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On the regularity of weak solutions to space-time fractional stochastic heat equations

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Abstract

This study is concerned with the space-time fractional stochastic heat-type equations driven by multiplicative noise, which can be used to model the anomalous heat diffusion in porous media with random effects with thermal memory. We first deduce the weak solutions to the given problem by means of the Laplace transform and Mittag-Leffler function. Using the fractional calculus and stochastic analysis theory, we further prove the pathwise spatial-temporal regularity properties of weak solutions to this type of SPDEs in the framework of Bochner spaces.

Keywords: Space-time fractional derivative, stochastic heat equations, weak solutions, regularity properties.

1. Introduction

We consider the following space-time fractional stochastic partial differential equations (SPDEs) on a bounded domain $D \subset \mathbb{R}^d (d \geq 1)$:

$$\begin{cases} \partial_t^\beta u(x,t) = -(-\triangle)^{\frac{\alpha}{2}} u(x,t) + I_t^{1-\beta}[\sigma(u(x,t))\dot{W}(x,t)],\ x \in D,\ t > 0, \\ u(x,t)|_{\partial D} = 0, t > 0, \\ u(x,0) = u_0(x), x \in D, \end{cases} \tag{1.1}$$

where ∂_t^β is the Caputo fractional derivative with $\beta \in (0,1)$, $(-\triangle)^{\frac{\alpha}{2}}$ is the fractional Laplacian with $\alpha \in (0,2]$, $I_t^{1-\beta}$ is the fractional integral operator will be given below. The dimension d and the parameters α and β in (1.1) satisfies that $d < \min\{2, \beta^{-1}\}\alpha$. Denote by $\dot{W}(x,t)$ space-time white noise modeling the random effects, and the function $\sigma : \mathbb{R} \to \mathbb{R}$ is a globally Lipschitz continuous function.

For any $\beta \geq 0$, we define the function $G_\beta(t) : \mathbb{R} \to \mathbb{R}$ by

$$G_\beta(t) = \begin{cases} \frac{1}{\Gamma(\beta)} t^{\beta-1},\ t > 0, \\ 0, \qquad\quad t \leq 0, \end{cases} \tag{1.2}$$

where $G_0(t) = 0$ and $\Gamma(\beta)$ denotes the gamma function. The Riemann-Liouville fractional integral operator I_t^β is defined by

$$I_t^\beta f(t) = (G_\beta * f)(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s)ds,\ t > 0, \tag{1.3}$$

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with $I_t^0 f(t) = f(t)$. For $\beta \in (0,1)$ and $t > 0$, then the expression

$$\partial_t^\beta f(t) = \frac{d}{dt}[I_t^{1-\beta}(f(t) - f(0))] = \frac{d}{dt}(\int_0^t G_{1-\beta}(t-s)(f(t) - f(0))ds) \tag{1.4}$$

is called the Caputo fractional derivative of order β of the function f (see [1]).

Note that the Eqs.(1.1) might be used to model the random effects on transport of particles in medium with thermal memory. Chen et al.[2] introduced a class of SPDEs with time-fractional derivatives and proved the existence and uniqueness of solutions to the equations. Mijena and Nane [3] proved the existence and uniqueness of mild solutions to non-linear space-time fractional SPDEs, and they also investigated the bounds for the intermittency fronts solutions of these equations [4]. Foondun and Nane [5] studied the asymptotic properties of space-time fractional SPDEs. Chen et al. [6] proved the existence and uniqueness of solutions to space-time fractional SPDEs in Gaussian noisy environment. It was worth mentioning that the above authors mainly focused on the mild solutions based on the Green function. As we known, the regularity of weak solutions to the fractional SPDEs has received less attention. The aim of this paper is to study the regularity of weak solutions to space-time fractional SPDEs, which are needed for the error analysis of numerical methods, for example, Zou et al.[16,17] investigated the finite element methods for solving a special case of fractional SPDEs in the given problem (1.1).

The remaining of this paper is organized as follows. In Section 2, some notations and preliminaries will be introduced, and we also deduce the weak solutions to the space-time fractional SPDEs. In Section 3, stochastic analysis techniques and fractional calculus are used to prove the spatial and temporal regularity properties of weak solutions to the equations (1.1) in Bochner spaces.

