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PERIODS AND MOTIVES IN THE SPECTRAL ACTION OF ROBERTSON–WALKER SPACETIMES

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Abstract. We show that, when considering the scaling factor as an affine variable, the coefficients of the asymptotic expansion of the spectral action on a (Euclidean) Robertson-Walker spacetime are periods of mixed Tate motives, involving relative motives of complements of unions of hyperplanes and quadric hypersurfaces and divisors given by unions of coordinate hyperplanes.

Contents

1. Introduction

Over the past decade, Grothendieck's theory of motives has come to play an increasingly important role in theoretical physics. While the existence of a relation between motives and periods of algebraic varieties and computations in high-energy physics might have seemed surprising and unexpected, the existence of underlying motivic structures in quantum field theory has now been widely established, see for instance [2], [3], [7], [15]. Typically, periods and motives occur in quantum field theory in the perturbative approach, through the asymptotic expansion in Feynman diagrams, where in the terms of the asymptotic expansion the renormalized Feynman integrals are identified with periods of certain hypersurface complements. The nature of the motive of the hypersurface constraints the class of numbers that can occur as periods. Similarly, a large body of recent work on amplitudes in $N = 4$ Supersymmetric Yang–Mills has uncovered another setting where the connection to periods and motives plays an important role, see [1], [11], [12].

In this paper, we present another surprising instance of the occurrences of periods and motives in theoretical physics, this time in a model of (modified) gravity based on the spectral action functional of [5]. The situation is somewhat similar to the one seen in the quantum field theory setting, with some important differences. As in the QFT framework, we deal with an asymptotic expansion, which in our case is given by the large energy expansion of the spectral action functional. We show in this paper that, in the case of (Euclidean) Robertson–Walker spacetimes, the terms of the asymptotic expansion of the spectral action functional can be expressed as periods of mixed Tate motives, given by complements of quadric hypersurfaces. An important difference, with respect to the case of a scalar massless quantum field theory of [2], is that here we need to consider only one quadric hypersurface for each term of the expansion, whereas in the quantum field theory case one has to deal with the much more complicated motive of a union of quadric hypersurfaces, associated to the edges of the Feynman graph. On the other hand, the algebraic differential form that is integrated on a semi-algebraic set in the hypersurface complement is much more complicated in the spectral action case considered here, than in the quantum field theory case: the terms in the algebraic differential form arise from the computation, via pseudo-differential calculus, of a parametrix for the square of the Dirac operator on the Robertson–Walker spacetime, after a suitable change of variables in the integral. While the explicit expression of the differential form, even for the simplest cases of the coefficients a_2 and a_4 can take up several pages, the structure of the terms can be understood, as we explain in the following sections, and the domain of definition is, in the case of the a_{2n} term, the complement of a union of two hyperplanes and a quadric hypersurfaces defined by a family of quadrics $Q_{\alpha,2n}$ in an affine space $\mathbb{A}^{2n+3}$.

In Section 2 we compute, using the Hopf coordinates on the sphere $\mathbb{S}^3$, the pseudo-differential symbol of the square D^2 of the Dirac operator on a (Euclidean) Robertson-Walker metric. In §2.3, we describe briefly how the Seeley-DeWitt coefficients of the heat kernel expansion can be computed in terms of Wodzicki residues, by taking products with auxiliary tori with flat metrics. We present in §2.4 the recursive formula for the terms $\sigma_{-2-n}(\Delta^{-1}_{r+2})$ of the heat kernel expansion of D^2. In §2.5 we introduce the integrals $\gamma_{2n}(\alpha, \alpha_1, \ldots, \alpha_{2n})$ and their densities $\Upsilon_{2n}(\alpha, \alpha_1, \ldots, \alpha_{2n})$ associated to the coefficients a_{2n} of the heat kernel expansion, treating the scaling factor $a(t)$ and its derivatives $a^{(k)}(t)$ as affine coordinates α, α_k. The integrals $\gamma_{2n}(\alpha, \alpha_1, \ldots, \alpha_{2n})$ are what we aim to express in terms of algebro-geometric period integrals. Section 3 contains the main results. We introduce in §3.1 as set of algebraic coordinates, and we show in §3.2 that the volume form is algebraic over $\mathbb{Q}$ in these coordinates. In §3.3 we show that the density $\Upsilon_2(\alpha, \alpha_1, \alpha_2)$ associated to the a_2 term, in the algebraic coordinates is a rational function on the complement in $\mathbb{A}^5$ of the union of a quadric hypersurface and two hyperplanes. In §3.4 we prove inductively a formula for the densities $\Upsilon_{2n}(\alpha, \alpha_1, \ldots, \alpha_{2n})$ in algebraic coordinates. The algebraic differential forms depend on $2n$ auxiliary affine parameters $\alpha_1, \ldots, \alpha_{2n}$, which correspond to the time derivatives of the scaling factor of the Robertson–Walker metric. In §3.5, passing to a homologous domain of integration in the cosphere bundle and using the symmetries of the Robertson–Walker metric, we prove that all terms in the expression of $\Upsilon_{2n}(\alpha, \alpha_1, \ldots, \alpha_{2n})$ with half-integer exponent have to cancel out, leaving an algebraic differential form, which is written more explicitly in §3.6. In §3.7 we show that, in the same choice of algebraic coordinates, the domain of integration in the integrals computing the terms $\gamma_{2n}(\alpha, \alpha_1, \ldots, \alpha_{2n})$ is a $\mathbb{Q}$-semialgebraic set. Together with the results of §3.6 about the algebraic differential form, this identifies the $\gamma_{2n}(\alpha, \alpha_1, \ldots, \alpha_{2n})$ with algebro-geometric period integrals. We identify explicitly the associated motives. The $\mathbb{Q}$-semialgebraic set in this hypersurface complement has boundary contained in a divisor given by a union of coordinate hyperplanes. Although the boundary divisor and the hypersurface intersect nontrivially, all the integrals are convergent and we do not have a renormalization problem, unlike what happens in the quantum field theory setting. In Section 4 we analyze more explicitly the motive, showing that, over a quadratic field extension $\mathbb{Q}(\sqrt{-1})$ where the quadrics become isotropic, it is a mixed Tate motive, while over $\mathbb{Q}$ it is a form of a Tate motive in the sense of [17], [19], [20]. We compute explicitly, by a simple inductive argument, the class in the Grothendieck ring of the relevant hypersurface complement. In §4.1, 4.2, and 4.3 we recall some general facts about pencils of quadrics, motives of quadrics, and Grothendieck classes of affine and projective cones. In §4.4 we compute the Grothendieck class and the motive for the case of the a_2 coefficient. In §4.5 we compute inductively the Grothendieck class of the complement $\mathbb{A}^{2n+3} \smallsetminus (H_0 \cup H_1 \cup \widehat{CZ}_{\alpha,2n})$ and in §4.6 we prove that the motive $\mathfrak{m}(\mathbb{A}^{2n+3} \smallsetminus (H_0 \cup H_1 \cup \widehat{CZ}_{\alpha,2n}), \Sigma)$ underlying the periods $\gamma_{2n}(\alpha, \alpha_1, \ldots, \alpha_{2n})$ is mixed Tate.

1.1. **The spectral model of gravity.** The *spectral action* functional, introduced in [5] is a regularized trace of the Dirac operator D given by

$$\mathcal{S}(\Lambda) = \mathrm{Tr}(f(D/\Lambda)) = \sum_{\lambda \in \mathrm{Spec}(D)} \mathrm{Mult}(\lambda) f(\lambda/\Lambda),$$

where the test function f is a smooth even rapidly decaying function, which should be thought of as a smooth approximation to a cutoff function. The parameter $\Lambda > 0$

is an energy scale. One of the main advantages of this action functional is that it is not only defined for smooth compact Riemannian spin manifolds, but also for a more general class of geometric objects that include the noncommutative analogs of Riemannian manifolds, finitely summable spectral triples, see [6]. In particular, the spectral action functional applied to almost commutative geometries (products of manifolds and finite noncommutative spaces) is used as a method to generate particle physics models with varying possible matter sectors depending on the finite geometry and with matter coupled to gravity, see [18] for a recent overview. It was shown in [5] that, in the case of commutative and almost commutative geometries, the spectral action functional has an asymptotic expansion for large energy Λ,

$$\mathrm{Tr}(f(D/\Lambda)) \sim \sum_{\beta \in \Sigma^+_{ST}} f_\beta \, \Lambda^\beta \, ⨍ |D|^{-\beta} \; + \; f(0) \, \zeta_D(0) + \cdots ,$$

where the coefficients depend on momenta $f_\beta = \int_0^\infty f(v) \, v^{\beta-1} \, dv$ and Taylor coefficients of the test function f and on residues

$$⨍ |D|^{-\beta} = \frac{1}{2} \mathrm{Res}_{s=\beta} \; \zeta_D(s)$$

at poles of the zeta function $\zeta_D(s)$ of the Dirac operator. The leading terms of the asymptotic expansion recover the usual local terms of an action functional for gravity, the Einstein-Hilbert action with cosmological term, with additional modified gravity terms given by Weyl conformal gravity and Gauss-Bonnet gravity. In the case of an almost commutative geometry the leading terms of the asymptotic expansion also determine the Lagrangian of the resulting particle physics model. The spectral action on ordinary manifold, as an action functional of modified gravity, was applied to cosmological models, see [16] for an overview. In the manifold case, the Mellin transform relation between zeta function and trace of the heat kernel expresses the coefficients of the spectral action expansion in terms of the Seeley-DeWitt coefficients a_{2n} of the heat kernel expansion,

$$\mathrm{Tr}(e^{-\tau D^2}) \sim_{\tau \to 0+} \quad \tau^{-m/2} \sum_{n=0}^\infty a_{2n}(D^2) \, \tau^n .$$

Pseudodifferential calculus techniques and the parametrix method can then be applied to the computation of the symbol and the Seeley-DeWitt coefficients. The resulting computations can easily become intractable, but a computationally more efficient method introduced in [8], based on Wodzicki residues and products by auxiliary flat tori can be applied to make the problem more easily tractable.

In the case of the Robertson-Walker spacetimes, it was conjectured in [4] and proved in [10] that all the terms in the expansion of the spectral action are polynomials with rational coefficients in the scaling factor and its derivatives. This rationality result suggests the existence of an underlying arithmetic structure. In the case of the Bianchi IX metrics, a similar rationality result was proved in [8] and the underlying arithmetic structure was analyzed in [9] for the Bianchi IX gravitational instantons, in terms of modular forms. Here we consider the case of the Robertson-Walker spacetimes and we look for arithmetic structures in the expansion of the spectral action in terms of periods and motives. A similar motivic analysis of the Bianchi IX case will be carried out in forthcoming work.

2. Robertson-Walker metric, Dirac operator, and heat kernel expansion

In this section we discuss some basic properties of the Dirac operator on a Euclidean Robertson–Walker spacetime, and of the coefficients of the corresponding heat kernel expansion, which we need for our main result.

2.1. **Robertson–Walker metric.** We consider the Robertson-Walker metric with the expansion factor $a(t)$,

$$ds^2 = dt^2 + a(t)^2 d\sigma^2,$$

where $d\sigma^2$ is the round metric on the 3-dimensional sphere $\mathbb{S}^3$. Using the Hopf coordinates for $\mathbb{S}^3$, we consider the following local chart

$$x = (t, \eta, \phi_1, \phi_2) \mapsto (t, \sin\eta\cos\phi_1, \sin\eta\sin\phi_2, \cos\eta\cos\phi_1, \cos\eta\sin\phi_2),$$

$$0 < \eta < \frac{\pi}{2}, \qquad 0 < \phi_1 < 2\pi, \qquad 0 < \phi_2 < 2\pi.$$

In this coordinate system, the Robertson-Walker metric is written as

$$ds^2 = dt^2 + a(t)^2 \left(d\eta^2 + \sin^2(\eta)\, d\phi_1^2 + \cos^2(\eta)\, d\phi_2^2\right), \tag{2.1}$$

or alternatively we write:

$$(g_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a(t)^2 & 0 & 0 \\ 0 & 0 & a(t)^2\sin^2(\eta) & 0 \\ 0 & 0 & 0 & a(t)^2\cos^2(\eta) \end{pmatrix},$$

with

$$(g^{\mu\nu}) = (g_{\mu\nu})^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{a(t)^2} & 0 & 0 \\ 0 & 0 & \frac{\csc^2(\eta)}{a(t)^2} & 0 \\ 0 & 0 & 0 & \frac{\sec^2(\eta)}{a(t)^2} \end{pmatrix}.$$

2.2. **Pseudodifferential symbol.** One can write the local expression for the Dirac operator D of the Robertson-Walker metric (2.1), as in §2 of [10], and one finds that the pseudodifferential symbol σ_D of D is given by

$$\sigma_D(x, \xi) = q_1(x, \xi) + q_0(x, \xi),$$

where the matrices q_1 and q_0 are as follows. Using the notation $\xi = (\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4$ for an element of the cotangent fibre $T_x^* M \simeq \mathbb{R}^4$ at the point $x = (t, \eta, \phi_1, \phi_2)$, we have

$$q_1(x,\xi) = \begin{pmatrix} 0 & 0 & \frac{i\sec(\eta)\xi_4}{a(t)} - \xi_1 & \frac{i\xi_2}{a(t)} + \frac{\csc(\eta)\xi_3}{a(t)} \\ 0 & 0 & \frac{i\xi_2}{a(t)} - \frac{\csc(\eta)\xi_3}{a(t)} & -\xi_1 - \frac{i\sec(\eta)\xi_4}{a(t)} \\ -\xi_1 - \frac{i\sec(\eta)\xi_4}{a(t)} & -\frac{i\xi_2}{a(t)} - \frac{\csc(\eta)\xi_3}{a(t)} & 0 & 0 \\ \frac{\csc(\eta)\xi_3}{a(t)} - \frac{i\xi_2}{a(t)} & \frac{i\sec(\eta)\xi_4}{a(t)} - \xi_1 & 0 & 0 \end{pmatrix},$$

$$q_0(x,\xi) = \begin{pmatrix} 0 & 0 & \frac{3ia'(t)}{2a(t)} & \frac{\cot(\eta)-\tan(\eta)}{2a(t)} \\ 0 & 0 & \frac{\cot(\eta)-\tan(\eta)}{2a(t)} & \frac{3ia'(t)}{2a(t)} \\ \frac{3ia'(t)}{2a(t)} & \frac{\tan(\eta)-\cot(\eta)}{2a(t)} & 0 & 0 \\ \frac{\tan(\eta)-\cot(\eta)}{2a(t)} & \frac{3ia'(t)}{2a(t)} & 0 & 0 \end{pmatrix}. \tag{2.2}$$

That is, the local formula of the action of the Dirac operator D on a spinor s is given by

$$\begin{aligned} Ds(x) &= (2\pi)^{-2} \int e^{ix\cdot\xi}\, \sigma(D)(x,\xi)\, \hat{s}(\xi)\, d\xi \\ &= (2\pi)^{-4} \int\int e^{i(x-y)\cdot\xi}\, \sigma(D)(x,\xi)\, s(y)\, dy\, d\xi, \end{aligned}$$

where $\hat{s}$ is the component-wise Fourier transform of s.

