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The hair-trigger effect for a class of nonlocal nonlinear equations

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Abstract

We prove the hair-trigger effect for a class of nonlocal nonlinear evolution equations on $\mathbb{R}^d$ which have only two constant stationary solutions, 0 and $\theta > 0$. The effect consists in that the solution with an initial condition non identical to zero converges (when time goes to ∞) to θ locally uniformly in $\mathbb{R}^d$. We find also sufficient conditions for existence, uniqueness and comparison principle in the considered equations.

Keywords: hair-trigger effect, nonlocal diffusion, reaction-diffusion equation, front propagation, monostable equation, nonlocal nonlinearity, long-time behavior, integral equation

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1 Introduction

We will deal with the following nonlinear nonlocal evolution equation on the Euclidean space $\mathbb{R}^d$, $d \geq 1$:

$$\frac{\partial u}{\partial t}(x,t) = \varkappa(a*u)(x,t) - mu(x,t) - u(x,t)(Gu)(x,t) \tag{1.1}$$

for $t > 0$, $x \in \mathbb{R}^d$, with an initial condition $u(x,0) = u_0(x)$, $x \in \mathbb{R}^d$. Here $m, \varkappa > 0$; a is a nonnegative probability kernel on $\mathbb{R}^d$, i.e. $0 \leq a \in L^1(\mathbb{R}^d)$ and

$$\int_{\mathbb{R}^d} a(x)\,dx = 1; \tag{1.2}$$

$(a*u)(x,t)$ means the convolution (in x) between a and u, namely,

$$(a*u)(x,t) = \int_{\mathbb{R}^d} a(x-y)u(y,t)dy; \tag{1.3}$$

and G is a mapping on a space of bounded on $\mathbb{R}^d$ functions.

We interpret $u(x,t)$ as a density of a population at the point $x \in \mathbb{R}^d$ at the moment of time $t \geq 0$. The probability kernel $a = a(x)$ describes distribution

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of the birth of new individuals with constant intensity $\varkappa > 0$. Individuals in the population may also die either with the constant mortality rate $m > 0$ or because of the competition, described by the density dependent rate Gu, where G is an (in general, also nonlinear) operator on a space of bounded functions (cf. the discussion in [53]).

The equation (1.1) can be also rewritten in a reaction-diffusion form

$$\frac{\partial u}{\partial t}(x,t) = \varkappa(a*u)(x,t) - \varkappa u(x,t) + (Fu)(x,t), \tag{1.4}$$

where

$$Fu := u(\varkappa - m - Gu) \tag{1.5}$$

plays the role of the so-called reaction term, whereas

$$Lu := \varkappa(a*u) - \varkappa u \tag{1.6}$$

describes the non-local diffusion generator, see e.g. [4] (note that L is also known as the generator of a continuous time random walk in $\mathbb{R}^d$ or of a compound Poisson process on $\mathbb{R}^d$). As a result, the solution u to the equation (1.4) may be interpreted as a density of a species which invades according to a nonlocal diffusion within the space $\mathbb{R}^d$ meeting a reaction F; see e.g. [28, 49, 55].

Below, we restrict ourselves to the case where (1.1) has two constant solutions $u \equiv 0$ and $u \equiv \theta > 0$ only. The main aim of the present paper is to find sufficient conditions for the so-called *hair-trigger effect*. The latter means that, unless $u_0 \equiv 0$, the corresponding solution to (1.1) achieves an arbitrary chosen level between 0 and θ uniformly on an arbitrary chosen domain of $\mathbb{R}^d$ after a finite time. In other words, $u(x,t)$ converges, as $t \to \infty$, locally uniformly in $x \in \mathbb{R}^d$ to the positive stationary solution $u \equiv \theta$. The latter constant solution, therefore, is globally asymptotically stable in the sense of the topology of local uniform convergence. Therefore, the equation (1.1) appears of the so-called monostable type; cf. also Remark 5.5 below.

Firstly, a reaction-diffusion equation of the form (1.4) was considered in the seminal paper [44] by Kolmogorov–Petrovsky–Piskunov (KPP). There, for the local reaction $Fu = f(u) = u(1-u)^2$ (that corresponds to $Gu = 2u - u^2$ in (1.5); we set also here $\varkappa - m = 1$), the equation (1.4) was derived from a model for the dispersion of a spatially distributed species. To analyze the model, the authors used a diffusion scaling, which led to the classical local diffusion generator $\varkappa\Delta u$ (for $d = 1$) instead of L in (1.4). Moreover, they proposed the method which covered more general local reactions $Fu = f(u)$ as well. We will say that such local reaction F has the *KPP-type* if $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous on $[0, \theta]$ and

$$f(0) = f(\theta) = 0; \qquad f'(0) > 0; \qquad 0 < f(r) \le f'(0)r, \quad r \in (0, \theta). \tag{1.7}$$

In particular, the logistic reaction $f(u) = u(\theta - u)$, that corresponds to the identical mapping $Gu = u$ in (1.5), satisfies (1.7). The corresponding model was considered early by Fisher [34], it described the advance of a favorable allele through a spatially distributed population. Note that the conditions for the mapping G (and hence, by product, for the reaction F) which we postulate in Section 2 below are reduced, in the case of a local reaction $Fu = f(u)$, to (1.7) (see Example 1 below).

Later, the significance of nonlocal terms in diffusion and/or reaction in (1.4) was stressed by many authors, in particular, in ecology and population biology, see e.g. [14, 16, 45]; see also recent papers [8, 50] where the importance and observed effects of nonlocal interactions in biological models are discussed.

A natural nonlocal analogue of the Fisher–KPP equation with the mentioned local reaction $f(u) = u(\theta - u)$ is the equation (1.4) with both nonlocal diffusion generator (1.6) and the linear nonlocal mapping $Gu = \varkappa^- a^- * u$ in (1.5), where $\varkappa^- > 0$, $0 \leq a^- \in L^1(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} a^-(x)\,dx = 1$, and the convolution is defined as in (1.3) (see Example 2 below). The corresponding equations (1.1), or (1.4), similarly to the classical Fisher–KPP equation, may be obtained from different models. In particular, for the case $\varkappa = \varkappa^-$, $a = a^-$, it was obtained, for $m = 0$ in [47, 48] from a model of simple epidemic, whereas, for $m > 0$, it was derived in [27] from a crabgrass model on the lattice $\mathbb{Z}^d$. For different kernels a and a^-, the equation (1.1) appeared in [11] from a population ecology model; see also [12, 23] and the rigorous derivation of (1.1) in [29, 35]

More generally, a nonlocal analogue of the local KKP-type reaction $f(u) = u(\theta - u)^n$ is, naturally, the reaction

$$Fu = \gamma_n u(\theta - a^- * u)^n, \quad n \in \mathbb{N}, \tag{1.8}$$

with a^- is as above and $\gamma_n > 0$ (see Example 3 below). Note also that the equation (1.4) with the nonlocal diffusion (1.6) and a local KPP-type reaction $Fu = f(u)$ was considered in [54] motivated by an analogy to Kendall's epidemic model [43].

The first (up to our knowledge) result about the hair-trigger effect described above, for a non-linear evolution equation with the local diffusion, was shown by Kanel [42], for the cases of the combustion and the Fisher–KPP reaction-diffusion equations in the dimension $d = 1$. Multidimensional analogues were shown by Aronson and Weinberger [6, 7]; in the latter reference the notion 'hair-trigger' was, probably, firstly used.

For the nonlocal diffusion (1.6), the first result about the hair-trigger effect for a solution to (1.4) was obtained in [46]: for the one-dimensional case $d = 1$, under additional restrictions on the probability kernel $a = a(x)$, and for a local reaction $Fu = f(u)$ of the KPP-type given by (1.7).

For the nonlocal diffusion in $\mathbb{R}^d$ with $d > 1$, the hair-trigger effect, for the local reaction term $f(u) = u^{1+p}(1-u)$ with $p > 0$, has been shown recently in [2], under additional assumptions on $a = a(x)$ (in particular, its radial symmetry was assumed). From this, by comparison-type arguments, it might be possible to show the hair-trigger effect for a local KPP-type reaction $Fu = f(u)$ described by (1.7), provided that, additionally, $f'(\theta) < 0$.

To the best of our knowledge, the present paper is the first one that shows the hair-trigger effect for non-local reactions. In particular, we allow the reaction (1.8) in (1.4)–(1.5), provided that an appropriate comparison between a and a^- is assumed (see Examples 2–3 below).

Another novelty of the present paper, even for the case of the local KPP-type reactions $Fu = f(u)$ given by (1.7) is that we allow general anisotropic probability kernels $a = a(x)$, $x \in \mathbb{R}^d$ (see Example 1 below). Note, that, however, we do not cover the local reaction $f(u) = u^{1+p}(1 - u)$ with $p > 0$, considered in [2].

