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NON-LINEAR GAUSSIAN SOVEREIGN CDS PRICING MODELS

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Abstract

Prior literature indicates that quadratic models and the Black-Karasinki model are very promising for CDS pricing. This paper extends these models and the Black (1995) model for pricing sovereign **CDS's. For** all ten sovereigns in the sample quadratic models best fit CDS spreads **in-sample, and a four factor** quadratic model can account for the joint **effects on CDS spreads of default** risk, default loss risk and liquidity risk **with no restriction to factors correlation**. **Liquidity risk appears to affect sovereign CDS spreads. However quadratic models tend to over-fit some CDS maturities at the expense of other maturities, while the BK model is particularly immune from this tendency. The Black model seems preferable because its out-of-sample performance in the time series dimension is the best.**

Key words: sovereign CDS pricing, discrete time quadratic model, Black model, Black-Karasinski model, method of lines, Extended Kalman Filter.

JEL classification: G12; G13.

1 Introduction

As the CDS market has grown, a number of reduced form CDS pricing models have been proposed and tested. These models have been extended to multiple factors driving default intensities, to jumps in default intensities, to stochastic recovery rates after default and also to stochastic liquidity risk of CDS contracts. All the while default and liquidity intensities have been modelled mainly through

Feller processes, which are tractable, but also through quadratic Gaussian models and Black-Karasinski models, which seem particularly promising and a step beyond Feller processes. **However past literature has focused on somewhat restrictive specifications of the quadratic and Black-Karasinski models, namely quadratic models that do not accommodate liquidity risk or single factor versions of the Black-Karasinski model with constant recovery rates.**

Other potentially interesting but relatively unexplored models for CDS pricing are extensions of the Black (1995) shadow rate model, with its ability to rule out negative default intensities in the same way as it rules out negative interest rates. The Black model has gained acceptance among interest rate term structure models, but not yet among credit risk pricing models. Then the value of extensions of the Black model for credit risk pricing seems an obvious question to address.

Not only quadratic, Black-Karasinki and Black models all seem promising for CDS pricing and could be further developed, but guidance is missing about their relative merits and comparative weaknesses. Therefore this paper addresses these issues by extending, testing and "racing" quadratic, Black-Karasinski and Black models for the purpose of sovereign CDS pricing. The focus on sovereign CDS's is due to the very size and economic relevance of the sovereign CDS market, and to the attention it has received by the academic literature.

All the said models have the common feature that the default intensity and the default loss are non-linear non-negative functions of Gaussian factors. For quadratic models the default intensity and default loss are **respectively** quadratic and exponential quadratic in the Gaussian factors. For the Black-Karasinski model the default intensity and default loss are exponential in the Gaussian factors. For the Black (1995) model the default intensity and default loss **are respectively powers and exponential powers** of Gaussian factors with zero lower bounds.

This "race" of models excludes Feller processes, since such processes require undesirable restrictions to the correlation between the factors driving CDS pricing, such as independent Wiener processes. For example the factors driving the default intensity and the default loss need to have no instantaneous the correlation. Another problem is that Feller processes cannot be negative, which requires some "trick" to keep Feller processes from turning negative during Kalman Filter or maximum likelihood estimation. Instead CDS pricing based on Gaussian factors does not suffer from these problems of Feller processes. Moreover Pan and Singleton (2008) concluded that a Black-Karasinski **model** is preferable to an affine model based on the Feller process for sovereign CDS pricing.

The empirical evidence in this study shows that "in-sample" Black-Karasinski, Black and quadratic models all fit sovereign CDS spreads quite well, with a four factor quadratic model best fitting CDS spreads

for each of the ten countries in the sample. However "out-of-sample" each of these models displays distinctive strengths and weaknesses.

Quadratic models have a marked tendency to over-fit some CDS spread maturities at the expense of other maturities. This tendency is weaker for the Black model. On the contrary the main strength of the BK model is that it is largely immune from this tendency. The Black model displays the best out-of-sample performance in the time series dimension, even superior to that of a quadratic model with liquidity risk that employs more parameters than the Black model. For these reasons the Black model seems the best compromise.

The empirical evidence also shows that liquidity risk affects sovereign CDS spreads jointly with default risk and default loss risk, thus confirming previous studies that separately detected the effects of default loss risk and liquidity risk on CDS spreads. Even when all these risks are correlated, quadratic models have convenient closed form solutions for CDS valuation, while the BK and Black models require numerical solutions that become almost prohibitive when more than two of the risk factors are correlated. This computational challenge also afflicts affine models based on correlated Feller processes. However, the correlation between more than two of the risk factors does not seem of primary importance for out-of-sample model performance. Finally the non-monotonic relationship between latent factors and CDS spreads in quadratic models makes it more difficult to give an economic interpretation to factors correlation parameters and to factors drift parameters.

The paper is organised as follows. The next section reviews the most relevant literature. The next two sections present the theoretical pricing models. Another section illustrates the empirical performance of the models. The conclusions follow. The Appendixes complete the paper.

2 Literature

The literature on CDS pricing has grown to such an extent that it cannot be here summarised. Pan and Singleton (2008) "race" a set of one factor sovereign CDS pricing models comprising an affine model and a Black-Karasinski model. Thereafter most CDS pricing models assume that one or more Feller processes drive the credit risk and liquidity risk of CDS contracts, as in Schneider, Soegner, Veza (2010), in Zinna (2013) or in Badaoui, Cathcart and El-Jahel (2013, 2015). Zinna (2013) provides evidence of sovereign default risk premia and their considerable variation and predictability. Badaoui, Cathcart and El-Jahel (2013) show that a large fraction of sovereign CDS spreads compensate liquidity risk rather than credit risk. Badaoui, Cathcart and El-Jahel (2015) study the link between liquidity premia driving sovereign CDS spreads and the underlying sovereign bonds yield spreads.

However other CDS pricing models assume that default intensities follow quadratic Gaussian processes, as in Chen, Cheng, Fabozzi, Liu (2008), in Doshi, Ericsson, Jacobs and Turnbull (2013), in Elkamhi, Jacobs and Pan (2014), or Black-Karasinski processes as in Rubia, Sanchis-Marco, Serrano (2016). It is to this second strand of literature that this paper contributes. In particular this paper tests variants of discrete time quadratic models for CDS pricing, first proposed by Realdon (2006), later extended by Doshi (2011) who accommodated stochastic bond recovery value, and later applied also by Elkamhi, Jacobs and Pan (2014) for corporate CDS pricing. Also Doshi, Ericsson, Jacobs and Turnbull (2013) use a somewhat similar discrete time quadratic model for pricing corporate CDS's with firm leverage and stock volatility as driving factors.

A feature of this paper is to compute **the** Black and Black-Karasinski CDS pricing models through the vertical method of lines (MOL). MOL is based on exponential time integration, according to which partial differential equations are only discretised in space, but not in time. Vertical MOL was proposed in finance by Khaliq, Voss and Yousuf (2007) to value exotic options with L-stable Pade' schemes. Vertical MOL proves particularly suitable for pricing problems involving default intensities. Apart from the finite difference approach of Pan and Singleton (2008), other recent papers employ radial basis functions (RBF) to solve partial differential equations for CDS pricing. Guarin, Liu and Ng (2011) use RBF to price CDS's with two stochastic factors driving default intensities. Guarin, Liu and Ng (2014) use RBF to price CDS's and even to solve multi-dimensional Fokker-Planck equations of a new non-linear filter for the latent factors. RBF are promising, but unreported computations with MOL proved faster than with RBF, therefore the present paper uses vertical MOL rather than RBF to solve partial differential equations and compute CDS spreads according to the Black and Black-Karasinski models.

In recent years Badaoui, Cathcart and El-Jahel (2013, 2015) among others proposed affine models based on Feller processes that account for liquidity risk and show that liquidity risk does affect sovereign CDS spreads. The common feature of CDS pricing models that have incorporated liquidity risk is that liquidity risk affects the values of the two legs of the CDS in an asymmetric way, meaning that liquidity risk alters the ratio between the value of the protection leg and the value of the fee leg of the CDS. According to Lovreta (2016) CDS spreads increase in periods of strong demand for default protection against limited supply of default protection, as protection sellers charge higher CDS spreads because it becomes more difficult to offset the taken position. Thus higher CDS spreads reflect liquidity risk premia due to limited supply from protection sellers. Also Rubia, Sanchis-Marco and Serrano (2016), who study the pricing errors of a Black-Karasinski CDS pricing model that only takes default risk into account, attribute much of the pricing errors to liquidity risk factors. The present paper too **studies CDS liquidity** risk, but in the context of quadratic models.

3 The Black-Karasinski (BK) and the Black Models for CDS pricing

This section presents the extended versions of the Black-Karasinksi model and of the Black (1995) model to be tested later on CDS spreads. At time t the sovereign default intensity is $\lambda_t^{\mathbb{Q}}$ under the risk-neutral measure $\mathbb{Q}$. The expected value of the recovery value at time t of the bond defaulted at the same time t is $R_t^{\mathbb{Q}}$ still under the risk-neutral measure $\mathbb{Q}$. $R_t^{\mathbb{Q}}$ is a kind of average of possible recovery values at time t of the bond just after default at time t, where the said average is taken under the measure $\mathbb{Q}$. $R_t^{\mathbb{Q}}$ is time varying. $R_t^{\mathbb{Q}}$ would be lower than average historical recovery rates under the real measure, however sovereign defaults are such rare events that historical recovery rates may be of little guidance. The bond actual recovery value is the market price of the bond immediately after default divided by the face value of the bond. The sovereign default event may be debt maturity acceleration, failure to pay, debt restructuring or debt repudiation. In the Black model to be tested, $\lambda_t^{\mathbb{Q}}$ is a function of the time t value of the two latent factors $x_{1,t}$ and $x_{2,t}$ so that

$$\lambda_t^{\mathbb{Q}} = \max(x_{1,t}, 0)^{q_1} + \max(x_{2,t}, 0)^{q_2}. \tag{1}$$

q_1 and q_2 are parameters to be estimated. The Black (1995) model for pricing default-free bonds assumes $r_t = \max(x_{1,t}, 0)$, where r_t is the instantaneous short interest rate at time t, hence the similarity with the model studied here. We assume that

$$R_t^{\mathbb{Q}} = 1 - e^{-\max(x_{3,t},0)^2}$$

where $x_{3,t}$ is a third factor independent of $x_{1,t}$ and $x_{2,t}$. This specification for $R_t^{\mathbb{Q}}$ will later fit CDS spreads well. In what follows we refer to the amount $1 - R_t^{\mathbb{Q}}$ as to the "default loss". Given a filtered probability space with the usual properties, we assume

$$\begin{aligned}
dx_{1,t} &= (p \cdot x_{2,t} + \kappa_1 \cdot (\theta_1 - x_{1,t})) \cdot dt + \sigma_1 \cdot dw_{1,t}^{\mathbb{Q}} \\
dx_{2,t} &= \kappa_2 \cdot (\theta_2 - x_{2,t}) \cdot dt + \sigma_2 \cdot dw_{2,t}^{\mathbb{Q}} \\
dx_{3,t} &= \kappa_3 \cdot (\theta_3 - x_{3,t}) \cdot dt + \sigma_3 \cdot dw_{3,t}^{\mathbb{Q}}.
\end{aligned}$$

$dx_{i,t}$ for $i = 1, 2, 3$ is the stochastic differential of the factor x_i and $dw_{i,t}^{\mathbb{Q}}$ the stochastic differential of a Wiener process in the risk-neutral measure $\mathbb{Q}$ over the infinitesimal time interval $[t, t + dt]$; $\kappa_i, \sigma_i, \theta_i$ and p are all constant parameters. We assume that

$$dw_{1,t}^{\mathbb{Q}} dw_{2,t}^{\mathbb{Q}} = dt \cdot \rho_{12}, \quad dw_{1,t}^{\mathbb{Q}} dw_{3,t}^{\mathbb{Q}} = dw_{2,t}^{\mathbb{Q}} dw_{3,t}^{\mathbb{Q}} = 0.$$

