Stabilised finite elements for compressible fluid flow.

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Stabilised Finite Elements for Compressible Fluid Flow

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Abstract

A finite element formulation for the Euler equations of fluid dynamics is developed and analysed in this work. An overview of fluid dynamics is presented along with an introduction to the finite element method. The Galerkin method is then applied to model problems to demonstrate its performance. Stabilisation in the form of the Galerkin Least-Squares method is added and different variations of stabilisation parameter are analysed for different governing equations. For temporal discretisation the generalised-α method is applied and studied. The Euler equations of fluid dynamics are transformed into primitive variable form and are stabilised so that the method can be tested for compressible flow problems. It is shown that certain stabilisation parameters can be successfully adapted for use with the Euler equations in primitive variables in order to simulate inviscid fluids.

Declarations

This work has not previously been accepted in substance for any degree and is not being concurrently submitted in candidature for any degree.

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Nomenclature

Roman Symbols

$\mathbf{a}$ Convective velocity vector in the advection-diffusion equation

c Speed of sound

c_p Specific heat at constant pressure

c_v Specific heat at constant volume

E Total energy per unit mass

e Internal energy per unit mass

$\mathscr{F}$ Force term

f Source term

H Enthalpy

k Thermal conductivity

$\mathscr{L}(\bullet)$ Differential operator

l Length scale

$\mathbf{n}$ Normal vector

p Pressure

R Gas constant

R Residual

u Scalar unknown, e.g. concentration or temperature, in the advection-diffusion equation

$\dot{u}$ Time derivative of u

Greek Symbols

α Péclet number, ratio of advection to diffusion

α_f Generalised-α time integration parameter

α_m Generalised-α time integration parameter

Δ Laplace operator

δ_{ij} Kronecker Delta

Γ Domain boundary

Γ_g Dirichlet boundary

γ Generalised-α time integration parameter

γ Ratio of specific heats

Γ_t Neumann boundary

μ Coefficient of diffusion

μ_{art} Artificial viscosity coefficient

∇ Gradient operator

ρ^h_∞ Spectral radius

Chapter 1

Introduction

The numerical modelling of Fluid-Structure Interaction (FSI) is an area of ongoing research with a wide degree of application to physical problems. The simulation of compressible fluids in the context of FSI is not particularly well explored, however. Examples of phenomena involving compressibility in FSI include high speed flow for aerospace applications, such as the dynamics of aircraft wings at transonic velocities, or the interaction of rocket nozzles and their exhaust. There are several challenges involved in accurately modelling such systems, and aspects of some of these will be analysed in this thesis.

1.1 Aims of the Thesis

The purpose of this thesis is to develop a numerical scheme based on the finite element method that can accurately model compressible flow, as well as being suited for use with fluid-structure interaction. The Euler equations are chosen to model the fluid, and they are described using a set of primitive variables. These variables are not commonly solved for but are desirable in the context of fluid-structure interaction. Semi-discrete time integration in the form of the generalised-α method is employed and Galerkin Least-Squares stabilised finite elements are used to discretise the equations. Initially, these methods are tested on simpler governing equations that mimic fluid behaviour and their accuracy and suitability is verified. Subsequently, the techniques developed are applied to the Euler equations and tested.

1.2 Layout of the Thesis

The thesis is divided into four main sections that cover the development and testing of the method. In the next chapter, the equations of fluid flow are derived from a set of conservation laws to produce a set of partial differential equations. An overview of the finite element method is then presented and applied to the heat equation to demonstrate the process.

In Chapter 3, a steady-state advection-diffusion model problem is described and the Galerkin method is applied to analyse its accuracy and deficiencies. Stabilisation in the form of Galerkin Least-Squares is added to compensate for errors. Time integration is then introduced using the generalised-*alpha* method and a transient example of the equations is studied, with an analysis of the effect of changing the stabilisation and time integration parameter.

Chapter 4 introduces nonlinearity in the form of Burgers' equation and discusses solution strategies. The impact of discontinuities in the solution is demonstrated with reference to exact solutions. Chapter 5 applies the methods developed in the preceding chapters to the Euler equations in primitive variables, in order to produce a complete stabilised finite element scheme. The method is then compared to known solutions for a shock tube in order to assess its performance. The complete scheme forms a basis for a finite element algorithm that may be employed to solve compressible inviscid fluid problems.

Chapter 2

Fluid Flow and the Finite Element Method

The physical behaviour of fluids and solids may be approximated by sets of Partial Differential Equations (PDEs); the solution of physical problems therefore depends upon solving these differential equations over a given domain. The Navier-Stokes equations are a set of PDEs that, with some modification, can be used to model a variety of different fluid types. They can be approximated by a number of numerical methods, though certain techniques are more suitable than others. The Euler equations are another set of PDEs that describe fluid flow, and can be defined similar to the Navier-Stokes equations but lacking viscous terms. In order to define them, derivation of the equations of fluid flow is first required. A complete derivation can be found in many fluid dynamics textbooks [22], [8].

2.1 Fluid Dynamics Equations

Consider a continuum of fluid within an infinitesimal control volume in a Cartesian coordinate system. Fluid may flow into and out of the volume, but certain quantities must be conserved across it. Given this restriction, a set of conservation laws can be formulated that describe the physics of a fluid within the volume. The overall domain of a problem may then be discretised into smaller volumes that can be modelled with these equations by applying a certain numerical scheme, such as finite elements.

2.1.1 Conservation of Mass

Given the fluid density ρ and velocity u_i, the mass flow ρu_i through the control volume is equal to the rate of change of density, and this can be stated

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0 \tag{2.1}$$

where the subscript i infers Einstein summation convention over spacial dimensions. This equation forms the first component of the fluid dynamics equations.

2.1.2 Conservation of Momentum

In addition to the mass flow in the fluid, the momentum ρu_j of the fluid entering and leaving the volume is equal to the stresses and body forces on the fluid

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_j u_i}{\partial x_i} - \frac{\partial \tau_{ij}}{\partial x_i} + \frac{\partial p}{\partial x_j} - \rho f_j = 0 \tag{2.2}$$

where τ_{ij} is the deviatoric stress and is defined as

$$\tau_{ij} = \mu \left[\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3} \delta_{ij} \frac{\partial u_k}{\partial x_k} \right] \tag{2.3}$$

where $j, k = 1, 2, 3$ indicate dimensions, and δ_{ij} is the Kronecker Delta.

2.1.3 Conservation of Energy

Given E, the total energy per unit mass, the rate of change of the total energy of a given volume ρE can be balanced against other energy terms in the volume, including heat flux q_i, internal energy per unit mass e, enthalpy H and viscous energy dissipation. Firstly, the equations of state are specified. The fluid is assumed to be an ideal gas with constant specific heats, with the following relations:

$$\rho = \frac{p}{RT} \tag{2.4}$$

$$H = E + \frac{p}{\rho} \tag{2.5}$$

$$E = e + \frac{1}{2}u_i u_i \tag{2.6}$$

$$e = \frac{1}{(\gamma - 1)}\frac{p}{\rho} = c_v T \tag{2.7}$$

$$R = c_p - c_v \tag{2.8}$$

$$\gamma = \frac{c_p}{c_v} \tag{2.9}$$

where R is the universal gas constant, c_p is the specific heat at constant pressure, c_v is the specific heat at constant volume, and γ is the ratio of specific heats. These equations combined with the energy terms can be used to express the conservation of energy in the volume, as follows:

$$\frac{\partial \rho E}{\partial t} + \frac{\partial \rho u_i H}{\partial x_i} - \frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right) - \frac{\partial \tau_{ij} u_j}{\partial x_i} - \rho g_i u_i - q_H = 0 \tag{2.10}$$

Here, k is the thermal conductivity.