The above matrices can be used to find the pseudodifferential symbol of the square of the Dirac operator:

$$\sigma_{D^2}(x,\xi) = p_2(x,\xi) + p_1(x,\xi) + p_0(x,\xi),$$

where, denoting the 4×4 identity matrix by $I_{4\times 4}$, we have:

$$\begin{aligned} p_2(x,\xi) &= q_1(x,\xi)\, q_1(x,\xi) = \left(\sum g^{\mu\nu}\xi_\mu\xi_\nu\right) I_{4\times 4} \\ &= \left(\xi_1^2 + \frac{\xi_2^2}{a(t)^2} + \frac{\csc^2(\eta)\xi_3^2}{a(t)^2} + \frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right) I_{4\times 4}, \end{aligned} \tag{2.3}$$

$$p_1(x,\xi) = q_0(x,\xi)\, q_1(x,\xi) + q_1(x,\xi)\, q_0(x,\xi) + \sum_{j=1}^{4} -i\frac{\partial q_1}{\partial \xi_j}(x,\xi)\, \frac{\partial q_1}{\partial x_j}(x,\xi), \tag{2.4}$$

$$p_0(x,\xi) = q_0(x,\xi)\, q_0(x,\xi) + \sum_{j=1}^{4} -i\frac{\partial q_1}{\partial \xi_j}(x,\xi)\, \frac{\partial q_0}{\partial x_j}(x,\xi). \tag{2.5}$$

2.3. **Heat expansion and the Wodzicki residue.** It is in general computationally difficult to obtain explicit expressions for the Seeley–DeWitt coefficients of the heat kernel expansions, even for nicely homogeneous and isotropic metrics like the Friedmann–Robertson–Walker case. A computationally more efficient method was introduced in [8], based on products with auxiliary flat tori and Wodzicki residues [22], [23]. We apply it here to calculate the coefficients a_{2n} that appear in the small time heat kernel expansion

$$\mathrm{Tr}(e^{-\tau D^2}) \ \sim_{\tau\to 0^+} \ \tau^{-2} \sum_{n=0}^{\infty} a_{2n}\, \tau^n. \tag{2.6}$$

In fact, it is proved in [8] that, for any non-negative even integer r, we have

$$a_{2+r} = \frac{1}{2^5\, \pi^{4+r/2}} \mathrm{Res}(\Delta^{-1}), \tag{2.7}$$

where

$$\Delta = D^2 \otimes 1 + 1 \otimes \Delta_{\mathbb{T}^r},$$

in which $\Delta_{\mathbb{T}^r}$ is the flat Laplacian on the r-dimensional torus $\mathbb{T}^r = (\mathbb{R}/\mathbb{Z})^r$. Here, the linear functional Res defined on the algebra of classical pseudodifferential operators is the Wodzicki residue, which is defined as follows. Assume that the dimension of the manifold is m, and that the symbol of a classical pseudodifferential operator is given in a local chart U by

$$\sigma(x,\xi) \sim \sum_{j=0}^{\infty} \sigma_{d-j}(x,\xi) \qquad (\xi \to \infty),$$

where each $\sigma_{d-j} : U \times (\mathbb{R}^m \setminus \{0\}) \to M_r(\mathbb{C})$ is positively homogeneous of order $d-j$ in ξ. Then one needs to consider the 1-density defined by

$$\mathrm{wres}_x P_\sigma = \left(\int_{|\xi|=1} \mathrm{tr}\left(\sigma_{-m}(x,\xi)\right) |\sigma_{\xi,\,m-1}| \right) |dx^0 \wedge dx^1 \wedge \cdots \wedge dx^{m-1}|, \tag{2.8}$$

in which $\sigma_{\xi,\,m-1}$ is the volume form of the unit sphere $|\xi| = 1$ in the cotangent fibre $\mathbb{R}^m \simeq T_x^*M$ given by

$$\sigma_{\xi,\,m-1} = \sum_{j=1}^{m} (-1)^{j-1} \xi_j \, d\xi_1 \wedge \cdots \wedge \widehat{d\xi_j} \wedge \cdots \wedge d\xi_m. \tag{2.9}$$

The Wodzicki residue of the pseudodifferential operator P_σ associated with the symbol σ is by definition the integral of the above 1-density associated to σ:

$$\mathrm{Res}\,(P_\sigma) = \int_M \mathrm{wres}_x P_\sigma. \tag{2.10}$$

One can find a detailed discussion of the Wodzicki residue in [22], [23] and in Chapter 7 of [13].

2.4. **Recursive formula for densities.** A recursive formula for the densities in (2.8) is the crucial property underlying our main result. It is obtained by performing symbolic calculations as explained in [8], to which we refer the reader for the details of the argument.

Lemma 2.1. *For a positive integer r, let Δ_{r+2} denote the operator*

$$\Delta_{r+2} = D^2 \otimes 1 + 1 \otimes \Delta_{\mathbb{T}^r},$$

where $\Delta_{\mathbb{T}^r}$ is the flat Laplacian on the r-dimensional torus $\mathbb{T}^r = (\mathbb{R}/\mathbb{Z})^r$. Then one has

$$\sigma_{-2}(\Delta_{r+2}^{-1}) = \big(p_2(x,\xi_1,\xi_2,\xi_3,\xi_4) + (\xi_5^2 + \cdots + \xi_{4+r}^2) I_{4\times 4}\big)^{-1}, \tag{2.11}$$

and, for $n > 0$, the terms $\sigma_{-2-n}(\Delta_{r+2}^{-1})$ are computed by the recursive formula

$$\sigma_{-2-n}(\Delta_{r+2}^{-1}) = -\left(\sum_{\substack{0\le j<n,\, 0\le k\le 2 \\ \alpha\in\mathbb{Z}^4_{\ge 0} \\ -2-j-|\alpha|+k=-n}} \frac{(-i)^{|\alpha|}}{\alpha!} \left(\partial_\xi^\alpha \sigma_{-2-j}(\Delta_{r+2}^{-1})\right) \left(\partial_x^\alpha p_k\right) \right) \sigma_{-2}(\Delta_{r+2}^{-1}). \tag{2.12}$$

In particular, when we express the term a_{2n} as the noncommutative residue (2.7),

$$a_{2n} = \frac{1}{2^5\,\pi^{3+n}} \mathrm{Res}(\Delta_{2n}^{-1}), \tag{2.13}$$

one needs to compute the term σ_{-2n-2} which is homogeneous of order $-2n-2$ in the expansion of the pseudodifferential symbol of Δ_{2n}^{-1}. This is obtained from the following recursion, which is a specialization of the previous lemma.

Corollary 2.2. *The densities $\mathrm{tr}(\sigma_{-2n-2}(\Delta_{2n}^{-1}))$ can be computed through the recursion*

$$\sigma_{-2n-2}(\Delta_{2n}^{-1}) = -\left(\sum_{\substack{0\le j<2n,\, 0\le k\le 2 \\ \alpha=(\ell_1,\ell_2)\in\mathbb{Z}^2_{\ge 0} \\ -2-j-|\alpha|+k=-2n}} \frac{(-i)^{\ell_1+\ell_2}}{\ell_1!\,\ell_2!} \left(\partial_{\xi_1}^{\ell_1}\partial_{\xi_2}^{\ell_2} \sigma_{-2-j}(\Delta_{2n}^{-1})\right) \left(\partial_t^{\ell_1}\partial_\eta^{\ell_2} p_k\right) \right) \sigma_{-2}(\Delta_{2n}^{-1}). \tag{2.14}$$

Proof. The expression (2.14) follows immediately from Lemma 2.1, upon observing that, by (2.2) and (2.3), (2.4), (2.5), the terms p_k only depend on the coordinates (t,η) and not on the angles (ϕ_1,ϕ_2) in the Hopf coordinates of S^3. $\square$

2.5. **Integrated densities and differential forms.** For the rest of this paper we focus on the form of the coefficients a_{2n}, written as residues as in (2.7), and we investigate the nature of the residue integral as a period in the sense of algebraic geometry.

To this purpose, we treat the scaling factor $a(t)$ and its derivatives $a^{(k)}(t)$ as independent affine variables $\alpha, \ldots, \alpha_k, \ldots$, so that the choice of a specific scaling factor $a(t)$ corresponds to restricting the variables $(\alpha, \alpha_1, \ldots, \alpha_{2n})$ to a real curve $(a(t), a'(t), \ldots, a^{(2n)}(t))$ inside the affine space $\mathbb{A}^{2n+1}$.

Lemma 2.3. *The coefficient a_{2n} is computed as an integral*

$$a_{2n} = \int_{\mathbb{R}} \gamma_{2n}(a(t), a'(t), \ldots, a^{(2n)}(t))\, dt,$$

with

$$\gamma_{2n}(\alpha, \alpha_1 \ldots, \alpha_{2n}) = \frac{1}{8\,\pi^{n+1}} \int_{0<\eta<\frac{\pi}{2}} \int_{|\xi|=1} \Upsilon_{2n}(\alpha, \alpha_1 \ldots, \alpha_{2n}, \eta, \xi)\; \widetilde{\sigma}_{2n+1}(\eta, \xi), \tag{2.15}$$

where the volume form is given by

$$\widetilde{\sigma}_{2n+1}(\eta, \xi) = d\eta \wedge \sigma_{\xi,\, 2n+1} \tag{2.16}$$

with $\sigma_{\xi,\, 2n+1}$ as in (2.9)*, and*

$$\Upsilon_{2n}(\alpha, \alpha_1 \ldots, \alpha_{2n}, \eta, \xi)|_{\alpha=a(t), \alpha_k=a^{(k)}(t)} = b_{-2n-2}(t, \eta, \xi), \tag{2.17}$$

where the density $b_{-2n-2}(t, \eta, \xi)$ satisfies

$$\int_{0<\eta<\frac{\pi}{2}} \int_{|\xi|=1} b_{-2n-2}(t, \eta, \xi)\, \widetilde{\sigma}_{2n+1}(\eta, \xi) = \int_{0<\eta<\frac{\pi}{2}} \int_{|\xi|=1} \mathrm{tr}(\sigma_{-2n-2})(t, \eta, \xi)\, \widetilde{\sigma}_{2n+1}(\eta, \xi) \tag{2.18}$$

and is obtained from $\mathrm{tr}(\sigma_{-2n-2})(t, \eta, \xi)$ by dropping all terms that have odd powers of some of the coordinates ξ_j in the numerator.

Proof. As observed in Corollary 2.2, by (2.2) and (2.3), (2.4), (2.5), the homogeneous components p_0, p_1, and p_2 of the pseudodifferential symbol of D^2 depend only on the variable η and are independent of the angles ϕ_1, ϕ_2 of the Hopf coordinates (η, ϕ_1, ϕ_2) on S^3. Thus, when writing the coefficient a_{2n} as a residue, using (2.7), (2.8), and (2.10), one finds

$$a_{2n} = \frac{1}{8\pi^{n+1}} \int_{\mathbb{R}} \int_{0<\eta<\frac{\pi}{2}} \int_{|\xi|=1} \mathrm{tr}(\sigma_{-2n-2})(t, \eta, \xi)\, \widetilde{\sigma}_{2n+1}(\eta, \xi)\; dt, \tag{2.19}$$

with $\widetilde{\sigma}_{2n+1}(\eta, \xi)$ as in (3.4). Using the recursions of Lemma 2.1 and Corollary 2.2, together with (2.11) and the explicit formula for the term $\mathrm{tr}(\sigma_{-4}((D^2)^{-1})$ given in the appendix, it follows that the terms $\mathrm{tr}(\sigma_{-2n-2}(\Delta_{2n}^{-1})(t, \eta, \xi)$ are a sum of fractions with monomials in the ξ_j coordinates, the scaling factor $a(t)$ and its derivatives, and trigonometric functions of η in the numerator and a power of a quadratic form in the ξ_j coordinates and trigonometric functions of η in the denominator. The more precise form of these terms will be discussed below. It suffices here to notice that all the terms that contain odd powers of coordinates ξ_j in the numerator necessarily vanish when the integration in (2.19) is performed. Thus, we can replace the expression $\mathrm{tr}(\sigma_{-2n-2}(\Delta_{2n}^{-1}))(t,\eta,\xi)$ by another density $b_{-2n-2}(t,\eta,\xi)$ obtained from $\mathrm{tr}(\sigma_{-2n-2}(\Delta_{2n}^{-1}))(t,\eta,\xi)$ by removing all summands with odd powers of ξ_j in the numerator. It is then clear that (2.18) is satisfied and that it is possible to define a density $\Upsilon_{2n}(\alpha,\alpha_1\ldots,\alpha_{2n},\eta,\xi)$ satisfying (2.17) so that (2.15) holds. □

3. Algebraic differential forms, semi-algebraic sets, and periods

In this section we study the functions $\gamma_{2n}(\alpha,\alpha_1,\ldots,\alpha_{2n})$ as periods of a family of algebraic differential forms $\Omega^{\alpha}_{(\alpha_1,\ldots,\alpha_{2n})}$ defined over $\mathbb{Q}$, integrated on a $\mathbb{Q}$-semi-algebraic set A_{2n} in an algebraic variety given by a family $\mathcal{X}_\alpha$ of hypersurfaces in the affine space $\mathbb{A}^{2n+3}$.

3.1. **Algebraic coordinates.** In order to interpret the terms $\gamma_{2n}(\alpha,\alpha_1,\ldots,\alpha_{2n})$ as periods, we introduce a simple change of coordinates that makes it possible to rewrite the integrand $\Upsilon_{2n}(\alpha,\alpha_1\ldots,\alpha_{2n},\eta,\xi)\,\widetilde{\sigma}_{2n+1}(\eta,\xi)$ as an algebraic differential form.