Therefore the Wiener processes are not correlated except for $dw_{1,t}^{\mathbb{Q}} dw_{2,t}^{\mathbb{Q}} = \rho_{12} \cdot dt$, while $\rho_{12}, \kappa_1, \sigma_1, \theta_1, \kappa_2, \sigma_2, \theta_2, \kappa_3, \sigma_3, \theta_3, p$ are all parameters. p links the drift of $x_{1,t}$ to $x_{2,t}$. Therefore even when $\rho = 0$, x_1 and x_2 are not unconditionally independent. All parameters are identifiable in estimation. Equation 1 implies that $\lambda_t^{\mathbb{Q}}$ cannot turn negative and CDS rates for maturities longer than the instantaneous maturity are guaranteed to be positive, even when $x_{1,t}$ and $x_{2,t}$ are both negative. This paper also tests an extension of the Black-Karasinski model whereby, all other things as in the Black model,

$$\begin{aligned}\lambda_t^{\mathbb{Q}} &= \exp(x_{1,t}) + \exp(x_{2,t}) \\ R_t^{\mathbb{Q}} &= 1 - e^{-\exp(x_{3,t})}.\end{aligned}$$

3.1 Processes in the real measure

We also assume that in the physical probability measure

$$\begin{aligned}&dx_{1,t} = (p^* \cdot x_{2,t} + \kappa_1^* \cdot (\theta_1^* - x_{1,t})) \cdot dt + \sigma_1 \cdot dw_{1,t}^* \\ &dx_{2,t} = \kappa_2^* \cdot (\theta_2^* - x_{2,t}) \cdot dt + \sigma_2 \cdot dw_{2,t}^* \\ &dx_{3,t} = \kappa_3^* \cdot (\theta_3^* - x_{3,t}) \cdot dt + \sigma_3 \cdot dw_{3,t}^* \\ &dw_{1,t}^* \cdot dw_{2,t}^* = \rho_{12} \cdot dt, \quad dw_{1,t}^* \cdot dw_{3,t}^* = dw_{2,t}^* \cdot dw_{3,t}^* = 0.\end{aligned}$$

$dw_{1,t}^*, dw_{2,t}^*, dw_{3,t}^*$ are differentials of Wiener processes in the physical probability measure. The superscript "*" indicates parameters and variables under the real probability measure. Let $t = 1, 2.., M$, denote the set of M dates on which we observe CDS spreads. $x_{1,t}, x_{2,t}, x_{3,t}$ denote the values of the three latent factors on day t. Δ is the time between consecutive observations and is approximately equal to one divided by the number trading days in one year. Therefore $\Delta = \frac{1}{260}$ since we observe about 260 daily prices per year in the data. Then, with little loss in accuracy, in the empirical tests we approximate the above stochastic differential equations using the Euler discretisation, so that the approximate physical conditional transition density of $\mathbf{x}_{t\cdot\Delta} = (x_{1,t\cdot\Delta}, x_{2,t\cdot\Delta}, x_{3,t\cdot\Delta})'$ given $\mathbf{x}_{(t-1)\cdot\Delta} = (x_{1,(t-1)\cdot\Delta}, x_{2,(t-1)\cdot\Delta}, x_{3,(t-1)\cdot\Delta})'$, which we denote as $l(\mathbf{x}_t \mid \mathbf{x}_{t-1})$, is

$$l(\mathbf{x}_t \mid \mathbf{x}_{t-1}) \sim N\left(\boldsymbol{\eta} + (\mathbf{I}_3 - \boldsymbol{\phi})\,\mathbf{x}_{t-1}, \boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right) \tag{2}$$

$$\boldsymbol{\phi} = \begin{pmatrix} \kappa_1^* & -p^* & 0 \\ 0 & \kappa_2^* & 0 \\ 0 & 0 & \kappa_3^* \end{pmatrix} \cdot \Delta, \ \boldsymbol{\eta} = \begin{pmatrix} \theta_1^* \cdot \kappa_1^* \\ \theta_2^* \cdot \kappa_2^* \\ \theta_3^* \cdot \kappa_3^* \end{pmatrix} \cdot \Delta,$$

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ \rho_{12} \cdot \sigma_2 & \sqrt{1-\rho_{12}^2} \cdot \sigma_2 & 0 \\ \rho_{13} \cdot \sigma_3 & \frac{\rho_{32} - \rho_{12}\cdot\rho_{13}}{\sqrt{1-\rho_{12}^2}} \cdot \sigma_3 & \sqrt{1 - \rho_{13}^2 - \frac{(\rho_{32} - \rho_{12}\cdot\rho_{13})^2}{1-\rho_{12}^2}} \cdot \sigma_3 \end{pmatrix} \sqrt{\Delta}$$

with $\rho_{13} = \rho_{32} = 0$. $N\left(\boldsymbol{\eta} + (\mathbf{I}_3 - \boldsymbol{\phi})\,\mathbf{x}_{t-1}, \boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)$ is the multivariate normal density with mean $\eta + (\mathbf{I}_3 - \boldsymbol{\phi})\,\mathbf{x}_{t-1}$ and covariance $\boldsymbol{\Sigma}\boldsymbol{\Sigma}'$. $\mathbf{I}_3$ is the 3×3 identity matrix.

3.2 The pricing equation

The Black and BK models are solved through the vertical method of lines presented in Khaliq, Voss and Yousuf (2007) and Realdon (2016). An Appendix describes the MOL scheme, which unreported simulations showed to be preferable to the implicit finite difference method and to be quicker then radial basis functions in order to price CDS's.

Let D denote the value at time t of a claim that pays $1 - R_T^{\mathbb{Q}}$ at time T provided the underlying bond does not default before T and that pays nothing otherwise. V denotes the value at time t of a defaultable discount bond with maturity at time T, no recovery in case of default and face value 1. $Z = \exp(-r \cdot (T-t))$ is the time time t value of a default-free bond with maturity T and face value 1. r is the instantaneous default-free interest rate, assumed constant over time. This simplifying assumption entails hardly any loss in CDS pricing accuracy and does not affect our conclusions about the relative performance of the CDS models in the "race". Absent arbitrage and dropping unnecessary time subscripts, from the above assumptions we obtain the pricing equation

$$D = Z \cdot V \cdot U \tag{3}$$

$$\frac{\partial V}{\partial \tau} = \frac{\partial^2 V}{\partial x_1^2}\frac{1}{2}\sigma_1^2 + \frac{\partial^2 V}{\partial x_1 \partial x_2}\rho_{12}\sigma_1\sigma_2 + \frac{\partial^2 V}{\partial x_2^2}\frac{1}{2}\sigma_2^2 + \frac{\partial V}{\partial x_1}\left(px_2 + \kappa_1\left(\theta_1 - x_1\right)\right) + \frac{\partial V}{\partial x_2}\kappa_2\left(\theta_2 - x_2\right) - V \cdot \lambda_t^{\mathbb{Q}} = 0 \tag{4}$$

$$\lim_{x_1 \to -\infty} \frac{\partial^2 V}{\partial x_1^2} \to 0, \ \lim_{x_1 \to \infty} \frac{\partial^2 V}{\partial x_1^2} \to 0, \ \lim_{x_2 \to -\infty} \frac{\partial^2 V}{\partial x_2^2} \to 0, \ \lim_{x_2 \to \infty} \frac{\partial^2 V}{\partial x_2^2} \to 0, \ V(\tau = 0) = 1 \tag{5}$$

$$\frac{\partial U}{\partial \tau} = \frac{\partial^2 U}{\partial x_3^2}\frac{1}{2}\sigma_3^2 + \frac{\partial U}{\partial x_3}\kappa_3\left(\theta_3 - x_3\right) = 0 \tag{6}$$

$$\lim_{x_3 \to -\infty} \frac{\partial^2 U}{\partial x_3^2} \to 0, \ \lim_{x_3 \to \infty} \frac{\partial^2 U}{\partial x_3^2} \to 0, \ U(\tau = 0) = 1 - R_T^{\mathbb{Q}}. \tag{7}$$

$\tau = T - t$, V is a function of x_1, x_2 and τ. U is a function of x_3 and τ. $V(\tau = 0) = 1$, $U(\tau = 0) = 1 - R_T^{\mathbb{Q}}$ are the values of V and U when $\tau = 0$, i.e. when $t = T$. V tends to be linear in the factors x_1, x_2 as these factors tend to plus infinity, in which case V tends to 0, or as the factors tend to minus infinity, in which case V tends to 1. Similarly U tends to be linear in the factor x_3 as this factor tends to plus infinity, in which case U tends to 0, or as x_3 tends to minus infinity, in which case U tends to 1. The assumption that x_3 is independent of x_1 and x_2 is needed to greatly simplify the numerical solution for D, and it is a restriction of the Black and BK models that is not needed by the quadratic models presented below. In the empirical part the partial differential equations for V and U are each solved through the vertical method of lines (MOL) illustrated in the Appendix.

3.3 Calculation of CDS spreads

Redefine with $Z_{k,t} = e^{-\delta\tau \cdot k \cdot r}$ the price at time t of a default-free discount bond with k time periods to maturity; each time period is of length $\delta\tau$, thus the discount bond matures at time $t + k \cdot \delta\tau$. Also redefine with $V_{k,t}$ the time t value of a defaultable discount bond with k periods to maturity, which becomes worthless in case of default.

Let t^* denote the time of default. If $t < t^* \leq t + \delta\tau$ we assume that the defaulted coupon bond is worth $R^{\mathbb{Q}}_{t+\delta\tau}$ at time $t + \delta\tau$. Similarly the protection-leg of a CDS written on the coupon bond makes a payment worth $1 - R^{\mathbb{Q}}_{t+k}$ at $t+k\cdot\delta\tau$ if $t+(k-1)\cdot\delta\tau < t^* \leq (t+k\cdot\delta\tau)$ with $0 \leq R^{\mathbb{Q}}_{t+k} \leq 1$ for $k = 1, 2, .., \mathfrak{N}$. $t + \mathfrak{N}\cdot\delta\tau$ is the time of the last payment of the CDS protection fee and the time when the CDS terminates.

Redefine $D_{k,t}$ as the value at time t of a defaultable claim that pays $1 - R^{\mathbb{Q}}_{t+k}$ at $t + k \cdot \delta\tau$ if $t^* > t + k \cdot \delta\tau$. Let $D^*_{k,t}$ be the value at time t of a defaultable claim that pays $1 - R^{\mathbb{Q}}_{t+k}$ at $t + k \cdot \delta\tau$ if $t^* > t + (k-1) \cdot \delta\tau$. Then the time t value of the CDS protection leg is

$$\textstyle\sum_{k=1}^{\mathfrak{N}} Z_{k,t} \cdot \left(D^*_{k,t} - D_{k,t}\right). \tag{8}$$

As $\delta\tau \rightarrow 0$ expression 8 becomes the "recovery of face" assumption, according to which the defaulted bond is worth $R^{\mathbb{Q}}_{t^*}$ at the exact time of default t^*. CDS fees are typically paid quarterly and in arrears, with a partial fee payment in case default occurs in between fee payment dates. We can approximate these contractual provisions by assuming that CDS fees, each equal to $CDS_t \cdot \delta\tau$, are paid at the times $t + k \cdot \delta\tau$ with $k = 1, 2, .., \mathfrak{N}$. CDS_t is the time t CDS spread for maturity $\mathfrak{N}\cdot\delta\tau$ such that

$$CDS_t = \frac{\sum_{k=1}^{\mathfrak{N}} Z_{k,t} \cdot \left(D^*_{k,t} - D_{k,t}\right)}{\sum_{k=1}^{\mathfrak{N}} \delta\tau \cdot Z_{k,t} \cdot V_{k,t}}.$$

In the empirical tests we set $\delta\tau = \frac{1}{26}$, which approximates the fact that a partial fee payment is due in case default occurs in between fee payment dates and also approximates the fact that the CDS pays off soon after default. The time step $\delta\tau = \frac{1}{26}$ is also used for solving the BK and Black models through vertical MOL, as explained in the Appendix, and such time step size proved quite accurate. $V_{k,t}$, $D^*_{k,t}$ and $D_{k,t}$ will be calculated according to the BK and Black models above.