2.1.4 The Assembled Equations

The above conservation laws can be assembled into a complete set of fluid dynamics equations as follows:

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0 \\
&\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_j u_i}{\partial x_i} - \frac{\partial \tau_{ij}}{\partial x_i} + \frac{\partial p}{\partial x_j} - \rho f_j = 0 \\
&\frac{\partial \rho E}{\partial t} + \frac{\partial \rho u_i H}{\partial x_i} - \frac{\partial}{\partial x_i}\left(k \frac{\partial T}{\partial x_i}\right) - \frac{\partial \tau_{ij} u_j}{\partial x_i} - \rho g_i u_i - q_H = 0
\end{aligned} \tag{2.11}$$

These equations are collectively known as the *Navier-Stokes equations.* Expressed in vector format, these become

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}_i}{\partial x_i} + \frac{\partial \mathbf{G}_i}{\partial x_i} + \mathbf{Q} = \mathbf{0} \tag{2.12}$$

where

$$\mathbf{U} = \left\{ \begin{array}{c} \rho \\ \rho u_j \\ \rho E \end{array} \right\} \tag{2.13}$$

$$\mathbf{F}_i = \left\{ \begin{array}{c} \rho u_i \\ \rho u_j u_i + p\delta_{ji} \\ \rho H u_i \end{array} \right\} \tag{2.14}$$

$$\mathbf{G}_i = \left\{ \begin{array}{c} 0 \\ -\tau_{ji} \\ -(\tau_{ij} u_j) - k\dfrac{\partial T}{\partial x_i} \end{array} \right\} \tag{2.15}$$

$$\mathbf{Q} = \left\{ \begin{array}{c} 0 \\ \rho g_j \\ \rho g_i u_i - q_H \end{array} \right\} \tag{2.16}$$

The second term in 2.12 is known as the *convective* or *advective* flux component, the third term represents the *diffusive* flux, and the final vector is the source term. In this derivation the control volume is fixed in space, meaning this is considered in the Eulerian formulation of the equations. Also note that as angular momentum is conserved, $\tau_{ij} = \tau_{ji}$. The vector of independent unknown variables can be stated as:

$$\mathbf{U} = \left\{ \begin{array}{c} \rho \\ \rho u_j \\ \rho E \end{array} \right\} \tag{2.17}$$

A complete set of partial differential equations is achieved with the addition of suitable boundary conditions. These may describe an inlet or outlet boundary, a wall, or some prescribed velocity or pressure, amongst others. Terms are added to the fluid equations that allow these types of boundaries to be described, and their construction is considered subsequently. Now that we have obtained a set of partial differential equations that describe fluid flow, they must be solved to analyse physical problems. There are many numerical techniques that may be used, and among them is the finite element method, which has been successfully applied in many contexts to fluid problems, and is described in the following section.

2.2 Finite Element Discretisation and the Classical Galerkin Method

Many physical problems (such as the behaviour of fluids) can be described by differential equations, but often it is impractical or impossible to solve the equations directly. Thus, it is necessary to apply a numerical approximation to the problem in order to obtain a solution. The Finite Element Method (FEM) is one such numerical scheme that involves dividing the problem domain into elements, applying an approximation to a function over the element and solving a system of equations constructed from each element. The finite element method has been extensively used in numerical simulations due to its ability to handle unstructured meshes and its capacity to be modified to solve a wide variety of physical problems. In the following section the Galerkin method, a classic and widely used approximation, is derived. It belongs to a wider class of weighted residual methods, schemes that work by finding a solution that minimises a residual equation multiplied by a weight; the choice of this weight can determine the particular method used. A complete derivation can be found in several textbooks on the finite element method, such as [21], [2] and [15].

2.2.1 Obtaining the Weak Form of the Equations

Consider a set of differential equations that may be symbolised by a differential operator $\mathscr{L}(\bullet)$ in the domain Ω with boundary Γ

$$\mathscr{L}(u) - \mathscr{F} = 0 \tag{2.18}$$

with the boundary conditions

$$u - g = 0 \quad \text{on} \quad \Gamma_g \tag{2.19}$$

$$\mathbf{q}(u) \cdot \mathbf{n} - t = 0 \quad \text{on} \quad \Gamma_t \tag{2.20}$$

Here, u is an unknown function that will be approximated over the domain, and $\mathscr{F}$ is a source term. Equation 2.19 describes the Dirichlet boundary conditions, where g is the prescribed value of the solution along the boundary Γ_g. The Neumann boundary conditions are represented by 2.20, where the flux $\mathbf{q}(u)$ acts on a normal vector $\mathbf{n}$ to the Neumann boundary Γ_t. Collectively, these are known as the *strong form* of the equations. Usually, it is impossible to solve the problem when the equations are in this form, so it is necessary to solve an integral form of the equations.

To obtain this expression, it is useful to examine the steady-state heat equation, which takes the form

$$k\Delta u = -f \quad \forall \mathbf{x} \in \Omega \tag{2.21}$$

with temperature flux

$$q = k\nabla u \tag{2.22}$$

where k is the thermal conductivity and f is a source term. Multiplying by an arbitrary function w and Integrating 2.18 over the domain Ω will yield a weakened form of the equation, as follows

$$\int_\Omega w(k\Delta u)\,\mathrm{d}\Omega = -\int_\Omega wf\,\mathrm{d}\Omega \tag{2.23}$$

Integration by parts (or Green's theorem in higher dimensions) is used to reduce the order of the equation and introduce a boundary term

$$\int_\Omega \nabla w(k\nabla u)\,\mathrm{d}\Omega = -\int_\Omega wf\,\mathrm{d}\Omega + \int_{\Gamma_t} w\mathbf{n}k\nabla u\,\mathrm{d}\Gamma \tag{2.24}$$

This is known as the *weak form* of the equation, as it only contains first order terms. It forms the basis for the subsequent finite element discretisation.

2.2.2 Shape functions and Isoparametric Elements

The domain Ω may be subdivided into elements Ω_e so that the solution may be better approximated. A function u over an element may be approximated by a function u^h that is constructed from the sum of a set of shape functions N_a multiplied by unknown values u_a at the nodes of the element.

$$u \approx u^h = \sum_{a=1}^{n} N_a(x_i)u_a \tag{2.25}$$

in a similar fashion, the arbitrary function w is also approximated as

$$w \approx w^h = \sum_{b=1}^{n} W_b(x_i)\delta \mathrm{w}_b \tag{2.26}$$

where w is the nodal values of w^h. The elements are finite dimensional subspaces W^h, usually with continuous piecewise polynomial shape functions. The shape functions sum to 1 at all points in the element, and each are set to have a value of 1 at their corresponding node and zero at all other nodes. As the shape functions are polynomial, the integration of the equation may be performed with the use of Gaussian quadrature, which allows the exact integration using n points of an order $2n-1$ polynomial (occasionally the integration is inexact, e.g. for high aspect ratio elements). Higher-order elements have more nodes and use higher-order shape functions in their construction, which requires a greater number of Gauss points in order to integrate. It is often necessary to introduce a transformation from geometric coordinates to parametric coordinates so that the shape functions may be easily defined and integrated. The transformation is dependent upon the geometry and type of the element, but involve a mapping to a coordinate system in the region $-1 \leq \zeta \leq 1$.

2.2.3 The Galerkin Method

The Galerkin method involves finding an approximate solution u^h to the strong form solution u by finding $u^h \in W^h$ such that

$$\int_\Omega \nabla w^h (k \nabla u) \, \mathrm{d}\Omega = -\int_\Omega w^h f \, \mathrm{d}\Omega + \int_{\Gamma_t} w^h \mathbf{n} k \nabla u \, \mathrm{d}\Gamma \quad \forall w^h \in W^h \tag{2.27}$$

Choosing the finite element spaces W^h to be spaces of piecewise continuous polynomials (i.e. the shape functions N_b) yields the standard finite element method

$$\int_\Omega k \nabla \mathbf{N} \cdot \nabla u^h \, \mathrm{d}\Omega = \int_{\Gamma_t} \mathbf{N} \mathbf{q}(u) \mathbf{n} \, \mathrm{d}\Gamma - \int_\Omega \mathbf{N} f \, \mathrm{d}\Omega \tag{2.28}$$

This method, along with some modification, can be used to model a variety of problems. However, it suffers from some limitations - notably instability for advection dominated cases. In order to use the Galerkin method to model such problems the method may have to be augmented with suitable stabilisation.

Chapter 3

The Advection-Diffusion Equation

3.1 The Steady-State Equation

The advection diffusion equation is a simple partial differential equation that is used to illustrate the basic properties of fluid flow. Consider an arbitrary scalar quantity, such as concentration or temperature, that is transported through a domain via advection (or convection) and diffusion. In its general steady-state form, the equation can be expressed as

$$\mathbf{a}\nabla u - \mu \Delta u = 0 \tag{3.1}$$

where u is the scalar variable, $\mathbf{a}$ is the convective velocity and μ is the diffusion coeficient. The two components of the equation, ∇u and Δu are analogous to the flux and diffusion terms respectively in the equations of fluid flow. Lacking the convective term, it becomes the heat equation, and the diffusion coefficient represents conductivity. The ratio of advection to diffusion in the equation is given by the Péclet number, a dimensionless value defined as

$$\alpha = \frac{|\mathbf{a}|l}{2\mu} \tag{3.2}$$

where l is a length scale associated with the problem, such as element size. When the Péclet number is large ($\alpha > 1$), advection dominates and errors can arise in certain approximations, the effects of which will be analysed in the following chapter.

3.1.1 Finite Element Discretisation

To demonstrate the derivation of the Galerkin method for the advection-diffusion equation it is useful to examine the simplest case of the one-dimensional version of the equation. it can be written as

$$au_{,x} - \mu u_{,xx} = 0 \tag{3.3}$$

In order to solve the problem, the weak (or variational) form of the equation is obtained by multiplying the strong form by an arbitrary function (or virtual perturbation) w and then integrating over the domain Ω, to yield

$$\int_{\Omega} w\left(au_{,x} - \mu u_{,xx}\right) \mathrm{d}\Omega = 0 \tag{3.4}$$

This may then be integrated by parts to reduce the order of the equations so that the may be solved using lower-order elements, producing

$$\int_{\Omega} awu_{,x} + \mu w_{,x}u_{,x} \,\mathrm{d}\Omega = \int_{\Gamma_t} \mu w \nabla u \cdot \mathbf{n} \,\mathrm{d}\Gamma \tag{3.5}$$

This is now in a form that allows calculation of the solution of the steady-state problem. In order to verify the method, the calculated result is compared against known exact solutions.

