μ*-almost everywhere on* A.

2.3.2 Definition and properties of classical Sobolev spaces

Let Ω be a domain in $\mathbb{R}^n$, $u \in C^1(\Omega)$ and $\phi \in C_0^{\infty}(\Omega)$. Then integration by parts yields

$$\int_{\Omega} (\partial_{x_i} u) \phi \, dx = - \int_{\Omega} u \partial_{x_i} \phi \, dx \quad \text{for all } \phi \in C_0^{\infty}(\Omega).$$

The boundary terms vanish because of $\operatorname{supp}(\phi) \subset \Omega$ compact. Generally, for $u \in C^m(\Omega)$ and multi-index $\alpha \in \mathbb{N}_0^n$

$$\int_{\Omega} (D^{\alpha} u) \phi \, dx = (-1)^{|\alpha|} \int_{\Omega} u \, D^{\alpha} \phi \, dx \quad \text{for all } \phi \in C_0^{\infty}(\Omega), \tag{2.7}$$

where $D^{\alpha} = \prod_{j=1}^{n} \partial_{x_j}^{\alpha_j} = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$. The relation (2.7) is well-defined for integrable $D^{\alpha} u$. However, it is sufficient to assume $D^{\alpha} u$ to be integrable on compact subsets of Ω, since ϕ has compact support. Hence, (2.7) is well-defined for $u, D^{\alpha} u \in L^1_{\mathrm{loc}}(\Omega)$. In general we have the following definition.

Definition 2.10 (Weak derivative). Let $\Omega \subset \mathbb{R}^n$ be a Lebesgue measurable subset of $\mathbb{R}^n$, $u, v \in L^1_{\mathrm{loc}}(\Omega)$ and $\alpha \in \mathbb{N}_0^n$. Then v is called *weak derivative of* u *of order* α, i.e. $v = D^{\alpha} u$, if

$$\int_{\Omega} v \, \phi \, dx = (-1)^{|\alpha|} \int_{\Omega} u \, D^{\alpha} \phi \, dx \quad \text{for all } \phi \in C_0^{\infty}(\Omega).$$

We call

$$W^m(\Omega) := \{u \in L^1_{\text{loc}}(\Omega) : \; u \text{ has weak derivative } v = D^\alpha u \text{ for all } |\alpha| \leqslant m\}$$

the *space of m-times weakly differentiable functions.*

Definition 2.11 (Sobolev space). Let $\Omega \subset \mathbb{R}^n$ be a Lebesgue measurable subset of $\mathbb{R}^n$, $u \in L^p(\Omega)$ and $\alpha \in \mathbb{N}_0^n$. We define the *Sobolev space*

$$W^{m,p}(\Omega) := \{u \in W^m(\Omega) : D^\alpha u \in L^p(\Omega) \text{ for all } |\alpha| \leqslant m\}$$

for $1 \leqslant p < \infty$, i.e. we say that u belongs to the Sobolev space $W^{m,p}(\Omega)$ if its weak derivatives of order $|\alpha| \leqslant m$ belong to $L^p(\Omega)$. For $p = \infty$ we define the Sobolev space $W^{m,\infty}(\Omega)$ as the space of all $u \in L^\infty(\Omega)$ which have weak derivatives $D^\alpha u \in L^\infty(\Omega)$ for all $|\alpha| \leqslant m$.

Remark 2.12. Similarly, one can define $W^{m,p}_{\text{loc}}(\Omega)$ for $u, D^\alpha u \in L^p_{\text{loc}}(\Omega)$. The Sobolev spaces $W^{m,p}(\Omega)$, $1 \leqslant p \leqslant \infty$, equipped with the norms

$$\|u\|_{W^{m,p}} = \Big(\sum_{|\alpha| \leqslant m} \|D^\alpha u\|^p_{L^p} \Big)^{1/p}$$

$$\|u\|_{W^{m,\infty}} = \max_{|\alpha| \leqslant m} \|D^\alpha u\|_{L^\infty}$$

are Banach spaces, whereas $W^{m,2}(\Omega)$ with the inner product

$$(u, v) := \sum_{|\alpha| \leqslant m} (D^\alpha u, D^\alpha v)_{L^2(\Omega)}$$

is a Hilbert space.

A fundamental result in the theory of classical Sobolev spaces is the Sobolev embedding theorem, which allows us to obtain bounds on a function using bounds on its derivative, see, e.g. [1, 22]. It says that for a function $u \in W^{1,p}(\mathbb{R}^n)$ and $1 \leqslant p < n$ we have

$$\|u\|_{L^{np/(n-p)}} \leqslant C(n,p) \, \|\nabla u\|_{L^p} \,, \tag{2.8}$$

i.e. $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n)$, where $p^* = \frac{np}{n-p}$ with n being the space dimension. Inequality (2.8) is also sometimes called *Gagliardo-Nirenberg-Sobolev inequality*, see [22]. For the case $p > n$ we have an embedding of the Sobolev space into the space of Hölder continuous functions,

$$\frac{|u(x) - u(y)|}{|x - y|^{1 - \frac{n}{p}}} \leqslant C(n,p) \, \|\nabla u\|_{L^p} \,,$$

i.e. $W^{1,p}(\mathbb{R}^n) \hookrightarrow C^{0,1-n/p}(\mathbb{R}^n)$. Now, (2.8) can be used to derive another important inequality in classical Sobolev spaces.

Consider a function $u \in C^\infty(B)$ defined on a ball $B := B_r^{|\cdot|} \subset \mathbb{R}^n$. Then the *Sobolev-Poincaré inequality*

$$\|u - u_B\|_{L^{np/(n-p)}} \leqslant C(n,p) \, \|\nabla u\|_{L^p} \tag{2.9}$$

holds, where u_B is defined as the mean value of u on the ball B, i.e.

$$u_B = \frac{1}{\lambda^{(n)}(B)} \int_B u \, dx = \fint_B u \, dx \,.$$

Multiplying both sides by $(\lambda^{(n)}(B))^{\frac{p-n}{np}}$ we obtain

$$\Big(\frac{1}{\lambda^{(n)}(B)}\int_B |u-u_B|^{\frac{np}{n-p}}\,dx\Big)^{\frac{n-p}{np}} \leqslant C(n,p)\,(\lambda^{(n)}(B))^{\frac{p-n}{np}}\Big(\int_B |\nabla u|^p\,dx\Big)^{\frac{1}{p}}$$

$$= C(n,p)\,(\lambda^{(n)}(B))^{\frac{1}{n}}\Big(\frac{1}{|B|}\int_B |\nabla u|^p\,dx\Big)^{\frac{1}{p}}.$$

As

$$(\lambda^{(n)}(B))^{\frac{1}{n}} = (r^n V_n)^{\frac{1}{n}} = 2r\Big(\frac{V_n}{2^n}\Big)^{\frac{1}{n}} = \operatorname{diam}(B)\Big(\frac{V_n}{2^n}\Big)^{\frac{1}{n}},$$

where $V_n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$ is the volume of the unit ball in $\mathbb{R}^n$, we arrive at

$$\Big(\fint_B |u-u_B|^{\frac{np}{n-p}}\,dx\Big)^{\frac{n-p}{np}} \leqslant \widetilde{C}(n,p)\operatorname{diam}(B)\Big(\fint_B |\nabla u|^p\,dx\Big)^{\frac{1}{p}}$$

and by applying Hölder's inequality

$$\fint_B |u-u_B|^p\,dx \leqslant \widetilde{C}(n,p)\,(\operatorname{diam} B)^p \fint_B |\nabla u|^p\,dx\,. \tag{2.10}$$

Inequality (2.10) is commonly called *Poincaré inequality*, see [40], Chapter 4, for this result on an arbitrary metric measure space.

The space $W^{1,\infty}(\Omega)$ with a domain $\Omega \subset \mathbb{R}^n$ is by Definition 2.11 the space of functions $u \in L^\infty(\Omega)$ with weak derivatives of first order that are also essentially bounded on Ω. Now, to cope with the difficulty of not being able to define a gradient on general metric spaces a generalisation of the classical theory to metric spaces is needed. Problems connected with metric spaces arise in applications such as analysis on graphs, fractals or questions in probability theory. For a domain $\Omega \subset \mathbb{R}^n$, the space $W^{1,\infty}(\Omega)$ can be identified with the space of Lipschitz functions on Ω, see e.g. Theorem 6.12 in [40]. Due to this identification a generalisation of $W^{1,\infty}$ to a general metric measure space makes sense. In § 2.3.4 we will focus on the definition of the Hajłasz-Sobolev space, which gives an appropriate analogon to $W^{1,p}(\Omega)$ on metric measure spaces for $1 \leqslant p < \infty$.

2.3.3 Sobolev spaces via upper gradients and the Poincaré inequality

There are several ways to generalise the notion of a Sobolev space to the setting of a metric space. Some methods are based on the introduction of upper gradients in this setting (see, e.g. [40], 7.22). First, we need some basic terminology in connection with upper gradients. For the following concepts, see [40], 7.1.

Definition 2.13. Let (X,d) be a metric space and $\gamma : [a,b] \to X$ a continuous map. The image $\gamma([a,b])$ in X is called a *curve*. Let $a = t_0 \leqslant t_1 \leqslant \ldots \leqslant t_n = b$ be a partition of the interval $[a,b]$. Then the *arc length* of γ is defined as

$$\ell(\gamma) = \ell(\gamma([a,b])) := \sup_{a=t_0\leqslant\ldots\leqslant t_n=b} \sum_{j=1}^{n} d(\gamma(t_j),\gamma(t_{j-1}))\,,$$

where the supremum is taken over all possible partitions of $[a,b]$. Then γ is said to be *rectifiable* if $\ell(\gamma) < \infty$. For a rectifiable curve γ with length $\ell(\gamma) = \ell(\gamma([a,b]))$ we can define a continuous map $s_\gamma : [a,b] \to [0,\ell(\gamma)]$ given by $s_\gamma(t) = \ell(\gamma([a,t]))$. Then there exists a unique map $\tilde{\gamma} : [0,\ell(\gamma)] \to X$ such that

$$\gamma = \tilde{\gamma}\circ s_\gamma \quad \text{with} \quad \ell(\tilde{\gamma}([0,t])) = t\,.$$

The curve $\tilde{\gamma}$ is then called *parametrised by arc length.*

Definition 2.14. Given a real-valued function u on a metric space X, then a measurable function $g \geqslant 0$ is called an *upper gradient* of the function u if

$$|u(x) - u(y)| \leqslant \int_\gamma g\, ds \tag{2.11}$$

holds for each pair x, y and all rectifiable curves γ joining x and y.

Every function has $g = \infty$ as an upper gradient. Upper gradients are not unique, but rather a local concept in the sense that the minimal upper gradient of u is zero almost everywhere in the set where the function u is constant. Then we can define a Sobolev space as a space consisting of those functions $u \in L^p(X)$ which have an upper gradient g belonging to $L^p(X)$, and we equip the space with the norm

$$\|u\|_{W^{1,p}} = \|u\|_{L^p} + \inf \|g\|_{L^p}\,,$$

where the infimum is taken over all functions $g \in L^p(X)$ satisfying (2.11). For the following example, cf. [40], 7.25.

Example 2.15. Consider a real-valued function $u \in C_0^{0,1}(X)$ and define for any point $x \in X$

$$L_u(x) := \liminf_{r \to 0} \sup_{d(x,y) \leqslant r} \frac{|u(x) - u(y)|}{r}\,. \tag{2.12}$$

Then $L_u(x)$ is an upper gradient in the sense of Definition 2.14. To see this, let $u : X \to \mathbb{R}$ be a Lipschitz continuous function on the metric space (X, d), $x, y \in X$ and $\gamma : [0, T] \to X$ be a rectifiable curve with $\gamma(0) = x$, $\gamma(T) = y$ and parametrised by its arc length. Then $u \circ \gamma : [0, T] \to \mathbb{R}$ is also Lipschitz continuous, hence, as $[0, T] \subset \mathbb{R}$, by Theorem 2.9 differentiable at almost every point $t \in [0, T]$ and we get

$$\begin{aligned}
|u(x) - u(y)| &= |u(\gamma(0)) - u(\gamma(T))| = \left| \int_0^T (u \circ \gamma)'(s))\, ds \right| \\
&\leqslant \int_0^T |(u \circ \gamma)'(s))|\, ds = \int_0^T \liminf_{r \to 0} \frac{|(u \circ \gamma)(s + r)) - (u \circ \gamma)(s))|}{r}\, ds \\
&\leqslant \int_0^T \liminf_{r \to 0} \sup_{\substack{\gamma(T) \\ d(\gamma(s), \gamma(T)) \leqslant r}} \frac{|(u \circ \gamma)(T)) - (u \circ \gamma)(s))|}{r}\, ds\,.
\end{aligned}$$

Thus, (2.12) is an upper gradient in the sense of Definition 2.14.

We note, however, that the technique of defining a Sobolev space using upper gradients fails if the underlying space does not admit rectifiable curves. This also involves the choice of the metric. The following example taken from [28] illustrates the phenomenon that some metrics lead to a rather degenerate behaviour of L_u given in (2.12).

Example 2.16. Let $X = \mathbb{R}^n$, $d(x, y) = |x - y|^\alpha$ for $0 < \alpha < 1$, and $u : \mathbb{R}^n \to \mathbb{R}$ be differentiable in $y \in X$ in the classical sense. Then

$$L_u(y) = \lim_{x \to y} \frac{|u(x) - u(y)|}{|x - y|^\alpha} = u'(y) \lim_{x \to y} |x - y|^{1-\alpha} = 0\,.$$

In particular, if $u : \mathbb{R}^n \to \mathbb{R}$ is a function that is differentiable almost everywhere in X, then $\|L_u\|_{L^p} = 0$.

For general metric measure spaces with Sobolev space constructed via upper gradients we have the following definition of a Poincaré inequality.

Definition 2.17. We say that (X, d, μ) supports a *weak* $(1,p)$*-Poincaré inequality* if there exist constants $C > 0$, $\sigma \geqslant 1$ such that whenever $B = B(x,r) \subset X$ and g is an upper gradient of u in the sense of (2.11) on $B(x, \sigma r)$, then

$$\fint_{B(x,r)} |u - u_B|^p \, \mu(dx) \leqslant \widetilde{C}(n,p)\, r \left(\fint_{B(x,\sigma r)} g^p \, \mu(dx) \right)^{1/p}. \tag{2.13}$$

Remark 2.18. The space (X, d^α) with $0 < \alpha < 1$ is a space which does not provide rectifiable paths, and therefore, is a space which does not support a Poincaré inequality. More on this can be found in, e.g., [38, 40, 41].

As in Definition 2.17 the notion of a *weak* Poincaré inequality is often used (cf. [38]) if both sides involve a ball and the radius of the ball on the right-hand side is greater than the radius of the ball on the left-hand side as it is the case in (2.13). In [38], Hajłasz and Koskela develop a theory of Sobolev spaces from inequality (2.13), which includes the study the of relationship between (2.13) and the construction due to P. Hajłasz [37] described below. We will come back to this in § 4.4.

2.3.4 The Hajłasz-Sobolev space

In [37] Hajłasz introduced a Lipschitz-type characterisation of a Sobolev space over a metric measure space (X, d, μ) with a Borel measure μ which is locally finite. Unlike the construction via upper gradients Hajłasz's approach can be used without imposing additional assumptions on the metric measure space. In the literature it is called a *universal non-local construction.* In fact, this construction was one of the early attempts to generalise the concept of a Sobolev space to the setting of a general metric measure space (X, d, μ). For $1 \leqslant p < \infty$, Hajłasz [37] defines the Sobolev space $M^{1,p}(X)$ as follows.

Definition 2.19 (Hajłasz-Sobolev space). Let (X, d, μ) be a metric measure space, $i \leqslant p < \infty$. Define the space

$$\begin{aligned} M^{1,p}(X) &:= M^{1,p}(X, d, \mu) \\ &:= \{u \in L^p(X) : \text{ there exists an } g \in L^p(X) \text{ such that} \\ &\qquad |u(x) - u(y)| \leqslant d(x,y)(g(x) + g(y)) \ \mu\text{-a.e.}\}. \end{aligned}$$

Then $M^{1,p}(X)$ is a vector space, and it becomes a normed linear space $(M^{1,p}(X), \|\cdot\|_{M^{1,p}})$ with

$$\|u\|_{M^{1,p}} := \|u\|_{L^p} + \inf_g \|g\|_{L^p},$$

where the infimum is taken over the set of functions $g \in L^p(X)$, $g \geqslant 0$, which satisfy the defining equation. More precisely, $u \in M^{1,p}(X)$ denotes an equivalence class of functions which may differ on a μ-null set. The following result is taken from [40], Theorem 5.7.

Theorem 2.20. *The space $M^{1,p}(X)$ is a Banach space for $1 \leqslant p < \infty$.*

In § 4.4 we prove this result in the context of the metric measure spaces we are working with.

In the metric measure space $(\mathbb{R}^n, |\cdot|, \lambda^{(n)})$, where $|\cdot|$ is the Euclidean distance and $\lambda^{(n)}$ the n-dimensional Lebesgue measure, the isomorphy of $M^{1,p}(B)$ and $W^{1,p}(B)$ is proved in [40], Chapter 5.

Proposition 2.21. *Consider the metric measure space $(\mathbb{R}^n, |\cdot|, \lambda^{(n)})$ and let $B = B^{|\cdot|} \subset \mathbb{R}^n$ be a ball in $\mathbb{R}^n$ with respect to the standard Euclidean metric $|\cdot|$. Then we have the continuous embeddings $W^{1,p}(B) \hookrightarrow M^{1,p}(B)$ and $M^{1,p}(B) \hookrightarrow W^{1,p}(B)$ for all $1 < p < \infty$.*

In fact, the embedding $M^{1,p} \hookrightarrow W^{1,p}$ holds for general domains $\Omega \subset \mathbb{R}^n$.

Corollary 2.22. *Let $B \subset \mathbb{R}^n$ be an open ball in the Euclidean space $\mathbb{R}^n$ and $1 < p < \infty$. Then $M^{1,p}(B)$ is isomorphic to $W^{1,p}(B)$ and the correspondig norms are comparable, i.e. $\|u\|_{W^{1,p}} \approx \|u\|_{M^{1,p}}$ in the sense that there exist constants $c_0, c_1 > 0$ such that*

$$c_0 \, \|u\|_{W^{1,p}} \leqslant \|u\|_{M^{1,p}} \leqslant c_1 \, \|u\|_{W^{1,p}}$$

on $B \subset \mathbb{R}^n$.

We can show that a Poincaré inequality is valid in the Hajłasz-Sobolev space with a general metric space X as a domain.

Theorem 2.23. *Let $(X, d,$ [illegible] ∞. Then for all $u \in M^{1,p}(X)$, $p \geqslant 1$, we have*

$$\int_X |u - u_X|^p \, \mu(dx) \leqslant 2^p \, (\operatorname{diam} X)^p \int_X g^p \, \mu(dx) \,,$$

$$\int_X |u - u_X|^p \, \mu(dx) \leqslant 2^p \, (\operatorname{diam} X)^p \int_X g^p \, \mu(dx) \,,$$

where $g \geqslant 0$ is an L^p function satisfying

$$|u(x) - u(y)| \leqslant d(x,y) \, (g(x) + g(y)) \,. \tag{2.14}$$

Proof. We first integrate inequality (2.14) with respect to y and obtain

$$\begin{aligned} |u(x) - u_X| &= \left| ⨍_X (u(x) - u(y)) \, \mu(dy) \right| \overset{(2.14)}{\leqslant} ⨍_X d(x,y) \, (g(x) + g(y)) \, dy \\ &\leqslant \operatorname{diam}(X) ⨍_X (g(x) + g(y)) \, \mu(dy) = \operatorname{diam}(X) \, (g(x) + g_X) \,, \end{aligned}$$

and then we integrate with respect to x which yields

$$\begin{aligned} \int_X |u(x) - u_X|^p \, \mu(dx) &\overset{(2.14)}{\leqslant} (\operatorname{diam} X)^p \int_X (g(x) + g_X)^p \, \mu(dx) \\ &\leqslant 2^{p-1} (\operatorname{diam} X)^p \Big(\int_X g(x)^p \, \mu(dx) + \int_X g_X^p \, \mu(dx) \Big) \\ &\leqslant 2^p (\operatorname{diam} X)^p \int_X g(x)^p \, \mu(dx) \,. \end{aligned}$$

□

Theorem 2.23 is only of interest in the case where $\operatorname{diam}(X)$ is finite. Alternatively, the theorem can also be formulated for open balls $B^d(x,r) \subset X$. However, as we do not need μ to be a doubling measure, it can be formulated to hold on the whole space X provided it is of finite measure.

Note that Corollary 2.22 only holds on a smooth and bounded domain in $\mathbb{R}^n$. For general domains $\Omega \subset \mathbb{R}^n$, $M^{1,p}(\Omega)$ is a smaller space than $W^{1,p}(\Omega)$, i.e. $M^{1,p}(\Omega) \subset W^{1,p}(\Omega)$, which already follows from the definition. The Hajłasz-Sobolev space is only determined up to a set of measure zero, whereas no similar result holds for the classical Sobolev space. But it is also clear from Theorem 2.23, since not all domains $\Omega \subset \mathbb{R}^n$ with finite measure support a Poincaré inequality. See Remark 5.17 in [40] for an example of such a domain.

Heat Kernel Estimates and Geometry Related to Lévy Processes

§3 Transition function estimation

Pseudodifferential operators are linked to certain stochastic processes, which can be characterised completely by these operators or, more precisely, by their symbols. In this section we establish this connection first for symmetric Markov processes before we move to the subclass of (symmetric) Lévy processes of jump-type which are closely related to (real-valued) continuous negative definite functions. Finding upper and lower bounds for the transition densities of specific Lévy processes has been of central interest in the past years. Several authors have studied relationships between decay estimates for norms of semigroup operators and some Sobolev- and Nash-type inequalities. In the case of symmetric Markov semigroups these inequalities can be formulated using the Dirichlet form. Among the authors who have contributed to this field are Davies [17, 18], Varopoulos [60, 61] and Varopoulos, Saloff-Coste and Coulhon [62].

3.1 Introduction and basic definitions

First, recall that a stochastic process $(X_t)_{t\geqslant 0}$ is a family of random variables which are all acting on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$, where $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{A})$. Unless otherwise stated we consider the measure space $(\mathbb{R}^n, \mathcal{B}^{(n)})$ as a state space of the process. Let $(X_t)_{t\geqslant 0}$ be a given process. Then for any bounded Borel measurable function u defined on the state space $\mathbb{R}^n$ we can define an operator

$$T_t u(x) = \mathbb{E}^x[u(X_t)] \tag{3.1}$$

for each $t \geqslant 0$, where the right-hand side denotes the expectation of $u(X_t)$ given that the process starts in $x \in \mathbb{R}^n$. In particular, setting $u = \chi_A$ for a Borel set $A \in \mathcal{B}^{(n)}$ we find

$$T_t \chi_A(x) = \mathbb{E}^x[\chi_A(X_t)] = \mathbb{P}^x(X_t \in A) =: p_t(x, A)\,. \tag{3.2}$$

This function denotes the probability of X_t hitting the set A at time t, provided it starts in x at time $t = 0$.

Definition 3.1 (Transition function). Let $(X_t)_{t\geqslant 0}$ be a stochastic process starting in x at time $t = 0$, and A be a Borel set. Then $p_t(x, A)$ as defined in (3.2) is called the *transition function* of $(X_t)_{t\geqslant 0}$.

Below we need the following concepts.

Definition 3.2. A. Let $(\Omega, \mathcal{A})$, $\Omega \subset \mathbb{R}^n$, be a measurable space. For any fixed $t \geqslant 0$ we call p_t a *Markovian kernel* if $x \mapsto p_t(x, A)$ is measurable for all $A \in \mathcal{A}$ and if $A \mapsto p_t(x, A)$ is a probability measure, i.e. $p_t(x, \Omega) = 1$ for all $x \in \Omega$. We call p_t a *sub-Markovian kernel* if $x \mapsto p_t(x, A)$ is measurable for all $A \in \mathcal{A}$ and $A \mapsto p_t(x, A)$ is a sub-probability measure, i.e. $p_t(x, \Omega) \leqslant 1$.

B. A stochastic process $(X_t)_{t\geqslant 0}$ is a *Markov process* on $\Omega \subset \mathbb{R}^n$ if its transition function satisfies the *Chapman-Kolmogorov equations*

$$p_{s+t}(x, A) = \int_\Omega p_s(y, A)\, p_t(x, dy)\,.$$

For general $u \in B_b(\mathbb{R}^n; \mathbb{R})$ the identity (3.1) becomes

$$T_t u(x) = \mathbb{E}^x[u(X_t)] = \int_{\mathbb{R}^n} u(y)\, \mathbb{P}^x(X_t \in dy) = \int_{\mathbb{R}^n} u(y)\, p_t(x, dy)\,, \tag{3.3}$$

and if p_t satisfies the Chapman-Kolmogorov equations then $(T_t)_{t\geqslant 0}$ is a semigroup of linear operators on $B_b(\mathbb{R}^n; \mathbb{R})$. If the restriction of the semigroup onto $C_\infty(\mathbb{R}^n; \mathbb{R})$ gives a Feller semigroup $(T_t^{(\infty)})_{t\geqslant 0}$ then the Markov process associated with $T_t^{(\infty)} : C_\infty(\mathbb{R}^n; \mathbb{R}) \to C_\infty(\mathbb{R}^n; \mathbb{R})$ is called a *Feller process.* A subclass of these processes is formed by the Lévy processes which we define below. In the following we will deal with transition functions of specific Lévy processes.

Definition 3.3 (Lévy process). A *Lévy process* is a stochastic process $(X_t)_{t\geqslant 0}$ which has *independent* and *stationary increments*, i.e. the increments $X_t - X_s$ and $X_{t'} - X_{s'}$ are independent random variables provided the time intervals $[s, t]$ and $[s', t']$ are disjoint, and the distribution of any increment $X_t - X_s$ only depends on the length $|t - s|$ of the interval, thus $X_t - X_s$ is independent from any X_u, $u < s$. The third requirement is the stochastic continuity $\lim_{t\to s+} \mathbb{P}(|X_t - X_s| > \varepsilon) = 0$ for all $\varepsilon > 0$.

Every Lévy process $(X_t)_{t\geqslant 0}$ with state space $\mathbb{R}^n$ is completely determined by its *characteristic function* which is related to the continuous negative definite function ψ by

$$\mathbb{E}^x[e^{i(X_t - x)\cdot\xi}] = e^{-t\,\overline{\psi(\xi)}}, \quad t \geqslant 0\,.$$

The function $\overline{\psi}$ is sometimes also called the *characteristic exponent*, and the function ψ is called the *symbol of the process* following the notion from the generator of the associated semigroup.

Definition 3.4 (Diffusion process). A continuous time Markov process with continuous sample paths is called a *diffusion process.*

The symbol of a diffusion process is in general a continuous negative definite function which has a Lévy-Khinchin representation without integral part. We will briefly discuss estimates for the transition function of diffusions on general metric measure spaces in § 3.3. In § 3.5 we will consider symmetric Lévy processes $(X_t)_{t\geqslant 0}$ which have no diffusion part. The corresponding symbol ψ of the process is then a real-valued continuous negative definite function of the form (1.8), and we will further restrict ourselves to those ψ with $\psi(0) = 0$.

Starting with a Lévy process $(X_t)_{t\geqslant 0}$ a convolution semigroup of sub-probability measures $(\mu_t)_{t\geqslant 0}$ on $\mathbb{R}^n$ is given by the distribution of the random variables $X_t - X_0$ with respect to $\mathbb{P}^x$ provided the process started in x at time $t = 0$ with initial distribution $\mathbb{P}^x_{X_0} = \varepsilon_x$, i.e. $\mu_t = \mathbb{P}^x_{X_t - X_0}$. Since a Lévy process has independent and stationary increments this distribution is the same whatever the starting point x. Thus the semigroup $(\mu_t)_{t\geqslant 0}$ is the same for all x. As $(\mu_t)_{t\geqslant 0}$ is a convolution semigroup of sub-probability measures on $\mathbb{R}^n$ the resulting Lévy process is always a Feller process. In the case of μ_t being probability measures we arrive at a Markov process due to the extendability from $C_\infty(\mathbb{R}^n; \mathbb{R})$ to $B_b(\mathbb{R}^n; \mathbb{R})$ as discussed above.

Remark 3.5. Let us make a remark about the actual construction of certain processes starting with (3.2). If $(T_t)_{t\geqslant 0}$ is a Feller semigroup then we can construct p_t by (3.2) and apply the theorem of A.N. Kolmogorov (see e.g. [44], Theorem A.1) to obtain a Feller process. However, using this approach with a sub-Markovian semigroup $(T_t)_{t\geqslant 0}$ of operators $T_t : L^p(\mathbb{R}^n;\mathbb{R}) \to L^p(\mathbb{R}^n;\mathbb{R})$ we face the problem that p_t can only be defined according to (3.2) for *almost every* point $x \in \mathbb{R}^n$. To be more precise, taking a representative of $x \mapsto p_t(x, A)$ then for every fixed $t \geqslant 0$ and every fixed Borel set A we obtain a different exceptional set of points. Therefore, it is not possible to say that $p_t(x, A)$ constructed by (3.2) gives a family of sub-Markovian kernels. Hence it is not possible to construct the process pointwise as in the Feller case.

However, a fundamental idea of Fukushima [27] was to introduce a refined notion of exceptional points which made it possible to overcome this problem. We have seen in § 1.5 that in the case of symmetric L^2-sub-Markovian semigroups $(T_t)_{t\geqslant 0}$ we may use the translation invariant Dirichlet form $(\mathcal{E}, \mathsf{D}(\mathcal{E}))$ to study the generator $Au = -\psi(D)u$ and hence also the process $(X_t)_{t\geqslant 0}$ associated to it. Fukushima introduced the concept of a *capacity* of an open set $A \subset \mathbb{R}^n$ which enabled him to associate to any (not necessarily symmetric) Dirichlet form a certain Markov process. Roughly, the idea is as follows.

Definition 3.6 (Capacity). Let $\mathcal{E}$ be a Dirichlet form defined on $\mathsf{D}(\mathcal{E}) \subset L^2(\mathbb{R}^n;\mathbb{R})$ equipped with the graph norm $\mathcal{E}_G := \|\cdot\|_{\mathcal{E}}$. For a Borel set $A \subset \mathbb{R}^n$ its capacity is defined as

$$\operatorname{cap}(A) := \inf\{\mathcal{E}_G(u,u) : u \in \mathsf{D}(\mathcal{E}) \text{ and } u \geqslant 1 \text{ on } A\}. \tag{3.4}$$

For an arbitrary set C we set

$$\operatorname{cap}(C) = \inf\{\operatorname{cap}(A) : C \subset A \text{ and } A \text{ open}\}.$$

The capacity $\operatorname{cap}(C)$ of an arbitrary set C has then the property that $\lambda^{(n)}(C) \leqslant \operatorname{cap}(C)$. Hence, whenever C has capacity 0 it also holds $\lambda^{(n)}(C) = 0$. This justifies to say that sets of capacity zero are *refined sets* of Lebesgue measure zero. The capacity has the further property that $\operatorname{cap}(C) \leqslant \operatorname{cap}(D)$ whenever $C \subset D$ and satisfies the subadditivity property $\operatorname{cap}(\bigcup_{j\in\mathbb{N}} C_j) \leqslant \sum_{j\in\mathbb{N}} \operatorname{cap}(C_j)$.

Definition 3.7. In this setting, we call a set N an *exceptional set* if $\operatorname{cap}(N) = 0$ and say that a function $u : \mathbb{R}^n \to \mathbb{R}$ defined on $\mathbb{R}^n \setminus N$ is *quasi-continuous* if for every $\varepsilon > 0$ there exists an open subset $E \subset \mathbb{R}^n$ such that $\operatorname{cap}(E) < \varepsilon$ and u is continuous on E^c. It is called *quasi-continuous in the restricted sense* if u is continuous on $(\mathbb{R}^n \cup \{\partial\}) \setminus E$, where $\mathbb{R}^n \cup \{\partial\}$ is the one-point compactification of $\mathbb{R}^n$. If there exists a function v which is quasi-continuous in the restricted sense, then v is called a *quasi-continuous modification* of u.

From Theorem 3.7 in [44] we know that such a quasi-continuous modification v exists for each $u \in \mathsf{D}(\mathcal{E})$. Fukushima's result was the observation that when defining the transition function p_t associated to a symmetric sub-Markovian semigroup on $L^2(\mathbb{R}^n;\mathbb{R})$, which is associated itself with a symmetric Dirichlet form $\mathcal{E}$ via quasi-continuous modifications of $u \in \mathsf{D}(\mathcal{E})$, then it is possible to construct a Markov process. Some results on Markov processes and their construction with the help of quasi-continuous versions of the associated sub-Markovian semigroups $(T_t)_{t\geqslant 0}$ can also be found in [52].

Now we turn to an essential issue arising both in Analysis and Probability Theory, namely the problem of estimating heat kernels or transition density functions, respectively. Seminal contributions to the field of finding upper and lower bounds for heat kernels have been made by J. Nash [54] and D.G. Aronson [2], both in the context of elliptic and parabolic partial differential equations.

Let A be a second-order elliptic differential operator on $\mathbb{R}^n$, then we can associate a diffusion process $(X_t)_{t\geqslant 0}$ with it, of which A is the infinitesimal generator. The fundamental solution for the associated diffusion equation is then the transition function of the process $(X_t)_{t\geqslant 0}$. For example, let $A = \Delta$ be the Laplace operator in $\mathbb{R}^n$. The associated diffusion process $(X_t)_{t\geqslant 0}$ is the Brownian motion, and the fundamental solution to the classical heat equation corresponding to $A = \Delta$ is the Gauß kernel

$$p_t(x,y) = (4\pi t)^{-n/2} \exp\Big(- \frac{|x-y|^2}{4t}\Big), \quad x,y \in \mathbb{R}^n,\, t>0\,. \tag{3.5}$$

More precisely, if $v \in C_b(\mathbb{R}^n;\mathbb{R})$, i.e. bounded and continuous on $\mathbb{R}^n$, then

$$u(x,t) = \int_{\mathbb{R}^n} p_t(x,y)\, v(y)\, dy$$

solves the heat equation $(\partial_t - \Delta)u = 0$ with initial condition $u(0,x) = v(x)$. This is a parabolic equation, and together with the initial condition, this problem is sometimes called the *Cauchy problem* for the diffusion equation. In this analytic context p_t is often called the *heat kernel* referring to the associated heat semigroup generated by the Laplace operator $A = \Delta$.

Now, upper and lower bounds for $p_t(x,y)$ serve to understand the behaviour of the kernel as $t\to\infty$ and $d(x,y) = |x-y| \to \infty$. We observe that for distances $|x-y| \leqslant \sqrt{ct}$ for $t>0$ and some constant $c>0$ the equation (3.5) reads as

$$p_t(x,y) \leqslant (4\pi t)^{-n/2}\, e^{-c/4}, \quad t>0\,, \tag{3.6}$$

i.e. the decay of p_t is of order $t^{-n/2}$. Hence, the exponential term becomes relevant for larger distances $|x-y| > \sqrt{ct}$. Note that an upper bound of the form (3.6) is especially satisfied if $x=y$. For this reason an estimate of the form

$$p_t(x,y) \leqslant C\, t^{-n/2}$$

is often called *on-diagonal estimate.* In § 3.2 we will focus on the derivation of on-diagonal estimates for the kernels of selfadjoint operators forming a symmetric L^2-sub-Markovian semigroup. These operators arise as generators of symmetric Markov processes.

Before we do this, we will turn to L^p-sub-Markovian semigroups of operators T_t having the representation (3.3) and ask under which conditions the kernels $p_t(x,dy)$ have a density $p_t(x,y)\,dy$ with respect to the Lebesgue measure. The answer is given by the Dunford-Pettis criterion, which we quote from [44], Theorem 6.2. In the following, for an operator $A: L^p(\mathbb{R}^n;\mathbb{R}) \to L^q(\mathbb{R}^n;\mathbb{R})$, $1\leqslant p,q\leqslant\infty$, we write

$$\|A\|_{L^p - L^q} := \inf\{c : \|Au\|_{L^q} \leqslant c\,\|u\|_{L^p} \text{ for all } u\in L^p(\mathbb{R}^n;\mathbb{R})\}$$

for its operator norm.

Theorem 3.8 (Dunford-Pettis). *Let $(T_t)_{t\geqslant 0}$ be a strongly continuous contraction semigroup on $L^p(\mathbb{R}^n;\mathbb{R})$, $1\leqslant p<\infty$, with $T_t : L^p(\mathbb{R}^n;\mathbb{R}) \to L^\infty(\mathbb{R}^n;\mathbb{R})$ bounded for each $t>0$. Then, for $\frac{1}{p}+\frac{1}{q}=1$, there exist kernels $k_t : \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$, $k_t(\cdot,y)\in L^\infty(\mathbb{R}^n;\mathbb{R})$, $k_t(x,\cdot)\in L^q(\mathbb{R}^n;\mathbb{R})$ such that the operators T_t can be represented with the help of these kernels having density with respect to the Lebesgue measure, i.e.*

$$T_t u(x) = \int_{\mathbb{R}^n} u(y)\, k_t(x,y)\, dy \qquad \text{a.e. for } u\in L^p(\mathbb{R}^n;\mathbb{R})\,. \tag{3.7}$$

Further, the kernel satisfies

$$k_{s+t}(x,y) = \int_{\mathbb{R}^n} k_s(x,z)\, k_t(z,y)\, dz\,, \qquad s,t > 0\,,$$

and for the operator norm of T_t we have

$$\|T_t\|_{L^p - L^\infty} = \operatorname{ess\,sup}_{x\in\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} |k_t(x,y)|^q\, dy \Big)^{1/q}\,.$$

The converse also holds true, i.e. if T_t has a representation (3.7) *then it is bounded from $L^p(\mathbb{R}^n;\mathbb{R})$ to $L^\infty(\mathbb{R}^n;\mathbb{R})$.*

In the particular case $p = 2$ and $(T_t)_{t\geqslant 0}$ being a symmetric sub-Markovian semigroup on $L^2(\mathbb{R}^n;\mathbb{R})$ which satisfies the condition of Theorem 3.8, we get $k_t(x,y) \geqslant 0$ and $k_t(x,y) = k_t(y,x)$. An essential aim is to obtain estimates for kernels k_t in cases where we have a representation (3.7). In the case of symmetric L^2-sub-Markovian semigroups, such estimates have been well-studied. In this Hilbert space setting it turns out that the estimates have a close connection to symmetric Dirichlet forms. In the following paragraph we will briefly present the essential ideas on how to obtain these estimates based on the monographs by Davies [18] and Varopoulos et al. [62]. An overview of these results can also be found in the paper by Carlen et al. [10].

3.2 Estimates for symmetric Markov processes

In this paragraph we will focus on symmetric sub-Markovian semigroups $(T_t)_{t\geqslant 0}$ of operators acting on the Hilbert space $L^2(\mathbb{R}^n;\mathbb{R})$. We will assume that they satisfy the assumptions of Theorem 3.8, i.e. they can be represented as

$$T_t u(x) = \int_{\mathbb{R}^n} u(y)\, k_t(x,y)\, dy$$

almost everywhere for $u \in L^2(\mathbb{R}^n;\mathbb{R})$. The fist result in this section is taken from Davies [18] and formulated adequately for our setting $L^2(\mathbb{R}^n;\mathbb{R})$. It relates upper bounds for the norms of symmetric operators to the existence and boundedness of the kernel functions.

