Regularity of stochastic nonlocal diffusion equations

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Abstract

In this paper, we are concerned with regularity of nonlocal stochastic partial differential equations of parabolic type. By using Companato estimates and Sobolev embedding theorem, we first show the Hölder continuity (locally in the whole state space $\mathbb{R}^d$) for mild solutions of stochastic nonlocal diffusion equations in the sense that the solutions u belong to the space $C^\gamma(D_T; L^p(\Omega))$ with the optimal Hölder continuity index γ (which is given explicitly), where $D_T := [0,T] \times D$ for $T > 0$, and $D \subset \mathbb{R}^d$ being a bounded domain. Then, by utilising tail estimates, we are able to obtain the estimates of mild solutions in $L^p(\Omega; C^{\gamma^*}(D_T))$. What's more, we give an explicit formula between the two index γ and γ^*. Moreover, we prove Hölder continuity for mild solutions on bounded domains. Finally, we present a new criteria to justify Hölder continuity for the solutions on bounded domains. The novelty of this paper is that our method are suitable to the case of time-space white noise.

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1 Introduction

Given $T > 0$ and $D \subset \mathbb{R}^d$, let $D_T := [0,T] \times D$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a given filtered probability space. In our previous paper [19], we obtained regularity of singular stochastic integrals in the following space

$$\mathscr{L}^{p,\theta}((D_T;\delta); L^p(\Omega))$$

for $p > 1, \theta > 0, \delta > 0$. Further, by virtue of the celebrated Sobolev embedding theorem $\mathscr{L}^{p,\theta}(D;\delta) \hookrightarrow C^\gamma(\bar{D};\delta)$ for $\theta > 1$, we succeeded in obtaining estimates of solutions in the Hölder space

$$C^\gamma(D_T; L^p(\Omega)),$$

where $\gamma = \frac{(d+2)(\theta-1)}{p}$. In the present paper, we aim to obtain the estimates of solutions in the space

$$L^p(\Omega; C^\gamma(D_T)).$$

The fundamental difficulty is the fact that usually

$$\mathbb{E} \sup_{t,x} \neq \sup_{t,x} \mathbb{E}.$$

In this paper, we are going to use the tail estimates to overcome the above mentioned difficulty. The idea is fairly easy to explicate. In fact, note that

$$\begin{aligned} \mathbb{E}(|X|^p) &= \int_\Omega |X|^p d\mathbb{P}(\omega) \\ &= p\int_0^\infty \mathbb{P}\{|X| > a\} a^{p-1} da \\ &= p\int_0^M \mathbb{P}\{|X| > a\} a^{p-1} da + p\int_M^\infty \mathbb{P}\{|X| > a\} a^{p-1} da \\ &\leq M^p + p\int_M^\infty \mathbb{P}\{|X| > a\} a^{p-1} da \end{aligned}$$

for any arbitrarily fixed constant $M > 0$. In order to obtain the L^p-boundedness, by the above inequality, we only need to show that the second integral is bounded. Further, by utilising Chebyshev's inequality, one can derive the desired results by means of the estimates in $\mathscr{L}^{p,\theta}((D_T;\delta); L^p(\Omega))$.

Let us recall some regularity results about stochastic partial differential equations (SPDEs). The earliest results about the L_p-theory of SPDEs appeared in the works of Krylov [17, 18]. Recently, Kim-Kim [11] considered the L_p-theory for SPDEs driven by Lévy processes, also see [5, 12, 14, 15]. Zhang [24] obtained the L_p-theory of semi-linear SPDEs on general measure spaces. Let us also mention Zhang [25] where very interestingly L_p-maximal regularity of (deterministic) nonlocal parabolic PDEs and Krylov estimate for SDEs driven by Cauchy processes are proved.

The Hölder estimate of SPDEs has been studied by many authors. Let us mention a few. Hsu-Wang-Wang [8] established the stochastic De Giorgi iteration and regularity of semilinear SPDEs. Du-Liu [6] obtained the Schauder estimate for SPDEs. Combining the deterministic theory and convolution properties, Debussche-de Moor-Hofmanová [4] established the regularity result for quasilinear SPDEs of parabolic type. Kuksin-Nadirashvili-Piatnitski [16] obtained Hölder estimates for solutions of parabolic SPDEs on bounded domains. Most recently, Tian-Ding-Wei [22] derived the local Hölder estimates of mild solutions of stochastic nonlocal diffusion equations by using tail estimates [16]. The results on Hölder estimate of PDEs with time-space white noise are few. Fortunately, our method is suitable the time-space white case.

There are two methods to deal with the Schauder estimate for SPDEs. One is using the smooth property of kernel, the other is using the iteration technique. In this paper, we use the Morrey-Campanato estimates and tail estimates to obtain the desired results. The advantage of Morrey-Campanato estimates is to use the properties of kernel function and Sobolev embedding theorem. Comparing with other methods to obtain the Hölder estimate, it is clear that this method is relatively simple.

The rest of this paper is organized as follows. Section 2 presents some preliminaries. In section 3, we state and prove our main results on Hölder estimate over the whole spatial space. Section 4 is concerned with Hölder estimate on bounded domains. Section 5 is devoted to some applications of our main results.

2 Preliminaries

Set, for $X = (t, x) \in \mathbb{R} \times \mathbb{R}^d$ and $Y = (s, y) \in \mathbb{R} \times \mathbb{R}^d$, the following

$$\delta(X,Y) := \max\left\{|x-y|,\, |t-s|^{\frac{1}{2}}\right\}.$$

Let $Q_c(X)$ be the ball centered in $X = (t, x)$ with radius $c > 0$, i.e.,

$$Q_c(X) := \{Y = (s,y) \in \mathbb{R} \times \mathbb{R}^d : \delta(X,Y) < R\} = (t-c^2, t+c^2) \times B_c(x).$$

Fix $T \in (0, \infty)$ arbitrarily. Denote

$$\mathcal{O}_T := (0,T) \times \mathbb{R}^d.$$

For a bounded domain $D \subset \mathbb{R}^d$, we denote $D_T := [0,T] \times D$. For a point $X \in D_T, D(X,r) := D_T \cap Q_r(X)$ and $d(D) := diam(D)$ (that is, the diameter of D). Let us first give the definition of Campanato space.

Definition 2.1 *(Campanato Space) Let $p \geq 1$ and $\theta \geq 0$. The Campanato space $\mathscr{L}^{p,\theta}(D;\delta)$ is a subspace of $L^p(D)$ such that*

$$[u]_{\mathscr{L}^{p,\theta}(D_T;\delta)} := \left(\sup_{X \in D_T, d(D) \geq \rho > 0} \frac{1}{|D(X,\rho)|^\theta} \int_{D(X,\rho)} |u(Y) - u_{X,\rho}|^p dY\right)^{1/p} < \infty, \ u \in L^p(D_T)$$

where $|D(X,\rho)|$ stands for the Lebesgue measure of the Borel set $D(X,\rho)$ and

$$u_{X,\rho} := \frac{1}{|D(X,\rho)|} \int_{D(X,\rho)} u(Y)dY.$$

For $u \in \mathscr{L}^{p,\theta}(D_T;\delta)$, we define

$$\|u\|_{\mathscr{L}^{p,\theta}(D_T;\delta)} := \left(\|u\|^p_{L^p(D_T)} + [u]^p_{\mathscr{L}^{p,\theta}(D_T;\delta)}\right)^{1/p}.$$

Next, we recall the definition of Hölder space.

Definition 2.2 *(Hölder Space) Let $0 < \alpha \leq 1$. A function u belongs to the Hölder space $C^\alpha(\bar{D}_T;\delta)$ if u satisfies the following condition*

$$[u]_{C^\alpha(\bar{D}_T;\delta)} := \sup_{X \in D_T, d(D) \geq \rho > 0} \frac{|u(X) - u(Y)|}{\delta(X,Y)^\alpha} < \infty.$$

For $u \in C^\alpha(\bar{D}_T;\delta)$, we define

$$\|u\|_{C^\alpha(\bar{D}_T;\delta)} := \sup_{D_T} |u| + [u]_{C^\alpha(\bar{D}_T;\delta)}.$$

Definition 2.3 *Let $D_T \subset \mathbb{R}^{d+1}$ be a domain. We call the domain D_T an A-type domain if there exists a constant $A > 0$ such that $\forall X \in D_T$ and $\forall\, 0 < \rho \leq d(D)$, it holds that*

$$|D_T(X,\rho)| = |D_T \cap Q_\rho(X)| \geq A|Q_\rho(X)|.$$

Recall that given two sets B_1 and B_2, the relation $B_1 \cong B_2$ means that both $B_1 \subseteq B_2$ and $B_2 \subseteq B_1$ hold. The notation $f(x) \approx g(x)$ means that there is a number $0 < C < \infty$ independent of x, i.e. a constant, such that for every x we have $C^{-1}f(x) \leq g(x) \leq Cf(x)$. We have then the following relation of the comparison of the two spaces defined above

Proposition 2.1 *Assume that D_T is an A-type bounded domain. Then, for $p \geq 1$ and for $1 < \theta \leq 1 + \frac{p}{d+2}$ (Recall that d is the dimension of the space),*

$$\mathscr{L}^{p,\theta}(D_T; \delta) \cong C^{\gamma}(\bar{D}_T; \delta)$$

with

$$\gamma = \frac{(d+2)(\theta-1)}{p}.$$

We want to use the tail estimate to deive the following boundedness results

$$\mathbb{E}\|u\|^p_{C^{\gamma}([0,T]\times D)} \leq C, \quad \forall p \geq 1$$

for solutions u of SPDEs. To this end, we need the following

Proposition 2.2 *[22, Lemma 2.1] Let $\rho_0 \in L^p(\mathbb{R}^d \times \Omega)$. Consider the Cauchy problem*

$$\partial_t \rho(t,x) = \Delta^{\alpha} \rho(t,x), \quad t > 0, \ x \in \mathbb{R}^d; \quad \rho(0,x) = \rho_0(x). \tag{2.1}$$

