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ON THE PATH-INDEPENDENCE OF THE GIRSANOV TRANSFORMATION FOR STOCHASTIC EVOLUTION EQUATIONS WITH JUMPS IN HILBERT SPACES

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Abstract. Based on a recent result in [13], in this paper, we extend it to stochastic evolution equations with jumps in Hilbert spaces. This is done via Galerkin type finite-dimensional approximations of the infinite-dimensional stochastic evolution equations with jumps in a manner that one could then link the characterisation of the path-independence for finite-dimensional jump type SDEs to that for the infinite-dimensional settings. Our result provides an intrinsic link of infinite-dimensional stochastic evolution equations with jumps to infinite-dimensional (nonlinear) integro-differential equations.

1. Introduction.

The object of this paper is to characterise the path-independent property of the density process (via the exponent process) of Girsanov transformation for stochastic evolution equations with jumps in Hilbert spaces, a class of semi-linear stochastic partial differential equations (SPDEs) with jumps. The latter class of SPDEs with jumps was studied analytically in [10], which is continuously to be a hot topic nowadays. As a result, we derive an intrinsic link of stochastic evolution equations with jumps in Hilbert spaces to infinite-dimensional (nonlinear) integro-differential equations. The derived nonlinear equations involve a Burgers-KPZ type nonlinearity, which should link very well to statistical physics, such as studies of infinite interacting systems with non-Gaussian noise driven stochastic dynamics (cf. e.g. the discussions in [15]).

The investigation of path-indepedent property of the density of Girsanov transformation for Itô type SDEs on $\mathbb{R}^d$ started in [14, 17], which was inspired by interesting considerations needed in economics and mathematical finance (cf. e.g. references in [17]). In [15], Wang and the second author considered stochastic evolution equations in Hilbert spaces driven by cylindrical Brownian motion, and obtained a characterisation theorem via Galerkin type finite-dimensional approximations developed in [16]. Moreover, in [13], we studied the characterising path-independence problem

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for non-Lipschnitz SDEs with jumps on $\mathbb{R}^d$, where we derived a link of SDEs with jumps to integro-differential equations via a proper setting of Girsanov transformation for SDEs with jumps.

Due to the complexity of SPDEs with jumps, our extension can not be a straightforward analogy to [15]. In fact, we have to overcome several difficulties arising from handling equations with jumps in infinite dimensions, as well as need to establish a suitable Girsanov transformation for such equations and most importantly the Itô formula for the solutions of our stochastic evolution equations with jumps in Hilbert spaces, which we could not find in the literature. The obtained results extend both characterisation results in [15] and in [13].

The rest of the paper is organized as follows. In the next section, we will formulate and prove a proper Girsanov theorem. The equations which we are concerned with are introduced in Section 3, where we follow [16] to develop a Galerkin type finite-dimensional approximation which extends the corresponding result in [16]. Moreover, we derive an Itô's formula for the solutions by utilising the Galerkin type finite-dimensional approximation. Section 4 is devoted to proving the main results of the characterisation for infinite-dimensional equations with jumps involving and not involving cylindrical Brownian motion, respectively.

2. **The Girsanov theorem.**

In the section, we state and show a Girsanov theorem for Brownian motions and random measures in a real separable Hilbert space. We introduce our framework first.

Give a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in[0,\infty)}, \mathbb{P})$. Let $\{\beta_i(t,\omega)\}_{i\geq 1}$ be a family of mutual independent one-dimensional Brownian motions on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in[0,\infty)}, \mathbb{P})$. For a real separable Hilbert space $(\mathbb{H}, \langle\cdot,\cdot\rangle_{\mathbb{H}}, \|\cdot\|_{\mathbb{H}})$, construct a cylindrical Brownian motion on $\mathbb{H}$ with respect to $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in[0,\infty)}, \mathbb{P})$ by

$$W_t := W_t(\omega) := \sum_{i=1}^{\infty} \beta_i(t,\omega) e_i, \quad \omega \in \Omega, \ t \in [0,\infty),$$

where $\{e_i\}_{i\geq 1}$ is a complete orthonormal basis for $\mathbb{H}$ which will be specified later. It is easy to justify that the covariance operator of the cylindrical Brownian motion W is the identity operator I on $\mathbb{H}$. Note that W is not a process on $\mathbb{H}$. It is convenient to realize W as a continuous process on an enlarged Hilbert space $\tilde{\mathbb{H}}$, the completion of $\mathbb{H}$ under the inner product

$$\langle x, y\rangle_{\tilde{\mathbb{H}}} := \sum_{i=1}^{\infty} 2^{-i} \langle x, e_i\rangle \langle y, e_i\rangle, \quad x, y \in \mathbb{H}.$$

Next, we introduce the jump measures. Let $(\mathbb{U}, \mathscr{U}, \nu)$ be a given σ-finite measure space (which is interpreted as a parameter space measuring jumps) and let $\lambda : [0,\infty) \times \mathbb{U} \to (0,1)$ be a measurable function. Then, following e.g. Theorem I.8.1 of [5], there exists an integer-valued random measure on $[0,\infty) \times \mathbb{U}$

$$N_\lambda : \mathscr{B}([0,\infty) \times \mathscr{U} \times \Omega \to \mathbb{N}_0 := \mathbb{N} \cup \{0\} \cup \{\infty\}$$

with intensity measure (i.e., its predictable compensator) $\lambda(t,u)\mathrm{d}t\nu(\mathrm{d}u)$:

$$\mathbb{E}N_\lambda(\mathrm{d}t, \mathrm{d}u, \cdot) = \lambda(t,u)\mathrm{d}t\nu(\mathrm{d}u).$$

Set

$$\tilde{N}_\lambda(\mathrm{d}t, \mathrm{d}u) := N_\lambda(\mathrm{d}t, \mathrm{d}u) - \lambda(t,u)\mathrm{d}t\nu(\mathrm{d}u),$$

and then $\tilde{N}_\lambda(\mathrm{d}t,\mathrm{d}u)$ is the associated compensated martingale measure of $N_\lambda(\mathrm{d}t,\mathrm{d}u)$. Moreover, we assume that W_t and N_λ are independent.

Fix arbitrarily $T>0$ and $\mathbb{U}_0\in\mathscr{U}$ with $\nu(\mathbb{U}\setminus\mathbb{U}_0)<\infty$. Set for any $t\in[0,T]$

$$\begin{aligned} Z_t &:= W_t+\int_0^t\gamma(s,x)\mathrm{d}s+\int_0^t\int_{\mathbb{U}_0}\varphi(s,x,u)\tilde{N}_\lambda(\mathrm{d}s,\mathrm{d}u) \\ &\quad +\int_0^t\int_{\mathbb{U}\setminus\mathbb{U}_0}\psi(s,x,u)N_\lambda(\mathrm{d}s,\mathrm{d}u), \end{aligned}$$

where $\gamma:[0,T]\times\Omega\times\mathbb{H}\mapsto\tilde{\mathbb{H}}$ is $\mathcal{P}\otimes\mathscr{B}(\mathbb{H})/\mathscr{B}(\tilde{\mathbb{H}})$-measurable, and $\varphi:[0,T]\times\Omega\times\mathbb{H}\times\mathbb{U}_0\mapsto\tilde{\mathbb{H}}$ is $\mathcal{P}\otimes\mathscr{B}(\mathbb{H})\otimes\mathscr{U}|_{\mathbb{U}_0}/\mathscr{B}(\tilde{\mathbb{H}})$-measurable and $\psi:[0,T]\times\Omega\times\mathbb{H}\times(\mathbb{U}\setminus\mathbb{U}_0)\mapsto\tilde{\mathbb{H}}$ is $\mathcal{P}\otimes\mathscr{B}(\mathbb{H})\otimes\mathscr{U}|_{\mathbb{U}\setminus\mathbb{U}_0}/\mathscr{B}(\tilde{\mathbb{H}})$-measurable, therein $\mathcal{P}$ stands for the predictable σ-algebra on $[0,T]\times\Omega$. Put

$$\begin{aligned} \Lambda_t: &= \exp\Big\{-\int_0^t\langle\gamma(s,x),\mathrm{d}W_s\rangle_{\tilde{\mathbb{H}}}-\frac{1}{2}\int_0^t\|\gamma(s,x)\|_{\tilde{\mathbb{H}}}^2\,\mathrm{d}s-\int_0^t\int_{\mathbb{U}_0}\log\lambda(s,u)N_\lambda(\mathrm{d}s,\mathrm{d}u) \\ &\quad -\int_0^t\int_{\mathbb{U}_0}(1-\lambda(s,u))\nu(\mathrm{d}u)\mathrm{d}s\Big\}. \end{aligned}$$

We will use Λ_t to define a new probability measure $\hat{\mathbb{P}}$ and show that under the measure $\hat{\mathbb{P}}$, Z_t has a simpler form, namely, we get the following Girsanov theorem in our present framework.

Theorem 2.1. *Assume that*

$$\mathbb{E}[\Lambda_T]=1. \tag{1}$$

Then under the probability $\mathrm{d}\hat{\mathbb{P}}:=\Lambda_T\mathrm{d}\mathbb{P}$, *the process* $Z_t,t\in[0,T]$, *has the following form*

$$Z_t=\hat{W}_t+\int_0^t\int_{\mathbb{U}_0}\varphi(s,x,u)\tilde{N}(\mathrm{d}s,\mathrm{d}u)+\int_0^t\int_{\mathbb{U}\setminus\mathbb{U}_0}\psi(s,x,u)N_\lambda(\mathrm{d}s,\mathrm{d}u),\quad t\in[0,T],$$

where, on the new filtered probability space $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\in[0,T]},\hat{\mathbb{P}})$, $\hat{W}_t:=W_t+\int_0^t\gamma(s,x)\mathrm{d}s$ *is a cylindrical Brownian motion, and the predictable compensator and the compensated martingale measure of* $N_\lambda(\mathrm{d}t,\mathrm{d}u)$ *are* $\mathrm{d}t\nu(\mathrm{d}u)$ *and* $\tilde{N}(\mathrm{d}t,\mathrm{d}u)$, *respectively.*

Proof. For the cylindrical Brownian motion W, one could use the method similar to that in [3, Theorem 10.14] with some slight modifications. The proof is then completed by directly applying Theorem 3.17 in [6] to the random measure $N_\lambda(\mathrm{d}t,\mathrm{d}u)$. □

In Section 4, the above theorem will be used to transform certain relevant processes.