Lemma 3.9 ([18], Lemma 2.1.2). *If $(T_t)_{t\geqslant 0}$ is a symmetric semigroup of bounded operators from $L^2(\mathbb{R}^n;\mathbb{R})$ to $L^\infty(\mathbb{R}^n;\mathbb{R})$ for all $t > 0$, then its operators have a kernel representation with kernel $k_t(x,y)$ for all $t > 0$ satisfying*

$$0 \leqslant k_t(x,y) \leqslant \|T_{t/2}\|^2_{L^2 - L^\infty} \quad a.e.$$

Conversely, if $(T_t)_{t\geqslant 0}$ admits a kernel representation with $0 \leqslant k_t(x,y) \leqslant a_t$ a.e. for some constant $a_t < \infty$, then $(T_t)_{t\geqslant 0}$ is a semigroup of bounded operators from $L^2(\mathbb{R}^n;\mathbb{R})$ to $L^\infty(\mathbb{R}^n;\mathbb{R})$ for all $t > 0$ with operator norm

$$\|T_t\|^2_{L^2 - L^\infty} \leqslant a_t\,.$$

In the following we will study estimates for the norms $\|T_t\|_{L^p - L^q}$ for symmetric sub-Markovian semigroups $(T_t)_{t\geqslant 0}$. For this we will need the notion of an *equicontinuous family* of linear operators.

Definition 3.10 (Equicontinuity). Let X, Y be Banach spaces and $\mathcal{L}$ be a family of linear operators mapping from X to Y. Then $\mathcal{L}$ is said to be *equicontinuous* if, and only if, $\sup_{T\in\mathcal{L}} \|T\|_{X-Y} < \infty$, i.e. the family $\mathcal{L}$ is uniformly bounded in the operator norm.

Definition 3.11 (Bounded analytic semigroup). Let X be a Banach space. A semigroup $(T_t)_{t\geqslant 0}$ of operators is called *bounded analytic on* X if

$$\|A\,T_t\|_{X-X} \leqslant \frac{C}{t} \quad \text{for all } t > 0\,,$$

where $-A$ is the generator of $(T_t)_{t\geqslant 0}$.

Definition 3.11 is in fact (see [62], p. 13) an equivalent formulation of T_t admitting an equicontinuous extension to a sector $\Sigma_\theta := \{z \in \mathbb{C} : \arg(z) < \theta\}$ with $\theta \in \left(0, \frac{\pi}{2}\right)$ for all $t > 0$. With the help of Plancherel's theorem one can show that symmetric L^2-sub-Markovian semigroups of operators are bounded analytic. Without proof we quote the following two theorems from the monograph by Varopoulos, Saloff-Coste and Coulhon [62].

Theorem 3.12 ([62], Theorem II.3.1). *Suppose that $(T_t)_{t\geqslant 0}$ is a semigroup whose operators T_t are equicontinuous on both $L^1(\mathbb{R}^n;\mathbb{R})$ and $L^\infty(\mathbb{R}^n;\mathbb{R})$. We assume there are $1 < p < q \leqslant \infty$ and a constant $\alpha > 0$ such that $\|u\|_{L^q} \leqslant C\,\|A^{\alpha/2}u\|_{L^p}$ is satisfied for $u \in L^q(\mathbb{R}^n;\mathbb{R})$ and, moreover, T_t is bounded analytic on $L^p(\mathbb{R}^n;\mathbb{R})$. Then*

$$\|T_t\|_{L^1-L^\infty} \leqslant C\,t^{-\upsilon/2}$$

for all $t > 0$, where υ is determined by $\upsilon := \alpha\left(\frac{1}{p} - \frac{1}{q}\right)^{-1}$.

In [62], the number $2 < \upsilon < \infty$ is called the *dimension of the semigroup* $(T_t)_{t\geqslant 0}$.

Theorem 3.13 ([62], Theorem II.3.2). *Suppose that $T_t : L^1(\mathbb{R}^n;\mathbb{R}) \to L^1(\mathbb{R}^n;\mathbb{R})$ are equicontinuous and $\upsilon > 0$. Assume there exist $C > 0$ such that $\|u\|_{L^2}^{2+4/\upsilon} \leqslant C\,\mathrm{Re}(Au,u)\,\|u\|_{L^1}^{4/\upsilon}$ with generator A of T_t. Then*

$$\|T_t\|_{L^1-L^2} \leqslant K\,t^{-\upsilon/2} \quad \text{for } t > 0\,.$$

If moreover T_t are also equicontinuous on $L^\infty(\mathbb{R}^n;\mathbb{R})$ then

$$\|T_t\|_{L^1-L^\infty} \leqslant K\,t^{-\upsilon/2} \quad \text{for } t > 0\,.$$

In the case of a symmetric L^2-sub-Markovian semigroups its generator A is self-adjoint and the condition in Theorem 3.12 for $p = 2$ and $q > 2$ reads as

$$\|u\|_{L^q} \leqslant C\,\|A^{\alpha/2}u\|_{L^2}$$

with $\alpha = \upsilon\left(\frac{1}{2} - \frac{1}{q}\right)$. In particular,

$$\|u\|_{L^q} \leqslant C\,\|A^{1/2}u\|_{L^2} = C\,\sqrt{\mathcal{E}(u,u)} \tag{3.8}$$

if $\upsilon = \frac{2q}{q-2} > 2$. According to Remark II.3.3 in [62] we also have the converse of Theorem 3.13 for symmetric L^2-sub-Markovian semigroups, i.e. the norm estimates for T_t imply that T_t is equicontinuous on both L^1 and L^∞ and

$$\|u\|_{L^2}^{2+4/\upsilon} \leqslant C\,\mathrm{Re}(Au,u)\,\|u\|_{L^1}^{4/\upsilon} \leqslant C\,(A^{1/2}u, A^{1/2}u)\,\|u\|_{L^1}^{4/\upsilon} \leqslant C\,\mathcal{E}(u,u)\,\|u\|_{L^1}^{4/\upsilon}\,. \tag{3.9}$$

In view of Theorem 3.12 we then find that (3.8) and (3.9) must be equivalent. This equivalence is shown in [10]. These results may be summarised in the following theorem (cf. Theorems 6.3 and 6.4 in [44]).

Theorem 3.14. *Let $(T_t)_{t\geqslant 0}$ be a symmetric sub-Markovian semigroup on $L^2(\mathbb{R}^n;\mathbb{R})$ with corresponding regular Dirichlet form $(\mathcal{E},\mathsf{D}(\mathcal{E}))$. Then the following estimates are equivalent.*

$$\|u\|_{L^q}^2 \leqslant C\,\mathcal{E}(u,u) \quad \text{for all } u\in\mathsf{D}(\mathcal{E}),\ q>2,\ \upsilon=\tfrac{2q}{q-2}>2 \qquad \text{(Sobolev-type inequality)}$$

$$\|u\|_{L^2}^{2+4/\upsilon} \leqslant C\,\mathcal{E}(u,u)\,\|u\|_{L^1}^{4/\upsilon} \quad \text{for all } u\in\mathsf{D}(\mathcal{E})\cap L^1(\mathbb{R}^n;\mathbb{R}),\ \upsilon>0 \qquad \text{(Nash-type inequality)}$$

$$\|T_t\|_{L^1-L^\infty} \leqslant C\,t^{-\upsilon/2} \quad \text{for all } t>0 \text{ and } \upsilon>0$$

$$\|T_t\|_{L^p-L^q} \leqslant C\,t^{-\frac{\upsilon}{2}\left(\frac1p-\frac1q\right)} \quad \text{for all } t>0,\ 1\leqslant p<q\leqslant\infty,\ \upsilon>0\,.$$

We note that the last estimate holds due to Theorem 1.45, i.e. the extendability of a symmetric L^2-sub-Markovian semigroup to an analytic sub-Markovian semigroup on $L^p(\mathbb{R}^n;\mathbb{R})$. For the following example see [44], Chapter 6.

Example 3.15. In this example we will now apply Theorem 3.14 to the pseudodifferential operator $A^{(2)}=-\psi(D)$ given by

$$-\psi(D)u(x) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\,\psi(\xi)\,\hat{u}(\xi)\,d\xi\,.$$

For $-\psi(D)$ to be the generator of a symmetric L^2-sub-Markovian semigroup we must have that $\mathsf{R}(\lambda-A^{(2)})=\mathsf{R}(\lambda+\psi(D))$ is dense in $L^2(\mathbb{R}^n;\mathbb{R})$ by the Hille-Yosida Theorem. Moreover, if $-\psi(D)$ generates the semigroup $(T_t^{(2)})_{t\geqslant 0}$ then $-(\lambda+\psi(D))$ generates the semigroup $(S_t)_{t\geqslant 0} := (e^{-\lambda t}T_t^{(2)})_{t\geqslant 0}$. Indeed, since the strongly continuous contraction semigroup $(T_t^{(2)})_{t\geqslant 2}$ consists of operators of the form $T_t^{(2)}=e^{tA^{(2)}}=e^{-t\psi(D)}$ with bounded linear operator $-\psi(D)$, we have

$$S_t = e^{-t(\lambda+\psi(D))} = e^{-\lambda t}e^{-t\psi(D)} = e^{-\lambda t}T_t^{(2)}$$

for $t\geqslant 0$. Hence, if $\|S_t\|_{L^p-L^q}\leqslant C$ for some constant $C>0$ then

$$\|T_t^{(2)}\|_{L^p-L^q} = \|e^{\lambda t}S_t\|_{L^p-L^q} \leqslant C\,e^{\lambda t}\,.$$

Now we may apply Theorem 3.14. As $(T_t^{(2)})_{t\geqslant 0}$ is a symmetric sub-Markovian semigroup on $L^2(\mathbb{R}^n;\mathbb{R})$ generated by the bounded linear operator $-\psi(D)$, it extends – by Theorem 1.45 – to an analytic sub-Markovian semigroup $(T_t^{(p)})_{t\geqslant 0}$ on $L^p(\mathbb{R}^n;\mathbb{R})$. Therefore, an application of Theorem 3.14 gives

$$\|T_t^{(p)}\|_{L^p-L^q} \leqslant C\,e^{\lambda t}\,t^{-\frac{\upsilon}{2}\left(\frac1p-\frac1q\right)}$$

for all $t>0$ and all $1\leqslant p<q\leqslant\infty$. In particular, setting $p=2$ and $q=\infty$ we obtain

$$\|T_t^{(2)}\|_{L^2-L^\infty} \leqslant C\,e^{\lambda t}\,t^{-\frac{\upsilon}{4}}\,.$$

But by Theorem 3.8 the operator norm of $T_t^{(2)}$ is given by

$$\|T_t^{(2)}\|_{L^2-L^\infty} = \operatorname{ess\,sup}_{x\in\mathbb{R}^n}\Big(\int_{\mathbb{R}^n}|k_t(x,y)|^2\,dy\Big)^{1/2} \leqslant C\,e^{\lambda t}\,t^{-\frac{\upsilon}{4}}\,. \tag{3.10}$$

Now, by the definition of k_t via the Chapman-Kolmogorov equation, the Cauchy-Schwarz inequality and the symmetry of any k_s, $s > 0$,

$$\begin{aligned} k_t(x,y) &= \int_{\mathbb{R}^n} k_{t/2}(x,z)\, k_{t/2}(z,y)\, dz \\ &\leqslant \left(\int_{\mathbb{R}^n} k_{t/2}^2(x,z)\, dz\right)^{1/2} \left(\int_{\mathbb{R}^n} k_{t/2}^2(z,y)\, dz\right)^{1/2} \\ &= \left(\int_{\mathbb{R}^n} k_{t/2}(x,z)\, k_{t/2}(z,x)\, dz\right)^{1/2} \left(\int_{\mathbb{R}^n} k_{t/2}(y,z)\, k_{t/2}(z,y)\, dz\right)^{1/2} \\ &= k_t(x,x)^{1/2}\, k_t(y,y)^{1/2}, \end{aligned}$$

and hence,

$$\begin{aligned} \operatorname{ess\,sup}_{x,y\in\mathbb{R}^n} k_t(x,y) &\leqslant \operatorname{ess\,sup}_{x,y\in\mathbb{R}^n} k_t^{1/2}(x,x)\, k_t^{1/2}(y,y) \\ &\leqslant \operatorname{ess\,sup}_{x\in\mathbb{R}^n} k_t^{1/2}(x,x)\, \operatorname{ess\,sup}_{y\in\mathbb{R}^n} k_t^{1/2}(y,y) \\ &= \operatorname{ess\,sup}_{x\in\mathbb{R}^n} k_t(x,x) \\ &\leqslant \operatorname{ess\,sup}_{x,y\in\mathbb{R}^n} k_t(x,y)\,. \end{aligned}$$

This gives the equality $\operatorname{ess\,sup}_{x,y\in\mathbb{R}^n} k_t(x,y) = \operatorname{ess\,sup}_{x\in\mathbb{R}^n} k_t(x,x)$ and hence the estimate (3.10) is in fact an on-diagonal estimate for the kernel $k_t(x,y)$.

In [17], Davies has constructed off-diagonal estimates for kernels related to elliptic operators of second order. In [10] his approach has been generalised to work for symmetric Markov processes which are not necessarily generated by a differential operator.

3.3 Remarks on heat kernel estimates on metric measure spaces

From the probabilistic point of view it has become of interest in the past few years to estimate the transition density of a process in more general contexts, such as on Riemannian manifolds (M, g) with Riemannian metric g, or on general metric measure spaces (X, d, μ), where μ is a Radon measure on (X, d). In this setting we mention the work by Sturm [58] who constructed a diffusion process on X for quasi-every starting point using the techniques of local Dirichlet forms that can be associated with a diffusion process. In the setting of (Riemannian) manifolds contributions have been made by Grigor'yan [29], Li and Yau [51] and Davies [17]. Works of Grigor'yan et al. [30, 31, 32, 33, 35, 36] deal with heat kernel estimates for diffusions on general metric measure spaces satisfying a certain volume growth condition.

On general metric measure spaces the idea is roughly as follows. Motivated by the classical Gauß kernel, the aim is to get an estimate of the form

$$p_t(x,y) \approx \frac{C_1}{\mu(B^d(x,\sqrt{t}))} \exp\left(-\frac{d^2(x,y)}{C_2\, t}\right), \tag{3.11}$$

for diffusion processes on (X, d, μ), where $B^d(x,r)$ denotes an open ball with respect to the metric $d(x,y)$ of radius $r > 0$ centred at the point $x \in X$. Here, $\approx$ stands for the existence of an upper and lower bound with in general different constants. Note that in the metric measure space $(\mathbb{R}^n, |\cdot|, \lambda^{(n)})$ the estimate (3.11) reads as

$$p_t(x,y) \approx C_1\, t^{-n/2} \exp\left(-\frac{|x-y|^2}{C_2\, t}\right) \tag{3.12}$$

and the constants with which we get equality in (3.12) can be given explicitly as $C_1 = (4\pi)^{-n/2}$ and $C_2 = 4$, see (3.5).

In a series of papers [3, 30, 31, 32, 36], Grigor'yan et al. discuss heat kernel estimates of self-similar type, as they appear in e.g. fractals. They study bounds of the form

$$p_t(x,y) \approx t^{-\alpha/\beta}\, \Phi\left(\frac{d(x,y)}{t^{1/\beta}}\right) \tag{3.13}$$

with parameter α and β, and a function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ on general metric measure spaces (X, d, μ). Let $p_t(x,y)$ be a heat kernel on a metric measure space (X, d, μ) which satisfies (3.13) for μ-almost all $x, y \in X$. Then in Theorem 6.7 of [31] it is shown that the following scenarios occur under certain assumptions on the metric measure space. In the case of diffusions, corresponding to a local operator generating the process, the function Φ and the parameter α, β satisfy the exponential estimate

$$\Phi(s) \approx c_1 \exp\left(-c_2\, s^{\frac{\beta}{\beta-1}}\right), \quad 2 \leqslant \beta \leqslant \alpha + 1,$$

whereas Φ is of the polynomial nature

$$\Phi(s) \approx (1+s)^{-(\alpha+\beta)}, \quad 0 < \beta \leqslant \alpha + 1,$$

in the non-local case. The following examples can also be found in [30].

Example 3.16 (Local case). For $\alpha = n$, $\beta = 2$ and $\Phi(s) = \exp(-s^2)$ the estimate (3.13) becomes the Gaussian estimate

$$p_t(x,y) \approx C\, t^{-n/2} \exp\left(-\frac{d(x,y)^2}{C_2\, t}\right)$$

for a diffusion process on the metric measure space (X, d, μ).

Example 3.17 (Non-local case). Let $A = \Delta^{1/2}$ be the fractional Laplace operator in the metric measure space $(\mathbb{R}^n, |\cdot|, \lambda^{(n)})$. The associated process is a Cauchy process, which belongs to the family of Lévy processes. The solution of the corresponding evolution equation $(\partial_t - A)u = (\partial_t - \Delta^{1/2})u = 0$ is the heat kernel

$$p_t(x,y) = \frac{C}{t^n}\left(1 + \frac{|x-y|^2}{t^2}\right)^{-\frac{n+1}{2}} = \frac{C\, t}{(t^2 + |x-y|^2)^{(n+1)/2}}$$

with the constant $C = \Gamma\left(\frac{n+1}{2}\right) \pi^{-\frac{n+1}{2}}$, which is called the *Poisson kernel* and is the transition density of the Cauchy process. Hence, if $\alpha = n$ and $\beta = 1$, the function $\Phi(s) = (1+s)^{-(\alpha+\beta)}$ in (3.13) leads to the Poisson heat kernel in the metric measure space $(\mathbb{R}^n, |\cdot|, \lambda^{(n)})$.

The parameter α and β given in the heat kernel estimate (3.13) do not only determine the actual estimate, but also reveal properties of the underlying metric measure space. While α turns out to be an exponent related to the volume growth of balls in (X, d, μ), the parameter β characterises properties of certain function spaces on (X, d, μ). More precisely, Grigor'yan et al. [34] introduced a family of Besov spaces $W^{\sigma,2}(X, d, \mu)$ on the metric measure space by

$$W^{\sigma,2}(X,d,\mu) := \left\{ u \in L^2(X) : \sup_{\sigma > 0} \frac{1}{r^{2\sigma}} \int_X \fint_{B_r^d(x)} |u(y) - u(x)|^2\, \mu(dy)\, \mu(dx) < \infty \right\}.$$

Concerning the parameter α we have the following result, which is taken from [31], Theorem 2.10.

Theorem 3.18. *Let (X, d, μ) be a metric measure space, $x, y \in X$ and $p_t(x, y)$ a heat kernel on X. Let $\alpha, \beta > 0$ and $\Phi_1, \Phi_2 : [0, \infty) \to [0, \infty)$ be monotone decreasing functions such that $\Phi_1(s) > 0$ for some $s > 0$ and Φ_2 satisfies the integrability condition*

$$\int_0^\infty s^{\alpha-1}\, \Phi_2(s)\, ds < \infty\,.$$

A. *If for μ-almost all $x, y \in X$ and all $t > 0$*

$$p_t(x, y) \geqslant t^{-\alpha/\beta}\, \Phi_1\left(\frac{d(x, y)}{t^{1/\beta}}\right)$$

then there exists a constant C such that

$$\mu(B^d(x, r)) \leqslant C\, r^\alpha$$

for all $x \in X$ and $r > 0$.

B. *If $\int_X p_t(x, y)\, \mu(dy) = 1$ for μ-almost all $x \in X$, and*

$$t^{-\alpha/\beta}\, \Phi_1\left(\frac{d(x, y)}{t^{1/\beta}}\right) \leqslant p_t(x, y) \leqslant t^{-\alpha/\beta}\, \Phi_2\left(\frac{d(x, y)}{t^{1/\beta}}\right)$$

for μ-almost all $x, y \in X$ and all $t > 0$, then there exists a constant C such that

$$C^{-1}\, r^\alpha \leqslant \mu(B^d(x, r)) \leqslant C\, r^\alpha\,.$$

As it is noted in [30], the functions $\Phi(s) = \exp(s^2)$ and $\Phi(s) = (1 + s)^{-(n+1)}$ satisfy the conditions on Φ_2 in Theorem 3.18.

In recent years, an increasing interest has taken place in finding estimates for transition densities of Lévy processes of jump-type using geometric properties of the underlying metric measure space (X, d, μ) such as the volume of metric balls. Before we study one of these proposed estimates in § 3.5 we first reveal an interesting link between the metric of the space and the process under consideration.

3.4 Embeddings of metric spaces into Hilbert space

In this paragraph we note a result concerned with the embedding of certain metric spaces (X, d) into a Hilbert space $(H, \langle\cdot,\cdot\rangle_H)$. It turns out that negative definite functions play a significant role in this context. The connection of these functions to isometric and uniform embeddings into a Hilbert space originates in the work of Schoenberg [55, 56]. However, we take the results from the monograph [8] by Benyamini and Lindenstrauss. First, we recall the following definitions.

Definition 3.19. Let (X_1, d_1) and (X_2, d_2) be two metric spaces.

A. An *isometric embedding* $f : (X_1, d_1) \hookrightarrow (X_2, d_2)$ is an injective map from X_1 into X_2 which preserves the distance, i.e. $d_2(f(x), f(y)) = d_1(x, y)$.

B. A *uniform embedding* $h : (X_1, d_1) \hookrightarrow (X_2, d_2)$ is a bijective mapping such that h and h^{-1} are uniformly continuous on their domain of definitions, i.e. h is a uniform homeomorphism between X_1 and X_2.

Definition 3.20. An Hermitian kernel K on a metric space (X,d) is said to be *positive definite* if $\sum_{i,j=1}^{n} K(x_i,x_j)\,\lambda_i\,\bar{\lambda}_j \geqslant 0$ for all $x_1,\ldots,x_n \in X$ and all scalars $\lambda_1,\ldots,\lambda_n \in \mathbb{C}$.

Note that the positive definiteness of K means that the matrix $(K(x_i,x_j))_{i,j=1,\ldots,n}$ is positive Hermitian. Therefore, K is diagonalisable and all subdeterminants are nonnegative. For positive definite kernels $K_1, K_2 : X \to \mathbb{C}$ we arrive at $K := K_1\,K_2$ being again positive definite, since

$$\sum_{i,j=1}^{n} K(x_i,x_j)\,\lambda_i\,\bar{\lambda}_j = \sum_{i,j=1}^{n} K_1(x_i,x_j)\,\tilde{\lambda}_i\,\bar{\tilde{\lambda}}_j \sum_{i,j=1}^{n} K_2(x_i,x_j)\,\tilde{\lambda}_i\,\bar{\tilde{\lambda}}_j \geqslant 0\,,$$

where $\tilde{\lambda}_k = \sqrt{\lambda_k}$. Further, if K is a positive definite kernel then so is its complex conjugate $\overline{K}$ and its real part $\mathrm{Re}(K)$.

Let $U \subset \mathbb{C}$ be an open set such that for some $\rho > 0$ the disk $D(0,\rho) := \{z \in \mathbb{C} : |z| \leqslant \rho\}$ is contained in U. Let f be analytic in U, hence it has a representation as a convergent power series $f(z) = \sum_{k=0}^{\infty} a_k z^k$ for $z \in D(0,\rho)$. Assume further that $a_k \geqslant 0$ and that $K : X \times X \to \mathbb{C}$ is a positive definite kernel with $\mathsf{R}(K) \subset D(0,\rho)$. Then $f(K)$ is also positive definite since

$$\begin{aligned}\sum_{i,j=1}^{n} f(K(x_i,x_j))\,\lambda_i\,\bar{\lambda}_j &= \sum_{i,j=1}^{n}\sum_{k=0}^{\infty} a_k K(x_i,x_j)^k\,\lambda_i\,\bar{\lambda}_j \\ &= \sum_{k=0}^{\infty} a_k \sum_{i,j=1}^{n} K(x_i,x_j)^k\,\lambda_i\,\bar{\lambda}_j \;\geqslant\; 0\,.\end{aligned}$$

Since $a_k \geqslant 0$ by assumption and the series is uniformly convergent on the compact set $D(0,\rho)$ and the set of positive definite kernels is closed under uniform convergence on compact sets. This result holds in particular for $f(z) = e^z$ as it is an entire function, i.e. analytic on the whole complex plane. In the special case $X = \mathbb{R}^n$, where the kernel is induced by a function in a way that $K(x,y) = g(x-y)$, we arrive at Definition 1.13 of a positive definite function. The above listed properties carry over to positive definite functions. The following proposition provides a relation of positive definite kernels K to mappings into a Hilbert space. Its proof is given in [8].

Proposition 3.21. *A kernel $K(x,y)$ on $X \times X$ such that $K(x,y) = \overline{K(y,x)}$ is positive definite if, and only if, there is a Hilbert space H and a mapping $T : X \to H$ such that $K(x,y) = \langle T(x), T(y)\rangle_H$, where $\langle\cdot,\cdot\rangle_H$ is the inner product on H.*

In particular, assume $K : X \times X \to \mathbb{R}$ symmetric, i.e. $K(x,y) = K(y,x)$, with $K(x,x) = 1$ for all $x \in X$, and $K(x,y)$ is induced by $g(x-y)$. Then Proposition 3.21 yields the following result.

Corollary 3.22. *Let g be a real-valued positive definite function with $g(0) = 1$. Then it holds*

(i) $|g(x)| \leqslant g(0)$,

(ii) $|g(x) - g(y)|^2 \leqslant 2(1 - g(x-y))$,

(iii) $1 - g(nx) \leqslant n^2(1 - g(x))$

for all $x, y \in X$ and $n \in \mathbb{N}$.

The proof of this result can be found in [8]. This corollary collects some properties of positive definite functions we introduced already in § 1.2. It turns out that real-valued positive definite kernels

K (induced by functions g) with the properties discussed in Corollary 3.22 play a fundamental role when studying uniform embeddings of metric spaces into a Hilbert space. There are conditions on K given in [8], Chapter 8.2, for this to be the case.

Closely related to the notion above are negative definite kernels N on general metric spaces (X, d). They are defined as follows.

Definition 3.23. An Hermitian kernel N on a metric space (X, d) is called *(conditionally) negative definite* if $\sum_{i,j=1}^{n} N(x_i, x_j)\, \lambda_i \bar{\lambda}_j \leqslant 0$ for all $x_1, \ldots, x_n \in X$ and all scalars $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ that satisfy $\sum_{j=1}^{n} \lambda_j = 0$.

It follows immediately from Definitions 3.20 and 3.23 that $1 - K(x, y)$ is negative definite whenever $K(x, y)$ is a positive definite one. As in the positive definite case a function ψ on $X = \mathbb{R}^n$ is negative definite if the kernel it induces on $\mathbb{R}^n$ by $N(x, y) = \psi(x - y)$ is negative definite. For real-valued negative definite kernels on X we have the following relation to mappings into a Hilbert space.

Proposition 3.24. *Let $N : X \times X \to \mathbb{R}$ be a real-valued kernel with the property $N(x, x) = 0$ for all $x \in X$. Then N is negative definite if, and only if, there exists a Hilbert space H and an injective mapping $T : X \to H$ such that $N(x, y) = \|T(x) - T(y)\|_H^2$ holds for all $x, y \in X$.*

The proof of Proposition 3.24 is given in [8]. Transferring this result into the context of a metric space $(\mathbb{R}^n, d)$, Proposition 3.24 gives the necessary and sufficient conditions for $(\mathbb{R}^n, d)$ to be isometrically embeddable into a Hilbert space $(H, \langle \cdot, \cdot \rangle_H)$, namely that $d^2(x, y) = \psi(x - y)$ is a real-valued negative definite function with $\psi(x - y) = 0$ if, and only if, $x = y$. By the properties of continuous negative definite functions $\psi : \mathbb{R}^n \to \mathbb{R}$, see § 1.2, we can deduce that $\psi^{1/2}$ indeed gives a metric, as it is non-negative, symmetric and satisfies the triangle inequality by Lemma 1.20 (i).

In [13] Chen and Kumagai introduced a heat kernel estimate for symmetric jump processes on a class of metric measure spaces. The following paragraph is concerned with the performance of their estimate in more detail. Motivated by Proposition 3.24 we will study the proposed upper bound for two examples of Lévy processes with state space $\mathbb{R}^n$ employing $\psi^{1/2}$ as a metric, where $\psi : \mathbb{R}^n \to \mathbb{R}$ is a continuous negative definite function.

3.5 Computation of a heat kernel estimate for a symmetric Lévy process

In § 3.3 we have given a brief overview about transition function estimation of diffusion processes on general metric measure spaces. In recent years research has been extended towards heat kernel estimates of symmetric jump processes. We will focus on an approach by Chen and Kumagai [13], who introduced the two-sided heat kernel estimate

$$p_t(x, y) \approx c_1 \left(\frac{1}{\mu(B^d(y, \phi^{-1}(t)))} \wedge \frac{t}{\mu(B^d(y, d(x, y)))\, \phi(c_2\, d(x, y))} \right) \tag{3.14}$$

on metric measure spaces (X, d, μ) for symmetric jump processes which correspond to fractional-like Laplacian operators, valid for any $x, y \in X$. They considered cases where the jump intensities J in the associated Dirichlet forms are comparable to radially symmetric functions. In this paragraph it is our aim to visualise the performance of the estimate (3.14) in the metric measure space

$(\mathbb{R}^n, \psi^{1/2}, \lambda^{(n)})$ in two explicit cases. To obtain (3.14) the authors impose the condition of uniform volume doubling in the metric measure space under consideration. This means that the balls $B^d(y,r)$ with respect to the metric d in the space satisfy a volume doubling property which only depends on their radius $r > 0$ and not on the points $y \in X$ at which the balls are centred. We will show in § 4 that the spaces $(\mathbb{R}^n, \psi^{1/2}, \lambda^{(n)})$ fulfil this condition.

We start with deriving the form of the transition functions we will be working with below. Consider again the semigroup $(T_t)_{t\geqslant 0}$ of operators given in (1.11), i.e.

$$T_t u(x) = \mu_t * u(x) = \int_{\mathbb{R}^n} u(x-y)\,\mu_t(dy)$$

with the convolution semigroup $(\mu_t)_{t\geqslant 0}$. In Example 1.43 we remarked that on $\mathcal{S}(\mathbb{R}^n;\mathbb{R})$ the operators T_t can be written in the form of a pseudodifferential operator involving a continuous negative definite function $\psi(\xi)$ emerging from Fourier transforming the measure μ_t, i.e. $\hat{\mu}_t(\xi) = (2\pi)^{-n/2} e^{-t\psi(\xi)}$. Using the definition of the Fourier transform of u we get

$$\begin{aligned} T_t u(x) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\, e^{-t\psi(\xi)}\, \hat{u}(\xi)\, d\xi \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i(y-x)\cdot\xi}\, e^{-t\psi(\xi)}\, d\xi\, u(y)\, dy \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\, e^{-t\psi(\xi)}\, d\xi\, u(y)\, dy \\ &= \int_{\mathbb{R}^n} p_t(x-y)\, u(y)\, dy \;=\; \int_{\mathbb{R}^n} u(x-y)\, p_t(y)\, dy\,, \end{aligned}$$

hence p_t is of the form

$$p_t(x-y) = \mathcal{F}^{-1}(\hat{\mu}_t)(x-y) = \mathcal{F}(\hat{\mu}_t)(y-x)\,.$$

If we assume that $\hat{\mu}_t \in L^1(\mathbb{R}^n;\mathbb{R}) \cap C_\infty(\mathbb{R}^n;\mathbb{R})$ then by the Lemma of Riemann-Lebesgue $p_t \in L^1(\mathbb{R}^n;\mathbb{R}) \cap C_\infty(\mathbb{R}^n;\mathbb{R})$ and

$$p_t(x-y) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\, e^{-t\psi(\xi)}\, d\xi\,. \tag{3.15}$$

We consider symmetric Lévy processes starting at a point $y \in \mathbb{R}^n$ which have no diffusion part, i.e. the respective continuous negative definite functions $\psi : \mathbb{R}^n \to \mathbb{R}$ are of the form

$$\psi(\xi) = \int_{\mathbb{R}^n\setminus\{0\}} (1-\cos(y\cdot\xi))\, \nu(dy)$$

with their Lévy measure ν. We will study two cases in which the Lévy process $(X_t)_{t\geqslant 0}$, $X_0 = y$, has a density (3.15) which is well-defined, i.e. ψ is such that $e^{-t\psi(\xi)} \in L^1(\mathbb{R}^n;\mathbb{R})$. The transition functions we consider are that of the α-symmetric stable and the relativistic symmetric stable processes, which have the real-valued continuous negative definite functions

$$\psi_1(\xi) = |\xi|^{2\alpha}, \quad 0 < \alpha < 1\,, \tag{3.16}$$

and

$$\psi_2(\xi) = (|\xi|^2 + m^2)^{1/2} - m\,, \quad m > 0\,, \tag{3.17}$$

as symbols, respectively. Note that both functions are radially symmetric and subordinate to the map $\xi \mapsto |\xi|^2$ with respect to certain Bernstein functions, hence, by Theorem 1.37, their Lévy-Khinchin representations take the form

$$\psi_i(\xi) = \int_{\mathbb{R}^n\setminus\{0\}} (1-\cos(y\cdot\xi))\, m_i(|y|^2)\, dy\,,$$

where the $m_i(\cdot)$, $i = 1, 2$, are the Laplace transforms of the respective Lévy measure. In each case we determine the corresponding non-local Dirichlet form, which reads as

$$\begin{aligned}\mathcal{E}(u,u) &= \frac{1}{2}\int_{\mathbb{R}^n\setminus\{0\}}\int_{\mathbb{R}^n\setminus\{0\}} (u(x+y)-u(x))^2\, m_i(|y|^2)\, dy\, dx \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (u(x)-u(y))^2\, J_i(x,y)\, dy\, dx\,.\end{aligned}$$

Here, J_i are the symmetric kernels $J_i(x,y) = \frac{1}{2}\, m_i(|x-y|^2)$ for $i = 1, 2$. For detailed calculations of the kernels J_i associated to (3.16) and (3.17) for general $\alpha \in (0,1)$ and $m > 0$ see Appendix B.

We are now able to compute an upper bound for each J_i and to visualise an estimate for each heat kernel $p_{t,i}(x,y)$, $i = 1, 2$, both given in [13], corresponding to the symbols (3.16) and (3.17). Following the approach by Chen and Kumagai we determine strictly increasing functions $\phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, 2$, satisfying $\phi_i(0) = 0$ and $\phi_i(1) = 1$ using the equality

$$J_i(x,y) = \frac{c_1}{\lambda^{(n)}(B^{\psi}(y, \psi_i^{1/2}(x-y)))\, \phi_i(\psi_i^{1/2}(x-y))} \tag{3.18}$$

(cf. the two-sided estimate for the symmetric jump kernel in [13], p. 280). In each case the constant c_1 in (3.18) can be chosen such that

$$\phi_i(\psi_i^{1/2}(x-y)) = \frac{c_1}{\lambda^{(n)}(B^{\psi}(y, \psi_i^{1/2}(x-y))\, J_i(x,y)} \tag{3.19}$$

satisfies the above conditions. The upper heat kernel estimates in the metric measure spaces $(\mathbb{R}^n, \psi_i^{1/2}, \lambda^{(n)})$ then read as

$$p_{t,i}(x,y) \leqslant C\left(\frac{1}{\lambda^{(n)}(B^{\psi_i}(y,\phi_i^{-1}(t)))} \wedge \frac{t}{\lambda^{(n)}(B^{\psi_i}(y,\psi_i^{1/2}(x-y)))\,\phi_i(\psi_i^{1/2}(x-y))}\right), \tag{3.20}$$

$i = 1, 2$, where $\phi_i : \mathbb{R}_+ \to \mathbb{R}_+$ are the strictly increasing functions from above and determined by (3.19), and $B^{\psi_i}(z,r)$ denotes the ball of radius $r > 0$ with respect to the metric $\psi_i^{1/2}$ centred at the point z.

In the following we assume that $y = 0$ which is possible due to the translation invariance of $\psi^{1/2}$ as we will show in § 4. This corresponds to the respective process starting in 0.

3.5.1 The estimate for the symmetric α-stable process

The process corresponding to (3.16) is a symmetric stable process of order 2α on $\mathbb{R}^n$, $0 < \alpha < 1$, and p_t can be given in concrete form if $\alpha = \frac{1}{2}$. In this case we have

$$p_{t,1}(x,y) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} e^{-t|\xi|}\, d\xi = \Gamma\left(\frac{n+1}{2}\right)\frac{t}{\pi(|x-y|^2+t^2)^{\frac{n+1}{2}}}\,.$$

Without loss of generality we set $y = 0$. The kernel $J_1(x) := J_1(x,0)$ for the symmetric stable process of order 1 is then given by

$$J_1(x) = \frac{\Gamma(\frac{n+1}{2})}{2\,\pi^{n/2}\,\Gamma(\frac{1}{2})}\,\frac{1}{|x|^{n+1}}\,,$$

and we use this to determine

$$\phi_1(\psi_1^{1/2}(x)) = \frac{c_1}{\lambda^{(n)}(B^{\psi_1}(0,\psi_1^{1/2}(x)))\, J_1(x)} = \frac{c_1}{\kappa_1\, |x|^{-1}}$$

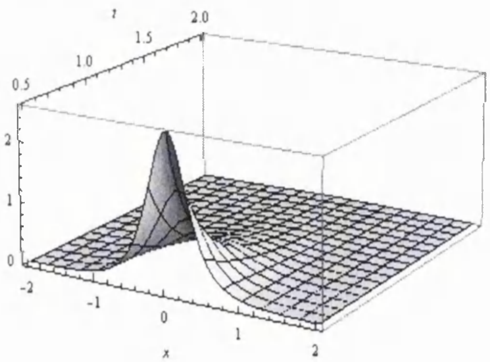


Figure 3.1. The Cauchy transition density and how it evolves for $0.5 \leqslant t \leqslant 2$.

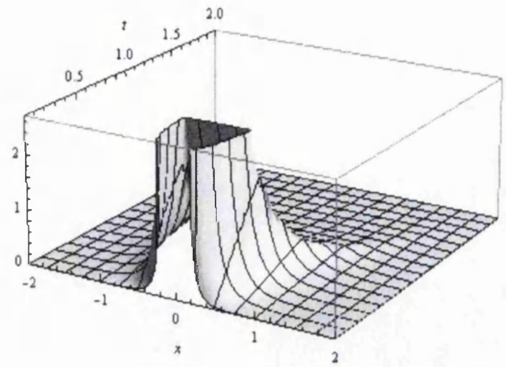


Figure 3.2. The upper bound by Chen and Kumagai [13] for $t > 0$.

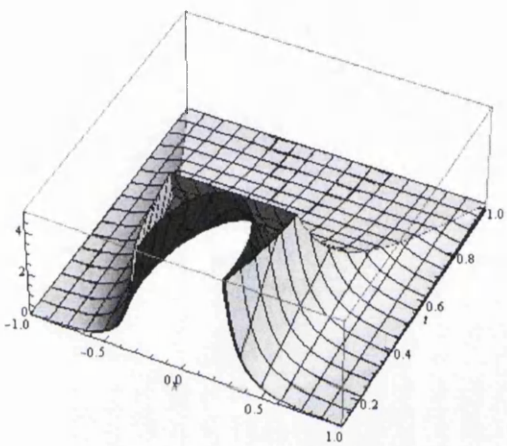


Figure 3.3. The upper estimate of the Cauchy transition for $0 < t \leqslant 1$.