3.1.2 Steady State Example

Consider a domain over the range $\forall x \in [0, L]$, with boundary conditions $u(0) = 0$ and $u(L) = 1$. The convective velocity a and diffusion coefficient μ are set as 1 and 0.01 respectively. The domain is subdivided into n_{elem} elements of width Δx. Recalling the Péclet number, α, we can say that the characteristic length in this case is equivalent to the element width, Δx. For the Galerkin method, the value of α is important to the stability of the solution, with loss of stability above a critical value $\alpha > 1$.

There exists an analytical solution to the 1D steady state advection-diffusion equation with Dirichlet boundary conditions, as it can be considered a simple homogeneous ordinary differential equation that can be solved via the use of characteristic equations. The solution, presented by Zienkiewicz [22], becomes

$$u(x) = u(0) + (u(L) - u(0))\frac{e^{\frac{a}{\mu}x} - 1}{e^{\frac{a}{\mu}L} - 1} \tag{3.6}$$

This can be used to verify the results of numerical approximations obtained from the use of finite elements. It is plotted against nodal data obtained from the solution of a system of equations constructed from the Galerkin method within each element.

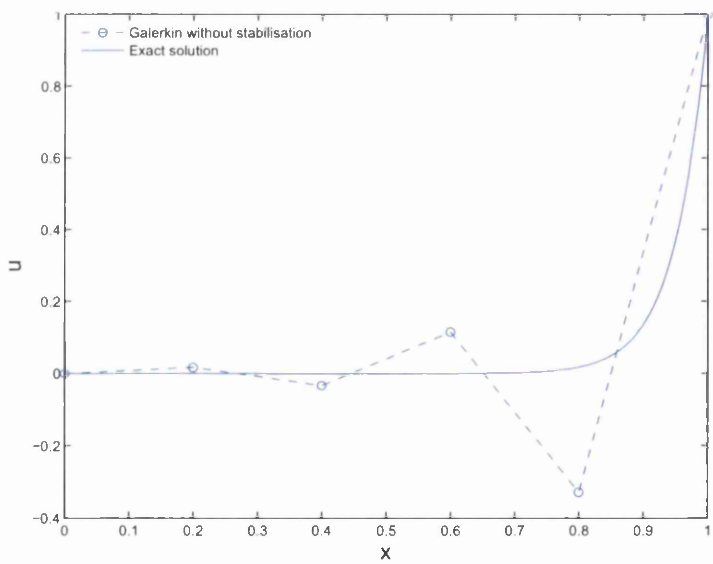


Figure 3.1: Comparison of Galerkin method to exact solution for the advection-diffusion equation. $a = 1$, $\mu = 0.05$, 5 elements.

3.1.3 Stabilisation

It can be seen in Figure 3.1 that the numerical results are very unstable, widely oscillating around the exact solution. The instability inherent in the Galerkin method can however be eliminated with the use of stabilisation. There are many techniques that are employed to stabilise convection-dominated problems, but most of them involve the introduction of higher-order terms that are dependent upon a characteristic element size. The Galerkin/Least-Squares (GLS) method will be used for the purpose of stabilising the equation as it has been shown that it is an effective tool for solving a variety of different problems in fluid flow [25], [19].

the GLS method involves the addition to the Galerkin formulation of the minimisation of the square of the residual, hence the name *least-squares.* If the residual can be stated $R = (\mathscr{L}(u^h) + \mathscr{F})$, then the added term can be stated as

$$\int_{\Omega} (\mathscr{L}^T \mathbf{W}) \tau (\mathscr{L}(u^h) + \mathscr{F}) \, \mathrm{d}\Omega \tag{3.7}$$

where

$$\mathscr{L} = \mathbf{A}_0 \frac{\partial}{\partial t} + \mathbf{A}_1 \frac{\partial}{\partial x} \tag{3.8}$$

and

$$\mathscr{L}^T = \mathbf{A}_0^T \frac{\partial}{\partial t} + \mathbf{A}_1^T \frac{\partial}{\partial x} \tag{3.9}$$

The term τ is stabilisation coefficient, which is proportional to the element size Δx. The choice of τ is crucial to the stability and accuracy of the scheme, and several variants have been derived by different authors. A stabilisation parameter that returns the exact solution for the steady-state advection-diffusion equation has been derived by [21] and may be stated as

$$\tau = \alpha \frac{\mathrm{sign}(a)}{a} \frac{h}{2} \tag{3.10}$$

where

$$\alpha = \alpha_{opt} = \coth|P_e| - \frac{1}{|P_e|} \tag{3.11}$$

The GLS term is then

$$\int_{\Omega} \tau \mathscr{L}(N_a) \left[\mathscr{L}(u^h) + \mathscr{F}\right] \, d\Omega \tag{3.12}$$

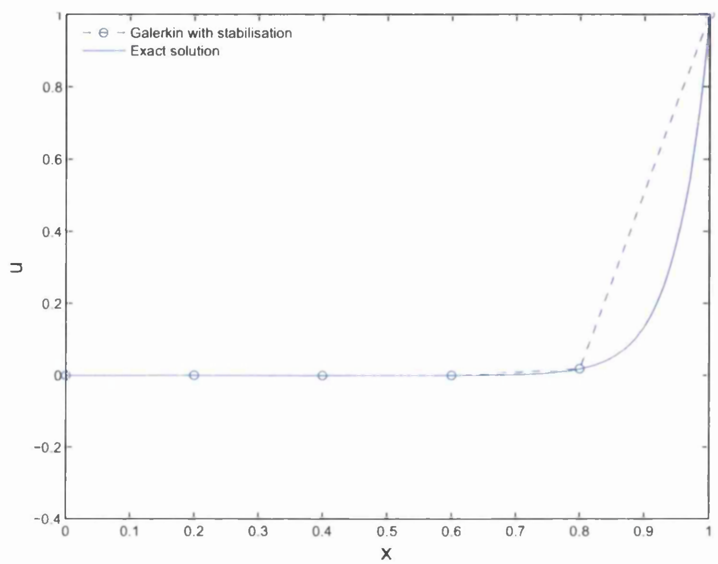


Figure 3.2: Galerkin method with Least-Squares stabilisation compared to exact solution, $a = 1$, $\mu = 0.05$, 5 elements.

As can be seen from Figure 3.2, the addition of Galerkin least-squares stabilisation returns the exact solution, thus compensating for the inability of the standard Galerkin method to model advection-dominated problems. Other formulations of τ for advection-diffusion problems are presented by Fries and Matthies[9]. This method of stabilisation only returns the exact solution for certain ideal cases where τ can be optimally determined. However, it is often not sufficient to completely stabilise the solution if other phenomena, such as discontinuities, are present.

3.2 The Transient Equation

In order to better describe the motion of a fluid it is necessary to simulate a change of state in time. Nearly all problems in numerical modelling of fluids are unsteady in their nature. The advection-diffusion equation can be modified from its steady-state form by adding a time derivative term, creating the unsteady advection-diffusion equation

$$\dot{u} + a\nabla u - \mu\Delta u = 0 \tag{3.13}$$

or, in one dimension,

$$\dot{u} + u_{,x} - u_{,xx} = 0 \tag{3.14}$$

where $u_{,t}$ is the derivative of u with respect to time. As was done with the steady state problem, this term must also be discretised. There are many strategies that can be employed to accomplish this; one such family is called *semi-discrete methods.* These involve applying a finite element approximation in space and discrete time integration scheme to the time derivative term. There are a number of discrete methods that can be used, such as the trapezoidal rule, the generalised midpoint rule, and the generalised-α method. The latter will be used to discretise the governing equation, and details of the method are found in the following section.

3.2.1 The Generalised-α Method of Time Integration

The generalised-α method, as described by G.M. Hubert and J. Chung [16], is often used in the solution of fluid and structural dynamics problems. It has been demonstrated to be very suited for use in the numerical solution of fluids by Jansen et. al. [18]. It is similar to the generalised midpoint rule in that it involves the choice of parameters that determine how 'implicit' the resulting time integration becomes, i.e. how much weight is given to information in the current time step versus information from the previous. It can be stated by substituting

u^h by it's time integrated term

$$\dot{u}^h_{n+\alpha_m} = \frac{\alpha_m}{\gamma \Delta t} u^h_{n+1} - \frac{\alpha_m}{\gamma \Delta t} u^h_n + \left(1 - \frac{\alpha_m}{\gamma}\right) \dot{u}^h_n \tag{3.15}$$

$$u^h_{n+\alpha_f} = \alpha_f u^h_{n+1} + (1 - \alpha_f) u^h_n \tag{3.16}$$

The two parameters, α_m and α_f, can be tuned to produce a time stepping method that is both second-order accurate and unconditionally stable. Unconditional stability is guaranteed by choosing the spectral radius $\rho^h_\infty \leq 1$, and second-order accuracy can be ensured by designating

$$\gamma = \frac{1}{2} + \alpha_m - \alpha_f \tag{3.17}$$

High frequency damping can be controlled by choosing α_m and α_f as

$$\alpha_m = \frac{1}{2} \frac{3 - \rho^h_\infty}{1 + \rho^h_\infty} \tag{3.18}$$

$$\alpha_f = \frac{1}{1 + \rho^h_\infty} \tag{3.19}$$

Note that the choice of $\rho^h_\infty = 1$ means that the generalised-α method becomes the trapezoidal rule. Choosing a lower value of ρ^h_∞ can introduce damping which may be desirable to create a more stable solution in exchange for a small loss in fidelity. this is analysed in more detail by Dettmer [6].