Then, for any $0 < \beta < 1$, the following estimates for the unique mild solution of (2.1)

$$\|\rho(t,\cdot)\|_{C^{\beta}(\mathbb{R}^d)} \leq C t^{-\frac{\beta}{2\alpha} - \frac{d}{2p\alpha}} \|\rho_0\|_{L^p(\mathbb{R}^d)}, \quad \mathbb{P} - a.s.\ \omega \in \Omega, \tag{2.2}$$

and

$$|\rho(t+\delta,x) - \rho(t,x)| \leq C t^{-\beta - \frac{d}{2p\alpha}} \|\rho_0\|_{L^p(\mathbb{R}^d)}, \quad \mathbb{P} - a.s.\ \omega \in \Omega. \tag{2.3}$$

We end this section with the following properties of kernel function K satisfying $K_t = \Delta^{\alpha} K$ (the reader is referred to [1, 2, 3, 9] for more details)

- for any $t > 0$,

$$\|K(t,\cdot)\|_{L^1(\mathbb{R}^d)} = 1 \text{ for all } t > 0.$$

- $K(t,x,y)$ is C^{∞} on $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$ for each $t > 0$;
- for $t > 0$, $x, y \in \mathbb{R}^d$, $x \neq y$, the sharp estimate of $K(t,x)$ is

$$K(t,x,y) \approx \min\left(\frac{t}{|x-y|^{d+2\alpha}}, t^{-d/(2\alpha)}\right);$$

- for $t > 0$, $x, y \in \mathbb{R}^d$, $x \neq y$, the estimate of the first order derivative of $K(t,x)$ is

$$|\nabla_x K(t,x,y)| \approx |y-x| \min\left\{\frac{t}{|y-x|^{d+2+2\alpha}}, t^{-\frac{d+2}{2\alpha}}\right\}. \tag{2.4}$$

The estimate (2.4) for the first order derivative of $K(t,x)$ was derived in [1, Lemma 5]. Xie et al. [23] obtained the estimate of the m-th order derivative of $p(t,x)$ by induction.

Proposition 2.3 [23, Lemma 2.1] *For any $m \geq 0$, we have*

$$\partial_x^m K(t,x) = \sum_{n=0}^{n=\lfloor \frac{m}{2} \rfloor} C_n |x|^{m-2n} \min\left\{\frac{t}{|x|^{d+2\alpha+2(m-n)}}, t^{-\frac{d+2(m-n)}{2\alpha}}\right\},$$

where $\lfloor \frac{m}{2} \rfloor$ means the largest integer that is less than $\frac{m}{2}$.

3 Hölder estimate locally over the whole spatial space

In this section, we establish the Morrey-Campanato estimates under different assumption on stochastic term. Set

$$\mathcal{K}g(t,x) := \int_0^t \int_{\mathbb{R}^d} K(t-r,y)g(r,x-y)dydW(r).$$

The first result is similar to the deterministic case. We consider the following equation

$$du_t = \Delta^\alpha u dt + g(t,x)dW_t, \quad u|_{t=0} = 0, \tag{3.1}$$

where $\Delta^\alpha = -(-\Delta)^\alpha$ and W_t is a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$.

Theorem 3.1 *Let D be an A-type bounded domain in $\mathbb{R}^{d+1}$ such that $\bar{D} \subset \mathcal{O}_T$. Suppose that $g \in L^\infty_{loc}(\mathbb{R}_+; L^p(\Omega \times \mathbb{R}^d))$ for $p > d/\alpha$ is $\mathcal{F}_t$-adapted process, and that $0 < \beta < \alpha$ satisfies $(\alpha-\beta)p - d \geq 0$. Then, there is a mild solution u of (3.1) and $u \in \mathscr{L}^{p,\theta}((D_T;\delta); L^p(\Omega)) \cap L^p(\Omega; C^\beta(D_T))$. Moreover, it holds that*

$$\|u\|_{\mathscr{L}^{p,\theta}((D_T;\delta);L^p(\Omega))} \leq C\|g\|_{L^\infty([0,T];L^p(\Omega\times\mathbb{R}^d))}, \tag{3.2}$$

$$\|u\|_{C^\beta(D_T;L^p(\Omega))} \leq C\|g\|_{L^\infty([0,T];L^p(\Omega\times\mathbb{R}^d))}, \tag{3.3}$$

where $\theta = 1 + \frac{\beta p}{d+2}$. Moreover, taking $0 < \delta < \beta p/2$ and $q > (d+2)/\delta$, we have for $0 < r < q$

$$\|u\|_{L^r(\Omega;C^{\beta^*}(D_T))} \leq C\|g\|_{L^\infty([0,T];L^p(\Omega\times\mathbb{R}^d))}, \tag{3.4}$$

where $\beta^* = \beta - 2\delta/p$.

Proof. The existence of mild solution of (3.1) is a classical result under the above assumptions. Now we prove the inequality (3.2). Due to the definition of Companato space, it suffices to show that

$$[u]_{\mathscr{L}^{p,\theta}((D_T;\delta);L^p(\Omega))} < \infty.$$

Direct calculus shows that

$$\begin{aligned}
[u]^p_{\mathscr{L}^{p,\theta}((D_T;\delta);L^p(\Omega))} \quad \leq & \sup_{D(X,c),X\in D_T,0<c\leq d(D)} \frac{1}{|D(X,c)|^{1+\theta}} \\
& \times \mathbb{E}\int_{D(X,c)}\int_{D(X,c)} |u(t,x)-u(s,y)|^p dtdxdsdy \\
\leq & \sup_{D(X,c),X\in D_T,0<c\leq d(D)} \frac{1}{|D(X,c)|^{1+\theta}} \\
& \times \mathbb{E}\int_{D(X,c)}\int_{D(X,c)} \Big| \int_0^t \int_{\mathbb{R}^d} K(t-r,x-z)g(r,z)dzdW(r) \\
& - \int_0^s \int_{\mathbb{R}^d} K(s-r,y-z)g(r,z)dzdW(r)\Big|^p \\
:= & \sup_{D(X,c),X\in D_T,0<c\leq d(D)} \frac{1}{|D(X,c)|^{1+\theta}} \int_{D(X,c)}\int_{D(X,c)} \mathbb{E}\Upsilon dtdxdsdy.
\end{aligned}$$

Set $t\geq s$. We have the following estimates

$$\begin{aligned}
\mathbb{E}\Upsilon &\leq C\mathbb{E}\Big|\int_0^s\int_{\mathbb{R}^d}(K(t-r,x-z)-K(s-r,y-z))g(r,z)dzdW(r)\Big|^p\\
&\quad +C\mathbb{E}\Big|\int_s^t\int_{\mathbb{R}^d}K(t-r,x-z)g(r,z)dzdW(r)\Big|^p\\
&\leq C\mathbb{E}\Big|\int_0^s\Big(\int_{\mathbb{R}^d}(K(t-r,x-z)-K(s-r,y-z))g(r,z)dz\Big)^2dr\Big|^{\frac{p}{2}}\\
&\quad +C\mathbb{E}\Big|\int_s^t\Big(\int_{\mathbb{R}^d}K(t-r,x-z)g(r,z)dz\Big)^2dr\Big|^{\frac{p}{2}}\\
&:= C(H_1+H_2).
\end{aligned}$$

Estimate of H_1.

Take $\beta>0$ satisfying $(\alpha-\beta)p-d\geq 0$. We first recall the following fractional mean value formula (see (4.4) of [10])

$$f(x+h)=f(x)+\Gamma^{-1}(1+\beta)h^{\beta}f^{(\beta)}(x+\theta h),$$

where $0<\beta<1$ and $0\leq\theta\leq 1$ depends on h satisfying

$$\lim_{h\downarrow 0}\theta^{\beta}=\frac{\Gamma^2(1+\beta)}{\Gamma(1+2\beta)},$$

By using the Propositions 2.2 and 2.3, the above fractional mean value formula and Hölder inequality, we have

$$\begin{aligned}
H_1 &= \mathbb{E}\Big|\int_0^s\Big(\int_{\mathbb{R}^d}(K(t-r,x-z)-K(s-r,y-z))g(r,z)dz\Big)^2dr\Big|^{\frac{p}{2}}\\
&\leq C\mathbb{E}\Big|\int_0^s\Big(\int_{\mathbb{R}^d}|K(t-r,x-z)-K(s-r,x-z)|\cdot|g(r,z)|dz\Big)^2dr\Big|^{\frac{p}{2}}\\
&\quad +C\mathbb{E}\Big|\int_0^s\Big(\int_{\mathbb{R}^d}(K(s-r,x-z)-K(s-r,y-z))\cdot g(r,z)dz\Big)^2dr\Big|^{\frac{p}{2}}\\
&\leq C(t-s)^{\frac{\beta p}{2}}\mathbb{E}\Big|\int_0^s\Big(\int_{\mathbb{R}^d}|\frac{\partial^{\frac{\beta}{2}}K}{\partial t^{\frac{\beta}{2}}}(\xi-r,x-z)|^q dz\Big)^{\frac{2}{q}}\|g(r)\|^2_{L^p(\mathbb{R}^d)}dr\Big|^{\frac{p}{2}}\\
&\quad +C|x-y|^{\beta p}\mathbb{E}\Big|\int_0^s(s-r)^{-\frac{\beta}{\alpha}-\frac{d}{p\alpha}}\|g(r)\|^2_{L^p(\mathbb{R}^d)}dr\Big|^{\frac{\delta}{2}}\\
&\leq C(t-s)^{\frac{\beta p}{2}}\|g\|^p_{L^p(\Omega;L^\infty([0,T];L^p(\mathbb{R}^d)))}\left[\int_0^s\Big(\int_{\mathbb{R}^d}|\frac{\partial^{\frac{\beta}{2}}K}{\partial t^{\frac{\beta}{2}}}(\xi-r,x-z)|^q dz\Big)^{\frac{2}{q}}dr\right]^{\frac{p}{2}}\\
&\quad +C|x-y|^{\beta p}\|g\|^p_{L^p(\Omega;L^\infty([0,T];L^p(\mathbb{R}^d)))}\left[\int_0^s(s-r)^{-\frac{\beta}{\alpha}-\frac{d}{p\alpha}}dr\right]^{\frac{p}{2}}\\
&\leq C((t-s)^{\frac{\beta p}{2}}+|x-y|^{\beta p}),
\end{aligned}$$

where $q=p/(p-1)$, $\xi=\theta t+(1-\theta)s$, and we used the following fact

$$\begin{aligned}
&\int_0^s\left(\int_{\mathbb{R}^d}|\frac{\partial^{\frac{\beta}{2}}K}{\partial t^{\frac{\beta}{2}}}(\xi-r,x-z)|^q dz\right)^{\frac{2}{q}}dr\\
\leq\ & C\int_0^s\Bigg(\int_0^{(\xi-r)^{\frac{1}{2\alpha}}}(\xi-r)^{-\frac{dq+2q\alpha\beta}{2\alpha}}|z|^{q\alpha\beta+d-1}d|z|\\
&+\int_{(\xi-r)^{\frac{1}{2\alpha}}}^{\infty}(\xi-r)^q|z|^{-(qd+2q\alpha+2q\alpha\beta)}|z|^{q\alpha\beta+d-1}d|z|\Bigg)^{\frac{2}{q}}dr\\
\leq\ & C\left[(\theta(t-s))^{\frac{d-dq+q\alpha(1-\beta)}{q\alpha}}+\xi^{\frac{d-dq+q\alpha(1-\beta)}{q\alpha}}\right]\\
\leq\ & C
\end{aligned}$$

because using $q=p/(p-1)$, we have

$$d-dq+q\alpha(1-\beta)>0\Leftrightarrow p(\alpha-\alpha\beta)>d\Leftarrow p(\alpha-\beta)>d.$$