Definition 3.1. *The algebraic coordinates $(u_0,\ldots,u_{2n+2})$ are defined by the change of variables*

$$\begin{aligned} &u_0=\sin^2(\eta), \qquad u_3=\csc(\eta)\,\xi_3, \qquad u_4=\sec(\eta)\,\xi_4,\\ &u_j=\xi_j, \qquad j=1,2,5,6,\ldots,2n+2. \end{aligned} \tag{3.1}$$

Lemma 3.2. *In the algebraic coordinates* (3.1) *the pseudodifferential symbol*

$$\sigma(D^2)=p_2+p_1+p_0$$

is given by

$$\begin{aligned} p_2 &= q_1^2=\left(u_1^2+\frac{1}{a(t)^2}(u_2^2+u_3^2+u_4^2)\right)I_{4\times 4},\\ p_1 &= q_0\,q_1+q_1\,q_0+\left(-i\frac{\partial q_1}{\partial \xi_1}\,\frac{\partial q_1}{\partial t}-i\frac{\partial q_1}{\partial \xi_2}\,\frac{\partial q_1}{\partial \eta}\right),\\ p_0 &= q_0^2+\left(-i\frac{\partial q_1}{\partial \xi_1}\,\frac{\partial q_0}{\partial t}-i\frac{\partial q_1}{\partial \xi_2}\,\frac{\partial q_0}{\partial \eta}\right), \end{aligned}$$

where q_0 and q_1 are given by

$$q_1=\begin{pmatrix} 0 & 0 & \frac{iu_4}{a(t)}-u_1 & \frac{iu_2}{a(t)}+\frac{u_3}{a(t)}\\ 0 & 0 & \frac{iu_2}{a(t)}-\frac{u_3}{a(t)} & -u_1-\frac{iu_4}{a(t)}\\ -u_1-\frac{iu_4}{a(t)} & -\frac{iu_2}{a(t)}-\frac{u_3}{a(t)} & 0 & 0\\ \frac{u_3}{a(t)}-\frac{iu_2}{a(t)} & \frac{iu_4}{a(t)}-u_1 & 0 & 0 \end{pmatrix}, \tag{3.2}$$

$$q_0 = \begin{pmatrix} 0 & 0 & \frac{3ia'(t)}{2a(t)} & \frac{1-2u_0}{2a(t)\sqrt{(1-u_0)u_0}} \\ 0 & 0 & \frac{1-2u_0}{2a(t)\sqrt{(1-u_0)u_0}} & \frac{3ia'(t)}{2a(t)} \\ \frac{3ia'(t)}{2a(t)} & -\frac{1-2u_0}{2a(t)\sqrt{(1-u_0)u_0}} & 0 & 0 \\ -\frac{1-2u_0}{2a(t)\sqrt{(1-u_0)u_0}} & \frac{3ia'(t)}{2a(t)} & 0 & 0 \end{pmatrix}. \tag{3.3}$$

Proof. In the coordinates (3.1) the pseudodifferential symbol of the Dirac operator D of the Robertson-Walker metric is given by

$$\sigma(D) = q_1 + q_0,$$

where q_1 and q_0 are now expressed as in (3.2) and (3.3). Since q_1 and q_0 depend only on t and u_0, or equivalently only on t and η, for the symbol of D^2, we have $\sigma(D^2) = p_2 + p_1 + p_0$, where p_2, p_1 and p_0 are as in the statement. □

3.2. **Algebraic volume form.** The volume form $\widetilde{\sigma}_{2n+1}$, when written in the algebraic coordinates (3.1), is an algebraic differential form on $\mathbb{A}^5$, defined over $\mathbb{Q}$.

Lemma 3.3. *The volume form $\widetilde{\sigma}_{2n+1}(\eta, \xi)$ in the algebraic coordinates is given by*

$$\widetilde{\sigma}_{2n+1}(u_0, \ldots, u_{2n+2}) = \frac{1}{2} \sum_{j=1}^{2n+2} (-1)^{j-1} u_j \, du_0 \, du_1 \wedge \cdots \wedge \widehat{du_j} \wedge \cdots \wedge du_{2n+2}. \tag{3.4}$$

Proof. Under the change of variables (3.1) we have

$$\begin{aligned} d\eta &= \frac{1}{2\sin(\eta)\cos(\eta)} \, du_0 = \frac{1}{2} \csc(\eta) \sec(\eta) \, du_0, \\ d\xi_3 &= \cos(\eta) u_3 \, d\eta + \sin(\eta) \, du_3, \\ d\xi_4 &= -\sin(\eta) u_4 \, d\eta + \cos(\eta) \, du_4, \\ d\xi_j &= du_j \quad \text{for} \quad j = 1, 2, 5, 6, \ldots, 2n+2. \end{aligned}$$

Thus we obtain

$$\begin{aligned} \widetilde{\sigma}_3 &:= d\eta \wedge \sigma_{\xi,3} \\ &= \sum_{j=1}^{4} (-1)^{j-1} \xi_j \, d\eta \wedge d\xi_1 \wedge \cdots \wedge \widehat{d\xi_j} \wedge \cdots \wedge d\xi_4 \\ &= \sin(\eta)\cos(\eta) \Big(u_1 \, d\eta \, du_2 \, du_3 \, du_4 - u_2 \, d\eta \, du_1 \, du_3 \, du_4 \\ &\qquad + u_3 \, d\eta \, du_1 \, du_2 \, du_4 - u_4 \, d\eta \, du_1 \, du_2 \, du_3 \Big) \\ &= \frac{1}{2} \Big(u_1 \, du_0 \, du_2 \, du_3 \, du_4 - u_2 \, du_0 \, du_1 \, du_3 \, du_4 + u_3 \, du_0 \, du_1 \, du_2 \, du_4 \\ &\qquad - u_4 \, du_0 \, du_1 \, du_2 \, du_3 \Big), \end{aligned}$$

and similarly, for all $n > 0$

$$\begin{aligned}
\widetilde{\sigma}_{2n+1} &:= d\eta \wedge \sigma_{\xi,\,2n+1} \\
&= \sum_{j=1}^{2n+2} (-1)^{j-1} \xi_j \, d\eta \wedge d\xi_1 \wedge \cdots \wedge \widehat{d\xi_j} \wedge \cdots \wedge d\xi_{2n+2} \\
&= \frac{1}{2} \sum_{j=1}^{2n+2} (-1)^{j-1} u_j \, du_0 \, du_1 \wedge \cdots \wedge \widehat{du_j} \wedge \cdots \wedge du_{2n+2}.
\end{aligned}$$

□

3.3. **The a_2 term and quadric surfaces in $\mathbb{P}^3$.** We consider here the first term $\gamma_2(\alpha, \alpha_1, \alpha_2)$, defined as in (2.15) in Lemma 2.3. We show that the differential form $\Upsilon_2(\alpha, \alpha_1, \alpha_2)\widetilde{\sigma}_3$, written in the algebraic coordinates of (3.1), is an algebraic differential form over $\mathbb{Q}$, defined on the complement of a quadric surface. We first introduce some preliminary notation.

Let Z be a projective hypersurface in $\mathbb{P}^{N-1}$. In the following we denote by $\hat{Z}$ the affine cone over Z in $\mathbb{A}^N$, and by CZ the projective cone over Z in $\mathbb{P}^N$. We also denote by $\widehat{CZ}$ the affine cone in $\mathbb{A}^{N+1}$ of CZ.

Consider the set of rational functions of the form

$$\frac{P(u_0, u_1, u_2, u_3, u_4, \alpha, \alpha_1, \alpha_2)}{\alpha^{2r} u_0^k (1-u_0)^m (u_1^2 + \alpha^{-2}(u_2^2 + u_3^2 + u_4^2))^\ell}, \tag{3.5}$$

where

$$P(u_0, u_1, u_2, u_3, u_4, \alpha, \alpha_1, \alpha_2) = P_{(\alpha_1,\alpha_2)}(u_0, u_1, u_2, u_3, u_4, \alpha)$$

are polynomials in $\mathbb{Q}[u_0, u_1, u_2, u_3, u_4, \alpha, \alpha_1, \alpha_2]$ and where r, k, m and ℓ are non-negative integers.

We then obtain the following characterization of the differential form $\Upsilon_2(\alpha, \alpha_1, \alpha_2)\widetilde{\sigma}_3$.

Theorem 3.4. *Consider affine coordinates $(u_0, u_1, u_2, u_3, u_4) \in \mathbb{A}^5$, $\alpha \in \mathbb{G}_m$, and $(\alpha_1, \alpha_2) \in \mathbb{A}^2$. Consider the complement*

$$\mathbb{A}^5 \smallsetminus (H_0 \cup H_1 \cup \widehat{CZ}_\alpha). \tag{3.6}$$

in the affine space $\mathbb{A}^5$ of the union of two affine hyperplanes

$$H_0 = \{u_0 = 0\} \quad \text{and} \quad H_1 = \{u_0 = 1\} \tag{3.7}$$

and the hypersurface $\widehat{CZ}_\alpha$ defined by the vanishing of the quadratic form

$$Q_{\alpha,2} = u_1^2 + \alpha^{-2}(u_2^2 + u_3^2 + u_4^2). \tag{3.8}$$

There is a 2-parameter (α_1, α_2) family of algebraic differential forms

$$\Omega^\alpha_{(\alpha_1,\alpha_2)}(u_0, u_1, u_2, u_3, u_4) = f_{(\alpha_1,\alpha_2)}(u_0, u_1, u_2, u_3, u_4, \alpha)\ \tilde{\sigma}_3(u_0, u_1, u_2, u_3, u_4), \tag{3.9}$$

defined on the complement (3.6)*, with $f_{(\alpha_1,\alpha_2)}$ $\mathbb{Q}$-linear combinations of rational functions of the form* (3.5)*, such that the differential form $\Upsilon_2(\alpha,\alpha_1,\alpha_2)\widetilde{\sigma}_3$, written in the coordinates* (3.1) *satisfies*

$$\Upsilon_2(\alpha,\alpha_1,\alpha_2,u_0,u_1,u_2,u_3,u_4)\widetilde{\sigma}_3(u_0,u_1,u_2,u_3,u_4) = \Omega^{\alpha}_{(\alpha_1,\alpha_2)}(u_0,u_1,u_2,u_3,u_4).$$

Proof. We have seen in Lemma 3.3 that the form $\widetilde{\sigma}_3(u_0,u_1,u_2,u_3,u_4)$ is an algebraic differential form on $\mathbb{A}^5$, defined over $\mathbb{Q}$. The explicit form of the density $\mathrm{tr}(\sigma_{-4}(t,\eta,\xi))$ is reported in (5.1) in the Appendix. The corresponding density $b_{-4}(t,\eta,\xi)$ is obtained from $\mathrm{tr}(\sigma_{-4}(t,\eta,\xi))$ of (5.1) by eliminating all the terms with odd exponents of ξ_j in the numerator. In particular, we see by direct inspection of (5.1) and of the associated density $b_{-4}(t,\eta,\xi)$, using elementary trigonometric identities for $\cot(2\eta)$, $\csc^2(\eta)$, $\tan^2(\eta)$ and $\cot^2(2\eta)$, that the density $\Upsilon_2(a(t),a'(t),a''(t),\eta,\xi)$ is a sum of fractions involving even powers of the ξ_j variables, and integer powers of the expressions

$$\xi_1^2 + \frac{\xi_2^2}{a(t)^2} + \frac{\xi_3^2\csc^2(\eta)}{a(t)^2} + \frac{\xi_4^2\sec^2(\eta)}{a(t)^2} = u_1^2 + \frac{1}{a(t)^2}(u_2^2+u_3^2+u_4^2),$$

$$\cot^2(\eta) = \frac{1-u_0}{u_0}, \quad \csc^2(\eta) = \frac{1}{u_0}, \quad \sec^2(\eta) = \frac{1}{1-u_0},$$

with the quadratic polynomial in the denominator. Thus, when expressed in the algebraic coordinates, each summand in

$$\Upsilon_2(\alpha,\alpha_1,\alpha_2,u_0,u_1,u_2,u_3,u_4)$$

is a rational function of the form (3.5), hence the result follows. □

The quadratic form (3.8) determines a quadric surface Z_α in $\mathbb{P}^3$, in fact a pencil of quadric surfaces depending on the parameter $\alpha \in \mathbb{G}_m$. The affine hypersurface $\widehat{CZ}_\alpha$ in $\mathbb{A}^5$ is the affine cone over the projective cone CZ_α in $\mathbb{P}^4$.

3.4. **Density Υ_{2n} in algebraic coordinates.** We now consider the following terms $\gamma_{2n}(\alpha,\alpha_1,\dots,\alpha_{2n})$ for all $n>1$, and we obtain inductively a general expression for the densities

$$\Upsilon_{2n}(\alpha,\alpha_1,\dots,\alpha_{2n},u_0,\dots,u_{2n+2}).$$

Theorem 3.5. *The term $\Upsilon_{2n}(\alpha,\alpha_1,\dots,\alpha_{2n})$, written in the algebraic coordinates of* (3.1)*, satisfies*

$$\Upsilon_{2n}(\alpha,\alpha_1,\dots,\alpha_{2n},u_0,\dots,u_{2n+2}) = \sum_{j=1}^{M_n} c_{j,2n}\, u_0^{\beta_{0,1,j}/2}\,(1-u_0)^{\beta_{0,2,j}/2}\, \frac{u_1^{\beta_{1,j}}\, u_2^{\beta_{2,j}}\,\cdots\, u_{2n+2}^{\beta_{2n+2,j}}}{Q_{\alpha,2n}^{\rho_{j,2n}}}\, \alpha^{k_{0,j}}\,\alpha_1^{k_{1,j}}\,\cdots\,\alpha_{2n}^{k_{2n,j}}, \tag{3.10}$$

where

$$Q_{\alpha,2n} = u_1^2 + \frac{1}{\alpha^2}(u_2^2+u_3^2+u_4^2) + u_5^2 + \cdots + u_{2n+2}^2,$$

and with coefficients and exponents

$$c_{j,2n}\in\mathbb{Q}, \qquad \beta_{0,1,j},\beta_{0,2,j},k_{0,j}\in\mathbb{Z}, \qquad \beta_{1,j},\dots,\beta_{2n+2,j},\rho_{j,2n},k_{1,j},\dots,k_{2n,j}\in\mathbb{Z}_{\geq 0}.$$

Proof. We need to compute the homogeneous term $\sigma_{-2n-2}(\Delta_{2n}^{-1})$. Using (2.14) and considering the independence of the symbols from the variables ϕ_1 and ϕ_2, we obtain

$$\sigma_{-2}(\Delta_{2n}^{-1}) = \big(p_2 + (u_5^2 + \cdots + u_{2n+2}^2)I_{4\times 4}\big)^{-1} = \frac{1}{Q_{\alpha,2n}} I_{4\times 4}, \tag{3.11}$$

with the quadratic form

$$Q_{\alpha,2n} = u_1^2 + \frac{1}{\alpha^2}(u_2^2 + u_3^2 + u_4^2) + u_5^2 + \cdots + u_{2n+2}^2. \tag{3.12}$$