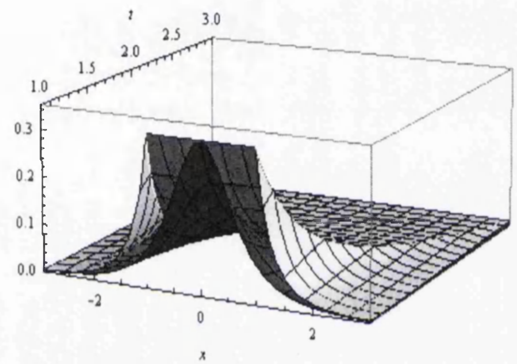


Figure 3.4. Cross-section of the estimate at time $t = 1$.

with $c_1, \kappa_1 > 0$. The constant c_1 can be chosen such that $\phi_1(1) = 1$. Then we can rewrite (3.20) as

$$p_{t,1}(x) := p_{t,1}(x, 0) \leqslant C_1 \left(\frac{1}{\lambda^{(n)}(B^{\psi_1}(0, \phi_1^{-1}(t)))} \wedge \frac{t\, J_1(x)}{c_1} \right) . \tag{3.21}$$

The calculations leading to estimate (3.21) are provided in Appendix B.1.

We use MATHEMATICA as a tool to carry out the computations necessary to visualise the upper transition function estimates. Figure 3.1 shows how the density of the symmetric stable process of order 1 evolves for $t > 0$, whereas Figure 3.2 shows the upper estimate

$$\frac{1}{\lambda^{(n)}(B^{\psi_1}(0, \phi_1^{-1}(t)))} \wedge \frac{t\, J_1(x)}{c_1}, \qquad c_1 > 0 .$$

The smallest constant $C_1 \geqslant 1$ for which (3.21) holds pointwise for all x and $t > 0$ can be calculated as $C_1 = \frac{4}{3}$. For small values of t on the left and a cross-section for larger t on the right-hand side, Figures 3.3 and 3.4 reveal the quality of the estimate (3.21). We can see that for both large t and x the estimate approximates the actual density very well. The same holds for very small values of

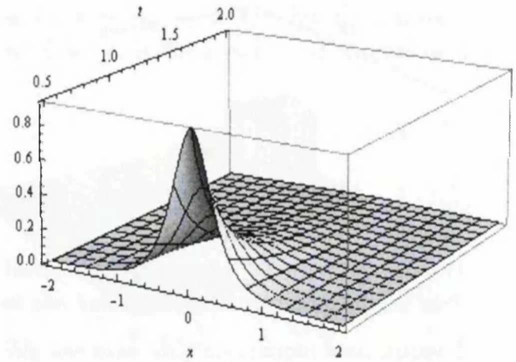


Figure 3.5: The transition function of the relativistic symmetric stable process for $0.5 \leqslant t \leqslant 2$.

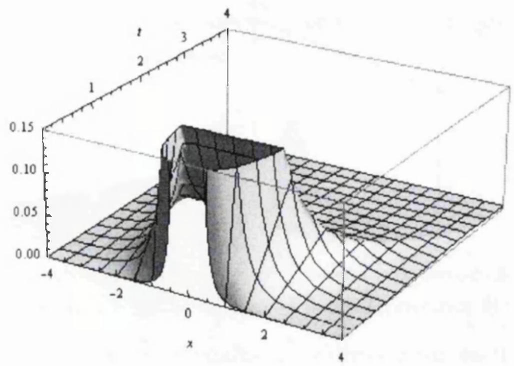


Figure 3.6: The upper bound for this density as proposed by Chen and Kumagai [13] for $t > 0$.

t and for x near the origin. However for moderate values of t the round shape of the curve is not modelled properly by the upper bound.

3.5.2 The estimate for the relativistic symmetric stable process

The Lévy process which corresponds to the continuous negative definite function (3.17) is a relativistic symmetric stable process on $\mathbb{R}^n$. It has the density

$$\begin{aligned} p_{t,2}(x,y) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\, e^{mt}\, e^{-t(|\xi|^2+m^2)^{1/2}}\, d\xi \\ &= 2(2\pi)^{-\frac{n+1}{2}} m^{\frac{n+1}{2}} e^{mt}\, t\,(|x-y|^2+t^2)^{-\frac{n+1}{2}} K_{\frac{n+1}{2}}(m\sqrt{|x-y|^2+t^2})\,. \end{aligned}$$

Here, $K_\tau(z)$ is the modified Bessel function of second kind and order $\tau > \frac{1}{2}$. We set $y = 0$ again which corresponds to the starting point 0 of the process. Then, for $m = \frac{1}{4}$, Figure 3.5 shows $p_{t,2}(\cdot)$ for variable time t, viewed as a function of the spatial variable x. The symmetric kernel $J_2(x) := J_2(x,0)$ in the corresponding Dirichlet form is given by

$$J_2(x) = \frac{1}{2\,\Gamma(\frac{1}{2})}\, 2^{-\frac{n+1}{2}}\, \pi^{-\frac{n}{2}}\, m^{\frac{n+1}{2}}\, \frac{K_{\frac{n+1}{2}}(m\,|x|)}{|x|^{\frac{n+1}{2}}}\,.$$

Then we obtain the function $\phi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ as

$$\phi_2(\psi_2^{1/2}(x)) = \frac{c_2\,|x|^{\frac{n+1}{2}}}{\kappa_2\,|x|^n\, K_{\frac{n+1}{2}}(m\,|x|)}$$

with positive constants c_2, κ_2, where c_2 can be chosen such that $\phi_2(1) = 1$. Note that ϕ_2 is not defined in $x = 0$, but we have $\lim_{x\to 0+} \phi_2(\psi_2^{1/2}(x)) = 0$. Figure 3.6 shows the upper bound

$$\frac{1}{\lambda^{(n)}(B^{\psi_2}(0,\phi_2^{-1}(t)))} \wedge \frac{t\,J_2(x)}{c_2}\,, \qquad c_2 > 0\,.$$

The constant $C_2 \geqslant 1$ such that the estimate

$$p_{t,2}(x) := p_{t,2}(x,0) \leqslant C_2 \left(\frac{1}{\lambda^{(n)}(B^{\psi_2}(0,\phi_2^{-1}(t)))} \wedge \frac{t\,J_2(x)}{c_2} \right)$$

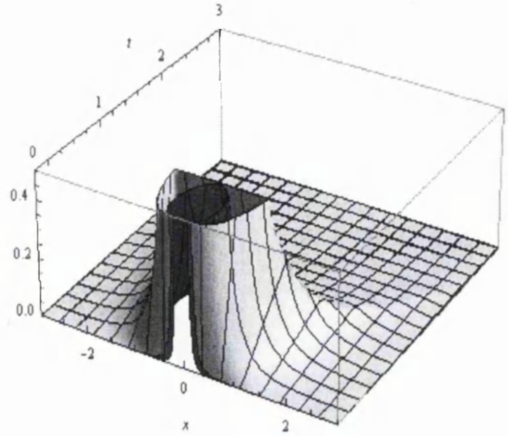


Figure 3.7: The upper estimate of the transition function of the relativistic symmetric stable process for $0 < t \leqslant 1$.

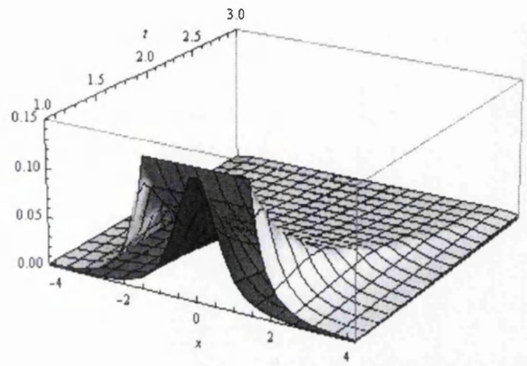


Figure 3.8: Cross-section of the estimate at time $t = 1$.

holds pointwise for all x and $t > 0$ can be determined to be $C_2 = 1.03203$. Figures 3.7 and 3.8 show the estimate for small values of t and a cross-section at time $t = 1$, respectively. Again, we can see that the estimate approximates the curve very well for large t and x, but it does deviate essentially from the actual density for moderate values of t near the origin. Hence the characteristic round shape of $p_{t,2}(\cdot)$ is not modelled by the upper bound.

All in all, we see that incorporating geometric devices such as the volume of balls with respect to certain metrics are indeed appropriate to get estimates. However, it seems essential to gain some more understanding of the geometry of the underlying metric measure space to be able to obtain somewhat refined estimates modelling the actual shape of the transition function. Therefore, in § 4 we will study spaces equipped with $\psi^{1/2}$ as a metric and discuss some of their geometrical aspects such as the convexity as well as volume doubling properties of balls $B^{\psi}(0, \rho)$, $\rho > 0$, in the setting $(\mathbb{R}^n, \psi^{1/2}, \lambda^{(n)})$. The overall aim is to provide a basis on the way to understand the behaviour of Lévy and Lévy-type processes in geometric terms. The key to this goal seems to be Schoenberg's result, Proposition 3.24. In view of this, it is an interesting question to what extent we can draw upon the theory of (possibly infinite dimensional) Hilbert spaces to gain a better understanding of Lévy processes on a metric measure space.

§4 Metric measure spaces $(\mathbb{R}^n, \psi^{1/2}, \mu)$

In this chapter we will study properties of metric measure spaces $(\mathbb{R}^n, \psi^{1/2}, \mu)$, where $(\mathbb{R}^n, \psi^{1/2})$ is a metric space, the function ψ is continuous negative definite, and μ a locally finite regular Borel measure. This work should provide the basis for studying transition functions of certain Lévy and Lévy-type processes by focussing on geometric aspects occuring in the associated metric measure spaces. As we have seen, § 3.4 and § 3.5 serve as a motivation to study these metric measure spaces, where it turns out that it is exactly a continuous negative definite function that gives the relevant metric to study the transition functions of a Lévy process in geometric terms. The following paragraphs now focus on some geometric aspects such as the volume doubling property, which turns out to be a helpful tool in the study and computation of heat kernel estimates, the convexity of metric balls and relations between metric measure spaces when switching to more complicated continuous negative definite functions as metrics or when moving from the Lebesgue measure to more general regular Borel measures. We finish this chapter by noting that it is possible to adopt the construction of a Sobolev space introduced by Hajłasz, see § 2.3.4, to our setting.

4.1 Some properties of the metric measure spaces $(\mathbb{R}^n, \psi^{1/2}, \mu)$

In this paragraph we state the basic setting first before we focus on properties such as the completeness of the metric space, the volume doubling of balls in $(\mathbb{R}^n, \psi^{1/2}, \lambda^{(n)})$, the study of the shape of certain metric balls in $(\mathbb{R}^n, \psi^{1/2})$ and on relations between doubling metric measure spaces $(\mathbb{R}^n, \psi^{1/2}, \mu)$ when varying the metric or the measure μ, which we assume to be a locally finite regular Borel measure.

Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a real-valued continuous negative definite function which vanishes only at the origin. By this assumption, together with the symmetry of ψ and the triangle inequality for $\psi^{1/2}$, see Lemma 1.20, we find that $d_0 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $d_0(x, y) := \psi^{1/2}(x - y)$ satisfies the conditions of a metric and hence $(\mathbb{R}^n, \psi^{1/2})$ is a metric space. The metric is translation invariant as for $z \in \mathbb{R}^n$ we get $d_0(x + z, y + z) = \psi^{1/2}((x + z) - (y + z)) = d_0(x, y)$, but not homogeneous in general.

Definition 4.1. For radii $\rho > 0$ we will denote balls in this metric by $B^{\psi}(y, \rho)$ when centred at a point $y \in \mathbb{R}^n$, i.e.

$$B^{\psi}(y, \rho) := \{x \in \mathbb{R}^n : \psi^{1/2}(x - y) < \rho\} = \{x \in \mathbb{R}^n : d_0(x, y) < \rho\}$$

for $\rho > 0$, $y \in \mathbb{R}^n$.

From the translation invariance of $\psi^{1/2}$ it follows

$$\begin{aligned} B^{\psi}(y,\rho) &= \{x \in \mathbb{R}^n : \psi^{1/2}(x-y) < \rho\} = \{x \in \mathbb{R}^n : d_0(x-y,0) < \rho\} \\ &= \{x+y \in \mathbb{R}^n : d_0(x,0) < \rho\} = y + \{x \in \mathbb{R}^n : d_0(x,0) < \rho\} \\ &= y + B^{\psi}(0,\rho). \end{aligned}$$

We consider the measurable space $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n, d_0))$ consisting of the state space $\mathbb{R}^n$ and the Borel-σ-algebra $\mathcal{B}(\mathbb{R}^n, d_0)$ on which we may define a locally finite regular Borel measure $\mu_0 : \mathcal{B}(\mathbb{R}^n, d_0) \to [0, \infty]$. We will mainly work with the n-dimensional Lebesgue measure $\lambda^{(n)}$ which has the additional property of being translation invariant. This yields

$$\lambda^{(n)}(B^{\psi}(y,\rho)) = \lambda^{(n)}(y + B^{\psi}(0,\rho)) = \lambda^{(n)}(B^{\psi}(0,\rho)),$$

hence the Lebesgue volume of the metric ball remains the same no matter where we locate the origin of the coordinate system. Therefore, from now on we shall always consider balls with respect to the metric $\psi^{1/2}$ to be centred at the origin in $\mathbb{R}^n$ unless stated otherwise.

4.1.1 Completeness of the metric space $(\mathbb{R}^n, \psi^{1/2})$

Before we study the fundamental property of completeness of the metric space $(\mathbb{R}^n, \psi^{1/2})$ we remind of the concepts of convergence and Cauchy sequences and of some fundamental properties and criteria for convergence in a general metric space (X, d_0).

Definition 4.2. A. We call a sequence $(x_k)_{k\in\mathbb{N}}$ of vectors in $\mathbb{R}^n$ *convergent to a limit* $x \in \mathbb{R}^n$, i.e. $\lim_{k\to\infty} x_k = x$ or $x_k \to x$ as $k \to \infty$, if for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $k \geqslant N$ we get $d_0(x_k, x) < \varepsilon$.

B. A sequence $(x_k)_{k\in\mathbb{N}}$ is called a *Cauchy sequence* if for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $d_0(x_k, x_m) < \varepsilon$ for all $k, m \geqslant N$.

The following theorem is valid in complete metric spaces only. In particular, it holds in the Euclidean setting $(\mathbb{R}^n, |\cdot|)$.

Theorem 4.3 (Cauchy Convergence Criterion). *In the metric space $(\mathbb{R}^n, |\cdot|)$ a sequence $(x_k)_{k\in\mathbb{N}}$, $x_k \in \mathbb{R}$, is convergent if, and only if, it is a Cauchy sequence.*

Note that a convergent sequence in any metric space (X, d_0) is always bounded.

Theorem 4.4 (Bolzano-Weierstrass). *Every bounded sequence $(x_k)_{k\in\mathbb{N}}$ of infinitely many real numbers contains at least one convergent subsequence.*

The following proposition guarantees the completeness of the metric space $(\mathbb{R}^n, \psi^{1/2})$ under certain assumptions.

Proposition 4.5. *Assume that there exists a constant $M > 0$ such that for all $\xi \in \mathbb{R}^n$ with $|\xi| \leqslant M$ we have $\psi^{1/2}(\xi) \geqslant c\,|\xi|$. Then the metric space $(\mathbb{R}^n, \psi^{1/2})$ is complete.*

Proof. We have to verify that every Cauchy sequence in $(\mathbb{R}^n, \psi^{1/2})$ converges to a limit in the same space. Let $(\xi_k)_{k\in\mathbb{N}}$ be a Cauchy sequence in $(\mathbb{R}^n, \psi^{1/2})$. Then for all $\varepsilon_0 > 0$ there exists $N = N(\varepsilon_0)$ such that

$$\psi^{1/2}(\xi_k - \xi_m) < \varepsilon_0 \quad \text{for all } k, m \geqslant N\,.$$

Set $\varepsilon := \frac{\varepsilon_0}{c}$, then by assumption

$$|\xi_k - \xi_m| < \frac{\varepsilon_0}{c} = \varepsilon \quad \text{for all } k, m \geqslant N\,.$$

Therefore, $(\xi_k)_{k\in\mathbb{N}}$ is also a Cauchy sequence in $(\mathbb{R}^n, |\cdot|)$. By Theorem 4.3 $(\xi)_{k\in\mathbb{N}}$ is convergent in $(\mathbb{R}^n, |\cdot|)$ to a limit $\xi \in \mathbb{R}^n$. Since $\psi : \mathbb{R}^n \to \mathbb{R}$ is continuous it follows that $\lim_{k\to\infty} \psi^{1/2}(\xi_k - \xi) = 0$, hence $\xi_k \to \xi$ in $(\mathbb{R}^n, \psi^{1/2})$. □

4.1.2 The volume doubling property in $(\mathbb{R}^n, \psi^{1/2}, \lambda^{(n)})$

In this paragraph we study the property of the metric measure spaces $(\mathbb{R}^n, \psi^{1/2}, \lambda^{(n)})$ of open balls satisfying a certain growth condition. The growth constants are of use in the computations of the upper heat kernel estimate by Chen and Kumagai [13], see § 3.5 and Appendix B.

Proposition 4.6. *Let $(\mathbb{R}^n, \psi^{1/2}, \lambda^{(n)})$ be a metric measure space and let $0 < r < R < \infty$. Assume there exist functions $c_r : \mathbb{R}_+ \setminus \{0\} \to \mathbb{R}_+ \setminus \{0\}$ and $c_R : \mathbb{R}_+ \setminus \{0\} \to \mathbb{R}_+ \setminus \{0\}$ only depending on r and R, respectively, such that $\psi^{1/2}(x) < R \Rightarrow \psi^{1/2}(\frac{x}{c_R}) < 1$ and $\psi^{1/2}(\frac{x}{c_r}) < 1 \Rightarrow \psi^{1/2}(x) < r$. Then we obtain the volume growth estimate*

$$\lambda^{(n)}(B^\psi(0,R)) \leqslant c(r,R;n)\,\lambda^{(n)}(B^\psi(0,r)) \tag{4.1}$$

in the metric measure space $(\mathbb{R}^n, \psi^{1/2}, \lambda^{(n)})$.

Proof. Fix two radii $0 < r < R < \infty$ of balls B^ψ around zero with respect to $\psi^{1/2}$ and assume the mapping c_R is such that $\psi^{1/2}(x) < R$ implies $\psi^{1/2}(\frac{x}{c_R}) < 1$. As R fixed, we have

$$\begin{aligned} \chi_{B^\psi(0,1)}\Big(\frac{x}{c_R}\Big) &= \begin{cases} 1 & \text{if } \frac{x}{c_R} \in B^\psi(0,1) \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & \text{if } \psi^{1/2}\Big(\frac{x}{c_R}\Big) < 1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Therefore

$$\lambda^{(n)}(B^\psi(0,R)) = \int_{\mathbb{R}^n} \chi_{B^\psi_R}(x)\,dx \leqslant \int_{\mathbb{R}^n} \chi_{B^\psi_1}\Big(\frac{x}{c_R}\Big)\,dx$$

and by the change of variable $\xi := \frac{x}{c_R}$, $d\xi = c_R^n\,dx$,

$$\lambda^{(n)}(B^\psi(0,R)) \leqslant c_R^n\,\lambda^{(n)}(B^\psi(0,1))\,.$$

Similarly, by assumption we can find a mapping $c_r : \mathbb{R}_+ \setminus \{0\} \to \mathbb{R}_+ \setminus \{0\}$ such that $\psi^{1/2}(\frac{x}{c_r}) < 1$ implies $\psi^{1/2}(x) < r$. Then again

$$\begin{aligned} \chi_{B^\psi(0,1)}\Big(\frac{x}{c_r}\Big) &= \begin{cases} 1 & \text{if } \frac{x}{c_r} \in B^\psi(0,1) \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & \text{if } \psi^{1/2}\Big(\frac{x}{c_r}\Big) < 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

$f(s) = s^\gamma$,	$0<\gamma<1$	$f(s) = \log\left(\frac{\gamma}{\delta}\frac{s+\delta}{s+\gamma}\right)$,	$\gamma>\delta>0$
$f(s) = \log(1+s)$,		$f(s) = \sqrt{s+m^2} - m$,	$m \geqslant 0$
$f(s) = 1 - e^{-\gamma s}$,	$\gamma > 0$	$f(s) = \frac{s}{s+\gamma}$,	$\gamma>0$
$f(s) = \log\left(\frac{s+\gamma}{s}\right)$,	$\gamma>0$	$f(s) = \frac{1}{\gamma} - \frac{1}{\gamma+s}$,	$\gamma>0$
$f(s) = \sqrt{s}\log(1+\sqrt{s})$,		$f(s) = \sqrt{s}(1-e^{-4\sqrt{s}})$.	

Table 4.1. List of (complete) Bernstein functions used for doubling constant computations.

which gives

$$\lambda^{(n)}(B^\psi(0,r)) = \int_{\mathbb{R}^n} \chi_{B_r^\psi}(x)\,dx \geqslant \int_{\mathbb{R}^n} \chi_{B_1^\psi}\left(\frac{x}{c_r}\right) dx = c_r^n\, \lambda^{(n)}(B^\psi(0,1)).$$

Hence, (4.1) follows with $c(r,R;n) := \left(\frac{c_R}{c_r}\right)^n$. □

In Appendix A we provide computations of the volume growth constant $c(r,R;n)$ in spaces where the metric is given by the composition $(f\circ\psi)^{1/2}$ of a continuous negative definite function with a (complete) Bernstein function $f:\mathbb{R}_+\to\mathbb{R}_+$ from Table 4.1. Moreover, not only $s\mapsto f(s)$, but also $s\mapsto g(s) := f^\alpha(s)$, $s\mapsto h(s) := f(s^\alpha)$ and $s\mapsto k(s) := f^{1/\alpha}(s^\alpha)$ for $0<|\alpha|<1$ are (complete) Bernstein functions and hence $g\circ\psi$, $h\circ\psi$ and $k\circ\psi$ are again continuous and negative definite. We show in Appendix A that (4.1) holds for metric measure spaces $(\mathbb{R}^n, (f\circ\psi)^{1/2}, \lambda^{(n)})$, $(\mathbb{R}^n, (h\circ\psi)^{1/2}, \lambda^{(n)})$ and $(\mathbb{R}^n, (k\circ\psi)^{1/2}, \lambda^{(n)})$ using the continuous negative definite functions $\psi(\xi) = |\xi|^2$ and $\psi(\xi,\eta) = |\xi|^\alpha + |\eta|^\beta$, for $(\xi,\eta)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$, $n = n_1+n_2$ and $0<\alpha,\beta<2$.

Examples 4.7. A. The study of the balls $B^{f\circ\psi}(0,\rho)$ of some radius ρ with respect to the metric $(f\circ\psi)^{1/2}(\xi) = |\xi|^\alpha$, $0<\alpha<1$, in the space $(\mathbb{R}^n, (f\circ\psi)^{1/2}, \lambda^{(n)})$ yields a doubling constant

$$\lambda^{(n)}(B^{f\circ\psi}(0,R)) = \left(\frac{R}{r}\right)^{n/\alpha} \lambda^{(n)}(B^{f\circ\psi}(0,r))$$

for any choice of radii $0<r<R<\infty$, see Example A.1.

B. For balls in the metric space obtained using the continuous negative definite function

$$f(|\xi|^2) = \sqrt{|\xi|^2+m^2} - m, \quad m>0,$$

we get the volume doubling constant

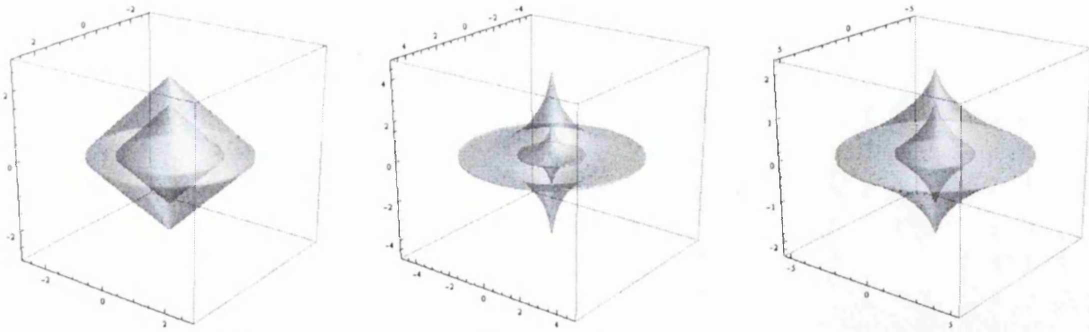
$$\lambda^{(n)}(B^{f\circ\psi}(0,R)) = \left(\frac{R^4+2mR^2}{r^4+2mr^2}\right)^{n/2} \lambda^{(n)}(B^{f\circ\psi}(0,r)),$$

for any choice of finite radii, such that $0<r<R<\infty$, see Example A.23.

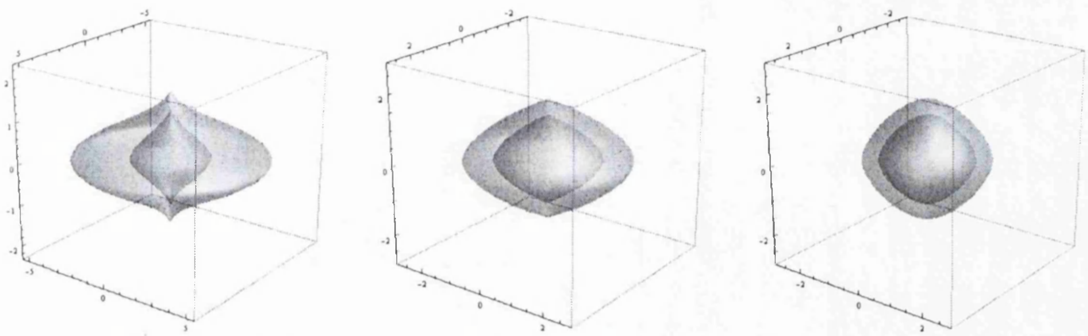
The functions $\xi\mapsto|\xi|^\alpha$, $0<\alpha<1$, and $\xi\mapsto\sqrt{|\xi|^2+m^2}-m$, $m>0$, are the symbols of the symmetric α-stable process and of the relativistic symmetric stable process, respectively, which play a role in § 3.5 together with the corresponding doubling constants computed in Examples 4.7.

Examples 4.8 (Local doubling). **A.** If we equip the space $(\mathbb{R}^n, (f\circ\psi)^{1/2}, \lambda^{(n)})$ with the metric emerging from $f(|\xi|^2) = 1-\exp(-\gamma|\xi|^2)$, $\gamma>0$, we find

$$\lambda^{(n)}(B^{f\circ\psi}(0,R)) = \left(\frac{\log(1-R^2)}{\log(1-r^2)}\right)^{n/2} \lambda^{(n)}(B^{f\circ\psi}(0,r)), \tag{4.2}$$



$\alpha = \beta = 1$, $c(r, R) = 3.815$ $\alpha = \beta = 0.5$, $c(r, R) = 14.552$ $\alpha = 0.5$, $\beta = 1$, $c(r, R) = 9.313$



$\alpha = 0.5$, $\beta = 1.5$, $c(r, R) = 8.026$ $\alpha = 1$, $\beta = 1.5$, $c(r, R) = 3.287$ $\alpha = \beta = 1.5$, $c(r, R) = 2.441$

Figure 4.1. Metric balls with respect to $\psi^{1/2}(\xi, \eta) = (|\xi|^\alpha + |\eta|^\beta)^{1/2}$ of radii $r = 1.2$ and $R = 1.5$ and various values for α and β. The exact volume growth constant $c(r, R) := c(r, R; 3)$ is computed numerically in each case.

see Example A.8. This gives an example of a *local volume doubling* property as we are restricted to choose r and R such that $0 < r < R < 1$. As this example shows we get local volume doubling if the inverse function f^{-1} of the Bernstein function under consideration has a restricted domain of definition. Due to the fact that

$$\begin{aligned} B^{f\circ\psi}(0, \rho) &= \{f \in \mathbb{R}^n : f(|\xi|^2) < \rho^2\} = \{f \in \mathbb{R}^n : |\xi|^2 < f^{-1}(\rho^2)\} \\ &= \left\{\xi \in \mathbb{R}^n : \left|\frac{\xi}{(f^{-1}(\rho^2))^{1/2}}\right|^2 < 1\right\}, \end{aligned}$$

whenever $\rho^2 \in \mathsf{R}(f)$, we can express the identity (4.2) as

$$\lambda^{(n)}(B^{f\circ\psi}(0, R)) = \left(\frac{f^{-1}(R^2)}{f^{-1}(r^2)}\right)^{n/2} \lambda^{(n)}(B^{f\circ\psi}(0, r))$$

with the inverse function $f^{-1}(s) = -\gamma^{-1}\log(1-s)$, which is only defined for radii r and R such that $r^2, R^2 \in \mathsf{R}(f)$.

B. A similar result holds for the Bernstein function $f(s) = \log\left(\frac{\gamma}{\delta}\frac{s+\delta}{s+\gamma}\right)$ for $\gamma > \delta > 0$, where we get the restriction $0 < r < R < \log\frac{\gamma}{\delta}$ on the radii, as well as for the Bernstein function $f(s) = \frac{s}{s+\gamma}$ for $\gamma > 0$, for which we have the restriction $0 < r < R < 1$ for the volume growth property to make sense, see Examples A.20 and A.29 in Appendix A, respectively.

Remark 4.9. From Examples 4.7 and 4.8 we observe that the mappings c_R and c_r introduced in Proposition 4.6 can in fact be determined explicitly as $f^{-1}(R^2)^{1/2}$ and $f^{-1}(R^2)^{1/2}$, respectively, whenever we consider metrics arising from $\psi(\xi) = f(|\xi|^2)$ with a (complete) Bernstein function f. However, in some cases f has an inverse that cannot be given in a closed form, see e.g. Examples A.16 and A.37. There, we make use of explicitly invertible Bernstein functions g and h which fulfil

$$\begin{cases} 0 < g(s) \leqslant f(s) \leqslant h(s) & \text{for } 0 < s \leqslant s_0 \,, \\ g(0) = f(0) = h(0) = 0 \,. \end{cases} \tag{4.3}$$

This implies that the volume doubling holds only locally, i.e. for radii $0 < r < R < \sqrt{s_0}$.

In the remainder of this paragraph, we focus again on the metric balls with respect to $\psi^{1/2}(\xi,\eta) = (|\xi|^\alpha + |\eta|^\beta)^{1/2}$, where $\psi : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, $n_1 + n_2 = n$, and visualise balls for different choices of $\alpha, \beta \in (0,2]$. The volume growth property in $(\mathbb{R}^n, \psi^{1/2}, \lambda^{(n)})$ reads as

$$\lambda^{(n)}(B^\psi(0,R)) = \left(\frac{R}{r}\right)^{2\left(\frac{n_1}{\alpha}+\frac{n_2}{\beta}\right)} \lambda^{(n)}(B^\psi(0,r)) \,, \tag{4.4}$$

for $0 < r < R < \infty$, see Example A.2 in Appendix A for the detailed calculation. In Figure 4.1 the effect of the volume doubling is shown for $r = 1.2$ and $R = 1.5$. Studying the shape of the balls the question emerges which metrics $(f \circ \psi)^{1/2}$ lead to convex balls. For $\psi^{1/2}(\xi,\eta) = (|\xi|^\alpha + |\eta|^\beta)^{1/2}$ this is obviously the case for $1 \leqslant \alpha, \beta \leqslant 2$. For general metrics this topic is addressed in the next paragraph.

4.1.3 On the convexity of metric balls in $(\mathbb{R}^n, \psi^{1/2})$

In this paragraph we study the aspect of convexity of balls in the metric spaces $(\mathbb{R}^n, \psi^{1/2})$. We first focus on conditions under which convexity is preserved when the metric is changed. The first result in this paragraph deals with metrics arising from the composition of a function ψ with a Bernstein function f. A direct consequence can be drawn from this for radially symmetric continuous negative definite functions. Furthermore, we investigate the convexity issue for balls in the metric $\psi^{1/2}(\xi,\eta) = (|\xi|^\alpha + |\eta|^\beta)^{1/2}$ for various values $\alpha, \beta \in (0,2]$.

Proposition 4.10. *Let $\psi^{1/2}$ be a metric such that $B^\psi(0,\rho)$ is a convex set for some $\rho > 0$. Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a Bernstein function, and $R > 0$ be such that $R^2 = f(\rho^2)$. Then $B^{f\circ\psi}(0,R)$ is also convex. Conversely, if $(f \circ \psi)^{1/2}$ is a metric with $B^{f\circ\psi}(0,R)$ convex, and $\psi^{1/2}$ is a metric then $B^\psi(0,\rho)$ is convex, where $\rho = ((f^{-1}(R^2))^{1/2}$.*

Proof. Let $\psi^{1/2}$ be a metric such that $B^\psi(0,\rho)$ is convex, $\rho > 0$. Taking a Bernstein function $f : \mathbb{R}_+ \to \mathbb{R}$, which has the property of being increasing, concave with $f(0) = 0$, we know that f^{-1} exists and is itself an increasing function. Hence $(f \circ \psi)^{1/2}$ is again a metric and the balls with respect to this read as

$$B^{f\circ\psi}(0,R) = \{\xi \in \mathbb{R}^n : (f\circ\psi)^{1/2}(\xi) < R\} = \{\xi \in \mathbb{R}^n : \psi^{1/2}(\xi) < (f^{-1}(R^2))^{1/2}\} = B^\psi(0,\rho) \,, \tag{4.5}$$

where $\rho = ((f^{-1}(R^2))^{1/2}$. Thus, if $B^\psi(0,\rho)$ is convex and ρ and R are related via $R^2 = f(\rho^2)$ then (4.5) holds and $B^{f\circ\psi}(0,R)$ is also convex. Due to the equality (4.5) of the sets the converse also holds true. More precisely, if $B^{f\circ\psi}(0,R)$ is convex with respect to $(f \circ \psi)^{1/2}$, where f is Bernstein and $\psi^{1/2}$ a metric itself then $B^\psi(0,\rho)$ is convex with $\rho = ((f^{-1}(R^2))^{1/2}$. □

An immediate consequence of Proposition 4.10 for $\psi(\xi) = |\xi|^2$ is the following result.

Corollary 4.11. *The balls with respect to the metric arising from the radially symmetric function $(f \circ \psi)(\xi) = f(|\xi|^2)$ with a Bernstein function f are convex.*

Proof. This follows immediately from Proposition 4.10, since balls $B^{|\cdot|}(0,\rho)$, $\rho > 0$, with respect to the Euclidean metric are convex. □

Focussing again on the balls with respect to the metric arising from the function $\psi : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, $\psi(\xi,\eta) = |\xi|^\alpha + |\eta|^\beta$, $n_1 + n_2 = n$, we observe that they are convex for $1 \leqslant \alpha, \beta \leqslant 2$. For α or β taking values in the interval $(0,1)$ we have the following result.

Proposition 4.12. *Let $B^\psi(0,\rho)$ be a ball of positive radius ρ with respect to $\psi : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, $\psi^{1/2}(\xi,\eta) = (|\xi|^\alpha + |\eta|^\beta)^{1/2}$.*
A. *If $0 < \alpha < 1$ or $0 < \beta < 1$ then*

$$\operatorname{conv}(B^\psi(0,\rho)) \subset B^\psi(0, C_{\alpha,\beta}\,\rho), \tag{4.6}$$

where $\operatorname{conv}(B)$ denotes the convex hull of a set B and $C_{\alpha,\beta} = 2^{\frac{1-\min\{\alpha,\beta\}}{2}}$.
B. *If $1 \leqslant \alpha, \beta \leqslant 2$ then $B^\psi(0,\rho)$ is itself a convex set.*

Proof. **A.** We have to show that for two points $(x,y), (\xi,\eta) \in B^\psi(0,\rho) \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ the straight line joining them in $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ lies in $B^\psi(0, C_{\alpha,\beta}\,\rho)$. Hence, we assume $\psi^{1/2}(x,y) < \rho$ and $\psi^{1/2}(\xi,\eta) < \rho$ and show that

$$\psi^{1/2}(tx + (1-t)\xi,\ ty + (1-t)\eta) < C_{\alpha,\beta}\,\rho \quad \text{for } 0 < t < 1\,.$$

First, let $0 < \alpha, \beta < 1$. Then by the triangle inequality for $|\cdot|^\alpha$ and $|\cdot|^\beta$,

$$\begin{aligned} \psi^{1/2}(t\,(x,y) + (1-t)\,(\xi,\eta)) &= \left(|tx + (1-t)\xi|^\alpha + |ty + (1-t)\eta|^\beta\right)^{1/2} \\ &\leqslant \left(t^\alpha |x|^\alpha + t^\beta |y|^\beta + (1-t)^\alpha |\xi|^\alpha + (1-t)^\beta |\eta|^\beta\right)^{1/2}. \end{aligned}$$

Since $t < 1$ it follows

$$\begin{aligned} \psi^{1/2}(t\,(x,y) + (1-t)\,(\xi,\eta)) &\leqslant \left(t^{\min\{\alpha,\beta\}}(|x|^\alpha + |y|^\beta) + (1-t)^{\min\{\alpha,\beta\}}(|\xi|^\alpha + |\eta|^\beta)\right)^{1/2} \\ &< \left(t^{\min\{\alpha,\beta\}} + (1-t)^{\min\{\alpha,\beta\}}\right)^{1/2} \rho\,. \end{aligned}$$

For $p := \min\{\alpha,\beta\} < 1$ we use that $a^p + b^p < 2^{1-p}(a+b)^p$ for $a, b > 0$, which implies

$$\begin{aligned} \psi^{1/2}(t\,(x,y) + (1-t)\,(\xi,\eta)) &< \left(t^{\min\{\alpha,\beta\}} + (1-t)^{\min\{\alpha,\beta\}}\right)^{1/2} \rho \\ &< \left(2^{1-\min\{\alpha,\beta\}}(t + (1-t))^{\min\{\alpha,\beta\}}\right)^{1/2} \rho \;=\; C_{\alpha,\beta}\,\rho\,. \end{aligned}$$

Now assume without loss of generality that $0 < \alpha < 1$ and $1 \leqslant \beta < 2$. Otherwise we may interchange the roles of α and β. Then

$$\begin{aligned} \psi^{1/2}(t\,(x,y) + (1-t)\,(\xi,\eta)) &= \left(|tx + (1-t)\xi|^\alpha + |ty + (1-t)\eta|^\beta\right)^{1/2} \\ &\leqslant \left(t^\alpha |x|^\alpha + (1-t)^\alpha |\xi|^\alpha + 2^{1-\beta}\, t^\beta |y|^\beta + 2^{1-\beta}\,(1-t)^\beta |\eta|^\beta\right)^{1/2} \\ &\leqslant 2^{\frac{1-\beta}{2}}\left(t^\alpha |x|^\alpha + (1-t)^\alpha |\xi|^\alpha + t^\beta |y|^\beta + (1-t)^\beta |\eta|^\beta\right)^{1/2} \\ &\leqslant 2^{\frac{1-\beta}{2}}\left(t^{\min\{\alpha,\beta\}}(|x|^\alpha + |y|^\beta) + (1-t)^{\min\{\alpha,\beta\}}(|\xi|^\alpha + |\eta|^\beta)\right)^{1/2} \\ &< 2^{\frac{1-\beta}{2}}\, 2^{\frac{1-\min\{\alpha,\beta\}}{2}}\,\rho \;\leqslant\; C_{\alpha,\beta}\,\rho \end{aligned}$$

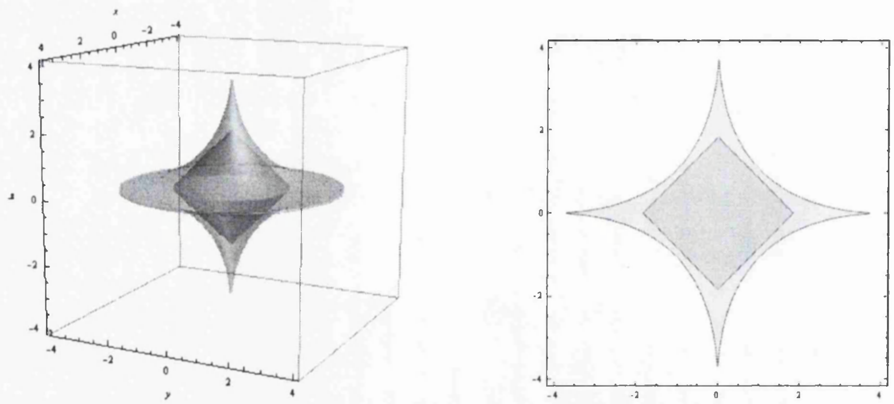


Figure 4.2. The convex hull of the ball of radius $r = 1.2$ with respect to $\psi^{1/2}$ and $\alpha = \beta = 0.5$ lying within the ball of radius $R = 2^{\frac{1-\min\{\alpha,\beta\}}{2}} r$. On the right-hand side a two-dimensional cross-section.

by a similar argument as above and since $2^{\frac{1-\beta}{2}} \leqslant 1$.