4 Discrete time quadratic model for CDS pricing

This section illustrates the discrete time quadratic CDS pricing model to be tested later. In discrete time the model requires fewer parameter constraints than in continuous time as explained below. For the quadratic models we set each time step equal to $\Delta = \frac{1}{260}$, not $\delta\tau = \frac{1}{26}$ as for solving the BK and Black models. The shorter time step $\Delta = \frac{1}{260}$ makes discrete time quadratic models approximate continuous time quadratic models, so that the comparison with continuous time BK and Black models becomes more meaningful.

For the quadratic model we redefine $V_{n,t}^{q}$ as the time t value of a defaultable discount bond with n trading days to maturity, so that the bond matures at $t + n \cdot \Delta$; such discount bond becomes worthless in case of default and is not perfectly liquid, meaning that the bond holder may have to sell the bond at some discount in order to be able to find a willing buyer at all times. Sometimes the liquidity discount may become a premium when there is strong demand for that bond, for example if the bond is a "special" bond for repo contracts. In such cases holding the bond gives a "convenience yield" to the bond holder similar to the "convenience yield" for holding a commodity. This liquidity discount or premium will later determine how CDS liquidity risk affects CDS spreads.

Let $\lambda_t^{\mathbb{Q}}$ be the default intensity for one trading day, i.e. for the period $(t, t+\Delta]$, under the risk-neutral pricing measure $\mathbb{Q}$. Let $l_t^{\mathbb{Q}}$ be the liquidity risk intensity for the same trading day under $\mathbb{Q}$. Again r is the continuously compounded default-free interest rate for the same trading day. The no-arbitrage risk-neutral valuation equation for $V_{n,t}^{q}$ is

$$V_{n,t}^{q} = E_t^{\mathbb{Q}}\left[e^{-\Delta\cdot\left(r+l_t^{\mathbb{Q}}+\lambda_t^{\mathbb{Q}}\right)} \cdot V_{n-1,t+1}^{q}\right] \tag{9}$$

where $E_t^{\mathbb{Q}}[..]$ denotes expectation under the risk-neutral measure $\mathbb{Q}$ conditional on time t information. We further assume that

$$l_t^{\mathbb{Q}} + \lambda_t^{\mathbb{Q}} = \boldsymbol{\beta}'\mathbf{x}_t + \mathbf{x}_t'\boldsymbol{\Psi}\mathbf{x}_t \tag{10}$$

$$\mathbf{x}_t = (x_{1,t}, .., x_{m,t})' \tag{11}$$

$$\mathbf{x}_{t+1} - \mathbf{x}_t = \boldsymbol{\phi}\left(\boldsymbol{\theta} - \mathbf{x}_t\right) + \boldsymbol{\Sigma}\boldsymbol{\xi}_{t+1}^{\mathbb{Q}} \tag{12}$$

$$\mathbf{x}_{t+1} - \mathbf{x}_t = \boldsymbol{\phi}^*\left(\boldsymbol{\theta}^* - \mathbf{x}_t\right) + \boldsymbol{\Sigma}\boldsymbol{\xi}_{t+1} \tag{13}$$

$$\boldsymbol{\xi}_{t+1}^{\mathbb{Q}} \backsim N\left(\mathbf{0}_{m\times1}, \mathbf{I}_m\right) \tag{14}$$

$$\boldsymbol{\xi}_{t+1} \backsim N\left(\mathbf{0}_{m\times1}, \mathbf{I}_m\right) \tag{15}$$

$$\boldsymbol{\Sigma} = \mathbf{S}\sqrt{\Delta} \tag{16}$$

$$\boldsymbol{\phi} = \Delta\cdot\boldsymbol{\kappa},\ \boldsymbol{\phi} = \Delta\cdot\boldsymbol{\kappa}^* \tag{17}$$

$$V_{n,t}^{q} = \exp\left(A_n + \mathbf{B}_n'\mathbf{x}_t + \mathbf{x}_t'\mathbf{C}_n\mathbf{x}_t\right). \tag{18}$$

$\mathbf{x}_t, \boldsymbol{\beta}, \theta, \theta^*, \boldsymbol{\xi}_{t+1}^{\mathbb{Q}}, \boldsymbol{\xi}_{t+1}, \mathbf{B}_n$ are $m\times1$ vectors. $\boldsymbol{\Psi}, \boldsymbol{\phi}, \boldsymbol{\phi}^*, \boldsymbol{\kappa}, \boldsymbol{\kappa}^*, \mathbf{C}_n, \boldsymbol{\Sigma}, \mathbf{S}$ are $m\times m$ matrices. A_n, A_0 are scalars. $\theta, \theta^*, \boldsymbol{\phi}, \boldsymbol{\phi}^*, \boldsymbol{\kappa}, \boldsymbol{\kappa}^*, \boldsymbol{\Sigma}$, $\mathbf{S}$ are parameters. The processes for $\mathbf{x}$ are specified under both the real measure and the risk-neutral measure $\mathbb{Q}$. $\mathbf{x}$ follows a first order Gaussian auto-regressive process. The time t conditional covariance matrix of $(\mathbf{x}_{t+1} - \mathbf{x}_t)$ is $E_t^{\mathbb{Q}}\left[\mathbf{x}_{t+1}\cdot\mathbf{x}_{t+1}'\right] = \boldsymbol{\Sigma}\boldsymbol{\Sigma}'$. $\boldsymbol{\xi}_t^{\mathbb{Q}} = \left(\varepsilon_{1,t+1}^{\mathbb{Q}}, .., \varepsilon_{m,t}^{\mathbb{Q}}\right)'$ and $\boldsymbol{\xi}_{t+1} = \left(\varepsilon_{1,t+1}, .., \varepsilon_{m,t+1}\right)'$. $\varepsilon_{1,t+1}, .., \varepsilon_{m,t+1}$ and $\varepsilon_{1,t+1}^{\mathbb{Q}}, .., \varepsilon_{m,t+1}^{\mathbb{Q}}$ are scalar Gaussian random shocks respectively in the real and risk-neutral measures. $N(\mathbf{0}_{m\times 1}, \mathbf{I}_m)$ denotes the multivariate normal density with mean $\mathbf{0}_{m\times 1}$ and covariance matrix $\mathbf{I}_m$. $\mathbf{0}_{m\times 1}$ is a $m \times 1$ vector of zeros. $\mathbf{I}_m$ is the $m \times m$ identity matrix. Under these assumptions Realdon (2006) showed that

$$A_n = -\Delta r_t + A_{n-1} + \mathbf{B}'_{n-1}\boldsymbol{\phi\theta} + (\boldsymbol{\phi\theta})' \mathbf{C}_{n-1}\boldsymbol{\phi\theta} + \ln\frac{|\boldsymbol{\gamma}|}{abs\,|\boldsymbol{\Sigma}|} + \frac{1}{2}\mathbf{G}_{n-1}\boldsymbol{\gamma\gamma}'\mathbf{G}'_{n-1} \tag{19}$$

$$\mathbf{B}'_n = -\Delta\boldsymbol{\beta}' + \mathbf{B}'_{n-1}(\mathbf{I}_3 - \boldsymbol{\phi}) + 2(\boldsymbol{\phi\theta})' \mathbf{C}_{n-1}(\mathbf{I}_3 - \boldsymbol{\phi}) + 2\cdot\mathbf{G}_{n-1}\boldsymbol{\gamma\gamma}'\mathbf{C}_{n-1}(\mathbf{I}_3 - \boldsymbol{\phi}) \tag{20}$$

$$\mathbf{C}_n = -\Delta\boldsymbol{\Psi} + (\mathbf{I}_3 - \boldsymbol{\phi})' \mathbf{C}_{n-1}(\mathbf{I}_3 - \boldsymbol{\phi}) + 2\cdot(\mathbf{I}_3 - \boldsymbol{\phi})' \mathbf{C}_{n-1}\boldsymbol{\gamma\gamma}'\mathbf{C}'_{n-1}(\mathbf{I}_3 - \boldsymbol{\phi}) \tag{21}$$

$$s.t.: A_0 = 0,\ \mathbf{B}_0 = \mathbf{0}_{m\times 1},\ \mathbf{C}_0 = \mathbf{0}_{m\times m}$$

where $\mathbf{0}_{m\times m}$ is an $m \times m$ square matrix of zeros, $\mathbf{G}_{n-1} = \mathbf{B}'_{n-1} + 2(\boldsymbol{\phi\theta})' \mathbf{C}_{n-1}$ and $\boldsymbol{\gamma} = \left((\boldsymbol{\Sigma\Sigma}')^{-1} - 2\mathbf{C}_{n-1}\right)^{-1/2}$.

Above we defined $D_{k,t}$ and here we define $D^q_{n,t}$ is a similar way for the quadratic model with time step $\Delta = \frac{1}{260}$ rather than $\delta\tau = \frac{1}{26}$. Then $D^q_{n,t}$ is the value at time t of a defaultable claim that pays $1 - R^{\mathbb{Q}}_{t+n}$ at time $t + \Delta n$ if $t^* > t + \Delta n$, where $R^{\mathbb{Q}}_{t+n}$ is again the expected value of recovery value of the defaulted bond. For the quadratic model we set $1 - R^{\mathbb{Q}}_{t+n} = D^q_{0,t+n} = e^{-x^2_{4,t+n}}$ which we refer to as the (expected) default loss. It can be shown that the absence of arbitrage implies

$$D^q_{n,t} = E^{\mathbb{Q}}_t\left[e^{-\Delta\cdot\left(r_t+\lambda^{\mathbb{Q}}_t\right)}\cdot D^q_{n-1,t+1}\right] = \exp\left(A^D_n + \mathbf{B}^{D\prime}_n\mathbf{x}_t + \mathbf{x}'_t\mathbf{C}^D_n\mathbf{x}_t\right)$$

$$s.t.: A^D_0 = 0,\ \mathbf{B}^D_n = \mathbf{0}_{m\times 1},\ \mathbf{C}^D_0 = \begin{pmatrix} 0 & 0 & .. & 0 \\ 0 & 0 & .. & 0 \\ .. & .. & .. & .. \\ 0 & 0 & .. & -1 \end{pmatrix}$$

$$A^D_n = -\Delta r_t + A^D_{n-1} + \mathbf{B}^{D\prime}_{n-1}\boldsymbol{\phi\theta} + (\boldsymbol{\phi\theta})' \mathbf{C}^D_{n-1}\boldsymbol{\phi\theta} + \ln\frac{|\boldsymbol{\gamma}|}{abs\,|\boldsymbol{\Sigma}|} + \frac{1}{2}\mathbf{G}^D_{n-1}\boldsymbol{\gamma}^D\boldsymbol{\gamma}^{D\prime}\mathbf{G}^{D\prime}_{n-1}$$

$$\mathbf{B}^{D\prime}_n = \mathbf{B}^{D\prime}_{n-1}(\mathbf{I}_3 - \boldsymbol{\phi}) + 2(\boldsymbol{\phi\theta})' \mathbf{C}^D_{n-1}(\mathbf{I}_3 - \boldsymbol{\phi}) + 2\cdot\mathbf{G}^D_{n-1}\boldsymbol{\gamma}^D\boldsymbol{\gamma}^{D\prime}\mathbf{C}^D_{n-1}(\mathbf{I}_3 - \boldsymbol{\phi})$$

$$\mathbf{C}^D_n = -\Delta\boldsymbol{\Psi}^D + (\mathbf{I}_3 - \boldsymbol{\phi})' \mathbf{C}^D_{n-1}(\mathbf{I}_3 - \boldsymbol{\phi}) + 2\cdot(\mathbf{I}_3 - \boldsymbol{\phi})' \mathbf{C}^D_{n-1}\boldsymbol{\gamma}^D\boldsymbol{\gamma}^{D\prime}\mathbf{C}^{D\prime}_{n-1}(\mathbf{I}_3 - \boldsymbol{\phi})$$

with $\mathbf{G}^D_{n-1} = \mathbf{B}^{D\prime}_{n-1} + 2\left(\boldsymbol{\phi}\boldsymbol{\theta}\right)' \mathbf{C}^D_{n-1}$ and $\boldsymbol{\gamma}^D = \left(\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1} - 2\mathbf{C}^D_{n-1}\right)^{-1/2}$. $A^D_n, \mathbf{B}^{D\prime}_n, \mathbf{C}^D_n$ satisfy the same equations as $A_n, \mathbf{B}'_n, \mathbf{C}_n$ with $\boldsymbol{\beta} = \mathbf{0}_{m\times 1}$, with $\boldsymbol{\Psi}^D$ replacing $\boldsymbol{\Psi}$ and with $\mathbf{C}^D_0$ set equal to an $m \times m$ matrix whose entries are all equal to 0 except for the entry in the m-th row and m-th column which is -1. $\boldsymbol{\Psi}^D$ and $\boldsymbol{\Psi}$ are defined below and vary with the model specifications. Given $D^q_{n,t}$ and $V^q_{n,t}$, for quadratic models we compute CDS spreads as

$$CDS_t = \frac{\sum_{k=1}^{\mathfrak{N}} D^q_{(k-1)\cdot 10,t} - D^q_{k\cdot 10,t}}{\sum_{k=1}^{\mathfrak{N}} \delta\tau \cdot V^q_{k\cdot 10,t}}.$$