Then the desired $\sigma_{-2n-2}(\Delta_{2n}^{-1})$ can be calculated recursively using Corollary 2.2. In expressing the result of (2.14) in the algebraic coordinates (3.1), note that in general, for a smooth function f of the variables (t, η, ξ), using the notation

$$f(t, \eta, \xi_1, \xi_2, \ldots, \xi_{2n+2}) = \tilde{f}(t, u_0, u_1, u_2, \ldots, u_{2n+2}),$$

we have the identities

$$\partial_t f = \partial_t \tilde{f}, \qquad \partial_{\xi_j} f = \partial_{u_j} \tilde{f}, \qquad j = 1, 2,$$

$$\partial_\eta f = 2\sqrt{u_0(1-u_0)}\, \partial_{u_0} \tilde{f} - u_3 \sqrt{\frac{1-u_0}{u_0}}\, \partial_{u_3} \tilde{f} + u_4 \sqrt{\frac{u_0}{1-u_0}}\, \partial_{u_4} \tilde{f}. \tag{3.13}$$

Combining (2.14), the result for the term σ_{-4} discussed in Theorem 3.4 and in the Appendix, and the change of variables of (3.13), one can see by induction that (3.10) holds as stated. $\square$

Theorem 3.5 above shows that $\Upsilon_{2n}(\alpha, \alpha_1, \ldots, \alpha_{2n})$, in the form (3.10) is a rational expression in $\sqrt{u_0}$, $\sqrt{1-u_0}$, $u_1, \ldots, u_{2n+2}$, $\alpha, \alpha_1, \ldots, \alpha_{2n}$. In order to prove that $\gamma_{2n}(\alpha, \alpha_1, \ldots, \alpha_{2n})$ is an integral of a rational differential form, we need to show that in fact only terms with even powers of $\sqrt{u_0}$ and $\sqrt{1-u_0}$ contribute nontrivially in the calculation of $\gamma_{2n}(\alpha, \alpha_1, \ldots, \alpha_{2n})$. This will then be used to show that the integral expression (2.15) for $\gamma_{2n}(\alpha, \alpha_1, \ldots, \alpha_{2n})$ is equal to the integral of a rational differential form in u_0, u_1, $\ldots$, u_{2n+2}, $\alpha, \alpha_1, \ldots, \alpha_{2n}$ over a $\mathbb{Q}$-semialgebraic set. We need a preliminary observation, which we state in the next subsection.

3.5. **Integration on the unit cosphere bundle.** The claim that only terms with $\beta_{0,1,j}, \beta_{0,2,j} \in 2\mathbb{Z}$ in the summation of (3.10) contribute nontrivially to the computation of the term $\Upsilon_{2n}(\alpha, \alpha_1, \ldots, \alpha_{2n})$ can be proved as follows.

Consider the unit cosphere of the metric in the cotangent fibre. This is given by the locus $\{\xi \,:\, |\xi|_g^2 = 1\}$, with

$$\begin{aligned} |\xi|_g^2 &= \xi_1^2 + \frac{\xi_2^2}{a(t)^2} + \frac{\csc^2(\eta)\xi_3^2}{a(t)^2} + \frac{\sec^2(\eta)\xi_4^2}{a(t)^2} + \xi_5^2 + \cdots + \xi_{2n+2}^2 \\ &= u_1^2 + \frac{1}{\alpha^2}(u_2^2 + u_3^2 + u_4^2) + u_5^2 + \cdots + u_{2n+2}^2. \end{aligned} \tag{3.14}$$

Proposition 3.6. *The integral of the density* $\mathrm{tr}(\sigma_{-2n-2})\cdot\sigma_{\xi,\,2n+1}$ *on the unit sphere is equal to the integral on the unit cosphere of the metric in the cotangent fibre,*

$$\int_{\sum_{j=1}^{2n+2}\xi_j^2=1}\mathrm{tr}(\sigma_{-2n-2})\cdot\sigma_{\xi,\,2n+1}=\int_{|\xi|_g^2=1}\mathrm{tr}(\sigma_{-2n-2})\cdot\sigma_{\xi,\,2n+1}.$$

Proof. Fixing a point $(x,x')=(t,\eta,\phi_1,\phi_2,x')\in M\times\mathbb{T}^{2n-2}$, the differential form $\mathrm{tr}(\sigma_{-2n-2})\,\sigma_{\xi,\,2n+1}$ on the Euclidean space $\mathbb{R}^{2n+2}\simeq T^*_{(x,x')}(M\times\mathbb{T}^{2n-2})$ is a closed differential form of degree $2n+1$, since $\mathrm{tr}(\sigma_{-2n-2})$ is homogeneous of order $-2n-2$ in $\xi\in\mathbb{R}^{2n+2}$, see Proposition 7.3, page 265 of [13]. Therefore, using the Stokes theorem, the integral of this differential form over the unit sphere $|\xi|=1$ is the same as its integral over the cosphere of the metric in the cotangent fibre given by $|\xi|_g^2=1$, since as closed cycles these two loci are homologous. $\square$

We parametrize the cosphere $|\xi|_g=1$ by writing

$$\begin{aligned}
\xi_1&=\sin(\psi_{2n+1})\sin(\psi_{2n})\cdots\sin(\psi_2)\cos(\psi_1),\\
\xi_2&=\alpha\sin(\psi_{2n+1})\sin(\psi_{2n})\cdots\sin(\psi_2)\sin(\psi_1),\\
\xi_3&=\frac{\alpha}{\csc(\eta)}\sin(\psi_{2n+1})\sin(\psi_{2n})\cdots\sin(\psi_3)\cos(\psi_2),\\
\xi_4&=\frac{\alpha}{\sec(\eta)}\sin(\psi_{2n+1})\sin(\psi_{2n})\cdots\sin(\psi_4)\cos(\psi_3),
\end{aligned}\tag{3.15}$$

$$\begin{aligned}
\xi_5&=\sin(\psi_{2n+1})\sin(\psi_{2n})\cdots\sin(\psi_5)\cos(\psi_4),\\
\xi_6&=\sin(\psi_{2n+1})\sin(\psi_{2n})\cdots\sin(\psi_6)\cos(\psi_5),\\
&\cdots\\
\xi_{2n+1}&=\sin(\psi_{2n+1})\cos(\psi_{2n}),\\
\xi_{2n+2}&=\cos(\psi_{2n+1}),
\end{aligned}$$

with the variables $\psi_1,\dots,\psi_{2n+1}$ having the following ranges:

$$0<\psi_1<2\pi,\qquad 0<\psi_2<\pi,\qquad 0<\psi_3<\pi,\qquad \dots,\qquad 0<\psi_{2n+1}<\pi.$$

Lemma 3.7. *In the parameterization* (3.15) *of the unit cosphere* $|\xi|_g=1$, *the density* $\mathrm{tr}(\sigma_{-2n-2})\,\sigma_{\xi,\,2n+1}$ *is given by the expression*

$$\begin{aligned}
&\sin(\eta)\cos(\eta)\sum_{j=1}^{M_n}\Big\{c_{j,2n}\,\alpha^{\beta_{2,j}+\beta_{3,j}+\beta_{4,j}+k_{0,j}}\;\alpha_1^{k_{1,j}}\cdots\alpha_{2n}^{k_{2n,j}}\sin^{\beta_{0,1,j}}(\eta)\,\cos^{\beta_{0,2,j}}(\eta)\\
&\cos^{\beta_{1,j}}(\psi_1)\sin^{\beta_{2,j}}(\psi_1)\prod_{\ell=2}^{2n+1}(\sin\psi_\ell)^{\ell-1+\sum_{i=1}^{\ell}\beta_{i,j}}\,(\cos\psi_\ell)^{\beta_{\ell+1,j}}\Big\}\,d\psi_1\,d\psi_2\cdots d\psi_{2n+1}.
\end{aligned}\tag{3.16}$$

Proof. Using the parameterization (3.15), over the cosphere $|\xi|_g = 1$ we have

$$\begin{aligned} \sigma_{\xi,\,2n+1} &= \sum_{j=1}^{2n+2} (-1)^{j-1} \xi_j \, d\xi_1 \wedge \cdots \wedge \widehat{d\xi_j} \wedge \cdots \wedge d\xi_{2n+2} \\ &= \alpha^3 \sin(\eta) \cos(\eta) \sin(\psi_2) \sin^2(\psi_3) \, \cdots \, \sin^{2n}(\psi_{2n+1}) \, d\psi_1 \, d\psi_2 \, \cdots \, d\psi_{2n+1}. \end{aligned}$$

Combining this form with the expression given by (3.10), we obtain (3.16). □

Proposition 3.8. *Only terms with even powers of* $\sqrt{u_0}$ *and* $\sqrt{1-u_0}$ *contribute non-trivially in the expression of* $\Upsilon_{2n}(\alpha, \alpha_1, \ldots, \alpha_{2n})$ *in* (3.10).

Proof. By exploiting symmetries of the Robertson-Walker metric and its consequent isometry group, it is shown in Lemma 1 of [10] that the local density that integrates to the term a_{2n} has a spatial independence. This fact, together with Lemma 3.7, implies that the following expression is independent of the variable η:

$$\begin{aligned} & \frac{1}{\sin(\eta)\cos(\eta)} \int_{|\xi|_g=1} \mathrm{tr}(\sigma_{-2n-2}) \, \sigma_{\xi,\,2n+1} \\ = & \sum_{j=1}^{M_n} c_{j,2n} \, d_{j,2n} \, \alpha^{\beta_{2,j}+\beta_{3,j}+\beta_{4,j}+k_{0,j}} \, \alpha_1^{k_{1,j}} \, \cdots \, \alpha_{2n}^{k_{2n,j}} \sin^{\beta_{0,1,j}}(\eta) \, \cos^{\beta_{0,2,j}}(\eta), \end{aligned} \tag{3.17}$$

where

$$\begin{aligned} d_{j,2n} = & \int_0^{2\pi} \cos^{\beta_{1,j}}(\psi_1) \sin^{\beta_{2,j}}(\psi_1) \, d\psi_1 \, \times \\ & \int_0^{\pi} d\psi_2 \cdots \int_0^{\pi} d\psi_{2n+1} \prod_{\ell=2}^{2n+1} (\sin\psi_\ell)^{\ell-1+\sum_{i=1}^{\ell}\beta_{i,j}} \, (\cos\psi_\ell)^{\beta_{\ell+1,j}} . \end{aligned}$$

We now exploit the independence from η of the sum in (3.17) to show that only the terms in (3.10) for which $\beta_{0,1,j}$ and $\beta_{0,2,j}$ are both even integers contribute in the computation of $\gamma_{2n}(\alpha, \alpha_1, \ldots, \alpha_{2n})$. We prove this by showing that, if for some coefficients c_j and some integers γ_j and ν_j, a finite summation of the form $\sum_j c_j \sin^{\gamma_j}(\eta)\cos^{\nu_j}(\eta)$ is identically equal to a non-zero constant, or without loss in generality equal to 1, then all the exponents γ_j and ν_j are even integers, and possible terms with odd exponents have to inevitably cancel each other out. Since η varies between 0 and $\pi/2$, this is equivalent to saying that if

$$\sum_j c_j \, s^{\gamma_j} \, (1-s^2)^{\nu_j/2} = 1, \qquad s \in (0,1),$$

then all γ_j and ν_j are even integers and all other terms cancel.

First observe that replacing s in the above equation by $s_1 = (1-s^2)^{1/2}$ shows that our claim is symmetric with respect to exchanging the γ_j and ν_j, hence it suffices to show that all γ_j are even integers. We decompose the summation on the left-hand-side of the above identity and write

$$\sum_{j_o} c_{j_o}\, s^{\gamma_{j_o}}\,(1-s^2)^{\nu_{j_o}/2} + \sum_{j_e} c_{j_e}\, s^{\gamma_{j_e}}\,(1-s^2)^{\nu_{j_e}/2} = 1, \qquad s\in(0,1),$$

where for each term in the first summation either γ_{j_o} or ν_{j_o} is odd, and in the second summation the γ_{j_e} and ν_{j_e} are even integers. Therefore we have

$$\sum_{j_o} c_{j_o}\, s^{\gamma_{j_o}}\,(1-s^2)^{\nu_{j_o}/2} = 1 - \sum_{j_e} c_{j_e}\, s^{\gamma_{j_e}}\,(1-s^2)^{\nu_{j_e}/2}, \qquad s\in(0,1),$$

and we proceed by considering the binomial series of the two sides of this equation. Since the series of the right-hand-side has only even powers of the variable $s\in(0,1)$, it follows that the terms on left-hand-side whose γ_{j_o} are odd cancel each other out, therefore with no loss in generality we can assume all the γ_{j_o} are even, which implies that all the ν_{j_o} have to be odd integers. Now by making the replacement $s_1 = (1-s^2)^{1/2}$ we are led to

$$\sum_{j_o} c_{j_o}\,(1-s_1^2)^{\gamma_{j_o}/2}\, s_1^{\nu_{j_o}} = 1 - \sum_{j_e} c_{j_e}\,(1-s_1^2)^{\gamma_{j_e}/2}\, s_1^{\nu_{j_e}}, \qquad s_1\in(0,1).$$

Finally we compare the binomial series in s_1 of the two sides of this equation: since the series of the right-hand-side has only even exponents and all the ν_{j_o} on the left side are odd integers, we conclude that

$$\sum_{j_o} c_{j_o}\,(1-s_1^2)^{\gamma_{j_o}/2}\, s_1^{\nu_{j_o}} = 0, \qquad s_1\in(0,1).$$

□

3.6. **Algebraic differential forms.** We obtain the following generalization of Theorem 3.4 for the densities $\Upsilon_{2n}(\alpha,\alpha_1,\ldots,\alpha_{2n},u_0,\ldots,u_{2n+2})$.

Definition 3.9. *Let $\mathcal{R}_{2n}$ be the set of rational functions given by $\mathbb{Q}$ linear combinations of terms of the form*

$$u_0^{\beta_{0,1,j}}\,(1-u_0)^{\beta_{0,2,j}}\,\frac{u_1^{\beta_{1,j}}\,u_2^{\beta_{2,j}}\,\cdots\,u_{2n+2}^{\beta_{2n+2,j}}}{Q_{\alpha,2n}^{\rho_{j,2n}}}\,\alpha^{k_{0,j}}\,\alpha_1^{k_{1,j}}\,\cdots\,\alpha_{2n}^{k_{2n,j}},$$

where

$$Q_{\alpha,2n} = u_1^2 + \frac{1}{\alpha^2}(u_2^2+u_3^2+u_4^2) + u_5^2 + \cdots + u_{2n+2}^2,$$

with $\beta_{0,1,j},\beta_{0,2,j},k_{0,j}\in\mathbb{Z}$ and $\beta_{1,j},\ldots,\beta_{2n+2,j},\rho_{j,2n},k_{1,j},\ldots,k_{2n,j}\in\mathbb{Z}_{\geq 0}$.

