Estimates for Invariant Probability Measures of Degenerate SPDEs with Singular and Path-Dependent Drifts*

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Abstract

In terms of a nice reference probability measure, integrability conditions on the path-dependent drift are presented for (infinite-dimensional) degenerate PDEs to have regular positive solutions. To this end, the corresponding stochastic (partial) differential equations are proved to possess the weak existence and uniqueness of solutions, as well as the existence, uniqueness and entropy estimates of invariant probability measures. When the reference measure satisfies the log-Sobolev inequality, Sobolev estimates are derived for the density of invariant probability measures. Some results are new even for non-degenerate SDEs with path-independent drifts. The main results are applied to nonlinear functional SPDEs and degenerate functional SDEs/SPDEs.

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1 Introduction

It is well known that hypoelliptic differential operators with smooth coefficients share similar properties with the elliptic ones. For instances, the Hörmander theorem [12] ensures the smoothness of heat kernels (see Malliavin [17] for a probabilistic proof), the Index theorem has been proved by Bismut [4]. See also [5, 22, 23] and references within for Harnack inequality for hypoelliptic equations and asymptotics of heat kernels. In this paper, we

investigate estimates of positive solutions to (infinite-dimensional) hypoelliptic equations with singular and path-dependent drifts.

Consider, for instance, the following second order differential operator in $\mathbb{R}^d$:

$$\mathscr{L} := \sum_{i=1}^{m} X_i^2 + X_0,$$

where $X_0, \cdots, X_m$ are locally bounded vector fields. A function $\rho \in L_{loc}(\mathbb{R}^d)$ is called a weak solution to $\mathscr{L}^* \rho = 0$ if

$$\int_{\mathbb{R}^d} (\rho \mathscr{L} f)(x) \mathrm{d}x = 0, \quad f \in C_0^\infty(\mathbb{R}^d).$$

Similarly, a locally finite signed measure ν is called a solution to the equation $\mathscr{L}^* \nu = 0$ if

$$\int_{\mathbb{R}^d} \mathscr{L} f \mathrm{d}\nu = 0, \quad f \in C_0^\infty(\mathbb{R}^d).$$

When $X_0, \cdots, X_m$ are smooth such that $\mathrm{Lie}\{X_1, \cdots, X_m\}$ has rank d (i.e. $\{X_1, \cdots, X_m\}$ satisfies the Hörmander condition), a positive solution ρ to the equation $\mathscr{L}^* \rho = 0$ is locally Hölder continuous with respect to the intrinsic distance and satisfies the local Harnack inequality, see [5] and references within.

When the operator is non-degenerate, i.e. $\mathrm{rank}\{X_1, \cdots, X_m\} = d$, the drift X_0 is allowed to be very singular. More precisely, rewrite $\mathscr{L} = \mathrm{tr}(a\nabla^2) + b \cdot \nabla$, where $b \in L_{loc}^1(\mathbb{R}^d \to \mathbb{R}^d)$ and $a = \sigma\sigma^*$ for some $\sigma \in L_{loc}^1(\mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d)$ such that $a \geq \lambda I$ for some $\lambda \in C(\mathbb{R}^d; (0, \infty))$. If a is differentiable in the distribution sense and $|\nabla a| + |b| \in L_{loc}^p(\mathbb{R}^d)$ for some $p > d$, then any positive solution to the equation $\mathscr{L}^* \nu = 0$ for measures has a strictly positive density $\rho \in W_{loc}^{1,p}(\mathbb{R}^d)$, see [2, 3] and references within for more results in the literature. Recently, explicit integrability conditions are presented in [33] to imply the existence, uniqueness and global regularity estimates on probability solutions to the equation $\mathscr{L}^* \nu = 0$.

Here, we investigate probability solutions to $\mathscr{L}^* \nu = 0$ for more general and more singular situations. For instance, we consider differential operators on the path space $\mathscr{C} := C([-\tau, 0]; \mathbb{R}^d)$ for some $\tau > 0$ (in the following, $\mathbb{R}^d$ will be extended to an Hilbert space). For any $h, g \in L^2([-\tau, 0]; \mathbb{R}^d)$, let $\langle h, g \rangle_2 = \int_{-\tau}^0 \langle h(\theta), g(\theta) \rangle \mathrm{d}\theta$. We introduce the class $\mathscr{F}C$ of cylindrical functions of type

$$\xi \mapsto f\big(\xi(0), \langle h_1, \xi \rangle_2, \cdots, \langle h_n, \xi \rangle_2\big), \quad n \geq 1, h_i \in C^1([-\tau, 0]; \mathbb{R}^d).$$

Consider the following path-dependent operator $\mathscr{L}_{a,b}$: for the above type function $f \in \mathscr{F}C$ and $\xi \in \mathscr{C}$,

$$\mathscr{L}_{a,b} f(\xi) := \Big(\sum_{i,j=1}^{d} a_{ij}(\xi(0)) \partial_i \partial_j + \sum_{i=1}^{d} b_i(\xi) \partial_i \Big) f\big(\cdot, \langle h_1, \xi \rangle_2, \cdots, \langle h_n, \xi \rangle_2 \big)(\xi(0))$$

$$+ \sum_{k=1}^{n} \big\{ \langle h_k(0), \xi(0) \rangle - \langle h_k(-\tau), \xi(-\tau) \rangle - \langle h_k', \xi \rangle_2 \big\} \big\{ \partial_k f(\xi(0), \cdot) \big\}(\langle h_1, \xi \rangle_2, \cdots, \langle h_1, \xi \rangle_2),$$

where $a := (a_{ij})_{1 \leq i,j \leq d} \geq 0$ (maybe degenerate) is C^2 but $b := (b_1, \cdots, b_d) : \mathscr{C} \to \mathbb{R}^d$ only satisfies an integrability condition with respect to a probability measure μ. So, b might be only μ-a.e. defined without any continuity. We will construct the Markov semigroup generated by $\mathscr{L}_{a,b}$ (i.e. weak solutions to the corresponding SDE), and investigate the invariant probability measures. In general, an invariant probability measure ν of the semigroup solves the equation $\mathscr{L}_{a,b}^* \nu = 0$ in the sense that

$$\int_{\mathscr{C}} (\mathscr{L}_{a,b} f)(\xi) \nu(\mathrm{d}\xi) = 0, \quad f \in \mathscr{F}C.$$

To explain how far we will go beyond the existing study, let us briefly recall the main result in [33]. Consider the following SDE on $\mathbb{R}^d$:

$$(1.1) \qquad \mathrm{d}X(t) = \{Z_0 + \sigma Z\}(X(t))\mathrm{d}t + \sigma(X(t))\mathrm{d}W(t),$$

where $W(t)$ is the d-dimensional Brownian motion; $Z : \mathbb{R}^d \to \mathbb{R}^d$ is measurable; $\sigma \in C^2(\mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d)$ such that $\sigma(x)$ is invertible for every $x \in \mathbb{R}^d$ and the intrinsic metric

$$\rho_\sigma(x,y) := \sup \left\{ |f(x) - f(y)| : \ f \in C^\infty(\mathbb{R}^d), |\sigma^* \nabla f| \leq 1 \right\}, \quad x, y \in \mathbb{R}^d$$

is complete; and

$$(1.2) \qquad Z_0 = \frac{1}{2} \sum_{i,j=1}^d \{\partial_j (\sigma \sigma^*)_{ij} - (\sigma \sigma^*)_{ij} \partial_j V\} e_i$$

for some $V \in C^2(\mathbb{R}^d)$ and the standard orthonormal basis $\{e_i\}_{i=1}^d$ of $\mathbb{R}^d$. Let $\mu_0(\mathrm{d}x) = \mathrm{e}^{-V(x)} \mathrm{d}x$, and let $H_\sigma^{1,2}(\mu_0)$ be the closure of $C_0^\infty(\mathbb{R}^d)$ under the norm

$$\|f\|_{H_\sigma^{1,2}(\mu_0)} := \left\{ \mu_0(|f|^2 + |\sigma^* \nabla f|^2) \right\}^{\frac{1}{2}}.$$

The following result is taken from [33, Theorem 2.1 and Theorem 2.3] where the constant κ is replaced by 2κ as the noise therein is $\sqrt{2}W(t)$ rather than $W(t)$.

Theorem 1.1 ([33]). *Let $\sigma \in C^2(\mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d)$ such that $\sigma(x)$ is invertible for every $x \in \mathbb{R}^d$, let Z_0 be in (1.2) for some $V \in C^2(\mathbb{R}^d)$, and let $\mu_0(\mathrm{d}x) = \mathrm{e}^{-V(x)} \mathrm{d}x$.*

(1) *If for some constant $\varepsilon \in (0,1)$*

$$\int_{\mathbb{R}^d} \mathrm{e}^{\varepsilon |Z(x)|^2 - \varepsilon^{-1} \rho_\sigma(0,x)^2} \mu_0(\mathrm{d}x) < \infty,$$

then for any initial points the SDE (1.1) has a unique non-explosive solution, and the associated Markov semigroup P_t^Z is strong Feller with at most one invariant probability measure.

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(2) *Let μ_0 be a probability measure satisfying the (defective) log-Sobolev inequality*

$$(1.3) \qquad \mu_0(f^2 \log f^2) \le \kappa \mu_0(|\sigma^* \nabla f|^2) + \beta, \quad f \in C_0^\infty(\mathbb{R}^d), \mu_0(f^2) = 1$$

for some constants $\kappa > 0, \beta \ge 0$. If

$$(1.4) \qquad \mu_0\big(e^{\varepsilon \|\sigma\|^2} + e^{\lambda |Z|^2}\big) := \int_{\mathbb{R}^d} \big\{ e^{\varepsilon \|\sigma\|^2} + e^{\lambda |Z|^2} \big\} d\mu_0 < \infty$$

holds for some constants $\varepsilon > 0$ and $\lambda > \kappa$, then P_t^Z has a unique invariant probability measure ν, which is absolutely continuous with respect to μ_0 and $\rho := \frac{d\nu}{d\mu_0}$ has a continuous, strictly positive version such that $\sqrt{\rho}, \log \rho \in H_\sigma^{1,2}(\mu_0)$ with

$$(1.5) \qquad \begin{aligned} \mu_0\big(|\sigma^* \nabla \sqrt{\rho}|^2\big) &\le \frac{1}{\lambda - \kappa} \big\{ \log \mu_0(e^{\lambda |Z|^2}) + \beta \big\} < \infty, \\ \mu_0(|\sigma^* \nabla \log \rho|^2) &\le 4\mu_0(|Z|^2) < \infty. \end{aligned}$$

Since an invariant probability measure ν of P_t^Z solves the equation $\mathscr{L}_Z^* \nu = 0$, due to the integration by parts formula, Theorem 1.1(2) provides regularity estimates on positive solutions to the singular elliptic PDE

$$\Big(\frac{1}{2} \mathrm{Tr}(\sigma\sigma^* \nabla^2) + (Z_0 - \sigma Z) \cdot \nabla \Big) \rho = 0.$$

We will improve and extend the above assertions in the following four aspects:

- The noise may be degenerate: $|\sigma^* \nabla f| = 0$ does not imply $\nabla f = 0$, so that the log-Sobolev inequality (1.3) is invalid. Moreover, σ is not necessarily C^2-smooth.

- The drift Z may be path-dependent, for which the corresponding SDE is called functional SDE.

- The state space may be infinite-dimensional such that the study applies to nonlinear or semilinear functional SPDEs.

- Derive stronger estimates on the density of the invariant probability measure.

We now introduce the framework of the present study in details.

Reference SDE. Let $\mathbb{H}$ and $\tilde{\mathbb{H}}$ be two separable Hilbert spaces, and let $\mathscr{L}(\tilde{\mathbb{H}}; \mathbb{H})$ be the class of bounded linear operators from $\tilde{\mathbb{H}}$ to $\mathbb{H}$. The cylindrical Brownian motion on $\tilde{\mathbb{H}}$ is formally defined by

$$W(t) = \sum_{i \ge 1} B_i(t) \tilde{e}_i,$$

where $\{\tilde{e}_i\}_{i \ge 1}$ is an orthonormal basis of $\tilde{\mathbb{H}}$, and $\{B_i(t)\}_{i \ge 1}$ are independent one-dimensional Brownian motions on a complete filtration probability space $(\Omega, \{\mathscr{F}_t\}_{t \ge 0}, \mathbb{P})$. Let $\mathbb{V}$ be a

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Banach space densely embedded into $\mathbb{H}$, and let $\mathbb{V}^*$ be its dual space with respect to $\mathbb{H}$. We call $\mathbb{V} \subset \mathbb{H} \subset \mathbb{V}^*$ the Gelfand triple. Consider the following reference SDE on $\mathbb{H}$:

$$\text{(1.6)} \qquad \mathrm{d}X(t) = Z_0(X(t))\mathrm{d}t + \sigma(X(t))\mathrm{d}W(t),$$

where $Z_0 : \mathbb{H} \to \mathbb{V}^*$ and $\sigma : \mathbb{H} \to \mathscr{L}(\tilde{\mathbb{H}}; \mathbb{H})$ are measurable. A continuous adapted process $X(t)$ on $\mathbb{H}$ is called a (variational) solution to (1.6) with initial value $X(0)$, if

$$\mathbb{E}\int_0^t \left\{ |_{\mathbb{V}^*}\langle Z_0(X(s)), v\rangle_{\mathbb{V}}| + |\{\sigma(X(s))\}^*v|^2 \right\}\mathrm{d}s < \infty$$

for all $t \in (0, \infty)$ and $v \in \mathbb{V}$, and $\mathbb{P}$-a.s.

$$\langle X(t), v\rangle_{\mathbb{H}} = \langle X(0), v\rangle_{\mathbb{H}} + \int_0^t {}_{\mathbb{V}^*}\langle Z_0(X(s)), v\rangle_{\mathbb{V}} \,\mathrm{d}s + \int_0^t \langle \{\sigma(X(s))\}^*v, \mathrm{d}W(s)\rangle_{\tilde{\mathbb{H}}}, \quad v \in \mathbb{V}, t \geq 0.$$

See [16] and reference within for the existence and uniqueness of variational solutions under framework of monotone SPDEs due to [21, 13]. When the initial value $X(0) = x$, we denote the solution by $X^x(t)$. When the initial value $X(0)$ has distribution ν_0 on $\mathbb{H}$, we also denote the solution by $X^{\nu_0}(t)$ to emphasize the initial distribution.

When $\mathbb{H} = \mathbb{V} = \mathbb{R}^d$, the variational solution reduces to the usual strong (i.e. pathwise) solution of SDEs. When $\mathbb{V} = \mathbb{H}$ and $Z_0(x) = Ax + \tilde{Z}_0(x)$, where $\tilde{Z}_0 \in C(\mathbb{H}; \mathbb{H})$ and A is a self-adjoint operator on $\mathbb{H}$ generating a C_0-contraction semigroup T_t such that

$$\mathbb{E}\int_0^t \left(\sum_{i \geq 1} \|T_{t-s}\sigma(X(s))\tilde{e}_i\|^2\mathrm{d}s + \|T_{t-s}\tilde{Z}_0(X(s))\| \right)\mathrm{d}s < \infty, \quad t \geq 0,$$

it coincides with the mild solution in the sense of [8, 9].

Throughout the paper, we assume:

(A) *Let $Z_0 : \mathbb{H} \to \mathbb{V}^*$ and $\sigma : \mathbb{H} \to \mathscr{L}(\tilde{\mathbb{H}}; \mathbb{H})$ be measurable such that for any $\mathscr{F}_0$-measurable random variable $X(0)$ on $\mathbb{H}$, the SDE (1.6) has a unique variational solution, and the associated Markov P_t^0 given by*

$$P_t^0 f(x) := \mathbb{E}\left[f(X^x(t))\right], \quad t \geq 0, x \in \mathbb{H}, f \in \mathscr{B}_b(\mathbb{H})$$

has a unique invariant probability measure μ_0.