B. Let $1 \leqslant \alpha, \beta \leqslant 2$. Then we have

$$\begin{aligned} &\psi^{1/2}(t\,(x,y) + (1-t)\,(\xi,\eta)) \\ &\leqslant \left(2^{1-\alpha} t^{\alpha}|x|^{\alpha} + 2^{1-\alpha}(1-t)^{\alpha}|\xi|^{\alpha} + 2^{1-\beta} t^{\beta}|y|^{\beta} + 2^{1-\beta}(1-t)^{\beta}|\eta|^{\beta}\right)^{1/2}. \end{aligned}$$

Since $2^{1-\alpha} \leqslant 1$ and $2^{1-\beta} \leqslant 1$ for $1 \leqslant \alpha, \beta \leqslant 2$,

$$\begin{aligned} \psi^{1/2}(t\,(x,y) + (1-t)\,(\xi,\eta)) &\leqslant \left(t^{\alpha}|x|^{\alpha} + t^{\beta}|y|^{\beta} + (1-t)^{\alpha}|\xi|^{\alpha} + (1-t)^{\beta}|\eta|^{\beta}\right)^{1/2} \\ &< 2^{\frac{1-\min\{\alpha,\beta\}}{2}} \rho \leqslant \rho \end{aligned}$$

analogous to part A and since $C_{\alpha,\beta} \leqslant 1$ for $1 \leqslant \alpha, \beta \leqslant 2$. It follows that $\operatorname{conv}(B^{\psi}(0,\rho)) = B^{\psi}(0,\rho)$, i.e. $B^{\psi}(0,\rho)$ is convex. □

Figure 4.2 illustrates the result of Proposition 4.12.A in the case $\alpha = \beta = \frac{1}{2}$. Obviously, $C_{\alpha,\beta} = 2^{\frac{1-\min\{\alpha,\beta\}}{2}}$ is not the optimal constant satisfying (4.6) for every value of α and β. In general, one can observe that $C_{\alpha,\beta}$ approaches optimality the closer α and β are to 1.

On the basis of the results in Proposition 4.12 we can now give explicit volume growth constants for the metric measure spaces $(\mathbb{R}^n, \psi^{1/2}, \lambda^{(n)})$ with $\psi^{1/2}(\xi,\eta) = (|\xi|^{\alpha} + |\eta|^{\beta})^{1/2}$, which ensure that the equality (4.4) is satisfied for $0 < r < R < \infty$ and imply that the convex hull $\operatorname{conv}(B^{\psi}(0,r))$ of the smaller balls are contained in the larger balls $B^{\psi}(0,R)$.

Corollary 4.13. *Let $(\mathbb{R}^n, \psi^{1/2}, \lambda^{(n)})$ be the metric measure space with $\psi^{1/2}(\xi,\eta) = (|\xi|^{\alpha} + |\eta|^{\beta})^{1/2}$, $\xi \in \mathbb{R}^{n_1}$, $\eta \in \mathbb{R}^{n_2}$, and $n_1 + n_2 = n$. Let $0 < \alpha < 1$ or $0 < \beta < 1$. If we choose r, R with $0 < r < R < \infty$ such that $\frac{R}{r} = C_{\alpha,\beta} = 2^{\frac{1-\min\{\alpha,\beta\}}{2}}$ in (4.4) then the convex hull of the ball $B^{\psi}(0,r)$ is contained in $B^{\psi}(0,R)$.*

4.2 Relations between metric measure spaces

In this paragraph we examine relations between metric measure spaces which satisfy the volume doubling property. More precisely, we assume that $(\mathbb{R}^n, d_0, \mu_0)$ is a volume doubling space with a metric $d_0 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. From now on, we assume $\mathcal{B}^{(n)} = \mathcal{B}(\mathbb{R}^n, d_0)$, i.e. that the σ-algebra of Borel sets with respect to d_0 is the same as $\mathcal{B}^{(n)}$, so d_0 and $|\cdot|$ induce the same topology on $\mathbb{R}^n$. The metric space is equipped with a locally finite regular Borel measure μ_0 on $\mathcal{B}(\mathbb{R}^n, d_0)$. We want to investigate, on the one hand, the relation of a locally finite regular Borel measure μ to μ_0 such that the doubling property also holds in $(\mathbb{R}^n, d_0, \mu)$. On the other hand, we study how a metric $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is related to d_0 for $(\mathbb{R}^n, d, \mu_0)$ to preserve volume doubling.

Let us start with fixing a volume doubling metric measure space $(\mathbb{R}^n, d_0, \mu_0)$ and focus on the question how we can associate a metric $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ with d_0 such that $(\mathbb{R}^n, d, \mu_0)$ also satisfies the volume growth condition. It turns out that the following two concepts play an essential role.

Definition 4.14 (Homeomorphism). Let X, Y be two topological spaces. A function $g : X \to Y$ is called a *homeomorphism* if it is bijective, continuous and the inverse g^{-1} is continuous. The function g is said to be an *open mapping* in this case, i.e. it maps open sets of X to open sets of Y. If such a g exists X and Y are called *homeomorphic* or *topologically equivalent*.

Definition 4.15 (Equivalence of metrics). Let d_1 and d_2 be two metrics on the same set X. We call them *topologically equivalent* if they induce the same topology on X, i.e. a subset $U \subset X$ is open with respect to d_1 if, and only if, it is open with respect to d_2.

Remark 4.16. If there exist constants $m, M > 0$ such that

$$M\, d_1(x, y) \leqslant d_2(x, y) \leqslant m\, d_1(x, y) \tag{4.7}$$

for all $x, y \in X$ then d_1 and d_2 are topological equivalent. Hence, (4.7) gives a sufficient condition for the equivalence of the two metrics. The converse only holds in normed spaces.

In this setting we can state the following result.

Proposition 4.17. *Let $(\mathbb{R}^n, d_0, \mu_0)$ be a volume doubling metric measure space with a locally finite regular Borel measure μ_0 on the measurable space $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n, d_0))$ that satisfies $\mu_0 \ll \lambda^{(n)}$. Assume that the density $g = \frac{d\mu_0}{d\lambda^{(n)}}$ is locally bounded and bounded away from zero. Let d be another metric on $\mathbb{R}^n$ which satisfies*

$$M\, d_0(x, y) \leqslant d(x, y) \leqslant m\, d_0(x, y) \tag{4.8}$$

for some $m, M > 0$ and all $x, y \in \mathbb{R}^n$, and $c_{d_0}(m, r), c_{d_0}(M, R) : \mathbb{R}_+ \setminus \{0\} \to \mathbb{R}_+ \setminus \{0\}$ such that

$$d(x, 0) < R \Rightarrow d_0\Big(\frac{x}{c_{d_0}(M, R)}, 0\Big) < 1 \quad \text{and} \quad d_0\Big(\frac{x}{c_{d_0}(m, r)}, 0\Big) < 1 \Rightarrow d(x, 0) < r\,.$$

Then $(\mathbb{R}^n, d, \mu_0)$ is a metric measure space that also satisfies the volume growth condition.

Proof. By assumption, $(\mathbb{R}^n, d)$ is a metric space which is topologically equivalent to $(\mathbb{R}^n, d_0)$ due to d and d_0 satisfying (4.8) for some $m, M > 0$ and $x, y \in \mathbb{R}^n$. Further, there exist $R > 0$ and a mapping $R \mapsto c_{d_0}(M, R)$ such that

$$\begin{aligned} B^d(0, R) &= \{x \in \mathbb{R}^n : d(x, 0) < R\} \subset \{x \in \mathbb{R}^n : M\, d_0(x, 0) < R\} \\ &\subset \Big\{x \in \mathbb{R}^n : d_0\Big(\frac{x}{c_{d_0}(M, R)}, 0\Big) < 1\Big\}. \end{aligned}$$

Note that for M and R fixed, we have

$$\chi_{B^{\psi}(0,1)}\Big(\frac{x}{c_{d_0}(M,R)}\Big) = \begin{cases} 1 & \text{if } \frac{x}{c_{d_0}(M,R)} \in B^{d_0}(0,1) \\ 0 & \text{if } \frac{x}{c_{d_0}(M,R)} \notin B^{d_0}(0,1) \end{cases} = \begin{cases} 1 & \text{if } d_0\Big(\frac{x}{c_{d_0}(M,R)}\Big) < 1 \\ 0 & \text{if } d_0\Big(\frac{x}{c_{d_0}(M,R)}\Big) \geqslant 1\,. \end{cases}$$

Since $\mu_0 \ll \lambda^{(n)}$ and $\lambda^{(n)}$ is a σ-finite measure, μ_0 has a density $g = \frac{d\mu_0}{d\lambda^{(n)}}$ with respect to the Lebesgue measure. As μ_0 is locally finite, we can compute

$$\begin{aligned}\mu_0(B^d(0,R)) &= \int_{\mathbb{R}^n} \chi_{B^d(0,R)}(x)\,\mu_0(dx) = \int_{\mathbb{R}^n} \chi_{B^d(0,R)}(x)\,g(x)\,dx \\ &\leqslant \int_{\mathbb{R}^n} \chi_{B^{d_0}(0,1)}\Big(\frac{x}{c_{d_0}(M,R)}\Big)\,g(x)\,dx\,.\end{aligned}$$

The density g is assumed to be locally bounded, therefore $g_R := \max\{g(x) : x \in B^{d_0}(0,1)\} < \infty$ for any $R > 0$. Then

$$\begin{aligned}\mu_0(B^d(0,R)) &\leqslant \int_{\mathbb{R}^n} \chi_{B^{d_0}(0,1)}\Big(\frac{x}{c_{d_0}(M,R)}\Big)\,g(x)\,dx \\ &\leqslant g_R \int_{\mathbb{R}^n} \chi_{B^{d_0}(0,1)}\Big(\frac{x}{c_{d_0}(M,R)}\Big)\,dx\end{aligned}$$

By the change of variable $\xi := \frac{x}{c_{d_0}(M,R)}$,

$$\begin{aligned}\mu_0(B^d(0,R)) &\leqslant g_R \int_{\mathbb{R}^n} \chi_{B^{d_0}(0,1)}\Big(\frac{x}{c_{d_0}(M,R)}\Big)\,dx \\ &= g_R\, c_{d_0}^n(M,R)\,\lambda^{(n)}(B^{d_0}(0,1))\,.\end{aligned}$$

On the other hand, by assumption there exists r with $0 < r < R$ and a mapping $r \mapsto c_{d_0}(m,r)$ such that

$$\begin{aligned}B^d(0,r) &= \{x \in \mathbb{R}^n : d(x,0) < r\} \supset \{x \in \mathbb{R}^n : m\,d_0(x,0) < r\} \\ &\supset \Big\{x \in \mathbb{R}^n : d_0\Big(\frac{x}{c_{d_0}(m,r)}, 0\Big) < 1\Big\}\,.\end{aligned}$$

Then, as before,

$$\mu_0(B^d(0,r)) = \int_{\mathbb{R}^n} \chi_{B^d(0,r)}(x)\,g(x)\,dx \geqslant \int_{\mathbb{R}^n} \chi_{B^{d_0}(0,1)}\Big(\frac{x}{c_{d_0}(m,r)}\Big)\,g(x)\,dx$$

and $g_r := \min\{g(x) : x \in B^{d_0}(0,1)\} > 0$ as g is assumed be bounded away from zero. It follows that

$$\begin{aligned}\mu_0(B^d(0,r)) &\geqslant g_r \int_{\mathbb{R}^n} \chi_{B^{d_0}(0,1)}\Big(\frac{x}{c_{d_0}(m,r)}\Big)\,dx \\ &= g_r\, c_{d_0}^n(m,r)\,\lambda^{(n)}(B^{d_0}(0,1))\end{aligned}$$

by a change of variable. Together, the above estimates give the volume growth

$$\mu_0(B^d(0,R)) \leqslant \frac{g_R}{g_r}\left(\frac{c_{d_0}(R)}{c_{d_0}(r)}\right)^n \mu_0(B^d(0,r))$$

and $(\mathbb{R}^n, d, \mu_0)$ is a volume doubling space, where the doubling constant depends on the metric d_0. □

Although it is not applicable to the metrics $\psi^{1/2}$ we are considering, this result has an obvious implication in cases where $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is induced by a norm on $\mathbb{R}^n$.

Corollary 4.18. *Let $(\mathbb{R}^n, |\cdot|, \mu_0)$ be a volume doubling metric measure space which satisfies the assumptions on Proposition 4.17. Let d be any metric on $\mathbb{R}^n$ induced by a norm on $\mathbb{R}^n$ and assume the existence of mappings $c_r, c_R : \mathbb{R}_+ \setminus \{0\} \to \mathbb{R}_+ \setminus \{0\}$ such that $d(x,0) < R \Rightarrow |\frac{x}{c_R}| < 1$ and $|\frac{x}{c_r}| < 1 \Rightarrow d(x,0) < r$. Then $(\mathbb{R}^n, d, \mu_0)$ is also a volume doubling metric measure space.*

Proof. Let d be any metric on $\mathbb{R}^n$ which is induced by a norm. As on $\mathbb{R}^n$ all norms are equivalent we can find $m, M > 0$ such that

$$M\,|x-y| \leqslant d(x,y) \leqslant m\,|x-y|$$

for all $x, y \in \mathbb{R}^n$. Then the result follows from Proposition 4.17. □

Our aim is to apply Proposition 4.17 to metrics arising from continuous negative definite functions $\psi : \mathbb{R}^n \to \mathbb{R}$. If we fix the standard Euclidean metric $d_0 = |\cdot|$ as reference metric and assume that μ_0 is such that $(\mathbb{R}^n, |\cdot|, \mu_0)$ satisfies the volume growth property, the question arises which metrics $d = \psi^{1/2}$ with a continuous negative definite ψ are actually topologically equivalent to d_0? An obvious result is the following.

Proposition 4.19. *Consider the metric space $(\mathbb{R}^n, |\cdot|)$ and a radially symmetric continuous negative definite function $\psi(\xi) = f(|\xi|^2)$. Then the balls $B^\psi(0,R)$ of radius $R > 0$ with respect to $\psi^{1/2}$ are homeomorphic to the unit ball $B^{|\cdot|}(0,1)$ with respect to the standard Euclidean metric.*

Proof. Note that we can write

$$B^\psi(0,R) = \{x \in \mathbb{R}^n : |x|^2 < f^{-1}(R^2)\} = \left\{x \in \mathbb{R}^n : \left|\frac{x}{f^{-1}(R^2)^{1/2}}\right| < 1\right\}$$

as every Bernstein function has a continuous inverse defined on $\mathbb{R}$. Hence, the homeomorphism $g : \mathbb{R}_+ \setminus \{0\} \to \mathbb{R}_+ \setminus \{0\}$ describing the deformation is given by $g(R) = \frac{1}{f^{-1}(R^2)^{1/2}}$. □

Let us now consider a continuous negative definite function $\psi : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, $n_1 + n_2 = n$, of the form $\psi(\xi,\eta) = \psi_1(\xi) + \psi_2(\eta)$. As before, we may ask under which conditions we obtain a metric $(\psi_1(\xi) + \psi_2(\eta))^{1/2}$ on $\mathbb{R}^n$ which is topologically equivalent to the standard Euclidean metric $d_0 = |\cdot|$ on $\mathbb{R}^n$.

Corollary 4.20. *Consider the metric space $(\mathbb{R}^n, |\cdot|)$ and a continuous negative definite function $\psi : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, $n_1 + n_2 = n$, $\psi(\xi,\eta) = \psi_1(\xi) + \psi_2(\eta)$. If we can find functions $c(\psi_1,\rho), c(\psi_2,\rho) : \mathbb{R}_+ \setminus \{0\} \to \mathbb{R}_+ \setminus \{0\}$ depending on the ψ_i, $i = 1,2$, such that for radii $0 < r, R < \infty$ we have*

$$B^\psi(0,R) \quad \subset \quad \left\{(\xi,\eta) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : \left|\frac{\xi}{c(\psi_1,R)}\right| + \left|\frac{\eta}{c(\psi_2,R)}\right| < 1\right\}$$

$$B^\psi(0,r) \quad \supset \quad \left\{(\xi,\eta) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : \left|\frac{\xi}{c(\psi_1,r)}\right| + \left|\frac{\eta}{c(\psi_2,r)}\right| < 1\right\}$$

and the mappings $\rho \mapsto \frac{1}{c(\psi_1,\rho)}$ and $\rho \mapsto \frac{1}{c(\psi_2,\rho)}$ are homeomorphisms, then the balls with respect to $\psi^{1/2}(\xi,\eta)$ are topologically equivalent to the unit ball in the standard Euclidean metric.

Corollary 4.20 is clearly valid for a continuous negative definite function $\psi(\xi,\eta) = \psi_1(\xi) + \psi_2(\eta) = f_1(|\xi|^2) + f_2(|\eta|^2)$, where f_1 and f_2 are complete Bernstein functions.

Example 4.21. Considering $\psi(\xi,\eta) = |\xi|^\alpha + |\eta|^\beta$ with $\xi \in \mathbb{R}^{n_1}$, $\eta \in \mathbb{R}^{n_2}$, $0 < \alpha, \beta \leqslant 2$, we know from Example A.2 that $c(\psi_1,\rho) = \rho^{2/\alpha}$ and $c(\psi_2,\rho) = \rho^{2/\beta}$, which fulfil the conditions of Corollary 4.20. Hence, the metric balls with respect to $\psi^{1/2}$ on $\mathbb{R}^n$ are topologically equivalent to the unit ball with respect to $|\cdot|$ on $\mathbb{R}^n$.

Assuming that $(\mathbb{R}^n, d_0, \mu_0)$ is a doubling metric measure space we now turn to the question how a measure μ has to be related to μ_0 in order to obtain a doubling space $(\mathbb{R}^n, d_0, \mu)$. As volume doubling is a property of the measure with which we equip the metric space, it is clear that changing the measure might result in a different behaviour in this respect. A rather obvious result in this setting is the following.

Proposition 4.22. *Let $(\mathbb{R}^n, d_0, \mu_0)$ be a volume doubling metric measure space with σ-finite measure μ_0 on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n, d_0))$. Let μ be a locally finite regular Borel measure on $\mathbb{R}^n$ with $\mu \ll \mu_0$. Assume that $g = \frac{d\mu}{d\mu_0}$ is locally bounded and bounded away from zero. Then $(\mathbb{R}^n, d_0, \mu)$ satisfies the volume doubling property.*

Proof. Since $(\mathbb{R}^n, d_0, \mu_0)$ satisfies the volume growth condition, there exists $C \geqslant 1$ such that for all $0 < r \leqslant R < \infty$

$$\mu_0(B^{d_0}(0,R)) \leqslant C\,\mu_0(B^{d_0}(0,r)).$$

Let μ be a locally finite Borel measure on $\mathbb{R}^n$ with $\mu \ll \mu_0$. By assumption μ_0 is σ-finite, hence $g = \frac{d\mu}{d\mu_0}$ exists. Moreover, we can find $R > 0$ such that g is locally bounded in $B^{d_0}(0,R)$, i.e. we find $g_R := \max\{g(x) : x \in B^{d_0}(0,R)\} < \infty$. It follows that

$$\mu(B^{d_0}(0,R)) \leqslant g_R \int_{\mathbb{R}^n} \chi_{B_R^{d_0}}(x)\,\mu_0(dx) = g_R\,\mu_0(B^{d_0}(0,R)).$$

Now, fix some r with $0 < r \leqslant R$. Then g is also locally bounded in $B^{d_0}(0,r) \subset B^{d_0}(0,R)$ and bounded away from zero by assumption. Hence $g_r := \min\{g(x) : x \in B^{d_0}(0,r)\} > 0$, and we get

$$\mu(B^{d_0}(0,r)) \geqslant g_r \int_{\mathbb{R}^n} \chi_{B_r^{d_0}}(x)\,\mu_0(dx) = g_r\,\mu_0(B^{d_0}(0,r)).$$

Therefore

$$\mu(B^{d_0}(0,R)) \leqslant g_R\,\mu_0(B^{d_0}(0,R)) \leqslant g_R\,C\,\mu_0(B^{d_0}(0,r)) \leqslant C\,\frac{g_R}{g_r}\,\mu(B^{d_0}(0,r)),$$

giving the result with a doubling constant $C\,\frac{g_R}{g_r} \geqslant 1$. □

Example 4.23. Let $\psi_i : \mathbb{R}^n \to \mathbb{R}$ be the negative definite functions $\psi_1(\xi) = \sqrt{|\xi|^2 + k^2} - k$, $k > 0$ and $\psi_2(\xi) = |\xi|$. Figure 4.3 shows the behaviour of $\frac{\sqrt{|\xi|^2+k^2}-k}{|\xi|}$ for the values $k = 1, 3, 5$. Both ψ_1 and ψ_2 are radially symmetric functions to which Theorem 1.37 applies. For $k = 1$, the associated Lévy measure is

$$\nu_1(dy) = m_1(|y|^2)\,dy = \pi^{-\frac{n+1}{2}}\,2^{-\frac{n-1}{2}}\,|y|^{-\frac{n+1}{2}}\,K_{\frac{n+1}{2}}(|y|)\,dy,$$

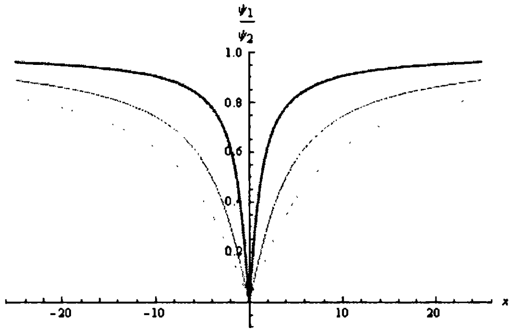


Figure 4.3: Graph of the ratio $\frac{\sqrt{|x|^2+k^2}-k}{|\xi|}$ for $k = 1, 3, 5$.

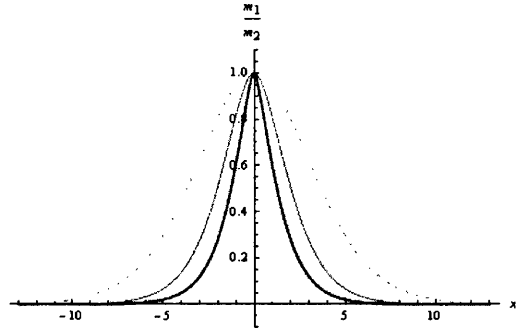


Figure 4.4: Graph of the ratio of the corresponding densities $\frac{m_1(|x|^2)}{m_2(|x|^2)}$ for the space dimensions $n = 1, 3, 10$.

see, e.g., the calculations in Appendix B.2, and the corresponding Lévy measure to ψ_2 is given by

$$\nu_2(dy) = m_2(|y|^2)\,dy = \pi^{-\frac{n+1}{2}}\,|y|^{-n-1}\,\Gamma\left(\frac{n+1}{2}\right)dy\,.$$

Then

$$\frac{m_1(|y|^2)}{m_2(|y|^2)} = \frac{K_{\frac{n+1}{2}}(|y|)\,|y|^{\frac{n+1}{2}}}{2^{\frac{n-1}{2}}\,\Gamma\left(\frac{n+1}{2}\right)}\,,$$

see Figure 4.4.

Let $M > 0$ be a fixed constant and define the set

$$\Omega_M := \left\{ y \in \mathbb{R}^n : \frac{K_{\frac{n+1}{2}}(|y|)\,|y|^{\frac{n+1}{2}}}{2^{\frac{n-1}{2}}\,\Gamma\left(\frac{n+1}{2}\right)} \geqslant M \right\} \subset \mathbb{R}^n\,.$$

then for $y \in \Omega_M$ we obviously have

$$M \leqslant \frac{K_{\frac{n+1}{2}}(|y|)\,|y|^{\frac{n+1}{2}}}{2^{\frac{n-1}{2}}\,\Gamma\left(\frac{n+1}{2}\right)} \leqslant 1\,.$$

Let $d_0 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a metric that is topologically equivalent to the Euclidean metric $|\cdot|$. So we assume the existence of functions $c_{d_0}(R), c_{d_0}(r) : \mathbb{R}_+ \setminus \{0\} \to \mathbb{R}_+ \setminus \{0\}$ such that for fixed $r, R > 0$

$$B^{d_0}(0,R) \subset B^{|\cdot|}(0, c_{d_0}(R)) \subset \Omega_M$$

and

$$\Omega_M \supset B^{d_0}(0,r) \supset B^{|\cdot|}(0, c_{d_0}(r))\,.$$

Setting $\mu(dy) := \frac{m_1(|y|^2)}{m_2(|y|^2)}\,dy$ this yields

$$\begin{aligned}
\mu(B^{d_0}(0,R)) &= \int_{\mathbb{R}^n} \chi_{B_R^{d_0}}(y)\,\frac{K_{\frac{n+1}{2}}(|y|)\,|y|^{\frac{n+1}{2}}}{2^{\frac{n-1}{2}}\,\Gamma\left(\frac{n+1}{2}\right)}\,dy \leqslant \int_{\mathbb{R}^n} \chi_{B_R^{d_0}}(y)\,dy \\
&\leqslant \int_{\mathbb{R}^n} \chi_{B_1^{|\cdot|}}\left(\frac{y}{c_{d_0}(R)}\right)dy = c_{d_0}^n(R)\,\lambda^{(n)}(B^{|\cdot|}(0,1))\,.
\end{aligned}$$

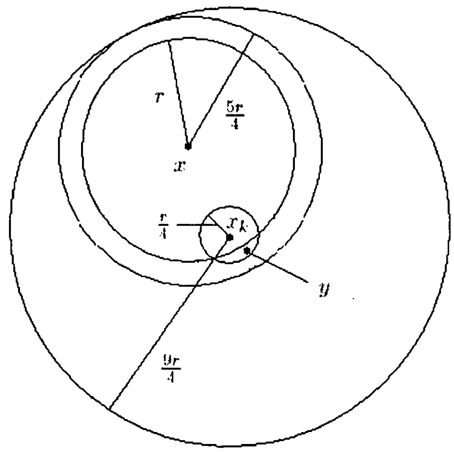


Figure 4.5. Sketch of the location of the balls used in the proof of Corollary 4.24.

On the other hand,

$$\begin{aligned}\mu(B^{d_0}(0,r)) &= \int_{\mathbb{R}^n} \chi_{B_r^{d_0}}(y) \frac{K_{\frac{n+1}{2}}(|y|)\,|y|^{\frac{n+1}{2}}}{2^{\frac{n-1}{2}}\,\Gamma\left(\frac{n+1}{2}\right)}\, dy \geqslant M \int_{\mathbb{R}^n} \chi_{B_r^{d_0}}(y)\, dy \\ &\geqslant M \int_{\mathbb{R}^n} \chi_{B_1^{|\cdot|}}\left(\frac{y}{c_{d_0}(r)}\right) dy = M\, c_{d_0}^n(r)\, \lambda^{(n)}(B^{|\cdot|}(0,1)),\end{aligned}$$

which gives the local doubling

$$\mu(B^{d_0}(0,R)) \leqslant \frac{c_{d_0}^n(R)}{M\, c_{d_0}^n(r)}\, \mu(B^{d_0}(0,r))$$

which holds for balls contained in $\Omega_M \subset \mathbb{R}^n$.

4.3 The metric measure spaces $(\mathbb{R}^n, \psi^{1/2}, \lambda^{(n)})$ as homogeneous spaces

In § 1, Definition 1.12, we introduced the notion of a space of homogeneous type as a topological space X equipped with a (quasi-)metric d such that the balls with respect to d form a basis of open neighbourhoods of $x \in X$ and that the homogeneity condition

there exists $N > 0$ such that for all $x \in X$ and all $r > 0$ the ball $B^d(x,r)$ contains at most N points $x_1, \ldots, x_N$ with distance $d(x_i, x_j) > \frac{r}{2}$ for $i \neq j$ (HC)

is satisfied, see also the monograph [15].

The following corollary shows that the metric measure spaces $(\mathbb{R}^n, \psi^{1/2}, \lambda^{(n)})$ with continuous negative definite functions ψ are homogeneous in the sense of Coifman and Weiss. According to [15], the homogeneity condition is satisfied if the metric space is equipped with a regular Borel measure satisfying the doubling property. This is satisfied in particular for the metric measure space $(\mathbb{R}^n, \psi^{1/2}, \lambda^{(n)})$ with a continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{R}$. For the sake of completeness we will provide the proof of the result in our setting. The general result for quasi-metric spaces is given in [15], p. 67.

Corollary 4.24. *The volume doubling metric measure spaces $(\mathbb{R}^n, \psi^{1/2}, \lambda^{(n)})$ satisfy the condition (HC) and are therefore spaces of homogeneous type.*

Proof. Consider the metric measure spaces $(\mathbb{R}^n, \psi^{1/2}, \lambda^{(n)})$ with volume growth condition

$$\lambda^{(n)}(B^\psi(0,R)) \leqslant c(r,R;n)\, \lambda^{(n)}(B^\psi(0,r))$$

for balls of radii $0 < r < R < \infty$ with respect to $\psi^{1/2}$. Using this we are now in a position to verify the homogeneity condition (HC). For some $N \in \mathbb{N}$ let $x_1, \dots, x_N$ be the points in $B^\psi(x,r)$, $x \in \mathbb{R}^n$, $r > 0$, such that $\psi^{1/2}(x_i - x_j) > \frac{r}{2}$ for $i \neq j$. Then the balls of radius $\frac{r}{4}$ around $x_1, \dots, x_N$ are disjoint, i.e.

$$\bigcap_{k=1}^{N} B^\psi(x_k, \tfrac{r}{4}) = \emptyset .$$

To see this, assume we found a point $y \in B^\psi(x_k, \frac{r}{4}) \cap B^\psi(x_j, \frac{r}{4})$, then due to the triangle inequality we would get

$$\psi^{1/2}(x_i - x_j) \leqslant \psi^{1/2}(x_i - y) + \psi^{1/2}(y - x_j) \leqslant \frac{r}{4} + \frac{r}{4} = \frac{r}{2}$$

in contradiction to the assumption $\psi^{1/2}(x_i - x_j) > \frac{r}{2}$. Hence the balls are disjoint.

Let $y \in B^\psi(x_k, \frac{r}{4})$ for a $k \in \{1, \dots, N\}$ and x the centre point of the ball $B^\psi(x,r)$. Then again by the triangle inequality

$$\psi^{1/2}(x - y) \leqslant \psi^{1/2}(x - x_k) + \psi^{1/2}(x_k - y) < r + \frac{r}{4} = \frac{5r}{4} .$$

This implies the inclusion $B^\psi(x, \frac{r}{4}) \subset B^\psi(x, \frac{5r}{4})$ for all $k = 1, \dots, N$ and hence

$$\bigcup_{k=1}^{N} B^\psi(x_k, \tfrac{r}{4}) \subset B^\psi(x, \tfrac{5r}{4}) .$$

As the balls $B^\psi(x, \frac{r}{4})$ are disjoint, and $\lambda^{(n)}$ is finitely additive, it follows

$$\lambda^{(n)}\Big(\bigcup_{k=1}^{N} B^\psi(x_k, \tfrac{r}{4})\Big) = \sum_{k=1}^{N} \lambda^{(n)}(B^\psi(x_k, \tfrac{r}{4})) \leqslant \lambda^{(n)}(B^\psi(x, \tfrac{5r}{4})) . \tag{4.9}$$

Since $y \in B^\psi(x_k, \frac{r}{4}) \subset B^\psi(x, \frac{5r}{4})$ for all $k = 1, \dots, N$ it holds

$$\psi^{1/2}(x_k - y) \leqslant \psi^{1/2}(x_k - x) + \psi^{1/2}(x - y) < \Big(r + \frac{5r}{4}\Big) = \frac{9r}{4} ,$$

and it follows that the ball $B^\psi(x, \frac{5r}{4})$ is itself contained in $B^\psi(x_k, \frac{9r}{4})$. All in all, for any $k \in \{1, \dots, N\}$ we get the inclusions

$$B^\psi(x_k, \tfrac{r}{4}) \subset B^\psi(x, \tfrac{5r}{4}) \subset B^\psi(x_k, \tfrac{9r}{4})$$

which lead to

$$\lambda^{(n)}(B^\psi(x, \tfrac{5r}{4})) \leqslant \lambda^{(n)}(B^\psi(x_k, \tfrac{9r}{4})) \leqslant c(r;n)\, \lambda^{(n)}(B^\psi(x_k, \tfrac{r}{4}))$$

due to the doubling property. Because of (4.9) we obtain

$$\sum_{k=1}^{N} \lambda^{(n)}(B^\psi(x_k, \tfrac{r}{4})) \leqslant \lambda^{(n)}(B^\psi(x, \tfrac{5r}{4})) \leqslant c(r;n)\, \lambda^{(n)}(B^\psi(x_k, \tfrac{r}{4}))$$

and therefore the number N of points x_k, $k = 1, \dots, N$, is bounded by $N \leqslant c(r;n)$. □

Example 4.25. If we consider the metric measure space $(\mathbb{R}^n, |\cdot|^{2\alpha}, \lambda^{(n)})$ with $\alpha < 1$. Then $\psi^{1/2}(x-y) := |x-y|^\alpha$ gives a metric. We have shown in Appendix A, Example A.1, that for radii R_1 and R_2 with $0 < R_1 < R_2 < \infty$ the doubling property

$$\lambda^{(n)}(B^\psi(0, R_2)) = \left(\frac{R_2}{R_1}\right)^{n/\alpha} \lambda^{(n)}(B^\psi(0, R_1))$$

holds. Setting $R_2 = \frac{9r}{4}$ and $R_1 = \frac{r}{4}$ we obtain the volume doubling constant $(\frac{R_2}{R_1})^{n/\alpha} = 9^{n/\alpha}$. By the argument in the proof of Corollary 4.24, $N \leqslant 9^{n/\alpha}$.

4.4 Remarks to the Hajłasz-Sobolev space over $(\mathbb{R}^n, \psi^{1/2}, \lambda^{(n)})$

In § 2.3.4 we introduced the Sobolev space $M^{1,p}$ due to Hajłasz based on a Lipschitz-type characterisation, which was constructed to meet the constraints of a metric measure space a priori not being equipped with a differential structure. We can adapt this construction to $(\mathbb{R}^n, \psi^{1/2}, \lambda^{(n)})$ and define a Sobolev space on this metric measure space analogously to the set-up in § 2.3.4.

Definition 4.26. We define the Hajłasz-Sobolev space $M^{1,p}(\mathbb{R}^n, \psi^{1/2}, \lambda^{(n)})$ of all p-integrable functions $u \in L^p(\mathbb{R}^n)$ such that there exists $g \in L^p(\mathbb{R}^n)$ that satisfies the inequality

$$|u(x) - u(y)| \leqslant \psi^{1/2}(x-y)\,(g(x) + g(y)) \tag{4.10}$$

for all $x, y \in \mathbb{R}^n \setminus N$, where N is a Lebesgue null set.

Theorem 2.20 can be transferred to the case $d(x,y) = \psi^{1/2}(x-y)$ giving that $M^{1,p}(\mathbb{R}^n) := M^{1,p}(\mathbb{R}^n, \psi^{1/2}, \lambda^{(n)})$ is a Banach space for $1 \leqslant p < \infty$.

Corollary 4.27. *The space $(M^{1,p}, \|\cdot\|_{M^{1,p}})$ is a Banach space for $1 \leqslant p < \infty$, where*

$$\|u\|_{M^{1,p}} := \|u\|_{L^p} + \inf_g \|g\|_{L^p}$$

and the infimum is taken over the set of functions $g \in L^p(\mathbb{R}^n)$ with $g \geqslant 0$ satisfying (4.10).

Proof. We adapt the proof given in [37] to the case $d(x,y) = \psi^{1/2}(x-y)$. Let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $M^{1,p}(\mathbb{R}^n)$. Since u_n are in $L^p(\mathbb{R}^n)$ by definition of $M^{1,p}(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$ is a complete normed linear space, it follows that $u_n \to u$ in L^p for some $u \in L^p(\mathbb{R}^n)$. Let $(u_{n_k})_{k\in\mathbb{N}}$ be a subsequence of $(u_n)_{n\in\mathbb{N}}$ such that

$$\|u_{n_{k+1}} - u_{n_k}\|_{M^{1,p}} < 2^{-k} \quad \text{and} \quad u_{n_k} \to u \text{ a.e. as } k \to \infty\,.$$

By definition of $M^{1,p}(\mathbb{R}^n)$ there exists a function $g_n \in L^p(\mathbb{R}^n)$ such that

$$|(u_{n_{k+1}} - u_{n_k})(x) - (u_{n_{k+1}} - u_{n_k})(y)| \leqslant \psi^{1/2}(x-y)\,(g_k(x) + g_k(y))$$

and $\|g_k\|_{L^p} < 2^{-k}$. Note that

$$\Big\|\sum_{k=0}^{\infty} g_k\Big\|_{L^p} \leqslant \sum_{k=0}^{\infty} \|g_k\|_{L^p} < \sum_{k=0}^{\infty} 2^{-k} = 2\,.$$

It follows that for all $j > k$

$$\begin{aligned} |(u_{n_j} - u_{n_k})(x) - (u_{n_j} - u_{n_k})(y)| &\leqslant \psi^{1/2}(x-y)\Big(\sum_{i=k}^{\infty} g_i(x) + \sum_{i=k}^{\infty} g_i(y)\Big) \\ {\scriptstyle j\to\infty} \downarrow {\scriptstyle \text{a.e.}} & \\ |(u - u_{n_k})(x) - (u - u_{n_k})(y)| &\leqslant \psi^{1/2}(x-y)\Big(\sum_{i=k}^{\infty} g_i(x) + \sum_{i=k}^{\infty} g_i(y)\Big). \end{aligned}$$

Further, we have

$$\sum_{i=n}^{\infty} \|g_i\|_{L^p} < 2\,2^{-n} = 2^{1-n}.$$

Therefore, $u \in M^{1,p}(\mathbb{R}^n)$ and $u_n \to u$ in $M^{1,p}$. □

Assume that $(\mathbb{R}^n, \psi^{1/2}, \lambda^{(n)})$ is a metric measure space satisfying the volume growth condition. Let $B_r^\psi := B^\psi(x,r)$ be the ball of radius $r > 0$ centred at $x \in \mathbb{R}^n$ with respect to the metric $\psi^{1/2}$. Define the average value of a function u on B_r^ψ by

$$u_{B_r^\psi} := \fint_{B_r^\psi} u\, d\lambda^{(n)} = \frac{1}{\lambda^{(n)}(B^\psi(x,r))} \int_{B_r^\psi(x)} u(y)\, dy\,.$$

Then we can adapt Theorem 7 (ii) in [39] to our case. Note that we do not need the volume doubling property for this result.