Theorem 3.10. *Consider affine coordinates $(u_0,\ldots,u_{2n+2})\in\mathbb{A}^{2n+3}$, $\alpha\in\mathbb{G}_m$, and $(\alpha_1,\ldots,\alpha_{2n})\in\mathbb{A}^{2n}$. Consider the algebraic variety, defined over $\mathbb{Q}$, given by the complement*

$$\mathbb{A}^{2n+3}\smallsetminus(H_0\cup H_1\cup\widehat{CZ}_{\alpha,2n}), \tag{3.18}$$

where H_0 and H_1 are hyperplanes defined as in (3.7) and $\widehat{CZ}_{\alpha,2n}$ is the hypersurface in $\mathbb{A}^{2n+3}$ defined by the vanishing of the quadratic form $Q_{\alpha,2n}$, with $\alpha \in \mathbb{G}_m(\mathbb{Q})$ regarded as a fixed parameter. There is a $2n$-parameter family of algebraic differential forms $\Omega^{\alpha}_{(\alpha_1,\dots,\alpha_{2n})}$, defined over $\mathbb{Q}$, with parameters $(\alpha_1,\dots,\alpha_{2n}) \in \mathbb{A}^{2n}(\mathbb{Q})$, such that

$$\Omega^{\alpha}_{(\alpha_1,\dots,\alpha_{2n})}(u_0,\dots,u_{2n+2}) = f_{(\alpha_1,\dots,\alpha_{2n})}(u_0,\dots,u_{2n+2},\alpha)\, \widetilde{\sigma}_{2n+1}(u_0,\dots,u_{2n+2}), \tag{3.19}$$

where the rational functions $f_{(\alpha_1,\dots,\alpha_{2n})}$ belong to the set $\mathcal{R}_{2n}$ of Definition 3.9, and with the property that

$$\Upsilon_{2n}(\alpha,\alpha_1,\dots,\alpha_{2n},u_0,\dots,u_{2n+2}) = f_{(\alpha_1,\dots,\alpha_{2n})}(u_0,\dots,u_{2n+2},\alpha). \tag{3.20}$$

Proof. The statement follows directly from Theorem 3.5 and Proposition 3.8. □

3.7. Semi-algebraic sets and Periods.

Let $\mathbb{K}$ be a number field. A $\mathbb{K}$-semialgebraic set is a subset S of some $\mathbb{R}^n$ that is of the form

$$S = \{(x_1,\dots,x_n) \in \mathbb{R}^n \,:\, P(x_1,\dots,x_n) \geq 0\}, \tag{3.21}$$

for some polynomial $P \in \mathbb{K}[x_1,\dots,x_n]$, or obtained from such sets by taking a finite number of complements, intersections, and unions. A semialgebraic set S in an algebraic variety X is a finite number of complements, intersections, and unions of subsets that, in a set of algebraic local coordinates have the form (3.21).

A *period* is an integral $\int_S \Omega$ of a $\mathbb{K}$-algebraic differential form Ω over a $\mathbb{K}$-semialgebraic set S in an algebraic variety X defined over the number field $\mathbb{K}$, see [14].

The theory of periods and motives of algebraic varieties constraints the type of numbers that can occur as periods on an algebraic variety X in terms of the motive $\mathfrak{m}(X)$, see [14]. In the rest of the paper we identify explicitly the periods and motives associated to the terms a_{2n} of the heat kernel expansion.

We first show that the density $\gamma_2(\alpha,\alpha_1,\alpha_2)$ associated to the coefficient a_2 of the heat kernel expansion is a period and we identify the corresponding motive.

Theorem 3.11. *The term $\gamma_2(\alpha,\alpha_1,\alpha_2)$ is a period integral given by*

$$\gamma_2(\alpha,\alpha_1,\alpha_2) = C \cdot \int_{A_4} \Omega^{\alpha}_{(\alpha_1,\alpha_2)}, \tag{3.22}$$

with the algebraic differential form of Theorem 3.4, with domain of integration the $\mathbb{Q}$-semialgebraic set

$$A_4 = \left\{ (u_0,u_1,u_2,u_3,u_4) \in \mathbb{A}^5(\mathbb{R}) \,:\, \begin{array}{l} u_1^2 + u_2^2 + u_0 u_3^2 + (1-u_0)u_4^2 = 1, \\ 0 < u_i < 1, \ \textit{for } i = 0,1,2 \end{array} \right\}, \tag{3.23}$$

and with a coefficient C in $\mathbb{Q}[(2\pi i)^{-1}]$. This integral is a period of the mixed motive

$$\mathfrak{m}(\mathbb{A}^5 \smallsetminus (\widehat{CZ}_\alpha \cup H_0 \cup H_1), \Sigma), \tag{3.24}$$

where $\widehat{CZ}_\alpha$ is the hypersurface in $\mathbb{A}^5$ defined by the vanishing of the quadric $Q_{\alpha,2}$ of (3.8), H_0, H_1 are the hyperplanes (3.7), and $\Sigma = \cup_{i,a} H_{i,a}$ is the divisor given by the union of the hyperplanes $H_{i,a} = \{u_i = a\}$, with $i \in \{0,1,2\}$ and $a \in \{0,1\}$.

Proof. We have

$$\begin{aligned}\gamma_2(a(t),a'(t),a''(t)) &= \frac{1}{2^3\pi^2}\int_0^{\pi/2} d\eta \int_{\xi_1^2+\xi_2^2+\xi_3^2+\xi_4^2=1} d^3\xi \cdot b_{-4}(t,\eta,\xi)\cdot \sigma_{\xi,3} \\ &= \frac{1}{2^3\pi^2}\int_{(0,\frac{\pi}{2})\times\mathbb{S}^3} \Upsilon_2(a(t),a'(t),a''(t),\eta,\xi)\,\tilde{\sigma}_3(\eta,\xi). \qquad (3.25)\end{aligned}$$

By Lemma 3.3 and Theorem 3.4, after changing coordinates as in (3.1), for the case $n=1$, we rewrite the form $\Upsilon_2(\alpha,\alpha_1,\alpha_2)\widetilde{\sigma}_3$ as the algebraic differential form $\Omega^{\alpha}_{(\alpha_1,\alpha_2)}$. Correspondingly, the domain of integration $(\eta,\xi)\in(0,\frac{\pi}{2})\times\mathbb{S}^3$ is transformed in the algebraic coordinates into the $\mathbb{Q}$-semialgebraic set (3.23). Thus, with a coefficient $C=(8\pi^2)^{-1}$ in $\mathbb{Q}[(2\pi i)^{-1}]$, we rewrite (3.25) as (3.22).

To identify the associated motive, notice that the forms $\Omega^{\alpha}_{(\alpha_1,\alpha_2)}$ are defined on the complement in $\mathbb{A}^5$ of the union of the hyperplanes H_0 and H_1 and the hypersurface $\widehat{CZ}_\alpha$ given by the vanishing of the quadric Q_α of (3.8). Thus, the $\Omega^{\alpha}_{(\alpha_1,\alpha_2)}$ are a two-parameter family (depending on the parameters (α_1,α_2) of algebraic differential forms on the algebraic variety $\mathbb{A}^5\smallsetminus(\widehat{CZ}_\alpha\cup H_0\cup H_1)$. The domain of integration A_4 is not a closed cycle: it has a boundary ∂A_4 which is contained in the union of the hyperplanes $H_{i,a}=\{u_i=a\}$, with $i\in\{0,1,2\}$ and $a\in\{0,1\}$. Thus, the period corresponds to the relative motive $\mathfrak{m}(\mathbb{A}^5\smallsetminus(\widehat{CZ}_\alpha\cup H_0\cup H_1),\Sigma)$, where the divisor Σ is the union of these hyperplanes, $\Sigma=\cup_{i,a}H_{i,a}$. □

Remark 3.12. The singular locus $\widehat{CZ}_\alpha\cup H_0\cup H_1$ of the algebraic differential form and the divisor Σ containing the boundary of the domain of integration A_4 have nonempty intersection along $H_0\cup H_1$. However, unlike the case of quantum field theory where the intersection of the boundary of the domain of integration with the graph hypersurface is the source of infrared divergences, here we know a priori that the integral (3.22) is convergent, and so are all the other analogous integrals for the higher order a_{2n} terms, as one can see by computing them in the original spherical coordinates. Thus, we do not have a renormalization problem for these integrals.

We have a similar result for the terms $\gamma_{2n}(\alpha,\alpha_1,\dots,\alpha_{2n})$.

Theorem 3.13. *The term $\gamma_{2n}(\alpha,\alpha_1,\dots,\alpha_{2n})$ is a period integral given by*

$$\gamma_{2n}(\alpha,\alpha_1,\dots,\alpha_{2n}) = C\cdot\int_{A_{2n}} \Omega^{\alpha}_{\alpha_1,\dots,\alpha_{2n}}$$

of the algebraic differential form $\Omega^{\alpha}_{\alpha_1,\dots,\alpha_{2n}}(u_0,u_1,\dots,u_{2n+2})$ of Theorem 3.10, defined on the algebraic variety $\mathbb{A}^{2n+3}\smallsetminus(\widehat{CZ}_{\alpha,2n}\cup H_0\cup H_1)$, with domain of integration the $\mathbb{Q}$-semialgebraic set

$$A_{2n+2} = \left\{(u_0,\dots,u_{2n+2})\in\mathbb{A}^{2n+3}(\mathbb{R}) \;:\; \begin{array}{l} u_1^2+u_2^2+u_0u_3^2+(1-u_0)u_4^2+\sum_{i=5}^{2n+2}u_i^2=1 \\ 0<u_i<1, \quad i=0,1,2,5,6,\dots,2n+2 \end{array}\right\}, \qquad (3.26)$$

and with a coefficient $C \in \mathbb{Q}[(2\pi i)^{-1}]$. The associated motive is the relative mixed motive

$$\mathfrak{m}(\mathbb{A}^{2n+3} \setminus (H_0 \cup H_1 \cup \widehat{CZ}_{\alpha,2n}), \Sigma)$$

where Σ is a divisor in $\mathbb{A}^{2n+3}$ consisting of a union of hyperplanes $\Sigma = \cup_{i,a} H_{i,a}$ with $i = 0, 1, 2, 5, 6, \ldots, 2n+2$ and $a = 0, 1$, with $H_{i,a} = \{u_i = a\}$. This divisor Σ contains the boundary ∂A_{2n} of the domain of integration.

Proof. We have

$$\gamma_{2n}(a(t), a'(t), \ldots, a^{(2n)}(t)) = \frac{1}{8\pi^{1+n}} \int_{(\eta,\,\xi)\in(0,\frac{\pi}{2})\times\mathbb{S}^{2n+1}} \mathrm{tr}(\sigma_{-2n-2})\, \widetilde{\sigma}_{2n+1}$$

$$= \frac{1}{8\pi^{1+n}} \int_{(\eta,\,\xi)\in(0,\frac{\pi}{2})\times\mathbb{S}^{2n+1}} \Upsilon_{2n}(a(t), a'(t), \ldots, a^{(2n)}(t), \eta,\, \xi)\, \widetilde{\sigma}_{2n+1}(\eta,\, \xi).$$

Passing to the algebraic coordinates of (3.1), the domain of integration

$$\{(\eta,\, \xi) \in (0, \frac{\pi}{2}) \times \mathbb{S}^{2n+1}\}$$

is transformed into the $\mathbb{Q}$-semialgebraic set (3.26), while by Theorem 3.5 the density $\Upsilon_{2n}(\alpha, \alpha_1, \ldots, \alpha_{2n}, \eta,\, \xi)\, \widetilde{\sigma}_{2n+1}(\eta,\, \xi)$ is transformed into the algebraic differential form $\Omega^{\alpha}_{(\alpha_1,\ldots,\alpha_{2n})}(u_0, \ldots, u_{2n+2})$. □

Again, as mentioned in Remark 3.12, the integrals are all convergent, hence there is no renormalization problem caused by the intersection of the boundary of the domain of integration with the singular set of the algebraic differential form.

4. The motives

In this section we analyze the motives associated to the periods obtained from the coefficients a_{2n} of the spectral action. We are considering a family of quadrics

$$Q_{\alpha,2n} = u_1^2 + \frac{1}{\alpha^2}(u_2^2 + u_3^2 + u_4^2) + u_5^2 + \cdots + u_{2n+2}^2, \tag{4.1}$$

where α is a (rational) parameter. These define quadric hypersurfaces $Z_{\alpha,2n}$ in $\mathbb{P}^{2n+1}$. We will also be considering the projective cone $CZ_{\alpha,2n}$ in $\mathbb{P}^{2n+2}$ and the affine cone $\widehat{CZ}_{\alpha,2n}$ in the affine space $\mathbb{A}^{2n+3}$.

4.1. Pencils of quadrics.

A quadratic form Q on a vector space V determines a quadric $Z_Q \subset \mathbb{P}(V)$. Given two quadratic forms Q_1 and Q_2 on V, a pencil $\mathcal{Z}_{\mathcal{Q}}$ of quadrics in $\mathbb{P}(V)$ is obtained by considering, for each $z = (\lambda : \mu) \in \mathbb{P}^1$, the quadric Z_{Q_z} defined by the quadratic form $\lambda Q_1 + \mu Q_2$. Let $\mathcal{Z}_{\mathcal{Q}} = \{(z, u) \in \mathbb{P}^1 \times \mathbb{P}(V) \,:\, u \in Z_{Q_z}\} \subset \mathbb{P}^1 \times \mathbb{P}(V)$.