2. Notations and preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ be a filtered probability space with the normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Recall that Q is a positive bounded linear operator on some Hilbert space U with finite trace. Let $W = \{W(t), t \geq 0\}$ be a U-valued Wiener process with covariance operator Q. We introduce the subspace $U_0 = Q^{1/2}(U)$, which endowed with the inner product:

$$(u, v)_{U_0} = (Q^{1/2}u, Q^{1/2}v), \ u, v \in U_0,$$

and induced norm $\|\cdot\|_{U_0}$, where $Q^{-1/2}$ denotes the pseudo-inverse of $Q^{1/2}$. Denote by $L_2^0 = L^2(U_0, H)$ the space of Hilbert-Schmidt operators $T : U_0 \to H$ endowed with the norm

$$\|\varphi\|_{L_2^0}^2 = Tr[(\varphi Q^{1/2})(\varphi Q^{1/2})^*] < \infty,$$

for any $\varphi \in L_2^0$. The details description of Wiener process should be referred to [7]. Let $p \geq 2$ and $\{v(t)\}_{t\in[0,T]}$ be an L_2^0-valued predictable stochastic process, the following generalized version of Itô isometry (including the Burkholder-Davis-Gundy inequality) are important for the stochastic integrals [8], that is

$$\mathbb{E}\|\int_0^t v(s)dW(s)\|^p \leq C(p)\mathbb{E}[(\int_0^t \|v(s)\|_{L_0^2}^2 ds)^{\frac{p}{2}}], \ t \in [0,T], \tag{2.1}$$

where $\mathbb{E}$ denotes the expectation and $C(p) > 0$ is a constant.

Next, we shall introduce fractional order spaces and norms. Define a linear operator $A := -\Delta$ with zero Dirichlet boundary condition on D. Denote by $\{\varphi_k\}_{k\geq 1}$ the complete orthonormal system of eigenfunctions in H for the operator A, i.e., for $k = 1, 2, \cdots$,

$$A\varphi_k = \lambda_k \varphi_k, \ \varphi_k|_{\partial D} = 0,$$

with $\lambda_1 \leq \lambda_2 \leq \cdots, \lambda_k \leq \cdots$.

For any $s > 0$, let $\dot{H}^s$ be the domain of the fractional power $A^{\frac{s}{2}} = (-\triangle)^{\frac{s}{2}}$, which can be defined by

$$A^{\frac{s}{2}} v = \sum_{k=1}^{\infty} \lambda_k^{\frac{s}{2}} (v, \varphi_k)\varphi_k, \ \dot{H}^s = \mathcal{D}(A^{\frac{s}{2}}) = \{v \in H : \|A^{\frac{s}{2}} v\|^2 = \sum_{k=1}^{\infty} \lambda_k^s (v, \varphi_k)^2 < \infty\},$$

with inner product $(u, v)_{\dot{H}^s} = (A^{\frac{s}{2}} u, A^{\frac{s}{2}} v)$ and induced norms $\|v\|^2_{\dot{H}^s} = \|A^{\frac{s}{2}} v\|^2 = \sum_{k=1}^{\infty} \lambda_k^s (v, \varphi_k)^2$. It is known that $\dot{H}^0 = H$, $\dot{H}^1 = H_0^1(D)$ and $\dot{H}^2 = H^2(D) \cap H_0^1(D)$ with equivalent norms and that $\dot{H}^{-s}$ can be identified with the dual space $(\dot{H}^s)^*$ for $s > 0$. In order to quantify the regularity we introduce the Bochner spaces $L^p(\Omega; B) = L^p((\Omega, \mathcal{F}, \mathbb{P}); B)$ endowed with the norm:

$$\|v\|_{L^p(\Omega;B)} = (\mathbb{E}[\|v\|_B^p])^{\frac{1}{p}}, \forall\ v \in L^p(\Omega; B),$$

where B being a Banach space and for any $p \geq 2$.