Since $\delta\tau = 10\cdot\Delta$, $D^q_{n=k\cdot 10,t}$ corresponds to $D_{k,t}$ and $V^q_{n=k\cdot 10,t}$ corresponds to $V_{k,t}$. The value of the default protection leg is approximated as $\sum_{k=1}^{\mathfrak{N}} D^q_{(k-1)\cdot 10,t} - D^q_{k\cdot 10,t}$ with little loss in accuracy.

Liquidity risk has been incorporated in CDS pricing models in various ways, in some cases by adjusting the CDS protection leg, as in Badaoui, Cathcart and El-Jahel (2013), in other cases by adjusting both the protection leg and the fee leg, as in Badaoui, Cathcart and El-Jahel (2015). However the common feature of CDS pricing models that incorporate liquidity risk is that liquidity risk affects the two legs of the CDS in an asymmetric way, meaning that liquidity risk alters the ratio between the values of the two legs. This is the case also in the model presented here, since $D^q_{n,t}$ and the CDS protection leg are unaffected by the **CDS** liquidity intensity, while $V^q_{n,t}$ and the fee leg of the CDS are. **The** stochastic default loss of the protection leg can account for the fact that bond illiquidity may affect the stochastic recovery value of the defaulted bond.

4.1 Quadratic models Q4, Q3

The empirical tests consider three and four factor quadratic models whereby $\boldsymbol{\Sigma} = \mathbf{S} \cdot \sqrt{\Delta}$ with

$$\begin{aligned}
\mathbf{S} &= \begin{pmatrix} \sigma_1 & 0 & 0 & 0 \\ \rho_{21}\cdot\sigma_2 & \sqrt{1-\rho_{21}^2}\cdot\sigma_2 & 0 & 0 \\ \rho_{31}\cdot\sigma_3 & \frac{\rho_{32}-\rho_{12}\cdot\rho_{31}}{\sqrt{1-\rho_{21}^2}}\cdot\sigma_3 & \sqrt{1-\rho_{31}^2-\frac{(\rho_{32}-\rho_{21}\cdot\rho_{31})^2}{1-\rho_{21}^2}}\cdot\sigma_3 & 0 \\ \rho_{41}\cdot\sigma_4 & \frac{\rho_{42}-\rho_{21}\cdot\rho_{41}}{\sqrt{1-\rho_{21}^2}}\cdot\sigma_4 & K\cdot\sigma_4 & \sqrt{1-\rho_{14}^2-\left(\frac{\rho_{42}-\rho_{12}\cdot\rho_{14}}{\sqrt{1-\rho_{12}^2}}\right)^2-K^2}\cdot\sigma_4 \end{pmatrix} \\
K &= \frac{\rho_{43}-\rho_{31}\rho_{41}-\frac{(\rho_{32}-\rho_{21}\cdot\rho_{31})(\rho_{42}-\rho_{21}\cdot\rho_{41})}{1-\rho_{21}^2}}{\sqrt{1-\rho_{31}^2-\frac{(\rho_{32}-\rho_{21}\cdot\rho_{31})^2}{1-\rho_{21}^2}}}.
\end{aligned}$$

ρ_{21} is the conditional correlation between $x_{2,t+1}$ and $x_{1,t+1}$, and $\rho_{31}, \rho_{32}, \rho_{41}, \rho_{42}, \rho_{43}$ have similar meaning, while $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ are volatility parameters.

For quadratic models, but not for Black and BK models, we consider also CDS liquidity risk and its impact on CDS spreads. We test model Q4, a quadratic model where $m = 4$, $l_t^{\mathbb{Q}} = x_{1,t}$, $\lambda_t^{\mathbb{Q}} = x_{2,t}^2 + x_{3,t}^2$ so that

$$\boldsymbol{\Psi} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \boldsymbol{\Psi}^D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ 1 - R_{t+n}^{\mathbb{Q}} = e^{-x_{4,t+n}^2}, \quad \boldsymbol{\beta}' = (1, 0, 0, 0)$$

$$\begin{pmatrix} x_{1,t+1} \\ x_{2,t+1} \\ x_{3,t+1} \\ x_{4,t+1} \end{pmatrix} = \begin{pmatrix} x_{1,t} \\ x_{2,t} \\ x_{3,t} \\ x_{4,t} \end{pmatrix} + \begin{pmatrix} \kappa_1 & 0 & 0 & 0 \\ 0 & \kappa_2 & 0 & 0 \\ 0 & \kappa_3 & \kappa_3 & 0 \\ 0 & 0 & 0 & \kappa_4 \end{pmatrix} \left(\begin{pmatrix} \theta_1 \\ 0 \\ \theta_3 \\ \theta_4 \end{pmatrix} - \begin{pmatrix} x_{1,t} \\ x_{2,t} \\ x_{3,t} \\ x_{4,t} \end{pmatrix} \right) \cdot \Delta + \mathbf{S} \cdot \begin{pmatrix} \varepsilon_{1,t+1}^{\mathbb{Q}} \\ \varepsilon_{2,t+1}^{\mathbb{Q}} \\ \varepsilon_{3,t+1}^{\mathbb{Q}} \\ \varepsilon_{4,t+1}^{\mathbb{Q}} \end{pmatrix} \sqrt{\Delta}$$

$$\boldsymbol{\theta} = \begin{pmatrix} \theta_1 \\ 0 \\ \theta_3 \\ \theta_4 \end{pmatrix}, \ \boldsymbol{\theta}^* = \begin{pmatrix} \theta_1^* \\ 0 \\ \theta_3^* \\ \theta_4^* \end{pmatrix}, \ \boldsymbol{\kappa} = \begin{pmatrix} \kappa_1 & 0 & 0 & 0 \\ 0 & \kappa_2 & 0 & 0 \\ 0 & \kappa_3 & \kappa_3 & 0 \\ 0 & 0 & 0 & \kappa_4 \end{pmatrix}, \ \boldsymbol{\kappa}^* = \begin{pmatrix} \kappa_1^* & 0 & 0 & 0 \\ 0 & \kappa_2^* & 0 & 0 \\ 0 & \kappa_3^* & \kappa_3^* & 0 \\ 0 & 0 & 0 & \kappa_4^* \end{pmatrix}.$$

In Q4 the one factor x_1 driving the liquidity intensity $l_t^{\mathbb{Q}}$ can be positive or negative **and has non-zero drift, unlike in Badaoui, Cathcart and El-Jahel (2013) who assume no drift for factors driving liquidity risk.**

In model Q3 $l_t^{\mathbb{Q}} = 0$ and $\lambda_t^{\mathbb{Q}} = x_{2,t}^2 + x_{3,t}^2$, $1 - R_{t+n}^{\mathbb{Q}} = e^{-x_{4,t+n}^2}$. Therefore Q3 is the same as Q4 except for $\boldsymbol{\beta}' = (0, 0, 0, 0)$, so that in Q3 the fee leg of the swap is not affected by the liquidity intensity. **Q3 is a restricted version of Q4 whereby $\sigma_1 = \kappa_1 = \kappa_1^* = x_{1,0} = 0$. Comparing Q4 with Q3 identifies the impact of the liquidity intensity on CDS spreads.**

As in Badaoui, Cathcart and El-Jahel (2013), in Q4 the liquidity intensity affecting the CDS fee leg may be positive or negative. Instead the analysis in Lovreta (2016) suggests that the liquidity intensity should be positive so as to increase CDS spreads. The reason is that demand from protection buyers may at times exceed supply from protection sellers, making it expensive for protection sellers to close out their position through an offsetting trade. Also Badaoui, Cathcart and El-Jahel (2015) assume that the liquidity intensity be non-negative. **However, if in Q4 we impose that $l_t^{\mathbb{Q}} = x_{1,t}^2$ so that the liquidity intensity cannot turn negative, all other things equal, the empirical perfomance of model Q4 tends to be slightly worse than when $l_t^{\mathbb{Q}} = x_{1,t}$, therefore the focus of this paper is on this latter specification**.

Badaoui, Cathcart and El-Jahel (2015) also state that one factor may suffice to model liquidity risk in sovereign CDS pricing. Their conclusion is based on a principal component analysis of CDS spread variations. Unreported results show that a four factor quadratic model with two factors driving the liquidity intensity still does not in general beat model Q4.

As the stochastic factors are latent, parameter identification restrictions are needed for estimation purposes. The above quadratic models are similar to the quadratic model canonical form in continuous time of Ahn, Dittmar and Gallant (2002), hereafter referred to as ADG, which ensures that parameters are identifiable. The ADG specification for the default intensity would be $\lambda_t^{\mathbb{Q}} = \mathbf{x}_t' \mathbf{\Psi} \mathbf{x}_t$ with $\boldsymbol{\kappa} \cdot \theta \geq 0$, $\boldsymbol{\kappa}^* \cdot \theta^* \geq 0$, $\mathbf{S}$ diagonal (triangular) and $\boldsymbol{\kappa}, \boldsymbol{\kappa}^*$ triangular (diagonal). According to ADG only one of the $\mathbf{S}$ and $\boldsymbol{\kappa}$ matrixes is triangular, while the other one is diagonal. Instead in the above quadratic models both $\mathbf{S}$ and $\boldsymbol{\kappa}$ are triangular at the same time. The reason is that in discrete time the conditional covariance of $\mathbf{x}_{t+1}$, i.e. $E_t^{\mathbb{Q}}\left[\mathbf{x}_{t+1} \cdot \mathbf{x}_{t+1}'\right] = \mathbf{\Sigma}\mathbf{\Sigma}'$, does not depend on $\boldsymbol{\kappa}$, unlike in continuous time. This entails that in discrete time model parameters remain identifiable even when both $\boldsymbol{\kappa}$ and $\mathbf{\Sigma}$ are triangular.

5 Empirical results

This section reports the empirical tests of the CDS models illustrated above.

5.1 The sample and the models

The data consists of daily sovereign CDS spread observations provided by Bloomberg. The daily quotes cover the periods and the sovereigns in Table 1, namely Brazil, Colombia, Hungary, Peru, Poland, Romania, Russia, South Africa, Turkey, Venezuela. For example, Brazil CDS spreads span the period from 14/3/2006 to 4/10/2013, which corresponds to 1968 trading days. For each country Table 1 reports the number of trading days. For all countries **there are** around 260 trading days per year. For each **country the analysis hereunder uses CDS spreads** of the one, two, three, five, seven and ten year maturities. The four year maturity is discarded because it often displays stale prices. CDS spreads are computed as the mid point between the bid and the ask quotes. For each country and CDS maturity Table 1 reports the minimum, maximum, mean and standard deviation of CDS spreads for all maturities.