Next, we would like to present a sufficient condition on γ and λ such that Λ_T fulfills the assumption (1). Note that $\Lambda_t,t\in[0,T]$, is the Doléans-Dade exponential of $M_t,t\in[0,T]$, i.e.,

$$M_t: = -\int_0^t\langle\gamma(s,x),\mathrm{d}W_s\rangle_{\tilde{\mathbb{H}}}+\int_0^t\int_{\mathbb{U}_0}\frac{1-\lambda(s,u)}{\lambda(s,u)}\tilde{N}_\lambda(\mathrm{d}s,\mathrm{d}u),\quad t\in[0,T].$$

Thus, we will analyze M_t to get the desired sufficient condition. Firstly, we have

$$\Delta M_t := M_t - M_{t-} = \frac{1-\lambda(t,u)}{\lambda(t,u)} = \frac{1}{\lambda(t,u)} - 1 > -1, \quad t \in [0,T].$$

Secondly, we assume the following

(H2.1)

$$\begin{aligned} &\mathbb{E}\Big[\exp\Big\{\frac{1}{2}\int_0^T \|\gamma(s,x)\|_{\tilde{\mathbb{H}}}^2 \mathrm{d}s + \int_0^T\int_{\mathbb{U}_0}\left(\frac{1-\lambda(s,u)}{\lambda(s,u)}\right)^2 \lambda(s,u)\nu(\mathrm{d}u)\mathrm{d}s\Big\}\Big] \\ < \ &\infty. \end{aligned}$$

Then, under **(H2.1)**, M_t is a locally square integrable martingale. Moreover, let M^c and M^d stand for continuous and purely discontinuous martingale parts of M, respectively, then

$$\begin{aligned} &\mathbb{E}\Big[\exp\Big\{\frac{1}{2} < M^c, M^c >_T + < M^d, M^d >_T \Big\}\Big] \\ = \ &\mathbb{E}\Big[\exp\Big\{\frac{1}{2}\int_0^T \|\gamma(s,x)\|_{\tilde{\mathbb{H}}}^2 \mathrm{d}s + \int_0^T\int_{\mathbb{U}_0}\left(\frac{1-\lambda(s,u)}{\lambda(s,u)}\right)^2 \lambda(s,u)\nu(\mathrm{d}u)\mathrm{d}s\Big\}\Big] \\ < \ &\infty. \end{aligned}$$

Thus, it follows from [11, Theorem 6] that $\Lambda_t, t \in [0,T]$ is a exponential martingale and satisfies the condition (1).

3. **Stochastic evolution equations with jumps on $\mathbb{H}$.** In the section, we consider stochastic evolution equations with jumps in our infinite dimensional setting. Let us begin with some notions and notations. For the Hilbert space $\mathbb{H}$ given in the previous section, $L(\mathbb{H})$ is the set of all bounded linear operators $\mathcal{L} : \mathbb{H} \to \mathbb{H}$ and $L_{HS}(\mathbb{H})$ is the collection of all Hilbert-Schmidt operator $\mathcal{L} : \mathbb{H} \to \mathbb{H}$ equipped with the Hilbert-Schmidt norm $\|\cdot\|_{HS}$.

Fix a linear, unbounded, negative definite and self-adjoint operator $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ on $\mathbb{H}$, where $\mathcal{D}(\mathcal{A})$ is the domain of the operator $\mathcal{A}$. Let $\{e^{t\mathcal{A}}\}_{t\geq 0}$ be the contraction C_0-semigroup generated by $\mathcal{A}$. Moreover, $L_{\mathcal{A}}(\mathbb{H})$ stands for the family of all densely defined closed linear operators $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ on $\mathbb{H}$ so that $e^{t\mathcal{A}}\mathcal{L}$ can extend uniquely to a Hilbert-Schmidt operator still denoted by $e^{t\mathcal{A}}\mathcal{L}$ for any $t > 0$. And then $L_{\mathcal{A}}(\mathbb{H})$, endowed with the σ-algebra induced by $\{\mathcal{L} \to \langle e^{tA}\mathcal{L}x, y\rangle_{\mathbb{H}} \mid t > 0, x, y \in \mathbb{H}\}$, becomes a measurable space.

Give $T > 0$. Consider the following stochastic evolution equation with jumps on $\mathbb{H}$

$$\left\{\begin{array}{l} dX_t = \{\mathcal{A}X_t + b(t,X_t)\}dt + \sigma(t,X_t)dW_t + \int_{\mathbb{U}_0} f(t,X_{t-},u)\tilde{N}_\lambda(\mathrm{d}t,\mathrm{d}u), \quad 0 < t \leq T \\ X_0 = x_0 \in \mathbb{H}, \end{array}\right. \tag{2}$$

where $b : [0,\infty) \times \mathbb{H} \to \tilde{\mathbb{H}}$, $\sigma : [0,\infty) \times \mathbb{H} \to L_{\mathcal{A}}(\mathbb{H})$ and $f : [0,\infty) \times \mathbb{H} \times \mathbb{U}_0 \mapsto \tilde{\mathbb{H}}$ are all Borel measurable mappings. Set $\|x\|_{\mathbb{H}} = \infty, x \notin \mathbb{H}$. Let us give a definition of mild solutions to Eq.(2), which will be used in the sequel.

Definition 3.1. A $\mathbb{H}$-valued predictable process $X_t, t \in [0,T]$ is called a mild solution of Eq.(2) if for any $t \in [0,T]$

$$\mathbb{E}\int_0^t \|e^{(t-s)\mathcal{A}}b(s,X_s)\|_{\mathbb{H}}^2 \mathrm{d}s + \mathbb{E}\int_0^t \|e^{(t-s)\mathcal{A}}\sigma(s,X_s)\|_{HS}^2 \mathrm{d}s$$
$$+\mathbb{E}\int_0^t\int_{\mathbb{U}_0} \|e^{(t-s)\mathcal{A}}f(s,X_{s-},u)\|_{\mathbb{H}}^2 \lambda(s,u)\nu(\mathrm{d}u)\mathrm{d}s < \infty,$$

and $\mathbb{P}$-a.s.

$$\begin{aligned} X_t &= e^{t\mathcal{A}}x_0 + \int_0^t e^{(t-s)\mathcal{A}}b(s,X_s)ds + \int_0^t e^{(t-s)\mathcal{A}}\sigma(s,X_s)dW_s \\ &\quad + \int_0^t\int_{\mathbb{U}_0} e^{(t-s)\mathcal{A}}f(s,X_{s-},u)\tilde{N}_\lambda(\mathrm{d}s,\mathrm{d}u). \end{aligned}$$

Next, let us derive existence and uniqueness for the mild solution of Eq.(2). To this end, we assume the following

(H3.1) There exists an integrable function $L_b : (0,T] \to (0,\infty)$ such that

$$\|e^{s\mathcal{A}}(b(t,x)-b(t,y))\|_{\mathbb{H}}^2 \le L_b(s)\|x-y\|_{\mathbb{H}}^2, \quad s \in (0,T], t \in [0,T], x,y \in \mathbb{H},$$

and

$$\int_0^T \sup_{r\in[0,T]} \|e^{s\mathcal{A}}b(r,0)\|_{\mathbb{H}}^2 \mathrm{d}s < \infty.$$

(H3.2) There exists an integrable function $L_\sigma : (0,T] \to (0,\infty)$ such that $\forall s \in (0,T], t \in [0,T]$ and $\forall x,y \in \mathbb{H}$

$$\|e^{s\mathcal{A}}\left(\sigma(t,x)-\sigma(t,y)\right)\|_{HS}^2 \le L_\sigma(s)\|x-y\|_{\mathbb{H}}^2$$

and

$$\int_0^T \sup_{r\in[0,T]} \|e^{s\mathcal{A}}\sigma(r,0)\|_{HS}^2 \mathrm{d}s < \infty.$$

(H3.3) There exists an integrable function $L_f : (0,T] \to (0,\infty)$ such that $\forall s \in (0,T], t \in [0,T]$ and $\forall x,y \in \mathbb{H}$

$$\int_{\mathbb{U}_0} \|e^{s\mathcal{A}}(f(t,x,u)-f(t,y,u))\|_{\mathbb{H}}^2 \lambda(t,u)\nu(\mathrm{d}u) \le L_f(s)\|x-y\|_{\mathbb{H}}^2$$

and

$$\int_0^T\int_{\mathbb{U}_0} \sup_{r\in[0,T]} \left(\|e^{s\mathcal{A}}f(r,0,u)\|_{\mathbb{H}}^2\lambda(r,u)\right)\nu(\mathrm{d}u)\mathrm{d}s < \infty.$$

Remark 1. (i) Under **(H3.1)**-**(H3.3)**, the following hold

$$\begin{aligned} \|e^{s\mathcal{A}}b(t,x)\|_{\mathbb{H}}^2 &\le 2L_b(s)\|x\|_{\mathbb{H}}^2 + 2\|e^{s\mathcal{A}}b(t,0)\|_{\mathbb{H}}^2, \\ \|e^{s\mathcal{A}}\sigma(t,x)\|_{HS}^2 &\le 2L_\sigma(s)\|x\|_{\mathbb{H}}^2 + 2\|e^{s\mathcal{A}}\sigma(t,0)\|_{HS}^2, \\ \int_{\mathbb{U}_0} \|e^{s\mathcal{A}}f(t,x,u)\|_{\mathbb{H}}^2\lambda(t,u)\nu(\mathrm{d}u) &\le 2L_f(s)\|x\|_{\mathbb{H}}^2 + 2\int_{\mathbb{U}_0} \|e^{s\mathcal{A}}f(t,0,u)\|_{\mathbb{H}}^2\lambda(t,u)\nu(\mathrm{d}u). \end{aligned}$$

These conditions are nothing but similar to linear growth conditions.

(ii) Comparing **(H3.1)**-**(H3.3)** with those in [7, Theorem 2.3], one could find that our conditions are more general.

We are now ready to give the existence and uniqueness result of the mild solution of Eq.(2) under **(H3.1)**-**(H3.3)**.