For results about the hair-trigger effect in other types of non-local equations see also [24].

The hair-trigger effect is an important tool in the study of the long-time behavior of evolution equations. In particular, it allows one to study the front propagation of the solutions to the equations [31, 32, 36]; it also yields the non-existence of other stationary solutions between the given two (see [30, Proposition 5.12] and cf. a discussion in [22]) and allows to demonstrate instability of non-monotonic traveling waves, cf. [39].

Since the hair-trigger effect means just that a level set for a solution to (1.1) is going to contain an arbitrary large compact in $\mathbb{R}^d$ when time grows, it is naturally based on a estimate from below for the solution. Note that, for the class of equations of the form (1.1) with a non-negative operator G (see the assumption (A2) below), one can estimate the corresponding non-negative solution from above by the solution to the linearization of (1.1) at zero. Indeed, by the Duhamel's principle, if $v(\cdot,0) \equiv u(\cdot,0)$ and $\partial_t v = \varkappa a * v - mv$, then $u(\cdot,t) \leq v(\cdot,t)$ point-wise for all $t \geq 0$. Then one can use estimates on v (see e.g. [4, 33, 38]) to estimate u. However, an estimate from below appears much more delicate problem, that seems to be typical for monostable-type evolution equations, since the nonlinear structure of the equation (1.1) is essential in this case.

Both nonlocal diffusion and, in general nonlocal, reaction in (1.4) require new methods in the proof of the hair-trigger effect. The mentioned results for the local diffusion (the Laplace operator instead of L in (1.4)–(1.6)) were based on the application of an auxiliary boundary-value problem [42] (which works for $d = 1$ only) or, in addition to properties of the Laplace operator, on the locality of the reaction term [6, 7]. These approaches are difficult (if possible at all) to repeat for (1.4) even for a local reaction in $\mathbb{R}^d$. Stress also that, in the case of a nonlocal reaction in (1.4), the comparison principle (which is necessary for the hair-trigger effect) requires additional restrictions (see Theorem 2.3 and also Remark 4.1).

Our approach is based on an extension of the classical Weinberger's result for discrete dynamical systems [57] to the continuous-time dynamics defined by (1.1). That result required, additionally, specific restrictions on the initial condition to (1.1) (see the beginning of Section 5 for details) or, equivalently, it requires an additional analysis for small level sets of the solution to (1.1) (which we provide in Propositions 5.15–5.16 below). A disadvantage of this approach is that we apply 'a black box', meaning that the result sacrifices the complete understanding for the behaviour of large level-sets of u. On the other hand, our approach is rather general and could be applied to other (nonlocal) evolution equations.

The paper is organized as follows. We prove the hair-trigger effect for (1.1) (Theorems 2.5, 2.7) in Section 5, applying Weinberger's results [57] and getting its time-continuous counterpart for (1.1) in Proposition 5.11; we also, in Propositions 5.15–5.16, get rid of the restrictions on the initial conditions imposed in Weinberger's paper. The proof is done under additional assumptions on G presented in Section 2, which, in particular, ensure the comparison principle (Theorem 2.3). In Sections 3 and 4, we prove the existence/uniqueness (Theorem 3.3) and the comparison principle (Theorem 4.2) for some generalizations of (1.1).

2 Assumptions and main results

Recall, that we treat $u = u(x, t)$ as the local density of a system at the point $x \in \mathbb{R}^d$ and at the moment of time $t \in \mathbb{R}_+ := [0, \infty)$. We assume that the initial condition u_0 to (1.1) is a bounded function on $\mathbb{R}^d$.

Namely, we will consider the following Banach spaces of real-valued functions on $\mathbb{R}^d$: the space $C_b(\mathbb{R}^d)$ of bounded continuous functions on $\mathbb{R}^d$ with sup-norm, the space $C_{ub}(\mathbb{R}^d)$ of bounded uniformly continuous functions on $\mathbb{R}^d$ with sup-norm, and the space $L^\infty(\mathbb{R}^d)$ of essentially bounded (with respect to the Lebesgue measure) functions on $\mathbb{R}^d$ with esssup-norm.

Let E be either of the spaces $C_{ub}(\mathbb{R}^d)$, $C_b(\mathbb{R}^d)$ or $L^\infty(\mathbb{R}^d)$ with the corresponding norm denoting by $\|\cdot\|_E$. For an interval $I \subset \mathbb{R}_+$, let $C(I \to E)$ and $C^1(I \to E)$ denote the sets of all continuous and, respectively continuously differentiable, E-valued functions on I.

Definition 2.1. Let I be either a finite interval $[0, T]$, for some $T > 0$, or the whole $\mathbb{R}_+ := [0, \infty)$. A function $u \in \mathcal{U}_I := C(I \to E) \cap C^1((I \setminus \{0\}) \to E)$ which satisfies (1.1) and such that $u(\cdot, 0) = u_0(\cdot)$ in E is said to be a *classical solution* to (1.1) on I. For brevity, we denote also

$$\mathcal{U}_T := \mathcal{U}_{[0,T]}, \quad T > 0; \qquad \mathcal{U}_\infty := \mathcal{U}_{\mathbb{R}_+}. \tag{2.1}$$

We will write $v \leq w$, for $v, w \in E$, if $v(x) \leq w(x)$, $x \in \mathbb{R}^d$. Here and below, for the case $E = L^\infty(\mathbb{R}^d)$, we will treat the latter inclusion a.e. only. Set also, for an $r > 0$,

$$E_r^+ := \{v \in E : 0 \leq v \leq r\}.$$

We denote by $T_y : E \to E$, $y \in \mathbb{R}^d$, the translation operator, given by

$$(T_y v)(x) = v(x - y), \quad x \in \mathbb{R}^d. \tag{2.2}$$

A sequence of functions $(v_n)_{n\in\mathbb{N}} \subset E$ is said to be convergent to a function $v \in E$ locally uniformly if $(v_n)_{n\in\mathbb{N}}$ converges to v uniformly on all compact subsets of $\mathbb{R}^d$. We denote this by

$$v_n \overset{\text{loc}}{\Longrightarrow} v, \quad n \to \infty,$$

Let also $B_r(x_0)$ denote the ball in $\mathbb{R}^d$ with the radius $r > 0$ centered at the $x_0 \in \mathbb{R}^d$. In the case $x_0 = 0 \in \mathbb{R}^d$, we will just write $B_r := B_r(0)$.

In Section 3, we prove an existence and uniqueness result for a more general equation than (1.1); it can be read in the case of (1.1) as follows

Theorem 2.2. *Let $0 \leq a \in L^1(\mathbb{R}^d)$ and (1.2) hold. Let $G : E \to E$ be such that $Gv \geq 0$ for all $0 \leq v \in E$ and, for some $\kappa > 0$,*

$$\|Gv - Gw\|_E \leq e^{\kappa r}\|v - w\|_E, \quad v, w \in E_r^+, \ r > 0.$$

Then, for any $T > 0$ and $0 \leq u_0 \in E$, there exists a unique nonnegative classical solution u to (1.1) on $[0, T]$. In particular, $u \in \mathcal{U}_\infty$.

To exclude the trivial case when $\|u(\cdot, t)\|_E$ converges to 0 uniformly in time, we assume that

$$\beta := \varkappa - m > 0. \tag{A1}$$

We suppose that there exist two constant solutions $u \equiv 0$ and $u \equiv \theta > 0$ to (1.1), more precisely,

$$\begin{gathered} \textit{there exists } \theta > 0 \textit{ such that} \\ 0 = G0 \le Gv \le G\theta = \beta, \quad v \in E_\theta^+. \end{gathered} \tag{A2}$$

We will also assume that G is (locally) Lipschitz continuous in E_θ^+, namely,

$$\begin{gathered} \textit{there exists } l_\theta > 0, \textit{ such that} \\ \|Gv - Gw\|_E \le l_\theta \|v - w\|_E, \quad v, w \in E_\theta^+. \end{gathered} \tag{A3}$$

We restrict ourselves to the case when the comparison principle for (1.1) holds. Namely, we assume that the right-hand side of (1.1) is a (quasi-)monotone operator:

$$\begin{gathered} \textit{for some } p \ge 0 \textit{ and for any } v, w \in E_\theta^+ \textit{ with } v \le w, \\ \varkappa a * v - v\, Gv + pv \le \varkappa a * w - w\, Gw + pw. \end{gathered} \tag{A4}$$

In Section 4, we also prove that the comparison principle holds for a more general equation than (1.1); in the case of (1.1) it gives the following result.