Under assumption **(A)**, for any probability measure ν_0 on $\mathbb{H}$, we have

$$\mathbb{E}[f(X^{\nu_0}(t))] = \int_{\mathbb{H}} \mathbb{E}[f(X^x(t))]\nu_0(\mathrm{d}x) = \nu_0(P_t^0 f).$$

In particular, $\mathbb{E}[f(X^{\mu_0}(t))] = \mu_0(f)$ for $t \geq 0$ and $f \in \mathscr{B}_b(\mathbb{H})$.

Let $\tau > 0$. For any $\xi \in C([-\tau, \infty); \mathbb{H})$ and $t \geq 0$, define $\xi_t \in \mathscr{C} := C([-\tau, 0]; \mathbb{H})$ by

$$\xi_t(\theta) := \xi(t + \theta), \quad \theta \in [-\tau, 0].$$

We call $(\xi_t)_{t\geq 0}$ the segment of $(\xi(t))_{t\geq -\tau}$. For an $\mathscr{F}_0$-measurable random variable ξ on $\mathscr{C}$, let

$$X^\xi(t) = \xi(t)1_{[-\tau,0]}(t) + X^{\xi(0)}(t)1_{(0,\infty)}(t), \quad t \geq 0,$$

recall that $X^{\xi(0)}(t)$ is the solution to (1.6) with initial value $\xi(0)$. Let $(X_t^\xi)_{t\geq 0}$ be the segment process of $(X^\xi(t))_{t\geq -\tau}$, i.e.

$$(1.7) \qquad X_t^\xi(\theta) = X^\xi(t+\theta), \quad \theta \in [-\tau,0].$$

When ξ has distribution ν, we also denote X_t^ξ by X_t^ν to emphasize the initial distribution. Then

$$(1.8) \qquad S_t^0 f(\xi) := \mathbb{E}[f(X_t^\xi)], \quad \xi \in \mathscr{C}, f \in \mathscr{B}_b(\mathscr{C}), t \geq 0$$

gives rise to a Markov semigroup S_t^0 on $\mathscr{B}_b(\mathscr{C})$.

Reference probability measure μ. Let μ be the distribution of the $\mathscr{C}$-valued random variable $X_\tau^{\mu_0}$ defined by $X_\tau^{\mu_0}(\theta) := X^{\mu_0}(\tau+\theta), \theta \in [-\tau,0]$. Recall that $X^{\mu_0}(t)$ is the solution to (1.6) with initial distribution μ_0. It is easy to see that μ is the unique invariant probability measure of the Markov semigroup S_t^0.

Since $X_t^\xi(0) := X^\xi(t) = X^{\xi(0)}(t)$ for $t \geq 0$,

$$(S_{t+\tau}f)(x) := (S_{t+\tau}^0 f)(\xi), \quad \xi \in \mathscr{C}, \xi(0) = x, f \in \mathscr{B}_b(\mathscr{C})$$

provides a family of contractive linear operators $(S_{t+\tau})_{t\geq 0}$ from $\mathscr{B}_b(\mathscr{C})$ to $\mathscr{B}_b(\mathbb{H})$. By the Markov property, this implies

$$(1.9) \qquad \begin{aligned} (S_{t+\tau}f)(\xi(0)) &= (S_{t+\tau}^0 f)(\xi) = \mathbb{E}[(S_\tau^0 f)(X_t^\xi)] = \mathbb{E}[(S_\tau f)(X_t^\xi(0))] \\ &= \mathbb{E}[(S_\tau f)(X^{\xi(0)}(t))] = P_t^0(S_\tau f)(\xi(0)), \quad \xi \in \mathscr{C}, t \geq 0, f \in \mathscr{B}_b(\mathscr{C}). \end{aligned}$$

Singular and path-dependent SDE. Consider the following SDE on $\mathbb{H}$:

$$(1.10) \qquad \mathrm{d}X(t) = \big\{Z_0(X(t)) + \sigma(X(t))Z(X_t)\big\}\,\mathrm{d}t + \sigma(X(t))\mathrm{d}W(t),$$

where $Z : \mathscr{C} \to \tilde{\mathbb{H}}$ and $(X_t)_{t\geq 0}$ is the segment process of $(X(t))_{t\geq -\tau}$. Even in the path-independent case, when σ is degenerate or $\mathbb{H}$ is infinite-dimensional, to ensure the strong existence and uniqueness one needs certain continuity conditions on the drift, see [6, 7, 30, 35, 38] and references within for details. So, to investigate (1.10) by using integrability conditions of Z with respect to the reference measure μ, we only look at the weak solution.

Definition 1.1. Let $\xi \in \mathscr{C}$. A $\mathscr{C}$-valued continuous process $(\tilde{X}_t^\xi)_{t\geq 0}$ under a complete filtration probability space $(\tilde{\Omega}, \{\tilde{\mathscr{F}}_t\}_{t\geq 0}, \tilde{\mathbb{P}})$ is called a weak solution of (1.10) starting at ξ, if it is $\tilde{\mathscr{F}}_t$-adapted with $\tilde{X}_0^\xi = \xi$ and for some $\tilde{\mathbb{H}}$-cylindrical Brownian motion $\tilde{W}(t)$ on the same probability space $\tilde{\mathbb{P}}$-a.s.

$$\langle \tilde{X}^\xi(t), v\rangle_{\mathbb{H}} = \langle \xi(0), v\rangle_{\mathbb{H}} + \int_0^t {}_{\mathbb{V}^*}\big\langle Z_0(\tilde{X}^\xi(s)) + \sigma(\tilde{X}^\xi(s))Z(\tilde{X}_s^\xi), v\big\rangle_{\mathbb{V}}\mathrm{d}s$$

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$$+ \int_0^t \left\langle \{\sigma(\tilde{X}^\xi(s))\}^* v, \mathrm{d}\tilde{W}(s)\right\rangle_{\tilde{\mathbb{H}}}, \quad t \geq 0, v \in \mathbb{V}.$$

The equation is said to have weak uniqueness if any two weak solutions with same initial point are equal in law.

When (1.10) has weak existence and uniqueness, let $\mathbb{P}^\xi$ denote the distribution of the weak solution starting at ξ, and define

$$(S_t^Z f)(\xi) := \int_{C([0,\infty);\mathscr{C})} f(\eta(t)) \, \mathbb{P}^\xi(\mathrm{d}\eta), \quad f \in \mathscr{B}_b(\mathscr{C}), \xi \in \mathscr{C}, t \geq 0.$$

A probability measure ν on $\mathscr{C}$ is called an invariant probability measure of S_t^Z, if $\nu(S_t^Z f) = \nu(f)$ holds for all $t > 0$ and $f \in \mathscr{B}_b(\mathscr{C})$.

The remainder of the paper is organized as follows. Under condition $\mu(\mathrm{e}^{\varepsilon|Z|^2}) < \infty$ for some $\varepsilon > 0$, we prove the weak existence and uniqueness of solutions (Section 2) as well as the uniqueness of invariant probability measure (Section 3). Moreover, the existence of invariant probability measures and entropy estimate of the density are proved in Section 4 using the hyperboundedness of P_t^0. Finally, the existence of invariant probability measures and Sobolev estimates on the density are addressed in Section 5 by using the log-Sobolev inequality (1.3), for which the Hörmander condition is adopted. The main results are applied to concrete models of degenerate functional SDEs/SPDEs. We emphasize that some estimates in Section 5 are new even for non-degenerate SDEs with path-independent drifts, see Theorem 5.1 and Theorem 5.2 below for details.

2 Weak solutions

Let $\mu(\mathrm{e}^{\varepsilon|Z|^2}) < \infty$ for some constant $\varepsilon > 0$. We will prove, for μ-a.e. $\xi \in \mathscr{C}$, that the equation (1.10) has a unique weak solution with distribution $\mathbb{P}^\xi$ satisfying

$$(2.1) \qquad \mathbb{P}^\xi\left(\left\{\gamma \in C([0,\infty);\mathscr{C}) : \int_0^T |Z(\gamma_s)|^2 \mathrm{d}s < \infty, T > 0\right\}\right) = 1.$$

When $\tau = 0$ and P_t^0 satisfies the Harnack inequality (2.5) below, the assertion holds for all initial point $x \in \mathbb{H}$ (in this case $\mathscr{C} = \mathbb{H}$).

To formulate the associated Markov semigroup, we introduce the process

$$(2.2) \qquad R^\xi(t) := \exp\left[\int_0^t \langle Z(X_s^\xi), \mathrm{d}W(s)\rangle - \frac{1}{2}\int_0^t |Z|^2(X_s^\xi)\mathrm{d}s\right], \quad t \geq 0,$$

where X_t^ξ is the segment solution to (1.6) with inital value $\xi \in \mathscr{C}$. By $\mu(\mathrm{e}^{\varepsilon|Z|^2}) < \infty$, this process is well defined for μ-a.e. ξ. We will use S_t^Z to denote the semigroup of segment solutions to (1.10). But when $\tau = 0$, we use P_t^Z to replace S_t^Z for the notation consistency with Theorem 1.1.

Theorem 2.1. *Assume* **(A)** *and* $\mu(e^{\varepsilon|Z|^2}) < \infty$ *for some constant* $\varepsilon > 0$.

(1) *For* μ-*a.s.* $\xi \in \mathscr{C}$, $R^{\xi}(t)$ *is a martingale and the equation* (1.10) *has a weak solution starting at* ξ *satisfying* (2.1). *Moreover, the associated Markov semigroup* S_t^Z *is given by*

$$(2.3) \qquad (S_t^Z f)(\xi) = \mathbb{E}\big[f(X_t^{\xi})R^{\xi}(t)\big], \quad t \geq 0, \quad f \in L^{\infty}(\mu).$$

(2) *For any* $\xi \in \mathscr{C}$, (1.10) *has at most one weak solution with distribution* $\mathbb{P}^{\xi}$ *satisfying* (2.1).

(3) *Let* $\tau = 0$. *If there exist* $p > 1$ *and* $\Phi_p \in C((0,\infty) \times \mathbb{H}^2)$ *such that*

$$(2.4) \qquad \int_0^t \frac{\mathrm{d}s}{\{\mu_0(\exp[-\Phi_p(s,x,\cdot)])\}^{\frac{1}{p}}} < \infty, \quad t > 0, x, y \in \mathbb{H},$$

and P_t^0 *satisfies the Harnack inequality*

$$(2.5) \qquad (P_t^0 f)^p(x) \leq e^{\Phi_p(t,x,y)} P_t^0 f^p(y), \quad t > 0, f \in \mathscr{B}^+(\mathbb{H}), x, y \in \mathbb{H},$$

then for any $x \in \mathbb{H}$, $R^x(t)$ *is a martingale, the equation* (1.10) *has a unique weak solution satisfying* (2.1) *starting at* x, *and the Markov semigroup is given by*

$$(2.6) \qquad (P_t^Z f)(x) = \mathbb{E}\big[f(X^x(t))R^x(t)\big], \quad t \geq 0, \ x \in \mathbb{H}, \ f \in \mathscr{B}_b(\mathbb{H}).$$

Remark 2.1. Although the reference measure μ is less explicit, the condition $\mu(e^{\varepsilon|Z|^2}) < \infty$ can be verified by using the marginal distribution μ_0, which is explicitly given in applications, for instance, in (1.1) $\mu_0(\mathrm{d}x) = e^{-V(x)}\mathrm{d}x$. Let, for instance,

$$Z(\xi) = \int_{-\tau}^0 h(\xi(\theta))\mathrm{d}\theta, \quad \xi \in \mathscr{C},$$

where h is a measurable function on $\mathbb{H}$. Then

$$\mu(e^{\varepsilon|Z|^2}) = \mathbb{E}e^{\varepsilon|Z|^2(X_{\tau}^{\mu})} = \mathbb{E}e^{\varepsilon\left|\int_{-\tau}^0 h(X^{\mu_0}(\tau+\theta))\mathrm{d}\theta\right|^2}$$

$$\leq \mathbb{E}e^{\varepsilon\tau\int_0^{\tau} h(X^{\mu_0}(s))^2\mathrm{d}s} \leq \frac{1}{\tau}\int_0^{\tau} \mathbb{E}e^{\varepsilon\tau^2 h(X^{\mu_0}(s))}\mathrm{d}s = \mu_0(e^{\varepsilon\tau^2 h^2}).$$

Therefore, $\mu(e^{\varepsilon|Z|^2}) < \infty$ follows from $\mu_0(e^{\varepsilon\tau^2 h^2}) < \infty$.

To prove Theorem 2.1, we first present the following lemma.

Lemma 2.2. *Assume* **(A)** *and* $\mu(e^{\varepsilon|Z|^2}) < \infty$ *for some constant* $\varepsilon > 0$.

(1) *The process*

$$R^{\mu}(t) := \exp\left[\int_0^t \langle Z(X_s^{\mu}), \mathrm{d}W(s)\rangle - \frac{1}{2}\int_0^t |Z|^2(X_s^{\mu})\mathrm{d}s\right], \quad t \geq 0$$

is a martingale. Consequently, for μ-*a.e.* $\xi \in \mathscr{C}$, $R^{\xi}(t)$ *is a martingale.*

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(2) *Let $\tau = 0$. If (2.5) holds for some $p > 1$ and Φ_p satisfying (2.4), then $R^x(t)$ is a martingale for any $x \in \mathbb{H}$.*

Proof. (1) By the stationarity of X_t^μ and $\mu(\mathrm{e}^{\varepsilon|Z|^2}) < \infty$, we have

$$\mathbb{E}\int_0^T |Z(X_s^\mu)|^2 \mathrm{d}s = T\mu(|Z|^2) < \infty, \quad T > 0.$$

So, $\mathbb{P}$-a.s.

$$\tau_n := \inf\left\{t \geq 0 : \int_0^t |Z(X_s^\mu)|^2 \mathrm{d}s \geq n\right\} \uparrow \infty \quad \text{as } n \uparrow \infty.$$

By Girsanov's theorem (see e.g. [18]), $(R^\mu(t \wedge \tau_n))_{t \geq 0}$ is a martingale for every $n \geq 1$, so that Fatou's lemma gives

$$\mathbb{E}(R^\mu(t)|\mathscr{F}_s) = \mathbb{E}\left(\liminf_{n\to\infty} R^\mu(t \wedge \tau_n)\Big|\mathscr{F}_s\right) \leq \liminf_{n\to\infty}\mathbb{E}\left(R^\mu(t \wedge \tau_n)\Big|\mathscr{F}_s\right)$$
$$= \liminf_{n\to\infty} R^\mu(s \wedge \tau_n) = R^\mu(s), \quad t \geq s \geq 0.$$

Thus, $(R^\mu(t))_{t \geq 0}$ is a supmartingale. Since $\mathbb{E}\int_0^T |Z(X_s^\mu)|^2 \mathrm{d}s < \infty$ implies $\mathbb{E}\int_0^T |Z(X_s^\xi)|^2 \mathrm{d}s < \infty$ for μ-a.e. ξ, the above argument also implies that $(R^\xi(t))_{t \geq 0}$ is a supmartingale for μ-a.e. ξ. Noting that $\mathbb{E}R^\mu(t) = \int_{\mathscr{C}}\mathbb{E}R^\xi(t)\mathrm{d}\mu$, we conclude that $(R^\mu(t))_{t \geq 0}$ and $(R^\xi(t))_{t \geq 0}$ for μ-a.e. ξ are martingales provided $\mathbb{E}R^\mu(t) = 1$ for all $t \geq 0$.