Similarly, we have

$$\int_0^s(s-r)^{-\frac{\beta}{\alpha}-\frac{d}{p\alpha}}dr=\frac{p\alpha}{(\alpha-\beta)p-d}s^{\frac{(\alpha-\beta)p-d}{p\alpha}}\leq C$$

provided that $(\alpha-\beta)p-d\geq 0$.

Estimate of H_2.

Similar to the estimate of H_1, we have

$$\begin{aligned}
H_2 &= \mathbb{E}\Big|\int_s^t\left(\int_{\mathbb{R}^d}K(t-r,x-z)g(r,z)dz\right)^2dr\Big|^{\frac{p}{2}}\\
&\leq \|g\|^p_{L^p(\Omega;L^\infty([0,T];L^p(\mathbb{R}^d)))}\left[\int_s^t\left(\int_{\mathbb{R}^d}|K(t-r,x-z)|^q dz\right)^{\frac{2}{q}}dr\right]^{\frac{p}{2}}\\
&\leq C\|g\|^p_{L^p(\Omega;L^\infty([0,T];L^p(\mathbb{R}^d)))}(t-s)^{\frac{q\alpha-(q-1)d}{q\alpha}\times\frac{p}{2}}
\end{aligned}$$

provided that $\alpha p>d$. Indeed, by using $1/p+1/q=1$, we have

$$q\alpha-(q-1)d>0\iff\alpha p>d.$$

Combining the assumption of p, we have

$$H_2\leq C(t-s)^{\frac{p\alpha-d}{2\alpha}}.$$

Assume that $D(X,c)=D_T\cap Q_c$ and $Q_c=Q_c(t_0,x_0)$. Noting that $(t,x)\in Q_c(t_0,x_0)$ and $(s,y)\in Q_c(t_0,x_0)$, we have

$$0\leq t-s\leq 2c^2\quad\text{and}\quad |x-y|\leq|x-x_0|+|y-x_0|\leq 2c.$$

By using the definition of A-type bounded domain, we have

$$\begin{aligned}
[u]_{\mathscr{L}^{p,\theta}((D_T;\delta);L^p(\Omega))} &\leq \sup_{D(X,c),X\in D_T,0<c\leq d(D)}\frac{1}{|D(X,c)|^{1+\theta}}\mathbb{E}\int_{D(X,c)}\int_{D(X,c)}\mathbb{E}\Upsilon dtdxdsdy\\
&\leq C\|g\|^p_{L^p(\Omega;L^\infty([0,T];L^p(\mathbb{R}^d)))},
\end{aligned}$$

where $\theta = 1 + \frac{\beta p}{d+2}$. This yields the inequality (3.2). Applying Proposition 2.1, one can obtain the inequality (3.3).

Next, we prove the inequality (3.4). In order to use the technique of tail estimates, we first consider the following estimates. Let $(t_0, x_0) \in D_T \subset \mathcal{O}_T$ and

$$Q_c(t_0, x_0) = (t_0 - c^2, t_0 + c^2) \times B_c(x_0).$$

Then we have $\bar{D}_T \subset Q_{d(D)}(t_0, x_0)$. Set $(t_1, x_1), (t_2, x_2) \in D_T$, $Q_i := D_T \cap Q_{c_i}(t_i, x_i)$, $i = 1, 2$ and

$$\begin{aligned} F(t_i, x_i, c_i) &= \frac{1}{|Q_i|^{1+\theta}} \int_{Q_i} \int_{Q_i} |u(t,x) - u(s,y)|^p dtdxdsdy \\ &= \frac{1}{|Q_i|^{1+\theta}} \int_{Q_i} \int_{Q_i} |\mathcal{K}g(t,x) - \mathcal{K}g(s,y)|^p dtdxdsdy. \end{aligned}$$

Notice that

$$\begin{aligned} F(t_1, x_1, c_1) - F(t_2, x_2, c_2) &= [F(t_1, x_1, c_1) - F(t_2, x_1, c_1)] \\ &\quad + [F(t_2, x_1, c_1) - F(t_2, x_2, c_1)] \\ &\quad + [F(t_2, x_2, c_1) - F(t_2, x_2, c_2)] \\ &:= I_1 + I_2 + I_3. \end{aligned}$$

Estimate of I_1:

$$\begin{aligned} I_1 &= F(t_1, x_1, c_1) - F(t_2, x_1, c_1) \\ &= \frac{1}{|Q_1|^{1+\theta}} \int_{Q_1} \int_{Q_1} |\mathcal{K}g(t,x) - \mathcal{K}g(s,y)|^p dtdxdsdy \\ &\quad - \frac{1}{|Q_{12}|^{1+\theta}} \int_{Q_{12}} \int_{Q_{12}} |\mathcal{K}g(t,x) - \mathcal{K}g(s,y)|^p dtdxdsdy \\ &= \frac{1}{|Q_1|^{1+\theta}} \left\{ \int_{Q_1 \setminus Q_{12}} \int_{Q_1 \setminus Q_{12}} |\mathcal{K}g(t,x) - \mathcal{K}g(s,y)|^p dtdxdsdy \right. \\ &\quad \left. + \int_{Q_{12} \setminus Q_1} \int_{Q_{12} \setminus Q_1} |\mathcal{K}g(t,x) - \mathcal{K}g(s,y)|^p dtdxdsdy \right\} \\ &\quad + \left[\frac{1}{|Q_1|^{1+\theta}} - \frac{1}{|Q_{12}|^{1+\theta}} \right] \int_{Q_{12}} \int_{Q_{12}} |\mathcal{K}g(t,x) - \mathcal{K}g(s,y)|^p dtdxdsdy \\ &:= I_{11} + I_{12}, \end{aligned}$$

where $Q_{12} = D_T \cap Q_{c_1}(t_2, x_1)$. For simplicity, we assume that $|Q_1| \geq |Q_{12}|$. Otherwise, we can chance the place of Q_1 and Q_{12}. And thus $I_{12} \leq 0$ almost surely. Now, we consider the term I_{11}. Before giving the estimates of I_{11}, we first recall our aim. In order to apply the tail estimate, we want to obtain the estimates of I_{11} like the followings:

$$\mathbb{E} I_{11} \leq C(t_1 - t_2)^{\delta} \quad \text{for some} \quad \delta > 0.$$

It is easy to see that

$$|Q_1 \setminus Q_{12}| \leq C(t_1 - t_2) c_1^d \quad \text{and} \quad |Q_1| \approx C c_1^{d+2}.$$

So we must put some assumption on g in order to get some help from it.

Set $t > s$. Denote

$$\begin{aligned} &\mathbb{E}\int_{Q_1\setminus Q_{12}}\int_{Q_1\setminus Q_{12}} |\mathcal{K}g(t,x)-\mathcal{K}g(s,y)|^p dtdxdsdy \\ = &\mathbb{E}\int_{Q_1\setminus Q_{12}}\int_{Q_1\setminus Q_{12}} \mathbb{E}\Upsilon dtdxdsdy. \end{aligned}$$

Similar to the proof of inequality (3.2), we have

$$\mathbb{E}\Upsilon \leq Cc_1^{\beta p}.$$

Noting that $(t,x)\in Q_1$ and $(s,y)\in Q_1$, we have

$$0 \leq t-s \leq 2{c_1}^2 \quad \text{and} \quad |x-y| \leq |x-x_1|+|y-x_1| \leq 2c_1.$$

Using the above inequalities and the properties of A-type domain, we deduce

$$\begin{aligned} &\mathbb{E}\int_{Q_1\setminus Q_{12}}\int_{Q_1\setminus Q_{12}} \mathbb{E}\Upsilon dtdxdsdy \\ \leq\ & C(p,T)c_1^{\beta p}|Q_1\setminus Q_{12}|^2\|g\|^p_{L^p(\Omega;L^\infty([0,T];L^p(\mathbb{R}^d)))}. \end{aligned}$$

Since D_T is a A-type bounded domain, we have for $2c_1 \leq diam D$,

$$\begin{aligned} &A|Q_{c_1}(t_1,x_1)| \leq |Q_1| \leq |Q_{c_1}(t_1,x_1)| \\ &A|Q_{c_1}(t_1,x_1)\setminus Q_{c_1}(t_2,x_1)| \leq |Q_1\setminus Q_{12}| \leq |Q_{c_1}(t_1,x_1)\setminus Q_{c_1}(t_2,x_1)|. \end{aligned}$$

We remark that

$$\begin{aligned} &|Q_{c_1}(t_1,x_1)| \approx Cc_1^{d+2}, \\ &|Q_{c_1}(t_1,x_1)\setminus Q_{c_1}(t_2,x_1)| \leq Cc_1^d[c_1^2\wedge(t_1-t_2)], \end{aligned}$$

where C is a positive constant which does not depend on c_1. Noting that $Q_1\setminus Q_{12}\subset Q_1$ and taking $0<\delta<\beta p/2$, we have

$$\begin{aligned} &\mathbb{E}\int_{Q_1\setminus Q_{12}}\int_{Q_1\setminus Q_{12}} |\mathcal{K}g(t,x)-\mathcal{K}g(s,y)|^p dtdxdsdy \\ \leq\ & C(C_0,D,d,T)\|g\|^p_{L^p(\Omega;L^\infty([0,T];L^p(\mathbb{R}^d)))}|Q_1|^{2+\frac{\beta p-2\delta}{d+2}}|t_1-t_2|^\delta. \end{aligned}$$

Similarly, we can get

$$\begin{aligned} &\mathbb{E}\int_{Q_{12}\setminus Q_1}\int_{Q_{12}\setminus Q_1} |u(t,x)-u(s,y)|^p dtdxdsdy \\ \leq\ & C(D,d,T)\|g\|^p_{L^p(\Omega;L^\infty([0,T];L^p(\mathbb{R}^d)))}|Q_1|^{2+\frac{\beta p-2\delta}{d+2}}|t_1-t_2|^\delta. \end{aligned}$$

Due to the fact that $I_{12}\leq 0$, we have

$$\mathbb{E}I_1 \leq C(D,d,T)\|g\|^p_{L^p(\Omega;L^\infty([0,T];L^p(\mathbb{R}^d)))}|t_1-t_2|^\delta,$$

where $\theta = 1+\frac{\beta p-2\delta}{d+2}$.