[Table 1 about here]

The two year maturity for Hungary displays short episodes of stale prices as does the one year maturity for Peru, which explains the relatively poor performance of all models for these two CDS maturities.

The sample of CDS spreads is split in two alternative ways. The first split uses all observed CDS spreads for "in-sample" estimation except for those of the last 400 trading days, which are used for "out-of-sample" testing. The second split uses all observed CDS spreads, even those of the last 400 trading days, for "in-sample" estimation except for the one year, two year and three year CDS spreads, which are used for out-of-sample testing.

The estimates and tests hereunder concern the following models:

- "Black-Karasinski" model (BK) whereby the MOL solution region in each "space" dimension is $[x_{i,1}, x_{i,200}]$ for $i = 1, 2, 3$ and whereby the MOL solution nodes are $x_{i,j} = -(20 - 0.1 \cdot j)$ with $j = 1, .., 200$;

- "Black" model whereby the MOL solution region in each "space" dimension is $[x_{i,1}, x_{i,200}]$ for $i = 1, 2, 3$ and whereby the MOL solution nodes are $x_{i,j} = -(1 - 0.01 \cdot j)$ with $j = 1, .., 200$;

- quadratic models Q4 **and Q3 illustrated** above.

The empirical tests of all models use Quasi Maximum Likelihood and the Extended Kalman Filter and require the maximisation of the log-likelihood function lk, as explained in the Appendix. h_j denotes the estimated standard deviation of the daily observation errors for the j-year CDS maturity with $j = 1, 2, 3, 5, 7, 10$. The observation error is the difference between model predicted CDS spread and observed CDS spread on a given day and for a given CDS maturity. **The Quasi-likelihood was maximised in Matlab by first using the simplex algorithm of the "fminsearch" routine and then by using the results as the input for another round of optimisation through the "patternsearch" routine. This approach seemed the most effective.**

5.2 The tables

The starting values of the latent factors at time $t = 0$, namely $x_{1,0}$, $x_{2,0}$, $x_{3,0}$, $x_{4,0}$ are parameters to be estimated, which avoids arbitrary assumptions about the prior probability density of $x_{1,0}$, $x_{2,0}$, $x_{3,0}$, $x_{4,0}$. Tables 2,3,4,5,**6 present the in-sample** estimation results **and out-of-sample performance** for **each** CDS pricing model and **country**. The BHHH estimator provides the estimates of the standard deviations of the parameter estimates. In each table the columns headed "param" provide the parameter estimates and the columns headed "stdev" provide the corresponding standard deviations of the parameter estimates. **The estimates in Tables 2,3,4,5 are computed by excluding the observations of the last 400 trading days of the overall sample.** The estimates of $\boldsymbol{\kappa}$,θ and of $\boldsymbol{\kappa}^*$,θ^* differ because risk premia demanded by the market drive the difference between the real measure and the risk-neutral measure. Since all CDS contracts are US dollar denominated, we assume that the US default-free term structure of interest rates is flat and constant at $3,16\%$, which is an average of the US Treasury term structure across maturities up to 10 years and across the whole sample period. Unreported numerical simulations show that such assumption of a flat and constant default-free term structure implies little loss in the accuracy of CDS pricing models.

Model results differ across countries **partly** because the sample periods differ across countries. Our main purpose is to compare the performance of different models for each single country. The rows **in Tables 2,3,4,5** named **AIC display** the Akaike information criterion **and the rows named SBIC display Schwartz' Bayesian Information Criterion** for each model and country. **AIC and SBIC compare** the empirical performance of non-nested models with different numbers of parameters. **SBIC uses a stronger penalty than**

AIC for adding parameters to a model.

5.3 Results for Black-Karasinski model

Table 2 summarises the estimation results for the Black-Karasinski (BK) model. For all countries except Venezuela, whose CDS spreads are extremely high, x_1 is more volatile than x_2 as $\sigma_1 > \sigma_2$. x_2 not only drives the default intensity $\lambda_t^{\mathbb{Q}}$, but also the drift of x_1. **Imposing** that $p = p^*$**,** p is positive and significant for all countries, therefore the drift of x_1 rises with x_2 **under** the real and risk-neutral measures.

Under the risk-neutral measure $\mathbb{Q}$, both x_1 and x_2 are mean reverting for all countries, as κ_1 and κ_2 are positive and **significant for** all countries, except for the **estimates** of κ_2 for **Poland and South Africa**, which **are** not significant. Under the real measure both x_1 and x_2 are still mean reverting for **most** countries**, since** κ_1^* and κ_2^* are **mostly** positive, **even though often not significant**.

The parameter ρ_{12} is the correlation between the Wiener processes driving x_1 and x_2. The estimates of ρ_{12} are significant and positive for all countries, **except for Romania and Venezuela, for which the correlation estimates are not significant**. The fact that both ρ_{12} and p are significant implies that simpler BK models whereby the latent factors driving $\lambda_t^{\mathbb{Q}}$ are independent seem mis-specified, **although** factors independence simplifies the pricing computations.

For six of the ten countries $\kappa_1\theta_1 > \kappa_1^*\theta_1^*$ **and/or** $\kappa_2\theta_2 > \kappa_2^*\theta_2^*$. **The results for these countries are consistent with** risk premia that make default probabilities under the risk-neutral measure $\mathbb{Q}$ higher than default probabilities under the real measure. **Such risk premia reward CDS investors for exposure to risk due to the uncertain dynamics of** $\lambda_t^{\mathbb{Q}}$**.**

The expected default loss expressed as a fraction of the bond face value is $1 - R_t^{\mathbb{Q}} = e^{-\exp(x_{3,t})}$ in the BK model and **it** decreases as $x_{3,t}$ rises. **For seven countries** κ_3 **is positive and for all countries it is strongly significant, while** κ_3^* **is positive and strongly significant for all countries except Turkey**. Therefore $x_{3,t}$ and the expected default loss $1 - R_t^{\mathbb{Q}}$ are mostly mean reverting **under the risk-neutral and under the real measures**.

θ_3^* and θ_3 are the long term mean reversion **levels** of $x_{3,t}$ under the real and risk-neutral measures **respectively**. **When** $x_{3,t} = \theta_3$ **the expected default loss under** $\mathbb{Q}$ **due to immediate default is** $e^{-\exp(-0.4318)} = 0.52$ **for Hungary,** $e^{-\exp(-8.1835)} = 0.999$ **for Brazil,** $e^{-\exp(-4.6437)} = 0.99$ **for Colombia,** $e^{-\exp(-1.223)} = 0.745$ **for Peru,** $e^{-\exp(-6.0698)} = 0.998$ **for Poland,** $e^{-\exp-(1.5114)} = 0.8$ **for Romania,** $e^{-\exp(-0.6643)} = 0.598$ **for Russia,** $e^{-\exp(-1.0642)} = 0.708$ **for South Africa,** $e^{-\exp(-0.9835)} = 0.688$ **for Turkey,** $e^{-\exp(-0.4181)} = 0.518$ **for Venezuela. Some of these estimates seem high, but expected losses (due to immediate default) should be higher under** $\mathbb{Q}$ **than under the real measure. Moreover, due to few recent sovereign defaults, even expected default losses under the real measure are hardly estimable.**

If the market prices risk due to the uncertain dynamics of the expected default loss over time, expected default losses (due to future default) should be higher under the risk-neutral measure $\mathbb{Q}$. Since expected default losses (due to immediate default) decrease in $x_{3,t}$, expected default losses (due to future default) decrease as the drift of $x_{3,t}$ increases. For five of the ten countries $\theta_3 \kappa_3 < \theta_3^* \kappa_3^*$, but this is not convincing evidence that under the real measure the drift of $x_{3,t}$ is higher, and expected default losses (due to future default) are lower, than under the risk-neutral measure $\mathbb{Q}$. Under this point of view, the Black model seems more convincing than the BK model, as explained below.

The expected default loss under $\mathbb{Q}$ is also volatile over time. The lowest value of σ_3 is 0.7294 for Turkey and σ_3 is greater than 1 for all the other countries. To simplify computations, which otherwise risk becoming prohibitive, x_3 is assumed independent of x_1 and x_2, so that the dynamics of the expected default loss are independent of the dynamics of **default probabilities.**

[**Table 2 about here**]

5.4 Results for Black model

Table 3 presents the results for the Black model illustrated above. In Table 3 q_1 and q_2 are parameters to be estimated, therefore the Black model has two more parameters than the BK model. The estimates of q_1 and q_2 are greater than 1 for all countries, except for q_2 for South Africa, and almost all such estimates are significantly great than 1. For some **countries the estimate** of q_2 **is** very high, so that the default intensity can be low even as $x_{2,t}$ is high. Poland is the only country for which the **estimate of q_2 is not significant, because the estimated standard deviation of q_2 is very high.** When $q_1 = q_2 = 1$ the survival probability reduces to the Black (1995) model. Estimation results for the model when $q_1 = q_2 = 1$ are not reported here because they were relatively disappointing. Estimation results for the model when $q_1 = q_2 = 2$ are not reported here for brevity, but such results tend to be only slightly worse than those reported, which assume no restrictions for q_1 and q_2.

As in the BK model, $x_{2,t}$ not only drives the default intensity $\lambda_t^{\mathbb{Q}}$, but also the drift of $x_{1,t}$. As in the BK model, also in the Black model the latent factors driving $\lambda_t^{\mathbb{Q}}$, namely x_1 and x_2, are not independent, since ρ_{12} and p are significantly different from 0 for all countries**, except for ρ_{12} of Venezuela**. As in the BK model, also in the Black model we impose $p = p^*$. As in the BK model, **also in the Black model** $p > 0$ for all countries, again implying that the drift of $x_{1,t}$ rises with $x_{2,t}$. ρ_{12} is positive for some countries and negative for others. As in the BK model, also in the Black model x_1 and x_2 are mean reverting for **most** countries under the risk-neutral measure $\mathbb{Q}$, since κ_1 and κ_2 are **positive for most countries and** significantly different from 0 for all countries. κ_2^* **shows that** under the real measure $x_{2,t}$ is **again** mean reverting for all countries**, except for Colombia**, **while κ_1^* shows that** $x_{1,t}$ does not revert toward θ_1^* **for six of the ten countries**. **For more than half of the countries $\kappa_1\theta_1 > \kappa_1^*\theta_1^*$ and/or $\kappa_2\theta_2 > \kappa_2^*\theta_2^*$. For these countries risk premia appear to** make default probabilities under the risk-neutral measure higher than under the real measure.

For the Black model the expected default loss at time t (**due to immediate default**) is $e^{-\max(x_{3,t},0)^2}$ under the risk-neutral measure $\mathbb{Q}$ and therefore it tends to decrease in $x_{3,t}$. σ_3 is significant**ly higher than one** for all countries **except Turkey**, which implies that the expected default loss is indeed volatile. In the Black model $x_{3,t}$ is mean reverting under both the real and risk-neutral measures for **most** countries **as** $\kappa_3, \kappa_3^* > 0$. As in the BK model, also in the Black model x_3 is assumed independent of x_1 and x_2, so that the expected default loss is independent of default probabilities. **For all countries $\theta_3 < \theta_3^*$. This result, coupled with the fact that κ_3 and κ_3^* are mostly positive and significant, means that the long term mean reversion level of $x_{3,t}$ tends to be higher in the real measure than in the risk-neutral measure $\mathbb{Q}$, which, since default losses decrease in $x_{3,t}$, in turn implies that expected future default losses tend to be higher in the risk-neutral measure $\mathbb{Q}$ than under the real measure, other things equal. This implies that the risk due to uncertain time variations in the expected default loss seems a risk priced by the market for most countries, according to the Black model**. **Under this point of view, the Black model seems more convincing than the BK model.**

The Akaike information criterion AIC and Schwartz's Bayesian Information Criterion SBIC in Tables 2 and 3 suggest that in-sample the Black model better fits CDS spreads than the BK model for six out of the ten countries, while BK performs better for Peru, Poland, Romania and South Africa. While the Black model has already gained acceptance in the literature on the term structure of interest rates, these results show its good empirical performance also for sovereign CDS pricing.