Corollary 4.28. *Let $(\mathbb{R}^n, \psi^{1/2}, \lambda^{(n)})$ and $r > 0$. Assume that for $u \in M^{1,p}(B_r^\psi)$, $B_r^\psi := B^\psi(x,r) \subset \mathbb{R}^n$, there exists a constant $C(n) > 0$ such that*

$$|u(x) - u(y)| \leqslant C(n)\, \psi^{1/2}(x-y)\,(g(x) + g(y)) \quad a.e. \tag{4.11}$$

for any two points $x, y \in B_r^\psi$. Then

$$\fint_{B_r^\psi} |u - u_{B_r^\psi}|\, d\lambda^{(n)} \leqslant C(n)\, r \fint_{B_r^\psi} g\, d\lambda^{(n)}.$$

Proof. We first integrate inequality (4.11) with respect to y and obtain

$$\begin{aligned} |u(x) - u_{B_r^\psi}| &= \fint_{B_r^\psi} |u(x) - u(y)|\, dy \overset{(4.11)}{\leqslant} C(n) \fint_{B_r^\psi} \psi^{1/2}(x-y)\,(g(x) + g(y))\, dy \\ &\leqslant C(n)\, r \fint_{B_r^\psi} (g(x) + g(y))\, dy = C(n)\, r\, (g(x) + g_{B_r^\psi}), \end{aligned}$$

and then we integrate with respect to x, which yields

$$\int_{B_r^\psi} |u(x) - u_{B_r^\psi}|\, dx \leqslant C(n)\, r \Big(\int_{B_r^\psi} g(x)\, dx + \int_{B_r^\psi} g_{B_r^\psi}\, dx \Big) \leqslant 2\, C(n)\, r \int_{B_r^\psi} g(x)\, dx$$

and therefore

$$\fint_{B_r^\psi} |u - u_{B_r^\psi}|\, d\lambda^{(n)} \leqslant C(n)\, r \fint_{B_r^\psi} g\, d\lambda^{(n)}.$$

□

Since $g \in L^p(B)$, we can apply the Hölder inequality and obtain the following result for a metric measure space equipped with a doubling measure. For this result see [23, 38].

Corollary 4.29. *Let $(\mathbb{R}^n, \psi^{1/2}, \lambda^{(n)})$ be a doubling metric measure space, $u \in M^{1,p}(\mathbb{R}^n, \psi^{1/2}, \lambda^{(n)})$. Let $0 < r < R$ be such that $\lambda^{(n)}(B^\psi(x,r)) \leqslant k(r,R;n)\,\lambda^{(n)}(B^\psi(x,R))$. Then there exists a constant $C(r,R;n) > 0$ and a nonnegative function $g \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, satisfying inequality (4.11) a.e. such that the following $(1,p)$-Poincaré-type inequality holds:*

$$\fint_{B_r^\psi} |u - u_{B_r^\psi}|\, d\lambda^{(n)} \leqslant C(r,R;n)\, r \left(\fint_{B_R^\psi} g^p\, d\lambda^{(n)}\right)^{1/p}.$$

Proof. From Corollary 4.28 and by the Hölder inequality for $\frac{1}{p} + \frac{1}{q} = 1$, we get

$$\begin{aligned}
\fint_{B_r^\psi} |u - u_{B_r}|\, d\lambda^{(n)} &\leqslant C(n)\, r \fint_{B_r^\psi} g\, d\lambda^{(n)} \leqslant C(n)\, r\, k(r,R;n) \fint_{B_R^\psi} g\, d\lambda^{(n)} \\
&= \frac{C(r,R;n)\, r}{\lambda^{(n)}(B_R^\psi)} \int_{B_R^\psi} g\, d\lambda^{(n)} \\
&\leqslant \frac{C(r,R;n)\, r}{\lambda^{(n)}(B_R^\psi)} \left(\int_{B_R^\psi} g^p\, d\lambda^{(n)}\right)^{1/p} \left(\int_{B_R^\psi} 1^q\, d\lambda^{(n)}\right)^{1/q} \\
&= \frac{C(r,R;n)\, r}{\lambda^{(n)}(B_R^\psi)} \left(\int_{B_R^\psi} g^p\, d\lambda^{(n)}\right)^{1/p} \lambda^{(n)}(B_R^\psi)^{1/q} \\
&= C(r,R;n)\, r\, \lambda^{(n)}(B_R^\psi)^{1/p} \left(\int_{B_R^\psi} g^p\, d\lambda^{(n)}\right)^{1/p} \\
&= C(r,R;n)\, r \left(\fint_{B_R^\psi} g^p\, d\lambda^{(n)}\right)^{1/p}.
\end{aligned}$$

□

Next, we show that under certain assumptions the Poincaré inequality implies that u and g satisfy an equality similar to (4.11). This result is adapted from [38], Theorem 3.2, to our case.

Proposition 4.30. *Let $(\mathbb{R}^n, \psi^{1/2}, \lambda^{(n)})$ satisfy the volume growth property. Assume $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $g \geqslant 0$ measurable such that there exist $r, R > 0$ and a constant $C(r,R;n) > 0$ such that*

$$\fint_{B_r^\psi} |u - u_{B_r^\psi}|\, d\lambda^{(n)} \leqslant C(r,R;n)\, r \left(\fint_{B_R^\psi} g^p\, d\lambda^{(n)}\right)^{1/p}$$

for $p > 0$. Then

$$|u(x) - u(y)| \leqslant C(r,R;n)\, \psi^{1/2}(x-y) \left[(M_{2\rho} g^p(x))^{1/p} + (M_{2\rho} g^p(y))^{1/p}\right]$$

for almost every x, $y \in \mathbb{R}^n$, where

$$(M_\rho f)(x) = \sup_{0<r<R<\rho} \fint_{B_R} |f(y)|\, dy$$

is the Hardy-Littlewood maximal function restricted to the ball $B^\psi(x,\rho) \subset \mathbb{R}^n$.

Proof. We consider the balls $B^\psi(x, r_j) \subset \mathbb{R}^n$ with

$$r_j := \frac{\psi^{1/2}(x-y)}{2^j} \to 0 \quad \text{as } j \to \infty.$$

Note that $r_{j-1} > r_j$ for $j \geqslant 0$ and $r_0 = \psi^{1/2}(x-y)$. Let x, y be Lebesgue points, i.e.

$$u(x) \;=\; \lim_{j\to\infty} \fint_{B^{\psi}_{r_j}(x)} u(s)\,ds\,, \qquad u(y) \;=\; \lim_{j\to\infty} \fint_{B^{\psi}_{r_j}(y)} u(s)\,ds\,,$$

where

$$\fint_{B^{\psi}_{r}(x)} u(s)\,ds = \frac{1}{\lambda^{(n)}(B^{\psi}_{r}(x))} \int_{B^{\psi}_{r}(x)} u(s)\,ds\,.$$

Let $\rho > 0$ and R_j be such that $0 < r_j < R_j < \rho$. For a measurable function h define the Hardy-Littlewood maximal function

$$(M_\rho h)(x) = \sup_{0<R_j<\rho} \fint_{B^{\psi}_{R_j}(x)} |h(y)|\,dy$$

restricted to a ball $B^{\psi}(x,\rho) \subset \mathbb{R}^n$. Then we obtain

$$\begin{aligned} |u(x) - u_{B^{\psi}_{r_0}}| &\leqslant \sum_{j=1}^{\infty} |u_{B^{\psi}_{r_{j-1}}} - u_{B^{\psi}_{r_j}}| \;\leqslant\; \sum_{j=1}^{\infty} \fint_{B^{\psi}_{r_{j-1}}} |u(s) - u_{B^{\psi}_{r_j}}|\,ds \\ &\leqslant \sum_{j=1}^{\infty} C(r_j, r_{j-1}; n) \fint_{B^{\psi}_{r_j}} |u(s) - u_{B^{\psi}_{r_j}}|\,ds \end{aligned}$$

by the volume growth condition and further

$$\begin{aligned} |u(x) - u_{B^{\psi}_{r_0}}| &\leqslant \sum_{j=1}^{\infty} C(r_j, r_{j-1}; n)\, r_{j-1} \left(\fint_{B^{\psi}_{R_j}} g^p(z)\,dz \right)^{1/p} \\ &\leqslant \sum_{j=1}^{\infty} C(r_j, r_{j-1}; n)\, r_{j-1} \left(\sup_{0<R_j<\rho} \fint_{B^{\psi}_{R_j}} g^p(z)\,dz \right)^{1/p} \\ &= \sum_{j=1}^{\infty} C(r_j, r_{j-1}; n)\, r_{j-1}\, (M_\rho g^p(x))^{1/p} \end{aligned}$$

by the definition of the restricted Hardy-Littlewood maximal function. Set

$$C(\rho; n) := \sup\{C(r_j, r_{j-1}; n) : r_j < r_{j-1} < \rho\}\,,$$

then

$$|u(x) - u_{B^{\psi}_{r_0}(x)}| \;\leqslant\; C(\rho; n) \sum_{j=0}^{\infty} r_j\, (M_\rho g^p(x))^{1/p} \;=\; C(\rho; n)\, \psi^{1/2}(x-y)\, (M_\rho g^p(x))^{1/p}\,.$$

Analogously, we get

$$|u(y) - u_{B^{\psi}_{r_0}(y)}| \;\leqslant\; C(\rho; n)\, \psi^{1/2}(x-y)\, (M_\rho g^p(y))^{1/p}\,.$$

Moreover we have

$$\begin{aligned} |u_{B^{\psi}_{r_0}(x)} - u_{B^{\psi}_{r_0}(y)}| &\leqslant |u_{B^{\psi}_{r_0}(x)} - u_{B^{\psi}_{2r_0}(y)}| + |u_{B^{\psi}_{r_0}(y)} - u_{B^{\psi}_{2r_0}(x)}| \\ &\leqslant 2 \fint_{B^{\psi}_{r_0}(x)} |u(s) - u_{B^{\psi}_{2r_0}(x)}|\,ds \\ &\leqslant 2\,c(r_0) \fint_{B^{\psi}_{2r_0}(x)} |u(s) - u_{B^{\psi}_{2r_0}(x)}|\,ds \end{aligned}$$

by the triangle inequality and the volume doubling property. By application of Corollary 4.29 and the Hardy-Littlewood maximal function we get with constant $C := 4c(r_0)$ and $r_0 = \psi^{1/2}(x-y)$

$$\begin{aligned}
|u_{B^\psi_{r_0}(x)} - u_{B^\psi_{r_0}(y)}| &\leqslant 2\,c(r_0) \fint_{B^\psi_{2r_0}(x)} |u(s) - u_{B^\psi_{2r_0}(x)}|\,ds \\
&\leqslant C\,\psi^{1/2}(x-y) \left(\fint_{B^\psi_{2R_0}(x)} g^p(s)\,ds \right)^{1/p} \\
&\leqslant C\,\psi^{1/2}(x-y)\,(M_{2\rho} g^p(x))^{1/p},
\end{aligned}$$

where M_ρ is defined as above. Combining the inequalities above we get

$$\begin{aligned}
|u(x) - u(y)| &\leqslant |u(x) - u_{B^\psi_{r_0}(x)}| + |u_{B^\psi_{r_0}(x)} - u_{B^\psi_{r_0}(y)}| + |u(y) - u_{B^\psi_{r_0}(y)}| \\
&\leqslant C(\rho;n)\,\psi^{1/2}(x-y)\,(M_\rho g^p(x))^{1/p} + C\,\psi^{1/2}(x-y)\,(M_{2\rho} g^p(x))^{1/p} \\
&\quad + C(\rho;n)\,\psi^{1/2}(x-y)\,(M_\rho g^p(y))^{1/p} \\
&\leqslant C(\rho;n)\,\psi^{1/2}(x-y)\left((M_{2\rho} g^p(x))^{1/p} + (M_{2\rho} g^p(y))^{1/p}\right).
\end{aligned}$$

□

Appendices

A Some doubling constant computations

In this appendix we will provide some examples of metric measure spaces $(\mathbb{R}^n, \psi^{1/2}, \lambda^{(n)})$ for continuous negative definite functions $\xi \mapsto \psi(\xi)$ and the computation of the doubling constants of the Lebesgue measure $\lambda^{(n)}$ in those spaces. In particular, we will study the spaces $(\mathbb{R}^n, (f \circ \psi)^{1/2}, \lambda^{(n)})$, where f are (complete) Bernstein functions. We consider the continuous negative definite functions $\xi \mapsto |\xi|^2$ for $\xi \in \mathbb{R}^n$ and also $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \ni (\xi, \eta) \mapsto |\xi|^\alpha + |\eta|^\beta$ for $0 < \alpha, \beta < 2$ and the following list of complete Bernstein functions $f : \mathbb{R}_+ \to \mathbb{R}$.

$f(s) = s^\gamma\,,$	$0 < \gamma < 1$	$f(s) = \log\left(\frac{\gamma}{\delta}\,\frac{s+\delta}{s+\gamma}\right),$	$\gamma > \delta > 0$
$f(s) = \log(1+s)\,,$		$f(s) = \sqrt{s+m^2} - m\,,$	$m \geqslant 0$
$f(s) = 1 - e^{-\gamma s}\,,$	$\gamma > 0$	$f(s) = \frac{s}{s+\gamma}\,,$	$\gamma > 0$
$f(s) = \log\left(\frac{s+\gamma}{s}\right),$	$\gamma > 0$	$f(s) = \frac{1}{\gamma} - \frac{1}{\gamma+s}\,,$	$\gamma > 0$
$f(s) = \sqrt{s}\,\log(1+\sqrt{s})\,,$		$f(s) = \sqrt{s}\,(1 - e^{-4\sqrt{s}})\,.$	

Then also $s \mapsto g(s) := f^\alpha(s)$, $s \mapsto h(s) := f(s^\alpha)$ and $s \mapsto k(s) := f^{1/\alpha}(s^\alpha)$ for $0 < |\alpha| < 1$ are complete Bernstein functions and further, $(\xi,\eta) \mapsto (g \circ \psi)(\xi,\eta)$, $(\xi,\eta) \mapsto (h \circ \psi)(\xi,\eta)$ and $(\xi,\eta) \mapsto (k \circ \psi)(\xi,\eta)$ are again continuous negative definite. Balls of a radius ρ with respect to a metric $\psi^{1/2}$ around 0 are denoted by $B^\psi(0,\rho)$ or $B^\psi_\rho(0)$ for short.

Example A.1. Let us first consider the Bernstein function $f(s) = s^\alpha$ for $0 < \alpha < 1$, and the continuous negative definite function $\psi(\xi) := f(|\xi|^2) = |\xi|^{2\alpha}$ with which we define the distance

$$\psi^{1/2}(\xi - \eta) = |\xi - \eta|^\alpha \quad \text{for } 0 < \alpha < 1\,.$$

Then for some positive radius $R > 0$ we define the balls with respect to ψ around the origin,

$$B^\psi(0,R) = \{x \in \mathbb{R}^n : |x|^\alpha < R\} = \left\{x \in \mathbb{R}^n : \left|\frac{x}{R^{1/\alpha}}\right|^\alpha < 1\right\}.$$

This leads to

$$\begin{aligned}\lambda^{(n)}(B^\psi(0,R)) &= \int_{\mathbb{R}^n} \chi_{B_1^\psi}(\xi)\, R^{n/\alpha}\, d\xi \;=\; R^{n/\alpha}\,\lambda^{(n)}(B^\psi(0,1)) \\ &= \int_{\mathbb{R}^n} \chi_{B_1^\psi}(\xi)\, R^{n/\alpha}\, d\xi \;=\; R^{n/\alpha}\,\lambda^{(n)}(B^\psi(0,1))\end{aligned}$$

by the change of variable $\xi := \frac{y}{R^{1/\alpha}}$, therefore $dy = R^{n/\alpha}\, d\xi$. Analogously, for a radius $0 < r < R$ and the ball $B^\psi(0,r)$ we get

$$\lambda^{(n)}(B^\psi(0,r)) \;=\; \int_{\mathbb{R}^n} \chi_{B_1^\psi}\!\left(\frac{y}{r^{1/\alpha}}\right) dy \;=\; r^{n/\alpha}\,\lambda^{(n)}(B^\psi(0,1))\,.$$

Therefore, we get the volume doubling

$$\lambda^{(n)}(B^\psi(0,R)) = \left(\frac{R}{r}\right)^{n/\alpha} \lambda^{(n)}(B^\psi(0,r))$$

in the metric measure space $(\mathbb{R}^n, \psi, \lambda^{(n)})$.

Example A.2. Let $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and $\psi : \mathbb{R}^n \to \mathbb{R}$ be the continuous negative definite function $\psi(\xi, \eta) = |\xi|^\alpha + |\eta|^\beta$ for $0 < \alpha,\ \beta < 2$. We consider balls of radius $R > 0$ centred at the origin with respect to the metric $\psi^{1/2}$, i.e.

$$\begin{aligned} B^\psi(0,R) &= \{(x,y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : (|x|^\alpha + |y|^\beta)^{1/2} < R\} \\ &= \left\{(x,y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : \left|\frac{x}{R^{2/\alpha}}\right|^\alpha + \left|\frac{y}{R^{2/\beta}}\right|^\beta < 1\right\}, \end{aligned}$$

for which we get

$$\begin{aligned} \lambda^{(n)}(B^\psi(0,R)) &= \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_2}} \chi_{B_1^\psi}\left(\frac{x}{R^{2/\alpha}}, \frac{y}{R^{2/\beta}}\right) dy\, dx \\ &= R^{2n_1/\alpha} R^{2n_2/\beta} \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_2}} \chi_{B_1^\psi}(\xi, \eta)\, d\eta\, d\xi \;=\; R^{2(\frac{n_1}{\alpha} + \frac{n_2}{\beta})} \lambda^{(n)}(B^\psi(0,1)), \end{aligned}$$

where we have used the change of variable $\xi := \frac{x}{R^{2/\alpha}}$, $\eta := \frac{y}{R^{2/\beta}}$, therefore $dx = R^{2n_1/\alpha}\, d\xi$ and $dy = R^{2n_2/\beta}\, d\eta$. Analogously, for balls $B^\psi(0,r)$ with radius $0 < r < R$ around the origin,

$$\lambda^{(n)}(B^\psi(0,r)) = r^{2(\frac{n_1}{\alpha} + \frac{n_2}{\beta})} \lambda^{(n)}(B^\psi(0,1)).$$

Hence, we obtain the volume doubling

$$\lambda^{(n)}(B^\psi(0,R)) = \left(\frac{R}{r}\right)^{2(\frac{n_1}{\alpha} + \frac{n_2}{\beta})} \lambda^{(n)}(B^\psi(0,r))$$

in the metric measure space $(\mathbb{R}^n, \psi^{1/2}, \lambda^{(n)})$.

Example A.3. Let f be the Bernstein function $f(s) = s^\gamma$ for $0 < \gamma < 1$ and the continuous negative definite function $\psi : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$ be given by $\psi(\xi, \eta) = |\xi|^\alpha + |\eta|^\beta$ for $0 < \alpha,\ \beta < 2$, $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Then we define the metric

$$(f \circ \psi)^{1/2}(\xi, \eta) = (|\xi|^\alpha + |\eta|^\beta)^{\gamma/2}.$$

Consider balls of radius $R > 0$, centred at the origin, with respect to $(f \circ \psi)^{1/2}$, i.e.

$$\begin{aligned} B^{f\circ\psi}(0,R) &= \{(x,y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : (|x|^\alpha + |y|^\beta)^\gamma < R^2\} \\ &= \left\{(x,y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : \left|\frac{x}{R^{2/\gamma}}\right|^\alpha + \left|\frac{y}{R^{2/\gamma}}\right|^\beta < 1\right\} \;=\; B^\psi(0, R^{1/\gamma}). \end{aligned}$$

Using the change of variable $\xi := \frac{x}{R^{2/\alpha\gamma}}$ and $\eta := \frac{y}{R^{2/\beta\gamma}}$ we get

$$\begin{aligned} \lambda^{(n)}(B^{f\circ\psi}(0,R)) &= \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_2}} \chi_{B_1^\psi}\left(\frac{x}{R^{2/\alpha\gamma}}, \frac{y}{R^{2/\beta\gamma}}\right) dy\, dx \\ &= R^{\frac{2n_1}{\alpha\gamma}} R^{\frac{2n_2}{\beta\gamma}} \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_2}} \chi_{B_1^\psi}(\xi, \eta)\, d\eta\, d\xi \\ &= R^{\frac{2}{\gamma}\left(\frac{n_1}{\alpha} + \frac{n_2}{\beta}\right)} \lambda^{(n)}(B^\psi(0,1)). \end{aligned}$$

Analogously, for ball with radius $0 < r < R$ we obtain

$$\lambda^{(n)}(B^{f\circ\psi}(0,r)) = r^{\frac{2}{\gamma}\left(\frac{n_1}{\alpha} + \frac{n_2}{\beta}\right)} \lambda^{(n)}(B^\psi(0,1)),$$

which implies the volume doubling

$$\lambda^{(n)}(B^{f\circ\psi}(0,R)) = \left(\frac{R}{r}\right)^{\frac{2}{\gamma}\left(\frac{n_1}{\alpha}+\frac{n_2}{\beta}\right)} \lambda^{(n)}(B^{f\circ\psi}(0,r)).$$

Example A.4. Let f be the Bernstein function $f(s) = \log(1+s)$ and $\psi : \mathbb{R}^n \to \mathbb{R}$ the continuous negative definite function $\psi(\xi) = |\xi|^2$ with which we define the metric

$$(f\circ\psi)^{1/2}(\eta) = (\log(1+|\xi|^2))^{1/2}.$$

Note that for balls of positive radius R,

$$\begin{aligned} B^{f\circ\psi}(0,R) &= \{x\in\mathbb{R}^n : \log(1+|x|^2) < R^2\} \\ &= \left\{x\in\mathbb{R}^n : \left|\frac{x}{(e^{R^2}-1)^{1/2}}\right|^2 < 1\right\} = B^{\psi}(0,(e^{R^2}-1)^{1/2}). \end{aligned}$$

Setting $\xi := \frac{x}{(e^{R^2}-1)^{1/2}}$ we get

$$\begin{aligned} \lambda^{(n)}(B^{f\circ\psi}(0,R)) &= \int_{\mathbb{R}^n} \chi_{B_1^{\psi}}\left(\frac{x}{(e^{R^2}-1)^{1/2}}\right) dx \\ &= (e^{R^2}-1)^{n/2}\int_{\mathbb{R}^n} \chi_{B_1^{\psi}}(\xi)\, d\xi = (e^{R^2}-1)^{n/2}\lambda^{(n)}(B^{\psi}(0,1)) \end{aligned}$$

and similarly, for any ball with radius $0 < r < R$ we obtain

$$\lambda^{(n)}(B^{f\circ\psi}(0,r)) = (e^{r^2}-1)^{n/2}\lambda^{(n)}(B^{\psi}(0,1)),$$

which implies

$$\lambda^{(n)}(B^{f\circ\psi}(0,R)) = \left(\frac{e^{R^2}-1}{e^{r^2}-1}\right)^{n/2} \lambda^{(n)}(B^{\psi}(0,r)).$$

Example A.5. Let f be the Bernstein function $f(s) = \log(1+s)$ and $\psi : \mathbb{R}^{n_1}\times\mathbb{R}^{n_2} \to \mathbb{R}$ be the continuous negative definite function $(\xi,\eta) \mapsto \psi(\xi,\eta) = |\xi|^\alpha + |\eta|^\beta$, $0 < \alpha,\beta < 2$ on $\mathbb{R}^n = \mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$, with which we define the metric

$$(f\circ\psi)^{1/2}(\xi,\eta) = (\log(1+|\xi|^\alpha+|\eta|^\beta))^{1/2}.$$

We consider balls of radius $R > 0$ with respect to $(f\circ\psi)^{1/2}$,i.e.

$$\begin{aligned} B^{f\circ\psi}(0,R) &= \{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : \log(1+|x|^\alpha+|y|^\beta) < R^2\} \\ &= \left\{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : \left|\frac{x}{(e^{R^2}-1)^{1/\alpha}}\right|^\alpha + \left|\frac{y}{(e^{R^2}-1)^{1/\beta}}\right|^\beta < 1\right\} \\ &= B^{\psi}(0,(e^{R^2}-1)^{1/2}). \end{aligned}$$

Then using the change of variable $\xi := \frac{x}{(e^{R^2}-1)^{1/\alpha}}$ and $\eta := \frac{y}{(e^{R^2}-1)^{1/\beta}}$ we obtain

$$\begin{aligned} \lambda^{(n)}(B^{f\circ\psi}(0,R)) &= \int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} \chi_{B_1^{\psi}}\left(\frac{x}{(e^{R^2}-1)^{1/\alpha}}, \frac{y}{(e^{R^2}-1)^{1/\beta}}\right) dy\, dx \\ &= (e^{R^2}-1)^{\frac{n_1}{\alpha}}(e^{R^2}-1)^{\frac{n_2}{\beta}} \int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} \chi_{B_1^{\psi}}(\xi,\eta)\, d\eta\, d\xi \\ &= (e^{R^2}-1)^{\frac{n_1}{\alpha}+\frac{n_2}{\beta}}\, \lambda^{(n)}(B^{\psi}(0,1)). \end{aligned}$$

Analogously, for any ball of radius $0 < r < R$ we get

$$\lambda^{(n)}(B^{f\circ\psi}(0,r)) = (e^{r^2}-1)^{\frac{n_1}{\alpha}+\frac{n_2}{\beta}}\,\lambda^{(n)}(B^{\psi}(0,1)),$$

and therefore the volume doubling to be

$$\lambda^{(n)}(B^{f\circ\psi}(0,R)) = \left(\frac{e^{R^2}-1}{e^{r^2}-1}\right)^{\frac{n_1}{\alpha}+\frac{n_2}{\beta}}\lambda^{(n)}(B^{f\circ\psi}(0,r)).$$

Example A.6. Let h be the Bernstein function $h(s) := f(s^\gamma) = \log(1+s^\gamma)$, $0<\gamma<1$ and $\psi : \mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to\mathbb{R}$ be the continuous negative definite function given by $\psi(\xi,\eta) = |\xi|^\alpha + |\eta|^\beta$, $0<\alpha,\ \beta<2$, $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}=\mathbb{R}^n$, with which we define the metric

$$(h\circ\psi)^{1/2}(\xi,\eta) = \left(\log(1+(|\xi|^\alpha+|\eta|^\beta)^\gamma)\right)^{1/2}.$$

Consider the ball of radius $R>0$ around the origin

$$\begin{aligned}
B^{h\circ\psi}(0,R) &= \{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : \log(1+(|\xi|^\alpha+|\eta|^\beta)^\gamma) < R^2\}\\
&= \left\{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : \left|\frac{x}{(e^{R^2}-1)^{\frac{1}{\alpha\gamma}}}\right|^\alpha + \left|\frac{y}{(e^{R^2}-1)^{\frac{1}{\beta\gamma}}}\right|^\beta < 1\right\}\\
&= B^\psi(0,(e^{R^2}-1)^{1/2\gamma}).
\end{aligned}$$

Using the change of variable $\xi := \frac{x}{(e^{R^2}-1)^{1/\alpha\gamma}}$ and $\eta := \frac{y}{(e^{R^2}-1)^{1/\beta\gamma}}$ we obtain

$$\begin{aligned}
\lambda^{(n)}(B^{h\circ\psi}(0,R)) &= \int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} \chi_{B_1^\psi}\left(\frac{x}{(e^{R^2}-1)^{\frac{1}{\alpha\gamma}}}, \frac{y}{(e^{R^2}-1)^{\frac{1}{\beta\gamma}}}\right) dy\,dx\\
&= (e^{R^2}-1)^{\frac{n_1}{\alpha\gamma}}(e^{R^2}-1)^{\frac{n_2}{\beta\gamma}}\int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}}\chi_{B_1^\psi}(\xi,\eta)\,d\eta\,d\xi\\
&= (e^{R^2}-1)^{\frac{n_1}{\alpha\gamma}+\frac{n_2}{\beta\gamma}}\,\lambda^{(n)}(B^\psi(0,1)).
\end{aligned}$$

Analogously, for $B^\psi(0,r)$ with $R>r>0$ we get

$$\lambda^{(n)}(B^{h\circ\psi}(0,r)) = (e^{r^2}-1)^{\frac{n_1}{\alpha\gamma}+\frac{n_2}{\beta\gamma}}\,\lambda^{(n)}(B^\psi(0,1)),$$

which gives the doubling constant

$$\lambda^{(n)}(B^{h\circ\psi}(0,R)) = \left(\frac{e^{R^2}-1}{e^{r^2}-1}\right)^{\frac{n_1}{\alpha\gamma}+\frac{n_2}{\beta\gamma}}\lambda^{(n)}(B^{h\circ\psi}(0,r)).$$

Example A.7. Let $k(s) := f^{1/\gamma}(s^\gamma)$ with f as in Example A.5, $0<|\gamma|<1$, be the Bernstein function $k(s) = (\log(1+s^\gamma))^{1/\gamma}$ and $\psi : \mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to\mathbb{R}$ be the continuous negative definite function given by $\psi(\xi,\eta) = |\xi|^\alpha+|\eta|^\beta$, $0<\alpha,\ \beta<2$, $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}=\mathbb{R}^n$. With respect to the metric

$$(k\circ\psi)^{1/2}(\xi,\eta) = \left((\log(1+(|\xi|^\alpha+|\eta|^\beta)^\gamma))^{1/\gamma}\right)^{1/2}$$

we consider the ball of radius $R>0$ centred at the origin

$$\begin{aligned}
B^{k\circ\psi}(0,R) &= \{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : (\log(1+(|\xi|^\alpha+|\eta|^\beta)^\gamma))^{1/\gamma} < R^2\}\\
&= \left\{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : \left|\frac{x}{(e^{R^{2\gamma}}-1)^{1/\alpha\gamma}}\right|^\alpha + \left|\frac{y}{(e^{R^{2\gamma}}-1)^{1/\beta\gamma}}\right|^\beta < 1\right\}\\
&= B^\psi(0,(e^{R^{2\gamma}}-1)^{1/2\gamma}).
\end{aligned}$$

We use the change of variable $\xi := \frac{x}{(e^{R^{2\gamma}}-1)^{1/\alpha\gamma}}$ and $\eta := \frac{y}{(e^{R^{2\gamma}}-1)^{1/\beta\gamma}}$ we get

$$\begin{aligned}\lambda^{(n)}(B^{k\circ\psi}(0,R)) &= \int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} \chi_{B_1^\psi}\left(\frac{x}{(e^{R^{2\gamma}}-1)^{1/\alpha\gamma}}, \frac{y}{(e^{R^{2\gamma}}-1)^{1/\beta\gamma}}\right) dy\, dx \\ &= (e^{R^{2\gamma}}-1)^{\frac{n_1}{\alpha\gamma}}(e^{R^{2\gamma}}-1)^{\frac{n_2}{\beta\gamma}} \int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} \chi_{B_1^\psi}(\xi,\eta)\, d\eta\, d\xi \\ &= (e^{R^{2\gamma}}-1)^{\frac{n_1}{\alpha\gamma}+\frac{n_2}{\beta\gamma}} \lambda^{(n)}(B^\psi(0,1)).\end{aligned}$$

Analogously, for a ball with radius $0<r<R$ we get

$$\lambda^{(n)}(B^{k\circ\psi}(0,r)) = (e^{r^{2\gamma}}-1)^{\frac{n_1}{\alpha\gamma}+\frac{n_2}{\beta\gamma}} \lambda^{(n)}(B^\psi(0,1)).$$

This implies

$$\lambda^{(n)}(B^{k\circ\psi}(0,R)) = \left(\frac{e^{R^{2\gamma}}-1}{e^{r^{2\gamma}}-1}\right)^{\frac{n_1}{\alpha\gamma}+\frac{n_2}{\beta\gamma}} \lambda^{(n)}(B^{k\circ\psi}(0,r)).$$

Example A.8. Let f be the Bernstein function $f(s) = 1-e^{-\gamma s}$, $\gamma>0$, and $\psi:\mathbb{R}^n\to\mathbb{R}$ be the continuous negative definite function $\psi(\xi) = |\xi|^2$, $\xi\in\mathbb{R}^n$. With the help of $f\circ\psi$ we define the metric

$$(f\circ\psi)^{1/2}(\xi) = (1-e^{-\gamma|\xi|^2})^{1/2}, \quad \gamma>0,$$

and consider the balls $B^{f\circ\psi}(0,R)$ of radius R with respect to this metric such that $0<R<1$, i.e.

$$\begin{aligned}B^{f\circ\psi}(0,R) &= \{x\in\mathbb{R}^n : 1-e^{-\gamma|\xi|^2} < R^2\} \\ &= \left\{x\in\mathbb{R}^n : \left|\frac{x}{(\frac{1}{\gamma}\log(1-R^2))^{1/2}}\right|^2 < 1\right\} = B^\psi(0,(\tfrac{1}{\gamma}\log(1-R^2))^{1/2}).\end{aligned}$$

Then we get

$$\begin{aligned}\lambda^{(n)}(B^{f\circ\psi}(0,R)) &= \int_{\mathbb{R}^n} \chi_{B_1^\psi}\left(\frac{x}{(\frac{1}{\gamma}\log(1-R^2))^{1/2}}\right) dx = (\tfrac{1}{\gamma}\log(1-R^2))^{n/2}\int_{\mathbb{R}^n}\chi_{B_1^\psi}(\xi)\, d\xi \\ &= (\tfrac{1}{\gamma}\log(1-R^2))^{n/2}\lambda^{(n)}(B^\psi(0,1)),\end{aligned}$$

and analogously for radii r with $0<r<R<1$

$$\lambda^{(n)}(B^{f\circ\psi}(0,r)) = (\tfrac{1}{\gamma}\log(1-r^2))^{n/2}\lambda^{(n)}(B^\psi(0,1)).$$

This implies the volume doubling

$$\lambda^{(n)}(B^{f\circ\psi}(0,R)) = \left(\frac{\log(1-R^2)}{\log(1-r^2)}\right)^{n/2} \lambda^{(n)}(B^{f\circ\psi}(0,r))$$

in the metric measure space $(\mathbb{R}^n, (f\circ\psi)^{1/2}, \lambda^{(n)})$ provided that the radii of the balls are such that $0<r<R<1$.

Example A.9. Let f be the Bernstein function $f(s) = 1-e^{-\gamma s}$ for $\gamma>0$ and $\psi:\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to\mathbb{R}$ be the continuous negative definite function $\psi(\xi,\eta) = |\xi|^\alpha + |\eta|^\beta$, $0<\alpha,\beta<2$. We define the metric

$$(f\circ\psi)^{1/2}(\xi,\eta) = \left(1-e^{-\gamma(|\xi|^\alpha+|\eta|^\beta)}\right)^{1/2}$$

and consider the balls of radius R satisfying $0 < R < 1$,

$$\begin{aligned}
B^{f\circ\psi}(0,R) &= \{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : 1-e^{-\gamma(|x|^\alpha+|y|^\beta)} < R^2\} \\
&= \left\{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : \left|\frac{x}{(\frac{1}{\gamma}\log(1-R^2))^{1/\alpha}}\right|^\alpha + \left|\frac{y}{(\frac{1}{\gamma}\log(1-R^2))^{1/\beta}}\right|^\beta < 1\right\} \\
&= B^\psi(0,(\tfrac{1}{\gamma}\log(1-R^2))^{1/2}).
\end{aligned}$$

Then we get by a change of variable

$$\begin{aligned}
\lambda^{(n)}(B^{f\circ\psi}(0,R)) &= \int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} \chi_{B_1^\psi}\left(\frac{x}{(\frac{1}{\gamma}\log(1-R^2))^{1/\alpha}}, \frac{y}{(\frac{1}{\gamma}\log(1-R^2))^{1/\beta}}\right) dy\,dx \\
&= (\tfrac{1}{\gamma}\log(1-R^2))^{\frac{n_1}{\alpha}}(\tfrac{1}{\gamma}\log(1-R^2))^{\frac{n_2}{\beta}} \int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} \chi_{B_1^\psi}(\xi,\eta)\,d\eta\,d\xi \\
&= (\tfrac{1}{\gamma}\log(1-R^2))^{\frac{n_1}{\alpha}+\frac{n_2}{\beta}}\,\lambda^{(n)}(B^\psi(0,1)),
\end{aligned}$$

and similarly for a radius r with $0 < r < 1$,

$$\lambda^{(n)}(B^{f\circ\psi}(0,r)) = (\tfrac{1}{\gamma}\log(1-r^2))^{\frac{n_1}{\alpha}+\frac{n_2}{\beta}}\,\lambda^{(n)}(B^\psi(0,1)).$$

Therefore, we obtain the doubling constant

$$\lambda^{(n)}(B^{f\circ\psi}(0,R)) = \left(\frac{\log(1-R^2)}{\log(1-r^2)}\right)^{\frac{n_1}{\alpha}+\frac{n_2}{\beta}} \lambda^{(n)}(B^{f\circ\psi}(0,r))$$

for radii $0 < r < R < 1$.

Example A.10. Consider the Bernstein function $h(s)$ defined by $h(s) := f(s^\delta) = 1-e^{-\gamma s^\delta}$ for $\gamma > 0$ and $0 < \delta < 1$. Further , let $\psi : \mathbb{R}^{n_1}\times\mathbb{R}^{n_2} \to \mathbb{R}$ be the continuous negative definite function $\psi(\xi,\eta) = |\xi|^\alpha + |\eta|^\beta$ with $0 < \alpha, \beta < 2$. We define the metric

$$(h\circ\psi)^{1/2}(\xi,\eta) = \left(1-e^{-\gamma(|\xi|^\alpha+|\eta|^\beta)^\delta}\right)^{1/2}$$

and study the balls $B^{h\circ\psi}(0,R)$ of radius R, where $0 < R < 1$, centred at the origin,

$$\begin{aligned}
B^{h\circ\psi}(0,R) &= \{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : 1-e^{-\gamma(|x|^\alpha+|y|^\beta)^\delta} < R^2\} \\
&= \left\{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : \left|\frac{x}{(\frac{1}{\gamma}\log(1-R^2))^{1/\alpha\delta}}\right|^\alpha + \left|\frac{y}{(\frac{1}{\gamma}\log(1-R^2))^{1/\beta\delta}}\right|^\beta < 1\right\} \\
&= B^\psi(0,(\tfrac{1}{\gamma}\log(1-R^2))^{1/2\delta}).
\end{aligned}$$

Then we obtain for the Lebesgue measure of these balls that

$$\begin{aligned}
\lambda^{(n)}(B^{h\circ\psi}(0,R)) &= \int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} \chi_{B_1^\psi}\left(\frac{x}{(\frac{1}{\gamma}\log(1-R^2))^{1/\alpha\delta}}, \frac{y}{(\frac{1}{\gamma}\log(1-R^2))^{1/\beta\delta}}\right) dy\,dx \\
&= (\tfrac{1}{\gamma}\log(1-R^2))^{\frac{n_1}{\alpha\delta}}(\tfrac{1}{\gamma}\log(1-R^2))^{\frac{n_2}{\beta\delta}} \int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} \chi_{B_1^\psi}(\xi,\eta)\,d\eta\,d\xi \\
&= (\tfrac{1}{\gamma}\log(1-R^2))^{\frac{n_1}{\alpha\delta}+\frac{n_2}{\beta\delta}}\,\lambda^{(n)}(B^\psi(0,1)).
\end{aligned}$$

Analogously, we get for a ball of radius $0 < r < 1$ that

$$\lambda^{(n)}(B^{h\circ\psi}(0,r)) = (\tfrac{1}{\gamma}\log(1-r^2))^{\frac{n_1}{\alpha\delta}+\frac{n_2}{\beta\delta}}\,\lambda^{(n)}(B^{\psi}(0,1)).$$

Therefore, we obtain the doubling constant

$$\lambda^{(n)}(B^{h\circ\psi}(0,R)) = \left(\frac{\log(1-R^2)}{\log(1-r^2)}\right)^{\frac{n_1}{\alpha\delta}+\frac{n_2}{\beta\delta}} \lambda^{(n)}(B^{h\circ\psi}(0,r))$$

for radii that satisfy $0 < r < R < 1$.