For the sake of convenient, the Eqs.(1.1) can be rewritten as the following abstract formulation:

$$\begin{cases} \partial_t^\beta u(t) + A_\alpha u(t) = I_t^{1-\beta}[\sigma(u(t))\dot{W}(t)], t > 0, \\ u(0) = u_0, \end{cases} \tag{2.2}$$

where we denote $u(t) = u(\cdot, t)$ and replace the fractional operator $(-\triangle)^{\frac{\alpha}{2}}$ by $A_\alpha := A^{\frac{\alpha}{2}}$. Indeed, it is convenient to treat u as a function of t taking value in H with the Fourier coefficients $u_k(t)$ as follows

$$u(t) = \sum_{k=1}^{\infty} u_k(t)\varphi_k, \text{ where } u_k(t) = (u(t), \varphi_k),$$

and we likewise put $G_k(t) = (\sigma(u(t))\dot{W}(t), \varphi_k)$ and $u_{0k} = (u_0, \varphi_k)$.

Taking the inner produce of φ_k with (2.2) gives the sequence of scalar initial value problem

$$\begin{cases} \partial_t^\beta u_k(t) + \lambda_k^{\frac{\alpha}{2}} u_k(t) = I_t^{1-\beta} G_k(t), t > 0, \\ u_k(0) = u_{0k}. \end{cases} \tag{2.3}$$

Next, we recall some facts about the theory of fractional calculus. Firstly, we introduce the Laplace transform of the function f with respect to t by

$$\widehat{f}(z) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-zt} f(t)dt,$$

then the Laplace transform of the Caputo derivative ∂_t^β in (1.4) and the Riemann-Liouville fractional integral operator I_t^β in (1.3) are given by (see [9,10])

$$\mathcal{L}\{\partial_t^\beta f(t)\} = z^\beta \widehat{f}(z) - z^{\beta-1} f(0), \ \mathcal{L}\{I_t^\beta h(t)\} = z^{-\beta}\widehat{h}(z). \tag{2.4}$$

Applying the Laplace transform to both sides of (2.3) and using (2.4) we deduce that

$$z^\beta \widehat{u_k}(z) - z^{\beta-1} u_k(0) + \lambda_k^{\frac{\alpha}{2}} \widehat{u_k}(z) = z^{\beta-1} \widehat{G_k}(z),$$

and so

$$\widehat{u_k}(z) = \frac{z^{\beta-1}}{z^\beta + \lambda_k^{\frac{\alpha}{2}}}[u_k(0) + \widehat{G_k}(z)]. \tag{2.5}$$

Now, we define the Mittag-Leffler function by (see [11,12])

$$E_\beta(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\beta k + 1)}, \tag{2.6}$$

and using the Laplace transform with respect to (2.6) yields

$$\mathcal{L}\{E_\beta(-\lambda t^\beta)\} = \int_0^\infty e^{-zt} E_\beta(-\lambda t^\beta) dt = \frac{z^{\beta-1}}{z^\beta + \lambda}.$$

Therefore, the representation (2.5) implies that

$$u_k(t) = E_\beta(-\lambda_k^{\frac{\alpha}{2}} t^\beta) u_{0k} + \int_0^t E_\beta(-\lambda_k^{\frac{\alpha}{2}} (t-s)^\beta) \sigma(u(s)) dW(s), t > 0,$$

leading us to define the operator $E(t)$:

$$E(t)v = \sum_{k=1}^{\infty} E_\beta(-\lambda_k^{\frac{\alpha}{2}} t^\beta)(v, \varphi_k)\varphi_k(x),$$

so that $u(t) = E(t)u_0$ is the solution of (2.2) when $\sigma(u) = 0$, and in general

$$u(t) = E(t)u_0 + \int_0^t E(t-s)\sigma(u(s))dW(s). \tag{2.7}$$

Under appropriate conditions, see the assumptions 3.1 and 3.2 below, the $\mathcal{F}_t$-adapted process $(u(t))_{t\in[0,T]}$ is called a unique weak solution to Eqs.(2.2) if it satisfies the integral equation (2.7) for almost surely $\omega \in \Omega$. On the other hand, under the assumption that $d < \min\{2, \beta^{-1}\}\alpha$, the existence of solutions to this type of space-time fractional equations (containing multi-fractional in space variable) in bounded domains have been established in [3,14,15], respectively. Comparing to the results obtained in those papers, what we constructed here is a new formulation of solutions to the space-time fractional equations, and we establish the new regularity properties of solutions to the equations in the framework of Bochner spaces.

3. Main results

This section we only focus on the pathwise spatial-temporal regularity properties of weak solutions (2.7) to (2.2). Throughout the paper, we will impose the following assumptions.