[Table 3 about here]

5.5 Results for model Q3

Table 4 presents the results for Q3, **i.e. the quadratic model where** $l_t^{\mathbb{Q}} = 0$, $\lambda_t^{\mathbb{Q}} = x_{2,t}^2 + x_{3,t}^2$ and $1 - R_t^{\mathbb{Q}} = e^{-x_{4,t}^2}$. x_2 and x_3 drive the default intensity and x_4 drives the expected default loss. All factors are correlated, so that **expected** default losses and survival probabilities are not independent, unlike in the BK and Black **models.** In Q3 liquidity risk is ignored, as it was ignored for the BK and Black models. **Table 4 shows that x_3 is mean reverting under the risk-neutral measure for all countries, but not under the real measure. x_2 is mean revering under both the real and risk-neutral measures, except for Peru and South Africa. x_4 is mostly mean averting.** x_1 plays no role in Q3. Correlation coefficients for the shocks driving x_2, x_3 and x_4 are **both positive and negative and often significantly different from zero.**

BK, Black and Q3 all assume two factors driving the default intensity and one factor driving the default loss. Both AIC and SBIC in Tables 2, 3 and 4 show that in-sample Q3 performs clearly better than BK and Black for all countries, with the exception of Venezuela where Black beats Q3. The relative strength of Q3 seems partly due to the fact that the factor driving the default loss is correlated with the factors driving the default intensity, while the BK and Black models rule out such correlation **to avoid prohibitive** numerical solutions for CDS spreads, which are not **an issue** for quadratic models. Taken together Tables 2, 3 and 4 provide **strong in-sample** evidence in favour of **model Q3** in the "race" with BK and Black for sovereign CDS pricing.

Tables 2, 3 and 4 assume that only credit risk affects sovereign CDS spreads, not liquidity risk. This **assumption does not seem to distort** the above "race" and the conclusions about the relative merits of BK, Black and Q3. However omitting liquidity risk can entail model mis-specification, as implied by recent studies, such **as Badaoui**, Cathcart and El-Jahel (2013, 2015). Therefore **we consider also quadratic model Q4, which "adds" liquidity risk to Q3. Q3** is a special case of Q4 as we impose the restricttions $\sigma_1 = \kappa_1 = \kappa_1^* = x_{1,0} = 0$. The p **values of the in-sample likelihood ratio test for these parameter restrictions are shown at the bottom of Table 4** in the row named "p value lk ratio". p values are virtually equal to 0 for all countries, therefore the restrictions of Q3 are clearly rejected in favour of Q4. **Liquidity risk appears to affect CDS spreads**.

[Table 4 about here]

5.6 Results for model Q4

Table 5 reports the results for **model** Q4 presented above. In Q4 the liquidity factor is x_1 and it can be positive or negative, while x_2 and x_3 drive **the** default **intensity** and x_4 drives the **expected default loss** as in Q3. **Unlike Q4**, **no model in previous literature seems to** account for both stochastic recovery and liquidity risk at the same time **in** pricing CDS's. According to both **AIC and SBIC**, **in-sample** Q4 performs significantly better than Q3 for all countries. This is evidence that liquidity risk **improves CDS pricing**, a result that confirms the findings in Badaoui, Cathcart and El-Jahel (2013, 2015) and in Corò, Dufour, Varotto (2013) among others.

For Q4 the volatility parameters σ_1, σ_2, σ_3, σ_4 are all significant for all countries. **The long term mean θ_1^* of the liquidity factor x_1 under the real measure is significant only for Hungary and Venezuela**. For most countries κ_1^* **appears not** significantly different from 0. **These results support the assumption of Badaoui, Cathcart and El-Jahel (2013) of no drift under the real measure for the factors driving liquidity risk. The evidence in favour of risk premia due to the uncertain dynamics of the liquidity risk factor seems weak.**

Under the risk-neutral measure the factors x_2 **and** x_3 **driving the default intensity are mean reverting for almost all countries, and so is the factor** x_4 **driving the default loss.** θ_4^* **is not significant for any country.** κ_3 **is positive and significant for all countries**, **while** κ_3^* **is significant for only four countries**. **As in Q3, also in Q4 most correlation coefficients are significantly different from 0 and can be positive or negative. Figure 1** shows how CDS spreads predicted by Q4 match Brazil's actual CDS spreads **over the in-sample period (the whole sample minus the last 400 trading days) and the out-of-sample period (the last 400 trading days)**.

[Table 5 about here]
[Figure 1 about here]

The "in-sample" evidence is clearly in favour of model Q4 and against the BK model. Table 6 summarises such evidence for all models in the rows named "Average AIC" and "Average SBIC" near the bottom of the table. Average AIC and Average SBIC are averages of AIC and SBIC across the ten countries. Even though they do not have a precise statistical meaning, they summarise "in-sample" performance: Q4 ranks first, Q3 second, Black third and BK fourth. Country level SBIC and AIC are largely consistent with this ranking. Unreported results confirm that quadratic models perform best even if we use the entire available data sample for "in-sample" estimation. However the "out-of-sample" evidence leads to different conclusions, as discussed below.

6 In-sample and out-of-sample pricing errors

To compare in-sample and out-of-sample model performance, Tables 2 to 6 display two measures of distance between model predicted CDS spreads and observed CDS spreads: MAPE (mean absolute percentage errors) and RMSE (root mean squared errors). MAPE and RMSE are calculated using daily CDS spreads, excluding the first observation date. Tables 2 to 5 present MAPE, which are the average absolute value of the daily difference between model predicted CDS spread and observed CDS spread divided by the observed CDS spread. Table 6 presents RMSE, which are the square root of the average of the square of the daily difference between model predicted CDS spread and observed CDS spread. During the 2008-2009 financial crisis there are short periods of stale CDS spreads for the two year maturity for Hungary and for the one year maturity for Peru, which explain the unusually high pricing errors for all models for these two maturities.

Tables 2 to 6 report two types of out-of-sample RMSE and MAPE in the grey shaded areas of the tables. The first type are time series

out-of-sample RMSE (hereafter TSOOS RMSE) and MAPE (hereafter TSOOS MAPE); these are computed by first estimating the models using all the CDS maturities, but excluding the observations for the last 400 trading days in the sample, and then by using the models so estimated to predict though the Kalman Filter the out-of-sample CDS spreads for the last 400 trading days. The "in-sample" counterparts of TSOOS RMSE are TSIS RMSE (time series in-sample RMSE) and the in-sample counterparts of TSOOS MAPE are TSIS MAPE.

The second type out-of-sample errors are cross sectional out-of-sample RMSE (hereafter CSOOS RMSE) and MAPE (hereafter CSOOS MAPE); these are computed by first estimating the models using only the CDS spreads of maturities of five, seven and ten years (this is the "in-sample" estimation) and then by using the models so estimated to predict the out-of-sample CDS spreads of maturities of one, two and three years. This severe test can highlight model deficiencies. A model may be able to fit long term CDS spreads well at the expense of implying unrealistic dynamics of the default intensity and of short term CDS spreads; this is indeed the case of the quadratic models as shown below. The "in-sample" counterparts of CSOOS RMSE are CSIS RMSE (cross sectional in-sample RMSE) and the in-sample counterparts of CSOOS MAPE are CSIS MAPE.

The rows named "Average" in Tables 2 to 6 compute the average MAPE (in-sample or out-of-sample) and the average RMSE (in-sample or out-of-sample) across all CDS maturities for each single country and model.

TSOOS MAPE and TSIS MAPE for each model and country are reported in Tables 2 to 5. The bottom row of Table 6 reports for each model Average* TSIS MAPE (time series in-sample MAPE), Average* TSOOS MAPE (time series out-of-sample MAPE) and Average* MAPE (across both TSIS and TSOOS MAPE). These averages are computed across all countries and all CDS maturities using the MAPE reported in Tables 2 to 5. It is striking that Q3 has the lowest Average* TSIS MAPE of 4.22%, but the highest Average* TSOOS MAPE of 4.688%, while Black has the highest Average* TSIS MAPE of 4.955%, but the lowest Average* TSOOS MAPE of $3,885\%$. In-sample and out-of sample relative model performance differs markedly. Q4 has the lowest overall Average* MAPE (across both TSIS and TSOOS MAPE) of 4.376%, but the out-of-sample perfomance of the Black model in the time series dimension seems the best.

Tables 2,3,4,5 show that for most models and countries TSOOS MAPE and TSOOS RMSE are smaller than their in-sample counterparts TSIS MAPE and TSIS RMSE. The reason for this unusual result is that the in-sample period spans the financial crisis of 2008 and 2009. Such market turbulence was absent in the out-of-sample

period. Tables 2,3,4,5 also show that for almost all models and countries MAPE tend to be highest for the 1 year and/or 2 year CDS maturities.

For each model, country and CDS maturity TSIS RMSE in Table 6 tend to be higher than the corresponding $h_1, h_2, h_3, h_5, h_7, h_{10}$ in Tables 2 to 5, which are the Kalman Filter estimates of the standard deviation of the daily difference between model predicted CDS spread and observed CDS spread for each CDS maturity. The reason is that these observation errors are not "white noise", while the Kalman Filter estimates assume that observation errors be Gaussian "white noise".

Tables 2 to 6 show that for all models Venezuela has the highest Average TSIS RMSE and the lowest Average TSIS MAPE, while Poland has the lowest Average TSIS RMSE and one of the highest Average TSIS MAPE. These averages are computed across the six CDS maturities. This in-sample evidence suggests that, for countries with higher CDS spreads, CDS pricing errors tend to be higher in absolute value (RMSE), but lower as a percentage of CDS spreads (MAPE). This feature of all the tested models shows that the models perform quite well even when credit risk is very high. For example, for the BK model the Average TSIS RMSE of 47 basis points of Venezuela are the highest, as shown in Table 6, but the Average TSIS MAPE of $2,384\%$ of Venezuela shown in Table 2 are the lowest. Instead for Poland the Average TSIS RMSE of 7 basis points are the lowest, as shown in Table 6, but the Average TSIS MAPE of $6,115\%$ are the highest, as shown in Table 2.

The row named "Avg* TSOOS RMSE" in the bottom part of Table 6 computes the average TSOOS RMSE across all countries and maturities for each model. The rows named "Avg* TSIS RMSE", "Avg* CSOOS RMSE", "Avg* CSIS RMSE" compute similar averages for TSIS RMSE, CSOOS RMSE and CSIS RMSE respectively. The smallest Avg* TSOOS RMSE are those of the Q4 and Black models (0.00081), followed by the Q3 model (0.00087) and by the BK model (0.0009). Thus time series out-of-sample evidence confirms the "in-sample" merits of Q4. Model Q4 again "beats" model Q3. Adding a liquidity factor improves out-of-sample perfomance. However Avg* TSOOS RMSE confirm the evidence based on MAPE: out-of-sample Q3 performs worse than Black, despite similar "fit" in-sample. Avg* TSIS RMSE are similar and around 0.00165-0.00166 for all the four models.

The out-of-sample deficiencies of quadratic models are best highlighted by cross sectional out-of-sample evidence. In the row "Avg* CSOOS RMSE" of Table 6, model Q4 scores by far the worst (0.0141), followed by Q3 (0.0077) and then by Black (0.0059), while BK scores best (0.0037). These average CSOOS RMSE are computed for each model across the one, two and three year CDS maturities and across

all countries. Cross sectional out-of-sample evidence is in strong favour of the BK model and against quadratic models and in particular against Q4. Q4 performs even worse than Q3 because its greater number of parameters (34 for Q4 instead 25 for Q3) causes Q4 to more strongly over-fit the "in-sample" CDS maturities of five, seven and ten years.