Theorem 3.2. *Suppose that b,σ,f satisfy* **(H3.1)-(H3.3)**. *Then there exists a unique mild solution X of Eq.(2) with the following property*

$$
\sup_{t\in[0,T]}\mathbb{E}\|X_t\|_{\mathbb{H}}^2<\infty.
$$

Proof. Denote by $\mathcal{H}$ the set of all $\mathbb{H}$-valued predictable processes $Y=(Y_t)_{t\in[0,T]}$ satisfying $\sup\limits_{t\in[0,T]}\mathbb{E}\|Y_t\|_{\mathbb{H}}^2<\infty$. For $Y\in\mathcal{H}$, set

$$
\begin{aligned}
J(Y)(t) \quad := \quad & e^{t\mathcal{A}}x_0+\int_0^t e^{(t-s)\mathcal{A}}b(s,Y_s)ds+\int_0^t e^{(t-s)\mathcal{A}}\sigma(s,Y_s)dW_s\\
&+\int_0^t\int_{\mathbb{U}_0} e^{(t-s)\mathcal{A}}f(s,Y_{s-},u)\tilde{N}_\lambda(\mathrm{d}s,\mathrm{d}u),
\end{aligned}
$$

and then $J(Y)\in\mathcal{H}$. In fact, by the isometry formula and Remark 1, it holds that for $0\leq s<t\leq T$,

$$
\begin{aligned}
&\mathbb{E}\|\int_0^t\int_{\mathbb{U}_0}e^{(t-r)\mathcal{A}}f(r,Y_{r-},u)\tilde{N}_\lambda(\mathrm{d}r,\mathrm{d}u)-\int_0^s\int_{\mathbb{U}_0}e^{(s-r)\mathcal{A}}f(r,Y_{r-},u)\tilde{N}_\lambda(\mathrm{d}r,\mathrm{d}u)\|_{\mathbb{H}}^2\\
\leq\ & 2\mathbb{E}\|\int_0^s\int_{\mathbb{U}_0}e^{(t-r)\mathcal{A}}f(r,Y_{r-},u)\tilde{N}_\lambda(\mathrm{d}r,\mathrm{d}u)-\int_0^s\int_{\mathbb{U}_0}e^{(s-r)\mathcal{A}}f(r,Y_{r-},u)\tilde{N}_\lambda(\mathrm{d}r,\mathrm{d}u)\|_{\mathbb{H}}^2\\
&+2\mathbb{E}\|\int_s^t\int_{\mathbb{U}_0}e^{(t-r)\mathcal{A}}f(r,Y_{r-},u)\tilde{N}_\lambda(\mathrm{d}r,\mathrm{d}u)\|_{\mathbb{H}}^2\\
=\ & 2\mathbb{E}\int_0^s\int_{\mathbb{U}_0}\|e^{(t-r)\mathcal{A}}f(r,Y_{r-},u)-e^{(s-r)\mathcal{A}}f(r,Y_{r-},u)\|_{\mathbb{H}}^2\lambda(r,u)\nu(\mathrm{d}u)\mathrm{d}r\\
&+2\mathbb{E}\int_s^t\int_{\mathbb{U}_0}\|e^{(t-r)\mathcal{A}}f(r,Y_{r-},u)\|_{\mathbb{H}}^2\lambda(r,u)\nu(\mathrm{d}u)\mathrm{d}r\\
\leq\ & 2\|e^{(t-s)\mathcal{A}}-I\|^2\mathbb{E}\int_0^s\int_{\mathbb{U}_0}\|e^{(s-r)\mathcal{A}}f(r,Y_{r-},u)\|_{\mathbb{H}}^2\lambda(r,u)\nu(\mathrm{d}u)\mathrm{d}r\\
&+2\mathbb{E}\int_s^t\int_{\mathbb{U}_0}\|e^{(t-r)\mathcal{A}}f(r,Y_{r-},u)\|_{\mathbb{H}}^2\lambda(r,u)\nu(\mathrm{d}u)\mathrm{d}r\\
\leq\ & 4\|e^{(t-s)\mathcal{A}}-I\|^2\Big(\sup_{t\in[0,T]}\mathbb{E}\|Y_t\|_{\mathbb{H}}^2\int_0^s L_f(r)\mathrm{d}r\Big)\\
&+4\|e^{(t-s)\mathcal{A}}-I\|^2\Big(\int_0^s\int_{\mathbb{U}_0}\sup_{r\in[0,T]}\Big(\|e^{v\mathcal{A}}f(r,0,u)\|_{\mathbb{H}}^2\lambda(r,u)\Big)\nu(\mathrm{d}u)\mathrm{d}v\Big)\\
&+4\sup_{t\in[0,T]}\mathbb{E}\|Y_t\|_{\mathbb{H}}^2\int_0^{t-s}L_f(r)\mathrm{d}r+4\int_0^{t-s}\int_{\mathbb{U}_0}\sup_{r\in[0,T]}\Big(\|e^{v\mathcal{A}}f(r,0,u)\|_{\mathbb{H}}^2\lambda(r,u)\Big)\nu(\mathrm{d}u)\mathrm{d}v.
\end{aligned}
$$

Taking the limits on two sides as $s\to t$, we obtain that $\int_0^t\int_{\mathbb{U}_0}e^{(t-r)\mathcal{A}}f(r,Y_{r-},u)\tilde{N}_\lambda(\mathrm{d}r,\mathrm{d}u)$ is mean square continuous in t. And mean square continuity of $e^{t\mathcal{A}}x_0$, $\int_0^t e^{(t-s)\mathcal{A}}b(s,Y_s)ds$, $\int_0^t e^{(t-s)\mathcal{A}}\sigma(s,Y_s)dW_s$ is easy to verify. Therefore, $J(Y)(t)$ is mean square continuous in t and then is predictable.

Moreover, it follows from the above definition, the Hölder inequality and the isometry formula that

$$\begin{aligned}
&\mathbb{E}\|J(Y)(t)\|_{\mathbb{H}}^2 \\
\leq\ & 4\|e^{t\mathcal{A}}x_0\|_{\mathbb{H}}^2 + 4t\mathbb{E}\int_0^t \|e^{(t-s)\mathcal{A}}b(s,Y_s)\|_{\mathbb{H}}^2 \mathrm{d}s + 4\mathbb{E}\|\int_0^t e^{(t-s)\mathcal{A}}\sigma(s,Y_s)dW_s\|_{\mathbb{H}}^2 \\
&+4\mathbb{E}\|\int_0^t\int_{\mathbb{U}_0} e^{(t-s)\mathcal{A}}f(s,Y_{s-},u)\tilde{N}_\lambda(\mathrm{d}s,\mathrm{d}u)\|_{\mathbb{H}}^2 \\
\leq\ & 4\|e^{t\mathcal{A}}x_0\|_{\mathbb{H}}^2 + 4t\mathbb{E}\int_0^t \|e^{(t-s)\mathcal{A}}b(s,Y_s)\|_{\mathbb{H}}^2 \mathrm{d}s + 4\mathbb{E}\int_0^t \|e^{(t-s)\mathcal{A}}\sigma(s,Y_s)\|_{HS}^2 \mathrm{d}s \\
&+4\mathbb{E}\int_0^t\int_{\mathbb{U}_0} \|e^{(t-s)\mathcal{A}}f(s,Y_{s-},u)\|_{\mathbb{H}}^2\lambda(s,u)\nu(\mathrm{d}u)\mathrm{d}s.
\end{aligned}$$

Remark 1 then let us to obtain further

$$\begin{aligned}
&\mathbb{E}\|J(Y)(t)\|_{\mathbb{H}}^2 \\
\leq\ & 4\|e^{t\mathcal{A}}x_0\|_{\mathbb{H}}^2 + 8t\int_0^t L_b(t-s)\mathbb{E}\|Y_s\|_{\mathbb{H}}^2\mathrm{d}s + 8\int_0^t L_\sigma(t-s)\mathbb{E}\|Y_s\|_{\mathbb{H}}^2\mathrm{d}s \\
&+8\int_0^t L_f(t-s)\mathbb{E}\|Y_s\|_{\mathbb{H}}^2\mathrm{d}s + 8\int_0^t \Big[t\|e^{(t-s)\mathcal{A}}b(s,0)\|_{\mathbb{H}}^2 + \|e^{(t-s)\mathcal{A}}\sigma(s,0)\|_{HS}^2 \\
&+\int_{\mathbb{U}_0}\|e^{(t-s)\mathcal{A}}f(s,0,u)\|_{\mathbb{H}}^2\lambda(s,u)\nu(\mathrm{d}u)\Big]\mathrm{d}s \\
\leq\ & 4\|e^{t\mathcal{A}}x_0\|_{\mathbb{H}}^2 + 8\sup_{t\in[0,T]}\mathbb{E}\|Y_t\|_{\mathbb{H}}^2\left(t\int_0^t L_b(s)\mathrm{d}s + \int_0^t L_\sigma(s)\mathrm{d}s + \int_0^t L_f(s)\mathrm{d}s\right) \\
&+8\int_0^t\Big[t\sup_{r\in[0,T]}\|e^{s\mathcal{A}}b(r,0)\|_{\mathbb{H}}^2 + \sup_{r\in[0,T]}\|e^{s\mathcal{A}}\sigma(r,0)\|_{HS}^2 \\
&+\int_{\mathbb{U}_0}\sup_{r\in[0,T]}\Big(\|e^{s\mathcal{A}}f(r,0,u)\|_{\mathbb{H}}^2\lambda(r,u)\Big)\nu(\mathrm{d}u)\Big]\mathrm{d}s.
\end{aligned}$$

By **(H3.1)** **(H3.2)** **(H3.3)** again, it holds that $\sup_{t\in[0,T]}\mathbb{E}\|J(Y)(t)\|_{\mathbb{H}}^2 < \infty$.

Next, let us calculate $\sup_{s\in[0,t]}\mathbb{E}\|J(Y^1)(s) - J(Y^2)(s)\|_{\mathbb{H}}^2$ for $Y^1, Y^2 \in \mathcal{H}$. By similar derivation to the above, one could have

$$\begin{aligned}
\sup_{s\in[0,t]}\mathbb{E}\|J(Y^1)(s)-J(Y^2)(s)\|_{\mathbb{H}}^2 \leq\ & 3\left(t\int_0^t L_b(r)\mathrm{d}r + \int_0^t L_\sigma(r)\mathrm{d}r + \int_0^t L_f(r)\mathrm{d}r\right) \\
&\cdot\sup_{s\in[0,t]}\mathbb{E}\|Y_s^1 - Y_s^2\|_{\mathbb{H}}^2.
\end{aligned}$$

Since $\lim_{t\to 0} 3\left(t\int_0^t L_b(r)\mathrm{d}r + \int_0^t L_\sigma(r)\mathrm{d}r + \int_0^t L_f(r)\mathrm{d}r\right) = 0$, there exists a $0 < t_0 \leq T$ such that $3\left(t_0\int_0^{t_0} L_b(r)\mathrm{d}r + \int_0^{t_0} L_\sigma(r)\mathrm{d}r + \int_0^{t_0} L_f(r)\mathrm{d}r\right) < 1$. Thus on $[0,t_0]$ the mapping J has a unique fixed point Y which is a unique mild solution of Eq.(2). If $t_0 = T$, the proof is finished. If $t_0 < T$, we repeat the above procedure to get a unique mild solution of Eq.(2) on $[t_0,t_1]$ for some $t_1 \in (t_0,T]$. The approach is further utilsed till $t_n = T$ so that a unique mild solution of Eq.(2) on the whole interval $[0,T]$ is obtained. We are done. □

Next, we will construct a finite dimensional approximation to Eq.(2) to set up a relation between Eq.(2) and a finite dimensional SDE with jumps. To be more precise, we will set up the Galerkin approximation to Eq.(2), for which we shall need the following assumption:

(H3.4) The operator $-\mathcal{A}$ has the following eigenvalues

$$0<\lambda_1\leq\lambda_2\leq\ldots\leq\lambda_j\leq\ldots$$

counting multiplicities such that

$$\sum_{j=1}^{\infty}\frac{1}{\lambda_j}<\infty\,.$$

We would like to emphasize here that the complete orthonormal basis $\{e_j\}_{j\in\mathbb{N}}$ is taken as the eigen-basis of $-\mathcal{A}$ throughout the rest of the paper.