Theorem 2.3. *Let* (A1)–(A4) *hold.*

1. *Let $T > 0$ be fixed and $u_1, u_2 \in \mathcal{U}_T$ be such that, for all $t \in (0, T]$, $x \in \mathbb{R}^d$,*

$$\begin{gathered} \frac{\partial u_1}{\partial t} - \varkappa a * u_1 + m u_1 + u_1 G u_1 \le \frac{\partial u_2}{\partial t} - \varkappa a * u_2 + m u_2 + u_2 G u_2, \\ 0 \le u_1(x,t) \le \theta, \qquad 0 \le u_2(x,t) \le \theta, \\ 0 \le u_1(x,0) \le u_2(x,0) \le \theta. \end{gathered}$$

   Then, for all $t \in [0, T]$, $x \in \mathbb{R}^d$,

$$0 \le u_1(x,t) \le u_2(x,t) \le \theta. \tag{2.3}$$

2. *Let $u \in \mathcal{U}_\infty$ be a classical solution to* (1.1)*, given by Theorem 2.2, such that $0 \le u_0 \le \theta$. Then, for all $t \in \mathbb{R}_+$, $x \in \mathbb{R}^d$,*

$$0 \le u(x,t) \le \theta.$$

   In particular, combining two previous parts, we get the following statement.

3. *Let functions $u_1, u_2 \in \mathcal{U}_\infty$ solve* (1.1) *and $0 \le u_1(x,0) \le u_2(x,0) \le \theta$, $x \in \mathbb{R}^d$. Then* (2.3) *holds for all $t \in \mathbb{R}_+$, $x \in \mathbb{R}^d$.*

We assume next that the kernel a is not degenerate at the origin, namely,

$$\textit{there exists } \varrho > 0 \textit{ such that } a(x) \ge \varrho \text{ for a.a. } x \in B_\varrho(0). \tag{A5}$$

Stability of the solution to (1.1) with respect to the initial condition in the topology of locally uniform convergence requires continuity of G in this topology:

$$\begin{gathered} \textit{for any } v_n, v \in E_\theta^+, \textit{ such that } v_n \overset{\text{loc}}{\Longrightarrow} v, \ n \to \infty, \textit{ one has} \\ Gv_n \overset{\text{loc}}{\Longrightarrow} Gv, \ n \to \infty. \end{gathered} \tag{A6}$$

We will consider the translation invariant case only:

$$\begin{aligned}&\textit{let } T_y,\ y\in\mathbb{R}^d, \textit{ be a translation operator, given by } (2.2), \textit{ then}\\ &\qquad (T_yGv)(x) = (GT_yv)(x), \quad v\in E_\theta^+,\ x\in\mathbb{R}^d.\end{aligned} \tag{A7}$$

Under (A7), for any $r\equiv const\in(0,\theta)$, $Gr\equiv const$. In this case, we assume also that

$$Gr<\beta, \qquad r\in(0,\theta). \tag{A8}$$

In Section 5, we prove the hair-trigger effect for the solutions to (1.1). For technical reasons, it will be done separately for kernels with and without the first moment. Namely, for the kernels which satisfy the condition

$$\int_{\mathbb{R}^d}|y|a(y)dy<\infty, \tag{A9}$$

we set

$$\mathfrak{m} := \varkappa\int_{\mathbb{R}^d} xa(x)\,dx\in\mathbb{R}^d, \tag{2.4}$$

and assume, additionally to (A4), that

$$\begin{aligned}&\textit{there exist } q\geq 0,\ \delta>0,\ 0\leq b\in C^\infty(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d), \textit{ such that}\\ &\qquad a(x)-b(x)\geq\delta 1\!\!1_{B_\delta(0)}(x), \quad x\in\mathbb{R}^d,\\ &\qquad w\,Gw\leq\varkappa b*w+qw, \quad w\in E_\theta^+.\end{aligned} \tag{A10}$$

Remark 2.4. We are going to formulate now our main results about the hair-trigger effect for a solution to (1.1). It requires that the initial condition to (1.1) is not degenerate: if E is a space of continuous functions, this means that u_0 is not identically equal to zero, $u_0\not\equiv 0$. For a brevity of notations, in the case $E=L^\infty(\mathbb{R}^d)$, we will treat $u_0\not\equiv 0$ as follows: there exists $\delta>0$ and $x_0\in\mathbb{R}^d$, such that $u_0(x)\geq\delta$ for a.a. $x\in B_\delta(x_0)$.

Then we can formulate the following

Theorem 2.5. *Let the conditions* (A1)–(A10) *hold. Let* $u_0\in E_\theta^+$, $u_0\not\equiv 0$ *(cf. Remark 2.4), and let* u *be the corresponding solution to* (1.1). *Then, for* $\mathfrak{m}$ *defined by* (2.4) *and any compact set* $K\subset\mathbb{R}^d$,

$$\lim_{t\to\infty}\operatorname*{essinf}_{x\in K} u(x+t\mathfrak{m},t)=\theta. \tag{2.5}$$

Remark 2.6. Note that the correction term $t\mathfrak{m}=t\varkappa\int_{\mathbb{R}^d}ya(y)dy$ in (2.5) equals to the expected value of the compound Poisson process with the probability density a and the intensity $\varkappa$.

An evident example of a probability kernel with an infinite first moment is the density $a(x)=c(1+|x|^2)^{-\frac{1+d}{2}}$, $x\in\mathbb{R}^d$ of the multivariate Cauchy distribution; here $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$, and c is the normalizing factor to ensure (1.2). To include this and other cases, for the kernels which do not satisfy (A9), we consider the following assumption:

$$\begin{gathered}\textit{for each } n\in\mathbb{N}, \textit{ let there exist}\\ 0\le a_n\in L^1(\mathbb{R}^d),\quad \varkappa_n>0,\quad G_n:E\to E,\quad \theta_n\in(0,\theta]\\ \textit{which satisfy } \text{(A1)–(A10)} \textit{ instead of } a,\ \varkappa,\ G,\ \theta,\\ \textit{correspondingly, such that}\\ \mathfrak{m}_n:=\varkappa_n\int_{\mathbb{R}^d} x a_n(x)dx\in\mathbb{R},\quad \theta_n\ge\theta-\frac{1}{n},\ n\in\mathbb{N},\\ \varkappa_n a_n*w-wG_nw\le\varkappa a*w-wGw,\quad w\in E^+_{\theta_n}.\end{gathered}\tag{A11}$$

Then the following counterpart of Theorem 2.5 holds.

Theorem 2.7. *Let the condition* (A11) *hold. Let* $u_0\in E^+_\theta$, $u_0\not\equiv 0$ *(cf. Remark 2.4), and let* u *be the corresponding solution to* (1.1). *Then, for any compact set* $K\subset\mathbb{R}^d$ *and for any* $n\in\mathbb{N}$,

$$\theta-\frac{1}{n}\le\liminf_{t\to\infty}\operatorname*{essinf}_{x\in K}u(x+t\mathfrak{m}_n,t)\le\limsup_{t\to\infty}\operatorname*{essinf}_{x\in K}u(x+t\mathfrak{m}_n,t)\le\theta.$$

In particular, if $\mathfrak{m}_n=\widetilde{\mathfrak{m}}\in\mathbb{R}$ *for all* $n\ge n_0\in\mathbb{N}$*, then*

$$\lim_{t\to\infty}\operatorname*{essinf}_{x\in K}u(x+t\widetilde{\mathfrak{m}},t)=\theta.$$

In particular, if (A1)–(A10) hold and $\mathfrak{m}=0\in\mathbb{R}^d$ or if (A11) holds and $\mathfrak{m}_n=0\in\mathbb{R}^d$ for all $n\ge n_0\in\mathbb{N}$, then one gets the desired hair-trigger effect described above.

Remark 2.8. Note that, indeed, for a properly ‘slanted’ anisotropic kernel a with $\mathfrak{m}\neq 0\in\mathbb{R}^d$, the solution to (1.1) may converge to 0 uniformly on any ball centered at the origin, whereas it will converge to θ on the ‘time-moving’ ball according to Theorems 2.5 or 2.7; see [30] for the corresponding result in the case of the Example 2 described below.

Examples

Example 1 (Reaction–diffusion equation with a local reaction). A particular example of (1.4), with $F(u)=f(u)$ for a function $f:\mathbb{R}\to\mathbb{R}$, was considered e.g. in [1,3,10,16–20,36,56,58]. We assume (A1) and (A5) as before, whereas the assumptions (A2)–(A4), (A6)–(A8), (A10) are fulfilled if only

$$\begin{gathered}f\ \textit{is Lipschitz continuous on } [0,\theta];\\ \lim_{r\to0+}\frac{f(r)}{r}=\beta;\\ f(0)=f(\theta)=0;\quad 0<f(r)\le\beta r,\ r\in(0,\theta).\end{gathered}$$

If (A9) does not hold, then, to fulfill (A11), it is enough to take $\varkappa_n=\varkappa$, $a_n(x):=1\!\!1_{\Lambda_n}(x)a(x)$, provided that $\Lambda_n\subset\mathbb{R}^d$ are such that $\Lambda_n\uparrow\mathbb{R}^d$ and $\int_{\Lambda_n}xa(x)dx=\widetilde{\mathfrak{m}}$. In particular, if $a(-x)=a(x)$, $x\in\mathbb{R}^d$, one can take $\Lambda_n:=B_n(0)$.