Assumption 3.1. The measurable function $\sigma : \Omega \times H \to L_0^2$ satisfies the following global Lipschitz and growth conditions:

$$\|\sigma(v)\|_{L_0^2} \leq C\|v\|, \|\sigma(u) - \sigma(v)\|_{L_0^2} \leq C\|u - v\|, \tag{3.1}$$

for any $u, v \in H$.

Assumption 3.2. Assume that the initial value $u_0 : \Omega \to \dot{H}^{\nu}$ is a $\mathcal{F}_0-$measurable random variable, it holds that

$$\|u_0\|_{L^p(\Omega;\dot{H}^{\nu})} < \infty, \tag{3.2}$$

for any $0 \le \nu < \alpha \le 2$.

Lemma 3.1. For any $t > 0$, $0 < \beta < 1$ and $0 \le \nu < \alpha \le 2$, there exists a constant $C > 0$ such that

$$\|(\frac{d}{dt})^m E(t)v\|_{\dot{H}^{\nu}} \le Ct^{-\frac{\beta\nu}{2}-m}\|v\|_{\dot{H}^{(2-\alpha)\nu/2}}, \tag{3.3}$$

with $m = 0, 1, 2, \cdots$.

Proof. The Mittag-Leffler function admits the asymptotic expansion (see [10,13]):

$$E_\beta(-t) = \sum_{k=1}^{N} \frac{(-1)^{k+1}t^{-k}}{\Gamma(1-\beta k)} + O(t^{-N-1}),\ 0 < \beta < 2,\ \text{as } t \to \infty, \tag{3.4}$$

For $0 < t < \infty$ and $-2 \le \mu \le 2$, we define $g(t) = E_\beta(-t^\beta)$ so that $g(\lambda^{\alpha/2\beta}t) = E_\beta(-\lambda^{\alpha/2}t^\beta)$, from the series definition of (2.6) and (3.4), we see that

$$t^m|(\frac{d}{dt})^m g(t)| \le C\min(t^\beta, t^{-\beta}) \le Ct^{-\frac{\beta\mu}{2}},\ m = 0, 1, 2, \cdots,$$

so by the chain rule

$$t^m|(\frac{d}{dt})^m g(\lambda^{\alpha/2\beta}t)| = (\lambda^{\alpha/2\beta}t)^m|(\frac{d}{dt})^m g(\lambda^{\alpha/2\beta}t)| \le C(\lambda^{\alpha/2\beta}t)^{-\frac{\beta\mu}{2}} = C\lambda^{-\frac{\alpha\mu}{4}}t^{-\frac{\beta\mu}{2}}. \tag{3.5}$$

Thus, for $t > 0$ and $0 \le \nu < \alpha \le 2$, by means of (3.5), we get

$$\begin{aligned}
\|t^m(\frac{d}{dt})^m E(t)v\|^2_{\dot{H}^{\nu}} &= \sum_{k=1}^{\infty} \lambda_k^{\nu}[t^m(\frac{d}{dt})^m E_\beta(-\lambda_k^{\frac{\alpha}{2}}t^\beta)(v,\varphi_k)]^2 \\
&\le Ct^{-\beta\nu}\sum_{k=1}^{\infty} \lambda_k^{\frac{(2-\alpha)\nu}{2}}(v,\varphi_k)^2 \\
&= Ct^{-\beta\nu}\|v\|^2_{\dot{H}^{(2-\alpha)\nu/2}}.
\end{aligned}$$

Lemma 3.2. For any $0 \le t_1 < t_2 \le T$ and $0 < \nu < \alpha \le 2$, there exists a constant $C > 0$ such that

$$\|[E(t_2) - E(t_1)]v\|_{\dot{H}^{\nu}} \le C(t_2 - t_1)^{\frac{\beta\nu}{2}}\|v\|_{\dot{H}^{(2-\alpha)\nu/2}}. \tag{3.6}$$