By the stationarity of X_t^μ and Jensen's inequality, we have

$$\mathbb{E}\mathrm{e}^{\frac{1}{2}\int_0^{2\varepsilon}|Z|^2(X_s^\mu)\mathrm{d}s} \leq \frac{1}{2\varepsilon}\int_0^{2\varepsilon}\mathbb{E}\mathrm{e}^{\varepsilon|Z|^2(X_s^\mu)}\mathrm{d}s = \mu(\mathrm{e}^{\varepsilon|Z|^2}) < \infty.$$

Then Girsanov's theorem ensures that $(R^\mu(t))_{t \in [0,2\varepsilon]}$ is a martingale. In particular,

(2.7) $$\mathbb{E}R^\mu(t) = \mathbb{E}R^\xi(t) = 1, \quad t \in [0, 2\varepsilon], \mu\text{-a.e. } \xi.$$

Assuming that $\mathbb{E}R^\mu(t) = 1$ for $t \in [0, 2k\varepsilon]$ and some $k \geq 1$, it remains to prove $\mathbb{E}R^\mu(t) = 1$ for $t \in [2k\varepsilon, 2(k+1)\varepsilon]$. Let $t_1 = 2\kappa\varepsilon$ and

$$\tilde{W}(t) = W(t + t_1) - W(t_1), \quad t \geq 0.$$

Then $\tilde{W}(t)$ is a cylindrical Brownian motion on the same probability space with respect to filtration $\mathscr{F}_{t_1+t}$. By **(A)**, (1.6) with $\tilde{W}(t)$ replacing $W(t)$ has existence and uniqueness as well. Let $\tilde{X}_t^\xi$ be the segment process of the solution with $\tilde{X}_0^\xi = \xi$ defined as in (1.7). By the Markov property,

$$\Xi(t) := \mathbb{E}\left(\mathrm{e}^{\int_{t_1}^t \langle Z(X_s^\mu), \mathrm{d}W(s)\rangle - \frac{1}{2}\int_{t_1}^t |Z|^2(X_s^\mu)\mathrm{d}s}\Big|\mathscr{F}_{t_1}\right)$$
$$= \left\{\mathbb{E}\left(\mathrm{e}^{\int_0^{t-t_1}\langle Z(\tilde{X}_s^\xi), \mathrm{d}\tilde{W}(s)\rangle - \frac{1}{2}\int_0^{t-t_1}|Z|^2(\tilde{X}_s^\xi)\mathrm{d}s}\right)\right\}\Big|_{\xi = X_{t_1}^\mu}$$
$$= \left\{\mathbb{E}R^\xi(t - t_1)\right\}\Big|_{\xi = X_{t_1}^\mu}, \quad t \geq t_1.$$

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Since the law of $X_{t_1}^\mu$ is μ, this and (2.7) imply $\Xi(t) = 1$ a.s. for all $t \in [t_1, t_1 + 2\varepsilon]$, so that

$$\mathbb{E}(R^\mu(t)|\mathscr{F}_{t_1}) = R^\mu(t_1)\Xi(t) = R^\mu(t_1).$$

So, by the assumption $\mathbb{E}R^\mu(t_1) = 1$, we obtain $\mathbb{E}R^\mu(t) = 1$ for $t \in [2k\varepsilon, 2(k+1)\varepsilon]$.

(2) By Girsanov's theorem and the Markov property, it suffices to find out a constant $t > 0$ such that

(2.8) $$\mathbb{E}e^{\frac{1}{2}\int_0^t |Z(X^x(s))|^2 ds} < \infty, \quad x \in \mathbb{H}.$$

By (2.5), we have

$$\mu_0\big(e^{-\Phi_p(s,x,\cdot)}\big)\big(P_s^0 e^{\frac{\varepsilon}{p}|Z|^2}\big)^p(x) \le \mu_0\big(e^{\varepsilon|Z|^2}\big) < \infty.$$

Combining this with (2.4) and Jensen's inequality, for any $\lambda > 0$ and $t_\lambda := \frac{\varepsilon}{p\lambda}$ we obtain

(2.9)
$$\begin{aligned} \mathbb{E}\big[e^{\lambda \int_0^{t_\lambda} |Z(X^x(s))|^2 ds}\big] &\le \frac{1}{t_\lambda}\int_0^{t_\lambda} P_s^0 e^{\frac{\varepsilon}{p}|Z|^2}(x)ds \\ &\le \frac{1}{t_\lambda}\int_0^{t_\lambda}\big\{\mu_0(e^{\varepsilon|Z|^2})\big\}^{\frac{1}{p}}\big\{\mu_0(e^{-\Phi_p(s,x,\cdot)})\big\}^{-\frac{1}{p}}ds < \infty, \quad x \in \mathbb{H}. \end{aligned}$$

In particular, (2.8) holds for some constant $t > 0$. $\qquad\square$

Proof of Theorem 2.1. (1) By Lemma 2.2 and Girsanov's theorem, for μ-a.s. ξ,

$$W^\xi(t) := W(t) - \int_0^t Z(X_s^\xi)ds, \quad t \in [0, T]$$

is a cylindrical Brownian motion on $\tilde{\mathbb{H}}$ under the probability measure $\mathbb{Q}^\xi$ defined on $\mathscr{F}_\infty$ by

$$\mathbb{Q}^\xi(A) = \mathbb{E}[1_A R^\xi(T)], \quad T > 0, A \in \mathscr{F}_T,$$

and $(X^\xi(t))_{t \ge 0}$ is a weak solution of (1.10) with respect to the cylindrical Brownian motion $W^\xi(t)$. Note that $\mathbb{Q}^\xi$ is well defined according to the martingale property of $R^\xi(t)$ and the Kolmogorov consistency theorem. Therefore, the associated Markov semigroup of the weak solution is given by (2.3).

(2) For $\xi \in \mathscr{C}$ and each $i = 1, 2$, let $(X^{(i)}(t))_{t \ge 0}$ be a weak solution to (1.10) starting at ξ with respect to the cylindrical Brownian motion $W^{(i)}(t)$ under a complete filtration probability space $(\Omega^{(i)}, \{\mathscr{F}_t^{(i)}\}_{t \ge 0}, \mathbb{P}^{(i)})$, such that the distribution $\mathbb{P}_i^\xi$ satisfies (2.1). We intend to prove $\mathbb{P}_1^\xi = \mathbb{P}_2^\xi$. By (2.1), we have

$$\tau_n^{(i)} := \inf\Big\{t \ge 0: \int_0^t |Z(X_s^{(i)})|^2 ds \ge n\Big\} \uparrow \infty \text{ as } n \uparrow \infty, \quad i = 1, 2.$$

For every $i = 1, 2$ and $n \ge 1$,

$$R_n^{(i)}(t) := \exp\Big[-\int_0^{\tau_n^{(i)}\wedge t}\langle Z(X_s^{(i)}), dW^{(i)}(s)\rangle - \frac{1}{2}\int_0^{\tau_n^{(i)}\wedge t}|Z|^2(X_s^{(i)})ds\Big], \quad t \ge 0$$

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is a $\mathbb{P}^{(i)}$-martingale. Define the probability measure $\mathbb{Q}_n^{(i)}$ on $\mathscr{F}_\infty^{(i)}$ by letting

$$\mathbb{Q}_n^{(i)}(A) = \mathbb{E}_{\mathbb{P}^{(i)}}[1_A R_n^{(i)}(T)], \quad T > 0, A \in \mathscr{F}_T^{(i)}.$$

By Girsanov's theorem,

$$\hat{W}^{(i)}(t) := W^{(i)}(t) + \int_0^{t \wedge \tau_n^{(i)}} Z(X_s^\xi) \mathrm{d}s, \quad t \geq 0$$

is a $\mathbb{Q}_n^{(i)}$-cylindrical Brwonian motion on $\tilde{\mathbb{H}}$. Therefore, up to time $\tau_n^{(i)}$, $X^{(i)}(t)$ solves the SDE (1.6) with the $\mathbb{Q}_n^{(i)}$-cylindrical Brownian motion $\hat{W}^{(i)}(t)$ replacing $W(t)$. By the pathwise (also weak) uniqueness of (1.6) according to **(A)**, $(X^{(i)}(t), \hat{W}^{(i)}(t))_{t \in [0, T \wedge \tau_n^{(i)}]}$ under $\mathbb{Q}_n^{(i)}$ coincides in law with $(X^\xi(t), W(t))_{t \in [0, T \wedge \tau_n^\xi]}$ under $\mathbb{P}$, where $T > 0$ and

$$\tau_n^\xi := \inf\left\{t \geq 0 : \int_0^t |Z(X_s^\xi)|^2 \mathrm{d}s \geq n\right\}.$$

Therefore, for any $F \in \mathscr{B}_b(C([0,T];\mathbb{H}) \times C([0,T];\mathbb{H}))$,

$$\mathbb{E}_{\mathbb{P}^{(i)}}\left[1_{\{\tau_n^{(i)} \geq T\}} F(X^{(i)}([0,T]), W^{(i)}([0,T]))\right]$$

$$= \mathbb{E}_{\mathbb{Q}_n^{(i)}}\left[1_{\{\sup_{t \in [0,T]} |X^{(i)}(t)| \leq n\}} \mathrm{e}^{\int_0^T \langle Z(X_s^{(i)}), \mathrm{d}\hat{W}^{(i)}(s)\rangle - \frac{1}{2}\int_0^T |Z|^2(X_s^{(i)})\mathrm{d}s}\right.$$

$$\left. \times F\left(X^{(i)}([0,T]), \left(\hat{W}^{(i)} - \int_0^\cdot Z(X_s^{(i)})\mathrm{d}s\right)([0,T])\right)\right]$$

$$= \mathbb{E}\left[1_{\{\sup_{t \in [0,T]} |X^\xi(t)| \leq n\}} \mathrm{e}^{\int_0^T \langle Z(X_s^\xi), \mathrm{d}W(s)\rangle - \frac{1}{2}\int_0^T |Z|^2(X_s^\xi)\mathrm{d}s}\right.$$

$$\left. \times F\left(X^\xi([0,T]), \left(W - \int_0^\cdot Z(X_s^\xi)\mathrm{d}s\right)([0,T])\right)\right], \quad i = 1, 2.$$

Consequently,

$$\mathbb{E}_{\mathbb{P}^{(1)}}\left[1_{\{\tau_n^{(1)} \geq T\}} F(X^{(1)}([0,T]), W^{(1)}([0,T]))\right] = \mathbb{E}_{\mathbb{P}^{(2)}}\left[1_{\{\tau_n^{(2)} \geq T\}} F(X^{(2)}([0,T]), W^{(2)}([0,T]))\right]$$

holds for any $n \geq 1$. Letting $n \to \infty$ we obtain

$$\mathbb{E}_{\mathbb{P}^{(1)}}\left[F(X^{(1)}([0,T]), W^{(1)}([0,T]))\right] = \mathbb{E}_{\mathbb{P}^{(2)}}\left[F(X^{(2)}([0,T]), W^{(2)}([0,T]))\right]$$

for any $T > 0$ and $F \in \mathscr{B}_b(C([0,T];\mathbb{H}) \times C([0,T];\mathbb{H}))$. Therefore, $\mathbb{P}_1^\xi = \mathbb{P}_2^\xi$.

(3) Let $\tau = 0$. By Lemma 2.2(2) and the Girsanov theorem, $(X^x(t))_{t \geq 0}$ is a weak solution to (1.10) satisfying (2.1) for the $\tilde{\mathbb{H}}$-cylindrical Brownian motion

$$W^x(t) := W(t) - \int_0^t Z(X^x(s))\mathrm{d}s, \quad t \geq 0$$

under the probability measure $\mathbb{Q}^x$, which is defined on $\mathscr{F}_\infty$ by

$$\mathbb{Q}^x(A) := \mathbb{E}[1_A R^x(T)], \quad T > 0, A \in \mathscr{F}_T.$$

Then the proof is finished by combining this with (2). $\qquad \square$

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3 Uniqueness of invariant probability measure

Since by Theorem 2.1 S_t^Z is a Markov semigroup on $L^\infty(\mu)$, it is meaningful to consider the class of invariant probability measures absolutely continuous with respect to μ:

$$\mathscr{P}_Z := \left\{ \rho\mu : \ \rho \geq 0, \mu(\rho) = 1, \mu(\rho S_t^Z f) = \mu(\rho f) \text{ for } t \geq 0, f \in \mathscr{B}_b(\mathscr{C}) \right\}.$$

Recall that when $\tau = 0$ we use P_t^Z to replace S_t^Z.

Theorem 3.1. *Assume* **(A)** *and* $\mu(\mathrm{e}^{\varepsilon|Z|^2}) < \infty$ *for some* $\varepsilon > 0$.

(1) *If there exists* $t > 0$ *such that* P_t^0 *has a strictly positive density* $p_t^0(x, y)$ *with respect to* μ_0, *then* $\nu \in \mathscr{P}_Z$ *implies that* $\rho := \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$ *has a strictly positive version, and* $\mathscr{P}_Z$ *contains at most one element.*

(2) *In the situation of Theorem 2.1(3), the Markov semigroup* P_t^Z *defined on* $\mathscr{B}_b(\mathbb{H})$ *has at most one invariant probability measure.*

Proof. (1) Let $\nu = \rho\mu \in \mathscr{P}_Z$. We first prove that ρ has a strictly positive version; i.e. μ is absolutely continuous with respect to ν. For measurable $A \subset \mathscr{C}$ with $\nu(A) = 0$, we intend to prove $\mu(A) = 0$. Since ν is S_t^Z-invariant, we have

$$\int_{\mathscr{C}} \mathbb{E}[R_{t+\tau}^\xi 1_A(X_{t+\tau}^\xi)]\nu(\mathrm{d}\xi) = \nu(S_t^Z 1_A) = \nu(A) = 0.$$

Noting that $R_{t+\tau}^\xi > 0$ for μ-a.e. (hence, ν-a.e.) ξ, this implies

$$\nu(S_{t+\tau}^0 1_A) = \int_{\mathscr{C}} \mathbb{E}[1_A(X_{t+\tau}^\xi)]\nu(\mathrm{d}\xi) = 0.$$

Letting $\rho_0(x) = \mu(\rho|\xi(0) = x)$ be the regular conditional expectation of ρ with respect to μ given $\xi(0)$, from this and (1.9) we obtain

$$\mu_0((\{P_t^0\}^*\rho_0)S_\tau 1_A) = \mu_0(\rho_0 P_t^0(S_\tau 1_A)) = \nu(S_{t+\tau}^0 1_A) = 0,$$

where due to $p_t^0 > 0$ and $\mu_0(\rho_0) = 1$,

$$(P_t^0)^*\rho_0 := \int_{\mathbb{H}} p_t^0(z, \cdot)\rho_0(z)\mu_0(\mathrm{d}z) > 0.$$

So, $\mu_0(S_\tau 1_A) = 0$. Combining this with (1.9) and that μ is S_τ^0-invariant, we obtain $\mu(A) = \mu(S_\tau^0 1_A) = \mu_0(S_\tau 1_A) = 0$.

Next, according to [34, Proof of Proposition 3.1(3)], the uniqueness follows if $S_{t+\tau}^Z$ has a strictly positive density with respect to ν. Since μ is equivalent to ν as proved above, and $S_{t+\tau}^Z$ is equivalent to $S_{t+\tau}^0$ according to (2.3), it suffices to prove that $S_{t+\tau}^0$ has a strictly positive density with respect to μ. Let $(S_\tau^0)^*$ be the adjoint operator of S_τ^0 in $L^2(\mu)$, and let $\hat{p}_t^0(\xi, \eta) = p_t^0(\xi(0), \eta(0))$. For any $f \in \mathscr{B}_b(\mathscr{C})$, (1.9) yields

$$S_{t+\tau}^0 f(\xi) = P_t^0(S_\tau f)(\xi(0)) = \int_{\mathbb{H}} (S_\tau f)(y)p_t^0(\xi(0), y)\mu_0(\mathrm{d}y)$$

$$= \int_{\mathscr{C}} (S_\tau^0 f)(\eta)\hat{p}_t^0(\xi,\eta)\mu(\mathrm{d}\eta) = \int_{\mathscr{C}} f(\eta)(S_\tau^0)^* \hat{p}_t^0(\xi,\cdot)(\eta)\mu(\mathrm{d}\eta).$$

Since $p_t^0 > 0$ implies $\hat{p}_t^0 > 0$, this implies that $S_{t+\tau}^0$ has a strictly positive density $(\xi,\eta) \mapsto (S_\tau^0)^* \hat{p}_t^0(\xi,\cdot)(\eta)$.