Example 2 (**Spatial logistic equation:** $\boldsymbol{Gu = \varkappa^- a^- * u}$). Let $\varkappa^- > 0$ and $a^-(x)$ be a probability kernel. We consider $Gu = \varkappa^- a^- * u$, i.e. (1.1) has the form

$$\frac{\partial u}{\partial t} = \varkappa(a * u) - \varkappa^- u(a^- * u) - mu.$$

This equation first appeared, for the case $\varkappa a = \varkappa^- a^-$, $m = 0$, in [47,48]; for the case $\varkappa a = \varkappa^- a^-$, $m > 0$ in [27], and for the different kernels in [11], where the so-called Bolker–Pacala model of spatial ecology was considered. The equation was rigorously derived from the Bolker–Pacala model in [35] for integrable u and in [29] for bounded u. The long-time behavior of this equation was studied in [30–32], see also [52].

We assume (A1) and (A5) as before. Under (A1), we have in this case $\theta = \dfrac{\varkappa - m}{\varkappa^-} > 0$. Then the conditions (A2)–(A3), (A6)–(A8) are satisfied. The condition (A4) holds if and only if

$$\varkappa a(x) \geq (\varkappa - m) a^-(x), \qquad x \in \mathbb{R}^d. \tag{2.6}$$

Condition (A10) holds if we additionally assume that there exists $\delta > 0$, such that

$$\varkappa a(x) - (\varkappa - m) a^-(x) \geq \delta 1\!\!1_{B_\delta(0)}(x), \qquad x \in \mathbb{R}^d.$$

In this case we can put, in (A10), $b(x) = (\varkappa - m) a^-(x)$, $q = 0$.

If (A9) does not hold, then, to fulfill (A11), one can proceed as in the previous example. Namely, we define a_n as before, and we set $G_n u = \varkappa^- a_n^- * u$, where $a_n^-(x) := 1\!\!1_{\Lambda_n}(x) a^-(x)$, $x \in \mathbb{R}^d$.

Example 3 (**The case** $\boldsymbol{Gu = \varkappa^- a^- * u - g_1(a^- * u)}$). Let $g(s) = \varkappa^- s - g_1(s)$, where $\varkappa^- > 0$, $g_1 : [0, \theta] \to \mathbb{R}_+$ is increasing and Lipschitz continuous, such that $g_1(s) = o(s)$, as $s \to 0$ and $\varkappa^- s \geq g_1(s)$, for $s \in (0, \theta)$. We define $Gv = g(a^- * v)$, where a^- is a probability kernel. Namely, we consider the following equation,

$$\frac{\partial u}{\partial t} = \varkappa(a * u) - \varkappa^- u(a^- * u) + u g_1(a^- * u) - mu.$$

As in the previous example, (A4) holds if and only if (2.6) holds. The rest of the assumptions can be characterized straightforward. Typical example is $g(s) = \beta\bigl(1 - \bigl(1 - \frac{s}{\theta}\bigr)^n\bigr)$. In this case, the corresponding reaction is

$$F(u) = \frac{\beta}{\theta^n} u(\theta - a^- * u)^n.$$

3 Existence and uniqueness

In this Section, we will show the existence and uniqueness of non-negative solutions to a generalized version of (1.1) on $\mathbb{R}_+$, see Theorem 3.3 below. Note that the equation (1.1) itself is a semi-linear evolution (parabolic) equation on E. The condition (A3) ensures that the nonlinear term $u\,Gu$ in (1.1) is locally Lipschitz. The general theory of semi-linear parabolic equations (see e.g. [51, Theorem 6.1.4]) provides existence and uniqueness of the so-called *mild* solution to (1.1) on the time interval $[0, t_{\max})$ for some $t_{\max} \leq \infty$. Since the operator (1.6) in (1.1) is bounded on E and G is continuous, this solution will be the classical one. Moreover, if $t_{\max} < \infty$, then, with necessity, $\|u(\cdot,t)\|_E \to \infty$, as $t \nearrow t_{\max}$. However, given $u_0 \geq 0$, the general theory does not ensure that $u(\cdot,t) \geq 0$, $t \in [0, t_{\max})$.

Remark 3.1. 1) Note that if we know *a priori* that u is non-negative on $[0, t_{\max})$, then $t_{\max} = \infty$, provided that $Gv \geq 0$ for all $0 \leq v \in E$ (cf. (A2) and the conditions of Theorem 2.2). Indeed, Duhamel's principle would imply then that $0 \leq u(x,t) \leq e^{-mt}e^{tA}u_0(x)$, where $(Av)(x) := (a * v)(x)$, and hence $\|u(\cdot,t)\|_E$ remains bounded on any finite time interval.

2) Another sufficient condition that would guarantee $t_{\max} = \infty$ is, therefore, the *a priori* global boundedness of u. In the case of the 'local' operator G, corresponding to the local reaction $Fu = f(u)$ in (1.4) (cf. Example 1), the global boundedness will follow from the comparison arguments considered in the Section 4 below (cf. Theorem 2.3). However, the case of a nonlocal operator G, and hence a nonlocal reaction F, would require a restrictive assumption (A4) for comparison. Moreover, one can modify the example in [40, pp. 2738–2739] to show that, in general, a solution to (1.1) does not need to be globally bounded on $\mathbb{R}_+$.

3) Note also, that any globally Lipschitz reaction F (and hence globally Lipschitz product $u\,Gu$) would lead to $t_{\max} = \infty$ (see e.g. [25, Theorem 3.2, 3.3], [24, Theorem 2.1]).

To avoid aforementioned additional assumptions for the non-local case of G and F, we consider here a direct proof of the existence and uniqueness of non-negative solutions to (a generalized version of) the equation (1.1). Our proof uses standard fixed point-arguments to get existence and uniqueness on consecutive time intervals $[\Upsilon_j, \Upsilon_{j+1}]$, $j \geq 0$, $\Upsilon_0 = 0$. Then, using Lemma 3.2 below, we will show that $\sum_{j\geq 0}(\Upsilon_{j+1} - \Upsilon_j) = \infty$ that implies the existence and uniqueness on an arbitrary time-interval.

Lemma 3.2. *Let $\{r_n\}_{n\in\mathbb{N}}$ be a sequence of numbers, such that $r_1 > 0$ and the following recurrence relation holds*

$$r_{n+1} = r_n + pe^{-qr_n}, \quad n \in \mathbb{N}, \tag{3.1}$$

where $p, q > 0$. Then the series $\sum\limits_{n\in\mathbb{N}} \dfrac{1}{r_n e^{qr_n}}$ is divergent.

Proof. By (3.1), r_n, $n \in \mathbb{N}$ is a positive increasing sequence. Passing to the limit in (3.1) when $n \to \infty$, one gets that $r_n \to \infty$, as $n \to \infty$. Hence, without loss of generality, one can assume that $b_n := e^{-qr_n} < (pq)^{-1}$, $n \in \mathbb{N}$. One can rewrite then (3.1) as follows: $b_{n+1} = b_n e^{-pqb_n}$. It is straightforward to check that

$$\frac{x}{1 + pqx(e-1)} \leq ye^{-pqy}, \quad 0 < x \leq y \leq \frac{1}{pq},$$

Therefore, if we set $c_1 := b_1$ and $c_{n+1} := \frac{c_n}{1+pq(e-1)c_n}$, $n \in \mathbb{N}$, we get $c_n \leq b_n$, $n \in \mathbb{N}$. On the other hand, $\frac{1}{c_{n+1}} = \frac{1}{c_n} + pq(e-1)$, that leads to

$$\frac{1}{c_{n+1}} = \frac{1}{c_1} + n(e-1)pq, \quad n \in \mathbb{N}. \tag{3.2}$$

Therefore,

$$\sum_{n\in\mathbb{N}} \frac{1}{r_n e^{qr_n}} = \sum_{n\in\mathbb{N}} \frac{b_n}{-q\ln b_n} \geq \sum_{n\in\mathbb{N}} \frac{c_n}{-q\ln c_n} = \infty,$$

since, by (3.2),

$$\frac{c_n}{-\ln c_n} \sim \frac{1}{pq(e-1)n\ln n}, \quad n\to\infty.$$

The statement is proved. □

Let $I \subset \mathbb{R}_+$ be a closed interval. The set $C_b(I \to E)$ of all continuous bounded E-valued functions on I becomes a Banach space being equipped with the norm

$$\|u\|_{C_b(I\to E)} := \sup_{t\in I} \|u(\cdot,t)\|_E.$$

For simplicity of notation, we denote also

$$\begin{aligned} \|u\|_{T_1,T_2} &:= \|u\|_{C_b([T_1,T_2]\to E)}, \quad 0<T_1<T_2; \\ \|u\|_T &:= \|u\|_{C_b([0,T]\to E)}, \qquad T>0. \end{aligned} \tag{3.3}$$

We are ready to prove now the existence and uniqueness result.