Next, we estimate I_2. By using the fact that

$$\Big|[D\cap Q_{c_1}(t_2,x_1)]\setminus[D\cap Q_{c_1}(t_2,x_2)]\Big| \leq Cc_1^{d+1}|x_1-x_2|,$$

similar to the estimates of I_1, we can take $0 < \delta < \beta p/2$ such that

$$\begin{aligned} \mathbb{E}I_2 &= \mathbb{E}[F(t_2, x_1, c_1) - F(t_2, x_2, c_1)] \\ &\leq C(D, d, T)\|g\|^p_{L^p(\Omega;L^\infty([0,T];L^p(\mathbb{R}^d)))}|x_1 - x_2|^\delta, \end{aligned}$$

where $\theta = 1 + \frac{\beta p - \delta}{d+2}$.

Next, we estimate I_3. By using the fact that

$$\Big|[D \cap Q_{c_1}(t_2, x_2)] \setminus [D \cap Q_{c_2}(t_2, x_2)]\Big| \leq Cc_1^{d+1}(c_1 - c_2), \quad \text{if} \quad c_1 \geq c_2,$$

similar to the estimates of I_1, we can estimate

$$\begin{aligned} \mathbb{E}I_3 &= \mathbb{E}[F(t_2, x_2, c_1) - F(t_2, x_2, c_2)] \\ &\leq C(D, d, T)\|g\|^p_{L^p(\Omega;L^\infty([0,T];L^p(\mathbb{R}^d)))}|c_1 - c_2|^\delta, \end{aligned}$$

where $\theta = 1 + \frac{\beta p - \delta}{d+2}$.

Therefore, we have

$$\begin{aligned} &\mathbb{E}|F(t_1, x_1, c_1) - F(t_2, x_2, c_2)|^q \\ \leq\ & C(D, d, T)\|g\|^{pq}_{L^p(\Omega;L^\infty([0,T];L^p(\mathbb{R}^d)))}(|t_1 - t_2| + |x_1 - x_2| + |c_1 - c_2|)^{\delta q}, \end{aligned}$$

where $\theta = 1 + \frac{\beta p - 2\delta}{d+2}$, $(t_i, x_i) \in D_T$ and $0 < c_i \leq d(D)$, $i = 1, 2$.

For simplicity, we set $D_T = [0, 1]^{d+1}$ and $c \in [0, 2]$. One introduces a sequence of sets:

$$\mathcal{S}_n = \{z \in \mathbb{Z}^{d+2} |\ z2^{-n} \in (0, 1)^{d+1} \times (0, 2)\}, \quad n \in \mathbb{N}.$$

For an arbitrary $e = (e_1, e_2, \cdots, e_{d+2}) \in \mathbb{Z}^{d+2}$ such that

$$|e|_\infty = \max_{1\leq j\leq d+2} |e_j| = 1,$$

and for every $z, z + e \in \mathcal{S}_n$, we define $v_z^{n,e} = |F((z + e)2^{-n}) - F(z2^{-n})|$. From the above discussion, we have

$$\mathbb{E}|v_z^{n,e}|^q \leq C(\beta, C_0, D, d, T)\|g\|^{pq}_{L^p(\Omega;L^\infty([0,T];L^p(\mathbb{R}^d)))}2^{-n\delta q} := \hat{C}2^{-n\delta q}.$$

For any $\tau > 0$ and $K > 0$, one sets a number of events

$$\mathcal{A}^{n,e}_{z,\tau} = \{\omega \in \Omega | v_z^{n,e} \geq K\tau^n,\ z, z + e \in \mathcal{S}_n\},$$

which yields that

$$\mathbb{P}(\mathcal{A}^{n,e}_{z,\tau}) \leq \frac{\mathbb{E}|v_z^{n,e}|^q}{K^q\tau^{qn}} \leq \frac{\hat{C}2^{-n\delta q}}{K^q\tau^{qn}}.$$

Noting that for each n, the total number of the events $\mathcal{A}^{n,e}_{z,\tau}$, $z, z+e \in \mathcal{S}_n$ is not larger than $2^{d+2}3^{d+2}$. Hence the probability of the union

$$\mathcal{A}^n_\tau = \cup_{z,z+e\in\mathcal{S}_n}(\cup_{\|e\|_\infty=1}\mathcal{A}^{n,e}_{z,\tau})$$

meets the estimate

$$\mathbb{P}(\mathcal{A}^n_\tau) \leq \frac{\hat{C}2^{-n\delta q}}{K^q\tau^{qn}}2^{(d+2)n} \leq \hat{C}K^{-q}\left(\frac{2^{d+2}}{(2^\delta\tau)^q}\right)^n.$$

Let $\tau = 2^{-\nu\delta}$, where $\nu > 0$ satisfies $(1-\nu)\delta q \geq d+2$. Then the probability of the event $\mathcal{A} = \cup_{n\geq 1}\mathcal{A}_\tau^n$ can be calculated that

$$\mathbb{P}(\mathcal{A}) \leq C\hat{C}K^{-q}. \tag{3.5}$$

For every point $\xi = (t,x,c) \in (0,1)^{d+1} \times (0,2)$, we have $\xi = \sum_{i=0}^{\infty} e_i 2^{-i}$ ($\|e_i\|_\infty \leq 1$). Denote $\xi_k = \sum_{i=0}^{k} e_i 2^{-i}$ and $\xi_0 = 0$. For any $\omega \notin \mathcal{A}$, we have $|F(\xi_{k+1}) - F(\xi_k)| < K\tau^{k+1}$, which implies that

$$|F(t,x,c)| \leq \sum_{k=0}^{\infty} |F(\xi_{k+1}) - F(\xi_k)| < K\sum_{k=1}^{\infty}\tau^k \leq K(2^{\nu\delta}-1)^{-1}. \tag{3.6}$$

Set $v_1 = \sup_{(t,x,c)\in(0,1)^{d+1}\times(0,2)} |F(t,x,c)|$, then $v_1 = \sup_{(t,x,c)\in[0,1]^{d+1}\times[0,2]} |F(t,x,c)|$ since F has a continuous version. For $0 < r < q$, we have

$$\mathbb{E}v_1^r = r\int_0^\infty a^{r-1}\mathbb{P}(v_1 \geq a)da = r\int_0^{\gamma K} a^{r-1}\mathbb{P}(v_1 \geq a)da + r\int_{\gamma K}^\infty a^{r-1}\mathbb{P}(v_1 \geq a)da. \tag{3.7}$$

If one chooses $\gamma \geq (2^{\nu\delta}-1)^{-1}$, using (3.5), (3.6) and (3.7), we get

$$\begin{aligned} \mathbb{E}v_1^r &\leq (\gamma K)^r + C\hat{C}^q r\int_{\gamma K}^\infty a^{r-1-q}da \\ &\leq (\gamma K)^r + C\hat{C}r(cK)^{r-q}, \end{aligned}$$

which yields that

$$\mathbb{E}v_1^r \leq C(D,d,T)\|g\|^{pr}_{L^p(\Omega;L^\infty([0,T];L^p(\mathbb{R}^d)))}2^{-n\delta q},$$

if we choose $K = \|g\|^p_{L^p(\Omega;L^\infty([0,T];L^p(\mathbb{R}^d)))}$. By using the following embed inequality

$$L^p(\Omega;\mathscr{L}^{p,\theta}(D_T;\delta)) \cong L^p(\Omega;C^\gamma(\bar{D}_T;\delta)),$$

we obtain the inequality (3.4). The proof is complete. □

Remark 3.1 *It follows from Theorem 3.1 that the index β and β^* satisfy $\beta > \beta^*$, which implies that if we want to change the places of $\mathbb{E}$ and $\sup_{t,x}$, we must pay it on the index.*

Comparing with the earlier results of [22] (Tian et al. obtained the Hölder estimate to equation (3.1) locally in $\mathbb{R}^d$), we find the Hölder continuous index in this paper is larger than that in [22]. More precisely, we obtain the index of time variable is closed to $1/2$. Since the index of Hölder continuous of Brownian motion is $\frac{1}{2}-$, maybe the index obtained in this paper is optimal.

Next, we consider another case. If g is a Hölder continuous function, the following theorem shows that what assumptions should be put on the kernel function K.