Example A.11. We consider the Bernstein function $k(s) := f^{1/\delta}(s^\delta) = (1-e^{-\gamma s^\delta})^{1/\delta}$ for $\gamma > 0$ and $0 < |\delta| < 1$ and let $\psi : \mathbb{R}^{n_1}\times\mathbb{R}^{n_2} \to \mathbb{R}$ be the continuous negative definite function given by $\psi(\xi,\eta) = |\xi|^\alpha + |\eta|^\beta$, $0 < \alpha, \beta < 2$, with which we define the metric

$$(k\circ\psi)^{1/2}(\xi,\eta) = \left(1-e^{-\gamma(|\xi|^\alpha+|\eta|^\beta)^\delta}\right)^{1/2\delta}.$$

Then we consider the ball $B^{k\circ\psi}(0,R)$ of radius R with $0 < R < 1$ with respect to this metric centred at the origin, i.e.

$$\begin{aligned} B^{k\circ\psi}(0,R) &= \left\{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : \left(1-e^{-\gamma(|x|^\alpha+|y|^\beta)^\delta}\right)^{1/\delta} < R^2\right\} \\ &= \left\{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : \left|\frac{x}{(\frac{1}{\gamma}\log(1-R^{2\delta}))^{1/\alpha\delta}}\right|^\alpha + \left|\frac{y}{(\frac{1}{\gamma}\log(1-R^{2\delta}))^{1/\beta\delta}}\right|^\beta < 1\right\} \\ &= B^{\psi}(0,(\tfrac{1}{\gamma}\log(1+R^{2\delta}))^{1/2\delta}). \end{aligned}$$

Then we obtain

$$\begin{aligned} \lambda^{(n)}(B^{k\circ\psi}(0,R)) &= \int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} \chi_{B_1^\psi}\left(\frac{x}{(\frac{1}{\gamma}\log(1-R^{2\delta}))^{\frac{1}{\alpha\delta}}}, \frac{y}{(\frac{1}{\gamma}\log(1-R^{2\delta}))^{\frac{1}{\beta\delta}}}\right) dy\,dx \\ &= (\tfrac{1}{\gamma}\log(1-R^{2\delta}))^{\frac{n_1}{\alpha\delta}}(\tfrac{1}{\gamma}\log(1-R^{2\delta}))^{\frac{n_2}{\beta\delta}} \int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} \chi_{B_1^\psi}(\xi,\eta)\,d\eta\,d\xi \\ &= (\tfrac{1}{\gamma}\log(1-R^{2\delta}))^{\frac{n_1}{\alpha\delta}+\frac{n_2}{\beta\delta}}\,\lambda^{(n)}(B^{\psi}(0,1)). \end{aligned}$$

Analogously, we get for a ball of radius r with $0 < r < R < 1$

$$\lambda^{(n)}(B^{k\circ\psi}(0,r)) = (\tfrac{1}{\gamma}\log(1-r^{2\delta}))^{\frac{n_1}{\alpha\delta}+\frac{n_2}{\beta\delta}}\,\lambda^{(n)}(B^{\psi}(0,1)),$$

which implies the volume doubling

$$\lambda^{(n)}(B^{k\circ\psi}(0,R)) = \left(\frac{\log(1-R^{2\delta})}{\log(1-r^{2\delta})}\right)^{\frac{n_1}{\alpha\delta}+\frac{n_2}{\beta\delta}} \lambda^{(n)}(B^{k\circ\psi}(0,r))$$

provided the radii of the balls are such that $0 < r < R < 1$.

Example A.12. Let f be the Bernstein function $f(s) = \log\frac{s+\gamma}{\gamma}$ for $\gamma > 0$ and the continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{R}$ be defined by $\psi(\xi) = |\xi|^2$. Then we define the metric

$$(f\circ\psi)^{1/2}(\xi) = \left(\log\frac{|\xi|^2+\gamma}{\gamma}\right)^{1/2}, \quad \gamma > 0,$$

and consider the balls $B^{f\circ\psi}(0,R)$ of radius $R>0$ with respect to this metric, i.e.

$$\begin{aligned} B^{f\circ\psi}(0,R) &= \left\{x\in\mathbb{R}^n : \log\frac{|x|^2+\gamma}{\gamma} < R^2\right\} \\ &= \left\{x\in\mathbb{R}^n : \left|\frac{x}{(\gamma e^{R^2}-\gamma)^{1/2}}\right|^2 < 1\right\} \;=\; B^{\psi}(0,(\gamma e^{R^2}-\gamma)^{1/2}). \end{aligned}$$

Setting $\xi := \gamma e^{R^2}-\gamma$ we get

$$\lambda^{(n)}(B^{f\circ\psi}(0,R)) = \int_{\mathbb{R}^n} \chi_{B_1^\psi}\left(\frac{x}{(\gamma e^{R^2}-\gamma)^{1/2}}\right)dx = (\gamma e^{R^2}-\gamma)^{n/2}\,\lambda^{(n)}(B^{\psi}(0,1)),$$

and similarly for radii $0<r<R$

$$\lambda^{(n)}(B^{f\circ\psi}(0,r)) = (\gamma e^{r^2}-\gamma)^{n/2}\,\lambda^{(n)}(B^{\psi}(0,1)),$$

which implies the doubling constant

$$\lambda^{(n)}(B^{f\circ\psi}(0,R)) = \left(\frac{e^{R^2}-1}{e^{r^2}-1}\right)^{\frac{n}{2}} \lambda^{(n)}(B^{f\circ\psi}(0,r))$$

for any positive radii $0<r<R$.

Example A.13. Considering the Bernstein function $f(s)=\log\frac{s+\gamma}{\gamma}$, $\gamma>0$, and $(\xi,\eta)\mapsto\psi(\xi,\eta) = |\xi|^\alpha+|\eta|^\beta$, where $0<\alpha,\ \beta<2$ and $\psi:\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to\mathbb{R}$, $n_1+n_2=n$, we define the metric

$$(f\circ\psi)^{1/2}(\xi,\eta) = \left(\log\frac{|\xi|^\alpha+|\eta|^\beta+\gamma}{\gamma}\right)^{1/2}.$$

Then the balls $B^{f\circ\psi}(0,R)$ of radius $R>0$ with respect to this metric are given by

$$\begin{aligned} B^{f\circ\psi}(0,R) &= \left\{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : \log\frac{|x|^\alpha+|y|^\beta+\gamma}{\gamma} < R^2\right\} \\ &= \left\{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : \left|\frac{x}{(\gamma e^{R^2}-\gamma)^{1/\alpha}}\right|^\alpha + \left|\frac{y}{(\gamma e^{R^2}-\gamma)^{1/\beta}}\right|^\beta < 1\right\} \\ &= B^{\psi}(0,(\gamma e^{R^2}-\gamma)^{1/2}). \end{aligned}$$

and we obtain by setting $\xi := \frac{x}{(\gamma e^{R^2}-\gamma)^{1/\alpha}}$ and $\eta := \frac{y}{(\gamma e^{R^2}-\gamma)^{1/\alpha}}$,

$$\begin{aligned} \lambda^{(n)}(B^{f\circ\psi}(0,R)) &= \int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} \chi_{B_1^\psi}\left(\frac{x}{(\gamma e^{R^2}-\gamma)^{1/\alpha}},\frac{y}{(\gamma e^{R^2}-\gamma)^{1/\beta}}\right)dy\,dx \\ &= (\gamma e^{R^2}-\gamma)^{\frac{n_1}{\alpha}}(\gamma e^{R^2}-\gamma)^{\frac{n_2}{\beta}} \int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} \chi_{B_1^\psi}(\xi,\eta)\,d\eta\,d\xi \\ &= (\gamma e^{R^2}-\gamma)^{\frac{n_1}{\alpha}+\frac{n_2}{\beta}}\,\lambda^{(n)}(B^{\psi}(0,1)). \end{aligned}$$

Analogously, for a ball $B^{f\circ\psi}(0,r)$ with radius $0<r<R$ we get

$$\lambda^{(n)}(B^{f\circ\psi}(0,r)) = (\gamma e^{r^2}-\gamma)^{\frac{n_1}{\alpha}+\frac{n_2}{\beta}}\,\lambda^{(n)}(B^{\psi}(0,1))$$

from which we conclude that the volume doubling

$$\lambda^{(n)}(B^{f\circ\psi}(0,R)) = \left(\frac{e^{R^2}-1}{e^{r^2}-1}\right)^{\frac{n_1}{\alpha}+\frac{n_2}{\beta}} \lambda^{(n)}(B^{f\circ\psi}(0,r))$$

holds for any positive radii $0<r<R$.

Example A.14. Consider the Bernstein function $h(s) := f(s^\delta) = \log \frac{s^\delta+\gamma}{\gamma}$ for $\gamma > 0$ and $0 < \delta < 1$ and the continuous negative definite function $\psi : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$ which is given by $\psi(\xi,\eta) = |\xi|^\alpha + |\eta|^\beta$, $0 < \alpha, \beta < 2$. Then define the metric

$$(h \circ \psi)^{1/2}(\xi,\eta) = \left(\log \frac{(|\xi|^\alpha + |\eta|^\beta)^\delta + \gamma}{\gamma}\right)^{1/2}$$

and the balls of radius $R > 0$ with respect to this metric,

$$\begin{aligned} B^{h\circ\psi}(0,R) &= \left\{(x,y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : \log \frac{(|x|^\alpha + |y|^\beta)^\delta + \gamma}{\gamma} < r^2\right\} \\ &= \left\{(x,y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : \left|\frac{x}{(\gamma e^{R^2} - \gamma)^{1/\alpha\delta}}\right|^\alpha + \left|\frac{y}{(\gamma e^{R^2} - \gamma)^{1/\beta\delta}}\right|^\beta < 1\right\} \\ &= B^\psi(0, (\gamma e^{R^2} - \gamma)^{1/2\delta}). \end{aligned}$$

With a change of variable we obtain

$$\begin{aligned} \lambda^{(n)}(B^{h\circ\psi}(0,R)) &= \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_2}} \chi_{B_1^\psi}\left(\frac{x}{(\gamma e^{R^2} - \gamma)^{1/\alpha\delta}}, \frac{y}{(\gamma e^{R^2} - \gamma)^{1/\beta\delta}}\right) dy\, dx \\ &= (\gamma e^{R^2} - \gamma)^{\frac{n_1}{\alpha\delta}} (\gamma e^{R^2} - \gamma)^{\frac{n_2}{\beta\delta}} \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_2}} \chi_{B_1^\psi}(\xi,\eta)\, d\eta\, d\xi \\ &= (\gamma e^{R^2} - \gamma)^{\frac{n_1}{\alpha\delta} + \frac{n_2}{\beta\delta}} \lambda^{(n)}(B^\psi(0,1)). \end{aligned}$$

Analogously, we get for balls $B^{h\circ\psi}(0,r)$ with radius $0 < r < R$

$$\lambda^{(n)}(B^{h\circ\psi}(0,r)) = (\gamma e^{r^2} - \gamma)^{\frac{n_1}{\alpha\delta} + \frac{n_2}{\beta\delta}} \lambda^{(n)}(B^\psi(0,1))$$

and therefore the doubling constant

$$\lambda^{(n)}(B^{h\circ\psi}(0,R)) = \left(\frac{e^{R^2} - 1}{e^{r^2} - 1}\right)^{\frac{n_1}{\alpha\delta} + \frac{n_2}{\beta\delta}} \lambda^{(n)}(B^{h\circ\psi}(0,r))$$

which holds for balls of any positive radii with $0 < r < R$.

Example A.15. Consider the Bernstein function $k(s) := f^{1/\delta}(s^\delta) = \left(\log \frac{s^\delta+\gamma}{\gamma}\right)^{1/\delta}$ for $\gamma > 0$, $0 < |\delta| < 1$ and let $\psi : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$ be the continuous negative definite function defined by $\psi(\xi,\eta) = |\xi|^\alpha + |\eta|^\beta$ for $0 < \alpha, \beta < 2$. Then we define the metric by

$$(k \circ \psi)^{1/2}(\xi,\eta) = \left(\log \frac{(|\xi|^\alpha + |\eta|^\beta)^\delta + \gamma}{\gamma}\right)^{1/2\delta}$$

and consider the balls $B^{k\circ\psi}(0,R)$ of radius $R > 0$ given by

$$\begin{aligned} B^{k\circ\psi}(0,R) &= \left\{(x,y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : \left(\log \frac{(|\xi|^\alpha + |\eta|^\beta)^\delta + \gamma}{\gamma}\right)^{1/\delta} < R^2\right\} \\ &= \left\{(x,y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : \left|\frac{x}{(\gamma e^{R^{2\delta}} - \gamma)^{1/\alpha\delta}}\right|^\alpha + \left|\frac{y}{(\gamma e^{R^{2\delta}} - \gamma)^{1/\beta\delta}}\right|^\beta < 1\right\} \\ &= B^\psi(0, (\gamma e^{R^{2\delta}} - \gamma)^{1/2\delta}). \end{aligned}$$

Then, by a change of variable,

$$\begin{aligned}\lambda^{(n)}(B^{k\circ\psi}(0,R)) &= \int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} \chi_{B_1^\psi}\left(\frac{x}{(\gamma\, e^{R^{2\delta}}-\gamma)^{1/\alpha\delta}}, \frac{y}{(\gamma\, e^{R^{2\delta}}-\gamma)^{1/\beta\delta}}\right) dy\, dx \\ &= (\gamma\, e^{R^{2\delta}}-\gamma)^{\frac{n_1}{\alpha\delta}} (\gamma\, e^{R^{2\delta}}-\gamma)^{\frac{n_2}{\beta\delta}} \int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} \chi_{B_1^\psi}(\xi,\eta)\, d\eta\, d\xi \\ &= (\gamma\, e^{R^{2\delta}}-\gamma)^{\frac{n_1}{\alpha\delta}+\frac{n_2}{\beta\delta}}\, \lambda^{(n)}(B^\psi(0,1)).\end{aligned}$$

Analogously, for balls $B^{k\circ\psi}(0,r)$ with radius $0<r<R$ around the origin we obtain

$$\lambda^{(n)}(B^{k\circ\psi}(0,r)) = (\gamma\, e^{r^{2\delta}}-\gamma)^{\frac{n_1}{\alpha\delta}+\frac{n_2}{\beta\delta}}\, \lambda^{(n)}(B^\psi(0,1)),$$

which leads to the doubling constant

$$\lambda^{(n)}(B^{k\circ\psi}(0,R)) = \left(\frac{e^{R^2\delta}-1}{e^{r^{2\delta}}-1}\right)^{\frac{n_1}{\alpha\delta}+\frac{n_2}{\beta\delta}} \lambda^{(n)}(B^{k\circ\psi}(0,r))$$

that holds for any $R>r>0$.

Example A.16. Let f be the Bernstein function $f(s)=\sqrt{s}\,\log(1+\sqrt{s})$ and $\psi:\mathbb{R}^n\to\mathbb{R}$ be the continuous negative definite function $\psi(\xi)=|\xi|^2$ which gives the metric

$$(f\circ\psi)^{1/2}(\xi) = (|\xi|\,\log(1+|\xi|))^{1/2}.$$

Note that for $s>0$

$$f(s)=\sqrt{s}\,\log(1+\sqrt{s}) > \log\left(1+\frac{s}{2}\right) =: f_{\text{low}}(s),$$

where $s\mapsto f_{\text{low}}(s)$ is again a Bernstein function, and thus $(f_{\text{low}}\circ\psi)^{1/2}$ gives a metric. Then

$$(f_{\text{low}}\circ\psi)^{1/2}(x) = \left(\log\left(1+\frac{|x|^2}{2}\right)\right)^{1/2} < R$$

implies

$$\left|\frac{x}{(2e^{R^2}-2)^{1/2}}\right|^2 < 1.$$

Thus, we consider open balls of radius $R>0$, i.e.

$$\begin{aligned}B^{f\circ\psi}(0,R) &= \{x\in\mathbb{R}^n : |x|\,\log(1+|x|) < R^2\} \\ &\subset \left\{x\in\mathbb{R}^n : \log\left(1+\frac{|x|^2}{2}\right) < R^2\right\} = B^{f_{\text{low}}\circ\psi}(0,R) \\ &= \left\{x\in\mathbb{R}^n : \left|\frac{x}{(2e^{R^2}-2)^{1/2}}\right|^2 < 1\right\} = B^\psi(0,(2e^{R^2}-2)^{1/2}).\end{aligned}$$

Setting $\xi := \frac{x}{(2e^{R^2}-2)^{1/2}}$ we get

$$\begin{aligned}\lambda^{(n)}(B^{f\circ\psi}(0,R)) &= \int_{\mathbb{R}^n} \chi_{B_R^{f\circ\psi}}(x)\, dx \leqslant \int_{\mathbb{R}^n} \chi_{B_1^\psi}\left(\frac{x}{(2e^{R^2}-2)^{1/2}}\right) dx \\ &= (2e^{R^2}-2)^{n/2}\int_{\mathbb{R}^n} \chi_{B_1^\psi}(\xi)\, d\xi = (2e^{R^2}-2)^{n/2}\, \lambda^{(n)}(B^\psi(0,1)).\end{aligned}$$

On the other hand note that for $0 < s < 1$

$$f(s) = \sqrt{s}\,\log(1+\sqrt{s}) < \log(1+s) = f_{\text{up}}(s)\,,$$

where $s \mapsto f_{\text{up}}(s)$ is again a Bernstein function and thus $(f_{\text{up}} \circ \psi)^{1/2}$ gives another metric. Now choose r such that $0 < r < R$ and $(f \circ \psi)^{1/2}(x) < (f_{\text{up}} \circ \psi)^{1/2}(x) < r^2$, then we have

$$\begin{aligned} B^{f\circ\psi}(0,r) &= \{x \in \mathbb{R}^n : |x|\,\log(1+|x|) < r^2\} \\ &\supset \left\{x \in \mathbb{R}^n : \log(1+|x|^2) < r^2\right\} = B^{f_{\text{up}}\circ\psi}(0,r) \\ &= \left\{x \in \mathbb{R}^n : \left|\frac{x}{(e^{r^2}-1)^{1/2}}\right|^2 < 1\right\} = B^{\psi}(0,(e^{r^2}-1)^{1/2})\,. \end{aligned}$$

By a change of variable

$$\begin{aligned} \lambda^{(n)}(B^{f\circ\psi}(0,r)) &= \int_{\mathbb{R}^n} \chi_{B_r^{f\circ\psi}}(x)\,dx \leqslant \int_{\mathbb{R}^n} \chi_{B_1^{\psi}}\left(\frac{x}{(e^{r^2}-1)^{1/2}}\right) dx \\ &= (e^{r^2}-1)^{n/2} \int_{\mathbb{R}^n} \chi_{B_1^{\psi}}(\xi)\,d\xi = (e^{r^2}-1)^{n/2}\,\lambda^{(n)}(B^{\psi}(0,1))\,. \end{aligned}$$

With the results above we get the estimates

$$\begin{aligned} \lambda^{(n)}(B^{f\circ\psi}(0,R)) &\leqslant (2e^{R^2}-2)^{n/2}\,\lambda^{(n)}(B^{\psi}(0,1)), \\ \lambda^{(n)}(B^{f\circ\psi}(0,r)) &\geqslant (e^{r^2}-1)^{n/2}\,\lambda^{(n)}(B^{\psi}(0,1)), \quad \text{for } \psi(\xi) = |\xi|^2 < 1\,. \end{aligned}$$

It follows that

$$\lambda^{(n)}(B^{f\circ\psi}(0,R)) \leqslant \left(2\,\frac{e^{R^2}-1}{e^{r^2}-1}\right)^{n/2} \lambda^{(n)}(B^{f\circ\psi}(0,r))$$

provided that $\psi(\xi) < 1$.

Example A.17. Let f be the Bernstein function $f(s) = \sqrt{s}\,\log(1+\sqrt{s})$, $\psi : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$ be the continuous negative definite function $\psi(\xi,\eta) = |\xi|^\alpha + |\eta|^\beta$, $0 < \alpha, \beta < 2$ and thus the metric given by

$$(f\circ\psi)^{1/2}(\xi,\eta) = \left[(|\xi|^\alpha + |\eta|^\beta)^{1/2}\,\log(1+(|\xi|^\alpha+|\eta|^\beta)^{1/2})\right]^{1/2}.$$

For $s > 0$ we have $f(s) > f_{\text{low}}(s)$ with $f_{\text{low}}(s)$ as in Example A.16. Using $(f_{\text{low}} \circ \psi)^{1/2}$ as a metric we consider the balls of radius $R > 0$

$$\begin{aligned} B^{f\circ\psi}(0,R) &= \{(x,y) \in \mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : (|\xi|^\alpha+|\eta|^\beta)^{1/2}\,\log(1+(|\xi|^\alpha+|\eta|^\beta)^{1/2} < R^2\} \\ &\subset \left\{(x,y) \in \mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : \log\left(1+\frac{|x|^\alpha+|y|^\beta}{2}\right) < R^2\right\} = B^{f_{\text{low}}\circ\psi}(0,R) \\ &= \left\{(x,y) \in \mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : \left|\frac{x}{(2e^{R^2}-2)^{1/\alpha}}\right|^\alpha + \left|\frac{y}{(2e^{R^2}-2)^{1/\beta}}\right|^\beta < 1\right\} \\ &= B^{\psi}(0,(2e^{R^2}-2)^{1/2})\,. \end{aligned}$$

By the change of variable $\xi := \frac{x}{(2e^{R^2}-2)^{1/\alpha}}$ and $\eta := \frac{y}{(2e^{R^2}-2)^{1/\beta}}$ we get

$$\begin{aligned} \lambda^{(n)}(B^{f\circ\psi}(0,R)) &\leqslant \int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} \chi_{B_1^{\psi}}\left(\frac{x}{(2e^{R^2}-2)^{1/\alpha}}, \frac{y}{(2e^{R^2}-2)^{1/\beta}}\right) dy\,dx \\ &= (2e^{R^2}-2)^{n_1/\alpha}(2e^{R^2}-2)^{n_2/\beta} \int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} \chi_{B_1^{\psi}}(\xi,\eta)\,d\eta\,d\xi \\ &= (2e^{R^2}-2)^{\frac{n_1}{\alpha}+\frac{n_2}{\beta}}\,\lambda^{(n)}(B^{\psi}(0,1))\,. \end{aligned}$$

On the other hand we have $f(s) < f_{\text{up}}(s)$ for $0 < s < 1$ with $s \mapsto f_{\text{up}}(s)$ given in Example A.16. Then we choose a radius r with $0 < r < R$ such that

$$\begin{aligned} B^{f\circ\psi}(0,r) &= \{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : (|\xi|^\alpha+|\eta|^\beta)^{1/2}\log(1+(|\xi|^\alpha+|\eta|^\beta)^{1/2}) < r^2\} \\ &\subset \left\{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : \log(1+|x|^\alpha+|y|^\beta) < r^2\right\} = B^{f_{\text{up}}\circ\psi}(0,r) \\ &= \left\{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : \left|\frac{x}{(e^{r^2}-1)^{1/\alpha}}\right|^\alpha + \left|\frac{y}{(e^{r^2}-1)^{1/\beta}}\right|^\beta < 1\right\} \\ &= B^\psi(0,(e^{r^2}-1)^{1/2}). \end{aligned}$$

It follows that

$$\begin{aligned} \lambda^{(n)}(B^{f\circ\psi}(0,r)) &\geqslant \int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} \chi_{B_1^\psi}\left(\frac{x}{(e^{r^2}-1)^{1/\alpha}}, \frac{y}{(e^{r^2}-1)^{1/\beta}}\right) dy\,dx \\ &= (e^{r^2}-1)^{\frac{n_1}{\alpha}+\frac{n_2}{\beta}}\,\lambda^{(n)}(B^\psi(0,1)) \end{aligned}$$

and hence for $\psi(\xi,\eta) = |\xi|^\alpha + |\eta|^\beta < 1$, we get the volume doubling

$$\lambda^{(n)}(B^{f\circ\psi}(0,R)) \leqslant \left(2\,\frac{e^{R^2}-1}{e^{r^2}-1}\right)^{\frac{n_1}{\alpha}+\frac{n_2}{\beta}} \lambda^{(n)}(B^{f\circ\psi}(0,r)).$$

Example A.18. Let h be the Bernstein function $h(s) := f(s^\gamma) = s^{\gamma/2}\log(1+s^{\gamma/2})$, $\gamma < 1$, and $\psi : \mathbb{R}^{n_1}\times\mathbb{R}^{n_2} \to \mathbb{R}$ be the continuous negative definite function $\psi(\xi,\eta) = |\xi|^\alpha + |\eta|^\beta$, $0 < \alpha,\beta < 2$. The metric is then defined by

$$(h\circ\psi^\gamma)^{1/2}(\xi,\eta) = \left[(|\xi|^\alpha+|\eta|^\beta)^{\gamma/2}\log(1+(|\xi|^\alpha+|\eta|^\beta)^{\gamma/2})\right]^{1/2}.$$

For $s > 0$ it holds $h(s) > h_{\text{low}}(s)$ with the Bernstein function $h_{\text{low}}(s) := f_{\text{low}}(s^\gamma) := \log(1+\frac{s^\gamma}{2})$, $0 < \gamma < 1$. We use the metric

$$(h_{\text{low}}\circ\psi)^{1/2}(\xi,\eta) = \left(\log\left(1+\frac{(|\xi|^\alpha+|\eta|^\beta)^\gamma}{2}\right)\right)^{1/2}, \quad 0<\gamma<1,$$

and consider the balls of radius $R > 0$ with respect to this metric, i.e.

$$\begin{aligned} B^{h\circ\psi}(0,R) &= \{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : (|\xi|^\alpha+|\eta|^\beta)^{\gamma/2}\log(1+(|\xi|^\alpha+|\eta|^\beta)^{\gamma/2}) < R^2\} \\ &\subset \left\{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : \log\left(1+\frac{(|x|^\alpha+|y|^\beta)^\gamma}{2}\right) < R^2\right\} = B^{h_{\text{low}}\circ\psi}(0,R) \\ &= \left\{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : \left|\frac{x}{(2e^{R^2}-2)^{1/\alpha\gamma}}\right|^\alpha + \left|\frac{y}{(2e^{R^2}-2)^{1/\beta\gamma}}\right|^\beta < 1\right\} \\ &= B^\psi(0,(2e^{R^2}-2)^{1/2\gamma}). \end{aligned}$$

This yields

$$\begin{aligned} \lambda^{(n)}(B^{h\circ\psi}(0,R)) &\leqslant \int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} \chi_{B_1^\psi}\left(\frac{x}{(2e^{R^2}-2)^{1/\alpha\gamma}}, \frac{y}{(2e^{R^2}-2)^{1/\beta\gamma}}\right) dy\,dx \\ &= (2e^{R^2}-2)^{n_1/\alpha\gamma}(2e^{R^2}-2)^{n_2/\beta\gamma}\int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} \chi_{B_1^\psi}(\xi,\eta)\,d\eta\,d\xi \\ &= (2e^{R^2}-2)^{\frac{n_1}{\alpha\gamma}+\frac{n_2}{\beta\gamma}}\,\lambda^{(n)}(B^\psi(0,1)). \end{aligned}$$

On the other hand we have $h(s) < h_{\text{up}}(s)$ for $0 < s < 1$ with $h_{\text{up}}(s) := f_{\text{up}}(s^\gamma) := \log(1+s^\gamma)$, $0<\gamma<1$. Then choose a radius r such that $0 < r < R$ and

$$\begin{aligned} B^{h\circ\psi}(0,r) &= \{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : (|\xi|^\alpha+|\eta|^\beta)^{\gamma/2}\log(1+(|\xi|^\alpha+|\eta|^\beta)^{\gamma/2}) < r^2\} \\ &\subset \left\{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : \log(1+(|x|^\alpha+|y|^\beta)^\gamma) < r^2\right\} = B^{h_{\text{up}}\circ\psi}(0,r) \\ &= \left\{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : \left|\frac{x}{(e^{r^2}-1)^{1/\alpha\gamma}}\right|^\alpha + \left|\frac{y}{(e^{r^2}-1)^{1/\beta\gamma}}\right|^\beta < 1\right\} \\ &= B^\psi(0,(e^{r^2}-1)^{1/2\gamma}). \end{aligned}$$

Then by a change of variable,

$$\begin{aligned} \lambda^{(n)}(B^{h\circ\psi}(0,r)) &\geqslant \int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} \chi_{B_1^\psi}\left(\frac{x}{(e^{r^2}-1)^{1/\alpha\gamma}}, \frac{y}{(e^{r^2}-1)^{1/\beta\gamma}}\right) dy\,dx \\ &= (e^{r^2}-1)^{\frac{n_1}{\alpha\gamma}+\frac{n_2}{\beta\gamma}}\,\lambda^{(n)}(B^\psi(0,1)) \end{aligned}$$

and therefore the doubling

$$\lambda^{(n)}(B^{h\circ\psi}(0,R)) \leqslant \left(2\,\frac{e^{R^2}-1}{e^{r^2}-1}\right)^{\frac{n_1}{\alpha\gamma}+\frac{n_2}{\beta\gamma}} \lambda^{(n)}(B^{h\circ\psi}(0,r))$$

for $\psi(\xi,\eta) < 1$.

Example A.19. Let k be the Bernstein function $k(s) := f^{1/\gamma}(s^\gamma) = (s^{\gamma/2}\log(1+s^{\gamma/2}))^{1/\gamma}$, $0 < |\gamma| < 1$, and $\psi : \mathbb{R}^{n_1}\times\mathbb{R}^{n_2} \to \mathbb{R}$ be given by $\psi(\xi,\eta) = |\xi|^\alpha + |\eta|^\beta$, $0<\alpha,\beta<2$. The metric is then defined by

$$(k\circ\psi)^{1/2}(\xi,\eta) = \left[(|\xi|^\alpha+|\eta|^\beta)^{\gamma/2}\log(1+(|\xi|^\alpha+|\eta|^\beta)^{\gamma/2})\right]^{1/2\gamma}.$$

For $s>0$ we have again $k(s) > k_{\text{low}}(s)$ with the Bernstein function

$$k_{\text{low}}(s) := f_{\text{low}}^{1/\gamma}(s^\gamma) := \left(\log\left(1+\frac{s^\gamma}{2}\right)\right)^{1/\gamma}, \quad 0<|\gamma|<1.$$

We use the metric

$$(k_{\text{low}}\circ\psi)^{1/2}(\xi,\eta) = \left(\log\left(1+\frac{(|\xi|^\alpha+|\eta|^\beta)^\gamma}{2}\right)\right)^{1/2\gamma}, \quad 0<|\gamma|<1,$$

and consider the balls of radius $R>0$ given by

$$\begin{aligned} B^{k\circ\psi}(0,R) &= \{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : (|\xi|^\alpha+|\eta|^\beta)^{\gamma/2}\log(1+(|\xi|^\alpha+|\eta|^\beta)^{\gamma/2}) < R^{2\gamma}\} \\ &\subset \left\{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : \log\left(1+\frac{(|x|^\alpha+|y|^\beta)^\gamma}{2}\right) < R^{2\gamma}\right\} = B^{k_{\text{low}}\circ\psi}(0,R) \\ &= \left\{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : \left|\frac{x}{(2e^{R^{2\gamma}}-2)^{1/\alpha\gamma}}\right|^\alpha + \left|\frac{y}{(2e^{R^{2\gamma}}-2)^{1/\beta\gamma}}\right|^\beta < 1\right\} \\ &= B^\psi(0,(2e^{R^{2\gamma}}-2)^{1/2\gamma}). \end{aligned}$$

With a change of variable this gives

$$\begin{aligned} \lambda^{(n)}(B^{k\circ\psi}(0,R)) &\leqslant \int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} \chi_{B_1^\psi}\left(\frac{x}{(2e^{R^{2\gamma}}-2)^{1/\alpha\gamma}}, \frac{y}{(2e^{R^{2\gamma}}-2)^{1/\beta\gamma}}\right) dy\,dx \\ &= (2e^{R^{2\gamma}}-2)^{\frac{n_1}{\alpha\gamma}+\frac{n_2}{\beta\gamma}}\,\lambda^{(n)}(B^\psi(0,1)). \end{aligned}$$

On the other hand we have $k(s) < k_{\text{up}}(s)$ for $0 < s < 1$ with

$$k_{\text{up}}(s) := f_{\text{up}}^{1/\gamma}(s^\gamma) := (\log(1+s^\gamma))^{1/\gamma}, \quad 0 < |\gamma| < 1 .$$

Then choose the radius r such that $0 < r < R$ such that

$$\begin{aligned} B^{k\circ\psi}(0,r) &= \{(x,y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : (|\xi|^\alpha + |\eta|^\beta)^{\gamma/2} \log(1 + (|\xi|^\alpha + |\eta|^\beta)^{\gamma/2}) < r^{2\gamma}\} \\ &\subset \left\{(x,y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : \log(1 + (|x|^\alpha + |y|^\beta)^\gamma) < r^{2\gamma}\right\} = B^{k_{\text{up}}\circ\psi}(0,r) \\ &= \left\{(x,y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : \left|\frac{x}{(e^{r^{2\gamma}}-1)^{1/\alpha\gamma}}\right|^\alpha + \left|\frac{y}{(e^{r^{2\gamma}}-1)^{1/\beta\gamma}}\right|^\beta < 1\right\} \\ &= B^\psi(0, (e^{r^{2\gamma}}-1)^{1/2\gamma}) . \end{aligned}$$

By a change of variable we obtain

$$\begin{aligned} \lambda^{(n)}(B^{k\circ\psi}(0,r)) &\geqslant \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_2}} \chi_{B_1^\psi}\left(\frac{x}{(e^{r^{2\gamma}}-1)^{1/\alpha\gamma}}, \frac{y}{(e^{r^{2\gamma}}-1)^{1/\beta\gamma}}\right) dy\, dx \\ &= (e^{r^{2\gamma}}-1)^{\frac{n_1}{\alpha\gamma}+\frac{n_2}{\beta\gamma}} \lambda^{(n)}(B^\psi(0,1)) , \end{aligned}$$

and therefore the doubling constant

$$\lambda^{(n)}(B^{k\circ\psi}(0,R)) \leqslant \left(2\,\frac{e^{R^{2\gamma}}-1}{e^{r^{2\gamma}}-1}\right)^{\frac{n_1}{\alpha\gamma}+\frac{n_2}{\beta\gamma}} \lambda^{(n)}(B^{k\circ\psi}(0,r))$$

for $\gamma > 0$ provided that $\psi(\xi,\eta) < 1$.

Example A.20. Let f be the Bernstein function $f(s) = \log\left(\frac{\gamma}{\delta}\,\frac{s+\delta}{s+\gamma}\right)$ for $\gamma > \delta > 0$ and let $\psi : \mathbb{R}^n \to \mathbb{R}$ be the continuous negative definite function $\psi(\xi) = |\xi|^2$ with which we define the metric

$$(f\circ\psi)^{1/2}(\xi) = \left[\log\left(\frac{\gamma}{\delta}\,\frac{|\xi|^2+\delta}{|\xi|^2+\gamma}\right)\right]^{1/2}$$

and the ball $B^{f\circ\psi}(0,R)$ of radius R with respect to this metric. We have to restrict ourselves to $0 < R < (\log(\gamma/\delta))^{1/2}$. Note that

$$\log\left(\frac{\gamma}{\delta}\,\frac{|x|^2+\delta}{|x|^2+\gamma}\right) < R^2 \Leftrightarrow |x|^2 < \frac{\delta\, e^{R^2}-\delta}{1-\frac{\delta}{\gamma}\, e^{R^2}} =: \rho(R) .$$

Thus,

$$B^{f\circ\psi}(0,R) = \left\{x \in \mathbb{R}^n : \log\left(\frac{\gamma}{\delta}\,\frac{|x|^2+\delta}{|x|^2+\gamma}\right) < R^2\right\} = \left\{x \in \mathbb{R}^n : \left|\frac{x}{\rho(R)^{1/2}}\right|^2 < 1\right\} = B^\psi(0,\rho(R)^{1/2}) ,$$

and using the change of variable

$$\xi := \frac{x}{\rho(R)^{1/2}}, \quad d\xi = \frac{dx}{\rho(R)^{1/2}} = \left(\frac{\delta\, e^{R^2}-\delta}{1-\frac{\delta}{\gamma}\, e^{R^2}}\right)^{2/n} dx$$

we get

$$\begin{aligned} \lambda^{(n)}(B^{f\circ\psi}(0,R)) &= \int_{\mathbb{R}^n} \chi_{B_R^{f\circ\psi}}(x)\, dx = \int_{\mathbb{R}^n} \chi_{B_1^\psi}\left(\frac{x}{\rho(R)^{1/2}}\right) dx \\ &= \rho(R)^{n/2} \int_{\mathbb{R}^n} \chi_{B_1^\psi}(\xi)\, d\xi = \rho(R)^{n/2} \lambda^{(n)}(B^\psi(0,1)) . \end{aligned}$$

Analogously, for the ball $B^{f\circ\psi}(0,r)$ with radius $0<r<R$ we get

$$\lambda^{(n)}(B^{f\circ\psi}(0,r)) = \rho(r)^{n/2}\,\lambda^{(n)}(B^{\psi}(0,1))$$

and hence

$$\lambda^{(n)}(B^{f\circ\psi}(0,R)) = \left(\frac{\rho(R)}{\rho(r)}\right)^{n/2}\lambda^{(n)}(B^{f\circ\psi}(0,r)),$$

provided that the radii of the balls satisfy $0<r<R<(\log(\gamma/\delta))^{1/2}$, where the doubling constant is

$$\frac{\rho(R)}{\rho(r)} = \frac{(\delta\, e^{R^2}-\delta)(1-\frac{\delta}{\gamma}\,e^{r^2})}{(\delta\, e^{r^2}-\delta)(1-\frac{\delta}{\gamma}\,e^{R^2})} = \frac{(\gamma-\delta\, e^{r^2})(1-e^{R^2})}{(\gamma-\delta\, e^{R^2})(1-e^{r^2})}.$$

Example A.21. Let f be the Bernstein function $f(s)=\log\left(\frac{\gamma}{\delta}\frac{s+\delta}{s+\gamma}\right)$, $\gamma>\delta>0$, $\psi:\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to\mathbb{R}$ be the continuous negative definite function $\psi(\xi,\eta)=|\xi|^\alpha+|\eta|^\beta$ with which we define the metric

$$(f\circ\psi)^{1/2}(\xi,\eta) = \left[\log\left(\frac{\gamma}{\delta}\,\frac{|\xi|^\alpha+|\eta|^\beta+\delta}{|\xi|^\alpha+|\eta|^\beta+\gamma}\right)\right]^{1/2}.$$

We consider the metric ball $B^{f\circ\psi}(0,R)$ of radius $R>0$ about the origin. Then, similarly to Example A.19 use the equivalence

$$\log\left(\frac{\gamma}{\delta}\,\frac{|x|^\alpha+|y|^\beta+\delta}{|x|^\alpha+|y|^\beta+\gamma}\right)<R^2 \;\Leftrightarrow\; |x|^\alpha+|y|^\beta<\frac{\delta\,e^{R^2}-\delta}{1-\frac{\delta}{\gamma}\,e^{R^2}}=\rho(R)$$

for $0<R<(\log\frac{\gamma}{\delta})^{1/2}$ to express $B^{f\circ\psi}(0,R)$ as

$$B^{f\circ\psi}(0,R)=\left\{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}:\left|\frac{x}{\rho(R)^{1/\alpha}}\right|^\alpha+\left|\frac{y}{\rho(R)^{1/\beta}}\right|^\beta<1\right\}=B^{\psi}(0,\rho(R)^{1/2}).$$

Employing the change of variable

$$\xi:=\frac{x}{\rho(R)^{1/\alpha}},\quad \eta:=\frac{y}{\rho(R)^{1/\beta}},$$

we obtain

$$\begin{aligned}\lambda^{(n)}(B^{f\circ\psi}(0,R)) &= \int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}}\chi_{B_1^\psi}\left(\frac{x}{\rho(R)^{1/\alpha}},\frac{y}{\rho(R)^{1/\beta}}\right)dy\,dx\\ &= \rho(R)^{n_1/\alpha}\rho(R)^{n_2/\beta}\int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}}\chi_{B_1^\psi}(\xi,\eta)\,d\eta\,d\xi\\ &= \rho(R)^{\frac{n_1}{\alpha}+\frac{n_2}{\beta}}\,\lambda^{(n)}(B^\psi(0,1)).\end{aligned}$$

Analogously, for a ball with respect to $(f\circ\psi)^{1/2}$ of radius r with $0<r<R$ we get

$$\lambda^{(n)}(B^{f\circ\psi}(0,r))=\rho(r)^{\frac{n_1}{\alpha}+\frac{n_2}{\beta}}\,\lambda^{(n)}(B^\psi(0,1)),$$

which implies

$$\lambda^{(n)}(B^{f\circ\psi}(0,R))=\left(\frac{\rho(R)}{\rho(r)}\right)^{\frac{n_1}{\alpha}+\frac{n_2}{\beta}}\lambda^{(n)}(B^{f\circ\psi}(0,r)),$$

where the doubling constant $\frac{\rho(R)}{\rho(r)}$ is as in Example A.20.