Theorem 3.3. *Let $A, G : E \to E$ be such that $Gv \geq 0$ and $Av \geq 0$ for all $0 \leq v \in E$, and, for some $\kappa, \varkappa > 0$,*

$$\|Av - Aw\|_E \leq \varkappa \|v-w\|_E, \qquad v,w\in E,\ v\geq 0,\ w\geq 0, \tag{3.4}$$

$$\|Gv - Gw\|_E \leq e^{\kappa r}\|v-w\|_E, \qquad v,w\in E_r^+,\ r>0. \tag{3.5}$$

Then, for any $T>0$ and $0\leq u_0 \in E$, there exists a unique nonnegative classical solution $u \in \mathcal{U}_T$ (cf. Definition 2.1) to the equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = (Au)(x,t) - mu(x,t) - u(x,t)(Gu)(x,t), \\ u(x,0) = u_0(x), \end{cases} \tag{3.6}$$

where $t \in (0,T]$, $x\in\mathbb{R}^d$.

Proof. First, we note that, by (3.4),

$$\|Av\|_E \leq \|A0\|_E + \varkappa\|v\|_E, \quad 0\leq v\in E. \tag{3.7}$$

We set $f_0 := \|A0\|_E$.

Let $T>0$ be arbitrary. Take any $0 \leq v \in C_b([0,T]\to E)$. For any $\tau\in[0,T)$, consider the following linear equation in the space E on the interval $[\tau, T]$:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = (Av)(x,t) - mu(x,t) - u(x,t)(Gv)(x,t), \quad t\in(\tau,T] \\ u(x,\tau) = u_\tau(x), \end{cases} \tag{3.8}$$

where $0\leq u_\tau \in E$, $\tau>0$, and u_0 is the same as in (3.6). By assumptions on A and G, we have that $Av, Gv \in C_b([0,T]\to E)$ for all $v\in C_b([0,T]\to E)$. In the right-hand side of (3.8), there is a time-dependent linear bounded operator (acting in u) in the space E whose coefficients are continuous on $[\tau, T]$. Therefore, there exists a unique solution to (3.8) in E on $[\tau, T]$, given by $u = \Phi_\tau v$ with

$$(\Phi_\tau v)(x,t) := (Bv)(x,\tau,t)u_\tau(x) + \int_\tau^t (Bv)(x,s,t)(Av)(x,s)\,ds, \tag{3.9}$$

for $x \in \mathbb{R}^d$, $t \in [\tau, T]$, where we set

$$(Bv)(x,s,t) := \exp\Bigl(-\int_s^t \bigl(m + (Gv)(x,p)\bigr)dp\Bigr), \tag{3.10}$$

for $x \in \mathbb{R}^d$, $t, s \in [\tau, T]$. Note that, in particular, $(\Phi_\tau v)(\cdot, t), (Bv)(\cdot, s, t) \in E$. Clearly, $(\Phi_\tau v)(x,t) \geq 0$ and, for any $\Upsilon \in (\tau, T]$,

$$\|\Phi_\tau v(\cdot,t)\|_E \leq \|u_\tau\|_E + (f_0 + \varkappa \|v\|_{\tau,\Upsilon})(\Upsilon - \tau), \qquad t \in [\tau, \Upsilon], \tag{3.11}$$

where we used (3.7) and the notation (3.3). Therefore, Φ_τ maps $\{0 \leq v \in C_b([\tau,\Upsilon] \to E)\}$ into itself, $\Upsilon \in (\tau, T]$.

For any $T_2 > T_1 \geq 0$ and $r > 0$, we define

$$\mathcal{X}^+_{T_1,T_2}(r) := \bigl\{v \in C_b([T_1,T_2] \to E) \bigm| v \geq 0, \|v\|_{T_1,T_2} \leq r\bigr\}. \tag{3.12}$$

Let now $0 \leq \tau < \Upsilon \leq T$, and take any $v, w \in \mathcal{X}^+_{\tau,\Upsilon}(r)$. By (3.9), one has, for any $x \in \mathbb{R}^d$, $t \in [\tau, \Upsilon]$,

$$\bigl|(\Phi_\tau v)(x,t) - (\Phi_\tau w)(x,t)\bigr| \leq J_1 + J_2, \tag{3.13}$$

where

$$\begin{aligned}
J_1 &:= \bigl|(Bv)(x,\tau,t) - (Bw)(x,\tau,t)\bigr|u_\tau(x),\\
J_2 &:= \int_\tau^t \bigl|(Bv)(x,s,t)(Av)(x,s) - (Bw)(x,s,t)(Aw)(x,s)\bigr|\,ds.
\end{aligned}$$

Clearly, for each $a \in L^1(\mathbb{R}^d)$, $f \in E$,

$$\bigl|(a * f)(x)\bigr| \leq \|f\|_E\, \|a\|_{L^1(\mathbb{R}^d)}. \tag{3.14}$$

Since $|e^{-a} - e^{-b}| \leq |a - b|$, for any constants $a, b \geq 0$, one has, by (3.10), (3.14),

$$J_1 \leq e^{\kappa r}(\Upsilon - \tau)\|u_\tau\|_E \|v - w\|_{\tau,\Upsilon}. \tag{3.15}$$

Next, for any constants $a, b, p, q \geq 0$,

$$\bigl|pe^{-a} - qe^{-b}\bigr| \leq e^{-a}|p - q| + q\max\bigl\{e^{-a}, e^{-b}\bigr\}|a - b|,$$

therefore, by (3.10), (3.14),

$$\begin{aligned}
J_2 &\leq \varkappa \int_\tau^t (Bv)(x,s,t)\,ds \|v - w\|_{\tau,\Upsilon}\\
&\quad + \int_\tau^t \max\bigl\{(Bv)(x,s,t), (Bw)(x,s,t)\bigr\}|(Aw)(x,s)|(t-s)e^{\kappa r}\|v - w\|_{\tau,\Upsilon}\,ds\\
&\leq \varkappa(\Upsilon - \tau)\|v - w\|_{\tau,\Upsilon} + e^{\kappa r}(f_0 + \varkappa\|w\|_{\tau,\Upsilon})\|v - w\|_{\tau,\Upsilon} \int_\tau^t e^{-m(t-s)}(t-s)\,ds\\
&\leq \Bigl(\varkappa + (f_0 + \varkappa\|w\|_{\tau,\Upsilon})\frac{e^{\kappa r}}{me}\Bigr)(\Upsilon - \tau)\|v - w\|_{\tau,\Upsilon},
\end{aligned} \tag{3.16}$$

as $re^{-r} \leq e^{-1}$, $r \geq 0$.

Take any $\mu \geq \|u_\tau\|_E$. By (3.11)–(3.16), one has,

$$\big|(\Phi_\tau v)(x,t) - (\Phi_\tau w)(x,t)\big| \leq \Big(\mu e^{\kappa r} + \varkappa + (f_0 + \varkappa r)\frac{e^{\kappa r}}{me}\Big)(\Upsilon - \tau)\|v - w\|_{\tau,\Upsilon},$$
$$\big|(\Phi_\tau v)(x,t)\big| \leq \mu + (f_0 + \varkappa r)(\Upsilon - \tau).$$

Therefore, Φ_τ will be a contraction mapping on the set $\mathcal{X}^+_{\tau,\Upsilon}(r)$ if only

$$\Big(\mu e^{\kappa r} + \varkappa + (f_0 + \varkappa r)\frac{e^{\kappa r}}{me}\Big)(\Upsilon - \tau) < 1 \quad \text{and} \quad \mu + (f_0 + \varkappa r)(\Upsilon - \tau) \leq r.$$

If $\frac{f_0}{\varkappa} \leq r$, it is sufficient to show

$$\Big(\mu e^{\kappa r} + \varkappa + 2\varkappa r\frac{e^{\kappa r}}{me}\Big)(\Upsilon - \tau) < 1 \quad \text{and} \quad \mu + 2\varkappa r(\Upsilon - \tau) \leq r. \tag{3.17}$$