Theorem 3.2 *Let $u = \mathcal{K}*g$ and D_T be an A-type bounded domain in $\mathbb{R}^{d+1}$ such that $\bar{D}_T \subset \mathcal{O}_T$. Suppose that $g \in C^\beta(\mathbb{R}_+ \times \mathbb{R}^d)$, $0 < \beta < 1$, is a non-random function and $g(0,0) = 0$. Assume that there exists positive constants γ_i ($i = 1,2$) such that the non-random kernel function satisfies that for any $t \in (0,T]$*

$$\int_0^s \left(\int_{\mathbb{R}^d} |K(t-r,z) - K(s-r,z)|(1+|z|^\beta)dz\right)^2 dr \leq C(T,\beta)(t-s)^{\gamma_1}, \tag{3.8}$$

$$\int_0^s \left(\int_{\mathbb{R}^d} |K(s-r,z)|dz\right)^2 dr \leq C_0, \tag{3.9}$$

$$\int_s^t \left(\int_{\mathbb{R}^d} |K(t-r,z)|(1+|z|^\beta)dz\right)^2 dr \leq C(T,\beta)(t-s)^{\gamma_2}, \tag{3.10}$$

where C_0 is a positive constant. Then we have, for $p \geq 1$ and $\beta < \gamma$,

$$\begin{aligned} \|u\|_{\mathscr{L}^{p,\theta}((D_T;\delta);L^p(\Omega))} &\leq C\|g\|_{C^\beta(\mathbb{R}_+\times\mathbb{R}^d))}, \\ \|u\|_{C^\beta(D_T;L^p(\Omega))} &\leq C\|g\|_{C^\beta(\mathbb{R}_+\times\mathbb{R}^d))}, \end{aligned} \tag{3.11}$$

where $\theta = 1 + \frac{\gamma p}{d+2}$ and $\gamma = \min\{\gamma_1, \gamma_2, \beta\}$. Moreover, taking $0 < \delta < \gamma p/2$ and $q > (d+2)/\delta$, we have for $0 < r < q$

$$\|u\|_{L^r(\Omega;C^{\beta^*}(D_T))} \leq C\|g\|_{C^\beta(\mathbb{R}_+\times\mathbb{R}^d))}, \tag{3.12}$$

where $\beta^* = \gamma - 2\delta/p$.

Proof. The proof of the (3.11) is contained in our paper [19]. And we only focus on the proof of (3.12).

Similar to the proof of Theorem 3.1, we need to estimate I_i, $i = 1, 2, 3$. Estimate of I_1:

$$\begin{aligned} I_1 &= F(t_1, x_1, c_1) - F(t_2, x_1, c_1) \\ &= \frac{1}{|Q_1|^{1+\theta}} \int_{Q_1} \int_{Q_1} |\mathcal{K}g(t,x) - \mathcal{K}g(s,y)|^p dtdxdsdy \\ &\quad - \frac{1}{|Q_{12}|^{1+\theta}} \int_{Q_{12}} \int_{Q_{12}} |\mathcal{K}g(t,x) - \mathcal{K}g(s,y)|^p dtdxdsdy \\ &= \frac{1}{|Q_1|^{1+\theta}} \left\{ \int_{Q_1 \setminus Q_{12}} \int_{Q_1 \setminus Q_{12}} |\mathcal{K}g(t,x) - \mathcal{K}g(s,y)|^p dtdxdsdy \right. \\ &\quad \left. + \int_{Q_{12} \setminus Q_1} \int_{Q_{12} \setminus Q_1} |\mathcal{K}g(t,x) - \mathcal{K}g(s,y)|^p dtdxdsdy \right\} \\ &\quad + \left[\frac{1}{|Q_1|^{1+\theta}} - \frac{1}{|Q_{12}|^{1+\theta}} \right] \int_{Q_{12}} \int_{Q_{12}} |\mathcal{K}g(t,x) - \mathcal{K}g(s,y)|^p dtdxdsdy \\ &:= I_{11} + I_{12}, \end{aligned}$$

where $Q_{12} = D \cap Q_{c_1}(t_2, x_1)$. For simplicity, we assume that $|Q_1| \geq |Q_{12}|$. Otherwise, we can chance the place of Q_1 and Q_{12}. And thus $I_{12} \leq 0$ almost surely.

It is easy to see that

$$|Q_1 \setminus Q_{12}| \leq C(t_1 - t_2)c_1^d \ \text{ and } \ |Q_1| \approx C c_1^{d+2}.$$

So we must put some assumption on g in order to get some help from it.

Set $t > s$. By the BDG inequality, we have

$$
\begin{aligned}
& \mathbb{E}\int_{Q_1\backslash Q_{12}}\int_{Q_1\backslash Q_{12}} |\mathcal{K}g(t,x)-\mathcal{K}g(s,y)|^p dtdxdsdy \\
= & \ \mathbb{E}\int_{Q_1\backslash Q_{12}}\int_{Q_1\backslash Q_{12}} \Big|\int_0^t\int_{\mathbb{R}^d} K(t-r,z)g(r,x-z)dzdW(r) \\
& -\int_0^s\int_{\mathbb{R}^d} K(s-r,z)g(r,y-z)dzdW(r)\Big|^p dtdxdsdy \\
\leq & \ 2^{p-1}\mathbb{E}\int_{Q_1\backslash Q_{12}}\int_{Q_1\backslash Q_{12}} \Big|\int_0^s\int_{\mathbb{R}^d}(K(t-r,z)-K(s-r,z))g(r,x-z)dzdW(r)\Big|^p \\
& +2^{p-1}\mathbb{E}\int_{Q_1\backslash Q_{12}}\int_{Q_1\backslash Q_{12}} \Big|\int_0^s\int_{\mathbb{R}^d} K(s-r,z)(g(r,x-z)-g(r,y-z))dzdW(r)\Big|^p \\
& +2^{p-1}\mathbb{E}\int_{Q_1\backslash Q_{12}}\int_{Q_1\backslash Q_{12}} \Big|\int_s^t\int_{\mathbb{R}^d} K(t-r,z)g(r,x-z)dzdW(r)\Big|^p dtdxdsdy \\
\leq & \ C(p)\int_{Q_1\backslash Q_{12}}\int_{Q_1\backslash Q_{12}} \left(\int_0^s \Big|\int_{\mathbb{R}^d} |K(t-r,z)-K(s-r,z)||g(r,x-z)|dz\Big|^2 dr\right)^{\frac{p}{2}} \\
& +C(p)\int_{Q_1\backslash Q_{12}}\int_{Q_1\backslash Q_{12}} \left(\int_0^s \Big|\int_{\mathbb{R}^d} |K(s-r,z)||g(r,x-z)-g(r,y-z)|dz\Big|^2 dr\right)^{\frac{p}{2}} \\
& +C(p)\int_{Q_1\backslash Q_{12}}\int_{Q_1\backslash Q_{12}} \left(\int_s^t \Big|\int_{\mathbb{R}^d} K(t-r,z)g(r,x-z)dz\Big|^2 dr\right)^{\frac{p}{2}} \\
=: & \int_{Q_1\backslash Q_{12}}\int_{Q_1\backslash Q_{12}} (J_1+J_2+J_3)dtdxdsdy.
\end{aligned}
$$

Estimate of J_1. By using the Hölder continuous of g, i.e.,

$$
\begin{aligned}
|g(r,x-z)-g(0,0)| & \leq C_g \max\left\{r^{\frac{1}{2}}, |x-z|\right\}^\beta \\
& \leq C(g,\beta)(T^{\frac{\beta}{2}}+|x-x_1|^\beta+|x_1|^\beta+|z|^\beta) \\
& \leq C(g,\beta)(T^{\frac{\beta}{2}}+c_1^\beta+|x_1|^\beta+|z|^\beta),
\end{aligned}
$$

and (3.8), we have

$$
\begin{aligned}
J_1 & = C(p)\left(\int_0^s \Big|\int_{\mathbb{R}^d} |K(t-r,z)-K(s-r,z)||g(r,x-z)|dz\Big|^2 dr\right)^{\frac{p}{2}} \\
& \leq C(p,\beta,T)\left(\int_0^s \Big|\int_{\mathbb{R}^d} |K(t-r,z)-K(s-r,z)|(1+|z|^\beta)dz\Big|^2 dr\right)^{\frac{p}{2}} \\
& \quad +c_1^{\beta p}C(p,\beta)\left(\int_0^s\int_{\mathbb{R}^d} |K(t-r,z)-K(s-r,z)|dr\right)^{\frac{p}{2}} \\
& \leq C(p,\beta,T)(1+c_1^{\beta p})(t-s)^{\frac{\gamma_1 p}{2}}.
\end{aligned}
$$

Here and in the rest part of the proof, we write the constant depending on $\|g\|_{C^\beta(\mathbb{R}_+\times\mathbb{R}^d))}$ as $C(\beta)$ for simplicity. The condition (3.9) and

$$
|g(r,x-z)-g(r,y-z)| \leq C_g|x-y|^\beta
$$

imply the following derivation

$$\begin{aligned} J_2 &= C(p)\int_Q\int_Q\left(\int_0^s|\int_{\mathbb{R}^d}|K(s-r,z)||g(r,x-z)-g(r,y-z)|dz|^2dr\right)^{\frac{p}{2}} \\ &\le C(p,g)\int_Q\int_Q\left(\int_0^s|\int_{\mathbb{R}^d}|K(r,z)||x-y|^\beta dz|^2dr\right)^{\frac{p}{2}} \\ &\le C(N_0,p,g,\beta)|x-y|^{\beta p}. \end{aligned}$$

Estimate of I_3. By using the property $g(0,0)=0$ and (3.10), we get

$$\begin{aligned} J_3 &= C(p)\left(\int_s^t|\int_{\mathbb{R}^d}K(t-r,z)g(r,x-z)dz|^2dr\right)^{\frac{p}{2}} \\ &\le C\left(\int_s^t\Big|\int_{\mathbb{R}^d}|K(r,z)|(T+|x-x_1|^\beta+|x_1|^\beta+|z|^\beta)dz\Big|^2dr\right)^{\frac{p}{2}} \\ &\le C(p,T,\beta)\left(\int_s^t\Big|\int_{\mathbb{R}^d}|K(t-r,z)|(1+|z|^\beta)dz\Big|^2dr\right)^{\frac{p}{2}} \\ &\quad +C(p,T,\beta)|x-y|^{\beta p}\left(\int_s^t\Big|\int_{\mathbb{R}^d}|K(t-r,z)|dz\Big|^2dr\right)^{\frac{p}{2}} \\ &\le C(p,T,\beta)(t-s)^{\frac{\gamma_2 p}{2}}(1+|x-y|^{\beta p}). \end{aligned}$$