In particular, we can view the quadrics $Z_{\alpha,2n}$ defined by the quadratic forms $Q_{\alpha,2n}$ of (4.1) as defining a pencil of quadrics in $\mathbb{P}^1 \times \mathbb{P}^{2n+1}$, with $\lambda/\mu = \alpha^2$. Namely, we regard the quadric $Z_{\alpha,2n}$ as part of the pencil of quadrics $\mathcal{Z}_{2n} = \{Z_{z,2n}\}_{z\in\mathbb{P}^1}$, defined by

$$Q_{z,2n} = \lambda(u_1^2 + u_5^2 + \cdots + u_{2n+2}^2) + \mu(u_2^2 + u_3^2 + u_4^2), \tag{4.2}$$

for $z = (\lambda : \mu) \in \mathbb{P}^1$. The quadric $Z_{z,2n}$ becomes degenerate over the set $X = \{0, 1\} \subset \mathbb{P}^1$, where it reduces, in the case $\lambda = 0$ to a projective cone $Z_{Q_1,2n} = C^{2n-1}B_1$ over the conic $B_1 = \{u_2^2 + u_3^2 + u_4^2 = 0\}$ in $\mathbb{P}^2$, and in the case $\mu = 0$ to a projective cone $Z_{Q_2,2n} = C^3B_2$ over the quadric $B_2 = \{u_1^2 + u_5^2 + \cdots + u_{2n+2}^2 = 0\}$ in $\mathbb{P}^{2n-2}$. There is a correspondence, as in §10 of [2],

$$\begin{array}{ccc} (\mathbb{P}^1 \times \mathbb{P}^{2n+1}) \smallsetminus \mathcal{Z}_n & \longrightarrow & \mathbb{P}^{2n+1} \smallsetminus (Z_{Q_1,2n} \cap Z_{Q_2,2n}) \\ \downarrow & & \\ \mathbb{P}^1 & & \end{array}$$

where the horizontal map is an $\mathbb{A}^1$-fibration and the vertical map is the projection to $z = (\lambda : \mu) \in \mathbb{P}^1$. By homotopy invariance, we can identify $H_c^{2n+2}((\mathbb{P}^1 \times \mathbb{P}^{2n+1}) \smallsetminus \mathcal{Z}_{2n})$ with the Tate twisted $H_c^{2n+1}(\mathbb{P}^{2n+1} \smallsetminus (Z_{Q_1,2n} \cap Z_{Q_2,2n}))(-1)$.

4.2. **Motives of quadrics.** The theory of motives of quadrics is a very rich and interesting topic, see [17], [19], [20]. We recall here only a few essential facts that we need in our specific case. Suppose given a quadratic form Q on an n-dimensional vector space V over a field $\mathbb{K}$ of characteristic not equal to 2. For our purposes, we will focus on the case where $\mathbb{K} = \mathbb{Q}$. We write $\langle a_1, \dots, a_n\rangle$ for the matrix of Q in diagonal form. The quadratic form $\mathbb{H} := \langle 1, -1\rangle$ is the elementary hyperbolic form. A quadratic form Q is isotropic if $\mathbb{H}$ is a direct summand, hence $Q = \mathbb{H} \perp Q'$. It is anisotropic otherwise. Any quadratic form can be written in the form $Q = d\cdot\mathbb{H} \perp Q'$, where Q' is a uniquely determined anisotropic quadratic form. The integer d is the Witt isotropy index of Q. Given an anisotropic quadratic form Q over the field $\mathbb{K}$, there is a tower of field extensions $\mathbb{K}_1 = \mathbb{K}(Q)$, $\mathbb{K}_2 = \mathbb{K}_1(Q_1)$, …, $\mathbb{K}_s = \mathbb{K}_{s-1}(Q_{s-1})$, such that over $\mathbb{K}_1$ the quadric $Q|_{\mathbb{K}_1} = d_1 \cdot \mathbb{H} \perp Q_1$, with Q_1 anisotropic; over $\mathbb{K}_2$ the quadric $Q_1|_{\mathbb{K}_2} d_2 \cdot \mathbb{H} \perp Q_2$, with Q_2 anisotropic, and so on, until $Q_s = 0$. The tower of extensions $\mathbb{K}_1, \dots, \mathbb{K}_s$ is the Knebusch universal splitting tower, and $d_1, \dots, d_s$ are the Witt numbers of Q.

Let Z_Q be the quadric defined by the quadratic form Q over $\mathbb{K}$. For a hyperbolic quadratic form $Q = d \cdot \mathbb{H}$ of dimension $2d$, the motive of Z_Q is given by (see [20])

$$\mathfrak{m}(Z_{d\mathbb{H}}) = \mathbb{Z}(d-1)[2d-2] \oplus \mathbb{Z}(d-1)[2d-2] \oplus \bigoplus_{i=0,\dots,d-2,d,\dots,2d-2} \mathbb{Z}(i)[2i], \tag{4.3}$$

where $\mathbb{Z} = \mathfrak{m}(\mathrm{Spec}(\mathbb{K}))$. In the case where $Q = d \cdot \mathbb{H} \perp \langle 1\rangle$ in dimension $2d + 1$, the motive of Z_Q is given by (see [20])

$$\mathfrak{m}(Z_{d\mathbb{H}\perp\langle 1\rangle}) = \bigoplus_{i=0,\dots,2d-1} \mathbb{Z}(i)[2i]. \tag{4.4}$$

Given a quadric Z_Q, we denote by Z_{Q^i} the variety of i-dimensional planes on the quadric Z_Q. As in [20], we write $\mathcal{X}_{Q^i}$ for the associated simplicial scheme (Definition 2.3.1 of [20]) and $\mathfrak{m}(\mathcal{X}_{Q^i})$ for the corresponding object in the category $\mathcal{DM}^{\rm eff}(\mathbb{K})$ of motives.

We also recall the following result (see Proposition 4.2 of [20]) that will be useful in our case. Let $Z_Q \subset \mathbb{P}^{m+1}$ be a quadratic form of dimension $m = 2n$ over $\mathbb{K}$, such that there exists a quadratic extension $\mathbb{K}(\sqrt{a})$ of $\mathbb{K}$ over which Q is hyperbolic. Then the motive $\mathfrak{m}(Z_Q)$ decomposes as a direct sum

$$\mathfrak{m}(Z_Q) = \begin{cases} \mathfrak{m}_1 \oplus \mathfrak{m}_1(1)[2] & m = 2 \mod 4 \\ \mathfrak{m}_1 \oplus \mathcal{R}_{Q,\mathbb{K}} \oplus \mathfrak{m}_1(1)[2] & m = 0 \mod 4 \end{cases} \tag{4.5}$$

where the motive $\mathfrak{m}_1$ is an extension of the motives $\mathfrak{m}(\mathcal{X}_{Q^i})(i)[2i]$ and $\mathfrak{m}(\mathcal{X}_{Q^\ell})(\dim(Q)-\ell)[2\dim(Q)-2\ell]$, for i (respectively, ℓ) ranging over all even (respectively, odd) numbers less than or equal to $2[\dim(Q)/4]$. The motive we denote by $\mathcal{R}_{Q,\mathbb{K}}$ is a form of a Tate motive, which is denoted by $\mathcal{R}_{Q,\mathbb{K}} = \mathbb{K}(\sqrt{\det(Q)})(\frac{\dim(Q)}{2})[\dim(Q)]$ in [20].

If Q is d-times isotropic, $Q = d \cdot \mathbb{H} \perp Q'$, then $\mathfrak{m}(\mathcal{X}_{Q^j}) = \mathbb{Z}$ for all $0 \leq j < i$. Thus, the motives $\mathfrak{m}(\mathcal{X}_{Q^j})$ become Tate motives in a field extension in which the quadric becomes isotropic, and one recovers the motivic decomposition into a sum of Tate motives mentioned above. The motives $\mathfrak{m}(\mathcal{X}_{Q^j})$ are therefore *forms* of the Tate motive, which means that over the algebraic closure $\mathfrak{m}(\mathcal{X}_{Q^j}|_{\bar{\mathbb{K}}}) = \mathbb{Z}$.

4.3. **Grothendieck classes.** It if often convenient, instead of working with objects in the category of mixed motives, to consider a simpler invariant given by the class in the Grothendieck ring of varieties, which can be regarded as a universal Euler characteristics. The Grothendieck ring $K_0(\mathcal{V}_\mathbb{K})$ of varieties over a field $\mathbb{K}$ is generated by the isomorphism classes $[X]$ of smooth quasi-projective varieties $X \in \mathcal{V}_\mathbb{K}$ with the inclusion-exclusion relations $[X] = [Y] + [X \smallsetminus Y]$ for closed embeddings $Y \subset X$ and the product $[X \times Y] = [X] \cdot [Y]$. The following simple identities will be useful in the computations of Grothendieck classes of the motives involved in the period computations described in the previous sections.

Lemma 4.1. *Let Z be a projective subvariety $Z \subset \mathbb{P}^{N-1}$, with $\hat{Z} \subset \mathbb{A}^N$ the affine cone. Let CZ denote the projective cone in $\mathbb{P}^N$ and $\widehat{CZ}$ the corresponding affine cone in $\mathbb{A}^{N+1}$. Let H and H' be two affine hyperplanes in $\mathbb{A}^{N+1}$ with $H \cap H' = \emptyset$ and such that the intersections $\widehat{CZ} \cap H$ and $\widehat{CZ} \cap H'$ are sections of the cone, given by copies of $\hat{Z}$. The Grothendieck classes of the projective and affine complements satisfy*

(1) $[\mathbb{A}^N \smallsetminus \hat{Z}] = (\mathbb{L}-1)[\mathbb{P}^{N-1} \smallsetminus Z]$
(2) $[\mathbb{A}^{N+1} \smallsetminus \widehat{CZ}] = (\mathbb{L}-1)[\mathbb{P}^N \smallsetminus CZ]$
(3) $[CZ] = \mathbb{L}\,[Z] + 1$
(4) $[\mathbb{A}^{N+1} \smallsetminus \widehat{CZ}] = \mathbb{L}^{N+1} - \mathbb{L}(\mathbb{L}-1)[Z] - \mathbb{L}$
(5) $[\mathbb{A}^{N+1} \smallsetminus (\widehat{CZ} \cup H \cup H')] = \mathbb{L}^{N+1} - 2\mathbb{L}^N - (\mathbb{L}-2)(\mathbb{L}-1)[Z] - (\mathbb{L}-2)$,

where $\mathbb{L} = [\mathbb{A}^1]$ is the Lefschetz motive, the class of the affine line.

Proof. The first and second identities follow from the fact that the class of the affine cone is given by $[\hat{Z}] = (\mathbb{L}-1)[Z]+1$, so that

$$[\mathbb{A}^N \smallsetminus \hat{Z}] = \mathbb{L}^N - (\mathbb{L}-1)[Z] - 1 = (\mathbb{L}-1)(\frac{(\mathbb{L}^N-1)}{(\mathbb{L}-1)} - [Z]) = (\mathbb{L}-1)[\mathbb{P}^{N-1} - Z].$$

The identity $[CZ] = \mathbb{L}\,[Z]+1$ follows by viewing the projective cone over Z as the union of a copy of Z and a copy of the affine cone $\hat{Z}$ over Z, and using the same identity $[\hat{Z}] = (\mathbb{L}-1)[Z]+1$ for the affine cone. The fourth identity follows from the second and the third,

$$(\mathbb{L}-1)[\mathbb{P}^N \smallsetminus CZ] = \mathbb{L}^{N+1} - 1 - (\mathbb{L}-1)[CZ]$$

$$= \mathbb{L}^{N+1} - 1 - (\mathbb{L}-1)(\mathbb{L}[Z]+1) = \mathbb{L}^{N+1} - (\mathbb{L}^2-\mathbb{L})[Z] - \mathbb{L}.$$

For the last identity, we write

$$[\mathbb{A}^{N+1} \smallsetminus (\widehat{CZ} \cup H \cup H')] = \mathbb{L}^{N+1} - [\widehat{CZ} \cup H \cup H'].$$

The class of the union is given by

$$[\widehat{CZ} \cup H \cup H'] = [\widehat{CZ}] + [H \cup H'] - [\widehat{CZ} \cap (H \cup H')].$$

Since $H \cap H' = \emptyset$, we have $[H \cup H'] = 2\mathbb{L}^N$ and $[\widehat{CZ} \cap (H \cup H')] = [\widehat{CZ} \cap H] + [\widehat{CZ} \cap H'] = 2[\hat{Z}] = 2(\mathbb{L}-1)[Z]+2$. Thus, we have

$$[\mathbb{A}^{N+1} \smallsetminus (\widehat{CZ} \cup H \cup H')] = \mathbb{L}^{N+1} - 2\mathbb{L}^N - [\widehat{CZ}] + 2(\mathbb{L}-1)[Z] + 2$$

$$= \mathbb{L}^{N+1} - 2\mathbb{L}^N - \mathbb{L}(\mathbb{L}-1)[Z] - \mathbb{L} + 2(\mathbb{L}-1)[Z] + 2 = \mathbb{L}^{N+1} - 2\mathbb{L}^N - (\mathbb{L}-2)(\mathbb{L}-1)[Z] - (\mathbb{L}-2).$$

□

4.4. **Pencils of quadrics in $\mathbb{P}^3$.** We look first at the case of the quadric $Z_\alpha = Z_{\alpha,2}$ in $\mathbb{P}^3$ that arises in the computation of the a_2 term of the heat kernel expansion.

Over $\mathbb{C}$, any quadric surface Z_Q in $\mathbb{P}^3$ can be put in the standard form $XY = ZW$ by a simple change of coordinates. Thus, over $\mathbb{C}$ any quadric surface in $\mathbb{P}^3$ is isomorphic to the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$. When we consider quadrics over $\mathbb{Q}$, this is no longer necessarily the case.