(2) By [34, Proposition 3.1], P_t^Z has at most one invariant probability measure if there exist $t > 0, q > 1$ and a measurable function $\Psi : \mathbb{H}^2 \to (0,\infty)$ such that

(3.1) $$P_t^Z f(x) \le (P_t^Z f^q(y))^{\frac{1}{q}} \Psi(x,y), \quad f \in \mathscr{B}^+(\mathbb{H}), x, y \in \mathbb{H}.$$

By (2.9), for any $r > 1$ there exists a constant $t(r) > 0$ such that

$$\Gamma_r(x,t) := \mathbb{E}\big[(R^x(t))^r + (R^x(t))^{-r}\big] < \infty, \quad t \in [0, t(r)], x \in \mathbb{H}.$$

Then (2.6) and (2.5) yield

$$P_t^Z f(x) = \mathbb{E}\big[f(X^x(t)R^x(t)\big] \le \sqrt{\Gamma_2(x,t)P_t^0 f^2(x)} \le \sqrt{\Gamma_2(x,t)}\big(P_t^0 f^{2p}(y)\big)^{\frac{1}{2p}} e^{\frac{1}{2p}\Phi_p(t,x,y)}$$

$$\le \sqrt{\Gamma_2(x,t)}\big(\mathbb{E}[f^{4p}(X^y(t))R^y(t)]\big)^{\frac{1}{4p}}\big(\mathbb{E}[(R^y(t))^{-1}]\big)^{\frac{1}{4p}} e^{\frac{1}{2p}\Phi_p(t,x,y)}$$

$$\le \sqrt{\Gamma_2(x,t)}\big(\Gamma_1(y,t)\big)^{\frac{1}{4p}} e^{\frac{1}{2p}\Phi_p(t,x,y)}\big(P_t^Z f^{4p}(y)\big)^{\frac{1}{4p}}.$$

Therefore, (3.1) holds for $q = 4p$ and some function Ψ, and the proof is thus finished. $\square$

4 Entropy estimate using hyperboundedness

In this section, we assume that P_t^0 is hyperbounded, i.e. there exist $p_0 > 1$ and $t_0 > 0$ such that

(4.1) $$\|P_{t_0}^0\|_{L^2(\mu_0) \to L^{2p_0}(\mu_0)} := \sup_{\mu_0(f^2) \le 1} \mu_0\big(|P_{t_0}^0 f|^{2p_0}\big)^{\frac{1}{2p_0}} < \infty.$$

According to Gross [10], when P_t^0 is symmetric in $L^2(\mu_0)$, for instance, Z_0 is given by (1.2) and $\mu_0(\mathrm{d}x) = e^{-V(x)}\mathrm{d}x$, (4.1) is equivalent to the defective log-Sobolev inequality (1.3) for some constants $\kappa > 0, \beta \ge 0$. However, in the non-symmetric case, the latter is strictly stronger than (4.1), see Examples 4.1 and 4.2 below for hypercontractive Markov semigroups without the log-Sobolev inequality. So, the following result is new even for $\tau = 0$ and $\mathbb{H} = \mathbb{R}^d$.

Theorem 4.1. *Assume* **(A)** *and* (4.1) *for some* $t_0 > 0$ *and* $p_0 > 1$. *If* $\mu(e^{\lambda|Z|^2}) < \infty$ *for some constant* $\lambda > \frac{(3p_0-1)(t_0+\tau)}{2(p_0-1)}$, *then there exists* $\nu := \rho\mu \in \mathscr{P}_Z$ *such that*

(4.2) $$\mu(\rho \log \rho) \le \frac{(t_0 + \tau)(3p_0 - 1)\log \mu(e^{\lambda|Z|^2}) + 4\lambda p_0 \log \|P_{t_0}^0\|_{L^2(\mu_0) \to L^{2p_0}(\mu_0)}}{2\lambda(p_0 - 1) - (3p_0 - 1)(t_0 + \tau)}.$$

Proof. Let $c = t_0 + \tau$ and

(4.3) $$\nu_n := \frac{1}{cn}\int_0^{cn} \mu S_t^Z \mathrm{d}t, \quad n \ge 1,$$

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where the probability measure μS_t^Z is defined by $(\mu S_t^Z)(A) := \mu(S_t^Z 1_A)$, $A \in \mathscr{B}(\mathscr{C})$. It suffices to find a subsequence $n_k \to \infty$ such that $\nu_{n_k} \to \nu$ weakly for some probability measure $\nu := \rho\mu$ with density ρ satisfying (4.2). We complete the proof by the following three steps.

(a) Let $c_0 = \|P_{t_0}^0\|_{L^2(\mu) \to L^{2p_0}(\mu)} < \infty$ and $p = 1 + \frac{p_0 - 1}{2p_0} > 1$. We first prove

(4.4)
$$\mathbb{E}\big[e^{\int_0^{cn} f(X_s^\mu)ds}\big] \le c_0^{\frac{n}{p}}\Big\{\mu\big(e^{\frac{c(3p_0-1)}{p_0-1}f}\big)\Big\}^{\frac{n(p_0-1)}{2p_0 p}}, \quad f \in \mathscr{B}_b(\mathscr{C}).$$

Since μ is an invariant probability measure of the segment process X_t, (1.9) implies

$$\mu_0(S_{t+\tau}f) = \int_{\mathscr{C}} \mathbb{E}[f(X_{t+\tau}^{\xi(0)})]\mu(d\xi) = \int_{\mathscr{C}} \mathbb{E}[f(X_{t+\tau}^\xi)]\mu(d\xi) = \mathbb{E}[f(X_{t+\tau}^\mu)] = \mu(f)$$

for $t \ge 0$ and $f \in \mathscr{B}_b(\mathscr{C})$. Combining this with (1.9) and using Jensen's inequality, we obtain

(4.5)
$$\begin{aligned}
\mu_0(|S_c f|^{2p_0}) &= \mu(|P_{t_0}^0(S_\tau f)|^{2p_0}) \le c_0^{2p_0}\mu_0((S_\tau f)^2)^{p_0} \\
&\le c_0^{2p_0}\{\mu_0(S_\tau f^2)\}^{p_0} = c_0^{2p_0}\{\mu(f^2)\}^{p_0}, \quad f \in L^2(\mu).
\end{aligned}$$

For any $F \in \mathscr{B}_b(\mathscr{C})$, we consider the Feymann-Kac semigroup

$$(\bar{S}_t^F f)(\xi) := \mathbb{E}\Big[f(X_t^\xi)e^{\int_0^t F(X_s^\xi)ds}\Big], \quad f \in \mathscr{B}_b(\mathscr{C}).$$

By the Hölder/Jensen inequalities, (1.9) and (4.5), we obtain

$$\begin{aligned}
\mu(|\bar{S}_c^F f|^{2p}) &= \int_{\mathscr{C}}\Big(\mathbb{E}\big[f(X_c^\xi)e^{\int_0^c F(X_s^\xi)ds}\big]\Big)^{2p}\mu(d\xi) \\
&\le \int_{\mathscr{C}}\Big\{\big(\mathbb{E}[f^p(X_c^\xi)]\big)^2\big(\mathbb{E}e^{\frac{p}{p-1}\int_0^c F(X_s^\xi)ds}\big)^{2(p-1)}\Big\}\mu(d\xi) \\
&\le \int_{\mathscr{C}}(S_c f^p)^2(\xi(0))\Big(\frac{1}{c}\int_0^c \mathbb{E}e^{\frac{cp}{p-1}F(X_s^\xi)}ds\Big)^{2(p-1)}\mu(d\xi) \\
&\le \Big(\int_{\mathbb{H}}(S_c f^p)^{2p_0}d\mu_0\Big)^{\frac{1}{p_0}}\Big\{\int_{\mathscr{C}}\Big(\frac{1}{c}\int_0^c \mathbb{E}\big[e^{\frac{cp}{p-1}F(X_s^\xi)}\big]ds\Big)^{\frac{2p_0(p-1)}{p_0-1}}\mu(d\xi)\Big\}^{\frac{p_0-1}{p_0}} \\
&\le c_0^2\mu(f^{2p})\Big\{\mu\big(e^{\frac{c(3p_0-1)}{p_0-1}F}\big)\Big\}^{\frac{p_0-1}{p_0}} < \infty.
\end{aligned}$$

So, $\bar{S}_c^F$ is bounded in $L^{2p_0}(\mu)$ with

$$\|\bar{S}_c^F\|_{L^{2p}(\mu)} \le c_0^{\frac{1}{p}}\Big\{\mu\big(e^{\frac{c(3p_0-1)}{p_0-1}F}\big)\Big\}^{\frac{p_0-1}{2p_0 p}}.$$

By the semigroup property, this leads to

$$\mathbb{E}e^{\int_0^{cn} F(X_s^\mu)ds} = \mu(\bar{S}_{cn}^F 1) \le \|\bar{S}_{cn}^F\|_{L^{2p}(\mu)} \le \|\bar{S}_c^F\|_{L^{2p}(\mu)}^n \le c_0^{\frac{n}{p}}\Big\{\mu\big(e^{\frac{c(3p_0-1)}{p_0-1}F}\big)\Big\}^{\frac{n(p_0-1)}{2p_0 p}}.$$

Therefore, (4.4) holds.

(b) For any $\varepsilon \in (0,1)$, we intend to prove

$$(4.6) \qquad \mathbb{E}\big[R^\mu(cn)\log R^\mu(cn)\big] \le \frac{n\varepsilon}{p(1-\varepsilon)}\log\Big[c_0\big\{\mu\big(\mathrm{e}^{\frac{c(3p_0-1)}{2\varepsilon(p_0-1)}|Z|^2}\big)\big\}^{\frac{p_0-1}{2p_0}}\Big].$$

We will apply the following Young inequality (see [1, Lemma 2.4]): for a probability measure Λ on a measurable space M,

$$(4.7) \qquad \Lambda(fg) \le \Lambda(f\log f) + \log\Lambda(\mathrm{e}^g), \quad f,g \in \mathscr{B}^+(M), \Lambda(f) = 1.$$

By the Young inequality and (4.4),

$$\begin{aligned}
\mathbb{E}\big[R^\mu(cn)\log R^\mu(cn)\big] &= \frac{1}{2}\mathbb{E}\Big[R^\mu(cn)\int_0^{cn}|Z(X_s^\mu)|^2\mathrm{d}s\Big] \\
&\le \varepsilon\mathbb{E}\big[R^\mu(cn)\log R^\mu(cn)\big] + \varepsilon\log\mathbb{E}\big[\mathrm{e}^{\frac{1}{2\varepsilon}\int_0^{cn}|Z(X_s^\mu)|^2\mathrm{d}s}\big] \\
&\le \varepsilon\mathbb{E}\big[R^\mu(cn)\log R^\mu(cn)\big] + \frac{n\varepsilon}{p}\log\Big[c_0\big\{\mu\big(\mathrm{e}^{\frac{c(3p_0-1)}{2\varepsilon(p_0-1)}|Z|^2}\big)\big\}^{\frac{p_0-1}{2p_0}}\Big].
\end{aligned}$$

When Z is bounded we have $\mathbb{E}\big[R^\mu(cn)\log R^\mu(cn)\big] \le \frac{cn}{2}\|Z\|_\infty^2 < \infty$, so that this implies (4.6). In general, let $Z_m = Z1_{\{|Z|\le m\}}, m \ge 1$. Define $R_m^\mu(t)$ as $R^\mu(t)$ using Z_m replacing Z. Then the assertion for bounded Z implies

$$\mathbb{E}\big[R_m^\mu(cn)\log R_m^\mu(cn)\big] \le \frac{n\varepsilon}{p(1-\varepsilon)}\log\Big[c_0\big\{\mu\big(\mathrm{e}^{\frac{c(3p_0-1)}{2\varepsilon(p_0-1)}|Z|^2}\big)\big\}^{\frac{p_0-1}{2p_0}}\Big], \quad m \ge 1.$$

Due to Fatou's lemma, we prove (4.6) by letting $m \to \infty$.

(c) By (2.3), (4.5), (4.6) and (4.7), for any $f \ge 0$ we have

$$\begin{aligned}
(4.8) \qquad \nu_n(f) &= \frac{1}{cn}\int_0^{cn}\mathbb{E}[R^\mu(cn)f(X_s^\mu)]\mathrm{d}s \\
&\le \frac{1}{cn}\mathbb{E}\big[R^\mu(cn)\log R^\mu(cn)\big] + \frac{1}{cn}\log\mathbb{E}\big[\mathrm{e}^{\int_0^{cn}f(X_s^\mu)\mathrm{d}s}\big] \\
&\le \frac{\varepsilon}{cp(1-\varepsilon)}\log\Big[c_0\big\{\mu\big(\mathrm{e}^{\frac{c(3p_0-1)}{2\varepsilon(p_0-1)}|Z|^2}\big)\big\}^{\frac{p_0-1}{2p_0}}\Big] + \frac{1}{cp}\log\Big[c_0\big\{\mu\big(\mathrm{e}^{\frac{c(3p_0-1)}{p_0-1}f}\big)\big\}^{\frac{p_0-1}{2p_0}}\Big].
\end{aligned}$$

This implies that ν_n is absolutely continuous with respect to μ, and (4.2) holds for $\rho_n := \frac{\mathrm{d}\nu_n}{\mathrm{d}\mu}$ replacing ρ. Indeed, taking $f = R1_A$ in (4.8) for $\mu(A) = 0$ and $R > 0$, we obtain

$$\nu_n(A) \le \frac{1}{R}\Big(\frac{\varepsilon}{cp(1-\varepsilon)}\log\Big[c_0\big\{\mu\big(\mathrm{e}^{\frac{c(3p_0-1)}{2\varepsilon(p_0-1)}|Z|^2}\big)\big\}^{\frac{p_0-1}{2p_0}}\Big] + \frac{1}{cp}\log c_0\Big), \quad R > 0.$$

Letting $R \to \infty$ we prove $\nu_n(A) = 0$ for $\mu(A) = 0$, so that ν_n has a density ρ_n with respect to μ. Next, applying (4.8) to $f = \frac{p_0-1}{c(3p_0-1)}\log(\rho_n \wedge m + m^{-1})$ and letting $m \to \infty$, we obtain

$$\frac{p_0-1}{c(3p_0-1)}\mu(\rho_n\log\rho_n) \le \frac{\varepsilon}{cp(1-\varepsilon)}\log\Big[c_0\big\{\mu\big(\mathrm{e}^{\frac{c(3p_0-1)}{2\varepsilon(p_0-1)}|Z|^2}\big)\big\}^{\frac{p_0-1}{2p_0}}\Big] + \frac{1}{cp}\log c_0.$$

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Taking $\varepsilon = \frac{c(3p_0-1)}{2\lambda(p_0-1)} \in (0,1)$ such that $\frac{c(3p_0-1)}{2\varepsilon(p_0-1)} = 1$, we arrive at

$$
\begin{aligned}
\mu(\rho_n \log \rho_n) &\leq \frac{\varepsilon(3p_0 - 1)}{p(1 - \varepsilon)(p_0 - 1)} \log\left[c_0\{\mu(e^{\lambda|Z|^2})\}^{\frac{p_0-1}{2p_0}}\right] + \frac{3p_0 - 1}{p(p_0 - 1)} \log c_0 \\
&= \frac{2cp_0(3p_0 - 1)}{[2\lambda(p_0 - 1) - c(3p_0 - 1)](p_0 - 1)} \log\left[c_0\{\mu(e^{\lambda|Z|^2})\}^{\frac{p_0-1}{2p_0}}\right] + \frac{2p_0}{p_0 - 1} \log c_0 \\
&= \frac{c(3p_0 - 1) \log \mu(e^{\lambda|Z|^2}) + 4\lambda p_0 \log c_0}{2\lambda(p_0 - 1) - c(3p_0 - 1)}, \quad n \geq 1.
\end{aligned}
$$

Therefore, $\{\rho_n\}_{n \geq 1}$ is uniformly integrable in $L^1(\mu)$, so that for some subsequence $n_k \to \infty$ we have $\rho_{n_k} \to \rho$ weakly in $L^1(\mu)$. Then $\nu_{n_k} \to \nu := \rho\mu$ strongly and ρ satisfies (4.2). □

We first consider a simple example to show that the integrability condition in Theorem 4.1 is asymptotically sharp for small t_0.