Proof. For any $0 < T_0 \le t_1 < t_2 \le T$, by virtue of Lemma 3.1 ($m = 1$) we have

$$\begin{aligned}
\|[E(t_2) - E(t_1)]v\|_{\dot{H}^{\nu}} &= \|\int_{t_1}^{t_2} \frac{d}{dt}E(t)v dt\|_{\dot{H}^{\nu}} \\
&\le C(\int_{t_1}^{t_2} t^{-\frac{\beta\nu}{2}-1}dt)\|v\|_{\dot{H}^{(2-\alpha)\nu/2}} \\
&= \frac{2C}{\beta\nu}(t_1^{-\frac{\beta\nu}{2}} - t_2^{-\frac{\beta\nu}{2}})\|v\|_{\dot{H}^{(2-\alpha)\nu/2}} \\
&\le \frac{2C}{\beta\nu T_0^{\beta\nu}}(t_2 - t_1)^{\frac{\beta\nu}{2}}\|v\|_{\dot{H}^{(2-\alpha)\nu/2}},
\end{aligned}$$

where we have used $t_2^a - t_1^a \leq C(t_2 - t_1)^a$ for $0 \leq t_1 < t_2 \leq T$ and $0 \leq a \leq 1$.

Now, we will prove the spatial and temporal regularity properties of weak solutions to time-space fractional SPDEs in the framework of Bochner spaces.

Theorem 3.1. Let Assumptions 3.1 and 3.2 hold with $0 \leq \nu < \alpha \leq 2$ and $p \geq 2$, let $u(t)$ be a unique weak solution of (2.2) with $\mathbb{P}(u(t) \in \dot{H}^\nu) = 1$ for any $t \in [0, T]$, then there exists a constant C such that

$$\sup_{t\in[0,T]} \|u(t)\|_{L^p(\Omega;\dot{H}^\nu)} \leq Ct^{-\frac{\beta\nu}{2}} \|u_0\|_{L^p(\Omega;\dot{H}^{(2-\alpha)\nu/2})}.$$

Proof. For any $0 \leq t \leq T$ and $0 \leq \nu < \alpha < 2$, from the weak solution (2.7) we have

$$\mathbb{E}\|u(t)\|^p_{\dot{H}^\nu} \leq 2^{p-1}\mathbb{E}\|E(t)u_0\|^p_{\dot{H}^\nu} + 2^{p-1}\mathbb{E}\|\int_0^t E(t-s)\sigma(u(s))dW(s)\|^p_{\dot{H}^\nu}. \tag{3.7}$$

Using Lemma 3.1 ($m = 0$), the first term on the right hand sides of (3.7) can be estimated by

$$\mathbb{E}\|E(t)u_0\|^p_{\dot{H}^\nu} \leq Ct^{-\frac{p\beta\nu}{2}}\mathbb{E}\|u_0\|^p_{\dot{H}^{(2-\alpha)\nu/2}} < \infty. \tag{3.8}$$

By means of the generalized version of Itô isometry (2.1), Hölder's inequality, Lemma 3.1 ($m = 0$) and Assumption 3.1, we can deduce

$$\begin{aligned}
&\mathbb{E}\|\int_0^t E(t-s)\sigma(u(s))dW(s)\|^p_{\dot{H}^\nu} \\
&\leq C(p)\mathbb{E}[(\int_0^t \|E(t-s)\sigma(u(s))\|^2_{L_2^0(U_0,\dot{H}^\nu)}ds)^{\frac{p}{2}}] \\
&\leq C(p)(\int_0^t \|E(t-s)\|^{\frac{2p}{p-2}}ds)^{\frac{p-2}{2}}(\int_0^t \sup_{s\in[0,T]} \mathbb{E}\|\sigma(u(s))\|^p_{L_2^0(U_0,\dot{H}^\nu)}ds) \\
&\leq C^\dagger(p)\int_0^t \sup_{s\in[0,T]} \mathbb{E}\|u(s)\|^p_{\dot{H}^\nu}ds.
\end{aligned} \tag{3.9}$$

Combining with the estimates (3.7)-(3.9) and Assumptions 3.2, a direct application of Gronwall's lemma yields

$$\sup_{t\in[0,T]} \mathbb{E}\|u(t)\|^p_{\dot{H}^\nu} \leq Ct^{-\frac{p\beta\nu}{2}}\mathbb{E}\|u_0\|^p_{\dot{H}^{(2-\alpha)\nu/2}} \leq CT_0^{-\frac{p\beta\nu}{2}}\mathbb{E}\|u_0\|^p_{\dot{H}^{(2-\alpha)\nu/2}} < \infty, \tag{3.10}$$

where $0 < T_0 \leq t \leq T$.