Theorem 4.2. *For $\alpha \in \mathbb{Q}$, over the quadratic extension $\mathbb{K} = \mathbb{Q}(\sqrt{-1})$, the quadric $Z_\alpha = Z_{\alpha,2}$ in $\mathbb{P}^3$ is isomorphic to the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$. The class of the complement in the Grothendieck ring is $[\mathbb{P}^3 \smallsetminus Z_\alpha] = \mathbb{L}^3 - \mathbb{L}$, while the class of the affine complement of $\widehat{CZ}_\alpha$ is $[\mathbb{A}^5 \smallsetminus \widehat{CZ}_\alpha] = \mathbb{L}^5 - \mathbb{L}^4 - \mathbb{L}^3 + \mathbb{L}^2$. The class of the complement $\mathbb{A}^5 \smallsetminus (\widehat{CZ}_\alpha \cup H_0 \cup H_1)$ with the affine hyperplanes $H_0 = \{u_0 = 0\}$ and $H_1 = \{u_0 = 1\}$ is given by*

$$[\mathbb{A}^5 \smallsetminus (\widehat{CZ}_\alpha \cup H_0 \cup H_1)] = \mathbb{L}^5 - 3\mathbb{L}^4 + \mathbb{L}^3 + 3\mathbb{L}^2 - 2\mathbb{L}.$$

Proof. Over the quadratic extension $\mathbb{K} = \mathbb{Q}(i)$ we can consider the change of variables

$$X = u_1 + \frac{i}{\alpha} u_2, \quad Y = u_1 - \frac{i}{\alpha} u_2, \quad Z = \frac{i}{\alpha}(u_3 + i u_4), \quad W = \frac{i}{\alpha}(u_3 - i u_4),$$

where we assume that $\alpha \in \mathbb{Q}$. This change of coordinates determines the identification of Z_α with the Segre quadric $\{XY - ZW = 0\} \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

The classes in the Grothendieck ring are then given by $[Z_\alpha] = [\mathbb{P}^1 \times \mathbb{P}^1] = (\mathbb{L}+1)^2 = \mathbb{L}^2 + 2\mathbb{L} + 1$, so that $[\mathbb{P}^3 \smallsetminus Z_\alpha] = \mathbb{L}^3 + \mathbb{L}^2 + \mathbb{L} + 1 - (\mathbb{L}^2 + 2\mathbb{L} + 1) = \mathbb{L}^3 - \mathbb{L}$. We then use Lemma 4.1 to compute the class $[\mathbb{A}^5 \smallsetminus \widehat{CZ}_\alpha]$. We have

$$[\mathbb{A}^5 \smallsetminus \widehat{CZ}_\alpha] = \mathbb{L}^5 - \mathbb{L}(\mathbb{L}-1)[Z_\alpha] - \mathbb{L} = \mathbb{L}^5 - \mathbb{L} - \mathbb{L}(\mathbb{L}-1)(\mathbb{L}+1)^2 = \mathbb{L}^5 - \mathbb{L}^4 - \mathbb{L}^3 + \mathbb{L}^2.$$

We then use the last identity of Lemma 4.1 to compute

$$[\mathbb{A}^5 \smallsetminus (\widehat{CZ}_\alpha \cup H_0 \cup H_1)] = \mathbb{L}^5 - 2\mathbb{L}^4 - (\mathbb{L}-2)(\mathbb{L}-1)[Z_\alpha] - (\mathbb{L}-2)$$
$$= \mathbb{L}^5 - 2\mathbb{L}^4 - (\mathbb{L}-2)(\mathbb{L}-1)(\mathbb{L}+1)^2 - (\mathbb{L}-2) = \mathbb{L}^5 - 3\mathbb{L}^4 + \mathbb{L}^3 + 3\mathbb{L}^2 - 2\mathbb{L}.$$

□

Theorem 4.3. *Over the quadratic extension $\mathbb{K} = \mathbb{Q}(\sqrt{-1})$, the motive*

$$\mathfrak{m}(\mathbb{A}^5 \smallsetminus (\widehat{CZ}_\alpha \cup H_0 \cup H_1), \Sigma)$$

is mixed Tate.

Proof. Over $\mathbb{K} = \mathbb{Q}(\sqrt{-1})$, the quadric Q_α, for $\alpha \in \mathbb{Q}$, satisfies

$$Q_\alpha|_{\mathbb{Q}(\sqrt{-1})} = 2 \cdot \mathbb{H}$$

hence the motive is given by (4.3) as

$$\mathfrak{m}(Z_\alpha) = \mathbb{Z} \oplus \mathbb{Z}(1)[2] \oplus \mathbb{Z}(1)[2] \oplus \mathbb{Z}(2)[4] = \mathfrak{m}(\mathbb{P}^1 \times \mathbb{P}^1)$$

where $\mathfrak{m}(\mathbb{P}^1) = \mathbb{Z} \oplus \mathbb{Z}(1)[2]$. This corresponds to the Grothendieck class $[Z_\alpha] = 1 + 2\mathbb{L} + \mathbb{L}^2$.

The Gysin distinguished triangle of the closed embedding $Z_\alpha \hookrightarrow \mathbb{P}^3$ of codimension one gives

$$\mathfrak{m}(\mathbb{P}^3 \smallsetminus Z_\alpha) \to \mathfrak{m}(\mathbb{P}^3) \to \mathfrak{m}(Z_\alpha)(1)[2] \to \mathfrak{m}(\mathbb{P}^3 \smallsetminus Z_\alpha)[1],$$

hence if two of the three terms are in the triangulated subcategory of mixed Tate motives, the third term also is. This implies that $\mathfrak{m}(\mathbb{P}^3 \smallsetminus Z_\alpha)$ is mixed Tate.

When passing to the projective cone CZ_α in $\mathbb{P}^4$, since $\mathbb{P}^4 \smallsetminus CZ_\alpha \to \mathbb{P}^3 \smallsetminus Z_\alpha$ is an $\mathbb{A}^1$-fibration, by homotopy invariance we have $\mathfrak{m}_c^j(\mathbb{P}^4 \smallsetminus CZ_\alpha) = \mathfrak{m}_c^{j-2}(\mathbb{P}^3 \smallsetminus Z_\alpha)(-1)$, where we consider here the motive $\mathfrak{m}_c^j$ with compact support that corresponds to the cohomology H_c^j. Thus, if the motive $\mathfrak{m}(\mathbb{P}^3 \smallsetminus Z_\alpha)$ is mixed Tate, then so is the motive $\mathfrak{m}(\mathbb{P}^4 \smallsetminus CZ_\alpha)$.

In passing from the motive $\mathfrak{m}(\mathbb{P}^4 \smallsetminus CZ_\alpha)$ to the motive $\mathfrak{m}(\mathbb{A}^5 \smallsetminus \widehat{CZ}_\alpha)$, consider the $\mathbb{P}^1$-bundle $\mathcal{P}$ compactification of the $\mathbb{G}_m$-bundle $\mathcal{T} = \mathbb{A}^5 \smallsetminus \widehat{CZ}_\alpha \to X = \mathbb{P}^4 \smallsetminus CZ_\alpha$ and the Gysin distinguished triangle

$$\mathfrak{m}(\mathcal{T}) \to \mathfrak{m}(\mathcal{P}) \to \mathfrak{m}_c(\mathcal{P} \smallsetminus \mathcal{T})^*(1)[2] \to \mathfrak{m}(\mathcal{T})[1],$$

see [21], p.197. The motive of a projective bundle satisfies $\mathfrak{m}(\mathcal{P})$ hence $\mathfrak{m}(\mathcal{P})$ is mixed Tate, since $\mathfrak{m}(X)$ is. The motive $\mathfrak{m}_c(\mathcal{P} \smallsetminus \mathcal{T})$ is also mixed Tate since $\mathcal{P} \smallsetminus \mathcal{T}$ consists of two copies of X, hence the remaining term $\mathfrak{m}(\mathcal{T})$ is also mixed Tate.

We then consider the union of $\widehat{CZ}_\alpha$ and the affine hyperplanes $H_0 = \{u_0 = 0\}$ and $H_1 = \{u_0 = 1\}$ in the affine space $\mathbb{A}^5$. In order to check that the motive of the union $\widehat{CZ}_\alpha \cup H_0 \cup H_1$ is mixed Tate suffices to know that the motives $\mathfrak{m}(\mathbb{A}^5 \smallsetminus (H_0 \cup H_1))$ and $\mathfrak{m}(\mathbb{A}^5 \smallsetminus \widehat{CZ}_\alpha)$ as well as the motive of the intersection $\mathfrak{m}(\widehat{CZ}_\alpha \cap (H_0 \cup H_1))$ are mixed Tate. This follows by applying the Mayer-Vietoris distinguished triangle

$$\mathfrak{m}(U \cap V) \to \mathfrak{m}(U) \oplus \mathfrak{m}(V) \to \mathfrak{m}(U \cup V) \to \mathfrak{m}(U \cap V)[1]$$

with $U = \mathbb{A}^5 \smallsetminus \widehat{CZ}_\alpha$ and $V = \mathbb{A}^5 \smallsetminus (H_0 \cup H_1)$. This shows that it suffices to know two of the three terms are mixed Tate to know the remaining one also is. The motive $\mathfrak{m}(\mathbb{A}^5 \smallsetminus \widehat{CZ}_\alpha)$ is mixed Tate by our previous argument. The motive $\mathfrak{m}(\mathbb{A}^5 \smallsetminus (H_0 \cup H_1))$ is also mixed Tate by a similar argument, since $\mathfrak{m}(H_0 \cup H_1)$ clearly is. Thus, it suffices to show that the motive $\mathfrak{m}(\mathbb{A}^5 \smallsetminus (\widehat{CZ}_\alpha \cap (H_0 \cup H_1))$ is mixed Tate, which can be shown by showing that the motive $\mathfrak{m}(\widehat{CZ}_\alpha \cap (H_0 \cup H_1))$ is mixed Tate. The intersection $\widehat{CZ}_\alpha \cap (H_0 \cup H_1)$ consists of two sections of the cone, hence one has two copies of the motive $\mathfrak{m}(\hat{Z}_\alpha)$ that is also a Tate motive.

The divisor Σ in $\mathbb{A}^5$ is a union of coordinate hyperplanes and their translates, and is also mixed Tate. Thus, the motive $\mathfrak{m}(\mathbb{A}^5 \smallsetminus (\widehat{CZ}_\alpha \cap (H_0 \cup H_1), \Sigma)$ sits in a distinguished triangle in the Voevodsky triangulated category of mixed motives over $\mathbb{Q}$, where two of the three terms, $\mathfrak{m}(\mathbb{A}^5 \smallsetminus (\widehat{CZ}_\alpha \cap (H_0 \cup H_1))$ and $\mathfrak{m}(\Sigma)$, are both mixed Tate. This implies that the remaining term $\mathfrak{m}(\mathbb{A}^5 \smallsetminus (\widehat{CZ}_\alpha \cap (H_0 \cup H_1), \Sigma)$ is also mixed Tate. □

4.5. **The Grothendieck class of $\mathbb{P}^{2n-1} \smallsetminus Z_{\alpha,2n}$ over $\mathbb{K} = \mathbb{Q}(\sqrt{-1})$.** We proceed with an inductive argument to compute the Grothendieck class $[\mathbb{P}^{2n-1} \smallsetminus Z_{\alpha,2n}]$ for all the quadrics $Z_{\alpha,n}$ determined by the quadratic forms

$$Q_{\alpha,2n} = u_1^2 + \frac{1}{\alpha^2}(u_2^2 + u_3^2 + u_4^2) + u_5^2 + u_6^2 + \cdots + u_{2n+1}^2 + u_{2n+2}^2, \tag{4.6}$$

for all $n \geq 3$.

Theorem 4.4. *Over the quadratic field extension $\mathbb{K} = \mathbb{Q}(\sqrt{-1})$ the quadric $Z_{\alpha,2n}$ has Grothendieck class $[\mathbb{P}^{2n+1} \smallsetminus Z_{\alpha,2n}] = \mathbb{L}^{2n+1} - \mathbb{L}^n$. The affine complement of $\widehat{CZ}_{\alpha,2n}$ has class*

$$[\mathbb{A}^{2n+3} \smallsetminus \widehat{CZ}_{\alpha,2n}] = \mathbb{L}^{2n+3} - \mathbb{L}^{2n+2} - \mathbb{L}^{n+2} + \mathbb{L}^{n+1}$$

and the affine complement of the union $\widehat{CZ}_{\alpha,2n} \cup H_0 \cup H_1$ has class

$$[\mathbb{A}^{2n+3} \smallsetminus (\widehat{CZ}_{\alpha,2n} \cup H_0 \cup H_1)] = \mathbb{L}^{2n+3} - 3\mathbb{L}^{2n+2} + 2\mathbb{L}^{2n+1} - \mathbb{L}^{n+2} + 3\mathbb{L}^{n+1} - 2\mathbb{L}^n.$$

Proof. Over the field $\mathbb{K} = \mathbb{Q}(\sqrt{-1})$ the change of coordinates

$$X = u_{2n+1} + iu_{2n+2}, \quad Y = u_{2n+1} - iu_{2n+2}$$

puts $Q_{\alpha,2n}$ in the form

$$Q_{\alpha,2n} = Q_{\alpha,2n-2}(u_1, \dots, u_{2n}) + XY.$$

Thus, the Grothendieck class $[\hat{Z}_{\alpha,2n}]$ is a sum of a contribution corresponding to $Y \neq 0$, which is of the form $(\mathbb{L}-1)\mathbb{L}^{2n}$ and a contribution from $Y = 0$, which is of the form $\mathbb{L}\,[\hat{Z}_{\alpha,n-1}]$. This gives

$$[\mathbb{A}^{2n+2} \smallsetminus \hat{Z}_{\alpha,2n}] = \mathbb{L}^{2n+2} - 2\mathbb{L}^{2n+1} + \mathbb{L}^{2n} + \mathbb{L}[\mathbb{A}^{2n} \smallsetminus \hat{Z}_{\alpha,2n-2}],$$

hence using the relation between the classes of the affine and projective complements,

$$[\mathbb{P}^{2n+1} \smallsetminus Z_{\alpha,2n}] = \mathbb{L}^{2n}(\mathbb{L}-1) + \mathbb{L}[\mathbb{P}^{2n-1} \smallsetminus Z_{\alpha,2n-2}].$$

Assuming inductively that $[\mathbb{P}^{2n-1} \smallsetminus Z_{\alpha,2n-2}] = \mathbb{L}^{2n-1} - \mathbb{L}^{n-1}$ we indeed obtain that the class of the complement is $[\mathbb{P}^{2n+1} \smallsetminus Z_{\alpha,2n}] = \mathbb{L}^{2n}(\mathbb{L}-1) + \mathbb{L}(\mathbb{L}^{2n-1} - \mathbb{L}^{n-1}) = \mathbb{L}^{2n+1} - \mathbb{L}^{n}$. We then have

$$[Z_{\alpha,2n}] = [\mathbb{P}^{2n+1}] - [\mathbb{P}^{2n+1} \smallsetminus Z_{\alpha,2n}] = \mathbb{L}^{2n} + \mathbb{L}^{2n-1} + \cdots + \mathbb{L}^{n+1} + 2\mathbb{L}^{n} + \mathbb{L}^{n-1} + \cdots + \mathbb{L}^2 + \mathbb{L} + 1.$$

Using Lemma 4.1, we obtain

$$[\mathbb{A}^{2n+3} \smallsetminus \widehat{CZ}_{\alpha,2n}] = \mathbb{L}^{2n-3} - \mathbb{L}(\mathbb{L}-1)[Z_{\alpha,2n}] - \mathbb{L}$$

$$= \mathbb{L}^{2n+3} - \sum_{j=2}^{n} \mathbb{L}^j - \mathbb{L}^{n+1} - 2\mathbb{L}^{n+2} - \sum_{j=n+3}^{2n+1} \mathbb{L}^j - \mathbb{L}^{2n+2} + \sum_{j=2}^{n} \mathbb{L}^j + 2\mathbb{L}^{n+1} + \mathbb{L}^{n+2} + \sum_{j=n+3}^{2n+1} \mathbb{L}^j$$

$$= \mathbb{L}^{2n+3} + \mathbb{L}^{n+1} - \mathbb{L}^{n+2} - \mathbb{L}^{2n+2}.$$

We proceed in the same way for the computation of the class of the affine complement of the union $\widehat{CZ}_{\alpha,2n} \cup H_0 \cup H_1$, using Lemma 4.1. We have

$$[\mathbb{A}^{2n+3} \smallsetminus (\widehat{CZ}_{\alpha,2n} \cup H_0 \cup H_1)] = \mathbb{L}^{2n+3} - 2\mathbb{L}^{2n+2} - (\mathbb{L}-2)(\mathbb{L}-1)[Z_{\alpha,2n}] - (\mathbb{L}-2)$$

and using again the expression

$$[Z_{\alpha,2n}] = \mathbb{L}^{2n} + \mathbb{L}^{2n-1} + \cdots + \mathbb{L}^{n+1} + 2\mathbb{L}^{n} + \mathbb{L}^{n-1} + \cdots + \mathbb{L}^2 + \mathbb{L} + 1$$

we obtain

$$(\mathbb{L}-2)(\mathbb{L}-1)[Z_{\alpha,2n}] = 2 - \mathbb{L} + 2\mathbb{L}^{n} - 3\mathbb{L}^{n+1} + \mathbb{L}^{n+2} - 2\mathbb{L}^{n+1} + \mathbb{L}^{2n+2}$$

due to cancellations of terms similar to the previous case. We then have

$$\mathbb{L}^{2n+3} - 2\mathbb{L}^{2n+2} - (\mathbb{L}-2)(\mathbb{L}-1)[Z_{\alpha,2n}] - (\mathbb{L}-2) =$$
$$\mathbb{L}^{2n+3} - 3\mathbb{L}^{2n+2} + 2\mathbb{L}^{2n+1} - \mathbb{L}^{n+2} + 3\mathbb{L}^{n+1} - 2\mathbb{L}^{n},$$

which agrees with the case $n = 1$ computed in Theorem 4.2. □

We then obtain an analog of Theorem 4.3, proved by a similar argument.