Noting that $(t,x)\in Q_1$ and $(s,y)\in Q_1$, we have

$$0\le t-s\le 2{c_1}^2 \quad \text{and} \quad |x-y|\le|x-x_1|+|y-x_1|\le 2c_1.$$

Using the above inequality and the properties of A-type domain, we deduce

$$\begin{aligned} \int_{Q_1\setminus Q_{12}}\int_{Q_1\setminus Q_{12}}J_1dtdxdsdy &\le C(p,T,\beta)(1+c_1^{\beta p}){c_1}^{\gamma_1 p}|Q_1\setminus Q_{12}|^2; \\ \int_{Q_1\setminus Q_{12}}\int_{Q_1\setminus Q_{12}}J_2dtdxdsdy &\le C(C_0,p,g,\beta){c_1}^{\beta p}|Q_1\setminus Q_{12}|^2; \\ \int_{Q_1\setminus Q_{12}}\int_{Q_1\setminus Q_{12}}J_3dtdxdsdy &\le C(p,T,\beta)|Q_1\setminus Q_{12}|^2{c_1}^{\gamma_2 p}(1+{c_1}^{\beta p}). \end{aligned}$$

Combining the estimates of J_1,J_2 and J_3, we get

$$\begin{aligned} &\mathbb{E}\int_{Q_1\setminus Q_{12}}\int_{Q_1\setminus Q_{12}}|u(t,x)-u(s,y)|^pdtdxdsdy \\ \le\ & C(\beta,C_0,T,p)|Q_1\setminus Q_{12}|^2(c_1^{\beta p}+1)(c_1^{\beta p}+c_1^{\gamma_1 p}+c_1^{\gamma_2 p}). \end{aligned}$$

Since D is a A-type bounded domain, we have for $2c_1\le diamD$,

$$\begin{aligned} &A|Q_{c_1}(t_1,x_1)|\le|Q_1|\le|Q_{c_1}(t_1,x_1)| \\ &A|Q_{c_1}(t_1,x_1)\setminus Q_{c_1}(t_2,x_1)|\le|Q_1\setminus Q_{12}|\le|Q_{c_1}(t_1,x_1)\setminus Q_{c_1}(t_2,x_1)|. \end{aligned}$$

We remark that

$$\begin{aligned} &|Q_{c_1}(t_1,x_1)|\approx Cc_1^{d+2}, \\ &|Q_{c_1}(t_1,x_1)\setminus Q_{c_1}(t_2,x_1)|\le Cc_1^d[c_1^2\wedge(t_1-t_2)], \end{aligned}$$

where C is a positive constant which does not depend on c_1. Noting that $Q_1 \setminus Q_{12} \subset Q_1$ and taking $0 < \delta < 1$, we have

$$\begin{aligned} & \mathbb{E} \int_{Q_1 \setminus Q_{12}} \int_{Q_1 \setminus Q_{12}} |\mathcal{K}g(t,x) - \mathcal{K}g(s,y)|^p dtdxdsdy \\ \leq \; & C(\beta, C_0, D, d, T) |Q_1|^{2+\frac{\gamma p - 2\delta}{d+2}} |t_1 - t_2|^{\delta}, \end{aligned}$$

where $\gamma = \min\{\gamma_1, \gamma_2, \beta\}$.

Similarly, we can get

$$\begin{aligned} & \mathbb{E} \int_{Q_{12} \setminus Q_1} \int_{Q_{12} \setminus Q_1} |u(t,x) - u(s,y)|^p dtdxdsdy \\ \leq \; & C(\beta, C_0, D, d, T) |Q_1|^{2+\frac{\gamma p - 2\delta}{d+2}} |t_1 - t_2|^{\delta}. \end{aligned}$$

Due to the fact that $I_{12} \leq 0$, we have

$$\mathbb{E} I_1 \leq C(\beta, C_0, D, d, T) |t_1 - t_2|^{\delta},$$

where $\theta = 1 + \frac{\gamma p - 2\delta}{d+2}$.

Next, similar to the proof of Theorem 3.1, one can estimate I_2 and I_3 as followings

$$\begin{aligned} \mathbb{E} I_2 &= \mathbb{E}[F(t_2, x_1, c_1) - F(t_2, x_2, c_1)] \leq C(\beta, C_0, D, d, T) |x_1 - x_2|^{\delta}, \\ \mathbb{E} I_3 &= \mathbb{E}[F(t_2, x_2, c_1) - F(t_2, x_2, c_2)] \leq C(\beta, C_0, D, d, T) |c_1 - c_2|^{\delta}, \end{aligned}$$

where $\theta = 1 + \frac{\gamma p - \delta}{d+2}$.

Therefore, we have

$$\begin{aligned} & \mathbb{E} |F(t_1, x_1, c_1) - F(t_2, x_2, c_2)|^q \\ \leq \; & C(C_0, D, d, T) \|g\|^q_{C^\beta(\mathbb{R}_+ \times \mathbb{R}^d))} (|t_1 - t_2| + |x_1 - x_2| + |c_1 - c_2|)^{\delta q}, \end{aligned}$$

where $\theta = 1 + \frac{\beta p - 2\delta}{d+2}$, $(t_i, x_i) \in D_T$ and $0 < c_i \leq d(D)$, $i = 1, 2$. The rest proof of this theorem is exactly similar to that of Theorem 3.1 and we omit it here. The proof of Theorem 3.2 is complete. $\square$

Next, we consider the following equation

$$\frac{\partial}{\partial t} u(t,x) = \Delta^\alpha u(t,x) + g(t,x) \dot{W}(t,x), \quad u|_{t=0} = 0, \tag{3.13}$$

where $\Delta^\alpha = -(-\Delta)^\alpha$ and W_t is a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$.

Theorem 3.3 *Let D be an A-type bounded domain in $\mathbb{R}^{d+1}$ such that $\bar{D} \subset \mathcal{O}_T$. Suppose that $g \in L^\infty_{loc}(\mathbb{R}_+; L^p(\Omega \times \mathbb{R}^d))$ is $\mathcal{F}_t$-adapted process. Set $d = 1$. Assume that $\frac{1}{2} < \alpha \leq 1$, $p > \frac{2}{2\alpha - 1}$. Let $\beta > 0$ be sufficiently small such that $p(2\alpha - 2\beta - 1) > 2$. Then, there is a mild solution u of (3.13) and $u \in \mathscr{L}^{p,\theta}((D_T; \delta); L^p(\Omega)) \cap L^p(\Omega; C^\beta(D_T))$. Moreover, it holds that*

$$\|u\|_{\mathscr{L}^{p,\theta}((D_T;\delta);L^p(\Omega))} \leq C \|g\|_{L^\infty([0,T];L^p(\Omega \times \mathbb{R}))}, \tag{3.14}$$

$$\|u\|_{C^\beta(D_T;L^p(\Omega))} \leq C \|g\|_{L^\infty([0,T];L^p(\Omega \times \mathbb{R}))}, \tag{3.15}$$

where $\theta = 1 + \frac{\beta p}{3}$. Moreover, taking $0 < \delta < \beta p/2$ and $q > 3/\delta$, we have for $0 < r < q$

$$\|u\|_{L^r(\Omega;C^{\beta^*}(D_T))} \leq C \|g\|_{L^\infty([0,T];L^p(\Omega \times \mathbb{R}))}, \tag{3.16}$$

where $\beta^* = \beta - 2\delta/p$.

Proof. The existence of mild solution of (3.13) is a classical result under the above assumptions. Now we prove the inequality (3.14). Due to the definition of Companato space, it suffices to show that

$$[u]_{\mathscr{L}^{p,\theta}((D_T;\delta);L^p(\Omega))} < \infty.$$

Direct calculus shows that

$$\begin{aligned}
[u]^p_{\mathscr{L}^{p,\theta}((D_T;\delta);L^p(\Omega))} &\leq \sup_{D(X,c),X\in D_T,0<c\leq d(D)} \frac{1}{|D(X,c)|^{1+\theta}} \\
&\quad \times \mathbb{E}\int_{D(X,c)}\int_{D(X,c)} |u(t,x)-u(s,y)|^p dtdxdsdy \\
&\leq \sup_{D(X,c),X\in D_T,0<c\leq d(D)} \frac{1}{|D(X,c)|^{1+\theta}} \\
&\quad \times \mathbb{E}\int_{D(X,c)}\int_{D(X,c)} \Big| \int_0^t\int_{\mathbb{R}} K(t-r,x-z)g(r,z)dzdW(r) \\
&\quad -\int_0^s\int_{\mathbb{R}} K(s-r,y-z)g(r,z)dW(dr,dz)\Big|^p \\
&:= \sup_{D(X,c),X\in D_T,0<c\leq d(D)} \frac{1}{|D(X,c)|^{1+\theta}} \int_{D(X,c)}\int_{D(X,c)} \mathbb{E}\Upsilon dtdxdsdy.
\end{aligned}$$

Set $t\geq s$. We have the following estimates

$$\begin{aligned}
\mathbb{E}\Upsilon &\leq C\mathbb{E}\Big|\int_0^s\int_{\mathbb{R}}(K(t-r,x-z)-K(s-r,y-z))g(r,z)W(dr,dz)\Big|^p \\
&\quad +C\mathbb{E}\Big|\int_s^t\int_{\mathbb{R}} K(t-r,x-z)g(r,z)W(dr,dz)\Big|^p \\
&\leq C\mathbb{E}\Big|\int_0^s\int_{\mathbb{R}}(K(t-r,x-z)-K(s-r,y-z))^2g^2(r,z)dzdr\Big|^{\frac{p}{2}} \\
&\quad +C\mathbb{E}\Big|\int_s^t\int_{\mathbb{R}} K^2(t-r,x-z)g^2(r,z)dzdr\Big|^{\frac{p}{2}} \\
&=: C(H_1+H_2).
\end{aligned}$$

Estimate of H_1.