This completes the proof of Theorem 3.1.

Next, we will devote to the Hölder regularity of the weak solutions (2.7) to (2.2).

Theorem 3.2. Let Assumptions 3.1 and 3.2 be fulfilled with $0 < \nu < \alpha \leq 2$ and $p \geq 2$, for any $0 \leq t_1 < t_2 \leq T$, the unique weak solution $u(t)$ to (2.2) is Hölder continuous with respect to the norm $\|\cdot\|_{L^p(\Omega;\dot{H}^\nu)}$ and satisfies

$$\|u(t_2) - u(t_1)\|_{L^p(\Omega;\dot{H}^\nu)} \leq C(t_2 - t_1)^{\min\{\frac{\beta\nu}{2},\frac{1}{2}\}}.$$

Proof. For any $0 \le t_1 < t_2 \le T$, from the expression (2.7), we have

$$\begin{aligned}
u(t_2) - u(t_1) &= E(t_2)u_0 - E(t_1)u_0 + \int_0^{t_2} E(t_2 - s)\sigma(u(s))dW(s) \\
&\quad - \int_0^{t_1} E(t_1 - s)\sigma(u(s))dW(s) \\
&= [E(t_2) - E(t_1)]u_0 + \int_0^{t_1} [E(t_2 - s) - E(t_1 - s)]\sigma(u(s))dW(s) \\
&\quad + \int_{t_1}^{t_2} E(t_2 - s)\sigma(u(s))dW(s) \\
&=: I_1 + I_2 + I_3. \qquad (3.11)
\end{aligned}$$

For any $0 < \nu < \alpha \le 2$, by virtue of Lemma 3.2, it follows that

$$\mathbb{E}\|I_1\|_{\dot{H}^\nu} = \mathbb{E}\|[E(t_2) - E(t_1)]u_0\|_{\dot{H}^\nu} \le C(t_2 - t_1)^{\frac{\beta\nu}{2}}\mathbb{E}\|u_0\|_{\dot{H}^{(2-\alpha)\nu/2}}. \qquad (3.12)$$

Applying the Hölder inequality, Assumptions 3.1, Lemma 3.2, the estimates (2.1) and (3.10), we have

$$\begin{aligned}
\mathbb{E}\|I_2\|^p_{\dot{H}^\nu} &= \mathbb{E}\|\int_0^{t_1} [E(t_2 - s) - E(t_1 - s)]\sigma(u(s))dW(s)\|^p_{\dot{H}^\nu} \\
&\le C(p)\mathbb{E}[(\int_0^{t_1} \|[E(t_2 - s) - E(t_1 - s)]\sigma(u(s))\|^2_{L_0^2(U_0,\dot{H}^\nu)}ds)^{\frac{p}{2}}] \\
&\le C(p)(\int_0^{t_1} ds)^{\frac{p-2}{2}}(\int_0^{t_1} \sup_{s\in[0,T]} \mathbb{E}\|[E(t_2 - s) - E(t_1 - s)]\sigma(u(s))\|^p_{L_0^2(U_0,\dot{H}^\nu)}ds) \\
&\le C(p)t_1^{\frac{p-2}{2}} \int_0^{t_1} (t_2 - t_1)^{\frac{p\beta\nu}{2}} \sup_{s\in[0,T]} \mathbb{E}\|u(s)\|^p_{\dot{H}^{(2-\alpha)\nu/2}}ds \\
&\le C^\dagger(p)T^{\frac{p}{2}}\mathbb{E}\|u_0\|^p_{\dot{H}^{(2-\alpha)^2\nu/4}}(t_2 - t_1)^{\frac{p\beta\nu}{2}}. \qquad (3.13)
\end{aligned}$$