Example 4.1   Let $\mathbb{H} = \mathbb{R}^d$, $\tau = 0$, $\sigma = \sqrt{2}I$ and $Z_0 = -x$. Then $\mu_0(dx) = (2\pi)^{-d/2}e^{-\frac{1}{2}|x|^2}dx$ is the standard Gaussian measure. It is well known by Nelson [20] (see also Gross [10]), we have

$$
\|P_{t_0}^0\|_{L^2(\mu_0) \to L^{2p_0}(\mu_0)} = 1, \quad t_0 > 0, p_0 = \frac{1}{2}(1 + e^{2t_0}).
$$

Since $\frac{3p_0-1)t_0}{2(p_0-1)} = \frac{t_0(3e^{2t_0}-1)}{2(e^{2t_0}-1)} \to \frac{1}{2}$ as $t_0 \to 0$, for any $\lambda > \frac{1}{2}$ there exists $t_0 > 0$ such that $\lambda > \frac{3p_0-1)t_0}{2(p_0-1)}$. By Theorems 3.1 and 4.1, if $\int_{\mathbb{R}^d} e^{\lambda|Z(x)|^2 - \frac{1}{2}|x|^2}dx < \infty$ then there exists a unique $\nu = \rho\mu_0 \in \mathscr{P}_Z$ and

$$
\mu_0(\rho \log \rho) \leq \inf\left\{\frac{t_0(3e^{2t_0} - 1) \log \mu_0(e^{\lambda|Z|^2})}{2\lambda(e^{2t_0} - 1) - t_0(3e^{2t_0} - 1)} : \frac{t_0(3e^{2t_0} - 1)}{2(e^{2t_0} - 1)} < \lambda\right\} < \infty.
$$

On the other hand, for any $\lambda < \frac{1}{2}$, there exists Z with $\mu_0(e^{\lambda|Z|^2}) < \infty$ but P_t^Z does not have any invariant probability measure. For instance, when $Z(x) = x$ we have $\mu_0(e^{\lambda|Z|^2}) < \infty$ for any $\lambda < \frac{1}{2}$ but $P_t^Z = e^{t\Delta}$ does not have invariant probability measure.

Below we consider three more examples. The first two are degenerate SDEs and semilinear SPDEs for which the defective log-Sobolev inequality does not hold, and the last belongs to monotone SPDEs where the defective log-Sobolev inequality is unknown. See [15, 26, 30] for more examples of hyperbounded Markov semigroups without the defective log-Sobolev inequality.

Example 4.2 (Infinite-dimensional stochastic Hamiltonian system).   Let $\mathbb{H}_1$ be a separable Hilbert space. Consider the following SPDE for $(X(t), Y(t))$ on $\mathbb{H} := \mathbb{H}_1 \times \mathbb{H}_1$:

$$
\begin{cases}
dX(t) = \{Y(t) - LX(t)\}dt, \\
dY(t) = \{Z(X_t, Y_t) - LY(t)\}dt + dW(t),
\end{cases}
$$

where $W(t)$ is the standard cylindrical Brownian motion on $\mathbb{H}_1$, $Z : \mathscr{C} \mapsto \mathbb{H}_1$ is measurable, $(L, \mathscr{D}(L))$ is a positive definite self-adjoint operator on $\mathbb{H}_1$ with discrete eigenvalues $0 < \lambda_1 \le \lambda_2 \le \cdots$ satisfying $\sum_{i=1}^{\infty} \lambda_i^{-\delta} < \infty$ for some constant $\delta \in (0,1)$. Then the reference SDE with $Z = 0$ has a unique invariant probability measure $\mu_0 = N(0, (2L)^{-1}) \times N(0, (2L)^{-1})$, where $N(0, (2L)^{-1})$ is the centered Gaussian measure on $\mathbb{H}_1$ with covariance $(2L)^{-1}$. By [32, Theorem 4.1] with $Z = 0, A = 0, B = I, L_1 = L_2 = L$ such that $\delta = 0$, the associated Markov semigroup P_t^0 is hypercontractive. So, Theorem 4.1 applies. Moreover, the Harnack inequality in [32, Lemma 4.2] implies that P_t^0 has a strictly positive density with respect to μ_0. Then Theorem 3.1 implies the uniqueness of invariant probability measure of S_t^Z.

Example 4.3 (Finite-dimensional stochastic Hamiltonian system). Consider the following degenerate SDE for $(X(t), Y(t))$ on $\mathbb{H} = \mathbb{R}^{2d}$:

$$
\begin{cases}
\mathrm{d}X(t) = Y(t)\mathrm{d}t, \\
\mathrm{d}Y(t) = \{Z(X_t, Y_t) - X(t) - Y(t)\}\mathrm{d}t + \mathrm{d}W(t),
\end{cases}
$$

Let P_t^0 be the Markov semigroup for the SDE with $Z = 0$. By [11, Theorem 4.4], for any $p > 1$ there exists a constant $c > 0$ such that

$$
(P_t f)^p(x) \le (P_t f^p)(y) \mathrm{e}^{\frac{c|x-y|^2}{1 \wedge t^3}}, \quad t > 0, x, y \in \mathbb{R}^{2d}, f \in \mathscr{B}^+(\mathbb{R}^{2d}).
$$

Since $\Phi_p(s, x, y) := \frac{c|x-y|^2}{1 \wedge s^3}$ satisfies

$$
\mu_0(\mathrm{e}^{-\Phi_p(s,x,y)}) \ge \mathrm{e}^{-c}\mu_0(B(x, 1 \wedge s^{\frac{3}{2}})) \ge \alpha(x)(1 \wedge s)^{\frac{3d}{2}}, \quad s > 0, x \in \mathbb{R}^{2d}
$$

for some positive $\alpha \in C(\mathbb{R}^{2d})$, (2.4) holds for $p > \frac{3d}{2}$. Therefore, when $\mu_0(\mathrm{e}^{\varepsilon|Z|^2}) < \infty$ for some $\varepsilon > 0$, Theorem 3.1(2) implies that P_t^Z is a Markov semigroup on $\mathscr{B}_b(\mathbb{R}^{2d})$ having at most one invariant probability measure.

Moreover, by [32, Example 5.1] with $W = 0$, P_t^0 has unique invariant probability measure $\mu_0(\mathrm{d}x) := (2\pi)^{-d}\mathrm{e}^{-\frac{1}{2}|x|^2}\mathrm{d}x$ on $\mathbb{R}^{2d}$, and

$$
\|P_{t_0}^0\|_{L^2(\mu_0) \to L^4(\mu_0)} = 1
$$

holds for some constant $t_0 > 0$. Therefore, by Theorem 4.1, if $\mu_0(\mathrm{e}^{\lambda|Z|^2}) < \infty$ for some $\lambda > 4t_0$ then P_t^Z has a (unique, as observed above) invariant probability measure ν with density $\rho := \frac{\mathrm{d}\nu}{\mathrm{d}\mu_0}$ satisfying

$$
\mu_0(\rho \log \rho) \le \frac{4t_0 \log \mu_0(\mathrm{e}^{\lambda|Z|^2})}{\lambda - 4t_0}.
$$

Example 4.4 (Monotone SPDE). Let $\sigma(0) \in \mathscr{L}_{HS}(\tilde{\mathbb{H}}, \mathbb{H})$, the class of Hilbert-Schmidt operators from $\tilde{\mathbb{H}}$ to $\mathbb{H}$, and let $\alpha \ge 2$ be a constant. Assume that $r \mapsto {}_{\mathbb{V}^*}\langle Z_0(rv_1 + v_2), v_3\rangle_{\mathbb{V}}$ is continuous in $r \in \mathbb{R}$ for any $v_1, v_2, v_3 \in \mathbb{V}$, and there exist constants $C, \delta > 0$ such that

$$
2{}_{\mathbb{V}^*}\langle Z_0(v_1) - Z_0(v_2), v_1 - v_2\rangle_{\mathbb{V}} + \|\sigma(v_1) - \sigma(v_2)\|_{L_{HS}}^2 \le C\|v_1 - v_2\|_{\mathbb{H}}^2 - \delta\|v_1 - v_2\|_{\mathbb{V}}^{\alpha},
$$

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$$\|Z_0(v)\|_{\mathbb{V}^*} \le C + C\|v\|_{\mathbb{V}}^{\alpha-1} \quad \text{for all } v_1, v_2, v \in \mathbb{V}.$$

According to [15, Theorem 1.4], the equation (1.6) has a unique variational solution for any initial value and the Markov semigroup P_t^0 is hyperbounded with respect to the unique invariant probability measure μ_0. Moreover, according to [34], the Harnack inequality in [15, Theorem 1.2] implies that P_t^0 has a positive density with respect to μ_0. So, Theorem 3.1(1) and Theorem 4.1 apply. When $\alpha > 2$, [15, Theorem 1.4] ensures (4.1) for any $t_0 > 0$ and $p_0 > 1 > 0$, so that by these results $\mu(\mathrm{e}^{\lambda|Z|^2}) < \infty$ for some $\lambda > 4\tau$ implies that $\mathscr{P}_Z$ contains a unique measure ν, which has a strictly positive density ρ with respect to μ, and $\mu(\rho \log \rho) < \infty$.

5   Sobolev estimates using log-Sobolev inequality

In this section, we aim to extend Theorem 1.1 to degenerate SDEs with path-dependent drifts. When $\tau > 0$, we will consider the Sobolev regularity of the marginal density of the invariant probability measure. For a probability measure ν on $\mathscr{C}$ and $\theta \in [-\tau, 0]$, let ν_θ be the θ-marginal distribution of ν, i.e.

$$\nu_\theta(A) := \nu(\{\xi \in \mathscr{C} : \xi(\theta) \in A\}), \quad A \in \mathscr{B}(\mathbb{H}).$$

In particular, by the stationarity of $X^{\mu_0}(t)$, we have $\mu_\theta = \mu_0$ for $\theta \in [-\tau, 0]$.

We mainly consider the finite-dimensional case, but make a simple infinite-dimensional extension in §5.4. Let $\mathbb{H} = \mathbb{R}^d$ and $\tilde{\mathbb{H}} = \mathbb{R}^m$ for some $d, m \ge 1$, and let $V \in C^2(\mathbb{R}^d)$ such that $\mu_0(\mathrm{d}x) := \mathrm{e}^{-V(x)}\mathrm{d}x$ is a probability measure on $\mathbb{R}^d$. Let $\sigma \in C^2(\mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^m)$ and let Z_0 be in (1.2). Then the operator

$$(5.1) \qquad \mathscr{L}_0 := \frac{1}{2}\sum_{i,j=1}^{d}(\sigma\sigma^*)_{ij}\partial_i\partial_j + \sum_{i=1}^{d}\langle Z_0, e_i\rangle\partial_i$$

defined on $C_0^\infty(\mathbb{R}^d)$ is symmetric in $L^2(\mu_0)$; namely,

$$-\mu_0(f\mathscr{L}_0 g) = \mathscr{E}_0(f, g) := \mu_0(\langle \sigma^*\nabla f, \sigma^*\nabla g\rangle), \quad f, g \in C_0^\infty(\mathbb{R}^d).$$

Let $H_\sigma^{1,2}(\mu_0)$ be the closure of $C_0^\infty(\mathbb{R}^d)$ with respect to the Sobolev norm

$$\|f\|_{H_\sigma^{1,2}(\mu_0)} := \{\mu_0(|f|^2 + |\sigma^*\nabla f|^2)\}^{\frac{1}{2}}.$$

Then $(\mathscr{E}_0, H_\sigma^{1,2}(\mu_0))$ is a symmetric Dirichlet form on $L^2(\mu_0)$ and the associated Markov process can be constructed as the solution to the SDE

$$(5.2) \qquad \mathrm{d}X(t) = Z_0(X(t))\mathrm{d}t + \sigma(X(t))\mathrm{d}W(t),$$

where $W(t)$ is the m-dimensional Brownian motion.

As in Section 4, we investigate the following functional SDE

$$(5.3) \qquad \mathrm{d}X(t) = \{\sigma(X(t))Z(X_t) + Z_0(X(t))\}\mathrm{d}t + \sigma(X(t))\mathrm{d}W(t)$$

by using integrability conditions on the measurable map $Z : \mathscr{C} \to \mathbb{R}^m$. Besides the existence of invariant probability measure and the entropy estimate presented in Theorem 4.1, we aim to derive more and stronger density estimates including those given in Theorem 1.1. To this end, we make the following assumption **(H)**, where the log-Sobolev inequality is essentially stronger than the hyperboundedness of P_t^0 used in Section 4.

Let $\mathrm{Lie}\{U_1, \cdots, U_m\}$ be the Lie algebra induced by vector fields $\{U_1, \cdots, U_m\}$. More precisely, let $\mathscr{A}_0 = \{U_i : 1 \leq i \leq m\}$ and

$$\mathscr{A}_k = \big\{[U, U'] := UU' - U'U : \ U, U' \in \cup_{0 \leq l \leq k-1}\mathscr{A}_l\big\}, \quad k \geq 1.$$

Then $\mathrm{Lie}\{U_1, \cdots, U_m\}$ is the linear space spanned by $\cup_{k \geq 0}\mathscr{A}_k$.

(H) *Let $V \in C^2(\mathbb{R}^d)$ such that $\mu_0(\mathrm{d}x) := \mathrm{e}^{-V(x)}\mathrm{d}x$ is a probability measure on $\mathbb{R}^d$. There exists $k \geq 2$ such that $\sigma \in C^k(\mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^m)$ and vector fields*

$$U_i := \sum_{j=1}^d \sigma_{ji}\partial_j, \quad 1 \leq i \leq m$$

satisfy the Hörmander condition up to the k-th order of Lie brackets. Let Z_0 be in (1.2). Moreover, $1 \in H_\sigma^{1,2}(\mu_0)$ with $\mathscr{E}_0(1,1) = 0$, and the log-Sobolev inequality (1.3) holds for some constants $\kappa > 0$ and $\beta \geq 0$.

This assumption implies that the solution to (5.2) is non-explosive, the associated Markov semigroup P_t^0 has strictly positive symmetric density $p_t^0(x, y)$ with respect to the unique invariant probability measure μ_0, and P_t^0 is hyperbounded. Since the Dirichlet form is irreducible, the hyperboundedness of P_t^0 is equivalent to $\|P_t^0\|_{L^2(\mu_0) \to L^4(\mu_0)} = 1$ for large $t > 0$ (i.e. the hypercontractivity), see [19, 28]. Consequently, the Poincaré inequality

$$(5.4) \qquad \mu_0(f^2) \leq C\mathscr{E}_0(f, f), \quad f \in H_\sigma^{1,2}(\mu_0), \mu_0(f) = 0$$

holds for some constant $C > 0$.