Combining this discrete time integration scheme with a Galerkin finite element approximation in space for the advection-diffusion equation will produce

$$\int_{\Omega_e} \mathbf{N} \dot{u}^h_{n+\alpha_m} + a \mathbf{N} u^h_{,x_{n+\alpha_f}} + \mu \mathbf{N}_{,x} u_{,x} \, \mathrm{d}\Omega = \int_{\Gamma_t} \mu \mathbf{N} t \, \mathrm{d}\Gamma \tag{3.20}$$

This can now be programmed and compared to exact solutions to verify its accuracy. Similar to the steady-state form, the presence of diffusion makes the solution stable and the Galerkin method is suitable for modelling the equation. An analytical solution based on an instantaneous point release (i.e. a Dirac impulse) can be considered. The result is a solution to the heat equation (diffusive component, solving an ODE) in a moving reference frame (advective component,

$x - at$):

$$u(x,t) = \frac{M}{\sqrt{4\pi\mu t}} e^{-\frac{(x-at)^2}{4\mu t}} \tag{3.21}$$

Where M is the magnitude of the initial impulse.

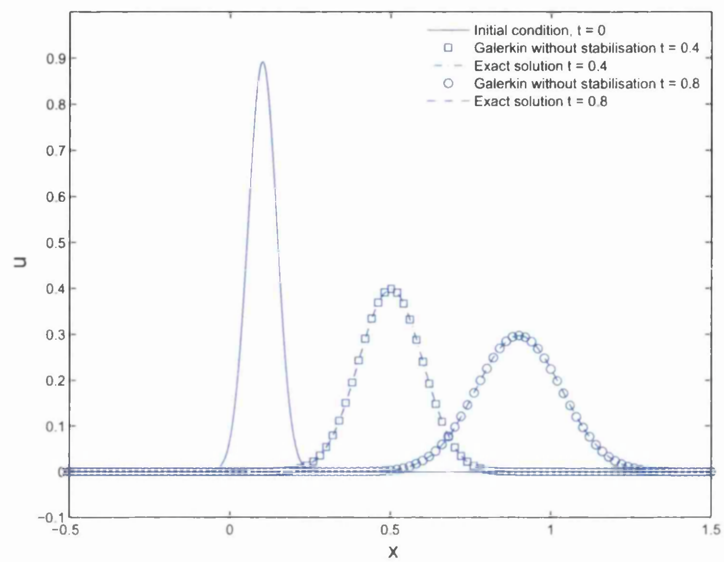


Figure 3.3: Solution of transient advection-diffusion equations, $a = 1$, $\mu = 0.1$, $\Delta t = 0.001$, $\rho_\infty^h = 0$, 1000 elements.

Figure 3.3 demonstrates that in the presence of diffusion the Galerkin method is suited for calculating a very accurate solution to the advection-diffusion equation. However, with low values of diffusion there are errors that may arise.

3.2.2 The Effect of Stabilisation on the Transient Equation

Consider the propagation of a square wave via the advection-diffusion equation through a domain with periodic boundary conditions. Setting the diffusion coefficient μ to be zero, it becomes a simple advection problem - the movement of a wave with convective velocity a. The Péclet number is therefore infinity and the standard Galerkin method will yield an unstable solution, but the application of a stabilisation scheme such as GLS should return a smoother result. The choice of stabilisation parameter τ is taken from Grohmann [10] given as

$$\tau = \frac{\Delta x}{2a}\xi \tag{3.22}$$

where

$$\xi = \frac{1}{\sqrt{1+\left(\frac{3}{\alpha}\right)^2}} \tag{3.23}$$

$$\alpha = \frac{a\Delta x}{2\mu} \tag{3.24}$$

and the overall GLS form of the equation is then

$$\begin{aligned} &\int_{\Omega_e} \mathbf{N}\dot{u}^h_{n+\alpha_m} + a\mathbf{N}u^h_{,x_{n+\alpha_f}} + \mu\mathbf{N}_{,x}u_{,x_{n+\alpha_f}}\,\mathrm{d}\Omega \\ &+ \int_{\Omega} (\mathscr{L}^T\mathbf{N})\tau(\mathscr{L}(u^h_{n+\alpha_f}) + \mathscr{F})\,\mathrm{d}\Omega = \int_{\Gamma_t} \mu\mathbf{N}t\,\mathrm{d}\Gamma \end{aligned} \tag{3.25}$$

The advantage of this new form of τ is that it will be effective even when the diffusion μ is zero. A variant of τ that is unconditionally stable for pure convection problems and third-order accurate in time is stated in [8], and has the definition

$$\tau = \left[\left(\frac{2}{\Delta t}\right)^2 + \left(\frac{2a}{\Delta x}\right)^2\right]^{-1/2} \tag{3.26}$$

Both versions stabilise the equations effectively. This problem is studied with different choices of ρ^h_∞ and advective velocity a, in order to assess the effect of stabilisation on the solution in combination with different time integration and Courant number. The choice of ρ^h_∞ can affect the stability and damping present in the solution.

As can be seen in Figure 3.4, increasing the advective velocity (without stabilisation) causes the resulting solution to be more unstable, which was also a feature of the steady-state equation. Changing the spectral radius ρ^h_∞ (and therefore α_m and α_f) can also effect the solution - less error is present when $\rho^h_\infty = 0$. Adding the stabilisation term to the equation results in a decrease of oscillatory modes

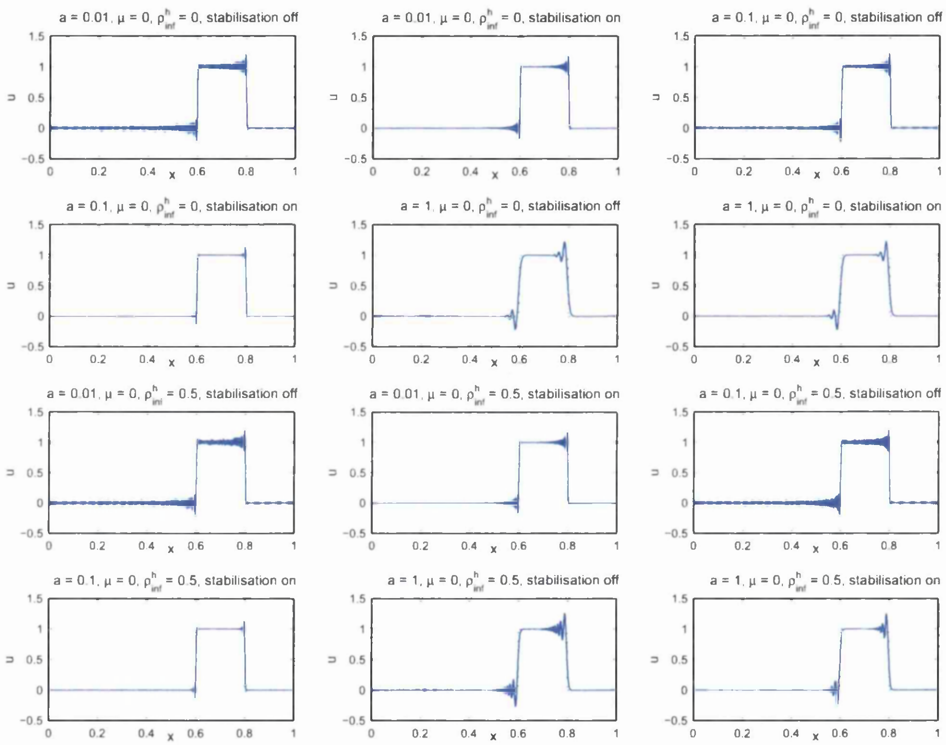


Figure 3.4: Comparison of results to exact solution with and without stabilisation for different velocities and time integration parameter ρ^h_∞ values. In all cases, $\Delta t = 0.01$, 1000 elements, time $t = 1.6$.

throughout the solution, similar to the steady-state problem, but some instability remains around steep gradients. These errors are inherent to the Galerkin Method, and need to be handled with a different stabilisation method, which will be discussed later.