Take for $\alpha \in (0,1)$,

$$r := \mu + \alpha m e^{1-\kappa\mu}, \qquad \Upsilon := \tau + \frac{\alpha m e}{2\varkappa r e^{\kappa r}}. \tag{3.18}$$

Then, the second inequality in (3.17) holds, since $e^{\kappa r}$ is increasing, namely,

$$\mu + 2\varkappa r(\Upsilon - \tau) = \mu + \alpha m e^{1-\kappa r} \leq \mu + \alpha m e^{1-\kappa\mu} = r.$$

Next

$$\Big(\mu e^{\kappa r} + \varkappa + \frac{2\varkappa r e^{\kappa r}}{me}\Big)(\Upsilon - \tau) = \frac{\alpha m e \mu}{2\varkappa r} + \frac{\alpha m e}{2 r e^{\kappa r}} + \alpha \leq \frac{\alpha m e}{2\varkappa} + \frac{\alpha m e}{2 r e^{\kappa r}} + \alpha.$$

In order to satisfy the second inequality in (3.17) it is sufficient to check,

$$\frac{\alpha m e}{2\varkappa} + \frac{\alpha m e}{2 r e^{\kappa\mu}} < 1 - \alpha,$$

but $re^{\kappa\mu} = \mu e^{\kappa\mu} + \alpha m e$, i.e. we need

$$\frac{\alpha m e}{2(\mu e^{\kappa\mu} + \alpha m e)} + \frac{\alpha m e}{2\varkappa} < 1 - \alpha. \tag{3.19}$$

Choose $\alpha \in (0,1)$, such that $\frac{\alpha m e}{2\varkappa} < 1 - \alpha$, and then choose $\mu > 0$ large enough to ensure (3.19). As a result, one gets that Φ_τ will be a contraction on the set $\mathcal{X}^+_{\tau,\Upsilon}(r)$ with Υ and r given by (3.18); the latter set naturally forms a complete metric space. Therefore, there exists a unique $u \in \mathcal{X}^+_{\tau,\Upsilon}(r)$ such that $\Phi_\tau u = u$. This u will be a solution to (3.6) on $[\tau, \Upsilon]$.

To fulfill the proof of the statement, one can do the following. Set $\tau := 0$, choose $r_0 > \max\{\|u_0\|_E, \frac{f_0}{\varkappa}\}$ and $\alpha \in (0,1)$ that satisfy (3.19) with $\mu = r_0$. One gets a solution u to (3.6) on $[0, \Upsilon_1]$ with $\|u\|_{\Upsilon_1} \leq r_0 + \alpha m e^{1-\kappa r_0} =: r_1$, $\Upsilon_1 = \frac{\alpha m e^{1-\kappa r_1}}{2\varkappa r_1}$.

Iterating this scheme, take sequentially, for each $n \in \mathbb{N}$, $\tau := \Upsilon_n$, $x \in \mathbb{R}^d$,

$$r_n := r_{n-1} + \alpha m e^{1-\kappa r_{n-1}} \geq \|u(\cdot, \Upsilon_n)\|_E.$$

Since $r_n > r_{n-1}$ and $e^{\kappa r}$ is increasing, the same α as before will satisfy (3.19) with $\mu = r_n$ as well. Then, one gets a solution u to (3.6) on $[\Upsilon_n, \Upsilon_{n+1}]$ with initial condition u_{Υ_n}, where

$$\Upsilon_{n+1} := \Upsilon_n + \frac{\alpha m e^{1-\kappa r_n}}{2\varkappa r_n}, \tag{3.20}$$

and

$$\|u\|_{\Upsilon_n,\Upsilon_{n+1}} \leq r_n + \alpha m e^{1-\kappa r_n} = r_{n+1}.$$

As a result, we will have a solution u to (3.6) on intervals $[0, \Upsilon_1]$, $[\Upsilon_1, \Upsilon_2]$, …, $[\Upsilon_n, \Upsilon_{n+1}]$, $n \in \mathbb{N}$. By (3.4)–(3.5), the right-hand side of (3.6), will be continuous on each of constructed time-intervals, therefore, one has that u is continuously differentiable on $(0, \Upsilon_{n+1}]$ and solves (1.1) there. By (3.20) and Lemma 3.2,

$$\Upsilon_{n+1} = \frac{\alpha m e}{2\varkappa} \sum_{j=0}^{n} \frac{1}{r_j e^{\kappa r_j}} \to \infty, \quad n \to \infty,$$

therefore, one has a solution to (3.6) on any $[0, T]$, $T > 0$.

To prove uniqueness, suppose that $v \in C_b([0, T] \to E)$ is a solution to (3.6) on $[0, T]$, with $v(x, 0) \equiv u_0(x)$, $x \in \mathbb{R}^d$. Choose $r_0 > \|v\|_T \geq \|u_0\|_E$. Since $\{r_n\}_{n\geq 0}$ above is an increasing sequence, v will belong to each of sets $\mathcal{X}^+_{\Upsilon_n,\Upsilon_{n+1}}(r_{n+1})$, $n \geq 0$, $\Upsilon_0 := 0$, considered above. Then, being solution to (3.6) on each $[\Upsilon_n, \Upsilon_{n+1}]$, v will be a fixed point for Φ_{Υ_n}. By the uniqueness of such a point, v coincides with u on each $[\Upsilon_n, \Upsilon_{n+1}]$ and, thus, on the whole $[0, T]$. As a result, $u(x, t) = (\Phi_0 u)(x, t)$, for $x \in \mathbb{R}^d$, $t \geq 0$. Since $u \in C_b([0, T] \to E)$, then $u = \Phi_0 u \in C^1((0, T] \to E)$. Thus u is a classical solution to (1.1). The proof is fulfilled. □

Remark 3.4. Since $Av := \varkappa a * v$, $v \in E$, evidently satisfies conditions of Theorem 3.3, one gets Theorem 2.2.

Proposition 3.5. *Let the conditions of Theorem 3.3 hold. Suppose, additionally, that A and G are continuous on $\{0 \leq v \in E\}$ in the topology of locally uniform convergence, i.e. for any $v_n, v \in E$, $v_n \geq 0$, $v \geq 0$, with $v_n \overset{\text{loc}}{\Longrightarrow} v$, one has*

$$Av_n \overset{\text{loc}}{\Longrightarrow} Av, \qquad Gv_n \overset{\text{loc}}{\Longrightarrow} Gv, \qquad n \to \infty.$$

Let $T > 0$ be fixed and, for some $\varrho > 0$, $\{u(\cdot, 0), u_n(\cdot, 0) : n \in \mathbb{N}\} \subset E^+_\varrho$ be the initial conditions to (3.6), and let $\{u(\cdot, t), u_n(\cdot, t) : n \in \mathbb{N}\}$ be the corresponding solutions to (3.6) on $[0, T]$. Assume that $u_n(\cdot, 0) \overset{\text{loc}}{\Longrightarrow} u(\cdot, 0)$, $n \to \infty$. Then $u_n(\cdot, t) \overset{\text{loc}}{\Longrightarrow} u(\cdot, t)$, $n \to \infty$ uniformly in $t \in [0, T]$.

Proof. By the proof of Theorem 3.3, there exist $0 = \tau_0 < \tau_1 < \ldots < \tau_N = T$ and $\varrho = r_0 \leq r_1 \leq \ldots \leq r_N =: r$, such that the following holds. Let, for any $\tau = \tau_k$, $\Upsilon = \tau_{k+1}$, $0 \leq k \leq N - 1$, the mapping Φ_τ be defined by (3.9) for $t \in [\tau, \Upsilon]$, with $u_\tau(x) = u(x, \tau)$, $x \in \mathbb{R}^d$; and, for each $n \in \mathbb{N}$, we set

$$(\Phi_{\tau,n} v)(x, t) := (Bv)(x, \tau, t) u_{\tau,n}(x) + \int_\tau^t (Bv)(x, s, t)(Av)(x, s)\, ds,$$

where $u_{\tau,n}(x) = u_n(x, \tau)$, $x \in \mathbb{R}^d$. Then $v \in \mathcal{X}^+_{\tau,\Upsilon}(r_{k+1})$, $\{u_\tau, u_{\tau,n} : n \in \mathbb{N}\} \subset E_{r_k}$ implies $\{\Phi_\tau v, \Phi_{\tau,n} v : n \in \mathbb{N}\} \subset \mathcal{X}^+_{\tau,\Upsilon}(r_{k+1})$, (cf. (3.12)).