5.1   Main result and examples

Let $\tau \geq 0$ and let μ on $\mathscr{C} := C([-\tau, 0]; \mathbb{R}^d)$ be the unique invariant probability measure of the segment solution to (5.2). We will need the condition $\mu(\mathrm{e}^{\lambda|Z|^2}) < \infty$ for λ big enough in terms of κ and τ. Let

$$\lambda_{\kappa,\tau} = \inf\Big\{\lambda > \kappa : \ \big(1 + \sqrt{1 + 8\lambda/\tau}\big)\big(1 - \sqrt{\kappa/\lambda}\big) \geq 16\Big\}.$$

When $\tau = 0$, we have $\lambda_{\kappa,0} = \kappa$. Then for any $\lambda > \lambda_{\kappa,\tau}$ we have

$$q_\lambda := \frac{2\sqrt{\lambda}}{\sqrt{\lambda} - \sqrt{\kappa} + \sqrt{(\sqrt{\lambda} - \sqrt{\kappa})^2 - \frac{16\sqrt{\lambda}(\sqrt{\lambda} - \sqrt{\kappa})}{1 + \sqrt{1 + 8\lambda/\tau}}}} \geq \frac{\sqrt{\lambda}}{\sqrt{\lambda} - \sqrt{\kappa}} > 1.$$

Theorem 5.1. *Assume* **(H)**. *If* $\mu(e^{\lambda|Z|^2}) < \infty$ *for some* $\lambda > \lambda_{\kappa,\tau}$, *then* $\mathscr{P}_Z$ *contains a unique probability measure* $\nu = \rho\mu$. *Moreover:*

(1) *For any* $p \in (1, \frac{q_\lambda}{q_\lambda - 1})$, *there exists a constant* $k = k(p, \lambda) \geq 1$ *such that*

$$(5.5) \qquad\qquad \mu(\rho^p) \leq k\mu(e^{\lambda|Z|^2})^k.$$

(2) *The marginal density* $\rho_\theta := \frac{d\nu_\theta}{d\mu_0}$ *does not depend on* $\theta \in [-\tau, 0]$, *and has a continuous, strictly positive version such that* $\log \rho_0, \rho_0^{\frac{p}{2}} \in H^{1,2}_\sigma(\mu_0)$ *for* $p \in (1, \frac{q_\lambda}{q_\lambda - 1})$ *with the following estimates holding for some constant* $k = k(p, \lambda) \geq 1$:

$$(5.6) \qquad \begin{aligned} \mu_0\big(|\sigma^*\nabla\sqrt{\rho_0}|^2\big) &\leq \frac{1}{\lambda - \kappa}\big\{\log\mu(e^{\lambda|Z|^2}) + \beta\big\} < \infty; \\ \mu_0\big(|\sigma^*\nabla\log\rho_0|^2\big) &\leq 4\mu(|Z|^2) < \infty; \\ \mu_0\big(|\sigma^*\nabla\rho_0^{\frac{p}{2}}|^2 + \rho_0^p\big) &\leq k\big\{\mu(e^{\lambda|Z|^2})\big\}^k. \end{aligned}$$

Since Theorem 1.1 does not imply $\rho^{\frac{p}{2}} \in H^{1,2}_\sigma(\mu_0)$, the last estimate is new even in the non-degenerate case without delay (i.e. $\tau = 0$). We present below two examples of degenerate diffusion processes satisfying the log-Sobolev inequality such that Theorem 5.1 applies.

Example 5.1 (Gruschin type diffusions). Let $d = 2$ and $l \in \mathbb{N}$. Let

$$U_1(x) = \partial_1, \qquad U_2(x) = x_1^l\partial_2, \quad x = (x_1, x_2) \in \mathbb{R}^2.$$

Then the Hörmander condition is satisfied. Let $m \geq 2, c_1 \in \mathbb{R}, c_2 \neq 0$ and $c_3, c_4 > 0$ such that $\mu_0(dx) := e^{-V(x)}dx$ is a probability measure for

$$V(x) := c_1 + (c_3|x_1 - c_1|^{l+1} + c_4 x_2^2)^m.$$

Obviously, $1 \in H^{1,2}_\sigma(\mu_0)$ with $\mathscr{E}_0(1,1) = 0$. Moreover, by [27, Proposition 4.1], (1.3) holds for some constant $\kappa > 0$ and $\beta = 0$. Therefore, assumption **(H)** is satisfied.

Example 5.2 (Diffusions on Heisenberg group). Consider the following vector fields on $\mathbb{R}^3$:

$$U_1(x) = \partial_1 - \frac{x_2}{2}\partial_3, \quad U_2(x) = \partial_2 + \frac{x_1}{2}\partial_3, \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3.$$

Then the Hörmander condition is satisfied. The Markov semigroup $e^{t\Delta_0}$ generated by the Kohn-Laplacian $\Delta_0 := U_1^2 + U_2^2$ has a strictly positive smooth density p_t:

$$(e^{t\Delta_0}f)(0) = \int_{\mathbb{R}^3} p_t(x)f(x)dx, \quad f \in \mathscr{B}_b(\mathbb{R}^3), t > 0.$$

For fixed $t_0 > 0$, let $V(x) = -\log p_{t_0}(x)$ so that $\mu_0(dx) := e^{-V(x)}dx = p_{t_0}(x)dx$. Obviously, $1 \in H^{1,2}_\sigma(\mu_0)$ with $\mathscr{E}_0(1,1) = 0$. Moreover, according to [14, Corollary 1.2], there exists a universal constant $c > 0$ such that the log-Sobolev inequality (1.3) holds for $\kappa = t_0 c$ and $\beta = 0$ (see [31] for more results on functional inequalities). So, assumption **(H)** holds.

In the next two subsections, we prove Theorem 5.1 for $\tau = 0$ and $\tau > 0$ respectively.

5.2 Proof of Theorem 5.1 for $\tau = 0$

When $\tau = 0$, Theorem 5.1 reduces to the following result where S_t^Z is replaced by P_t^Z for notation consistency with Theorem 1.1.

Theorem 5.2. *Assume* **(H)** *and let* $\tau = 0$. *For any* $\lambda > \kappa$ *and* $p \in \left(1, \frac{\sqrt{\lambda}}{\sqrt{\kappa}}\right)$, *there exists a constant* $C_{p,\lambda}$ *such that* $\mu_0(e^{\lambda|Z|^2}) < \infty$ *implies that* $\mathscr{P}_Z$ *has a unique element* $\nu = \rho\mu_0$, *where* ρ *is continuous, strictly positive satisfying* $\log\rho, \rho^{p/2} \in H_\sigma^{1,2}(\mu_0)$, (1.5) *and*

$$
(5.7) \qquad \mu_0\left(|\sigma^*\nabla\rho^{\frac{p}{2}}|^2 + \rho^p\right) \leq C_{p,\lambda}\{\mu_0(e^{\lambda|Z|^2})\}^{C_{p,\lambda}}.
$$

We first prove this result for bounded Z with compact support. Let $\mathscr{L}_Z = \mathscr{L}_0 + (\sigma Z)\cdot\nabla$, where $\mathscr{L}_0$ is in (5.1). Then an invariant probability measure ν of P_t^Z solves the equation $L_Z^*\nu = 0$ in the sense that

$$
\int_{\mathbb{R}^d} L_Z f \mathrm{d}\nu = 0, \quad f \in C_0^\infty(\mathbb{R}^d).
$$

The following lemma extends [33, Lemma 4.2] to the present degenerate case.

Lemma 5.3. *Assume* **(H)** *and let* $\tau = 0$. *If* $|Z| \in \cap_{p\in[1,\infty)}L_{loc}^p(\mathbb{R}^d)$ *and* $\nu := \rho\mu_0$ *is a probability measure such that* $L_Z^*\nu = 0$, *then* ρ *has a continuous, strictly positive version. If moreover* Z *is bounded and has compact support, then* $\rho \in H_\sigma^{1,2}(\mu_0)$ *and*

$$
(5.8) \qquad \int_{\mathbb{R}^d}\langle\sigma^*\nabla f, \sigma^*\nabla\rho\rangle\mathrm{d}\mu_0 = 2\int_{\mathbb{R}^d}\langle Z, \sigma^*\nabla f\rangle\rho\mathrm{d}\mu_0, \quad f \in H_\sigma^{1,2}(\mu_0).
$$

Proof. We first prove that ρ has a continuous, strictly positive version using results in [5]. Let v_i stand for the i-th component of a vector v, and let

$$
A_i(x, u, \xi) = \xi_i - 2uZ_i(x), \quad x \in \mathbb{R}^d, u \in \mathbb{R}, \xi \in \mathbb{R}^m, \quad 1 \leq i \leq m.
$$

It follows from the integration by parts formula and $L_Z^*\nu = 0$ that

$$
(5.9) \begin{aligned}
&\int_{\mathbb{R}^d}\sum_{i=1}^m A_i(\cdot, \rho, (U_1\rho, \cdots, U_m\rho))U_i f\mathrm{d}\mu_0 = \sum_{i=1}^m\int_{\mathbb{R}^d}(U_i\rho - 2\rho Z_i)U_i f\mathrm{d}\mu_0 \\
&=: -\sum_{i=1}^m\int_{\mathbb{R}^d}\left(\mathrm{div}\{(e^{-V}U_i f)U_i\} + 2e^{-V}Z_i U_i f\right)\rho\mathrm{d}x = -2\nu(L_Z f) = 0, \quad f \in C_0^\infty(\mathbb{R}^d).
\end{aligned}
$$

Obviously,

$$
|A(x, u, \xi)| := \sum_{i=1}^m|A_i(x, u, \xi)| \leq 2|u|\cdot|Z|(x) + |\xi|,
$$

$$
A(x, u, \xi)\cdot\xi \geq |\xi|^2 - |Z|(x)|u|\cdot|\xi| \geq \frac{1}{2}|\xi|^2 - 2|u|^2|Z|^2(x),
$$

where $|Z| \in L_{loc}^p(\mu)$ for any $p \in [1, \infty)$. Then by [5, Theorem 3.1 and Theorem 3.35], ρ has a locally Hölder continuous version (denoted again by ρ) with respect to the intrinsic

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distance induced by σ. By the Hörmander condition, the intrinsic distance induces the classical topology in $\mathbb{R}^d$, so that this version ρ is continuous. Moreover, for any compact set K there exists a constant $C(K) > 0$ such that

$$\sup_K \rho \leq C(K) + C(K) \inf_K \rho.$$

Since the equation (5.9) is linear in ρ, this inequality also holds for $n\rho$ replacing ρ, so that

$$\sup_K \rho \leq \frac{C(K)}{n} + C(K) \inf_K \rho, \quad n \geq 1.$$

Letting $n \to \infty$ we obtain

$$\sup_K \rho \leq C(K) \inf_K \rho,$$

which implies that ρ is strictly positive since $\mu_0(\rho) = 1$.

Now, let Z be bounded with compact support. Since ρ is locally bounded due to the continuity, ρZ is globally bounded. In particular, $\rho Z \in L^2(\mathbb{R}^d \to \mathbb{R}^d; \mu_0)$. By (5.4) and the completion of $H^{1,2}_\sigma(\mu_0)$, $\{\sigma^* \nabla h : h \in H^{1,2}_\sigma(\mu_0)\}$ is a closed subspace of $L^2(\mathbb{R}^d \to \mathbb{R}^d; \mu_0)$. Let $h \in H^{1,2}_\sigma(\mu_0)$ such that $\sigma^* \nabla h$ is the orthogonal projection of ρZ on this subspace. Since $L^*_Z \nu = 0$, we have

$$\mu_0(\rho \mathscr{L}_0 f) = \nu(L_Z f) - \nu(\langle Z, \sigma^* \nabla f \rangle)$$
$$= -\mu_0(\langle \rho Z, \sigma^* \nabla f \rangle) = -\mathscr{E}_0(h, f) = \mu_0(h \mathscr{L}_0 f), \quad f \in C_0^\infty(\mathbb{R}^d).$$

By (5.4), $\{\mathscr{L}_0 f : f \in C_0^\infty(\mathbb{R}^d)\}$ is dense in $\{f \in L^2(\mu_0) : \mu_0(f) = 0\}$, so this implies $\rho = h + c$ for some constant c. Hence, $\rho \in H^{1,2}_\sigma(\mu_0)$ and

$$\frac{1}{2} \mu_0(\langle \sigma^* \nabla \rho, \sigma^* \nabla f \rangle) = \mu_0(\rho \langle Z, \sigma^* \nabla f \rangle), \quad f \in C_0^\infty(\mathbb{R}^d).$$

Due to the boundedness of ρZ, this is equivalent to (5.8). $\qquad\square$

Lemma 5.4. *Assume* **(H)** *and let* $\tau = 0$. *For any* $\lambda > \kappa$ *and* $p \in \left(1, \frac{\sqrt{\lambda}}{\sqrt{\kappa}}\right)$, *there exists a constant* $C_{p,\lambda}$ *such that for any bounded* Z *with compact support, if a probability measure* $\nu := \rho \mu_0$ *solves* $L^*_Z \nu = 0$, *then* ρ *has a continuous, strictly positive version such that* $\log \rho, \rho^{p/2} \in H^{1,2}_\sigma(\mu_0)$, *and* (1.5), (5.7) *hold.*

Proof. By Lemma 5.3, ρ has a continuous, strictly positive version such that $\rho \in H^{1,2}_\sigma(\mu_0)$ and (5.8) holds. According to step (a) in the proof of [33, Theorem 2.3], (5.8) implies $\log \rho, \sqrt{\rho} \in H^{1,2}_\sigma(\mu_0)$ and (1.5). To prove (5.7), let $f_n := (n^{-1} + \rho \wedge n)^{p-1}$, $n \geq 1$. Then $f_n \in H^{1,2}_\sigma(\mu_0)$ and by (5.8),

$$I_n := \mu_0\big(\big|\sigma^* \nabla (\rho \wedge n + n^{-1})^{\frac{p}{2}}\big|^2\big) = \frac{p^2}{4(p-1)} \int_{\mathbb{R}^d} \langle \sigma^* \nabla f_n, \sigma^* \nabla \rho \rangle \mathrm{d}\mu_0$$

(5.10)
$$= \frac{p^2}{2(p-1)} \int_{\mathbb{R}^d} \langle Z, \sigma^* \nabla f_n \rangle \rho \, \mathrm{d}\mu_0 \leq \frac{p^2}{2(p-1)} \mu_0\big(\{\rho \wedge n\} |Z| \cdot |\sigma^* \nabla f_n|\big)$$
$$\leq p\mu_0\big(|Z|(\rho \wedge n + n^{-1})^{\frac{p}{2}} |\sigma^* \nabla (\rho \wedge n + n^{-1})^{\frac{p}{2}}|\big)$$
$$\leq p\sqrt{I_n \mu_0\big(|Z|^2 (\rho \wedge n + n^{-1})^p\big)}.$$

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Next, it follows from (4.7) and (1.3) that

(5.11)
$$\mu_0\big(|Z|^2(\rho \wedge n + n^{-1})^p\big) - \frac{\mu_0((\rho \wedge n + n^{-1})^p)}{\lambda} \log \mu_0(e^{\lambda|Z|^2})$$
$$\leq \frac{1}{\lambda}\mu_0\bigg((\rho \wedge n + n^{-1})^p \log \frac{(\rho \wedge n + n^{-1})^p}{\mu_0((\rho \wedge n + n^{-1})^p)}\bigg)$$
$$\leq \frac{\kappa}{\lambda}I_n + \frac{\beta}{\lambda}\mu_0\big((\rho \wedge n + n^{-1})^p\big).$$

Noting that the log-Sobolev inequality (1.3) implies the super Poincaré inequality (see [24] or [25])

(5.12)
$$\mu_0(f^2) \leq r\mu_0(|\sigma^*\nabla f|^2) + e^{c(1+r^{-1})}\mu_0(|f|)^2, \quad r > 0, f \in H_\sigma^{1,2}(\mu_0)$$

for some constant $c > 0$, we obtain

(5.13)
$$\mu_0\big((\rho \wedge n + n^{-1})^p\big) \leq rI_n + e^{c(1+r^{-1})}\mu_0\big((\rho \wedge n + n^{-1})^{\frac{p}{2}}\big)^2, \quad r > 0.$$

Combining (5.10)-(5.13), we arrive at

(5.14)
$$I_n \leq p^2\mu_0(|Z|^2(\rho \wedge n + n^{-1})^p)$$
$$\leq p^2\bigg(\frac{\kappa}{\lambda}I_n + \frac{\beta + \log\mu_0(e^{\lambda|Z|^2})}{\lambda}\mu_0\big((\rho \wedge n + n^{-1})^p\big)\bigg)$$
$$\leq p^2\bigg(\frac{\kappa}{\lambda}I_n + \frac{\beta + \log\mu_0(e^{\lambda|Z|^2})}{\lambda}\Big\{rI_n + e^{c(1+r^{-1})}\mu_0\big((\rho \wedge n + n^{-1})^{\frac{p}{2}}\big)^2\Big\}\bigg)$$
$$= \frac{p^2}{\lambda}\bigg[\{\kappa + r(\beta + \log\mu_0(e^{\lambda|Z|^2}))\}I_n$$
$$+ \{e^{c(1+r^{-1})}(\beta + \log\mu_0(e^{\lambda|Z|^2}))\}\mu_0\big((\rho \wedge n + n^{-1})^{\frac{p}{2}}\big)^2\bigg], \quad r > 0.$$

Now, we are ready to complete the proof by induction in p as follows.