Remark 2. Note that by **(H3.4)**, there are invertible operators on $\mathbb{H}$ in $L_{\mathcal{A}}(\mathbb{H})$, such as the identity operator I.

Recall that from now on we have the fixed complete orthonormal basis $\{e_j\}_{j\in\mathbb{N}}$ for $\mathbb{H}$ as specified in **(H3.4)**. Set

$$\pi_n:\mathbb{H}\rightarrow\mathbb{H}_n:=\text{span}\{e_1,\cdots,e_n\},\quad n\in\mathbb{N},$$
$$\pi_n x:=\sum_{i=1}^{n}\langle x,e_i\rangle_{\mathbb{H}}e_i,\quad x\in\mathbb{H},$$

and then π_n is the orthogonal project operator from $\mathbb{H}$ to $\mathbb{H}_n$. Moreover, $\pi_n e^{t\mathcal{A}}=e^{t\mathcal{A}}\pi_n$ for $t\geq 0$. Again set $\mathcal{A}_n:=\mathcal{A}\mid_{\mathbb{H}_n}, b_n:=\pi_n b$, $\sigma_n:=\pi_n\sigma$ and $f_n:=\pi_n f$. And then consider the following SDE with jumps in $\mathbb{H}_n$

$$\begin{cases}dX_t^n=\{\mathcal{A}_nX_t^n+b_n(t,X_t^n)\}dt+\sigma_n(t,X_t^n)dW_t+\int_{\mathbb{U}_0}f_n(t,X_{t-}^n,u)\tilde{N}_\lambda(\mathrm{d}t,\mathrm{d}u),\\ X^n(0)=\pi_n x_0.\end{cases}\tag{3}$$

Under **(H3.1)**-**(H3.4)**, one can justify that the coefficients b_n, σ_n and f_n are Lipschitz continuous and linearly growing. For example, for σ_n, noting that $W_t=\sum_{j=1}^{\infty}\beta_j(t,\omega)e_j$, we deduce that

$$\begin{aligned}&\sum_{j=1}^{\infty}\|\sigma_n(t,x)e_j\|_{\mathbb{H}_n}^2\\ =\;&\sum_{j=1}^{\infty}\sum_{i=1}^{n}|\langle\sigma(t,x)e_j,e_i\rangle_{\mathbb{H}}|^2=\sum_{j=1}^{\infty}\sum_{i=1}^{n}e^{2\lambda_iT}|\langle\sigma(t,x)e_j,e^{-\lambda_iT}e_i\rangle_{\mathbb{H}}|^2\\ =\;&\sum_{j=1}^{\infty}\sum_{i=1}^{n}e^{2\lambda_iT}|\langle\sigma(t,x)e_j,e^{T\mathcal{A}}e_i\rangle_{\mathbb{H}}|^2\leq e^{2\lambda_nT}\sum_{j=1}^{\infty}\sum_{i=1}^{n}|\langle e^{T\mathcal{A}}\sigma(t,x)e_j,e_i\rangle_{\mathbb{H}}|^2\\ \leq\;&e^{2\lambda_nT}\|e^{T\mathcal{A}}\sigma(t,x)\|_{HS}^2\leq e^{2\lambda_nT}\left(2L_\sigma(T)\|x\|_{\mathbb{H}}^2+2\|e^{T\mathcal{A}}\sigma(t,0)\|_{HS}^2\right),\end{aligned}$$

i.e. σ_n is linearly growing. Thus, by [5, Theorem 9.1], there exists a unique strong solution $X_t^n\in\mathbb{H}_n, t\in[0,T]$ to Eq.(3). Moreover, we have the following result.

Lemma 3.3. *Under* **(H3.1)**-**(H3.4)**,

$$\lim_{n\rightarrow\infty}\mathbb{E}\|X_t^n-X_t\|_{\mathbb{H}}^2=0,\quad t\in[0,T].\tag{4}$$

Proof. Note that X^n, the unique strong solution to Eq.(3), also satisfies the following equation

$$
\begin{aligned}
X_t^n &= e^{t\mathcal{A}_n}\pi_n x_0 + \int_0^t e^{(t-s)\mathcal{A}_n} b_n(s, X_s^n) ds + \int_0^t e^{(t-s)\mathcal{A}_n}\sigma_n(s, X_s^n) dW_s \\
&\quad + \int_0^t \int_{\mathbb{U}_0} e^{(t-s)\mathcal{A}_n} f_n(s, X_{s-}^n, u)\tilde{N}_\lambda(\mathrm{d}s, \mathrm{d}u), \quad t \in [0, T]. \qquad (5)
\end{aligned}
$$

Based on this and Definition 3.1, we compute $\mathbb{E}\|X_t^n - X_t\|_{\mathbb{H}}^2$. By the Hölder inequality and the isometry formula, it holds that

$$
\begin{aligned}
& \mathbb{E}\|X_t^n - X_t\|_{\mathbb{H}}^2 \\
\leq\ & 4\|e^{t\mathcal{A}_n}\pi_n x_0 - e^{t\mathcal{A}}x_0\|_{\mathbb{H}}^2 + 4t\mathbb{E}\int_0^t \|e^{(t-s)\mathcal{A}_n} b_n(s, X_s^n) - e^{(t-s)\mathcal{A}} b(s, X_s)\|_{\mathbb{H}}^2 \mathrm{d}s \\
& +4\mathbb{E}\int_0^t \|e^{(t-s)\mathcal{A}_n}\sigma_n(s, X_s^n) - e^{(t-s)\mathcal{A}}\sigma(s, X_s)\|_{HS}^2 \mathrm{d}s \\
& +4\mathbb{E}\int_0^t \int_{\mathbb{U}_0} \|e^{(t-s)\mathcal{A}_n} f_n(s, X_{s-}^n, u) - e^{(t-s)\mathcal{A}} f(s, X_{s-}, u)\|_{\mathbb{H}}^2 \lambda(s, u)\nu(\mathrm{d}u)\mathrm{d}s \\
=:\ & I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

For I_1, since $e^{t\mathcal{A}_n}\pi_n x_0 = \pi_n e^{t\mathcal{A}}x_0$,

$$
I_1 \leq 4\|\pi_n - I\|^2 \|e^{t\mathcal{A}}x_0\|_{\mathbb{H}}^2. \qquad (6)
$$

And then we deal with I_2. It follows from **(H3.1)** that

$$
\begin{aligned}
I_2 &\leq 4t\mathbb{E}\int_0^t \Big[2\|e^{(t-s)\mathcal{A}_n} b_n(s, X_s^n) - e^{(t-s)\mathcal{A}_n} b_n(s, X_s)\|_{\mathbb{H}}^2 \\
&\quad +2\|e^{(t-s)\mathcal{A}_n} b_n(s, X_s) - e^{(t-s)\mathcal{A}} b(s, X_s)\|_{\mathbb{H}}^2\Big] \mathrm{d}s \\
&\leq 4t\mathbb{E}\int_0^t \Big[2\|e^{(t-s)\mathcal{A}} b(s, X_s^n) - e^{(t-s)\mathcal{A}} b(s, X_s)\|_{\mathbb{H}}^2 \\
&\quad +2\|\pi_n - I\|^2 \|e^{(t-s)\mathcal{A}} b(s, X_s)\|_{\mathbb{H}}^2\Big] \mathrm{d}s \\
&\leq 8t\int_0^t L_b(t-s)\mathbb{E}\|X_s^n - X_s\|_{\mathbb{H}}^2 \mathrm{d}s \\
&\quad +8t\|\pi_n - I\|^2 \int_0^t \mathbb{E}\|e^{(t-s)\mathcal{A}} b(s, X_s)\|_{\mathbb{H}}^2 \mathrm{d}s. \qquad (7)
\end{aligned}
$$

By the similar deduction to the above, we obtain that

$$
\begin{aligned}
I_3 + I_4 &\leq 8\int_0^t \left(L_\sigma(t-s) + L_f(t-s)\right)\mathbb{E}\|X_s^n - X_s\|_{\mathbb{H}}^2 \mathrm{d}s \\
&\quad +8\|\pi_n - I\|^2 \int_0^t \mathbb{E}\|e^{(t-s)\mathcal{A}}\sigma(s, X_s)\|_{HS}^2 \mathrm{d}s \\
&\quad +8\|\pi_n - I\|^2 \int_0^t \int_{\mathbb{U}_0} \mathbb{E}\|e^{(t-s)\mathcal{A}} f(s, X_{s-}, u)\|_{HS}^2 \lambda(s, u)\nu(\mathrm{d}u)\mathrm{d}s. \qquad (8)
\end{aligned}
$$

Combining (6) (7) with (8), we have further that

$$\begin{aligned}\mathbb{E}\|X_t^n-X_t\|_{\mathbb{H}}^2 &\leq 4\|\pi_n-I\|^2\|e^{t\mathcal{A}}x_0\|_{\mathbb{H}}^2+8\|\pi_n-I\|^2U_t\\ &\quad+8\int_0^t\left(TL_b(t-s)+L_\sigma(t-s)+L_f(t-s)\right)\mathbb{E}\|X_s^n-X_s\|_{\mathbb{H}}^2\mathrm{d}s,\end{aligned}\tag{9}$$

where

$$\begin{aligned}U_t &:= t\int_0^t\mathbb{E}\|e^{(t-s)\mathcal{A}}b(s,X_s)\|_{\mathbb{H}}^2\mathrm{d}s+\int_0^t\mathbb{E}\|e^{(t-s)\mathcal{A}}\sigma(s,X_s)\|_{HS}^2\mathrm{d}s\\ &\quad+\int_0^t\int_{\mathbb{U}_0}\mathbb{E}\|e^{(t-s)\mathcal{A}}f(s,X_{s-},u)\|_{HS}^2\lambda(s,u)\nu(\mathrm{d}u)\mathrm{d}s.\end{aligned}$$

By Definition 3.1 and Theorem 3.2, it holds that $U_t<\infty$ and $\sup\limits_{t\in[0,T]}\mathbb{E}\|X_t\|_{\mathbb{H}}^2<\infty$.