Proposition 4.5. *Over the field extension $\mathbb{K} = \mathbb{Q}(\sqrt{-1})$, the mixed motive*

$$\mathfrak{m}(\mathbb{A}^{2n+3} \smallsetminus (\widehat{CZ}_{\alpha,2n} \cup H_0 \cup H_1), \Sigma)$$

is mixed Tate.

Proof. The argument is completely analogous to Theorem 4.3, using the fact that, over $\mathbb{K} = \mathbb{Q}(\sqrt{-1})$ the quadratic form is

$$Q_{\alpha,2n}|_{\mathbb{Q}(\sqrt{-1})} = (n+1)\cdot \mathbb{H},$$

with (4.3) giving the motive $\mathfrak{m}(Q_{\alpha,2n}|_{\mathbb{Q}(\sqrt{-1})})$. The motives of complements, and projective and affine cones and the relative motives $\mathfrak{m}(\mathbb{A}^{2n+3} \smallsetminus \widehat{CZ}_{\alpha,2n}, \Sigma)$ and $\mathfrak{m}(\mathbb{A}^{2n+3} \smallsetminus (\widehat{CZ}_{\alpha,2n} \cup H_0 \cup H_1), \Sigma)$ are then obtained as in Theorem 4.3. $\square$

4.6. **The motive of $Z_{\alpha,2n}$ over $\mathbb{Q}$.** Over the rationals, the quadratic form $Q_{\alpha,2n}$ is anisotropic, although, as we have seen, it becomes isotropic over the field extension $\mathbb{K} = \mathbb{Q}(\sqrt{-1})$, with $Q_{\alpha,2n}|_{\mathbb{Q}(\sqrt{-1})} = (n+1)\cdot \mathbb{H}$. The motive of $Z_{\alpha,2n}$ over $\mathbb{Q}(\sqrt{-1})$ is a sum of Tate motives

$$\mathfrak{m}(Z_{\alpha,2n}|_{\mathbb{K}}) = \mathbb{Z}(n)[2n] \oplus \mathbb{Z}(n)[2n] \oplus \bigoplus_{i=0,\dots,n-1,n+1,\dots 2n} \mathbb{Z}(i)[2i],$$

which corresponds to the Grothendieck class $[Z_{\alpha,2n}] = [\mathbb{P}^{2n+1}] - [\mathbb{P}^{2n+1} \smallsetminus Z_{\alpha,2n}] = 1 + \cdots + \mathbb{L}^{2n+1} - (\mathbb{L}^{2n+1} - \mathbb{L}^n) = 1 + \mathbb{L} + \cdots + \mathbb{L}^{n-1} + 2\mathbb{L}^n + \mathbb{L}^{n+1} + \cdots + \mathbb{L}^{2n}$. Over the field $\mathbb{Q}$, the motive of $Z_{\alpha,2n}$ is given by (4.5), with

$$\mathfrak{m}(Z_{\alpha,2n}|_{\mathbb{Q}}) = \mathfrak{m}_1 \oplus \mathfrak{m}_1(1)[2]$$

when n is odd and

$$\mathfrak{m}(Z_{\alpha,2n}|_{\mathbb{Q}}) = \mathfrak{m}_1 \oplus \mathcal{R}_{Q,\mathbb{Q},n} \oplus \mathfrak{m}_1(1)[2]$$

when n is even, where $\mathcal{R}_{Q,\mathbb{Q},n}$ is a form of a Tate motive denoted by $\mathcal{R}_{Q,\mathbb{Q},n} = \mathbb{Q}(\sqrt{\det(Q_{\alpha,2n})})(n)[2n]$ in [20]. When passing to the quadratic field extension $\mathbb{Q}(\sqrt{-1})$ these motivic decompositions become the decomposition into Tate motives given above.

5. Appendix: explicit density for the a_2 coefficient

We use the formula (2.7) in the special case of $r = 0$ to calculate the term a_2 appearing in the heat kernel expansion (2.6). In this case we have

$$a_2 = \frac{1}{2^5\,\pi^4}\,\mathrm{Res}\left((D^2)^{-1}\right),$$

where $(D^2)^{-1}$ denotes the parametrix of D^2. In order to use the formula (2.8), since the dimension of the manifold is 4, we need to calculate the term $\sigma_{-4}(x,\xi)$ that is homogeneous of order -4 in the expansion of the symbol of $(D^2)^{-1}$. By performing symbolic calculations we find the following explicit expression.

$$\mathrm{tr}(\sigma_{-4}(t,\eta,\xi)) = \tag{5.1}$$

$$\frac{32\cot^2(\eta)\xi_3^4\csc^4(\eta)}{a(t)^6\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^4} + \frac{32\xi_2^2\xi_3^2\csc^4(\eta)}{a(t)^6\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^4} +$$
$$\frac{32\xi_3^4 a'(t)^2\csc^4(\eta)}{a(t)^6\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^4} - \frac{8\xi_3^2\csc^4(\eta)}{a(t)^4\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^3} -$$

$$\frac{192\xi_1^2\xi_3^4a'(t)^2\csc^4(\eta)}{a(t)^6\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^5}-\frac{384\cot(\eta)\xi_1\xi_2\xi_3^4a'(t)\csc^4(\eta)}{a(t)^7\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^5}-$$

$$\frac{192\cot^2(\eta)\xi_2^2\xi_3^4\csc^4(\eta)}{a(t)^8\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^5}-\frac{384\sec(\eta)\xi_1\xi_2\xi_3^2\xi_4^2a'(t)\csc^3(\eta)}{a(t)^7\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^5}+$$

$$\frac{64\cot^2(\eta)\xi_2^2\xi_3^2\csc^2(\eta)}{a(t)^6\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^4}+\frac{16\cot(\eta)\cot(2\eta)\xi_3^2\csc^2(\eta)}{a(t)^4\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^3}+$$

$$\frac{384\sec^2(\eta)\xi_2^2\xi_3^2\xi_4^2\csc^2(\eta)}{a(t)^8\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^5}+\frac{64\xi_2^2\xi_3^2a'(t)^2\csc^2(\eta)}{a(t)^6\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^4}+$$

$$\frac{4\xi_3^2a'(t)^2\csc^2(\eta)}{a(t)^4\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^3}+\frac{64\sec^2(\eta)\xi_3^2\xi_4^2a'(t)^2\csc^2(\eta)}{a(t)^6\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^4}+$$

$$\frac{48\cot(\eta)\xi_1\xi_2\xi_3^2a'(t)\csc^2(\eta)}{a(t)^5\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^4}+\frac{8\xi_3^2a''(t)\csc^2(\eta)}{a(t)^3\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^3}-$$

$$\frac{\csc^2(\eta)}{a(t)^2\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^2}-\frac{12\cot^2(\eta)\xi_3^2\csc^2(\eta)}{a(t)^4\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^3}-$$

$$\frac{32\xi_1^2\xi_3^2a''(t)\csc^2(\eta)}{a(t)^3\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^4}-\frac{48\xi_1^2\xi_3^2a'(t)^2\csc^2(\eta)}{a(t)^4\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^4}-$$

$$\frac{96\cot(2\eta)\xi_1\xi_2\xi_3^2a'(t)\csc^2(\eta)}{a(t)^5\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^4}-\frac{96\cot(\eta)\cot(2\eta)\xi_2^2\xi_3^2\csc^2(\eta)}{a(t)^6\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^4}-$$

$$\frac{64\sec^2(\eta)\xi_3^2\xi_4^2\csc^2(\eta)}{a(t)^6\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^4}-\frac{384\xi_1^2\xi_2^2\xi_3^2a'(t)^2\csc^2(\eta)}{a(t)^6\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^5}-$$

$$\frac{384\sec^2(\eta)\xi_1^2\xi_3^2\xi_4^2a'(t)^2\csc^2(\eta)}{a(t)^6\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^5}-\frac{384\cot(\eta)\xi_1\xi_2^3\xi_3^2a'(t)\csc^2(\eta)}{a(t)^7\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^5}+$$

$$\frac{384\sec^3(\eta)\xi_1\xi_2\xi_3^2\xi_4^2a'(t)\csc(\eta)}{a(t)^7\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^5}+\frac{32\csc^2(2\eta)\xi_2^2}{a(t)^4\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^3}+$$

$$\frac{32\sec^4(\eta)\xi_2^2\xi_4^2}{a(t)^6\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^4}+\frac{32\sec^4(\eta)\xi_4^4\tan^2(\eta)}{a(t)^6\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^4}+$$

$$\frac{64\sec^2(\eta)\xi_2^2\xi_4^2\tan^2(\eta)}{a(t)^6\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^4}+\frac{32\xi_2^4a'(t)^2}{a(t)^6\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^4}+$$

$$\frac{32\sec^4(\eta)\xi_4^4a'(t)^2}{a(t)^6\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^4}+\frac{4\xi_2^2a'(t)^2}{a(t)^4\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^3}+$$

$$\frac{4\sec^2(\eta)\xi_4^2a'(t)^2}{a(t)^4\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^3}+\frac{64\sec^2(\eta)\xi_2^2\xi_4^2a'(t)^2}{a(t)^6\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^4}+$$

$$\frac{3a'(t)^2}{a(t)^2\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^2}+\frac{96\cot(2\eta)\sec^2(\eta)\xi_2^2\xi_4^2\tan(\eta)}{a(t)^6\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^4}+$$

$$\frac{384\sec^4(\eta)\xi_1\xi_2\xi_4^4\tan(\eta)a'(t)}{a(t)^7\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^5}+\frac{384\sec^2(\eta)\xi_1\xi_2^3\xi_4^2\tan(\eta)a'(t)}{a(t)^7\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^5}+$$

$$\frac{8\xi_2^2a''(t)}{a(t)^3\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^3}+\frac{8\sec^2(\eta)\xi_4^2a''(t)}{a(t)^3\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^3}+$$

$$\frac{6a''(t)}{a(t)\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^2}-\frac{\sec^2(\eta)}{a(t)^2\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^2}-$$

$$
\begin{gathered}
\frac{4}{a(t)^2\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^2}-\frac{24\xi_1^2a''(t)}{a(t)\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^3}- \\
\frac{12\xi_1^2a'(t)^2}{a(t)^2\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^3}-\frac{16\cot(2\eta)\xi_1\xi_2a'(t)}{a(t)^3\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^3}- \\
\frac{16\cot^2(2\eta)\xi_2^2}{a(t)^4\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^3}-\frac{8\sec^4(\eta)\xi_4^2}{a(t)^4\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^3}- \\
\frac{12\sec^2(\eta)\xi_4^2\tan^2(\eta)}{a(t)^4\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^3}-\frac{16\cot(2\eta)\sec^2(\eta)\xi_4^2\tan(\eta)}{a(t)^4\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^3}- \\
\frac{32\xi_1^2\xi_2^2a''(t)}{a(t)^3\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^4}-\frac{32\sec^2(\eta)\xi_1^2\xi_4^2a''(t)}{a(t)^3\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^4}- \\
\frac{48\xi_1^2\xi_2^2a'(t)^2}{a(t)^4\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^4}-\frac{48\sec^2(\eta)\xi_1^2\xi_4^2a'(t)^2}{a(t)^4\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^4}- \\
\frac{96\cot(2\eta)\xi_1\xi_2^3a'(t)}{a(t)^5\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^4}-\frac{96\cot(2\eta)\sec^2(\eta)\xi_1\xi_2\xi_4^2a'(t)}{a(t)^5\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^4}- \\
\frac{48\sec^2(\eta)\xi_1\xi_2\xi_4^2\tan(\eta)a'(t)}{a(t)^5\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^4}-\frac{192\xi_1^2\xi_2^4a'(t)^2}{a(t)^6\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^5}- \\
\frac{192\sec^4(\eta)\xi_1^2\xi_4^4a'(t)^2}{a(t)^6\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^5}-\frac{384\sec^2(\eta)\xi_1^2\xi_2^2\xi_4^2a'(t)^2}{a(t)^6\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^5}- \\
\frac{192\sec^4(\eta)\xi_2^2\xi_4^4\tan^2(\eta)}{a(t)^8\left(\xi_1^2+\frac{\xi_2^2}{a(t)^2}+\frac{\csc^2(\eta)\xi_3^2}{a(t)^2}+\frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right)^5}.
\end{gathered}
$$

The density $b_{-4}(t,\eta,\xi)$ is obtained from $\mathrm{tr}(\sigma_{-4}(t,\eta,\xi))$ above by eliminating all terms with an odd exponent of ξ_j in the numerator.

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