Making use of Hölder inequality, the Assumptions 3.1, Lemma 3.1 ($m = 0$), the estimates (2.1) and (3.10), we obtain

$$\begin{aligned}
\mathbb{E}\|I_3\|^p_{\dot{H}^\nu} &= \mathbb{E}\|\int_{t_1}^{t_2} E(t_2 - s)\sigma(u(s))dW(s)\|^p_{\dot{H}^\nu} \\
&\le C(p)\mathbb{E}[(\int_{t_1}^{t_2} \|E(t_2 - s)\sigma(u(s))\|^2_{L_2^0(U_0,\dot{H}^\nu)}ds)^{\frac{p}{2}}] \\
&\le C(p)(\int_{t_1}^{t_2} \|E(t_2 - s)\|^{\frac{2p}{p-2}}ds)^{\frac{p-2}{2}}(\int_{t_1}^{t_2} \sup_{s\in[0,T]} \mathbb{E}\|\sigma(u(s))\|^p_{L_2^0(U_0,\dot{H}^\nu)}ds) \\
&\le C(p)(t_2 - t_1)^{\frac{p-2}{2}} \int_{t_1}^{t_2} T_0^{-\frac{p\beta\nu}{2}}\mathbb{E}\|u_0\|^p_{\dot{H}^{(2-\alpha)\nu/2}}ds \\
&\le C^\ddagger(p)\mathbb{E}\|u_0\|^p_{\dot{H}^{(2-\alpha)\nu/2}}(t_2 - t_1)^{\frac{p}{2}}. \qquad (3.14)
\end{aligned}$$

Taking expectation on both side of (3.11), and in view of the estimates (3.12)-(3.14), we conclude that

$$\|u(t_2) - u(t_1)\|_{L^p(\Omega;\dot{H}^\nu)} \le C(t_2 - t_1)^{\min\{\frac{\beta\nu}{2},\frac{1}{2}\}}.$$

This completes the proof of Theorem 3.2.

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References

[1] H.M. Srivastava, J.J. Trujillo, Theory and applications of fractional differential equations, Elsevier, 2006.

[2] Z.Q. Chen, K.H. Kim, P. Kim, Fractional time stochastic partial differential equations, Stoch. Process. Appl. 125 (2015) 1470-1499.

[3] J.B. Mijena, E. Nane, Space-time fractional stochastic partial differential equations, Stoch. Proc. Appl. 125 (2015) 3301-3326.

[4] J.B. Mijena, E. Nane, Intermittence and space-time fractional stochastic partial differential equations, Potential Anal. 44 (2016) 295-312.

[5] M. Foondun, E. Nane, Asymptotic properties of some space-time fractional stochastic equations, Math. Z. (2015) 1-27.

[6] L. Chen, G. Hu, Y. Hu, J. Huang, Space-time fractional diffusions in Gaussian noisy environment, Stochastics (2016) 1-36.

[7] C. Prévôt, M. Röckner, A concise course on stochastic partial differential equations, Springer, 2007.

[8] R. Kruse, Strong and weak approximation of semilinear stochastic evolution equations, Springer, 2014.

[9] H.M. Srivastava, J.J. Trujillo, Theory and applications of fractional differential equations, Elsevier, 2006.

[10] F. Mainardi, Fractional calculus and waves in linear viscoelasticity: an introduction to mathematical models, World Scientific, 2010.

[11] H.J. Haubold, A.M. Mathai, R.K. Saxena, Mittag-Leffler functions and their applications, J. Appl. Math. 2011 (2011) 51. Article ID 298628.

[12] F. Mainardi, The fundamental solutions for the fractional diffusion-wave equation, Appl. Math. Lett. 9(6) (1996) 23-28.

[13] W. Magnus, F. Oberhettinger, R. Soni, Formulas and theorems for the special functions of mathematical physics, Springer, 2013.

[14] V. Anh, N. Leonenko, M. Ruiz-Medina, Fractional-in-time and multifractional-in-space stochastic partial differential equations. Fract. Calc. Appl. Anal. 19(6) (2016) 1434-1459.

[15] M. Foondun, J. Mijena, E. Nane, Non-linear noise excitation for some space-time fractional stochastic equations in bounded domains. Fract. Calc. Appl. Anal. 19(6) (2016) 1527-1553.

[16] G. Zou, A. Atangana, Y. Zhou, Error estimates of a semidiscrete finite element method for fractional stochastic diffusion-wave equations. Numer. Methods Partial Differential Eq. (2018). https://doi.org/10.1002/num.22252.

[17] G. Zou, A Galerkin finite element method for time-fractional stochastic heat equation. Comput. Math. Appl. (2018). https://doi.org/10.1016/j.camwa.2018.03.019.