Example A.22. Let h be the Bernstein function $h(s) := f(s^\zeta) = \log\left(\frac{\gamma}{\delta}\,\frac{s^\zeta+\delta}{s^\zeta+\gamma}\right)$, for $\gamma > \delta > 0$ and $0 < \zeta < 1$. Further, let ψ be the continuous negative definite function as given in Example A.21, i.e. $\psi(\xi,\eta) = |\xi|^\alpha + |\eta|^\beta$, which we use to define the metric

$$(h\circ\psi)^{1/2}(\xi,\eta) = \left[\log\left(\frac{\gamma}{\delta}\,\frac{(|\xi|^\alpha+|\eta|^\beta)^\zeta+\delta}{(|\xi|^\alpha+|\eta|^\beta)^\zeta+\gamma}\right)\right]^{1/2}.$$

Let $B^{h\circ\psi}(0,R)$ be the ball with respect to $(h\circ\psi)^{1/2}$ of radius R with $0 < R < (\log\frac{\gamma}{\delta})^{1/2}$ about the origin. Note that

$$\log\left(\frac{\gamma}{\delta}\,\frac{(|x|^\alpha+|y|^\beta)^\zeta+\delta}{(|x|^\alpha+|y|^\beta)^\zeta+\gamma}\right) < R^2 \;\Leftrightarrow\; |x|^\alpha+|y|^\beta < \left(\frac{\delta\, e^{R^2}-\delta}{1-\frac{\delta}{\gamma}\,e^{R^2}}\right)^{1/\zeta} =: \rho(R)^{1/\zeta}.$$

Then,

$$B^{h\circ\psi}(0,R) = \left\{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : \left|\frac{x}{\rho(R)^{1/\alpha\zeta}}\right|^\alpha + \left|\frac{y}{\rho(R)^{1/\beta\zeta}}\right|^\beta < 1\right\} = B^\psi(0,\rho(R)^{1/2\rho})$$

and by a change of variable

$$\begin{aligned}\lambda^{(n)}(B^{h\circ\psi}(0,R)) &= \int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} \chi_{B_1^\psi}\left(\frac{x}{\rho(R)^{1/\alpha\zeta}},\frac{y}{\rho(R)^{1/\beta\zeta}}\right)dy\,dx\\ &= \rho(R)^{n_1/\alpha\zeta}\,\rho(R)^{n_2/\beta\zeta}\int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}}\chi_{B_1^\psi}(\xi,\eta)\,d\eta\,d\xi\\ &= \rho(R)^{\frac{n_1}{\alpha\zeta}+\frac{n_2}{\beta\zeta}}\,\lambda^{(n)}(B_1^\psi(0,1))\end{aligned}$$

and analogously, for a radius r with $0 < r < R$ we get

$$\lambda^{(n)}(B^{h\circ\psi}(0,r)) = \rho(r)^{\frac{n_1}{\alpha\zeta}+\frac{n_2}{\beta\zeta}}\,\lambda^{(n)}(B^\psi(0,1)),$$

hence

$$\lambda^{(n)}(B^{h\circ\psi}(0,R)) = \left(\frac{\rho(R)}{\rho(r)}\right)^{\frac{n_1}{\alpha\zeta}+\frac{n_2}{\beta\zeta}}\lambda^{(n)}(B^{h\circ\psi}(0,r))$$

with $\frac{\rho(R)}{\rho(r)}$ as in Example A.20.

Example A.23. Let k be the Bernstein function $k(s) := f^{1/\zeta}(s^\zeta) = \left(\log\left(\frac{\gamma}{\delta}\,\frac{s^\zeta+\delta}{s^\zeta+\gamma}\right)\right)^{1/\zeta}$ for $\gamma > \delta > 0$ and $0 < |\zeta| < 1$ and $\psi : \mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to\mathbb{R}$ be the continuous negative definite function $\psi(\xi,\eta) = |\xi|^\alpha + |\eta|^\beta$. Then we define the distance

$$(k\circ\psi)^{1/2}(\xi,\eta) = \left[\log\left(\frac{\gamma}{\delta}\,\frac{(|\xi|^\alpha+|\eta|^\beta)^\zeta+\delta}{(|\xi|^\alpha+|\eta|^\beta)^\zeta+\gamma}\right)\right]^{1/2\zeta}$$

and consider the balls with respect to the metric of radius R about the origin, where we have to restrict ourselves to $0 < R < (\log\frac{\gamma}{\delta})^{1/2\zeta}$. We have

$$\log\left(\frac{\gamma}{\delta}\,\frac{(|x|^\alpha+|y|^\beta)^\zeta+\delta}{(|x|^\alpha+|y|^\beta)^\zeta+\gamma}\right) < R^{2\zeta} \;\Leftrightarrow\; |x|^\alpha+|y|^\beta < \left(\frac{\delta\, e^{R^{2\zeta}}-\delta}{1-\frac{\delta}{\gamma}\,e^{R^{2\zeta}}}\right)^{1/\zeta} =: \rho(R,\zeta)^{1/\zeta}$$

and hence

$$\begin{aligned}B^{k\circ\psi}(0,R) &= \{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : (k\circ\psi)^{1/2}(x,y) < R\}\\ &= \left\{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : \left|\frac{x}{\rho(R,\zeta)^{1/\alpha\zeta}}\right|^\alpha + \left|\frac{y}{\rho(R,\zeta)^{1/\beta\zeta}}\right|^\beta < 1\right\} = B^\psi(0,\rho(R,\zeta)^{1/2\zeta}).\end{aligned}$$

By a change of variable

$$\begin{aligned}\lambda^{(n)}(B^{k\circ\psi}(0,R)) &= \int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} \chi_{B_1^\psi}\Big(\frac{x}{\rho(R,\zeta)^{1/\alpha\zeta}}, \frac{y}{\rho(R,\zeta)^{1/\beta\zeta}}\Big)\,dy\,dx \\ &= \rho(R,\zeta)^{n_1/\alpha\zeta}\,\rho(R,\zeta)^{n_2/\beta\zeta} \int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} \chi_{B_1^\psi}(\xi,\eta)\,d\eta\,d\xi \\ &= \rho(R,\zeta)^{\frac{n_1}{\alpha\zeta}+\frac{n_2}{\beta\zeta}}\,\lambda^{(n)}(B^\psi(0,1))\end{aligned}$$

and analogously, for a radius r with $0<r<R$,

$$\lambda^{(n)}(B^{k\circ\psi}(0,r)) = \rho(r,\zeta)^{\frac{n_1}{\alpha\zeta}+\frac{n_2}{\beta\zeta}}\,\lambda^{(n)}(B^\psi(0,1)).$$

Hence

$$\lambda^{(n)}(B^{k\circ\psi}(0,R)) = \Big(\frac{\rho(R,\zeta)}{\rho(r,\zeta)}\Big)^{\frac{n_1}{\alpha\zeta}+\frac{n_2}{\beta\zeta}}\,\lambda^{(n)}(B^{k\circ\psi}(0,r)),$$

where

$$\frac{\rho(R,\zeta)}{\rho(r,\zeta)} = \frac{(\gamma-\delta\, e^{r^{2\zeta}})(1-e^{R^{2\zeta}})}{(\gamma-\delta\, e^{R^{2\zeta}})(1-e^{r^{2\zeta}})}, \qquad 0<r<R<(\log\tfrac{\gamma}{\delta})^{1/2\zeta}.$$

Example A.24. Consider the Bernstein function $f(s)=\sqrt{s+m^2}-m$ for $m\geqslant 0$ and the continuous negative definite function $\psi:\mathbb{R}^n\to\mathbb{R}$, $\psi(\xi)=|\xi|^2$, with which we define the metric

$$(f\circ\psi)^{1/2}(\xi) = (\sqrt{|\xi|^2+m^2}-m)^{1/2}.$$

We consider the ball

$$\begin{aligned}B^{f\circ\psi}(0,R) &= \{x\in\mathbb{R}^n : \sqrt{|x|^2+m^2}-m<R^2\} \\ &= \Big\{x\in\mathbb{R}^n : \Big|\frac{x}{(R^4+2mR^2)^{1/2}}\Big|^2<1\Big\} \;=\; B^\psi(0,(R^4+2mR^2)^{1/2}).\end{aligned}$$

By a change of variable we obtain the volume

$$\lambda^{(n)}(B^{f\circ\psi}(0,R)) = \int_{\mathbb{R}^n}\chi_{B_1^\psi}\Big(\frac{x}{(R^4+2mR^2)^{1/2}}\Big)\,dx = (R^4+2mR^2)^{n/2}\,\lambda^{(n)}(B^\psi(0,1)).$$

Similarly, for balls of radius r with $0<r<R$

$$\lambda^{(n)}(B^{f\circ\psi}(0,r)) = (r^4+2mr^2)^{n/2}\,\lambda^{(n)}(B^\psi(0,1)),$$

hence

$$\lambda^{(n)}(B^{f\circ\psi}(0,R)) = \Big(\frac{R^4+2mR^2}{r^4+2mr^2}\Big)^{n/2}\,\lambda^{(n)}(B^{f\circ\psi}(0,r)).$$

Example A.25. Consider the Bernstein function $f(s)=(s+m^{2/\alpha})^{\alpha/2}-m$, $0<\alpha\leqslant 1$, $m>0$, and the continuous negative definite function $\psi:\mathbb{R}^n\to\mathbb{R}$, $\psi(\xi)=|\xi|^2$. Define the metric

$$(f\circ\psi)^{1/2}(\xi) = \Big((|\xi|^2+m^{2/\alpha})^{\alpha/2}-m\Big)^{1/2}$$

and consider the ball

$$\begin{aligned} B^{f\circ\psi}(0,R) &= \{x\in\mathbb{R}^n : (|x|^2+m^{2/\alpha})^{\alpha/2}-m<R^2\} \\ &= \{x\in\mathbb{R}^n : |x|^2+m^{2/\alpha}<(R^2+m)^{2/\alpha}\} \\ &= \left\{x\in\mathbb{R}^n : \left|\frac{x}{((R^2+m)^{2/\alpha}-m^{2/\alpha})^{1/2}}\right|^2<1\right\} \\ &= B^{\psi}(0,((R^2+m)^{2/\alpha}-m^{2/\alpha})^{1/2}). \end{aligned}$$

By a change of variable we obtain the Lebesgue volume

$$\begin{aligned} \lambda^{(n)}(B^{f\circ\psi}(0,R)) &= \int_{\mathbb{R}^n}\chi_{B_1^{\psi}}\Big(\frac{x}{((R^2+m)^{2/\alpha}-m^{2/\alpha})^{1/2}}\Big)\,dx \\ &= ((R^2+m)^{2/\alpha}-m^{2/\alpha})^{n/2}\int_{\mathbb{R}^n}\chi_{B_1^{\psi}}(\xi)\,d\xi \\ &= ((R^2+m)^{2/\alpha}-m^{2/\alpha})^{n/2}\,\lambda^{(n)}(B^{\psi}(0,1)). \end{aligned}$$

Similarly, for any ball with radius r, $0<r<R$,

$$\lambda^{(n)}(B^{f\circ\psi}(0,r))=((r^2+m)^{2/\alpha}-m^{2/\alpha})^{n/2}\,\lambda^{(n)}(B^{\psi}(0,1)),$$

which gives the general doubling constant

$$\lambda^{(n)}(B^{f\circ\psi}(0,R))=\left(\frac{(R^2+m)^{2/\alpha}-m^{2/\alpha}}{(r^2+m)^{2/\alpha}-m^{2/\alpha}}\right)^{\frac{n}{2}}\lambda^{(n)}(B^{f\circ\psi}(0,r))$$

for balls with radii such that $0<r<R$.

Example A.26. We consider the Bernstein function $f(s)=\sqrt{s+m^2}-m$ for $m\geqslant 0$ and $\psi:\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to\mathbb{R}$, $\psi(\xi,\eta)=|\xi|^{\alpha}+|\eta|^{\beta}$, $0<\alpha,\beta<2$, and define the metric

$$(f\circ\psi)^{1/2}(\xi,\eta)=(\sqrt{|\xi|^{\alpha}+|\eta|^{\beta}+m^2}-m)^{1/2}.$$

Then it holds for the balls of radius $R>0$ with respect to this metric

$$\begin{aligned} B^{f\circ\psi}(0,R) &= \{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : \sqrt{|x|^{\alpha}+|y|^{\beta}+m^2}-m<R^2\} \\ &= \left\{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : \left|\frac{x}{(R^4+2mR^2)^{1/\alpha}}\right|^{\alpha}+\left|\frac{x}{(R^4+2mR^2)^{1/\beta}}\right|^{\beta}<1\right\} \\ &= B^{\psi}(0,(R^4+2mR^2)^{1/2}). \end{aligned}$$

Employing a change of variable gives

$$\begin{aligned} \lambda^{(n)}(B^{f\circ\psi}(0,R)) &= \int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}}\chi_{B_1^{\psi}}\Big(\frac{x}{(R^4+2mR^2)^{1/\alpha}},\frac{y}{(R^4+2mR^2)^{1/\beta}}\Big)\,dy\,dx \\ &= (R^4+2mR^2)^{n_1/\alpha}(R^4+2mR^2)^{n_2/\beta}\int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}}\chi_{B_1^{\psi}}(\xi,\eta)\,d\eta\,d\xi \\ &= (R^4+2mR^2)^{\frac{n_1}{\alpha}+\frac{n_2}{\beta}}\,\lambda^{(n)}(B^{\psi}(0,1)). \end{aligned}$$

Analogously,

$$\lambda^{(n)}(B^{f\circ\psi}(0,r)) = (r^4+2mr^2)^{\frac{n_1}{\alpha}+\frac{n_2}{\beta}}\,\lambda^{(n)}(B^{\psi}(0,1))$$

for $0<r<R$, which implies

$$\lambda^{(n)}(B^{f\circ\psi}(0,R)) = \left(\frac{R^4+2mR^2}{r^4+2mr^2}\right)^{\frac{n_1}{\alpha}+\frac{n_2}{\beta}}\lambda^{(n)}(B^{f\circ\psi}(0,r))\,.$$

Example A.27. We consider the Bernstein function $h(s) := f(s^\gamma) = \sqrt{s^\gamma+m^2}-m$ for $m\geqslant 0$, $0<\gamma<1$, and let $\psi:\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to\mathbb{R}$ be the continuous negative definite function $\psi(\xi,\eta) = |\xi|^\alpha+|\eta|^\beta$, $0<\alpha,\beta<2$, with which we define the metric

$$(h\circ\psi)^{1/2}(\xi,\eta) = \left(\sqrt{(|\xi|^\alpha+|\eta|^\beta)^\gamma+m^2}-m\right)^{1/2}.$$

Then for the metric ball of radius $R>0$ with respect to this metric we get

$$\begin{aligned} B^{h\circ\psi}(0,R) &= \{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : \sqrt{(|x|^\alpha+|y|^\beta)^\gamma+m^2}-m<R^2\} \\ &= \left\{x,y\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : \left|\frac{x}{(R^4+2mR^2)^{1/\alpha\gamma}}\right|^\alpha+\left|\frac{x}{(R^4+2mR^2)^{1/\beta\gamma}}\right|^\beta<1\right\} \\ &= B^\psi(0,(R^4+2mR^2)^{1/2\gamma})\,, \end{aligned}$$

and therefore by a change of variable

$$\begin{aligned} \lambda^{(n)}(B^{h\circ\psi}(0,R)) &= \int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}}\chi_{B_1^\psi}\Big(\frac{x}{(R^4+2mR^2)^{1/\alpha\gamma}},\frac{y}{(R^4+2mR^2)^{1/\beta\gamma}}\Big)\,dy\,dx \\ &= (R^4+2mR^2)^{n_1/\alpha\gamma}(R^4+2mR^2)^{n_2/\beta\gamma}\int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}}\chi_{B_1^\psi}(\xi,\eta)\,d\eta\,d\xi \\ &= (R^4+2mR^2)^{\frac{n_1}{\alpha\gamma}+\frac{n_2}{\beta\gamma}}\,\lambda^{(n)}(B^\psi(0,1))\,. \end{aligned}$$

Analogously,

$$\lambda^{(n)}(B^{h\circ\psi}(0,r)) = (r^4+2mr^2)^{\frac{n_1}{\alpha\gamma}+\frac{n_2}{\beta\gamma}}\,\lambda^{(n)}(B^\psi(0,1))$$

for a ball of radius r, such that $0<r<R$. Hence we get the volume doubling

$$\lambda^{(n)}(B^{h\circ\psi}(0,R)) = \left(\frac{R^4+2mR^2}{r^4+2mr^2}\right)^{\frac{n_1}{\alpha\gamma}+\frac{n_2}{\beta\gamma}}\lambda^{(n)}(B^{h\circ\psi}(0,r))\,.$$

Example A.28. Let k be the Bernstein function $k(s) := f^{1/\gamma}(s^\gamma) = (\sqrt{s^\gamma+m^2}-m)^{1/\gamma}$ for $m\geqslant 0$, $0<|\gamma|<1$, and ψ be the continuous negative definite function on $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ given by $\psi(\xi,\eta) = |\xi|^\alpha+|\eta|^\beta$, $0<\alpha,\beta<2$. We define the metric

$$(k\circ\psi)^{1/2}(\xi,\eta) = \left(\sqrt{(|\xi|^\alpha+|\eta|^\beta)^\gamma+m^2}-m\right)^{1/2\gamma}$$

and note that

$$\begin{aligned} B^{k\circ\psi}(0,R) &= \{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : \sqrt{(|x|^\alpha+|y|^\beta)^\gamma+m^2}-m<R^{2\gamma}\} \\ &= \left\{x,y\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : \left|\frac{x}{(R^{4\gamma}+2mR^{2\gamma})^{1/\alpha\gamma}}\right|^\alpha+\left|\frac{x}{(R^{4\gamma}+2mR^{2\gamma})^{1/\beta\gamma}}\right|^\beta<1\right\} \\ &= B^\psi(0,(R^{4\gamma}+2mR^{2\gamma})^{1/2\gamma})\,, \end{aligned}$$

By the change of variable $\xi := x(R^{4\gamma}+2mR^{2\gamma})^{-1/\alpha_\gamma}$ and $\eta := y(R^{4\gamma}+2mR^{2\gamma})^{-1/\beta_\gamma}$ we obtain

$$\begin{aligned}\lambda^{(n)}(B^{k\circ\psi}(0,R)) &= \int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} \chi_{B_1^\psi}\Big(\frac{x}{(R^{4\gamma}+2mR^{2\gamma})^{1/\alpha_\gamma}}, \frac{y}{(R^{4\gamma}+2mR^{2\gamma})^{1/\beta_\gamma}}\Big)\,dy\,dx \\ &= (R^{4\gamma}+2mR^{2\gamma})^{n_1/\alpha_\gamma}(R^{4\gamma}+2mR^{2\gamma})^{n_2/\beta_\gamma}\int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}}\chi_{B_1^\psi}(\xi,\eta)\,d\eta\,d\xi \\ &= (R^{4\gamma}+2mR^{2\gamma})^{\frac{n_1}{\alpha_\gamma}+\frac{n_2}{\beta_\gamma}}\,\lambda^{(n)}(B^\psi(0,1)).\end{aligned}$$

Analogously, for balls of radius r with $0<r<R$ we get

$$\lambda^{(n)}(B^{k\circ\psi}(0,r)) = (r^{4\gamma}+2mr^{2\gamma})^{\frac{n_1}{\alpha_\gamma}+\frac{n_2}{\beta_\gamma}}\,\lambda^{(n)}(B^\psi(0,1))$$

and therefore the volume doubling

$$\lambda^{(n)}(B^{k\circ\psi}(0,R)) = \left(\frac{R^{4\gamma}+2mR^{2\gamma}}{r^{4\gamma}+2mr^{2\gamma}}\right)^{\frac{n_1}{\alpha_\gamma}+\frac{n_2}{\beta_\gamma}}\lambda^{(n)}(B^{k\circ\psi}(0,r))$$

for $0<r<R$.

Example A.29. Let f be the Bernstein function $f(s)=\frac{s}{s+\gamma}$ for $\gamma>0$ and the continuous negative definite function $\psi:\mathbb{R}^n\to\mathbb{R}$ be given by $\psi(\xi)=|\xi|^2$. Then we consider the metric

$$(f\circ\psi)^{1/2}(\xi) = \left(\frac{|\xi|^2}{|\xi|^2+\gamma}\right)^{1/2}$$

and the ball of radius R with $0<R<1$ centred at the origin,

$$\begin{aligned}B^{f\circ\psi}(0,R) &= \Big\{x\in\mathbb{R}^n : \frac{|x|^2}{|x|^2+\gamma}<R^2\Big\} \\ &= \{x\in\mathbb{R}^n : (1-R^2)\,|x|^2<\gamma\,R^2\} \\ &= \Big\{x\in\mathbb{R}^n : \Big|x\Big(\frac{\gamma R^2}{1-R^2}\Big)^{-1/2}\Big|^2<1\Big\} = B^\psi\Big(0,\Big(\frac{\gamma R^2}{1-R^2}\Big)^{1/2}\Big).\end{aligned}$$

It follows by change of variable

$$\begin{aligned}\lambda^{(n)}(B^{f\circ\psi}(0,R)) &= \int_{\mathbb{R}^n}\chi_{B_1^\psi}\Big(x\Big(\frac{\gamma R^2}{1-R^2}\Big)^{-1/2}\Big)\,dx \\ &= \Big(\frac{\gamma R^2}{1-R^2}\Big)^{n/2}\lambda^{(n)}(B^\psi(0,1))\end{aligned}$$

and analogously

$$\lambda^{(n)}(B^{f\circ\psi}(0,r)) = \Big(\frac{\gamma r^2}{1-r^2}\Big)^{n/2}\lambda^{(n)}(B^\psi(0,1))$$

for a radius r with $0<r<R<1$. This implies

$$\begin{aligned}\lambda^{(n)}(B^{f\circ\psi}(0,R)) &= \Big(\frac{\gamma R^2(1-r^2)}{\gamma r^2(1-R^2)}\Big)^{n/2}\lambda^{(n)}(B^{f\circ\psi}(0,r)) \\ &= \Big(\frac{R}{r}\Big)^n\Big(\frac{1-r^2}{1-R^2}\Big)^{n/2}\lambda^{(n)}(B^{f\circ\psi}(0,r)),\end{aligned}$$

where the doubling constant $\big(\frac{R}{r}\big)^n\big(\frac{1-r^2}{1-R^2}\big)^{n/2}>1$.

Example A.30. Consider the Bernstein function $f(s) = \frac{s}{s+\gamma}$ for $\gamma > 0$ and the continuous negative definite function $\psi : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$ with $n_1 + n_2 = n$ given by $\psi(\xi) = |\xi|^\alpha + |\eta|^\beta$, where $0 < \alpha, \beta < 2$. Use these functions to define the metric

$$(f \circ \psi)^{1/2}(\xi, \eta) = \left(\frac{|\xi|^\alpha + |\eta|^\beta}{|\xi|^\alpha + |\eta|^\beta + \gamma} \right)^{1/2} .$$

and the balls $B^{f\circ\psi}(0, R)$ of radius $0 < R < 1$ with respect to this metric. For these we get the equality

$$\begin{aligned} B^{f\circ\psi}(0,R) &= \left\{(x,y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : \frac{|x|^\alpha + |y|^\beta}{|x|^\alpha + |y|^\beta + \gamma} < R^2\right\} \\ &= \{(x,y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : (1 - R^2)\,(|x|^\alpha + |y|^\beta) < \gamma\, R^2\} \\ &= \left\{(x,y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : \left|\frac{x}{\rho(\gamma,R)^{1/\alpha}}\right|^\alpha + \left|\frac{y}{\rho(\gamma,R)^{1/\beta}}\right|^\beta < 1\right\} \\ &= B^\psi(0, \rho(\gamma,R)^{1/2}) . \end{aligned}$$

with $\rho(\gamma, R) = \frac{\gamma R^2}{1-R^2}$. Then we get for their n-dimensional Lebesgue volume

$$\begin{aligned} \lambda^{(n)}(B^{f\circ\psi}(0,R)) &= \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_2}} \chi_{B_1^\psi}\Big(\frac{x}{\rho(\gamma,R)^{1/\alpha}}, \frac{y}{\rho(\gamma,R)^{1/\beta}}\Big)\, dy\, dx \\ &= \rho(\gamma,R)^{\frac{n_1}{\alpha}+\frac{n_2}{\beta}}\, \lambda^{(n)}(B^\psi(0,1)) \end{aligned}$$

and analogously for the balls of radius r with $0 < r < R < 1$,

$$\lambda^{(n)}(B^{f\circ\psi}(0,r)) = \rho(\gamma,r)^{\frac{n_1}{\alpha}+\frac{n_2}{\beta}}\, \lambda^{(n)}(B^\psi(0,1))$$

leading to the volume doubling

$$\begin{aligned} \lambda^{(n)}(B^{f\circ\psi}(0,R)) &= \left(\frac{\rho(\gamma,R)}{\rho(\gamma,r)}\right)^{\frac{n_1}{\alpha}+\frac{n_2}{\beta}} \lambda^{(n)}(B^{f\circ\psi}(0,r)) \\ &= \left(\frac{R}{r}\right)^{2\left(\frac{n_1}{\alpha}+\frac{n_2}{\beta}\right)} \left(\frac{1-r^2}{1-R^2}\right)^{\frac{n_1}{\alpha}+\frac{n_2}{\beta}} \lambda^{(n)}(B^{f\circ\psi}(0,r)) . \end{aligned}$$

Example A.31. Now we consider the Bernstein function $h(s) := f(s^\delta) = \frac{s^\delta}{s^\delta+\gamma}$ for $\gamma > 0$ and $0 < \delta < 1$, composed with the continuous negative definite function $\psi : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, $\psi(\xi, \eta) = |\xi|^\alpha + |\eta|^\beta$, $0 < \alpha, \beta < 2$, which gives the metric

$$(h \circ \psi)^{1/2}(\xi, \eta) = \left(\frac{(|\xi|^\alpha + |\eta|^\beta)^\delta}{(|\xi|^\alpha + |\eta|^\beta)^\delta + \gamma} \right)^{1/2} .$$

Then it holds for the open balls of radius $0 < R < 1$ in this metric that

$$\begin{aligned} B^{h\circ\psi}(0,R) &= \left\{(x,y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : \frac{(|x|^\alpha + |y|^\beta)^\delta}{(|x|^\alpha + |y|^\beta)^\delta + \gamma} < R^2\right\} \\ &= \{(x,y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : (1 - R^2)\,(|x|^\alpha + |y|^\beta)^\delta < \gamma\, R^2\} \\ &= \left\{(x,y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : \left|\frac{x}{\rho(\gamma,R)^{1/\alpha\delta}}\right|^\alpha + \left|\frac{y}{\rho(\gamma,R)^{1/\beta\delta}}\right|^\beta < 1\right\} \\ &= B^\psi(0, \rho(\gamma,R)^{1/2\delta}) , \end{aligned}$$

where $\rho(\gamma, R)$ is the same constant independent of x and y as in Example A.30. Then it follows for the Lebesgue volume

$$\begin{aligned}\lambda^{(n)}(B^{h\circ\psi}(0,R)) &= \int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} \chi_{B_1^\psi}\Big(\frac{x}{\rho(\gamma,R)^{1/\alpha\delta}}, \frac{y}{\rho(\gamma,R)^{1/\beta\delta}}\Big)\,dy\,dx \\ &= \rho(\gamma,R)^{\frac{n_1}{\alpha\delta}+\frac{n_2}{\beta\delta}}\,\lambda^{(n)}(B^\psi(0,1))\end{aligned}$$

and analogously for the balls of radius r with $0 < r < R < 1$,

$$\lambda^{(n)}(B^{h\circ\psi}(0,r)) = \rho(\gamma,r)^{\frac{n_1}{\alpha\delta}+\frac{n_2}{\beta\delta}}\,\lambda^{(n)}(B^\psi(0,1))\,.$$

This yields the volume doubling for the Lebesgue measure

$$\begin{aligned}\lambda^{(n)}(B^{h\circ\psi}(0,R)) &= \Big(\frac{\rho(\gamma,R)}{\rho(\gamma,r)}\Big)^{\frac{n_1}{\alpha\delta}+\frac{n_2}{\beta\delta}}\lambda^{(n)}(B^{h\circ\psi}(0,r)) \\ &= \Big(\frac{R}{r}\Big)^{2\left(\frac{n_1}{\alpha\delta}+\frac{n_2}{\beta\delta}\right)}\Big(\frac{1-r^2}{1-R^2}\Big)^{\frac{n_1}{\alpha\delta}+\frac{n_2}{\beta\delta}}\lambda^{(n)}(B^{h\circ\psi}(0,r))\end{aligned}$$

in the metric measure space $(\mathbb{R}^n, (h\circ\psi)^{1/2}, \lambda^{(n)})$.

Example A.32. Consider the Bernstein function $k(s) := f^{1/\delta}(s^\delta) = \big(\frac{s^\delta}{s^\delta+\gamma}\big)^{1/\delta}$ for $\gamma > 0$ and $0 < |\delta| < 1$, composed with the continuous negative definite function $\psi : \mathbb{R}^{n_1}\times\mathbb{R}^{n_2} \to \mathbb{R}$, $\psi(\xi,\eta) = |\xi|^\alpha + |\eta|^\beta$, where $0 < \alpha, \beta < 2$, which gives the metric

$$(k\circ\psi)^{1/2}(\xi,\eta) = \Big(\frac{(|\xi|^\alpha+|\eta|^\beta)^\delta}{(|\xi|^\alpha+|\eta|^\beta)^\delta+\gamma}\Big)^{1/2}.$$

Then it holds for the open balls of radius $0 < R < 1$ with respect to this metric that

$$\begin{aligned}B^{k\circ\psi}(0,R) &= \Big\{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : \frac{(|x|^\alpha+|y|^\beta)^\delta}{(|x|^\alpha+|y|^\beta)^\delta+\gamma} < R^2\Big\} \\ &= \{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : (1-R^2)\,(|x|^\alpha+|y|^\beta)^\delta < \gamma R^2\} \\ &= \Big\{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : \Big|\frac{x}{\rho(\gamma,\delta,R)^{1/\alpha\delta}}\Big|^\alpha + \Big|\frac{y}{\rho(\gamma,\delta,R)^{1/\beta\delta}}\Big|^\beta < 1\Big\} \\ &= B^\psi(0, \rho(\gamma,\delta,R)^{1/2\delta}),\end{aligned}$$

where $\rho(\gamma,\delta,R) = \frac{\gamma R^{2\delta}}{1-R^{2\delta}}$. Then we get for the n-dimensional volume of the balls,

$$\begin{aligned}\lambda^{(n)}(B^{k\circ\psi}(0,R)) &= \int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} \chi_{B_1^\psi}\Big(\frac{x}{\rho(\gamma,\delta,R)^{1/\alpha\delta}}, \frac{y}{\rho(\gamma,\delta,R)^{1/\beta\delta}}\Big)\,dy\,dx \\ &= \rho(\gamma,\delta,R)^{\frac{n_1}{\alpha\delta}+\frac{n_2}{\beta\delta}}\,\lambda^{(n)}(B^\psi(0,1))\end{aligned}$$

and analogously for the balls of radius r with $0 < r < R < 1$,

$$\lambda^{(n)}(B^{k\circ\psi}(0,r)) = \rho(\gamma,\delta,r)^{\frac{n_1}{\alpha\delta}+\frac{n_2}{\beta\delta}}\,\lambda^{(n)}(B^\psi(0,1))\,,$$

that leads to the volume doubling

$$\begin{aligned}\lambda^{(n)}(B^{k\circ\psi}(0,R)) &= \Big(\frac{\rho(\gamma,\delta,R)}{\rho(\gamma,\delta,r)}\Big)^{\frac{n_1}{\alpha\delta}+\frac{n_2}{\beta\delta}}\lambda^{(n)}(B^{k\circ\psi}(0,r)) \\ &= \Big(\frac{R}{r}\Big)^{2\left(\frac{n_1}{\alpha}+\frac{n_2}{\beta}\right)}\Big(\frac{1-r^{2\delta}}{1-R^{2\delta}}\Big)^{\frac{n_1}{\alpha\delta}+\frac{n_2}{\beta\delta}}\lambda^{(n)}(B^{k\circ\psi}(0,r))\end{aligned}$$

in the metric measure space $(\mathbb{R}^n, (k\circ\psi)^{1/2}, \lambda^{(n)})$.

Example A.33. Consider the Bernstein function $f(s) = \frac{1}{\gamma} - \frac{1}{\gamma+s} = \frac{s}{\gamma(\gamma+s)}$, $\gamma > 0$, and the continuous negative definite function $\psi(\xi) = |\xi|^2$, $\xi \in \mathbb{R}^n$. This gives the metric

$$(f \circ \psi)^{1/2}(\xi) = \left(\frac{|\xi|^2}{\gamma(\gamma + |\xi|^2)}\right)^{1/2}.$$

Then for the balls $B^{f\circ\psi}(0, R)$ of radius $0 < R < \frac{1}{\sqrt{\gamma}}$ in this metric we get

$$\begin{aligned} B^{f\circ\psi}(0, R) &= \left\{x \in \mathbb{R}^n : \frac{|x|^2}{\gamma(\gamma + |x|^2)} < R^2\right\} \\ &= \{x \in \mathbb{R}^n : (1 - R^2\gamma)\,|x|^2 < \gamma^2 R^2\} \\ &= \left\{x \in \mathbb{R}^n : \left|\frac{x}{\rho(\gamma, R)^{1/2}}\right|^2 < 1\right\} = B^{\psi}(0, \rho(\gamma, R)^{1/2}), \end{aligned}$$

where $\rho(\gamma, R) = \frac{R^2\gamma^2}{1-R^2\gamma}$, implying by a change of variable

$$\lambda^{(n)}(B^{f\circ\psi}(0, R)) = \int_{\mathbb{R}^n} \chi_{B_1^{\psi}}\left(\frac{x}{\rho(\gamma, R)^{1/2}}\right) dx = \rho(\gamma, R)^{n/2}\lambda^{(n)}(B^{\psi}(0, 1)).$$

Analogously, for balls of radius r such that $0 < r < R < \frac{1}{\sqrt{\gamma}}$

$$\lambda^{(n)}(B^{f\circ\psi}(0, r)) = \rho(\gamma, r)^{n/2}\lambda^{(n)}(B^{\psi}(0, 1))$$

and therefore

$$\begin{aligned} \lambda^{(n)}(B^{f\circ\psi}(0, R)) &= \left(\frac{\rho(\gamma, R)}{\rho(\gamma, r)}\right)^{\frac{n}{2}} \lambda^{(n)}(B^{f\circ\psi}(0, r)) \\ &= \left(\frac{R}{r}\right)^n \left(\frac{1 - r^2\gamma}{1 - R^2\gamma}\right)^{\frac{n}{2}} \lambda^{(n)}(B^{f\circ\psi}(0, r)) \end{aligned}$$

whenever the radii satisfy $0 < r < R < \frac{1}{\sqrt{\gamma}}$.

Example A.34. We consider the Bernstein function $f(s) = \frac{1}{\gamma} - \frac{1}{\gamma+s} = \frac{s}{\gamma(\gamma+s)}$, $\gamma > 0$, and the continuous negative definite function $\psi : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$ given by $\psi(\xi, \eta) = |\xi|^\alpha + |\eta|^\beta$, $0 < \alpha, \beta < 2$. This gives the metric

$$(f \circ \psi)^{1/2}(\xi, \eta) = \left(\frac{|\xi|^\alpha + |\eta|^\beta}{\gamma(\gamma + |\xi|^\alpha + |\eta|^\beta)}\right)^{1/2}.$$

Then for balls $B^{f\circ\psi}(0, R)$ of radius $0 < R < \frac{1}{\sqrt{\gamma}}$ in this metric we get

$$\begin{aligned} B^{f\circ\psi}(0, R) &= \left\{(x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : \frac{|x|^\alpha + |y|^\beta}{\gamma(\gamma + |x|^\alpha + |y|^\beta)} < R^2\right\} \\ &= \{(x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : (1 - R^2\gamma)\,(|x|^\alpha + |y|^\beta) < \gamma^2 R^2\} \\ &= \left\{(x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : \left|\frac{x}{\rho(\gamma, R)^{1/\alpha}}\right|^\alpha + \left|\frac{y}{\rho(\gamma, R)^{1/\beta}}\right|^\beta < 1\right\} \\ &= B^{\psi}(0, \rho(\gamma, R)^{1/2}), \end{aligned}$$

where $\rho(\gamma, R)$ is the same as in Example A.32. By a change of variable this implies

$$\begin{aligned}\lambda^{(n)}(B^{f\circ\psi}(0,R)) &= \int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} \chi_{B_1^\psi}\Big(\frac{x}{\rho(\gamma,R)^{1/\alpha}}, \frac{y}{\rho(\gamma,R)^{1/\beta}}\Big)\, dy\, dx \\ &= \rho(\gamma,R)^{\frac{n_1}{\alpha}+\frac{n_2}{\beta}}\lambda^{(n)}(B^\psi(0,1)).\end{aligned}$$

Analogously, for balls $B^{f\circ\psi}(0,r)$ of radius r that satisfies $0 < r < R < \frac{1}{\sqrt{\gamma}}$ we have

$$\lambda^{(n)}(B^{f\circ\psi}(0,r)) = \rho(\gamma,r)^{\frac{n_1}{\alpha}+\frac{n_2}{\beta}}\lambda^{(n)}(B^\psi(0,1)),$$

and therefore

$$\begin{aligned}\lambda^{(n)}(B^{f\circ\psi}(0,R)) &= \left(\frac{\rho(\gamma,R)}{\rho(\gamma,r)}\right)^{\frac{n_1}{\alpha}+\frac{n_2}{\beta}} \lambda^{(n)}(B^{f\circ\psi}(0,r)) \\ &= \left(\frac{R}{r}\right)^{2\left(\frac{n_1}{\alpha}+\frac{n_2}{\beta}\right)} \left(\frac{1-r^2\gamma}{1-R^2\gamma}\right)^{\frac{n_1}{\alpha}+\frac{n_2}{\beta}} \lambda^{(n)}(B^{f\circ\psi}(0,r))\end{aligned}$$

whenever the radii satisfy $0 < r < R < \frac{1}{\sqrt{\gamma}}$.