Chapter 4

Nonlinear Problems: Burgers' Equation

4.1 Discretising Burgers' Equation

Burgers' equation is a partial differential equation similar in form to the advection-diffusion equation that is often used to demonstrate the effects of *shocks*, or *discontinuities*, in numerical simulations, as well as being an example of a nonlinear problem. Shocks are regions of the domain where when the solution is potentially multi-valued, and discontinuities occur as a result. These can lead to instability in certain numerical schemes, such as the Galerkin Method. In order to obtain a stable solution, it may be necessary to employ a shock capturing method. The reason that Burgers' equation develops shocks is that instead of possessing a fixed convective velocity, the flux component is instead dependent on velocity u, thus allowing a wave to 'overtake' itself. The equation has both a viscid and inviscid form, and shocks can arise in either, though the presence of diffusion in the viscous form can reduce their effects. The general form is as follows:

$$\dot{u} + u\nabla u - \mu\Delta u = 0 \tag{4.1}$$

with $\mu = 0$ for the inviscid case. The convective velocity has been substituted for velocity u creating a nonlinear equation. The Navier-Stokes and Euler equations are nonlinear in a similar fashion, and so it is useful to study the behaviour at a simpler level in Burgers' equation.

4.1.1 The Equation in 1D

Reducing the equation to one dimension allows the study of the behaviour of solitons, and the equation takes the form

$$u_{,t} + uu_{,x} - \mu u_{,xx} = 0 \tag{4.2}$$

Multiplying this by an arbitrary function and integrating gives the weak form

$$\int_{\Omega} w(u_{,t} + uu_{,x} + \mu u_{,xx}) \, \mathrm{d}\Omega = \int_{\Gamma_t} wt \, \mathrm{d}\Gamma \tag{4.3}$$

Discretising, applying shape functions over an element and setting $\mathbf{W} = \mathbf{N}$ yields the Galerkin formulation

$$\int_{\Omega_e} \mathbf{N}u^h_{,t} + \mathbf{N}u^h u^h_{,x} + \mu \mathbf{N}_{,x} u^h_{,x} \, \mathrm{d}\Omega = \int_{\Gamma_t} \mathbf{N}t\mathbf{n} \, \mathrm{d}\Gamma \tag{4.4}$$

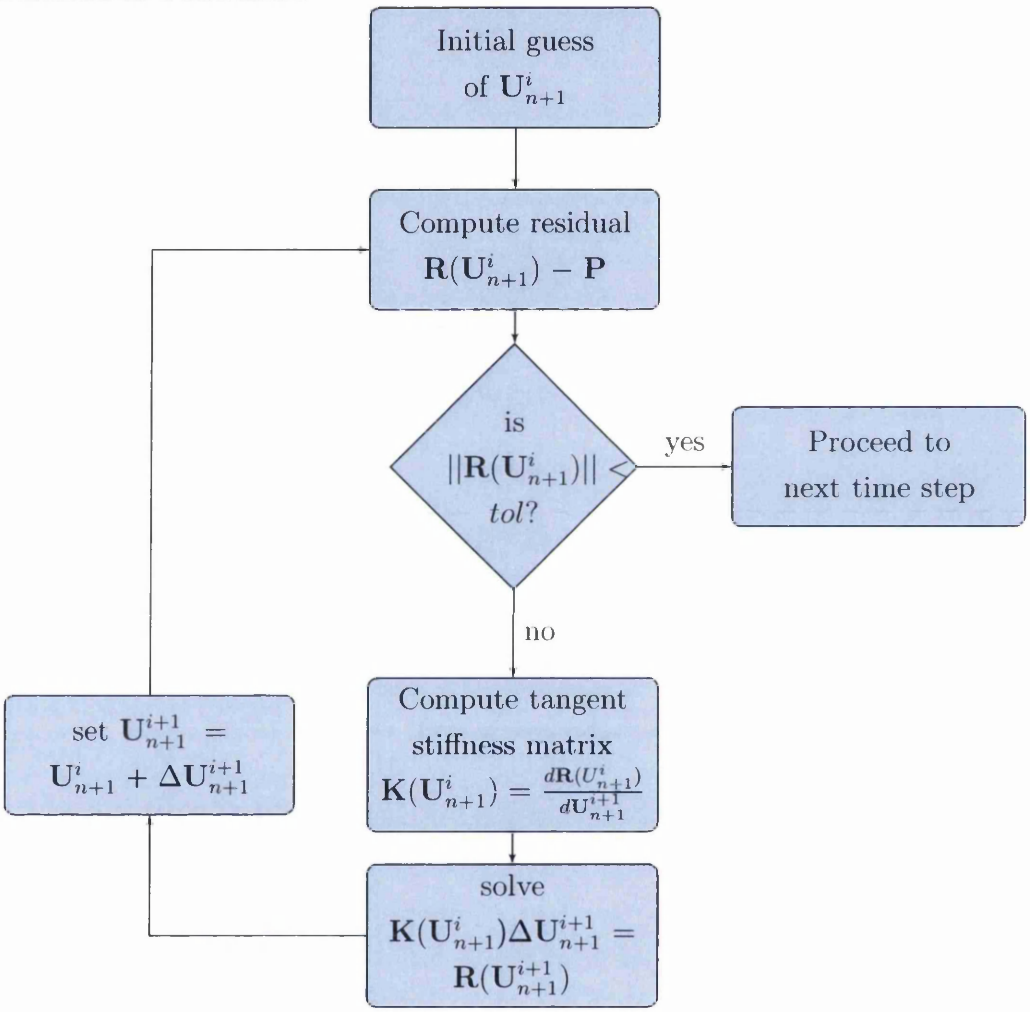
Time integration in the form of the generalised-α method is applied to the transient term, as in the case of advection-diffusion. Galerkin Least squares must also be added to the formulation, but because the advective velocity is not constant, τ is different. a value of τ from [6] is given as

$$\tau = \frac{\Delta x^2}{12\mu} \tag{4.5}$$

This is not an optimal parameter, but it is sufficient in stabilising Burgers' equation. However, the subsequent solution of the equation is not straightforward as in the advection-diffusion case due to the nonlinearity of Burgers' equation.

4.2 Solving a Nonlinear System

When the governing equations in a problem are nonlinear, the assembled system of equations cannot be solved for the unknowns at once. In order to obtain a solution, the equations must be linearised and solved iteratively. One of the most widely used techniques for this process is the Newton-Raphson method, which finds a solution at an asymptotically quadratic rate (in most cases) by calculating the unknowns for a given time step based on an initial guess and substituting them back into the equations to be solved over and over again. The algorithm can be represented as a flowchart:



$\mathbf{U}$ is the vector of unknowns across all nodes. The initial guess of $\mathbf{U}^i_{n+1}$ is chosen as the solution vector from the previous time step $\mathbf{U}_n$. The tolerance is chosen as a number sufficiently small enough to guarantee a degree of accuracy but without adding unnecessary computation time. To create the tangent stiffness matrix $\mathbf{K}$ the equations must be linearised by differentiating the residual with respect to the nodal unknowns. This may result in complex equations, especially when considering stabilisation terms. It may therefore be necessary to not linearise the stabilisation factor, instead constructing it using values of U from time t_n instead of t_{n+1}. This will have a small effect on stability, but should not affect results too much as long as the time step is small.

4.3 Examples

There exist some analytical solutions of Burgers' equation in 1D, allowing a comparison of the finite element approximations to exact results for some problems. A solution based on a method of characteristics has been derived by P.J. Olver [23] and this is useful to assess accuracy of numerical results. A Hopf-Cole transformation ([5], [14]) is used linearise the Burgers' equation into a form that can be explicitly solved. The resulting expression can be used to describe the motion of a wave or step input in a domain, and takes the form

$$u(t,x) = 2\sqrt{\frac{\mu}{\pi t}}\frac{e^{-x^2/(4\mu t)}}{\coth\left(\frac{1}{4\mu}\right) - \operatorname{erf}\left(\frac{x}{2\sqrt{\mu t}}\right)} \tag{4.6}$$

However, this analytical solution breaks down when the diffusion is small ($\mu < 0.03$). The result of plotting the Galerkin method with GLS stabilisation against this analytical solution can be found in Figure 4.1. This verifies the accuracy of using GLS to solve Burgers' equation with some viscosity, and agrees well with similar results obtained by Dogan[7]. The pure inviscid case will prove problematic, however, due to the lack of damping.