Take $\beta>0$ satisfying $(2\alpha-2\beta-1)p-2\geq 0$. By using the Proposition 2.3, and Hölder inequality, we have

$$\begin{aligned}
H_1 &= \mathbb{E}\Big|\int_0^s\int_{\mathbb{R}}(K(t-r,x-z)-K(s-r,y-z))^2g^2(r,z)dzdr\Big|^{\frac{p}{2}} \\
&\leq C\mathbb{E}\Big|\int_0^s\int_{\mathbb{R}}|K(t-r,x-z)-K(s-r,x-z)|^2\cdot|g^2(r,z)|dzdr\Big|^{\frac{p}{2}} \\
&\quad +C\mathbb{E}\Big|\int_0^s\int_{\mathbb{R}}(K(s-r,x-z)-K(s-r,y-z))^2\cdot g^2(r,z)dzdr\Big|^{\frac{p}{2}} \\
&=: H_{11}+H_{12}.
\end{aligned}$$

For H_{11}, we have

$$\begin{aligned}
H_{11} &\leq C(t-s)^{\frac{\beta p}{2}}\mathbb{E}\Big|\int_0^s\left(\int_{\mathbb{R}}|\frac{\partial^{\frac{\beta}{2}}K}{\partial t^{\frac{\beta}{2}}}(\xi-r,x-z)|^qdz\right)^{\frac{2}{q}}\|g(r)\|^2_{L^p(\mathbb{R})}dr\Big|^{\frac{p}{2}} \\
&\leq C(t-s)^{\frac{\beta p}{2}}\|g\|^p_{L^p(\Omega;L^\infty([0,T];L^p(\mathbb{R})))}\left[\int_0^s\left(\int_{\mathbb{R}}|\frac{\partial^{\frac{\beta}{2}}K}{\partial t^{\frac{\beta}{2}}}(\xi-r,x-z)|^qdz\right)^{\frac{2}{q}}dr\right]^{\frac{p}{2}},
\end{aligned}$$

where $q = 2p/(p-2)$, $\xi = \theta t + (1-\theta)s$, $0 < \theta < 1$ and we used the following fact

$$\begin{aligned}
&\int_0^s \left(\int_{\mathbb{R}} |\frac{\partial^{\frac{\beta}{2}} K}{\partial t^{\frac{\beta}{2}}}(\xi - r, x - z)|^q dz \right)^{\frac{2}{q}} dr \\
\leq\ & C \int_0^s \left(\int_0^{(\xi-r)^{\frac{1}{2\alpha}}} (\xi - r)^{-\frac{q+2q\alpha\beta}{2\alpha}} |z|^{q\alpha\beta} d|z| \right. \\
&\left. + \int_{(\xi-r)^{\frac{1}{2\alpha}}}^{\infty} (\xi - r)^q |z|^{-(q+2q\alpha+2q\alpha\beta)} |z|^{q\alpha\beta} d|z| \right)^{\frac{2}{q}} dr \\
\leq\ & C \left[(\theta(t-s))^{\frac{1-q+q\alpha(1-\beta)}{q\alpha}} + \xi^{\frac{1-q+q\alpha(1-\beta)}{q\alpha}} \right] \\
\leq\ & C
\end{aligned}$$

because using $q = 2p/(p-2)$, we have

$$1 - q + q\alpha(1-\beta) > 0 \Leftrightarrow p(2\alpha - 2\alpha\beta - 1) > 2 \Leftarrow p(2\alpha - 2\beta - 1) > 2.$$

For H_{12}, by using the fractional mean value formula again, we have

$$\begin{aligned}
H_{12} &\leq C|x-y|^{\beta p} \|g\|^p_{L^p(\Omega; L^\infty([0,T]; L^p(\mathbb{R})))} \Big| \int_0^s \left(\int_{\mathbb{R}} [K^{(\beta)}(s-r, \xi - z)]^q dz dr \right)^{\frac{2}{q}} dr \Big|^{\frac{p}{2}} \\
&\leq C|x-y|^{\beta p} \|g\|^p_{L^p(\Omega; L^\infty([0,T]; L^p(\mathbb{R})))} \left[\int_0^s (s-r)^{-\frac{d(q-1)+\beta q}{q\alpha}} dr \right]^{\frac{p}{2}} \\
&\leq C|x-y|^{\beta p},
\end{aligned}$$

where $q = 2p/(p-2)$, $\xi = \theta x + (1-\theta)y$ and we used the following inequality

$$\int_0^s (s-r)^{-\frac{d(q-1)+\beta q}{q\alpha}} dr = \frac{q\alpha}{q(\alpha-\beta)-(q-1)} s^{\frac{q(\alpha-\beta)-(q-1)}{q\alpha}} \leq C$$

provided that $(2\alpha - 2\beta - 1)p - 2 \geq 0$.

Estimate of H_2.

Similar to the estimate of H_1, we have

$$\begin{aligned}
H_2 &= \mathbb{E} \Big| \int_s^t \int_{\mathbb{R}^d} K^2(t-r, x-z) g^2(r,z) dz dr \Big|^{\frac{p}{2}} \\
&\leq \|g\|^p_{L^p(\Omega; L^\infty([0,T]; L^p(\mathbb{R}^d)))} \left[\int_s^t \left(\int_{\mathbb{R}^n} |K(t-r, x-z)|^q dz \right)^{\frac{2}{q}} dr \right]^{\frac{p}{2}} \\
&\leq C\|g\|^p_{L^p(\Omega; L^\infty([0,T]; L^p(\mathbb{R}^d)))} (t-s)^{\frac{q\alpha-(q-1)d}{q\alpha} \times \frac{p}{2}}
\end{aligned}$$

provided that $p(2\alpha - 1) > 2$. Indeed, by using $q = \frac{2p}{p-2}$, we have

$$q\alpha - (q-1) > 0 \iff p(2\alpha - 1) > 2.$$

Combining the assumption of p, we have

$$H_2 \leq C(t-s)^{\frac{p(2\alpha-1)-2}{2\alpha}}.$$

The rest proof is similar to that of 3.1 and we omit it here. □

4 Hölder estimate on a bounded domain

In this section, we consider the SPDEs of the following form

$$\begin{cases} du = Audt + g(t,x)dW_t, \quad (t,x) \in (0,\infty) \times D, \\ u|_{\partial D} = 0, \\ u_{t=0} = 0, \end{cases} \tag{4.1}$$

where D is a smooth bounded domain in $\mathbb{R}^d$, W_t is standard one-dimensional Brownian motion, and g is progressively measurable L^∞- or L^p-function.

Throughout this section, we assume that A is a uniformly elliptic second-order differential operator of the form

$$A = a_{ij}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} + b_i(x)\frac{\partial}{\partial x_i} + c(x)$$

with smooth coefficients. Furthermore, we assume that at least one of the following two assumptions holds:

$$\begin{aligned} B^\infty : \quad & \|g\|_{L^\infty([0,T];L^p(\Omega;L^\infty(D)))} < \infty, \\ B^p : \quad & \|g\|_{L^\infty([0,T];L^p(\Omega\times D)))} < \infty. \end{aligned}$$

In order to obtain the Hölder estimate, we need the following Lemma. Consider the following initial-boundary problem:

$$\frac{\partial v}{\partial t} - Av = 0, \quad v|_{t=0} = F(x), \quad v|_{\partial D} = 0, \tag{4.2}$$

and denote by S_t the corresponding semigroup:

$$v(t,\cdot) = (S_tF)(\cdot), \quad F = F(\cdot).$$

Lemma 4.1 *[16, Lemma 1] Let $|F(x)| < M$. Then, for any $\theta < 1$, the following estimates hold with $c > 0$:*

$$\begin{aligned} &\|v(t,\cdot)\|_{C^\theta(D)} \le c(\theta)Mt^{-\theta/2}\exp(-ct), \\ &|v(t+\delta,x) - v(t,x)| \le c(\theta)Mt^{-\theta}\delta^\theta \exp(-ct). \end{aligned}$$

Moreover, if $\|F\|_{L^p(D)} \le M$ and $p > 1$, then

$$\begin{aligned} &\|v(t,\cdot)\|_{C^\theta(D)} \le c(\theta)Mt^{-\theta/2-d/(2p)}\exp(-ct), \\ &|v(t+\delta,x) - v(t,x)| \le c(\theta)Mt^{-\theta-d/(2p)}\delta^\theta \exp(-ct). \end{aligned}$$

Theorem 4.1 *Let D_T be an A-type bounded domain in $\mathbb{R}^{d+1}$.*

(i) Suppose that B^p holds for $p > d$ and that $0 < \beta < 1$ satisfies $(1-\beta)p - d \ge 0$. Then, there is a mild solution u of (4.1) and $u \in \mathscr{L}^{p,\theta}((D_T;\delta); L^p(\Omega)) \cap L^p(\Omega; C^\beta(D_T))$. Moreover, it holds that

$$\begin{aligned} &\|u\|_{\mathscr{L}^{p,\theta}((D_T;\delta);L^p(\Omega))} \le C\|g\|_{L^\infty([0,T];L^p(\Omega\times D))}, \\ &\|u\|_{C^\beta(D_T;L^p(\Omega))} \le C\|g\|_{L^\infty([0,T];L^p(\Omega\times D))}, \end{aligned}$$

where $\theta = 1 + \frac{\beta p}{d+2}$. Moreover, taking $0 < \delta < \beta p/2$ and $q > (d+2)/\delta$, we have for $0 < r < q$

$$\|u\|_{L^r(\Omega;C^{\beta^*}(D_T))} \le C\|g\|_{L^\infty([0,T];L^p(\Omega\times\mathbb{R}^d))},$$

where $\beta^* = \beta - 2\delta/p$.

(ii) Suppose that B^∞ *holds for* $p > 1$. *Then, there is a mild solution* u *of (4.1) and* $u \in \mathscr{L}^{p,\theta}((D_T;\delta); L^p(\Omega)) \cap L^p(\Omega; C^\beta(D_T))$. *Moreover, it holds that*

$$\|u\|_{\mathscr{L}^{p,\theta}((D_T;\delta);L^p(\Omega))} \le C\|g\|_{L^\infty([0,T];L^p(\Omega\times D))},$$
$$\|u\|_{C^\beta(D_T;L^p(\Omega))} \le C\|g\|_{L^\infty([0,T];L^p(\Omega\times D))},$$

where $\theta = 1 + \frac{p}{d+2}$. *Moreover, taking* $0 < \delta < p/2$ *and* $q > (d+2)/\delta$, *we have for* $0 < r < q$

$$\|u\|_{L^r(\Omega;C^{\beta^*}(D_T))} \le C\|g\|_{L^\infty([0,T];L^p(\Omega\times\mathbb{R}^d))},$$

where $\beta^* = 1 - 2\delta/p$.