Next, we compute $\sup\limits_{n\geq 1}\sup\limits_{t\in[0,T]}\mathbb{E}\|X_t^n\|_{\mathbb{H}}^2$. For Eq.(5), by the similar calculation to that in the proof of Theorem 3.2, one could have that

$$\begin{aligned}\mathbb{E}\|X_t^n\|_{\mathbb{H}}^2 &\leq 4\|e^{t\mathcal{A}}x_0\|_{\mathbb{H}}^2+8t\int_0^tL_b(t-s)\mathbb{E}\|X_s^n\|_{\mathbb{H}}^2\mathrm{d}s+8\int_0^tL_\sigma(t-s)\mathbb{E}\|X_s^n\|_{\mathbb{H}}^2\mathrm{d}s\\ &\quad+8\int_0^tL_f(t-s)\mathbb{E}\|X_s^n\|_{\mathbb{H}}^2\mathrm{d}s+8\int_0^t\Big[t\|e^{(t-s)\mathcal{A}}b(s,0)\|_{\mathbb{H}}^2+\|e^{(t-s)\mathcal{A}}\sigma(s,0)\|_{HS}^2\\ &\quad+\int_{\mathbb{U}_0}\|e^{(t-s)\mathcal{A}}f(s,0,u)\|_{\mathbb{H}}^2\lambda(s,u)\nu(\mathrm{d}u)\Big]\mathrm{d}s\\ &\leq 4\|e^{t\mathcal{A}}x_0\|_{\mathbb{H}}^2+8\sup_{s\in[0,t]}\mathbb{E}\|X_s^n\|_{\mathbb{H}}^2\left(t\int_0^tL_b(s)\mathrm{d}s+\int_0^tL_\sigma(s)\mathrm{d}s+\int_0^tL_f(s)\mathrm{d}s\right)\\ &\quad+8\int_0^t\Big[t\sup_{r\in[0,T]}\|e^{s\mathcal{A}}b(r,0)\|_{\mathbb{H}}^2+\sup_{r\in[0,T]}\|e^{s\mathcal{A}}\sigma(r,0)\|_{HS}^2\\ &\quad+\int_{\mathbb{U}_0}\sup_{r\in[0,T]}\left(\|e^{s\mathcal{A}}f(r,0,u)\|_{\mathbb{H}}^2\lambda(r,u)\right)\nu(\mathrm{d}u)\Big]\mathrm{d}s.\end{aligned}$$

Furthermore, by taking t_0 with $8\left(t_0\int_0^{t_0}L_b(s)\mathrm{d}s+\int_0^{t_0}L_\sigma(s)\mathrm{d}s+\int_0^{t_0}L_f(s)\mathrm{d}s\right)<1/2$, it holds that

$$\begin{aligned}\sup_{s\in[0,t_0]}\mathbb{E}\|X_s^n\|_{\mathbb{H}}^2 &\leq 8\|e^{t_0\mathcal{A}}x_0\|_{\mathbb{H}}^2+16\int_0^{t_0}\Big[t_0\sup_{r\in[0,T]}\|e^{s\mathcal{A}}b(r,0)\|_{\mathbb{H}}^2+\sup_{r\in[0,T]}\|e^{s\mathcal{A}}\sigma(r,0)\|_{HS}^2\\ &\quad+\int_{\mathbb{U}_0}\sup_{r\in[0,T]}\left(\|e^{s\mathcal{A}}f(r,0,u)\|_{\mathbb{H}}^2\lambda(r,u)\right)\nu(\mathrm{d}u)\Big]\mathrm{d}s.\end{aligned}$$

On $[t_0,2t_0],[2t_0,3t_0],\dots,[mt_0,T]$ for $m\in\mathbb{N}$, by the same way to the above we deduce and conclude that

$$\begin{aligned}\sup_{s\in[0,T]}\mathbb{E}\|X^n_s\|^2_{\mathbb{H}} &\leq 8^{m+1}\|e^{t_0\mathcal{A}}x_0\|^2_{\mathbb{H}}+16\sum_{i=1}^{m}8^i\int_{(i-1)t_0}^{it_0}\Big[t_0\sup_{r\in[0,T]}\|e^{s\mathcal{A}}b(r,0)\|^2_{\mathbb{H}}\\ &\quad+\sup_{r\in[0,T]}\|e^{s\mathcal{A}}\sigma(r,0)\|^2_{HS}+\int_{\mathbb{U}_0}\sup_{r\in[0,T]}\|e^{s\mathcal{A}}f(r,0,u)\|^2_{\mathbb{H}}\lambda(r,u)\nu(\mathrm{d}u)\Big]\mathrm{d}s\\ &\quad+16\int_{mt_0}^{T}\Big[t_0\sup_{r\in[0,T]}\|e^{s\mathcal{A}}b(r,0)\|^2_{\mathbb{H}}+\sup_{r\in[0,T]}\|e^{s\mathcal{A}}\sigma(r,0)\|^2_{HS}\\ &\quad+\int_{\mathbb{U}_0}\sup_{r\in[0,T]}\Big(\|e^{s\mathcal{A}}f(r,0,u)\|^2_{\mathbb{H}}\lambda(r,u)\Big)\nu(\mathrm{d}u)\Big]\mathrm{d}s.\end{aligned}$$

This shows $\sup_{n\geq 1}\sup_{t\in[0,T]}\mathbb{E}\|X^n_t\|^2_{\mathbb{H}}<\infty$.

Thus, taking the super limit on two sides of the inequality (9) as $n\to\infty$, by the Fatou lemma we obtain that

$$\begin{aligned}\limsup_{n\to\infty}\mathbb{E}\|X^n_t-X_t\|^2_{\mathbb{H}} &\leq \int_0^t (TL_b(t-s)+L_\sigma(t-s)\\ &\qquad +L_f(t-s))\limsup_{n\to\infty}\mathbb{E}\|X^n_s-X_s\|^2_{\mathbb{H}}\mathrm{d}s.\end{aligned}$$

Based on the proof in [16, Theorem 3.1.2], one could get

$$\limsup_{n\to\infty}\mathbb{E}\|X^n_t-X_t\|^2_{\mathbb{H}}=0.$$

Thus, the proof is completed. □

In the following, we will apply the Galerkin finite-dimensional approximation in the above lemma to deduce an Itô formula for real-valued functions of the solution $X_t, t\in[0,T]$ to Eq.(2). Now, there exist some Itô formulas for real-valued functions of these solutions processes for these infinite-dimensional semi-linear SDEs with jumps containing Eq.(2), such as [1, Theorem 2.4] and [8, Theorem 27.1]. Unfortunately, they don't work here because of two requirements in them that the diffusion coefficient σ in Eq.(2) is a Hilbert-Schmidt operator and that the solution X_t to Eq.(2) is a strong solution. In present, we will prove an Itô formula for Eq.(2) with $\sigma\in L_{\mathcal{A}}(\mathbb{H})$ and a unique mild solution X_t. Therefore, the result is independently interesting.

Proposition 1. *Assume* **(H3.1)**-**(H3.4)**, *and let* $v:[0,T]\times\mathbb{H}\to\mathbb{R}$ *be in* $C^{1,2}_b([0,T]\times\mathbb{H})$ *such that* $[\nabla v(t,x)]\in\mathcal{D}(\mathcal{A})$ *for any* $(t,x)\in[0,T]\times\mathbb{H}$ *and* $\|\mathcal{A}\nabla v(t,\cdot)\|_{\mathbb{H}}$ *is* *bounded locally and uniformly in* $t \in [0,T]$*. Then we have*

$$\begin{aligned} v(t,X_t) &= v(0,x_0)+\int_0^t\left[\frac{\partial}{\partial s}v(s,X_s)+\langle\nabla v(s,X_s),b(s,X_s)\rangle_{\tilde{\mathbb{H}}}+\langle\mathcal{A}\nabla v(s,X_s),X_s\rangle_{\mathbb{H}}\right]ds\\ &\quad+\int_0^t\langle\sigma^*(s,X_s)\nabla v(s,X_s),dW_s\rangle_{\tilde{\mathbb{H}}}+\frac{1}{2}\int_0^t Tr[(\sigma\sigma^*)(s,X_s)\nabla^2 v(s,X_s)]ds\\ &\quad+\int_0^t\int_{\mathbb{U}_0}[v(s,X_{s-}+f(s,X_{s-},u))-v(s,X_{s-})]\,\tilde{N}_\lambda(\mathrm{d}s,\mathrm{d}u)\\ &\quad+\int_0^t\int_{\mathbb{U}_0}\Big[v(s,X_{s-}+f(s,X_{s-},u))-v(s,X_{s-})\\ &\qquad-\langle f(s,X_{s-},u),\nabla v(s,X_{s-})\rangle_{\tilde{\mathbb{H}}}\Big]\lambda(s,u)\nu(\mathrm{d}u)\mathrm{d}s, \end{aligned} \tag{10}$$

where $\sigma^*(t,x)$ *stands for the transposed matrix of* $\sigma(t,x)$*,* ∇ *and* ∇^2 *stand for the first and second Fréchet operators with respect to the second variable, respectively.*