4.4 Shock Capturing

Although stabilisation can eliminate instability in the Galerkin method with convection dominated problems, there may still be errors generated around regions of steep gradients in the solution. The inviscid Burgers' equation can exhibit this behaviour as steep gradients are not smoothed by diffusion. Oscillatory modes that form at the shock can propagate upstream in the solution, decreasing accuracy in the larger domain. This can be illustrated by the inviscid case of Burgers' equation in a one dimensional domain. Starting with a sine wave as the ini-

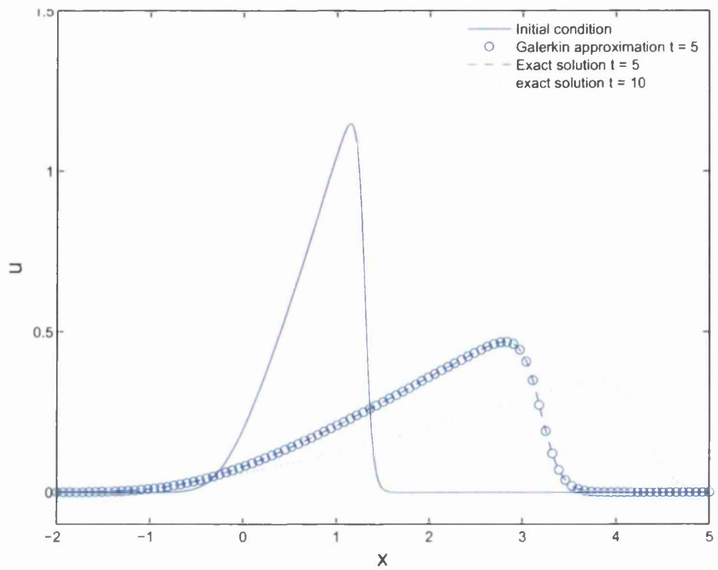


Figure 4.1: Verification of GLS solution of Burgers' equation against analytical solution. 1000 elements, $\Delta t = 0.001$, $\mu = 0$, $\rho^h_\infty = 0.5$.

tial condition, the wave travels through the domain, increasing in steepness as it moves. At some point, the equation is nearly multi-valued and errors appear that can make the scheme unstable. This can be demonstrated in Figure 4.2.

To prevent this behaviour, it is necessary to 'capture' the shock by smoothing the solution with an artificial viscosity term. This artificial viscosity is similar to the diffusive term in the advection-diffusion equation but the viscosity μ is replaced by a coefficient that is proportional to the residual of the solution for each element, thus ensuring that it does not alter the results where no discontinuities are present. The artificial viscosity term can be added to both the Galerkin and GLS formulations, and takes the form

$$\int_\Omega \mu_{art} \nabla \mathbf{W} \cdot \nabla u \, d\Omega \tag{4.7}$$

The choice of μ_{art} is crucial to the stability and accuracy of the solution, and many variants have been proposed. The simplest method, described by Donea [8], is for mu_{art} to be a scalar function of the residual. This is not always an

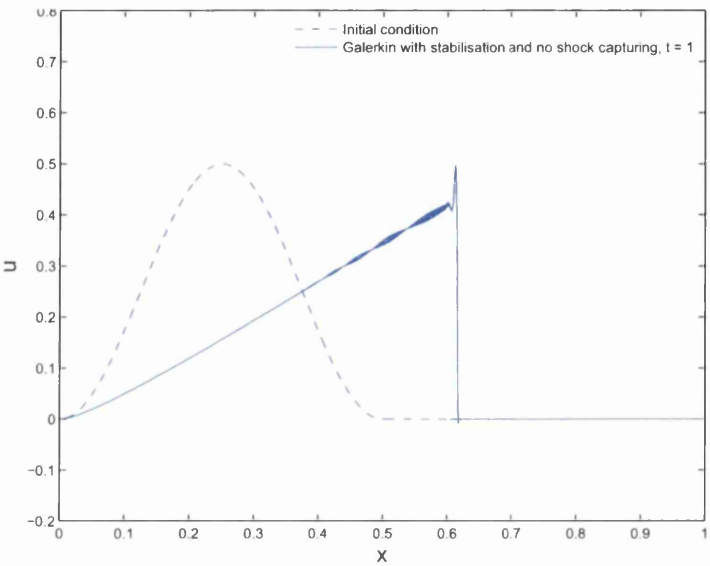


Figure 4.2: Instability around steep gradients for the Galerkin method in Burgers' equation. 1000 elements, $\Delta t = 0.001$, $\mu = 0$, $\rho_{\infty}^{h} = 0.5$.

optimum value for the coefficient, but does reduce oscillatory modes around the shock. μ_{art} should be proportional to Δx so that it vanishes as the element size decreases. Smaller elements are able to capture the profile of the shock more accurately, removing the need for diffusion. The application of a scalar μ_{art} to Burgers' equation results in Figure 4.3. Some instability remains but there are less oscillations at the discontinuity. Better choices for μ_{art} would produce a solution with a smoother shock profile, as demonstrated by Juanes and Patzek [17].

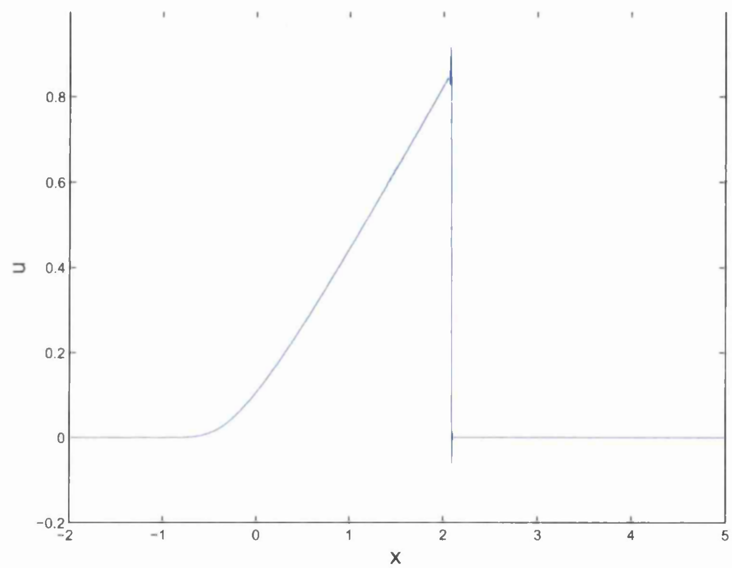


Figure 4.3: GLS with artificial viscosity for Burgers' equation at t = 1.5. 1000 elements, $\Delta t = 0.001$, $\mu = 0$, $\rho_{\infty}^{h} = 0.5$.

Chapter 5

The Euler Equations of Fluid Flow

5.1 Formulation of Euler Equations

The Euler equations of fluid dynamics are a series of equations that describe the behaviour of an inviscid fluid, and correspond to the Navier-Stokes equations without viscous and heat conduction terms. They are useful for solving high-speed flow where the inertial forces of the fluid far outweigh viscous forces within it. The lack of diffusion terms means that the Galerkin method alone will not return a stable solution (similar to problems with the advection-diffusion equation), and problems can arise with discontinuities in the solution in a similar fashion to shock formation in Burgers' equation. Adding suitable stabilisation is therefore essential to accurately model high speed flows.

5.1.1 Obtaining the Euler equations

As the Euler equations describe inviscid flow, the viscous term of the Navier-Stokes equations 2.12 can be discarded. Additionally, heat conduction may be neglected, leaving only the transient and convective flux terms:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}_i}{\partial x_i} = 0 \tag{5.1}$$

This describes the Euler equations in conservative vector form, with

$$\mathbf{U} = \left\{ \begin{array}{c} \rho \\ \rho u_k \\ \rho E \end{array} \right\} \tag{5.2}$$

$$\mathbf{F}_i = \left\{ \begin{array}{c} \rho u_i \\ \rho u_k \\ \rho E \end{array} \right\} \tag{5.3}$$

These equations are the most common form of the Euler equations

5.1.2 Euler Equations in Primitive Variables

Though it is possible to solve the Euler equations using conservation variables, it is desirable to introduce a change of variables. A vector of *primitive variables* is employed, which consists of pressure, velocity and temperature terms. These are chosen for both conceptual ease and usefulness in a fluid-structure interaction problem. Information in the form of pressure, velocity and temperature is shared between the fluid and the solid domains in every time step, so the ability to manipulate the data directly instead of calculating the values for each point along an interface is advantageous. The finite element method, when applied to the Euler equations, maintains global conservation for any set of variables, and is variationally consistent (the exact solution satisfies the weak form). This means the Euler equations can be solved with any set of arbitrary variables, but the choice of variables has an impact on the stability of the solution. This is due to the entropy production inequality that is produced when non-entropy variables are selected. to compensate for this entropy production (i.e. instability), the equations must be amended with suitable dissipative stabilisation. The GLS method will be able to achieve this with careful selection of the parameter τ. The primitive variables take the form

$$\mathbf{Y} = \left\{ \begin{array}{c} p \\ u_k \\ RT \end{array} \right\} \tag{5.4}$$

where p is pressure and u is velocity. The gas constant R and temperature T always occur together in the equations, and so RT is considered an independent variable in order to simplify the resulting equations. Each term in the equations must be expressed as a combination of these variables, so the vector of conservation variables can be redefined as:

$$\mathbf{U}(\mathbf{Y}) = \left\{ \begin{array}{c} \dfrac{p}{RT} \\ \dfrac{pu_j}{RT} \\ p\left(\dfrac{1}{\gamma - 1} + \dfrac{u^2}{2RT}\right) \end{array} \right\} \tag{5.5}$$

We can substitute the variables into the governing equations by introducing transformation matrices. By employing the chain rule of differentiation, we can write

$$\frac{\partial \mathbf{U}}{\partial t} = \frac{\partial \mathbf{U}}{\partial \mathbf{Y}} \frac{\partial \mathbf{Y}}{\partial t} \tag{5.6}$$

where

$$\frac{\partial \mathbf{U}}{\partial \mathbf{Y}} = \mathbf{A}_0 \tag{5.7}$$

$\mathbf{A}_0$ is defined as:

$$\mathbf{A}_0 = \begin{bmatrix} \dfrac{1}{RT} & 0 & -\dfrac{p}{RT^2} \\ \dfrac{u}{RT} & \dfrac{p}{RT}\delta_{ij} & -\dfrac{pu}{RT^2} \\ \dfrac{u^2(\gamma-1) - 2RT}{2p} & \dfrac{u(1-\gamma)}{p} & \dfrac{\gamma - 1}{p} \end{bmatrix} \tag{5.8}$$

where δ_{ij} is the Kronecker Delta.