(i) Assume that $p \leq 2$. Since $\mu_0(\rho) = 1$, we have

(5.15)
$$\mu_0\big((\rho \wedge n + n^{-1})^{\frac{p}{2}}\big)^2 \leq \{\mu_0(\rho + 1)\}^p \leq 2^2 = 4, \quad n \geq 1.$$

So, (5.14) implies

$$I_n \leq \frac{p^2}{\lambda}\bigg[\{\kappa + r(\beta + \log\mu_0(e^{\lambda|Z|^2}))\}I_n + 4\{e^{c(1+r^{-1})}(\beta + \log\mu_0(e^{\lambda|Z|^2}))\}\bigg], \quad r > 0.$$

Since $p^2 < \frac{\lambda}{\kappa}$, letting

$$r = r_{p,\lambda} := \frac{\lambda - p^2\kappa}{2p^2(\beta + \log\mu_0(e^{\lambda|Z|^2}))} > 0,$$

we obtain

$$I_n \leq \frac{8p^2e^{c(1+r_{p,\lambda})}(\beta + \log\mu_0(e^{\lambda|Z|^2}))}{\lambda - p^2\kappa} \leq \alpha_{p,\lambda}\mu_0(e^{\lambda|Z|^2})^{\alpha_{p,\lambda}}, \quad n \geq 1$$

for some constant $\alpha_{p,\lambda} > 0$. Combining this with (5.13) for e.g. $r = 1$ and (5.15), we may find out a constant $C_{p,\lambda} > 0$ such that

$$(5.16) \qquad\qquad I_n + \mu((\rho \wedge n)^p) \leq C_{p,\lambda} \mu_0(e^{\lambda |Z|^2})^{C_{p,\lambda}}, \quad n \geq 1.$$

Letting $n \to \infty$ we conclude that $\rho^{p/2} \in H_\sigma^{1,2}(\mu_0)$ and (5.7) holds.

(ii) Assume that (5.7) holds with $p \in (1, \frac{\sqrt{\lambda}}{\sqrt{\kappa}}) \cap [0, k]$ for some $k \geq 2$, we aim to prove it for $p \in (1, \frac{\sqrt{\lambda}}{\sqrt{\kappa}}) \cap [0, 2k]$. It suffices to consider $p \in (1, \frac{\sqrt{\lambda}}{\sqrt{\kappa}}) \cap (k, 2k]$ and $k < \frac{\sqrt{\lambda}}{\sqrt{\kappa}}$. In this case, by the assumption there exists a constant $\alpha_{k,\lambda} \geq 1$ such that

$$\mu_0(\rho^k) \leq \alpha_{k,\lambda} \mu_0(e^{\lambda |Z|^2})^{\alpha_{k,\lambda}}.$$

Since $p \leq 2k$, we have

$$\mu_0((\rho \wedge n + n^{-1})^{p/2})^2 \leq \{\mu_0(\rho^k)\}^{p/k} \leq \alpha_{k,\lambda}^p \mu_0(e^{\lambda |Z|^2})^{p\alpha_{k,\lambda}}.$$

Substituting this into (5.14) and repeating the argument in (i), we prove (5.7) for some constant $C_{p,\lambda} > 0$. □

Proof of Theorem 5.2. By the Hörmander theorem, **(H)** implies that P_t^0 has a strictly positive density with respect to μ_0. So, the uniqueness of $\nu \in \mathscr{P}_Z$ follows from Theorem 3.1(1).

To prove the existence, for any $n \geq 1$, let $Z_n = Z 1_{\{|Z|+|\cdot| \leq n\}}$. Then Z_n is bounded with compact support. By Theorem 4.1 and Lemma 5.4, $P_t^{Z_n}$ has an invariant probability measure $\nu_n = \rho_n \mu_0$, where ρ_n is continuous, strictly positive such that $\log \rho_n, \rho_n^{p/2} \in H_\sigma^{1,2}(\mu_0)$ and (1.5), (5.7) hold for ρ_n replacing ρ. In particular, $\{\rho_n^{p/2}\}_{n \geq 1}$ is bounded in $H_\sigma^{1,2}(\mu_0)$. Then, as explained in step (b) in the proof of [33, Theorem 2.3(1)], the defective log-Sobolev inequality (1.3) implies that $\{\rho_n^{p/2}\}_{n \geq 1}$ is relatively compact in $L^2(\mu_0)$. So, up to a subsequence, $\rho_n^{p/2} \to \rho^{p/2}$ in $L^2(\mu_0)$ for some probability density ρ with respect to μ_0. Moreover, $\log \rho, \rho^{p/2} \in H_\sigma^{1,2}(\mu_0)$ and (1.5), (5.7) hold. It remains to prove that $\nu := \rho \mu_0$ is an invariant probability measure of P_t^Z, where ρ has a continuous, strictly positive version according to Lemma 5.3.

For any $f \in \mathscr{B}_b(\mathbb{R}^d)$, by $\nu_n(P_t^{Z_n} f) = \nu_n(f)$, $\rho_n \to \rho$ in $L^1(\mu_0)$ and the boundedness of $\mu_0(\rho_n^p)$, we obtain

$$
\begin{aligned}
\left| \nu(P_t^Z f) - \nu(f) \right| &= \left| \lim_{n \to \infty} \mu_0(\rho_n(P_t^Z f - f)) \right| \\
&\leq \limsup_{n \to \infty} \mu_0\big(\rho_n |P_t^Z f - P_t^{Z_n} f|\big) \leq C \limsup_{n \to \infty} \mu_0\big(|P_t^Z f - P_t^{Z_n} f|^q\big)^{\frac{1}{q}}
\end{aligned}
$$

(5.17)

for some constant $C > 0$ and $q := \frac{p}{p-1}$. By (2.3) and $\tau = 0$, we have

$$
\mu_0\big(|P_t^Z f - P_t^{Z_n} f|^q\big)
$$

$$
= \int_{\mathbb{R}^d} \Big|\mathbb{E}\big[f(X^x(t))\big\{\mathrm{e}^{\int_0^t \langle Z(X^x(s)),\mathrm{d}W(s)\rangle - \frac{1}{2}\int_0^t |Z(X^x(s))|^2 \mathrm{d}s}
$$

$$
- \mathrm{e}^{\int_0^t \langle Z_n(X^x(s)),\mathrm{d}W(s)\rangle - \frac{1}{2}\int_0^t |Z_n(X^x(s))|^2 \mathrm{d}s}\big\}\big]\Big|^q \mu_0(\mathrm{d}x)
$$

(5.18)

$$
\leq \mathbb{E}\big[|f(X^{\mu_0}(t))|\big\{\mathrm{e}^{\int_0^t \langle Z(X^{\mu_0}(s)),\mathrm{d}W(s)\rangle - \frac{1}{2}\int_0^t |Z(X^{\mu_0}(s))|^2 \mathrm{d}s}
$$

$$
- \mathrm{e}^{\int_0^t \langle Z_n(X^{\mu_0}(s)),\mathrm{d}W(s)\rangle - \frac{1}{2}\int_0^t |Z_n(X^{\mu_0}(s))|^2 \mathrm{d}s}\big\}|^q\big]
$$

$$
\leq \|f\|_\infty^q \big(\mathbb{E}\mathrm{e}^{2q\int_0^t \langle Z(X^{\mu_0}(s)),\mathrm{d}W(s)\rangle - q\int_0^t |Z(X^{\mu_0}(s))|^2 \mathrm{d}s}\big)^{\frac{1}{2}}
$$

$$
\times \Big(\mathbb{E}\big|\mathrm{e}^{\int_0^t \langle (Z-Z_n)(X^{\mu_0}(s)),\mathrm{d}W(s)\rangle - \frac{1}{2}\int_0^t |(Z-Z_n)(X^{\mu_0}(s))|^2 \mathrm{d}s} - 1\big|^{2q}\Big)^{\frac{1}{2}}.
$$

Since $\mu_0(\mathrm{e}^{\lambda|Z|^2}) < \infty$, for any $\alpha > 1$ and measurable function g with $|g| \leq |Z|$,

$$
\mathbb{E}\mathrm{e}^{2\alpha\int_0^t \langle g(X^{\mu_0}(s)),\mathrm{d}W(s)\rangle - \alpha\int_0^t |g(X^{\mu_0}(s))|^2 \mathrm{d}s}
$$

$$
\leq \Big\{\big(\mathbb{E}\mathrm{e}^{4\alpha\int_0^t \langle g(X^{\mu_0}(s)),\mathrm{d}W(s)\rangle - 8\alpha^2\int_0^t |g(X^{\mu_0}(s))|^2 \mathrm{d}s}\big)\mathbb{E}\mathrm{e}^{(8\alpha^2-2\alpha)\int_0^t |g(X^{\mu_0}(s))|^2 \mathrm{d}s}\Big\}^{\frac{1}{2}}
$$

$$
\leq \big(\mu_0(\mathrm{e}^{(8\alpha^2-2\alpha)t|Z|^2})\big)^{\frac{1}{2}} \leq \sqrt{\mu_0(\mathrm{e}^{\lambda|Z|^2})} < \infty, \quad t \leq \frac{\lambda}{8\alpha^2 - 2\alpha}.
$$

So, for small enough $t > 0$, by (5.18) and the dominated convergence theorem we prove $\lim_{n\to\infty} \mu_0\big(|P_t^Z f - P_t^{Z_n} f|^q\big) = 0$. Then (5.17) implies $\nu(P_t^Z f) = \nu(f)$ for small $t > 0$ and all $f \in \mathscr{B}_b(\mathbb{R}^d)$. Therefore, ν is an invariant probability measure of P_t^Z. $\square$

5.3 Proof of Theorem 5.1 for $\tau > 0$

Again we start from bounded Z.

Lemma 5.5. *Assume* **(H)** *and let Z be bounded. Then S_t^Z has an invariant probability measure $\nu = \rho\mu$ such that $\rho \in \cap_{p\in(1,\infty)} L^p(\mu)$. Moreover, $\nu_\theta = \nu_0$ for $\theta \in [-\tau, 0]$, and it has a continuous, strictly positive density ρ_0 with respect to μ_0 such that $\log \rho_0, \rho_0^{p/2} \in H_\sigma^{1,2}(\mu_0)$ for any $p > 1$. Moreover, for any $\lambda > \kappa$ and $p \in (1, \frac{\sqrt{\lambda}}{\sqrt{\kappa}})$, there exists $k = k(p, \lambda) > 0$ such that* (5.6) *holds.*

Proof. Since (1.3) implies the hyperboundedness of P_t^0, the existence of invariant probability measure $\nu = \rho\mu$ is ensured by Theorem 4.1, which is the weak limit of a subsequence of $\{\nu_n\}_{n\geq 1}$ in (4.3). Below we first prove the assertion on the marginal density, then prove $\rho \in L^p(\mu)$ for all $p > 1$.

(a) The marginal density. Let $\theta \in [-\tau, 0]$. Since ν is S_t^Z-invariant, for any $f \in \mathscr{B}_b(\mathbb{R}^d)$ and $f_\theta(\xi) := f(\xi(\theta))$, we have

$$
\nu_\theta(f) = \bar{\mu}(f_\theta) = \nu(P_{-\theta}^Z f_\theta) = \int_{\mathscr{C}} \mathbb{E}\big[f_\theta(X_{-\theta}^\xi)R^\xi(-\theta)\big]\nu(\mathrm{d}\xi)
$$

$$= \int_{\mathscr{C}} f(X^\xi(-\theta + \theta))\nu(\mathrm{d}\xi) = \int_{\mathscr{C}} f(\xi(0))\nu(\mathrm{d}\xi) = \nu_0(f).$$

Therefore, $\nu_\theta = \nu_0$.

Moreover, for any $f \in C_0^\infty(\mathbb{R}^d)$ and $f_0(\xi) := f(\xi(0))$, we have

$$\mathbb{E}\big[f(X^\nu(t))R^\nu(t)\big] = \nu(S_t^Z f_0) = \nu(f_0) = \nu_0(f), \quad t \geq 0.$$

On the other hand, let $Z|_{\xi(0)} = \nu(Z|\xi(0))$ be the regular conditional expectation of Z under probability ν given $\xi(0)$. By Itô's formula,

$$\mathbb{E}\big[f(X^\nu(t))R^\nu(t)\big] - \nu_0(f) = \int_0^t \mathbb{E}\big[(L_{Z|_{\xi(0)}}f)(X^\nu(s))R^\nu(s)\big]\mathrm{d}s = \int_0^t \nu_0(L_{Z|_{\xi(0)}}f)\mathrm{d}s.$$

Therefore, $\nu_0(L_{Z|_{\xi(0)}}f) = 0$ for all $f \in C_0^\infty(\mathbb{R}^d)$, i.e. $L^*_{Z|_{\xi(0)}}\nu_0 = 0$. We then finish the proof by considering the following two situations.

(i) Z is supported on a bounded subset of $\mathscr{C}$. Then $Z|_{\xi(0)}$ has compact support. Since by Jensen's inequality

$$\mu_0(\mathrm{e}^{\lambda|Z|_{\xi(0)}|^2}) = \mu\big(\mathrm{e}^{\lambda|\mu(Z|\xi(0))|^2}\big) \leq \mu\big[\mu(\mathrm{e}^{\lambda|Z|^2}|\xi(0))\big] = \mu(\mathrm{e}^{\lambda|Z|^2}) < \infty,$$

the desired assertion on ρ_0 follows from Lemma 5.4.