Prove that if, for some $\{w, w_n : n \in \mathbb{N}\} \subset \mathcal{X}^+_{\tau,\Upsilon}(r_{k+1})$, we have that $w_n(\cdot,t) \overset{\text{loc}}{\Longrightarrow} w(\cdot,t)$, $n \to \infty$, uniformly in $t \in [\tau, \Upsilon]$, then

$$\Phi_{\tau,n} w_n(\cdot,t) \overset{\text{loc}}{\Longrightarrow} \Phi_\tau w(\cdot,t), \quad n \to \infty, \tag{3.21}$$

uniformly in $t \in [\tau, \Upsilon]$. Indeed, applying the inequalities,

$$|e^{-a} - e^{-b}| \leq |a-b|, \qquad \bigl|pe^{-a} - qe^{-b}\bigr| \leq |p-q| + q|a-b|,$$

for $a, b, p, q \geq 0$, we get, for any bounded $\Lambda \subset \mathbb{R}^d$,

$$\begin{aligned}
&\mathbb{1}_\Lambda(x)\bigl|(\Phi_{\tau,n} w_n)(x,t) - (\Phi_\tau w)(x,t)\bigr| \\
&\leq \mathbb{1}_\Lambda(x)\bigl|(\Phi_{\tau,n} w_n)(x,t) - (\Phi_{\tau,n} w)(x,t)\bigr| + \mathbb{1}_\Lambda(x)\bigl|(\Phi_{\tau,n} w)(x,t) - (\Phi_\tau w)(x,t)\bigr| \\
&\leq \mathbb{1}_\Lambda(x)\bigl|u_{\tau,n}(x) - u_\tau(x)\bigr| + r_k \int_\tau^t \mathbb{1}_\Lambda(x)\bigl|(Gw_n)(x,p) - (Gw)(x,p)\bigr|\, dp \\
&\quad + \int_\tau^t \mathbb{1}_\Lambda(x)\bigl|(Aw_n)(x,s) - (Aw)(x,s)\bigr|\, ds \\
&\quad + \int_\tau^t \mathbb{1}_\Lambda(x)\bigl|(Aw)(x,s)\bigr| \int_s^t \bigl|(Gw_n)(x,p) - (Gw)(x,p)\bigr|\, dp\, ds \\
&\leq \bigl\|\mathbb{1}_\Lambda\bigl(u_{\tau,n} - u_\tau\bigr)\bigr\|_E + r_k \int_\tau^\Upsilon \bigl\|\mathbb{1}_\Lambda\bigl((Gw_n)(\cdot,p) - (Gw)(\cdot,p)\bigr)\bigr\|_E dp \\
&\quad + \int_\tau^\Upsilon \bigl\|\mathbb{1}_\Lambda\bigl((Aw_n)(\cdot,s) - (Aw)(\cdot,s)\bigr)\bigr\|_E\, ds \\
&\quad + (\|A(0)\|_E + \varkappa r) \int_\tau^\Upsilon \int_s^\Upsilon \bigl\|\mathbb{1}_\Lambda\bigl((Gw_n)(\cdot,p) - (Gw)(\cdot,p)\bigr)\bigr\|_E\, dp\, ds.
\end{aligned}$$

Hence (3.21) holds. Iterating this scheme, one gets that, for each $m \in \mathbb{N}$, $v \in \mathcal{X}^+_{\tau,\Upsilon}(r_{k+1})$,

$$(\Phi_{\tau,n})^m v(\cdot,t) \overset{\text{loc}}{\Longrightarrow} (\Phi_\tau)^m v, \quad n \to \infty, \tag{3.22}$$

uniformly in $t \in [\tau, \Upsilon]$. Therefore, for any bounded $\Lambda \subset \mathbb{R}^d$,

$$\begin{aligned}
&\bigl|\mathbb{1}_\Lambda(x)(u_n(x,t) - u(x,t))\bigr| \\
&\leq \bigl|\mathbb{1}_\Lambda(x)\bigl(u_n(x,t) - (\Phi_{\tau,n})^m v(x,t)\bigr)\bigr| + \bigl|\mathbb{1}_\Lambda(x)\bigl((\Phi_{\tau,n})^m v(x,t) - (\Phi_\tau)^m v(x,t)\bigr)\bigr| \\
&\quad + \bigl|\mathbb{1}_\Lambda(x)\bigl(u(x,t) - (\Phi_\tau)^m v(x,t)\bigr)\bigr| \\
&\leq \bigl\|u_n - (\Phi_{\tau,n})^m v\|_{\tau,\Upsilon} + \sup_{t\in[\tau,\Upsilon]} \bigl\|\mathbb{1}_\Lambda\bigl((\Phi_{\tau,n})^m v(\cdot,t) - (\Phi_\tau)^m v(\cdot,t)\bigr)\bigr\|_E \\
&\quad + \bigl\|u - (\Phi_\tau)^m v\|_{\tau,\Upsilon},
\end{aligned}$$

for any $m \in \mathbb{N}$. Passing m to ∞, one gets then the statement by (3.22). $\square$

4 Comparison principle

The comparison principle is a standard tool in studying parabolic- and elliptic-type equations, see e.g. [21, 37]. For instance, it allows to estimate an unknown solution, constructing explicit sub- and super-solutions [5–7]. See also [17,18,54] for comparison results and its applications in studying traveling waves for nonlocal equations. To the best of our knowledge, the first detailed proof of the comparison principle for the parabolic equation in the case of nonlocal diffusion (1.6) in (1.4), was done by Yagisita [58] in the case of globally Lipschitz KPP-type reaction $Fu = f(u)$ (see also [46, Lemma D.1]). The comparison principle is often used in other articles without any reference on the proof. Also we do not know any result on the comparison principle in the case of a non-local reaction.

We will get in Theorem 4.2 the comparison principle related to an abstract evolution equation

$$\frac{\partial u}{\partial t}(x,t) = (Hu)(x,t),$$

where $H : E \to E$ is locally Lipschitz continuous and such that the operator $H + p$ is monotone on E for some $p > 0$. Here and below we use the same notation for a constant and for the operator of multiplication by this constant in the space E.

Remark 4.1. For the equation (1.1), the monotonicity of $H + p$ has the form (A4). Note that in the case of a local operator G (cf. Example 1), there exists $p > 0$, such that (A3) implies (A4), and hence the comparison indeed does not require any additional assumptions. However, for a nonlocal G the assumption (A4) is restrictive. For instance, in Example 2, (A4) is necessary and sufficient (and hence optimal) condition to ensure the comparison principle in E_θ^+, see [30, Remark 3.6].

We introduce some additional notations. For any $v \in E$, $r \in \mathbb{R}$, we set

$$(v \wedge r)(x) := \min\{v(x), r\}, \quad (v \vee r)(x) := \max\{v(x), r\}.$$

Let $H : E \to E$. For any $u \in \mathcal{U}_T$, cf. (2.1), and $r > 0$, we define

$$(\mathcal{F}_r u)(x,t) := \frac{\partial u}{\partial t}(x,t) - H(0 \vee u \wedge r)(x,t), \qquad t \in (0,T],\ x \in \mathbb{R}^d. \tag{4.1}$$

Here and below we consider the left derivative at $t = T$ only.

Theorem 4.2. *Let $H : E \to E$ and $h, p, r > 0$ be such that H is Lipschitz continuous on E_r^+ with the Lipschitz constant $h > 0$, and $H + p$ is monotone on E_r^+, namely,*

$$\|Hw - Hv\|_E \le h\|w - v\|_E, \qquad w, v \in E_r^+, \tag{4.2}$$

$$Hv + pv \le Hw + pw, \qquad v \le w,\ w, v \in E_r^+. \tag{4.3}$$

Let $T > 0$ be fixed. Suppose that $u_1, u_2 \in \mathcal{U}_T$ are such that

$$0 \le u_1(x,t), \quad u_2(x,t) \le r, \qquad (x,t) \in \mathbb{R}^d \times (0,T], \tag{4.4}$$

$$(\mathcal{F}_r u_1)(x,t) \le (\mathcal{F}_r u_2)(x,t), \qquad (x,t) \in \mathbb{R}^d \times (0,T], \tag{4.5}$$

$$u_1(x,0) \le u_2(x,0), \qquad x \in \mathbb{R}^d. \tag{4.6}$$

Then $u_1(x,t) \le u_2(x,t)$ for all $(x,t) \in \mathbb{R}^d \times [0,T]$.