5.1.3 1D form of the Equations

Reducing the equations to one dimension allows the verification of the numerical scheme by comparing it to analytical solutions that can be derived for simple test cases. Exact solutions are difficult to obtain for the Euler equations in higher dimensions.

In one dimension, all subscripts are equal to 1 and the equations simplify to some degree. In this case, the vector of conservation variables reduces to

$$\mathbf{U} = \left\{ \begin{array}{c} \rho \\ \rho u \\ \rho E \end{array} \right\} \tag{5.9}$$

and the vector of primitive variables becomes

$$\mathbf{Y} = \left\{ \begin{array}{c} p \\ u \\ RT \end{array} \right\} \tag{5.10}$$

The vector of conservation variables can be expressed in terms of the primitive variables as

$$\mathbf{U}(\mathbf{Y}) = \left\{ \begin{array}{c} \dfrac{p}{RT} \\ \dfrac{pu}{RT} \\ \dfrac{p}{(\gamma - 1)} + \dfrac{pu^2}{2RT} \end{array} \right\} \tag{5.11}$$

And the same can be done for the flux vector:

$$\mathbf{F}(\mathbf{Y}) = \left\{ \begin{array}{c} \dfrac{pu}{RT} \\ \dfrac{pu^2}{RT} + p \\ pu\left(\dfrac{1}{(\gamma - 1)} + \dfrac{u^2}{2RT} + 1\right) \end{array} \right\} \tag{5.12}$$

with the transformation matrix

$$\mathbf{A}_0 = \begin{bmatrix} \frac{1}{RT} & 0 & -\frac{p}{RT^2} \\ \frac{u}{RT} & \frac{p}{RT} & -\frac{pu}{RT^2} \\ \frac{1}{RT}\left(\frac{RT}{\gamma-1}+\frac{u^2}{2}\right) & \frac{pu}{RT} & -\frac{pu^2}{RT} \end{bmatrix} \tag{5.13}$$

and its inverse is given by

$$\mathbf{A_0}^{-1} = RT \begin{bmatrix} \frac{u^2(\gamma-1)}{2RT} & \frac{u(1-\gamma)}{RT} & \frac{(\gamma-1)}{RT} \\ -\frac{u}{p} & \frac{1}{p} & 0 \\ \frac{u^2(\gamma-1)-2RT}{2p} & \frac{u(1-\gamma)}{p} & \frac{\gamma-1}{p} \end{bmatrix} \tag{5.14}$$

the flux jacobian $\mathbf{A}_i$ is necessary for contructing the residual in the GLS term, and is given by

$$\frac{\partial \mathbf{F}_i}{\partial x} = \frac{\partial \mathbf{F}_i}{\partial \mathbf{U}}\frac{\partial \mathbf{U}}{\partial x_i} = \mathbf{A}_i \frac{\partial \mathbf{U}}{\partial x_i} \tag{5.15}$$

and in one dimension becomes

$$\mathbf{A}_1 = \frac{1}{RT} \begin{bmatrix} u & p & -\frac{pu}{RT} \\ RT+u^2 & 2pu & -\frac{pu^2}{RT} \\ \frac{u^3}{2}+\frac{RTu\gamma}{\gamma-1} & p\left(\frac{3u^2}{2}+\frac{RT\gamma}{\gamma-1}\right) & -\frac{pu^3}{2RT} \end{bmatrix} \tag{5.16}$$

With the equations in this form, they can be discretised using finite elements.

5.2 Discretising the Euler equations

Now that we have a formulation of the Euler equations in primitive variables, we can solve them by applying the Galerkin method. The Euler equations, as opposed to the Navier-Stokes equations, present problems for the standard Galerkin approximation due to their inviscid nature. The lack of dissipation will lead to instability in the solution, which may necessitate stabilisation of the Galerkin scheme. To obtain the discretised form of the equations, a similar procedure used for the advection-diffusion and Burgers' equations is applied.

5.2.1 Obtaining the Weak Form of the Equations

Multiplying by an arbitrary function and integrating over the domain, and applying a weighted residual approximation gives, in accordance with [12]

$$\int_{\Omega^e} \mathbf{W}^h \mathbf{A}_0 \mathbf{Y}^h_{,t} + \mathbf{W}^{\mathbf{h}} \mathbf{F}(\mathbf{Y}) \mathbf{Y}_{,x} \, \mathrm{d}x = \int_{\Gamma_t} \mathbf{W}(-\mathbf{F}(\mathbf{Y})) n \, \mathrm{d}\Gamma \tag{5.17}$$

Setting the weighting functions equal to the shape functions results in the Galerkin form

$$\int_{\Omega^e} \mathbf{N} \mathbf{A}_0 \mathbf{Y}^h_{,t} + \mathbf{N} \mathbf{F}(\mathbf{Y}) \mathbf{Y}^h_{,x} \, \mathrm{d}x = \int_{\Gamma_t} \mathbf{N}(-\mathbf{F}(\mathbf{Y})) n \, \mathrm{d}\Gamma \tag{5.18}$$

Linear shape functions may be used for the finite element approximation as this equation only contains first order terms. However, this form of the equations will produce an unstable solution due to the choice of variables, and requires additional stabilisation.

5.2.2 Stabilising the Equations

Similar to the earlier examples, a Galerkin-Least Squares method is used to stabilise the Euler Equations. GLS has mostly been applied to finite element approximations of the Navier-Stokes equations, but it should be suitable for use with the Euler equations as it effectively adds an oriented viscous term which will damp out instability. However, most formulations of GLS include terms which are dependent on terms present only in the Navier-Stokes equations, and so they may not provide ideal stability. To implement GLS, a term is added to the Galerkin formulation that takes the form

$$\int_{\Omega^e} (\mathscr{L}^T \mathbf{W}) \cdot \tau(\mathscr{L}(\mathbf{Y} - \mathscr{F}) \, \mathrm{d}x \tag{5.19}$$

where

$$\mathscr{L} = \mathbf{A}_0 \frac{\partial}{\partial t} + \mathbf{A}_1 \frac{\partial}{\partial x} \tag{5.20}$$

and

$$\mathscr{L}^T = \mathbf{A}_0^T \frac{\partial}{\partial t} + \mathbf{A}_1^T \frac{\partial}{\partial x} \tag{5.21}$$

This is similar to the GLS scheme applied to the advection-diffusion and Burgers' equations. However, the stabilisation parameter τ (also known as the matrix of intrinsic time scales) is more complex. Hauke [11] defines, for the compressible Navier-Stokes equations, a non-diagonal τ_{nd} that can be used with primitive variables:

$$\tau_{nd} = \mathbf{A}_0^{-1} \hat{\tau} \tag{5.22}$$

$$\hat{\tau} = \text{diag}(\hat{\tau}_c, \hat{\tau}_m, \hat{\tau}_e) \tag{5.23}$$

where, for one dimension, $\tau_c = \tau_m = \tau e$

$$\hat{\tau}_c = \min\left(\frac{\Delta t}{2}, \frac{1}{\lambda^e}\right) \tag{5.24}$$

$$\lambda^e = \frac{4}{\Delta t^2} + \frac{u + c}{\Delta x^2} \tag{5.25}$$

where c is the speed of sound, defined as

$$c = \sqrt{\gamma R T} \tag{5.26}$$

This form of τ was used in [4] successfully with the Euler equations. Additionally, stabilisation terms that have been used with Streamline-Upwind Petrov-Galerkin (SUPG) methods can be adapted for use with GLS, due to their similarity [24]. Some examples of these τ are presented by Tezduyar and Osawa [27]. When the advective components of the equation are much greater than diffusive parts, the matrix of intrinsic time scales represents the transit times for information to be propagated over one half of the element length [28]. Diffusive phenomena act instantaneously through the domain, and therefore τ tends to zero as diffusion dominates.