(ii) In general, let $Z^{\langle n \rangle} = Z1_{\{|\cdot| \leq n\}}$ for $n \geq 1$. Then for every $n \geq 1$, $S_t^{Z^{\langle n \rangle}}$ has an invariant probability measure $\nu^{\langle n \rangle} = \rho^{\langle n \rangle}\mu$ with the marginal density $\rho_0^{\langle n \rangle}$ satisfying (5.6) in place of ρ. As shown in the proof of Theorem 4.1 that up to a subsequence $\rho^{\langle n \rangle} \to \rho$ weakly in $L^1(\mu)$, $\nu := \rho\mu$ is an invariant probability measure of S_t^Z, and ρ_0 satisfies (5.6). Note that by Lemma 5.3, ρ_0 has a continuous, strictly positive version, so that the Poincaré inequality (5.4) implies $\log \rho_0 \in L^2(\mu_0)$, see step (a) in the proof of [33, Theorem 2.3(1)] for details. Therefore, after proving $\rho \in L^p(\mu)$ (hence, $\rho_0 \in L^p(\mu_0)$) in the next step, we conclude that $\log \rho_0, \rho_0^{\frac{p}{2}} \in H_\sigma^{1,2}(\mu_0)$.

(b) $\rho \in L^p(\mu)$ for $p > 1$. Let $f \geq 0$ with $\mu(f^{\frac{p}{p-1}}) \leq 1$. Since Z is bounded, for any $\alpha > 1$ there exists a constant $c(\alpha) > 0$ such that $\mathbb{E}[(R^\nu(\tau))^\alpha] \leq c(\alpha) < \infty$. Let $q \in (1, \frac{p}{p-1})$. Combining this with (1.9) and (2.3) and using $\rho_0 \in L^\alpha(\mu_0)$ for any $\alpha > 1$, we obtain

$$\nu(f) = \nu(S_\tau^Z f) = \mathbb{E}[f(X_\tau^\nu)R^\nu(\tau)] \leq c_1\big\{\mathbb{E}[f^q(X_\tau^\nu)]\big\}^{\frac{1}{q}}$$

(5.19)
$$= c_1\big\{\mu_0(\rho_0 S_\tau f^q)\big\}^{\frac{1}{q}} \leq c_1\big\{\mu_0\big((S_\tau f^q)^{\frac{p}{q(p-1)}}\big)\big\}^{\frac{p-1}{p}}\big\{\mu_0(\rho_0^{\frac{p}{p-(p-1)q}})\big\}^{\frac{p-(p-1)q}{pq}}$$

$$\leq c_2\big\{\mu(f^{\frac{p}{p-1}})\big\}^{\frac{p-1}{p}}, \quad f \in \mathscr{B}^+(\mathscr{C})$$

for some constants $c_1, c_2 > 0$. Therefore, $\rho \in L^p(\mu)$. □

Proof of Theorem 5.1. By Hörmander's theorem, **(H)** implies that P_t^0 has a strictly positive density with respect to μ_0. So, the uniqueness of ν follows from Theorem 3.1(1). Below, we prove the existence and assertions (1) and (2).

(a) We first assume that Z is bounded. By Lemma 5.5, it remains to prove (5.5) for $p < p_\lambda := \frac{q_\lambda}{q_\lambda - 1}$ but close enough to p_λ. Since $\lambda > \lambda_{\kappa,\tau}$, we have

$$p_3 := \frac{1}{4}\left(1 + \sqrt{1 + 8\lambda/\tau}\right) > \frac{4\sqrt{\lambda}}{\sqrt{\lambda} - \sqrt{\kappa}}$$

and

$$q_\lambda = \frac{1}{2}\left(p_3 - \sqrt{p_3^2 - \frac{4p_3\sqrt{\lambda}}{\sqrt{\lambda} - \sqrt{\kappa}}}\right) < \frac{p_3}{2}.$$

Since $\frac{p_\lambda}{p_\lambda - 1} = q_\lambda$, when $p \in (1, p_\lambda)$ is close enough to p_λ we have

$$q_\lambda < \frac{p}{p - 1} < \frac{1}{2}\left(p_3 + \sqrt{p_3^2 - \frac{4p_3\sqrt{\lambda}}{\sqrt{\lambda} - \sqrt{\kappa}}}\right) < p_3,$$

so that

(5.20)
$$\left(\frac{p}{p - 1}\right)^2 - \frac{p_3 p}{p - 1} + \frac{p_3\sqrt{\lambda}}{\sqrt{\lambda} - \sqrt{\kappa}} < 0$$

and

$$p_1 := p_3(p - 1) > p, \quad p_2 := \frac{p_1}{p_1 - p} = \frac{p_3(p - 1)}{p_3(p - 1) - p} > 1.$$

It is easy to see that (5.20) is equivalent to

$$\frac{\frac{\sqrt{\lambda}}{\sqrt{\kappa}} - 1}{p_2\frac{\sqrt{\lambda}}{\sqrt{\kappa}}} = \frac{\sqrt{\lambda} - \sqrt{\kappa}}{\sqrt{\lambda}}\left(1 - \frac{p}{p_3(p - 1)}\right) > \frac{p - 1}{p}.$$

Then there exists $\theta \in (1, \frac{\sqrt{\lambda}}{\sqrt{\kappa}})$ depending on p, λ such that $\frac{\theta - 1}{p_2\theta} = \frac{p - 1}{p}$. By (1.9) and Lemma 5.5, there exists a constant $k = k(p, \lambda)$ such that

(5.21)
$$\int_{\mathscr{C}} \rho(\xi)\mathbb{E}[f^{p_2}(X_\tau^\xi)]\mu(\mathrm{d}\xi) = \mu_0(\rho_0 S_\tau f^{p_2}) \leq \{\mu_0(\rho_0^\theta)\}^{\frac{1}{\theta}}\{\mu_0(S_\tau f^{\frac{p_2\theta}{\theta - 1}})\}^{\frac{\theta - 1}{\theta}}$$

$$\leq \{k\mu(\mathrm{e}^{\lambda|Z|^2})^k\}^{p_2}\{\mu(f^{\frac{p_2\theta}{\theta - 1}})\}^{\frac{\theta - 1}{\theta}} \leq \{k\mu(\mathrm{e}^{\lambda|Z|^2})^k\}^{p_2}, \quad f \geq 0, \mu(f^{\frac{p}{p - 1}}) \leq 1.$$

Noting that

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1 + p_3(p - 1) - p + p - 1}{p_3(p - 1)} = 1,$$

by Hölder's inequality and (5.21), we obtain

$$\nu(f) = \nu(P_\tau^Z f) = \int_{\mathscr{C}} \rho(\xi)\mathbb{E}[f(X_\tau^\xi)R^\xi(\tau)]\mu(\mathrm{d}\xi)$$

(5.22)
$$\leq \mu(\rho^p)^{\frac{1}{p_1}}\left(\int_{\mathscr{C}} \rho(\xi)\mathbb{E}[f^{p_2}(X_\tau^\xi)]\mu(\mathrm{d}\xi)\right)^{\frac{1}{p_2}}\left(\mathbb{E}[(R^\mu(\tau))^{p_3}]\right)^{\frac{1}{p_3}}$$

$$\leq k\mu(\mathrm{e}^{\lambda|Z|^2})^k\mu(\rho^p)^{\frac{1}{p_1}}\left(\mathbb{E}[(R^\mu(\tau))^{p_3}]\right)^{\frac{1}{p_3}}, \quad f \geq 0, \mu(f^{\frac{p}{p - 1}}) \leq 1.$$

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To estimate $\mathbb{E}[(R^\mu(\tau))^{p_3}]$, recall that for any continuous martingale $M(t)$ we have

$$\mathbb{E}\mathrm{e}^{p_3 M(\tau) - \frac{p_3}{2}\langle M\rangle(\tau)} \leq \left(\mathbb{E}\mathrm{e}^{2p_3 M(\tau) - 2p_3^2\langle M\rangle(\tau)}\right)^{\frac{1}{2}}\left(\mathbb{E}\mathrm{e}^{p_3(2p_3-1)\langle M\rangle(\tau)}\right)^{\frac{1}{2}} \leq \left(\mathbb{E}\mathrm{e}^{p_3(2p_3-1)\langle M\rangle(\tau)}\right)^{\frac{1}{2}}.$$

Taking $M(t) = \int_0^t \langle Z(X_s^\mu), \mathrm{d}W(s)\rangle$ and noting that $p_3(2p_3 - 1)\tau = \lambda$ by the definition of p_3, we obtain

$$\left(\mathbb{E}(R^\mu(\tau))^{p_3}\right)^2 \leq \mathbb{E}\left[\mathrm{e}^{p_3(2p_3-1)\int_0^\tau |Z(X_s^\mu)|^2 \mathrm{d}s}\right]$$
$$\leq \frac{1}{\tau}\int_0^\tau \mathbb{E}\left[\mathrm{e}^{p_3(2p_3-1)\tau|Z(X_s^\mu)|^2}\right]\mathrm{d}s = \mu(\mathrm{e}^{\lambda|Z|^2}).$$

Combining this with (5.22), we arrive at

$$\mu(\rho f) = \nu(f) \leq \mu(\rho^p)^{\frac{1}{p_1}} k\mu(\mathrm{e}^{\lambda|Z|^2})^k, \quad f \geq 0, \mu(f^{\frac{p}{p-1}}) \leq 1$$

for some constant $k = k(p, \lambda)$. Since $p_1 > p$ and $\rho \in L^p(\mu)$ due to Lemma 5.5, this implies the desired estimate (5.5).

(b) In general, for any $n \geq 1$, let $Z^{\langle n\rangle} = Z1_{\{|Z| \leq n\}}$. Then $S_t^{Z^{\langle n\rangle}}$ has an invariant probability measure $\nu^{\langle n\rangle} := \rho^{\langle n\rangle}\mu$ such that (5.5) and (5.6) hold for $\rho^{\langle n\rangle}$ and $\rho_0^{\langle n\rangle}$ replacing ρ and ρ_0 with constants independent of n. In particular, $\rho^{\langle n\rangle}$ converges weakly in $L^1(\mu)$ to some ρ, and as shown in the proof of Theorem 4.1 that $\nu := \rho\mu$ is an invariant probability measure of S_t^Z satisfying (5.5) and (5.6). Moreover, applying Lemma 5.3 to the marginal distribution ν_0 (recall that $L_{Z|_{\xi(0)}}^* \nu_0 = 0$), we conclude that ρ_0 has a continuous, strictly positive version.

$\square$

5.4 The infinite-dimensional case

By finite-dimensional approximations, it is easy to extend Theorem 5.1 to the infinite-dimensional case. For simplicity, here we only consider an Ornstein-Uhlenbeck type reference process on $\mathbb{H}$.

Let $W(t)$ be the cylindrical Brownian motion on $\mathbb{H}$, and let L, σ be self-adjoint operators such that for some orthonormal basis $\{e_i\}_{i \geq 1}$ of $\mathbb{H}$

$$Le_i = \lambda_i e_i, \quad \sigma e_i = q_i e_i, \quad i \geq 1$$

holds for some constants λ_i, q_i satisfying

$$\lambda_1 = \inf_{i \geq 1}\lambda_i > 0, \quad \inf_{i \geq 1}q_i^2 > 0, \quad \sum_{i \geq 1}\frac{q_i^2}{\lambda_i^\delta} < \infty \text{ for some } \delta \in (0, 1).$$

Then for any initial point, the SDE

(5.23)
$$\mathrm{d}X(t) = -LX(t)\mathrm{d}t + \sigma\mathrm{d}W(t)$$

has a unique continuous mild solution, and the associated Markov semigroup P_t^0 is symmetric in $L^2(\mu_0)$ for μ_0 being the centered Gauss measure on $\mathbb{H}$ of covariance operator Q with $Qe_i := \frac{q_i^2}{2\lambda_i}e_i$, $i \geq 1$. When $\tau > 0$, let μ be the distribution of $X_\tau^{\mu_0}$ as introduced in Section 1.

Next, according to [10], we have the following log-Sobolev inequality

$$(5.24) \qquad \mu_0(f^2 \log f^2) \le \frac{1}{\lambda_1} \mu_0(|\sigma^* \nabla f|^2), \quad f \in \mathscr{F}C_0^\infty, \mu_0(f^2) = 1,$$

where $\mathscr{F}C_0^\infty := \{x \mapsto f(\langle x, e_1 \rangle, \cdots, \langle x, e_n \rangle) : n \ge 1, f \in C_0^\infty(\mathbb{R}^n)\}$ is the class of smooth cylindrical functions.

Below we extend Theorem 5.1 to the SDE

$$(5.25) \qquad \mathrm{d}X(t) = \{\sigma Z(X_t) - LX(t)\}\mathrm{d}t + \sigma \mathrm{d}W(t)$$

by using finite-dimensional approximations.

For any $n \ge 1$, let $\pi_n : \mathbb{H} \to \mathbb{H}_n := \mathrm{span}\{e_1, \cdots, e_n\}$ be the orthogonal projection. Then $\pi_n X(t)$ is a Markov process on $\mathbb{H}_n$ which is symmetric with respect to $\mu_0^{(n)} := \mu_0 \circ \pi_n^{-1}$, and (5.24) implies the same log-Sobolev inequality for $\mu_0^{(n)}$ on $\mathbb{H}_n$ replacing μ_0 on $\mathbb{H}$. Let $Z : \mathscr{C} \to \mathbb{H}$ be measurable satisfying conditions in Theorem 5.1. Then $Z_n = Z|_{\mathscr{C}_n}$ also satisfies these conditions, where $\mathscr{C}_n := C([-\tau, 0]; \mathbb{H}_n) \subset \mathscr{C}$. So, letting $\mu^{(n)}$ be the marginal distribution of μ on $\mathscr{C}_n$, the corresponding finite-dimensional Markov semigroup $S_t^{Z_n}$ has an invariant probability measure $\nu^{(n)} = \rho^{(n)} \mu^{(n)}$ with $\rho^{(n)}$ and $\rho_0^{(n)}$ satisfying (5.5) and (5.6) respectively. Thus, up to a subsequence, $\rho^{(n)} \circ \pi_n \to \rho$ weakly in $L^p(\mu)$, and $(S_t^{Z_n} f) \circ \pi_n \to S_t^Z f$ in $L^{\frac{p}{p-1}}(\mu)$ for any bounded cylindrical function f on $\mathbb{H}$. Therefore, $\nu := \rho\mu \in \mathscr{P}_Z$ with ρ and ρ_0 satisfying (5.5) and (5.6) respectively.

Moreover, let P_t^0 be the Markov semigroup of the linear equation (5.23). According to e.g. [29, Theorem 3.2.1], P_t^0 satisfies the following Harnack inequality for some constant $C > 0$:

$$(P_t^0 f(x))^p \le (P_t f^p(y)) \exp\left[\frac{Cp|x - y|^2}{(p - 1)t}\right], \quad t > 0, p > 1, f \in \mathscr{B}_b^+(\mathbb{H}).$$

By [29, Theorem 1.4.1], this implies that P_t^0 has a strictly positive density with respect to μ_0. Therefore, by Theorem 3.1, $\nu \in \mathscr{P}_Z$ is unique, and the density ρ has a strictly positive version. In particular, the marginal density ρ_0 has a strictly positive version as well. This together with the Poincaré inequality (5.4) implies $\log \rho_0 \in L^2(\mu_0)$, see step (a) in the proof of [33, Theorem 2.3(1)].

In conclusion, we have the following result, where $\lambda_{\kappa,\tau}$ and q_λ are given before Theorem 5.1.

Theorem 5.6. *In the above framework, let $\kappa = \frac{1}{\lambda_1}$. If $\mu(e^{\lambda|Z|^2}) < \infty$ for some $\lambda > \lambda_{\kappa,\tau}$, the $\mathscr{P}_Z$ contains a unique measure $\nu = \rho\mu$, where ρ is strictly positive such that (5.5) and (5.6) hold for any $p \in (1, \frac{q_\lambda}{q_\lambda - 1})$ and some constant $k = k(\lambda, p) > 0$.*

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