The columns headed "CSOOS&CSIS RMSE" in Table 6 display for each model, each country and each maturity:

- the CSIS RMSE for the maturities of five, seven and ten years in the grey areas;

- the CSOOS RMSE for the maturities of one, two and three years;

- the Average CSOOS&CSIS RMSE across all six maturities in the rows named "Average"; remarkably the Average CSOOS&CSIS RMSE of the BK model are the lowest of all models in almost every country; for example for Brazil Average RMSE are 0.0015 for BK, 0.0019 for Black, 0.0028 for Q3 and 0.0022 for Q4; the BK model is the most immune from the tendency to over-fit "in-sample" CDS maturities at the expense of "out-of-sample" CDS maturities.

Quadratic models can well fit the "in-sample" long term CDS spreads, while at the same time predicting completely unrealistic short term CDS spreads. For example note the one year CSOOS RMSE of 0.0979 in the "CSOOS&CSIS RMSE" column for Venezuela for Q4. Q4 also performs much worse then Q3 along the same metric, and again this seems due to the higher number of parameters of Q4, i.e. to more over-fitting of the "in-sample" maturities. Q3 has about the same number of parameters as the BK and Black models. This failure of quadratic models in the out-of-sample "cross section" of CDS spreads is notable since these models assume the absence of arbitrage across contemporaneous CDS spreads of different maturities, i.e. they consistently price of all CDS maturities. The reason for the failure may be that in quadratic models the default intensity, the expected default loss and the CDS spreads are quadratic, and therefore non-monotonic, functions of the factors.

This "non-monotonicity" of quadratic models also entails that the CDS spreads are non-monotonic in the drift parameters θ_3, θ_4 even while $\kappa_3, \kappa_4 > 0$. In the BK and Black models default intensity and expected default loss are monotonic in the factors, therefore if $\kappa_1, \kappa_2, \kappa_3 > 0$ CDS spreads cannot decrease as $\theta_1, \theta_2, \theta_3$ increase.

A similar "non-monotonicity" concerns also the correlation parameters of quadratic models. For example, as ρ_{34} rises in model Q4, the instantaneous correlation between $x_{3,t}$ and $x_{4,t}$ rises, but the correlation between the expected default loss, which is quadratic in $x_{4,t}$, and the default intensity, which is quadratic in $x_{3,t}$, can either rise or fall, and so can CDS spreads too.

These "non-monotonicities" make it more difficult to give an economic interpretation to the risk-premia, drift and correlation parame-

ters of quadratic models than to the risk-premia, drift and correlation parameters of the BK and Black models.

7 Conclusion

This paper has tested new specifications of promising non-linear Gaussian CDS pricing models using the sovereign CDS spreads of ten quite diverse **countries**. The models are quadratic Gaussian models, extensions of the Black (1995) model and extensions of the Black-Karasinski model. **All these models prove to have relative strengths in fitting different aspects of** sovereign CDS spreads, with the quadratic models performing best **of all** in-sample. Quadratic models are also more tractable than Black and Black-Karasinky models. **The empirical evidence supports the view that liquidity risk affects CDS pricing jointly with default risk and default loss risk and quadratic models can easily model the correlations between all these risks, unlike the BK and Black models. However quadratic models have a marked tendency to over-fit some CDS spread maturities at the expense of other maturities. The BK model is particularly immune from this tendency and this seems the main strength of the BK model.**

The Black model seems the best compromise; it has less of a tendency than quadratic models to over-fit some CDS maturities and displays the best out-of-sample performance in the time series dimension, even slightly superior to that of a quadratic model with liquidity risk, which has a few more parameters.

The choice of model may depend on out-of-sample errors in the cross section or in the time series of CDS spreads, but out-of-sample errors in the time series seem more relevant, as we estimate a model on past CDS spreads of all maturities and then use the model to predict current and future term structures of CDS spreads. For this reason the Black model seems the best compromise.

A Vertical method of lines

Vertical MOL that is employed in this paper to price CDS according to the Black and Black-Karasinski models is taken from Realdon (2016) and is hereunder summarised for the reader. Realdon reports that MOL has advantages over the finite difference method for solving pricing PDE's. Here we only show how MOL with sequential splitting is used to compute survival probabilities for the Black-Karasinki, while the solution for the Black model is almost the same. Only in this Appendix, we write x to mean x_1 and y to mean x_2. Thus $V(x, y, \tau)$ or more simply V is a function of x, y and τ. Then, dropping time subscripts equation 4 becomes for $x, y \in (-\infty, \infty)$

$$\frac{\partial V}{\partial \tau} = \frac{\partial^2 V}{\partial x \partial y}\rho_{12}\sigma_1\sigma_2 + \frac{\partial^2 V}{\partial x^2}\frac{\sigma_1^2}{2} + \frac{\partial V}{\partial x}\left(py + \kappa_1\left(\theta_1 - x\right)\right) + \frac{\partial^2 V}{\partial y^2}\frac{\sigma_2^2}{2} + \frac{\partial V}{\partial y}\kappa_2\left(\theta_2 - y\right) + \left(\exp\left(x\right) + \exp\left(y\right)\right)V \tag{22}$$

$$V\left(\tau = 0\right) = 1, \quad \lim_{x \to -\infty}\frac{\partial^2 V}{\partial x^2} \to 0, \quad \lim_{x \to \infty}\frac{\partial^2 V}{\partial x^2} \to 0, \lim_{y \to -\infty}\frac{\partial^2 V}{\partial y^2} \to 0, \quad \lim_{y \to \infty}\frac{\partial^2 V}{\partial y^2} \to 0,$$

PDE 22 is solved over the finite region $[x_1, x_I] \times [y_1, y_I]$. We define the following grid

$$x_i = i \cdot \delta x + x_0, \quad \delta x = \frac{x_I - x_0}{I}, \qquad y_j = j \cdot \delta y + y_0, \quad \delta y = \frac{y_I - y_0}{I}$$

for $i, j = 1, 2, .., I$.

We now use vertical MOL to discretise the PDE in both "space" dimensions, but not in the time dimension, and then we "sequentially split" the discretised PDE. Define $\tau_k = k \cdot \delta\tau$ for $k = 0, 1, 2, .., K$. $\delta\tau$ is the size of a time step. $[0, K \cdot \delta\tau]$ is the time interval over which PDE 22 is solved. Using vertical MOL with sequential splitting, we approximate PDE 22 during the interval $[\tau_k, \tau_{k+1}]$ as

$$\begin{aligned}
\frac{\partial u_{i,j}^{k+1}}{\partial \tau} &= \frac{\rho_{12}\sigma_1\sigma_2}{2}\frac{\partial^2 u_{i,j}^{k+1}}{\partial x \partial y} + \frac{u_{i+1,j}^{k+1} - 2u_{i,j}^{k+1} + u_{i-1,j}^{k+1}}{\left(\delta x\right)^2}\frac{\sigma_1^2}{2} + \frac{u_{i+1,j}^{k+1} - u_{i-1,j}^{k+1}}{2 \cdot \delta x}\left(py_j + \kappa_1\left(\theta_1 - x_i\right)\right) + \exp\left(\delta x \cdot i\right)u_{i,j}^{k+1} \\
\frac{\partial v_{i,j}^{k+1}}{\partial \tau} &= \frac{\rho_{12}\sigma_1\sigma_2}{2}\frac{\partial^2 v_{i,j}^{k+1}}{\partial x \partial y} + \frac{v_{i,j+1}^{k+1} - 2v_{i,j}^{k+1} + v_{i,j-1}^{k+1}}{\left(\delta y\right)^2}\frac{\sigma_2^2}{2} + \frac{v_{i,j+1}^{k+1} - v_{i,j-1}^{k+1}}{2 \cdot \delta y}\kappa_2\left(\theta_2 - y_j\right) + \exp\left(\delta y \cdot j\right)v_{i,j}^{k+1} \\
\frac{\partial^2 u_{i,j}^{k+1}}{\partial x \partial y} &\backsimeq \frac{u_{i+1,j+1}^{k+1}\left(\tau_k\right) - u_{i-1,j+1}^{k+1}\left(\tau_k\right) - u_{i+1,j-1}^{k+1}\left(\tau_k\right) + u_{i-1,j-1}^{k+1}\left(\tau_k\right)}{4\delta x \delta y} \\
\frac{\partial^2 v_{i,j}^{k+1}}{\partial x \partial y} &\backsimeq \frac{v_{i+1,j+1}^{k+1}\left(\tau_k\right) - v_{i-1,j+1}^{k+1}\left(\tau_k\right) - v_{i+1,j-1}^{k+1}\left(\tau_k\right) + v_{i-1,j-1}^{k+1}\left(\tau_k\right)}{4\delta x \delta y} \\
u_{i,j}^{k+1}(\tau_k) &= v_{i,j}^{k}(\tau_k) \\
v_{i,j}^{k+1}(\tau_k) &= u_{i,j}^{k+1}(\tau_{k+1})
\end{aligned}$$

for $i, j = 1, 2, .., I$.

We can rewrite this system of equations as

$$\begin{aligned}
&\frac{\partial \mathbf{u}_j^{k+1}}{\partial \tau} = \mathbf{M}_u\left(j\right) \cdot \mathbf{u}_j^{k+1} + \mathbf{q}_j^{k+1}\left(u\right) \\
&\mathbf{u}_j^{k+1}(\tau_k) = \mathbf{v}_j^{k}(\tau_k) \\
&\left(\frac{\partial \mathbf{v}^{k+1,i}}{\partial \tau}\right)' = \mathbf{M}_v \cdot \left(\mathbf{v}^{k+1,i}\right)' + \left(\mathbf{q}^{k+1,i}\left(v\right)\right)' \\
&\mathbf{v}^{k+1,i}(\tau_k) = \mathbf{u}^{k+1,i}(\tau_{k+1})
\end{aligned} \tag{23}$$

for $i, j = 1, 2, .., I$ and $k = 0, 1, 2, .., K$ with

$$\mathbf{u}_j^{k+1} = \begin{bmatrix} u_{1,j}^{k+1} \\ .. \\ u_{I,j}^{k+1} \end{bmatrix} \quad \mathbf{v}_j^{k+1} = \begin{bmatrix} v_{1,j}^{k+1} \\ .. \\ v_{I,j}^{k+1} \end{bmatrix} \quad \frac{\partial \mathbf{u}_j^{k+1}}{\partial \tau} = \begin{bmatrix} \frac{\partial u_{1,j}^{k+1}}{\partial \tau} \\ .. \\ \frac{\partial u_{I,j}^{k+1}}{\partial \tau} \end{bmatrix} \quad \frac{\partial \mathbf{v}_j^{k+1}}{\partial \tau} = \begin{bmatrix} \frac{\partial v_{1,j}^{k+1}}{\partial \tau} \\ .. \\ \frac{\partial v_{I,j}^{k+1}}{\partial \tau} \end{bmatrix}$$

$$\mathbf{M}_u(j) = \begin{bmatrix} 2\mathcal{A}_{x,1,j} + \mathcal{B}_{x,1,j} & \mathcal{C}_{x,1,j} - \mathcal{A}_{x,1,j} & 0 & .. & 0 & 0 & 0 \\ \mathcal{A}_{x,2,j} & \mathcal{B}_{x,2,j} & \mathcal{C}_{x,2,j} & .. & 0 & 0 & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & .. & \mathcal{A}_{x,(I-1),j} & \mathcal{B}_{x,(I-1),j} & \mathcal{C}_{x,(I-1),j} \\ 0 & 0 & 0 & .. & 0 & \mathcal{A}_{x,I,j} - \mathcal{C}_{x,I,j} & \mathcal{B}_{x,I,j} + 2\mathcal{C}_{x,I,j} \end{bmatrix}$$

$$\mathbf{M}_v = \begin{bmatrix} 2\mathcal{A}_{y,1} + \mathcal{B}_{y,1} & \mathcal{C}_{y,1} - \mathcal{A}_{y,1} & 0 & .. & 0 & 0 & 0 \\ \mathcal{A}_{y,2} & \mathcal{B}_{y,2} & \mathcal{C}_{y,2} & .. & 0 & 0 & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & .. & \mathcal{A}_{y,(I-1)} & \mathcal{B}_{y,(I-1)} & \mathcal{C}_{y,(I-1)} \\ 0 & 0 & 0 & .. & 0 & \mathcal{A}_{y,I} - \mathcal{C}_{y,I} & \mathcal{B}_{y,I} + 2\mathcal{C}_{y,I} \end{bmatrix}$$