Proof. First of all, take the approximation sequence $\{X_t^n, t\in[0,T]\}_{n\in\mathbb{N}}$ for the solution $\{X_t, t\in[0,T]\}$ of Eq.(2). Note that $\{X_t^n, t\in[0,T]\}$ is a n-dimensional process. Applying the Itô formula in [5] to $v(t,X_t^n)$ for any $t\in[0,T]$, one could obtain that

$$\begin{aligned} v(t,X_t^n) &= v(0,\pi_n x_0)+\int_0^t\langle\nabla_n v(s,X_s^n),\sigma_n(s,X_s^n)dW_s\rangle_{\mathbb{H}}\\ &\quad+\int_0^t\int_{\mathbb{U}_0}\left[v(s,X_{s-}^n+f_n(s,X_{s-}^n,u))-v(s,X_{s-}^n)\right]\tilde{N}_\lambda(\mathrm{d}s,\mathrm{d}u)\\ &\quad+\int_0^t\Big[\frac{\partial}{\partial s}v(s,X_s^n)+\langle\nabla_n v(s,X_s^n),\mathcal{A}_nX_s^n+b_n(s,X_s^n)\rangle_{\mathbb{H}}\Big]ds\\ &\quad+\frac{1}{2}\int_0^t Tr[\nabla_n^2 v(s,X_s^n)(\sigma_n(s,X_s^n)(Id)^{\frac{1}{2}})(\sigma_n(s,X_s^n)(Id)^{\frac{1}{2}})^*]ds\\ &\quad+\int_0^t\int_{\mathbb{U}_0}\Big[v(s,X_{s-}^n+f_n(s,X_{s-}^n,u))-v(s,X_{s-}^n)\\ &\qquad-\langle f_n(s,X_{s-}^n,u),\nabla_n v(s,X_{s-}^n)\rangle_{\mathbb{H}}\Big]\lambda(s,u)\nu(\mathrm{d}u)\mathrm{d}s\\ &= v(0,\pi_n x_0)+\int_0^t\langle\sigma_n^*(s,X_s^n)\nabla_n v(s,X_s^n),dW_s\rangle_{\mathbb{H}}\\ &\quad+\int_0^t\int_{\mathbb{U}_0}\left[v(s,X_{s-}^n+f_n(s,X_{s-}^n,u))-v(s,X_{s-}^n)\right]\tilde{N}_\lambda(\mathrm{d}s,\mathrm{d}u)\\ &\quad+\int_0^t\Big[\frac{\partial}{\partial s}v(s,X_s^n)+\langle\nabla_n v(s,X_s^n),\mathcal{A}_nX_s^n+b_n(s,X_s^n)\rangle_{\mathbb{H}}\Big]ds\\ &\quad+\frac{1}{2}\int_0^t Tr[(\sigma_n\sigma_n^*)(s,X_s^n)\nabla_n^2 v(s,X_s^n)]ds\\ &\quad+\int_0^t\int_{\mathbb{U}_0}\Big[v(s,X_{s-}^n+f_n(s,X_{s-}^n,u))-v(s,X_{s-}^n)\\ &\qquad-\langle f_n(s,X_{s-}^n,u),\nabla_n v(s,X_{s-}^n)\rangle_{\mathbb{H}}\Big]\lambda(s,u)\nu(\mathrm{d}u)\mathrm{d}s, \end{aligned} \tag{11}$$

where $\nabla_n\cdot := \sum_{j=1}^n\langle\nabla\cdot,e_j\rangle_{\mathbb{H}}e_j$.

Firstly, by continuity of $v(t,x)$, $\frac{\partial}{\partial s}v(s,x)$ with respect to x and Lemma 3.3, it is clear that

$$\begin{aligned}\lim_{n\to\infty} v(t,\pi_n x_0) &= v(t,x_0),\\ \lim_{n\to\infty} v(t,X_t^n) &= v(t,X_t),\\ \lim_{n\to\infty}\frac{\partial}{\partial s}v(s,X_s^n) &= \frac{\partial}{\partial s}v(s,X_s) \quad a.s..\end{aligned}$$

Those assumptions on v and self-adjoint property of the operator $\mathcal{A}$ admit us to obtain that

$$\begin{aligned}&\lim_{n\to\infty}\int_0^t \langle[\sigma_n^*\nabla_n v](s,X_s^n), dW_s\rangle_{\mathbb{H}} = \int_0^t \langle[\sigma^*\nabla v](s,X_s), dW_s\rangle_{\tilde{\mathbb{H}}}\,,\\ &\lim_{n\to\infty}\int_0^t\int_{\mathbb{U}_0}\left[v(s,X_{s-}^n+f_n(s,X_{s-}^n,u))-v(s,X_{s-}^n)\right]\tilde{N}_\lambda(\mathrm{d}s,\mathrm{d}u)\\ =\;&\int_0^t\int_{\mathbb{U}_0}\left[v(s,X_{s-}+f(s,X_{s-},u))-v(s,X_{s-})\right]\tilde{N}_\lambda(\mathrm{d}s,\mathrm{d}u),\end{aligned}$$

in the mean square sense and

$$\begin{aligned}&\lim_{n\to\infty}\int_0^t \langle\nabla_n v(s,X_s^n), \mathcal{A}_n X_s^n\rangle_{\mathbb{H}} ds = \int_0^t \langle\mathcal{A}\nabla v(s,X_s), X_s\rangle_{\mathbb{H}} ds\,,\\ &\lim_{n\to\infty}\int_0^t \langle\nabla_n v(s,X_s^n), b_n(s,X_s^n)\rangle_{\mathbb{H}} ds = \int_0^t \langle\nabla v(s,X_s), b(s,X_s)\rangle_{\tilde{\mathbb{H}}} ds\,,\\ &\lim_{n\to\infty}\int_0^t Tr[(\sigma_n\sigma_n^*)(s,X_s^n)\nabla_n^2 v(s,X_s^n)]ds = \int_0^t Tr[(\sigma\sigma^*)(s,X_s)\nabla^2 v(s,X_s)]ds,\\ &\lim_{n\to\infty}\int_0^t\int_{\mathbb{U}_0}\Big[v(s,X_{s-}^n+f_n(s,X_{s-}^n,u))-v(s,X_{s-}^n)\\ &\qquad -\langle f_n(s,X_{s-}^n,u),\nabla_n v(s,X_{s-}^n)\rangle_{\mathbb{H}}\Big]\lambda(s,u)\nu(\mathrm{d}u)\mathrm{d}s\\ =\;&\int_0^t\int_{\mathbb{U}_0}\Big[v(s,X_{s-}+f(s,X_{s-},u))-v(s,X_{s-})\\ &\qquad -\langle f(s,X_{s-},u),\nabla v(s,X_{s-})\rangle_{\tilde{\mathbb{H}}}\Big]\lambda(s,u)\nu(\mathrm{d}u)\mathrm{d}s, \quad a.s..\end{aligned}$$

Finally, taking the limits on two sides of (11) as $n\to\infty$, by the above equalities we have (10). The proof is completed. □

4. **The characterization theorem on $\mathbb{H}$.**

In the section, we shall state and prove two characterization theorems, which are the main results in the paper. First of all, consider Eq.(2), i.e.

$$\left\{\begin{array}{l}\mathrm{d}X_t = (\mathcal{A}X_t + b(t,X_t))\mathrm{d}t + \sigma(t,X_t)dW_t + \int_{\mathbb{U}_0} f(t,X_{t-},u)\tilde{N}_\lambda(\mathrm{d}t,\mathrm{d}u), \quad t\in[0,T],\\ X_0 = x_0\in\mathbb{H}.\end{array}\right.$$

Moreover, we assume:

(H3.3') There exists an integrable function $L'_f : [0,T]\to(0,\infty)$ such that

$$\int_{\mathbb{U}_0}\|e^{s\mathcal{A}}(f(t,x,u)-f(t,y,u))\|_{\mathbb{H}}^2\lambda(t,u)\nu(\mathrm{d}u) \le L'_f(s)\|x-y\|_{\mathbb{H}}^2, \quad s,t\in[0,T], x,y\in\mathbb{H},$$

and for $q=2$ and 4

$$\int_{\mathbb{U}_0}\|e^{s\mathcal{A}}f(t,x,u)\|_{\mathbb{H}}^q\lambda(t,u)\nu(\mathrm{d}u)\leq L'_f(s)(1+\|x\|_{\mathbb{H}})^q.$$

(H4.1) (Non-degeneracy) $\sigma(t,x)$ is invertible for any $(t,x)\in[0,T]\times\mathbb{H}$ and the inverse operator of $\sigma(t,x)$ is uniformly bounded on $(t,x)\in[0,T]\times\mathbb{H}$.

It is obvious that the assumption **(H3.3')** is stronger than **(H3.3)**. In the following, we define the support of a $\mathbb{H}$-valued random variable ([9]) and give out a support theorem under these assumptions.

Definition 4.1. The support of a $\mathbb{H}$-valued random variable Y is defined to be

$$supp(Y):=\{x\in\mathbb{H}|(\mathbb{P}\circ Y^{-1})(B(x,r))>0,\ \text{for all } r>0\}$$

where $B(x,r):=\{y\in\mathbb{H}|\|y-x\|_{\mathbb{H}}<r\}$.

Lemma 4.2. *Under* **(H3.1)**-**(H3.2)** **(H3.3')** **(H3.4)** *and* **(H4.1)**, $supp(X_t)=\mathbb{H}$ *for* $t\in[0,T]$.

Proof. Since it is easy to see $supp(X_t)\subset\mathbb{H}$, we only prove $supp(X_t)\supset\mathbb{H}$. Moreover, from Definition 4.1, we only need to show that for any $x\in\mathbb{H}$ and $r>0$,

$$\mathbb{P}\{\|X_t-x\|_{\mathbb{H}}<r\}>0,$$

or equivalently,

$$\mathbb{P}\{\|X_t-x\|_{\mathbb{H}}\geq r\}<1.$$

On one hand, by Lemma 3.3 and the Chebyshev inequality, it holds that for any small $0<\varepsilon<r$ and $0<\eta<1$, there exists a $N\in\mathbb{N}$ such that for $n>N$,

$$\mathbb{P}\{\|X_t-X_t^n\|_{\mathbb{H}}\geq\varepsilon/2\}<\eta/2,\quad \mathbb{P}\{\|\pi_n x-x\|_{\mathbb{H}}\geq\varepsilon/2\}<\eta/2.$$

On the other hand, for X_t^n, [12, Proposition 2.4] admits us to obtain that

$$\mathbb{P}\{\|X_t^n-\pi_n x\|_{\mathbb{H}}\geq r-\varepsilon\}<1-\eta.$$

Thus, combining these inequalities, we furthermore have that

$$\begin{aligned}\mathbb{P}\{\|X_t-x\|_{\mathbb{H}}\geq r\} &\leq \mathbb{P}\{\|X_t-X_t^n\|_{\mathbb{H}}\geq\varepsilon/2\}+\mathbb{P}\{\|X_t^n-\pi_n x\|_{\mathbb{H}}\geq r-\varepsilon\}\\ &\quad+\mathbb{P}\{\|\pi_n x-x\|_{\mathbb{H}}\geq\varepsilon/2\}\\ &< 1.\end{aligned}$$

So, the proof is completed. □

In order to give main results, we also need the following assumption.