5.2.3 Discontinuity Capturing

Although the GLS stabilisation is effective in reducing oscillatory modes throughout most of the solution, additional terms are needed to eliminate errors at discontinuites in the fluid. This is most commonly done via the addition of an artificial viscosity that smooths out the solution in the region of discontinuities, but tends to zero where the solution is smooth. Therefore, it is constructed as a function of the residual. The following can be added to the Galerkin formulation in conjunction with the GLS term to enable shock capturing:

$$\int_{\Omega_e} \nu^h g^{ij} \mathbf{W}_{,x} \cdot \mathbf{A}_0 \mathbf{Y}_{,x} \, \mathrm{d}\Omega \tag{5.27}$$

An example of the operator ν^h is given by Hughes and Mallet [29] as

$$\nu^h_{\text{quad}} = 2\frac{(\mathscr{L}\mathbf{Y} - \mathscr{F}) \cdot \tilde{\tau}(\mathscr{L}\mathbf{Y} - \mathscr{F})}{g^{ij}\mathbf{Y}_{,x_i} \cdot \mathbf{A}_0^{DC}\mathbf{Y}_{x_j}} \tag{5.28}$$

where

$$\mathbf{A}_0^{DC} = \mathbf{V}_{,\mathbf{Y}}^T \mathbf{A}_0 \tag{5.29}$$

$$\tilde{\mathbf{A}}_0 \mathbf{V}_{,\mathbf{Y}} = \mathbf{A}_0 \tag{5.30}$$

$$\tau = \mathbf{Y}_{,\mathbf{V}}\tilde{\tau} \tag{5.31}$$

$\mathbf{V}$ is the vector of entropy variables and g^{ij} is the contravariant metric tensor. Unfortunately this term was unable to be tested successfully in conjunction with the GLS stabilisation, due to its complex construction. See Bater and Darmofal [1] for more examples of artificial viscosity parameter construction for the Euler Equations.

5.3 One-Dimensional example: Riemann shock tube

This problem, also known as a Sod shock tube (introduced by Sod [26]), is commonly used to analyse the accuracy of numerical solutions of fluid problems, particularly the Euler equations. It consists of two states of an ideal gas initially separated by a membrane in a tube, each at different pressures and temperatures of sufficient difference from each other to cause shocks to form when the gases are allowed to mix. It can be modelled in one dimension using the Euler equations, with exact solutions recovered by solving a Riemann problem across discontinuities. The problem is stated with the following initial condition, according to Hirsch [13]:

$$\begin{aligned} p_L &= 10^5, \quad \rho_L = 1, \qquad u_L = 0; \\ p_R &= 10^4, \quad \rho_R = 0.125, \quad u_R = 0; \end{aligned} \tag{5.32}$$

where the subscripts L and R denote left and right sides of the domain, respectively. This gives an initial pressure ratio $P = 10$, which will cause three distinct behaviours in the solution: an expansion fan, a contact discontinuity and a shock wave. Both the contact discontinuity and shock wave will propagate right in the fluid, whereas the expansion fan travels left. The beginning of the expansion fan, and the shock wave, both move with the speed of sound in the undisturbed medium, therefore following characteristics of the system. Across each feature certain properties are conserved. Entropy is constant along the expansion fan, and across the contact discontinuity, pressure and velocity of the fluid normal to the surface are constant, but there is a discontinuity in density. The density of the fluid across the shock wave can be calculated using the Rankine-Hugoniot shock jump conditions. Given this information, the exact solution for the problem can be attained.

5.3.1 Discussion of results

Figures 5.1, 5.2 and 5.3 show the resultant state of the fluid at time $t = 1.2$ with time integration parameter $\rho^h_\infty = 0$. The problem was also attempted with

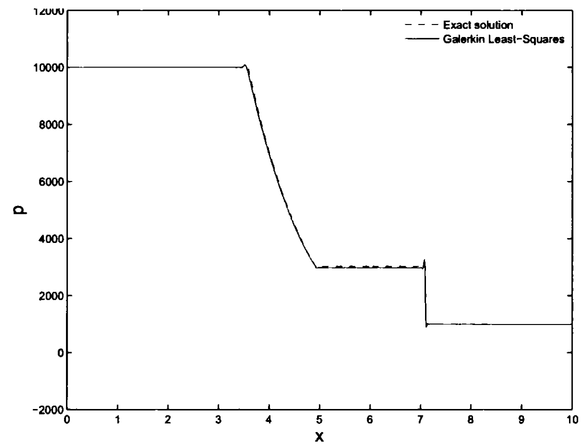


Figure 5.1: Pressure in solution to the shock tube problem at time t = 1.2

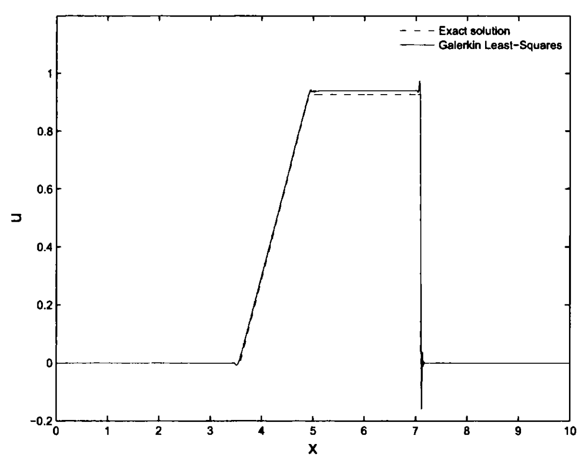


Figure 5.2: Velocity in solution to the shock tube problem at time t = 1.2

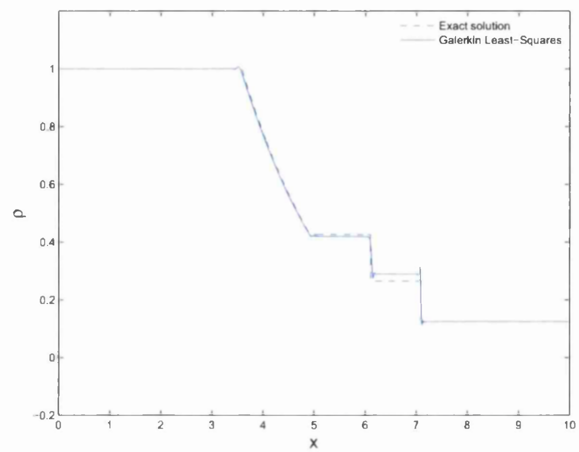


Figure 5.3: Density in solution to the shock tube problem at time t = 1.2

different ρ_∞^h values, yielding very similar results; however, values of ρ_∞^h greater than 0.5 did not allow the solution to converge. The same was true for higher initial pressure ratios. Stabilisation has been added in each case, as without GLS the Galerkin method does not converge to a solution for even low pressure ratios. The difference in pressure and density in the initial condition has caused a shock wave to form, which can be observed as the rightmost discontinuity in the density plot (Figure 5.3). The numerical solution seems to match the analytical solution relatively closely, especially in the expansion fan region (the slope in the pressure and density plots). The contact discontinuity (second step in the density plot) also displays a close match to the analytical solution. Therefore, the GLS stabilisation seems to work very well with a non-diagonal τ as the stabilisation parameter, not exhibiting any over-diffusive behaviour. This would manifest by decreasing the solution gradient around the shock and contact discontinuity, but they retain a steep profile.

Chapter 6

Conclusions

The aim of this thesis was to derive a finite element framework for compressible fluid-structure interaction by applying a stabilised finite element method to a primitive variable formulation of the Euler equations. This was achieved by building on the basic concept of the Galerkin method to yield a scheme that was able to effectively model a compressible fluid.

The Galerkin Least-Squares stabilisation has been shown to be effective for use with the Euler equations, and the matrix τ does reduce instability. However, the composition of τ is not optimal, and so the resulting term does not guarantee unconditional stability. For some very large values of pressure ratio ($P \sim 100$) in the shock tube example the solution was not able to converge. In the literature, there are several different constructions of τ, mostly for the Navier-Stokes equations in conservation variables, but each variant is a compromise. The choice made in this thesis is sufficient but could be improved so that higher pressure ratios do not present a problem. A construction of a stabilisation operator by Polner et. al. [20] would be a suitable choice if further work was undertaken.

6.1 Recommendations for Future Work

Creating a solver that incorporates a linear elastic model for solid mechanics is the obvious continuation of the result of this thesis. The piston problem, as studied by Blom [3] would be suitable for preliminary fluid-structure interaction studies. Extending this stabilised finite element formulation to higher dimensions would be another next step in the research, so that it can be applied to a wider variety of physical problems. Ideally, an optimum stabilisation parameter τ should be developed, although it is difficult to derive from first principles. However, there are other versions of τ constructed by different authors that may be preferable for GLS, and further research should yield a more optimised parameter. Additionally, a shock capturing algorithm will need to be refined such that small errors around discontinuities are minimised, which would also allow for the method to solve problems with much higher pressure gradients. A mesh refinement method would be a useful tool in this regard. The techniques applied to the Euler equations in Chapter 5 would also be applicable to the Navier-Stokes equations, and so it could be desirable to adapt them for modelling viscous fluids. Finally, formulating the Euler equations using entropy variables is a promising avenue of inquiry as it may result in a more robust method due to less dependency on stabilisation.

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