$$\begin{aligned} \mathcal{A}_{x,i,j} &= \tfrac{1}{2}\left(\left(\tfrac{\sigma_1}{\delta x}\right)^2 - \tfrac{(py_j + \kappa_1(\theta_1 - x_i))}{\delta x}\right), & \mathcal{A}_{y,j} &= \tfrac{1}{2}\left(\left(\tfrac{\sigma_2}{\delta y}\right)^2 - \tfrac{\kappa_2(\theta_2 - y_j)}{\delta y}\right), \\ \mathcal{B}_{x,i,j} &= \exp(\delta x \cdot i) - \left(\tfrac{\sigma_1}{\delta x}\right)^2, & \mathcal{B}_{y,j} &= \exp(\delta y \cdot j) - \left(\tfrac{\sigma_2}{\delta y}\right)^2, \\ \mathcal{C}_{x,i,j} &= \tfrac{1}{2}\left(\left(\tfrac{\sigma_1}{\delta x}\right)^2 + \tfrac{(py_j + \kappa_1(\theta_1 - x_i))}{\delta x}\right), & \mathcal{C}_{y,j} &= \tfrac{1}{2}\left(\left(\tfrac{\sigma_2}{\delta y}\right)^2 + \tfrac{\kappa_2(\theta_2 - y_j)}{\delta y}\right), \end{aligned}$$

$$\mathbf{Q}^{k+1}(u) = \left[.., \mathbf{q}_j^{k+1}(u), ..\right] \quad \mathbf{Q}^{k+1}(v) = \begin{bmatrix} .. \\ \mathbf{q}^{k+1,i}(v) \\ .. \end{bmatrix} \quad \mathbf{U}^{k+1} = \left[.., \mathbf{u}_j^{k+1}, ..\right] \quad \mathbf{V}^{k+1} = \begin{bmatrix} .. \\ \mathbf{v}^{k+1,i} \\ .. \end{bmatrix}$$

$$\mathbf{q}_j^{k+1}(u) = \begin{bmatrix} 0 \\ \frac{\rho_{12}\sigma_1\sigma_2}{2} \frac{v_{3,j+1}^{k+1}(\tau_k) - v_{1,j+1}^{k+1}(\tau_k) - v_{3,j-1}^{k+1}(\tau_k) + v_{1,j-1}^{k+1}(\tau_k)}{4\delta x \delta y} \\ .. \\ \frac{\rho_{12}\sigma_1\sigma_2}{2} \frac{v_{i+1,j+1}^{k+1}(\tau_k) - v_{i-1,j+1}^{k+1}(\tau_k) - v_{i+1,j-1}^{k+1}(\tau_k) + v_{i-1,j-1}^{k+1}(\tau_k)}{4\delta x \delta y} \\ .. \\ \frac{\rho_{12}\sigma_1\sigma_2}{2} \frac{v_{I,j+1}^{k+1}(\tau_k) - v_{I-2,j+1}^{k+1}(\tau_k) - v_{I,j-1}^{k+1}(\tau_k) + v_{I-2,j-1}^{k+1}(\tau_k)}{4\delta x \delta y} \\ 0 \end{bmatrix}$$

$$
\mathbf{q}^{k+1,i}(v)=\begin{bmatrix} 0 \\ \frac{\rho_{12}\sigma_1\sigma_2}{2}\frac{u^{k+1}_{i+1,3}(\tau_{k+1})-u^{k+1}_{i-1,3}(\tau_{k+1})-u^{k+1}_{i+1,1}(\tau_{k+1})+u^{k+1}_{i-1,1}(\tau_{k+1})}{4\delta x\delta y} \\ .. \\ \frac{\rho_{12}\sigma_1\sigma_2}{2}\frac{u^{k+1}_{i+1,j+1}(\tau_{k+1})-u^{k+1}_{i-1,j+1}(\tau_{k+1})-u^{k+1}_{i+1,j-1}(\tau_{k+1})+u^{k+1}_{i-1,j-1}(\tau_{k+1})}{4\delta x\delta y} \\ .. \\ \frac{\rho_{12}\sigma_1\sigma_2}{2}\frac{u^{k+1}_{i+1,I}(\tau_{k+1})-u^{k+1}_{i-1,I}(\tau_{k+1})-u^{k+1}_{i+1,I-2}(\tau_{k+1})+u^{k+1}_{i-1,I-2}(\tau_{k+1})}{4\delta x\delta y} \\ 0 \end{bmatrix}.
$$

$\mathbf{u}^{k+1}_j$ is the j-th column of $\mathbf{U}^{k+1}$. $\mathbf{v}^{k+1,i}$ is the i-th row of $\mathbf{V}^{k+1}$. The solution to system 23 is

$$
\mathbf{u}^{k+1}_j=-\mathbf{M}_u(j)^{-1}(\mathbf{I}_I-\exp(\delta\tau\cdot\mathbf{M}_u(j)))\cdot\mathbf{q}^{k+1}_j(u)+\exp(\delta\tau\cdot\mathbf{M}_u(j))\cdot\mathbf{v}^{k+1}_j(\tau_k) \tag{24}
$$

$$
\mathbf{v}^{k+1,i}(\tau_{k+1})'=-\mathbf{M}_v^{-1}(\mathbf{I}_I-\exp(\delta\tau\cdot\mathbf{M}_v))\cdot\left(\mathbf{q}^{k+1,i}(v)\right)'+\exp(\delta\tau\cdot\mathbf{M}_v)\cdot\left(\mathbf{u}^{k+1,i}\right)'
$$

for $i,j=1,2,..,I$. I_I is the $I\times I$ identity matrix. $\mathbf{q}^{k+1}_j(u)$ is known from the previous time step. It is expedient to assume that the cross derivative be 0 on the boundaries, so that $\mathbf{q}^{k+1}_1(u)=\mathbf{q}^{k+1}_I(u)=[0,..,0]'$ and $\mathbf{q}^{k+1,1}(v)=\mathbf{q}^{k+1,I}(v)=[0,..,0]$. If $\rho_{12}\neq 0$, solution 24 may be unstable, unless $\delta\tau\to 0$.

B Extended Kalman filter (EKF)

This Appendix describes how EKF is implemented to estimate models Q3 and Q4, The other models are estimated in a similar way. We introduce the following notation and assumptions:

- $\mathbf{x}_t=(x_{1,t},x_{2,t},x_{3,t},x_{4,t})'$ are the latent factors; $\mathbf{x}_t$ is a 4×1 vector**;**

- $\widehat{\mathbf{x}}_{t^-}$ is the estimator of $\mathbf{x}_t$ conditional on information on date $t-1$**;** $\widehat{\mathbf{x}}_t$ is the estimator of $\mathbf{x}_t$ conditional on information at time t**;**

- $\mathbf{P}_{t^-}=E_{t-1}\left[(\mathbf{x}_t-\widehat{\mathbf{x}}_{t^-})(\mathbf{x}_t-\widehat{\mathbf{x}}_{t^-})'\right]$**;** $E_{t-1}[..]$ is the real measure expectation operator conditional on time $t-1$ information**;**

- $\mathbf{o}_t=(o_{1,t},o_{2,t},o_{3,t},o_{5,t},o_{7,t},o_{10,t})'$ are the CDS spreads observed in the market on date t for maturities of $1,2,3,5,7,10$ years;

- $\mathbf{z}(\mathbf{x}_t)=(z_{1,t}(\mathbf{x}_t),z_{2,t}(\mathbf{x}_t),z_{3,t}(\mathbf{x}_t),z_{5,t}(\mathbf{x}_t),z_{7,t}(\mathbf{x}_t),z_{10,t}(\mathbf{x}_t))'$ is the time t vector of CDS spreads computed using a model; each spread being a function of $\mathbf{x}_t$**;** $\mathbf{z}(\mathbf{x}_t)$ is a 6×1 vector;

- $\boldsymbol{\epsilon}_t$ is the vector of observation errors at time t, which is normally distributed such that $\epsilon_t\sim N(\mathbf{0}_{6\times 1},\mathbf{H}_6)$**;** $\mathbf{0}_{6\times 1}$ is a 6×1 column vector of zeroes; $\mathbf{H}_6$ is a 6×6 diagonal matrix;

- the observation errors $\boldsymbol{\epsilon}_t$ are uncorrelated with the $\mathbf{x}_t$ and with all lags of $\mathbf{x}_t$**;** $\mathbf{x}_0$ are the initial values of the latent factors and are parameters to be estimated.

The EKF equations are

$$\widehat{\mathbf{o}}_{t^-} = E_{t-1}\left[\mathbf{o}_t\right] = \mathbf{z}\left(\widehat{\mathbf{x}}_{t^-}\right) \tag{25}$$

$$\widehat{\mathbf{x}}_{t^-} = \boldsymbol{\phi}^*\boldsymbol{\theta}^* + \left(\mathbf{I} - \boldsymbol{\phi}^*\right)\mathbf{x}_{t-1} \tag{26}$$

$$\widehat{\mathbf{P}}_{t^-} = \boldsymbol{\phi}^*\widehat{\mathbf{P}}'_{t-1}\boldsymbol{\phi}^{*\prime} + \boldsymbol{\Sigma}\boldsymbol{\Sigma}' \tag{27}$$

$$\mathbf{F}_t = \mathbf{D}_t\mathbf{P}_{t^-}\mathbf{D}'_t + \mathbf{H} \tag{28}$$

$$\widehat{\mathbf{x}}_t = \widehat{\mathbf{x}}_{t^-} + \widehat{\mathbf{P}}_{t^-}\mathbf{D}'_t\mathbf{F}_t^{-1}\left(\mathbf{o}_t - \widehat{\mathbf{o}}_{t^-}\right) \tag{29}$$

$$\widehat{\mathbf{P}}_t = \widehat{\mathbf{P}}_{t^-} - \widehat{\mathbf{P}}_{t^-}\mathbf{D}'_t\mathbf{F}_t^{-1}\mathbf{D}_t\widehat{\mathbf{P}}_{t^-} \tag{30}$$

$$\mathbf{D}_t = \left[\frac{\partial\mathbf{z}\left(\mathbf{x}_t\right)}{\partial\mathbf{x}'_t}\right]_{\mathbf{x}_t=\widehat{\mathbf{x}}_{t^-}}. \tag{31}$$

$\mathbf{D}_t$ is a 6×4 matrix. The **quasi-likelihood function of $\mathbf{o}_t$ conditional on time $t-1$ information is $l_{t-1}\left(\mathbf{o}_t\right) \sim N\left(\widehat{\mathbf{o}}_{t^-}, \mathbf{F}_t\right)$, where** $N\left(\widehat{\mathbf{o}}_{t^-}, \mathbf{F}_t\right)$ denotes the multivariate normal density with mean $\widehat{\mathbf{o}}_{t^-}$ and covariance $\mathbf{F}_t$**, therefore**

$$\ln\left(l_{t-1}\left(\mathbf{o}_t\right)\right) = -\frac{6}{2}\ln 2\pi - \frac{1}{2}\ln\left(abs\left(\left|\mathbf{F}_t\right|\right)\right) - \frac{1}{2}\left(\mathbf{o}_t - \widehat{\mathbf{o}}_{t^-}\right)'\mathbf{F}_t^{-1}\left(\mathbf{o}_t - \widehat{\mathbf{o}}_{t^-}\right) \tag{32}$$

$$lk = \Sigma_{t=1}^{M}\ln\left(l_{t-1}\left(\mathbf{o}_t\right)\right). \tag{33}$$

$abs\left(\left|\mathbf{F}_t\right|\right)$ denotes the absolute value of the determinant of $\mathbf{F}_t$**. lk is the log of the quasi-likelihood to be maximised to estimate the model parameters.** M is the number of CDS observation dates for a given country**.** Δ is the time between consecutive observations. On average there are about 260 daily prices per year in the data, therefore $\Delta = 1/260$.

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