(H4.2)

$$\begin{aligned}&\mathbb{E}\Big[\exp\Big\{\frac{1}{2}\int_0^T\big\|\sigma^{-1}(s,X_s)b(s,X_s)\big\|_{\mathbb{H}}^2\,\mathrm{d}s+\int_0^T\int_{\mathbb{U}_0}\left(\frac{1-\lambda(s,u)}{\lambda(s,u)}\right)^2\lambda(s,u)\nu(\mathrm{d}u)\mathrm{d}s\Big\}\Big]\\ <&\ \infty.\end{aligned}$$

Taking

$$\begin{aligned}\Lambda_t &= \exp\Big\{-\int_0^t\langle\sigma^{-1}(s,X_s)b(s,X_s),\mathrm{d}W_s\rangle_{\tilde{\mathbb{H}}}-\frac{1}{2}\int_0^t\big\|\sigma^{-1}(s,X_s)b(s,X_s)\big\|_{\mathbb{H}}^2\,\mathrm{d}s\\ &\qquad -\int_0^t\int_{\mathbb{U}_0}\log\lambda(s,u)N_\lambda(\mathrm{d}s,\mathrm{d}u)-\int_0^t\int_{\mathbb{U}_0}(1-\lambda(s,u))\nu(\mathrm{d}u)\mathrm{d}s\Big\},\end{aligned}$$

by Section 2 we know that Λ_t is a exponential martingale under **(H4.2)** and satisfies the condition (1). Thus, by Theorem 2.1, one can obtain that under the measure $\hat{\mathbb{P}}$ Eq.(2) is transformed as

$$
\mathrm{d}X_t = \mathcal{A}X_t\mathrm{d}t + \sigma(t, X_t)\mathrm{d}\tilde{W}_t + \int_{\mathbb{U}_0} f(t, X_{t-}, u)\tilde{N}(\mathrm{d}t, \mathrm{d}u),
$$

where

$$
\tilde{W}_t := W_t + \int_0^t \sigma^{-1}(s, X_s)b(s, X_s)\mathrm{d}s.
$$

Next, we observe Λ_t. Set

$$
\begin{aligned}
Y_t &:= -\log \Lambda_t \\
&= \int_0^t \langle \sigma^{-1}(s, X_s)b(s, X_s), \mathrm{d}W_s\rangle_{\tilde{\mathbb{H}}} + \frac{1}{2}\int_0^t \left\|\sigma^{-1}(s, X_s)b(s, X_s)\right\|_{\mathbb{H}}^2 \mathrm{d}s \\
&\quad + \int_0^t \int_{\mathbb{U}_0} \log \lambda(s, u) N_\lambda(\mathrm{d}s, \mathrm{d}u) + \int_0^t \int_{\mathbb{U}_0} (1 - \lambda(s, u))\nu(\mathrm{d}u)\mathrm{d}s.
\end{aligned}
$$

Clearly, Y_t is a one-dimensional stochastic process with the stochastic differential form

$$
\begin{aligned}
\mathrm{d}Y_t &= \langle \sigma^{-1}(t, X_t)b(t, X_t), \mathrm{d}W_t\rangle_{\tilde{\mathbb{H}}} + \frac{1}{2}\left\|\sigma^{-1}(t, X_t)b(t, X_t)\right\|_{\mathbb{H}}^2 \mathrm{d}t \\
&\quad + \int_{\mathbb{U}_0} \log \lambda(t, u) N_\lambda(\mathrm{d}t, \mathrm{d}u) + \int_{\mathbb{U}_0} (1 - \lambda(t, u))\nu(\mathrm{d}u)\mathrm{d}t.
\end{aligned}
$$

Now, we state and prove the first result of the section.

Theorem 4.3. *Assume* **(H3.1)**-**(H3.2) (H3.3') (H3.4)** *and* **(H4.1)**-**(H4.2)**. *Let* $v : [0, T] \times \mathbb{H} \to \mathbb{R}$ *be a scalar function which is* C^1 *with respect to the first variable and* C^2 *with respect to the second variable such that* $[\nabla v(t, x)] \in \mathcal{D}(\mathcal{A})$ *for any* $(t, x) \in [0, T] \times \mathbb{H}$ *and* $\|\mathcal{A}\nabla v(t, \cdot)\|_{\mathbb{H}}$ *is bounded locally and uniformly in* $t \in [0, T]$, *and* $\|\mathcal{A}\nabla v(t, \cdot)\|_{\mathbb{H}} : \mathbb{H} \to [0, \infty)$ *is continuous (in the variable* $x \in \mathbb{H}$*) for each* $t \in [0, T]$. *Then the Girsanov density* Λ_t *for Eq.(2) has the following path-independent property:*

$$
\Lambda_t = \exp\{v(0, x_0) - v(t, X_t)\}, \quad t \in [0, T], \tag{12}
$$

if and only if

$$
\begin{aligned}
b(t, x) &= (\sigma\sigma^*\nabla v)(t, x), && (t, x) \in [0, T] \times \mathbb{H}, \quad (13) \\
\lambda(t, u) &= \exp\{v(t, x + f(t, x, u)) - v(t, x)\}, && (t, x, u) \in [0, T] \times \mathbb{H} \times \mathbb{U}_0, \quad (14)
\end{aligned}
$$

and v *satisfies the following time-reversed integro-differential equation(IDE),*

$$
\begin{aligned}
&\frac{\partial}{\partial t}v(t, x) \\
= &-\frac{1}{2}[Tr(\sigma\sigma^*)\nabla^2 v](t, x) - \frac{1}{2}\|\sigma(t, x)^*\nabla v(t, x)\|_{\mathbb{H}}^2 - \langle x, \mathcal{A}\nabla v(t, x)\rangle_{\mathbb{H}} \\
&-\int_{\mathbb{U}_0} \left[e^{v(t, x+f(t,x,u))-v(t,x)} - 1 - \langle f(t, x, u), \nabla v(t, x)\rangle_{\tilde{\mathbb{H}}} e^{v(t,x+f(t,x,u))-v(t,x)}\right]\nu(\mathrm{d}u).
\end{aligned}
\tag{15}
$$

Proof. Firstly, let us show sufficiency. Assume that there exists a $C^{1,2}$-function $v(t,x)$ satisfying (13)(14)(15). For the composition process $v(t,X_t)$, the Itô formula in Proposition 1 admits us to get

$$\begin{aligned}
\mathrm{d}v(t,X_t) &= \frac{\partial}{\partial t}v(t,X_t)\mathrm{d}t + \langle \mathcal{A}X_t, \nabla v(t,X_t)\rangle_{\mathbb{H}}\mathrm{d}t \\
&\quad + \langle b(t,X_t), \nabla v(t,X_t)\rangle_{\tilde{\mathbb{H}}}\mathrm{d}t + \frac{1}{2}[Tr(\sigma\sigma^*)\nabla^2 v](t,X_t)\mathrm{d}t \\
&\quad + \int_{\mathbb{U}_0}\Big[v(t,X_{t-}+f(t,X_{t-},u)) - v(t,X_{t-}) \\
&\qquad - \langle f(t,X_{t-},u), \nabla v(t,X_{t-})\rangle_{\tilde{\mathbb{H}}}\Big]\lambda(t,u)\nu(\mathrm{d}u)\mathrm{d}t \\
&\quad + \int_{\mathbb{U}_0}[v(t,X_{t-}+f(t,X_{t-},u)) - v(t,X_{t-})]\,\tilde{N}_\lambda(\mathrm{d}t,\mathrm{d}u) \\
&\quad + \langle (\sigma^*\nabla v)(t,X_t), \mathrm{d}W_t\rangle_{\tilde{\mathbb{H}}}. \qquad (16)
\end{aligned}$$

Combining (13)(14)(15) with (16), one could have

$$\begin{aligned}
&\mathrm{d}v(t,X_t) \\
&= \left[\frac{1}{2}\left\|\sigma^{-1}(t,X_t)b(t,X_t)\right\|_{\mathbb{H}}^2 + \int_{\mathbb{U}_0}\Big(\big(\log\lambda(t,u)\big)\lambda(t,u) + \big(1-\lambda(t,u)\big)\Big)\nu(\mathrm{d}u)\right]\mathrm{d}t \\
&\quad + \int_{\mathbb{U}_0}\log\lambda(t,u)\tilde{N}_\lambda(\mathrm{d}t,\mathrm{d}u) + \langle \sigma^{-1}(t,X_t)b(t,X_t), \mathrm{d}W_t\rangle_{\tilde{\mathbb{H}}} \\
&= \langle \sigma^{-1}(t,X_t)b(t,X_t), \mathrm{d}W_t\rangle_{\tilde{\mathbb{H}}} + \frac{1}{2}\left\|\sigma^{-1}(t,X_t)b(t,X_t)\right\|_{\mathbb{H}}^2\mathrm{d}t \\
&\quad + \int_{\mathbb{U}_0}\log\lambda(t,u)N_\lambda(\mathrm{d}t,\mathrm{d}u) + \int_{\mathbb{U}_0}(1-\lambda(t,u))\nu(\mathrm{d}u)\mathrm{d}t.
\end{aligned}$$

Integrating the above equality from 0 to $t\in[0,T]$, we know that

$$v(t,X_t) - v(t,x_0) = Y_t = -\log\Lambda_t.$$

By simple calculation, that is exactly (12).

Next, we prove necessity. On one side, there exists a $C^{1,2}$-function $v(t,x)$ such that $v(t,X_t)$ satisfies (12), i.e.

$$\begin{aligned}
\mathrm{d}v(t,X_t) &= -\mathrm{d}\log\Lambda_t = \mathrm{d}Y_t = \left[\frac{1}{2}\left\|\sigma^{-1}(t,X_t)b(t,X_t)\right\|_{\mathbb{H}}^2\right. \\
&\quad \left. + \int_{\mathbb{U}_0}\Big(\big(\log\lambda(t,u)\big)\lambda(t,u) + \big(1-\lambda(t,u)\big)\Big)\nu(\mathrm{d}u)\right]\mathrm{d}t \\
&\quad + \int_{\mathbb{U}_0}\log\lambda(t,u)\tilde{N}_\lambda(\mathrm{d}t,\mathrm{d}u) + \langle \sigma^{-1}(t,X_t)b(t,X_t), \mathrm{d}W_t\rangle_{\tilde{\mathbb{H}}}. \quad (17)
\end{aligned}$$

Moreover, based on (17) we conclude that $v(t,X_t)$ is a càdlàg semimartingale with a predictable finite variation part. On the other side, note that X_t solves Eq.(2) and $v(t,x)$ is a $C^{1,2}$-function. Applying Proposition 1 to the composition process

$v(t, X_t)$, one could obtain (16), i.e.

$$\begin{aligned}
\mathrm{d}v(t,X_t) &= \frac{\partial}{\partial t}v(t,X_t)\mathrm{d}t + \langle \mathcal{A}X_t, \nabla v(t,X_t)\rangle_{\mathbb{H}}\mathrm{d}t \\
&\quad + \langle b(t,X_t), \nabla v(t,X_t)\rangle_{\tilde{\mathbb{H}}}\mathrm{d}t + \frac{1}{2}[Tr(\sigma\sigma^*)\nabla^2 v](t,X_t)\mathrm{d}t \\
&\quad + \int_{\mathbb{U}_0}\Big[v(t,X_{t-}+f(t,X_{t-},u)) - v(t,X_{t-}) \\
&\qquad - \langle f(t,X_{t-},u), \nabla v(t,X_{t-})\rangle_{\tilde{\mathbb{H}}}\Big]\lambda(t,u)\nu(\mathrm{d}u)\mathrm{d}t \\
&\quad + \int_{\mathbb{U}_0}\left[v(t,X_{t-}+f(t,X_{t-},u)) - v(t,X_{t-})\right]\tilde{N}_\lambda(\mathrm{d}t,\mathrm{d}u) \\
&\quad + \langle (\sigma^*\nabla v)(t,X_t), \mathrm{d}W_t\rangle_{\tilde{\mathbb{H}}}.
\end{aligned}$$