Example A.35. Consider the Bernstein function $h(s) := f(s^\delta) = \frac{1}{\gamma} - \frac{1}{\gamma+s^\delta} = \frac{s^\delta}{\gamma(\gamma+s^\delta)}$, $\gamma > 0$, $0 < \delta < 1$, and the continuous negative definite function $\psi : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$ given by $\psi(\xi,\eta) = |\xi|^\alpha + |\eta|^\beta$, $0 < \alpha, \beta < 2$. This gives the metric

$$(h\circ\psi)^{1/2}(\xi,\eta) = \left(\frac{(|\xi|^\alpha + |\eta|^\beta)^\delta}{\gamma(\gamma + (|\xi|^\alpha+|\eta|^\beta)^\delta)}\right)^{1/2}.$$

Then for the balls $B^{h\circ\psi}(0,R)$ of radius $0 < R < \frac{1}{\sqrt{\gamma}}$ with respect to this metric we obtain

$$\begin{aligned}B^{h\circ\psi}(0,R) &= \Big\{(x,y) \in \mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : \frac{(|x|^\alpha+|y|^\beta)^\delta}{\gamma(\gamma+(|x|^\alpha+|y|^\beta)^\delta)} < R^2\Big\} \\ &= \{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : (1-R^2\gamma)\,(|x|^\alpha+|y|^\beta)^\delta < \gamma^2 R^2\} \\ &= \Big\{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : \Big|\frac{x}{\rho(\gamma,R)^{1/\alpha\delta}}\Big|^\alpha + \Big|\frac{y}{\rho(\gamma,R)^{1/\beta\delta}}\Big|^\beta < 1\Big\} \\ &= B^\psi(0, \rho(\gamma,R)^{1/2\delta}),\end{aligned}$$

where $\rho(\gamma, R)$ is the same constant as in Example A.33. This implies by a change of variable that

$$\begin{aligned}\lambda^{(n)}(B^{h\circ\psi}(0,R)) &= \int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} \chi_{B_1^\psi}\Big(\frac{x}{\rho(\gamma,R)^{1/\alpha\delta}}, \frac{y}{\rho(\gamma,R)^{1/\beta\delta}}\Big)\, dy\, dx \\ &= \rho(\gamma,R)^{\frac{n_1}{\alpha\delta}+\frac{n_2}{\beta\delta}}\lambda^{(n)}(B^\psi(0,1)).\end{aligned}$$

Analogously, for balls $B^{f\circ\psi}(0,r)$ of radius r such that $0 < r < R < \frac{1}{\sqrt{\gamma}}$ we have

$$\lambda^{(n)}(B^{h\circ\psi}(0,r)) = \rho(\gamma,r)^{\frac{n_1}{\alpha\delta}+\frac{n_2}{\beta\delta}}\lambda^{(n)}(B^\psi(0,1)),$$

which implies

$$\begin{aligned}\lambda^{(n)}(B^{h\circ\psi}(0,R)) &= \left(\frac{\rho(\gamma,R)}{\rho(\gamma,r)}\right)^{\frac{n_1}{\alpha\delta}+\frac{n_2}{\beta\delta}} \lambda^{(n)}(B^{h\circ\psi}(0,r)) \\ &= \left(\frac{R}{r}\right)^{2\left(\frac{n_1}{\alpha\delta}+\frac{n_2}{\beta\delta}\right)} \left(\frac{1-r^2\gamma}{1-R^2\gamma}\right)^{\frac{n_1}{\alpha\delta}+\frac{n_2}{\beta\delta}} \lambda^{(n)}(B^{h\circ\psi}(0,r))\end{aligned}$$

for radii that satisfy $0 < r < R < \frac{1}{\sqrt{\gamma}}$.

Example A.36. Consider the Bernstein function $k(s) := f^{1/\delta}(s^\delta) = \left(\frac{1}{\gamma} - \frac{1}{\gamma+s^\delta}\right)^{1/\delta} = \left(\frac{s^\delta}{\gamma(\gamma+s^\delta)}\right)^{1/\delta}$, $\gamma > 0$, $0 < |\delta| < 1$, and the continuous negative definite function $\psi : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$ given by $\psi(\xi, \eta) = |\xi|^\alpha + |\eta|^\beta$, $0 < \alpha, \beta < 2$. This gives the metric

$$(k \circ \psi)^{1/2}(\xi, \eta) = \left(\frac{(|\xi|^\alpha + |\eta|^\beta)^\delta}{\gamma(\gamma + (|\xi|^\alpha + |\eta|^\beta)^\delta)}\right)^{1/2\delta}.$$

Then for the balls $B^{h\circ\psi}(0, R)$ of radius $0 < R < \left(\frac{1}{\gamma}\right)^{1/2\delta}$ in this metric we obtain

$$\begin{aligned} B^{k\circ\psi}(0,R) &= \left\{(x,y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : \frac{(|x|^\alpha + |y|^\beta)^\delta}{\gamma(\gamma + (|x|^\alpha + |y|^\beta)^\delta)} < R^{2\delta}\right\} \\ &= \{(x,y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : (1 - \gamma R^{2\delta})\,(|x|^\alpha + |y|^\beta)^\delta < \gamma^2 R^{2\delta}\} \\ &= \left\{(x,y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : \left|\frac{x}{\rho(\gamma,\delta,R)^{1/\alpha\delta}}\right|^\alpha + \left|\frac{y}{\rho(\gamma,\delta,R)^{1/\beta\delta}}\right|^\beta < 1\right\} \\ &= B^\psi(0, \rho(\gamma,\delta,R)^{1/2\delta}), \end{aligned}$$

where $\rho(\gamma, \delta, R) = \frac{\gamma^2 R^{2\delta}}{1-\gamma R^{2\delta}}$, implying that

$$\begin{aligned} \lambda^{(n)}(B^{k\circ\psi}(0,R)) &= \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_2}} \chi_{B_1^\psi}\left(\frac{x}{\rho(\gamma,\delta,R)^{1/\alpha\delta}}, \frac{y}{\rho(\gamma,\delta,R)^{1/\beta\delta}}\right) dy\, dx \\ &= \rho(\gamma,\delta,R)^{\frac{n_1}{\alpha\delta}+\frac{n_2}{\beta\delta}} \lambda^{(n)}(B^\psi(0,1)). \end{aligned}$$

Analogously, for balls of radius r with $0 < r < R < \left(\frac{1}{\gamma}\right)^{1/2\delta}$,

$$\lambda^{(n)}(B^{k\circ\psi}(0,r)) = \rho(\gamma,\delta,r)^{\frac{n_1}{\alpha\delta}+\frac{n_2}{\beta\delta}} \lambda^{(n)}(B^\psi(0,1)),$$

which leads to the volume doubling

$$\begin{aligned} \lambda^{(n)}(B^{k\circ\psi}(0,R)) &= \left(\frac{\rho(\gamma,\delta,R)}{\rho(\gamma,\delta,r)}\right)^{\frac{n_1}{\alpha\delta}+\frac{n_2}{\beta\delta}} \lambda^{(n)}(B^{k\circ\psi}(0,r)) \\ &= \left(\frac{R}{r}\right)^{2\left(\frac{n_1}{\alpha}+\frac{n_2}{\beta}\right)} \left(\frac{1-\gamma r^{2\delta}}{1-\gamma R^{2\delta}}\right)^{\frac{n_1}{\alpha\delta}+\frac{n_2}{\beta\delta}} \lambda^{(n)}(B^{k\circ\psi}(0,r)) \end{aligned}$$

for radii $0 < r < R < \left(\frac{1}{\gamma}\right)^{1/2\delta}$.

Example A.37. Consider the Bernstein function $f(s) = \sqrt{s}\,(1 - e^{-4\sqrt{s}})$ and the continuous negative definite function $\psi(\xi) = |\xi|^2$, $\xi \in \mathbb{R}^n$. This gives the metric

$$(f \circ \psi)^{1/2}(\xi) = \left(|\xi|\,(1 - e^{-4|\xi|})\right)^{1/2}.$$

Note that for $s > 0$

$$f(s) = \sqrt{s}\,(1 - e^{-4\sqrt{s}}) > 1 - e^{-2s} =: f_{\text{low}}(s),$$

where $s \mapsto f_{\text{low}}(s)$ is again a Bernstein function and thus $(f_{\text{low}} \circ \psi)^{1/2}$ a metric. Then for a radius R that satisfies $0 < R < 1$,

$$\begin{aligned} B^{f\circ\psi}(0,R) &= \{x \in \mathbb{R}^n : |x|\,(1 - e^{-4|x|}) < R^2\} \\ &\subset \{x \in \mathbb{R}^n : 1 - e^{-2|x|^2} < R^2\} = B^{f_{\text{low}}\circ\psi}(0,R) \\ &= \left\{x \in \mathbb{R}^n : \left|\frac{x}{(\frac{1}{2}\log(1-R^2))^{1/2}}\right|^2 < 1\right\} = B^\psi(0, (\tfrac{1}{2}\log(1-R^2))^{1/2}). \end{aligned}$$

By a change of variable we obtain the volume estimate

$$\begin{aligned}\lambda^{(n)}(B^{f\circ\psi}(0,R)) &= \int_{\mathbb{R}^n} \chi_{B_R^{f\circ\psi}}(x)\,dx \leqslant \int_{\mathbb{R}^n} \chi_{B_1^\psi}\Big(\frac{x}{(\frac{1}{2}\log(1-R^2))^{1/2}}\Big)\,dx \\ &= \Big(\frac{1}{2}\log(1-R^2)\Big)^{\frac{n}{2}} \lambda^{(n)}(B^\psi(0,1)).\end{aligned}$$

On the other hand, for $0 < s < 1$

$$f(s) = \sqrt{s}\,(1-e^{-4\sqrt{s}}) < 1 - e^{-4s} =: f_{\text{up}}(s)$$

with the Bernstein function $s \mapsto f_{\text{up}}(s)$ giving the metric $(f_{\text{up}}\circ\psi)^{1/2}$. Now choose the radius r with $0 < r < R < 1$ such that $(f\circ\psi)^{1/2}(x) < (f_{\text{up}}\circ\psi)^{1/2}(x) < r$, then it follows

$$\begin{aligned}B^{f\circ\psi}(0,r) &= \{x\in\mathbb{R}^n : |x|\,(1-e^{-4|x|}) < r^2\} \\ &\supset \{x\in\mathbb{R}^n : 1 - e^{-4|x|^2} < r^2\} = B^{f_{\text{up}}\circ\psi}(0,r) \\ &= \Big\{x\in\mathbb{R}^n : \Big|\frac{x}{(\frac{1}{4}\log(1-r^2))^{1/2}}\Big|^2 < 1\Big\} = B^\psi(0,(\tfrac{1}{4}\log(1-r^2))^{1/2}).\end{aligned}$$

Then we get a lower estimate of the Lebesgue volume

$$\begin{aligned}\lambda^{(n)}(B^{f\circ\psi}(0,r)) &= \int_{\mathbb{R}^n} \chi_{B_r^{f\circ\psi}}(x)\,dx \geqslant \int_{\mathbb{R}^n} \chi_{B_1^\psi}\Big(\frac{x}{(\frac{1}{4}\log(1-r^2))^{1/2}}\Big)\,dx \\ &= \Big(\frac{1}{4}\log(1-r^2)\Big)^{\frac{n}{2}} \lambda^{(n)}(B^\psi(0,1)).\end{aligned}$$

Hence, the two estimates give the volume doubling

$$\lambda^{(n)}(B^{f\circ\psi}(0,R)) \leqslant 2^{n/2}\left(\frac{\log(1-R^2)}{\log(1-r^2)}\right)^{\frac{n}{2}} \lambda^{(n)}(B^{f\circ\psi}(0,r))$$

provided $\psi(\xi) < 1$ and $0 < r < R < 1$.

Example A.38. Consider now the Bernstein function $f(s) = \sqrt{s}\,(1-e^{-4\sqrt{s}})$ and the continuous negative definite function $\psi : \mathbb{R}^{n_1}\times\mathbb{R}^{n_2} \to \mathbb{R}$ given by $\psi(\xi,\eta) = |\xi|^\alpha + |\eta|^\beta$ with $0 < \alpha,\beta < 2$. Using this we define the metric

$$(f\circ\psi)^{1/2}(\xi,\eta) = \Big(\sqrt{|\xi|^\alpha + |\eta|^\beta}\,(1-e^{-4\sqrt{|\xi|^\alpha+|\eta|^\beta}})\Big)^{1/2}.$$

We use the Bernstein functions $f_{\text{low}}(s)$ and $f_{\text{up}}(s)$ from Example A.37 to determine a volume doubling constant. We consider the balls of radius $0 < R < 1$,

$$\begin{aligned}B^{f\circ\psi}(0,R) &= \{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : \sqrt{|\xi|^\alpha+|\eta|^\beta}\,(1-e^{-4\sqrt{|\xi|^\alpha+|\eta|^\beta}}) < R^2\} \\ &\subset \{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : 1 - e^{-2(|x|^\alpha+|y|^\beta)} < R^2\} = B^{f_{\text{low}}\circ\psi}(0,R) \\ &= \Big\{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : \Big|\frac{x}{(\frac{1}{2}\log(1-R^2))^{1/\alpha}}\Big|^\alpha + \Big|\frac{y}{(\frac{1}{2}\log(1-R^2))^{1/\beta}}\Big|^\beta < 1\Big\} \\ &= B^\psi(0,(\tfrac{1}{2}\log(1-R^2))^{1/2}).\end{aligned}$$

Then, by a change of variables,

$$\lambda^{(n)}(B^{f\circ\psi}(0,R)) \leqslant \int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} \chi_{B_1^{\psi}}\Big(\frac{x}{(\frac{1}{2}\log(1-R^2))^{1/\alpha}}, \frac{y}{(\frac{1}{2}\log(1-R^2))^{1/\beta}}\Big)\,dy\,dx$$
$$= \Big(\frac{1}{2}\log(1-R^2)\Big)^{\frac{n_1}{\alpha}+\frac{n_2}{\beta}} \lambda^{(n)}(B^{\psi}(0,1)).$$

Similarly, using the Bernstein function $f_{\mathrm{up}}(s)$, $0<s<1$, from Example A.37 we obtain for the balls of radius r with respect to $(f_{\mathrm{up}}\circ\psi)^{1/2}$, such that $0<r<R<1$ and $(f\circ\psi)^{1/2}(x) < (f_{\mathrm{up}}\circ\psi)^{1/2}(x) < r$,

$$\begin{aligned} B^{f\circ\psi}(0,r) &= \{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : \sqrt{|\xi|^\alpha+|\eta|^\beta}\,(1-e^{-4\sqrt{|\xi|^\alpha+|\eta|^\beta}}) < r^2\} \\ &\supset \{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : 1-e^{-4(|x|^\alpha+|y|^\beta)} < r^2\} = B^{f_{\mathrm{low}}\circ\psi}(0,r) \\ &= \Big\{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : \Big|\frac{x}{(\frac{1}{4}\log(1-r^2))^{1/\alpha}}\Big|^\alpha + \Big|\frac{y}{(\frac{1}{4}\log(1-r^2))^{1/\beta}}\Big|^\beta < 1\Big\} \\ &= B^{\psi}(0,(\tfrac{1}{4}\log(1-r^2))^{1/2}). \end{aligned}$$

By a change of variables it follows

$$\lambda^{(n)}(B^{f\circ\psi}(0,r)) \geqslant \int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} \chi_{B_1^{\psi}}\Big(\frac{x}{(\frac{1}{4}\log(1-r^2))^{1/\alpha}}, \frac{y}{(\frac{1}{4}\log(1-r^2))^{1/\beta}}\Big)\,dy\,dx$$
$$= \Big(\frac{1}{4}\log(1-r^2)\Big)^{\frac{n_1}{\alpha}+\frac{n_2}{\beta}} \lambda^{(n)}(B^{\psi}(0,1)).$$

The two estimates then give the volume doubling

$$\lambda^{(n)}(B^{f\circ\psi}(0,R)) \leqslant 2^{\frac{n_1}{\alpha}+\frac{n_2}{\beta}} \left(\frac{\log(1-R^2)}{\log(1-r^2)}\right)^{\frac{n_1}{\alpha}+\frac{n_2}{\beta}} \lambda^{(n)}(B^{f\circ\psi}(0,r))$$

provided that $\psi(\xi,\eta)<1$ and $0<r<R<1$.

Example A.39. We take the Bernstein function $h(s) := f(s^\delta) = \sqrt{s^\delta}\,(1-e^{-4\sqrt{s^\delta}})$ for $0<\delta<1$ and the continuous negative definite function $\psi(\xi,\eta) = |\xi|^\alpha+|\eta|^\beta$, where $0<\alpha,\beta<2$. We consider the open balls with respect to the metric

$$(h\circ\psi)^{1/2}(\xi,\eta) = \Big((|\xi|^\alpha+|\eta|^\beta)^{\delta/2}\,(1-e^{-4(|\xi|^\alpha+|\eta|^\beta)^{\delta/2}})\Big)^{1/2}$$

and determine a volume doubling constant using the Bernstein functions $h_{\mathrm{low}}(s) := f_{\mathrm{low}}(s^\delta) = 1-e^{-2s^\delta}$ and $h_{\mathrm{up}}(s) := f_{\mathrm{up}}(s^\delta) = 1-e^{-4s^\delta}$ which have the property $h_{\mathrm{low}}(s) < h(s) < h_{\mathrm{up}}(s)$ provided that $0<s<1$. For a radius R that satisfies $0<R<1$ we study

$$\begin{aligned} B^{h\circ\psi}(0,R) &= \{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : (|\xi|^\alpha+|\eta|^\beta)^{\delta/2}\,(1-e^{-4(|\xi|^\alpha+|\eta|^\beta)^{\delta/2}}) < R^2\} \\ &\subset \{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : 1-e^{-2(|x|^\alpha+|y|^\beta)^\delta} < R^2\} = B^{h_{\mathrm{low}}\circ\psi}(0,R) \\ &= \Big\{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : \Big|\frac{x}{(\frac{1}{2}\log(1-R^2))^{1/(\alpha\delta)}}\Big|^\alpha + \Big|\frac{y}{(\frac{1}{2}\log(1-R^2))^{1/(\beta\delta)}}\Big|^\beta < 1\Big\} \\ &= B^{\psi}(0,(\tfrac{1}{2}\log(1-R^2))^{1/2\delta}). \end{aligned}$$

By a change of variables we obtain

$$\begin{aligned}\lambda^{(n)}(B^{h\circ\psi}(0,R)) &\leqslant \int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} \chi_{B_1^\psi}\Big(\frac{x}{(\frac{1}{2}\log(1-R^2))^{1/(\alpha\delta)}}, \frac{y}{(\frac{1}{2}\log(1-R^2))^{1/(\beta\delta)}}\Big)\,dy\,dx \\ &= \Big(\frac{1}{2}\log(1-R^2)\Big)^{\frac{n_1}{\alpha\delta}+\frac{n_2}{\beta\delta}} \lambda^{(n)}(B^\psi(0,1)).\end{aligned}$$

Similarly, using the Bernstein function $h_{\text{up}}(s)$, $0<s<1$, we obtain for the balls of radius r with respect to $(h_{\text{up}}\circ\psi)^{1/2}$ such that $0<r<R<1$ and $(h\circ\psi)^{1/2}<(h_{\text{up}}\circ\psi)^{1/2}<r$, that

$$\begin{aligned}B^{h\circ\psi}(0,r) &= \big\{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : (|\xi|^\alpha+|\eta|^\beta)^{\delta/2}\,(1-e^{-4\,(|\xi|^\alpha+|\eta|^\beta)^{\delta/2}})<r^2\big\} \\ &\supset \big\{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : 1-e^{-4\,(|x|^\alpha+|y|^\beta)^\delta}<r^2\big\} \;=\; B^{h_{\text{up}}\circ\psi}(0,r) \\ &= \Big\{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : \Big|\frac{x}{(\frac{1}{4}\log(1-r^2))^{1/(\alpha\delta)}}\Big|^\alpha+\Big|\frac{y}{(\frac{1}{4}\log(1-r^2))^{1/(\beta\delta)}}\Big|^\beta<1\Big\} \\ &= B^\psi(0,(\tfrac{1}{4}\log(1-r^2))^{1/2\delta}).\end{aligned}$$

By a change of variable it follows

$$\begin{aligned}\lambda^{(n)}(B^{h\circ\psi}(0,r)) &\geqslant \int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} \chi_{B_1^\psi}\Big(\frac{x}{(\frac{1}{4}\log(1-r^2))^{1/(\alpha\delta)}}, \frac{y}{(\frac{1}{4}\log(1-r^2))^{1/(\beta\delta)}}\Big)\,dy\,dx \\ &= \Big(\frac{1}{4}\log(1-r^2)\Big)^{\frac{n_1}{\alpha\delta}+\frac{n_2}{\beta\delta}} \lambda^{(n)}(B^\psi(0,1)).\end{aligned}$$

The two estimates above them give the volume doubling

$$\lambda^{(n)}(B^{h\circ\psi}(0,R)) \leqslant 2^{\frac{n_1}{\alpha\delta}+\frac{n_2}{\beta\delta}}\left(\frac{\log(1-R^2)}{\log(1-r^2)}\right)^{\frac{n_1}{\alpha\delta}+\frac{n_2}{\beta\delta}} \lambda^{(n)}(B^{h\circ\psi}(0,r))$$

provided that $\psi(\xi,\eta)<1$ and $0<r<R<1$.

Example A.40. Take the Bernstein function $k(s) := f^{1/\delta}(s^\delta) = \big(\sqrt{s^\delta}\,(1-e^{-4\sqrt{s^\delta}})\big)^{1/\delta}$ for $0<|\delta|<1$ and the continuous negative definite function $\psi(\xi,\eta)=|\xi|^\alpha+|\eta|^\beta$, where $0<\alpha,\beta<2$. We consider balls of radii r and R satisfying $0<r<R<1$ with respect to the metric

$$(k\circ\psi)^{1/2}(\xi,\eta) = \Big((|\xi|^\alpha+|\eta|^\beta)^{\delta/2}\,(1-e^{-4\,(|\xi|^\alpha+|\eta|^\beta)^{\delta/2}})\Big)^{1/2\delta}$$

and determine a volume doubling constant using the Bernstein functions $k_{\text{low}}(s) := \big(1-e^{-2s^\delta}\big)^{1/\delta}$ and $k_{\text{up}}(s) := \big(1-e^{-4s^\delta}\big)^{1/\delta}$ having the property that $k_{\text{low}}(s)<k(s)<k_{\text{up}}(s)$ provided that $0<s<1$. For a radius R that satisfies $0<R<1$ we study

$$\begin{aligned}B^{k\circ\psi}(0,R) &= \big\{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : (|\xi|^\alpha+|\eta|^\beta)^{\delta/2}\,(1-e^{-4\,(|\xi|^\alpha+|\eta|^\beta)^{\delta/2}})<R^{2\delta}\big\} \\ &\subset \big\{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : 1-e^{-2\,(|x|^\alpha+|y|^\beta)^\delta}<R^{2\delta}\big\} \;=\; B^{k_{\text{low}}\circ\psi}(0,R) \\ &= \Big\{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : \Big|\frac{x}{(\frac{1}{2}\log(1-R^{2\delta}))^{\frac{1}{\alpha\delta}}}\Big|^\alpha+\Big|\frac{y}{(\frac{1}{2}\log(1-R^{2\delta}))^{\frac{1}{\beta\delta}}}\Big|^\beta<1\Big\} \\ &= B^\psi(0,(\tfrac{1}{2}\log(1-R^{2\delta}))^{1/2\delta}).\end{aligned}$$

By a change of variable we obtain

$$\begin{aligned}
\lambda^{(n)}(B^{k\circ\psi}(0,R)) &\leqslant \int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} \chi_{B_1^\psi}\Big(\frac{x}{(\frac{1}{2}\log(1-R^{2\delta}))^{1/(\alpha\delta)}}, \frac{y}{(\frac{1}{2}\log(1-R^{2\delta}))^{1/(\beta\delta)}}\Big)\, dy\, dx \\
&= \Big(\frac{1}{2}\log(1-R^{2\delta})\Big)^{\frac{n_1}{\alpha\delta}+\frac{n_2}{\beta\delta}} \lambda^{(n)}(B^\psi(0,1)).
\end{aligned}$$

Similarly, using the Bernstein function $k_{\text{up}}(s)$, $0 < s < 1$, we obtain for the balls of radius r with respect to $(k_{\text{up}}\circ\psi)^{1/2}$ such that $0<r<R<1$ and $(k\circ\psi)^{1/2} < (k_{\text{up}}\circ\psi)^{1/2} < r$, that

$$\begin{aligned}
B^{k\circ\psi}(0,r) &= \big\{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : (|\xi|^\alpha+|\eta|^\beta)^{\delta/2}\,(1-e^{-4\,(|\xi|^\alpha+|\eta|^\beta)^{\delta/2}}) < r^{2\delta}\big\} \\
&\subset \big\{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : 1-e^{-4\,(|x|^\alpha+|y|^\beta)^\delta} < r^{2\delta}\big\} = B^{k_{\text{up}}\circ\psi}(0,r) \\
&= \Big\{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : \Big|\frac{x}{(\frac{1}{4}\log(1-r^{2\delta}))^{\frac{1}{\alpha\delta}}}\Big|^\alpha + \Big|\frac{y}{(\frac{1}{4}\log(1-r^{2\delta}))^{\frac{1}{\beta\delta}}}\Big|^\beta < 1\Big\} \\
&= B^\psi(0,(\tfrac{1}{4}\log(1-r^{2\delta}))^{1/2\delta}).
\end{aligned}$$

By a change of variable it follows

$$\begin{aligned}
\lambda^{(n)}(B^{k\circ\psi}(0,r)) &\geqslant \int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} \chi_{B_1^\psi}\Big(\frac{x}{(\frac{1}{4}\log(1-r^{2\delta}))^{\frac{1}{\alpha\delta}}}, \frac{y}{(\frac{1}{4}\log(1-r^{2\delta}))^{\frac{1}{\beta\delta}}}\Big)\, dy\, dx \\
&= \Big(\frac{1}{4}\log(1-r^{2\delta})\Big)^{\frac{n_1}{\alpha\delta}+\frac{n_2}{\beta\delta}} \lambda^{(n)}(B^\psi(0,1)).
\end{aligned}$$

These two estimates finally yield the volume doubling

$$\lambda^{(n)}(B^{k\circ\psi}(0,R)) \leqslant 2^{\frac{n_1}{\alpha\delta}+\frac{n_2}{\beta\delta}} \left(\frac{\log(1-R^{2\delta})}{\log(1-r^{2\delta})}\right)^{\frac{n_1}{\alpha\delta}+\frac{n_2}{\beta\delta}} \lambda^{(n)}(B^{k\circ\psi}(0,r))$$

provided that $\psi(\xi,\eta) < 1$ and $0<r<R<1$.

B Jump intensities and transition function estimates

B.1 Estimates for the Cauchy process

Let f be the Bernstein function $f(x) = x^\alpha$, $0 < \alpha < 1$, with the integral representation

$$f(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_{0+}^{\infty} (1 - e^{-sx})\, s^{-\alpha-1}\, ds\,.$$

Then it follows for the continuous negative definite function $\xi \mapsto f(|\xi|^2)$ that

$$\begin{aligned} |\xi|^{2\alpha} &= \frac{\alpha}{\Gamma(1-\alpha)} \int_{0+}^{\infty} (1 - e^{-s|\xi|^2})\, s^{-\alpha-1}\, ds \\ &= \frac{\alpha}{\Gamma(1-\alpha)} \int_{0+}^{\infty} \int_{\mathbb{R}^n} (1 - e^{-iy\cdot\xi})(4\pi s)^{-n/2} e^{-\frac{|y|^2}{4s}}\, dy\, s^{-\alpha-1}\, ds \\ &= \frac{\alpha}{\Gamma(1-\alpha)} \int_{\mathbb{R}^n} (1 - e^{-iy\cdot\xi}) \int_{0+}^{\infty} (4\pi s)^{-n/2} e^{-\frac{|y|^2}{4s}}\, s^{-\alpha-1}\, ds\, dy\,. \end{aligned}$$

using the Fourier transform of $(4\pi s)^{-n/2} e^{-\frac{|y|^2}{4s}}$ being equal to $e^{-s|\xi|^2}$. Therefore, $\psi(\xi) = f(|\xi|^2)$ has the form $\psi(\xi) = \int_{\mathbb{R}^n} (1 - e^{-iy\cdot\xi})\, m(|y|^2)\, dy$ with

$$m(|y|^2) = \frac{\alpha}{\Gamma(1-\alpha)} \int_{0+}^{\infty} (4\pi s)^{-n/2} e^{-\frac{|y|^2}{4s}} \frac{ds}{s^{\alpha+1}} = \frac{\alpha\, 4^\alpha\, \Gamma(\frac{n}{2}+\alpha)}{\pi^{n/2}\, \Gamma(1-\alpha)} \frac{1}{|y|^{n+2\alpha}}\,,$$

and hence the symmetric kernel J is given by

$$J(x,y) = \frac{1}{2}\, m(|x-y|^2) = \frac{\alpha\, 4^\alpha\, \Gamma(\frac{n}{2}+\alpha)}{2\pi^{n/2}\Gamma(1-\alpha)} \frac{1}{|x-y|^{n+2\alpha}}\,.$$

As we are especially interested in the case $\alpha = \frac{1}{2}$, the kernel used for the computations reads as

$$J(x,y) = \frac{\Gamma(\frac{n}{2}+\frac{1}{2})}{2\pi^{n/2}\Gamma(\frac{1}{2})} \frac{1}{|x-y|^{n+1}}\,.$$

In order to compute $J(x,y)$ and the heat kernel estimate for $p_t(x,y)$ according to [13] we need the volume of the open ball with respect to the metric ψ. Note that due to translation invariance we have the equality

$$\lambda^{(n)}(B^\psi(y, \psi^{1/2}(x-y))) = \lambda^{(n)}(B^\psi(0,R))$$

when setting $y = 0$ and $\psi^{1/2}(x) = R$. Hence, the two-sided estimate (1.8) in [13] and also the uniform volume doubling assumption (1.7) are satisfied in our case. More precisely, we are able to reduce the computation of the volume of B^ψ to that of the unit ball centred at the origin in the Euclidean space by

$$\lambda^{(n)}(B^\psi(0,R)) = R^{n/\alpha} \lambda^{(n)}(B^{|\cdot|}(0,1)) = R^{n/\alpha} \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} = [f^{-1}(R^2)]^{n/2} \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$$

with the Bernstein function $f(x) = x^\alpha$. Using this we determine the function ϕ (see p. 280 in [13]) as

$$\begin{aligned}\phi(\psi^{1/2}(x-y)) &= \frac{c_1}{\lambda^{(n)}(B^\psi(y,\psi^{1/2}(x-y)))\,J(x,y)} \\ &= \frac{2\,c_1\,\Gamma(\frac{n}{2}+1)\,\Gamma(1-\alpha)\,|x-y|^{(n+2\alpha)/2}}{\alpha\,4^\alpha\,\Gamma(\frac{n}{2}+\alpha)\,(f^{-1}\circ\psi)^{n/2}(x-y)} \\ &= \frac{c_1\,|x-y|^{n+2\alpha}}{\kappa_1\,(f^{-1}\circ\psi)^{1/2}(x-y)}\end{aligned}$$

with

$$\kappa_1 = \frac{\alpha\,4^\alpha\,\Gamma(\frac{n}{2}+\alpha)}{2\,\Gamma(\frac{n}{2}+1)\,\Gamma(1-\alpha)}\,.$$

Note that with $|x-y| = (f^{-1}\circ\psi)^{1/2}(x-y)$ we can rewrite this as

$$\phi(\psi^{1/2}(x-y)) = \frac{c_1}{\kappa_1\,|x-y|^{2\alpha}}$$

and solve

$$1 = \phi(1) = \frac{c_1\,(f^{-1}(1))^{(n+2\alpha)/2}}{\kappa_1\,(f^{-1}(1))^{n/2}}$$

with respect to c_1 which gives the normalising constant $c_1 = \kappa_1\,[f^{-1}(1)]^{-\alpha}$, which ensures that ϕ satisfies the conditions $\phi(0) = 0$ and $\phi(1) = 1$.

In order to visualise the estimate

$$p_t(x,y) \leqslant C\left(\frac{1}{\lambda^{(n)}(B^\psi(x,\phi^{-1}(t)))} \wedge \frac{t}{\lambda^{(n)}(B^\psi(x,\psi^{1/2}(x-y)))\,\phi(\psi^{1/2}(x-y))}\right) \tag{B.1}$$

with $C \geqslant 1$ as provided in [13], p. 282, we restrict ourselves to $\alpha = \frac{1}{2}$, i.e. p is the transition function

$$p_t(x,u) = \Gamma\left(\frac{n}{2}+\frac{1}{2}\right)\,\frac{t}{\pi\,(|x-y|^2+t^2)^{(n+1)/2}}\,.$$

Using the definition of J note that

$$\frac{t}{\lambda^{(n)}(B^\psi(y,\psi^{1/2}(x-y)))\,\phi(\psi^{1/2}(x-y))} = \frac{t\,J(x,y)}{c_1}\,,$$

hence we arrive at (B.1) having the form

$$p_t(x,y) \leqslant C\left(\frac{1}{\lambda^{(n)}(B^\psi(x,\phi^{-1}(t)))} \wedge \frac{t\,J(x,y)}{c_1}\right).$$

Now it is possible to determine the constant $C \geqslant 1$ such that the inequality holds pointwise for all $x, y \in \mathbb{R}^n$ and $t > 0$.

B.2 Estimates for the relativistic symmetric stable process

Let f be the Bernstein function $f(x) = (x+m^2)^{1/2} - m$, $m > 0$, with the integral representation

$$f(x) = \frac{1}{2\Gamma(\frac{1}{2})}\int_{0+}^\infty (1-e^{-sx})\,e^{-sm^2}s^{-\frac{3}{2}}\,ds\,.$$

Then the continuous negative definite function $\psi_2(\xi)$ in (3.17) is of the form $\xi \mapsto f(|\xi|^2)$ for which we can derive its integral form as

$$\begin{aligned}\psi_2(\xi) &= \frac{1}{2\Gamma(\frac{1}{2})}\int_{0+}^{\infty}(1-e^{-s|\xi|^2})\,e^{-sm^2}\,s^{-3/2}\,ds\\ &= \frac{1}{2\Gamma(\frac{1}{2})}\int_{0+}^{\infty}\int_{\mathbb{R}^n}(1-e^{-iy\cdot\xi})\,(4\pi s)^{-n/2}\,e^{-\frac{|y|^2}{4s}}\,e^{-sm^2}\,dy\,s^{-3/2}\,ds\\ &= \frac{1}{2\Gamma(\frac{1}{2})}\int_{\mathbb{R}^n}(1-e^{-iy\cdot\xi})\int_{0+}^{\infty}(4\pi s)^{-n/2}\,e^{-\frac{|y|^2}{4s}}\,e^{-sm^2}\,s^{-3/2}\,ds\,dy\end{aligned}$$

by using the inverse Fourier transform of $e^{-s|\xi|^2}$. Thus, $\psi_2(\xi)$ can be expressed in the integral form $\psi_2(\xi) = \int_{\mathbb{R}^n}(1-e^{-iy\cdot\xi})\,m(|y|^2)\,dy$ with

$$\begin{aligned}m(|y|^2) &= \frac{1}{2\Gamma(\frac{1}{2})}\int_{0+}^{\infty}(4\pi s)^{-n/2}\,e^{-\frac{|y|^2}{4s}}\,e^{-sm^2}\,s^{-3/2}\,ds\\ &= \pi^{-\frac{1}{2}}\,2^{-\frac{n+1}{2}}\,\pi^{-\frac{n}{2}}\,m^{\frac{n+1}{2}}\,\frac{K_{\frac{n+1}{2}}(m\,|y|)}{|y|^{\frac{n+1}{2}}}\,.\end{aligned}$$

This implies for the symmetric kernel

$$J(x,y) = \frac{1}{2}\,m(|x-y|^2) = \frac{1}{2}\,(2\pi)^{-\frac{n+1}{2}}\,m^{\frac{n+1}{2}}\,\frac{K_{\frac{n+1}{2}}(m\,|x-y|)}{|x-y|^{\frac{n+1}{2}}}\,.$$

For the computation of the estimate for the transition function we need the volume of the ball $B^{\psi}(y,\psi^{1/2}(x-y))$ with respect to the metric $\psi^{1/2}$ centred at the point $y \in \mathbb{R}^n$, with fixed radius $\psi^{1/2}(x-y)$. Again, due to translation invariance of the metric we may shift the ball to the origin and set $\psi^{1/2}(x) = R$. Then it follows

$$\lambda^{(n)}(B^{\psi}(0,R)) = \left((R^2+m)^2-m^2\right)^{n/2}\lambda^{(n)}(B^{|\cdot|}(0,1)) = \left(f^{-1}(R^2)\right)^{n/2}\frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}\,,$$

where f is the Bernstein function given above. For the doubling constant computation, see Example A.24 in Appendix A. We determine the function ϕ as

$$\begin{aligned}\phi(\psi^{1/2}(x-y)) &= \frac{c_2}{\lambda^{(n)}(B^{\psi}(y,\psi^{1/2}(x-y)))\,J(x,y)}\\ &= \frac{c_2\,|x-y|}{\kappa_2\,(f^{-1}(\psi(x-y)))^{1/2}(x-y)\,K_{\frac{n+\alpha}{2}}(m\,|x-y|)}\,,\end{aligned} \tag{B.2}$$

where κ_2 is given by

$$\kappa_2 = \frac{1}{2}\,2^{-\frac{n+1}{2}}\,\pi^{-\frac{n+1}{2}}\,m^{\frac{n+1}{2}}\,\frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} = \frac{m^{\frac{n+1}{2}}}{2^{\frac{n+1}{2}+1}\sqrt{\pi}\,\Gamma(\frac{n}{2}+1)}\,.$$

At this point we choose c_2 such that $\phi(1) = 1$. Note that $|x| = f^{-1}(\psi(x))^{1/2}$, which we use to solve

$$1 = \phi(1) = \frac{c_2\,f^{-1}(1)^{\frac{n+1}{2}}}{\kappa_2\,f^{-1}(1)^{n/2}\,K_{\frac{n+1}{2}}(mf^{-1}(1)^{1/2})}$$

with respect to c_2. This yields

$$c_2 = \kappa_2\,f^{-1}(1)^{\frac{n-1}{4}}\,K_{\frac{n+1}{2}}(m\,f^{-1}(1)^{1/2})$$

as the normalising constant. However, the function ϕ as determined in (B.2) does not satisfy $\phi(0) = 0$, but $\lim_{t\to 0+}\phi(t) = 0$.

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