Proof. The proof of this Theorem is exactly similar to that of Theorem 3.1 by using Lemma 4.1. We omit it to the readers. The proof is complete. □

Remark 4.1 *Theorem 4.1 does not hold for the nonlocal operator because we did not have the similar properties of kernel function on bounded domain.*

Comparing Theorem 4.1 with [16, Theorems 1 and 2], we find the index of [16] is $\beta < \frac{1}{2} - \frac{d}{2p}$ *for the case* B^p *and the index in this paper is larger than that of [16].*

5 Applications and further discussions

We first give an example for Theorem 3.2. Consider the equation (3.1). In our paper [20], by using Proposition 2.3, we got the following result.

Lemma 5.1 *Let* $0 \le \epsilon < \alpha$. *The following estimates hold.*

$$\int_0^s \left(\int_{\mathbb{R}^d} |\nabla^\epsilon p(t-r,z) - \nabla^\epsilon p(s-r,z)|(1+|z|^\beta)dz\right)^2 dr \le N(T,\beta)(t-s)^\gamma,$$
$$\int_0^s \left(\int_{\mathbb{R}^d} |\nabla^\epsilon p(s-r,z)|dz\right)^2 dr \le N_0,$$
$$\int_s^t \left(\int_{\mathbb{R}^d} |\nabla^\epsilon p(t-r,z)|(1+|z|^\beta)dz\right)^2 dr \le N(T,\beta)(t-s)^\gamma,$$

where $\gamma = \frac{\alpha-\epsilon}{\alpha}$.

Then applying Theorem 3.2, we have the following result.

Theorem 5.1 *Let* $0 \le \epsilon < \alpha$ *and* D_T *be an A-type bounded domain in* $\mathbb{R}^{d+1}$ *such that* $\bar{D}_T \subset \mathcal{O}_T$. *Suppose that* $g \in C^\beta(\mathbb{R}_+ \times \mathbb{R}^d)$, $0 < \beta < 1$, *is a non-random function and* $g(0,0) = 0$. *Then we have, for* $p \ge 1$ *and* $\beta < \gamma$,

$$\|\nabla^\epsilon u\|_{\mathscr{L}^{p,\theta}((D_T;\delta);L^p(\Omega))} \le C\|g\|_{C^\beta(\mathbb{R}_+\times\mathbb{R}^d))},$$
$$\|\nabla^\epsilon u\|_{C^\beta(D_T;L^p(\Omega))} \le C\|g\|_{C^\beta(\mathbb{R}_+\times\mathbb{R}^d))},$$

where $\theta = 1 + \frac{\gamma p}{d+2}$ *and* $\gamma = \frac{\alpha-\epsilon}{\alpha}$. *Moreover, taking* $0 < \delta < \gamma p/2$ *and* $q > (d+2)/\delta$, *we have for* $0 < r < q$

$$\|\nabla^\epsilon u\|_{L^r(\Omega;C^{\beta^*}(D_T))} \le C\|g\|_{C^\beta(\mathbb{R}_+\times\mathbb{R}^d))},$$

where $\beta^* = \gamma - 2\delta/p$.

In fact, one can use the factorization method to obtain the Hölder estimates of solutions to the following equation

$$du_t = [\Delta^\alpha u + f(t,x,u)]dt + g(t,x)dW_t, \quad u|_{t=0} = u_0(x),$$

where $\Delta^\alpha = -(-\Delta)^\alpha$, $\alpha \in (0,1]$ and W_t is a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. About the factorization method, see [4].

In addition, one can use the Kunita's first inequality to deal with a general case. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete probability space such that $\{\mathcal{F}_t\}_{t\in[0,T]}$ is a filtration on Ω containing all P-null subsets of Ω and $\mathbb{F}$ be the predictable σ-algebra associated with the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$. We are given a σ-finite measure space $(Z, \mathcal{Z}, \nu)$ and a Poisson random measure μ on $[0,T]\times Z$, defined on the stochastic basis. The compensator of μ is Leb$\otimes\nu$, and the compensated martingale measure $\tilde{N} := \mu - Leb\otimes\nu$. The method used here is also suitable to the case that

$$\begin{aligned}\mathcal{G}g(t,x) &= \int_0^t \int_Z K(t,s,\cdot) * g(s,\cdot,z)(x)\tilde{N}(dz,ds)\\ &= \int_0^t \int_Z \int_{\mathbb{R}^d} K(t-s,x-y)g(s,y,z)dy\tilde{N}(dz,ds) \qquad (5.1)\end{aligned}$$

for $\mathbb{F}$-predictable processes $g : [0,T]\times\mathbb{R}^d\times Z\times\Omega \to \mathbb{R}$.

In the end of this section, we give a new criteria based on the following Proposition.

Proposition 5.1 *[21, Theorem 2.1] Let $\{X_t, t\in[0,1]\}$ be a Banach-valued stochastic field for which there exist three strictly positive constants γ, c, ε such that*

$$E[\sup_{0\le t\le 1} |X_t(x) - X_t(y)|^\gamma] \le c|x-y|^{d+\varepsilon},$$

then there is a modification $\tilde{X}$ of X such that

$$E\Big[\Big(\sup_{s\neq t}\frac{|\tilde{X}_t - \tilde{X}_s|}{|t-s|^\alpha}\Big)^\gamma\Big] < \infty$$

for every $\alpha \in [0, \varepsilon/\gamma)$. In particular, the paths of $\tilde{X}$ are Hölder continuous in x of order α.

For applications, we need prove the Kolmogorov criterion with the following form.

Theorem 5.2 *Let $\{X_t(x), x\in[0,1]^d, t\in[0,1]\}$ be a Banach-valued stochastic field for which there exist three strictly positive constants γ, c, ε such that*

$$E[\sup_{0\le t\le 1} |X_t(x) - X_t(y)|^\gamma] \le c|x-y|^{d+\varepsilon},$$

then there is a modification $\tilde{X}$ of X such that

$$E\Big[\sup_{0\le t\le 1}\Big(\sup_{x\neq y}\frac{|\tilde{X}_t(x) - \tilde{X}_t(y)|}{|x-y|^\alpha}\Big)^\gamma\Big] < \infty$$

for every $\alpha \in [0, \varepsilon/\gamma)$. In particular, the paths of $\tilde{X}$ are Hölder continuous in x of order α.

Proof. Let D_m be the set of points in $[0,1]^d$ whose components are equal to $2^{-m}i$ for some integral $i \in [0, 2^m]$. The set $D = \cup_m D_m$ is the set of dyadic numbers. Let further Δ_m be the set of pairs (x,y) in D_m such that $|x-y| = 2^{-m}$. There are $2^{(m+1)d}$ such pairs in Δ_m.

Let us finally set $K_i(t) = \sup_{(x,y)\in\Delta_i} |X_t(x) - X_t(y)|$. The hypothesis entails that for a constant J,

$$E[\sup_{0\le t\le 1} K_i(t)^\gamma] \le \sum_{(x,y)\in\Delta_i} E[\sup_{0\le t\le 1} |X_t(x) - X_t(y)|^\gamma] \le c2^{(i+1)d}2^{-i(d+\varepsilon)} = J2^{-i\varepsilon}.$$

For a point x (resp. y) in D, there is an increasing sequences $\{x_m\}$ (resp. $\{y_m\}$) of points in D such that x_m (resp. y_m) is in D_m for each m, $x_m \le x$ ($y_m \le y$) and $x_m = x$ ($y_m = y$) from some m on. If $|x - y| \le 2^{-m}$, then either $x_m = y_m$ or $(x_m, y_m) \in \Delta_m$ and in any case

$$X_t(x) - X_t(y) = \sum_{i=m}^{\infty}(X_t(x_{i+1}) - X_t(x_i)) + X_t(x_m) - X_t(y_m) - \sum_{i=m}^{\infty}(X_t(y_{i+1}) - X_t(y_i)),$$

where the series are actually finite sums. It follows that

$$|X_t(x) - X_t(y)| \le K_m + 2\sum_{i=m+1}^{\infty} K_i(t) \le 2\sum_{i=m}^{\infty} K_i(t).$$

As a result, setting $M_\alpha(t) = \sup\{|X_t(x) - X_t(y)|/|x-y|^\alpha,\ x, y \in D,\ x \ne y\}$, we have

$$\begin{aligned} M_\alpha(t) &\le \sup_{m\in N}\Big\{2^{m\alpha}\sup_{|x-y|\le 2^{-m}} |X_t(x) - X_t(y)|,\ x, y \in D,\ x \ne y\Big\} \\ &\le \sup_{m\in N}\Big\{2^{m\alpha+1}\sum_{i=m}^{\infty} K_i(t)\Big\} \\ &\le 2\sum_{i=0}^{\infty} 2^{i\alpha}K_i(t). \end{aligned}$$

For $\gamma \ge 1$ and $\alpha < \varepsilon/\gamma$, we get with $J' = 2J$,

$$[E\sup_{0\le t\le 1} M_\alpha(t)^\gamma]^{1/\gamma} \le 2\sum_{i=0}^{\infty} 2^{i\alpha}[E\sup_{0\le t\le 1} K_i(t)^\gamma]^{1/\gamma} \le J'\sum_{i=0}^{\infty} 2^{i(\alpha-\varepsilon/\gamma)} < \infty.$$

For $\gamma < 1$, the same reasoning applies to $[E\sup_{0\le t\le 1} M_\alpha(t)^\gamma]$ instead of $[E\sup_{0\le t\le 1} M_\alpha(t)^\gamma]^{1/\gamma}$.

It follows in particular that for almost every ω, $X_t(\cdot)$ is uniformly continuous on D and it is uniformly in t, so it make sense to set

$$\tilde{X}_t(x, \omega) = \lim_{y\in D, y\to x} X_t(y, \omega).$$

By Fatou's lemma and the hypothesis, $\tilde{X}_t(x) = X_t(x)$ a.s. and $\tilde{X}$ is clearly the desired modification. □

It is easy to see that one can use Theorem 5.2 to consider the equation (3.1) and (5.1)

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