Thus, the above equality is another decomposition of the semimartingale $v(t, X_t)$. By uniqueness for decomposition of the semimartingale ([4]), it holds that for $t \in [0, T]$,

$$\begin{aligned}
\sigma^{-1}(t,X_t)b(t,X_t) &= \sigma(t,X_t)^*\nabla v(t,X_t), \\
\log\lambda(t,u) &= v(t,X_{t-}+f(t,X_{t-},u)) - v(t,X_{t-}), \quad u \in \mathbb{U}_0,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2}\left\|\sigma^{-1}(t,X_t)b(t,X_t)\right\|^2_{\mathbb{H}} + \int_{\mathbb{U}_0}\Big(\big(\log\lambda(t,u)\big)\lambda(t,u) + \big(1-\lambda(t,u)\big)\Big)\nu(\mathrm{d}u) \\
= &\frac{\partial}{\partial t}v(t,X_t) + \langle \mathcal{A}X_t, \nabla v(t,X_t)\rangle_{\mathbb{H}} + \langle b(t,X_t), \nabla v(t,X_t)\rangle_{\tilde{\mathbb{H}}} + \frac{1}{2}[Tr(\sigma\sigma^*)\nabla^2 v](t,X_t) \\
&+ \int_{\mathbb{U}_0}\Big[v(t,X_{t-}+f(t,X_{t-},u)) - v(t,X_{t-}) - \langle f(t,X_{t-},u), \nabla v(t,X_{t-})\rangle_{\tilde{\mathbb{H}}}\Big]\lambda(t,u)\nu(\mathrm{d}u), \quad a.s..
\end{aligned}$$

Based on Lemma 4.2 and our assumptions on $\mathcal{A}\nabla v(t,x)$, we have that

$$\sigma^{-1}(t,x)b(t,x) = \sigma(t,x)^*\nabla v(t,x), \quad (t,x) \in [0,T]\times\mathbb{H}, \tag{18}$$

$$\log\lambda(t,u) = v(t,x+f(t,x,u)) - v(t,x), \quad (t,x,u) \in [0,T]\times\mathbb{H}\times\mathbb{U}_0, \tag{19}$$

and

$$\begin{aligned}
&\frac{1}{2}\left\|\sigma^{-1}(t,x)b(t,x)\right\|^2_{\mathbb{H}} + \int_{\mathbb{U}_0}\Big(\big(\log\lambda(t,u)\big)\lambda(t,u) + \big(1-\lambda(t,u)\big)\Big)\nu(\mathrm{d}u) \\
= &\frac{\partial}{\partial t}v(t,x) + \langle \mathcal{A}x, \nabla v(t,x)\rangle_{\mathbb{H}} + \langle b(t,x), \nabla v(t,x)\rangle_{\tilde{\mathbb{H}}} + \frac{1}{2}[Tr(\sigma\sigma^*)\nabla^2 v](t,x) \\
&+ \int_{\mathbb{U}_0}\Big[v(t,x+f(t,x,u)) - v(t,x) - \langle f(t,x,u), \nabla v(t,x)\rangle_{\tilde{\mathbb{H}}}\Big]\lambda(t,u)\nu(\mathrm{d}u).
\end{aligned} \tag{20}$$

By simple computation, (18)(19) correspond to (13)(14), respectively. Moreover, (18)(19) together with (20) yield to (15). The proof is completed. □

The above theorem gives a necessary and sufficient condition, and hence a characterization of path-independence for the density Λ_t of the Girsanov transformation for a SEE with jumps in terms of a IDE. Namely, we establish a bridge from Eq.(2) to a IDE.

Remark 3. Let $f(t,x,u)=0$, and then Eq.(2) has no jumps. In Theorem 4.3, based on (14), we know that $\lambda(t,u)=1$ for $u\in\mathbb{U}_0$. Thus, Eq.(12) becomes

$$\begin{aligned}\Lambda_t &= \exp\{-\int_0^t\langle\sigma^{-1}(s,X_s)b(s,X_s),\mathrm{d}W_s\rangle_{\tilde{\mathbb{H}}}-\frac{1}{2}\int_0^t\left\|\sigma^{-1}(s,X_s)b(s,X_s)\right\|_{\mathbb{H}}^2\mathrm{d}s\}\\ &= \exp\{v(0,x_0)-v(t,X_t)\}.\end{aligned}$$

By Theorem 4.3, the above equation holds if and only if

$$\begin{aligned}b(t,x) &= (\sigma\sigma^*\nabla v)(t,x), \qquad (t,x)\in[0,T]\times\mathbb{H},\\ \frac{\partial}{\partial t}v(t,x) &= -\frac{1}{2}[Tr(\sigma\sigma^*)\nabla^2 v](t,x)-\frac{1}{2}\|\sigma(t,x)^*\nabla v(t,x)\|_{\mathbb{H}}^2-\langle x,\mathcal{A}\nabla v(t,x)\rangle_{\mathbb{H}}.\end{aligned}$$

This is exactly Theorem 3.1 in [15]. That is, our result is more general.

Remark 4. Let $f(t,x,u)=0$, and then Eq.(15) becomes the following time-reversed partial differential equation,

$$\frac{\partial}{\partial t}v(t,x)=-\frac{1}{2}[Tr(\sigma\sigma^*)\nabla^2 v](t,x)-\frac{1}{2}\|\sigma(t,x)^*\nabla v(t,x)\|_{\mathbb{H}}^2-\langle x,\mathcal{A}\nabla v(t,x)\rangle_{\mathbb{H}}.$$

The above type of equations has been analyzed in [15]. Moreover, in some special cases (cf. Example 3.1 and 3.2 in [15]) it is an infinite-dimensional analogy of the Burgers-KPZ equation that is well known in statistics physics. If $f(t,x,u)\neq 0$, the special kind of Eq.(15) appears in [2]. There its classical solution and viscosity solution are defined and studied. Furthermore, it is worthwhile to mention that a family of option prices is its viscosity solution.

Next, we consider Eq.(2) with $\sigma(t,x)=0$, i.e.

$$\left\{\begin{array}{l}\mathrm{d}\bar{X}_t=(\mathcal{A}\bar{X}_t+b(t,\bar{X}_t))\mathrm{d}t+\int_{\mathbb{U}_0}f(t,\bar{X}_{t-},u)\tilde{N}_\lambda(\mathrm{d}t,\mathrm{d}u), \quad t\in[0,T],\\ \bar{X}_0=x_0,\end{array}\right. \tag{21}$$

where $b:[0,\infty)\times\mathbb{H}\to\mathbb{H}$ and $f:[0,\infty)\times\mathbb{H}\times\mathbb{U}_0\mapsto\mathbb{H}$ are two Borel measurable mappings. Since Eq.(21) is driven by a purely jump process, some conclusions about it will be different from that about Eq.(2). Let us describe them in details. By Theorem 3.2, Eq.(21) has a unique mild solution denoted by $\bar{X}_t$. Assume:

(H4.3)

$$\exp\Big\{\int_0^T\int_{\mathbb{U}_0}\left(\frac{1-\lambda(s,u)}{\lambda(s,u)}\right)^2\lambda(s,u)\nu(\mathrm{d}u)\mathrm{d}s\Big\}<\infty.$$

Set

$$\bar{\Lambda}_t:=\exp\Big\{-\int_0^t\int_{\mathbb{U}_0}\log\lambda(s,u)N_\lambda(\mathrm{d}s,\mathrm{d}u)-\int_0^t\int_{\mathbb{U}_0}(1-\lambda(s,u))\nu(\mathrm{d}u)\mathrm{d}s\Big\},$$

and then by similar deduction to the above, $\bar{\Lambda}_t$ is an exponential martingale. Define a measure $\bar{\mathbb{P}}_t$ via

$$\frac{\mathrm{d}\bar{\mathbb{P}}_t}{\mathrm{d}\mathbb{P}}=\bar{\Lambda}_t.$$

Under $\bar{\mathbb{P}}_t$, by Theorem 2.1, the system (21) is transformed as

$$\mathrm{d}\bar{X}_t=(\mathcal{A}\bar{X}_t+b(t,\bar{X}_t))\mathrm{d}t+\int_{\mathbb{U}_0}f(t,\bar{X}_{t-},u)\tilde{N}(\mathrm{d}t,\mathrm{d}u).$$

Note that the drift term still exists.

Now, we study path-independence of $\bar{\Lambda}_t$. By the similar proof to that in Theorem 4.3, we obtain the following result.

Theorem 4.4. *Assume* **(H3.1) (H3.3') (H3.4)** *and* **(H4.3)**. *Let $\bar{v}:[0,T]\times\mathbb{H}\to\mathbb{R}$ be a scalar function which is C^1 with respect to the first variable and C^2 with respect to the second variable such that $[\nabla\bar{v}(t,x)]\in\mathcal{D}(\mathcal{A})$ for any $(t,x)\in[0,T]\times\mathbb{H}$ and $\|\mathcal{A}\nabla\bar{v}(t,\cdot)\|_{\mathbb{H}}$ is bounded locally and uniformly in $t\in[0,T]$. Then the Girsanov density $\bar{\Lambda}_t$ for Eq.(21) has the following path-indenpendent property:*

$$\bar{\Lambda}_t=\exp\{\bar{v}(0,\bar{x}_0)-\bar{v}(t,\bar{X}_t)\},\quad t\in[0,T],$$

if and only if

$$\lambda(t,u)=\exp\{\bar{v}(t,x+f(t,x,u))-\bar{v}(t,x)\},\quad (t,x,u)\in[0,T]\times\mathbb{R}^d\times\mathbb{U}_0,$$

and $\bar{v}$ satisfies the following time-reversed equation,

$$\begin{aligned}\frac{\partial}{\partial t}\bar{v}(t,x) &= -\langle\mathcal{A}x+b(t,x),\nabla\bar{v}(t,x)\rangle_{\mathbb{H}}-\int_{\mathbb{U}_0}\Big[e^{\bar{v}(t,x+f(t,x,u))-\bar{v}(t,x)}-1\\ &\qquad -\langle f(t,x,u),\nabla\bar{v}(t,x)\rangle_{\mathbb{H}}e^{\bar{v}(t,x+f(t,x,u))-\bar{v}(t,x)}\Big]\nu(\mathrm{d}u).\end{aligned}$$

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