Proof. Define, cf. (4.5), the following function

$$\phi_r(x,t) := (\mathcal{F}_r u_2)(x,t) - (\mathcal{F}_r u_1)(x,t) \geq 0, \tag{4.7}$$

for $(x,t) \in \mathbb{R}^d \times [0,T]$. For a constant $K > 0$, which will be specified later, consider the mapping

$$\begin{aligned}\Theta(t,w) :&= Kw + e^{Kt}\bigl(H\bigl(0 \vee (e^{-Kt}w + u_1) \wedge r\bigr) - H(u_1 \wedge r)\bigr)\\ &\quad + e^{Kt}\phi_r(x,t), \qquad w \in C_b([0,T] \to E).\end{aligned} \tag{4.8}$$

We have, for $w \geq 0$,

$$0 \leq u_1 \wedge r \leq (e^{-Kt}w + u_1) \wedge r \leq r.$$

Since, for any $x \geq y \geq 0$, $z \geq 0$,

$$0 \leq x \wedge z - y \wedge z \leq x - y,$$

one has, by (4.3), (4.7), that $0 \leq w \in C_b([0,T] \to E)$ yields

$$\Theta(t,w) \geq (K-p)w + e^{Kt}\phi_r(x,t) \geq 0,$$

if only $K \geq p$ that we will assume in the following.

Next, applying (4.2) to (4.8), we will get that $w \in C_b([0,T] \to E)$ implies, for all $t \in [0,T]$,

$$\|\Theta(t,w)\|_T \leq (K+h)\|w\|_T + e^{Kt}\|\phi_r\|_T.$$

Therefore, since $u_1, u_2 \in \mathcal{U}_T$ implies, by (4.1), (4.7), that $\phi_r \in C_b([0,T] \to E)$, one gets that $\Theta(t,w) \in C_b([0,T] \to E)$.

Define also the function

$$v(x,t) := e^{Kt}(u_2(x,t) - u_1(x,t)), \quad x \in \mathbb{R}^d, \ t \in [0,T].$$

Clearly, $v \in \mathcal{U}_T$, and it is straightforward to check that

$$\Theta(t, v(x,t)) = \frac{\partial}{\partial t} v(x,t), \quad x \in \mathbb{R}^d, \ t \in (0,T].$$

Therefore, v solves the following integral equation in E:

$$\begin{cases} v(x,t) = v(x,0) + \displaystyle\int_0^t \Theta(s, v(x,s))ds, & (x,t) \in \mathbb{R}^d \times (0,T], \\ v(x,0) = u_2(x,0) - u_1(x,0), & x \in \mathbb{R}^d, \end{cases} \tag{4.9}$$

where $v(x,0) \geq 0$ by (4.6).

Consider also another integral equation in E:

$$\widetilde{v}(x,t) = (\Psi\widetilde{v})(x,t) \tag{4.10}$$

where, for $w \in C_b([0,T] \to E)$,

$$(\Psi w)(x,t) := v(x,0) + \int_0^t \max\bigl\{\Theta(s, w(x,s)), 0\bigr\}\, ds. \tag{4.11}$$

If we take $\widetilde{T} < T$ such that the following inequality holds

$$q_1 := 2r(K+h) + e^{KT}\|\phi_r\|_T \le \frac{r}{\widetilde{T}},$$

then, $w \in \mathcal{X}^+_{\widetilde{T}}(2r)$ yields $\Psi w \in \mathcal{X}^+_{\widetilde{T}}(2r)$, where, cf. (3.12), $\mathcal{X}^+_{\widetilde{T}}(2r) := \mathcal{X}^+_{0,\widetilde{T}}(2r)$. Let $w_1, w_2 \in \mathcal{X}_{\widetilde{T}}(2r)$; by (4.2), (4.8), we have, for all $(t,x) \in [0,\widetilde{T}] \times \mathbb{R}^d$,

$$|\Theta(t,w_1) - \Theta(t,w_2)| \le (K+h)\|w_1 - w_2\|_{\widetilde{T}} =: q_2\|w_1 - w_2\|_{\widetilde{T}}.$$

Therefore, using the elementary inequality $|\max\{a,0\} - \max\{b,0\}| \le |a-b|$, $a,b \in \mathbb{R}$, we obtain from (4.11), that

$$\|\Psi w_1 - \Psi w_2\|_{\widetilde{T}} \le q_2\widetilde{T}\|w_2 - w_1\|_{\widetilde{T}}.$$

Therefore, for $\widetilde{T} < \max\{\frac{r}{2q_1}, \frac{1}{q_2}\}$, Ψ is a contraction on $\mathcal{X}^+_{\widetilde{T}}(2r)$. Thus, there exists a unique solution $\widetilde{v}$ to (4.10) on $[0,\widetilde{T}]$. By (4.10), (4.11),

$$\widetilde{v}(x,t) \ge v(x,0) \ge 0, \qquad x \in \mathbb{R}^d,\ t \in [0,\widetilde{T}]. \tag{4.12}$$

By the considerations above, $0 \le w \in C_b([0,T] \to E)$ yields $0 \le \Theta(s, w(x,s)) \in C_b([0,T] \to E)$. Hence $\widetilde{v}$ is a solution to (4.9) on $[0,\widetilde{T}]$ as well. Namely,

$$\widetilde{v}(x,t) = v(x,0) + \int_0^t \Theta(s, \widetilde{v}(x,s))\, ds =: \Xi(\widetilde{v})(x,t).$$

By the same arguments as the above, Ξ is a contraction on $\mathcal{X}_{\widetilde{T}}(2r)$, for the same $\widetilde{T}$. We deduce that $v = \widetilde{v}$ on $\mathbb{R}^d \times [0,\widetilde{T}]$. Then, by (4.12), $v(x,t) \ge 0$ on $\mathbb{R}^d \times [0,\widetilde{T}]$, that yields

$$0 \le u_1(x,\widetilde{T}) \le u_2(x,\widetilde{T}) \le r,\ x \in \mathbb{R}^d.$$

In the same way, the proof can be extended on $[\widetilde{T}, 2\widetilde{T}], [2\widetilde{T}, 3\widetilde{T}], \ldots$, keeping the same q_1 and q_2, and, therefore, on the whole $[0,T]$. Then $v(x,t) \ge 0$ on $\mathbb{R}^d \times [0,T]$, that yields the statement. □

Clearly, Theorem 4.2 in the case $r = \theta$, $Hv = \varkappa a * v - vGv - mv$, $v \in E$, implies the first statement of Theorem 2.3. The following Proposition yields the second statement of Theorem 2.3.

Proposition 4.3. *Let* (A1)–(A4) *hold and* $0 \le u_0 \le \theta$. *Then there exists a unique (classical) solution* u *to* (1.1), *and* $0 \le u(x,t) \le \theta$ *for any* $x \in \mathbb{R}^d$, $t \ge 0$.

Proof. We set $Hv := \varkappa a * v - vGv - mv$ for $v \in E^+_\theta$, and $Hv := H(0 \vee v \wedge \theta)$ for $v \in E \setminus E^+_\theta$. Prove, first, that H is (globally) Lipschitz continuous on E. Indeed, for any $x, y \in \mathbb{R}$,

$$|x \wedge \theta - y \wedge \theta| = \frac{1}{2}\big|(x + \theta - |x-\theta|) - (y + \theta - |y - \theta|)\big| \le |x - y|$$

and, similarly, $|x \vee 0 - y \vee 0| \le |x-y|$. Therefore, denoting $v_\theta := 0 \vee v \wedge \theta$ for $v \in E$, one gets that $\|v_\theta - w_\theta\|_E \le \|v - w\|_E$ for $w, v \in E$, and hence

$$\begin{aligned}\|Hw - Hv\|_E &\le (\varkappa + m + \sup_{v\in E}\|G(0 \vee v \wedge \theta)\|_E + \theta l_\theta)\|w - v\|_E\\ &= (2\varkappa + \theta l_\theta)\|w - v\|_E.\end{aligned}$$

As a result, for any $T > 0$, the initial value problem

$$\frac{\partial \widetilde{u}}{\partial t}(x,t) = (H\widetilde{u})(x,t), \quad \widetilde{u}(x,0) = u_0(x), \quad x \in \mathbb{R}^d,\ t \in (0,T],$$

has a unique classical solution $\widetilde{u}$, i.e., for $\mathcal{F}_\theta$ defined by (4.1), $\mathcal{F}_\theta\widetilde{u} \equiv 0$.

Note that, for any $r \geq \theta$, $v \in E_r^+$ implies $Hv = H(v \wedge \theta)$. In particular, applying this for $v = 0 \vee \widetilde{u} \wedge r$, one gets

$$\mathcal{F}_r\widetilde{u} = \mathcal{F}_\theta\widetilde{u} \equiv 0. \tag{4.13}$$

Moreover, by (A4), there exists $p \geq 0$, such that, for any $r \geq \theta$, $v, w \in E_r^+$, $v \leq w$,

$$p(w - v) + Hw - Hv \geq p(w \wedge \theta - v \wedge \theta) + H(w \wedge \theta) - H(v \wedge \theta) \geq 0.$$

Assume that $\|\widetilde{u}\|_T > \theta$. Then, by the arguments above and (4.13), we may apply Theorem 4.2 for the case $r = \|\widetilde{u}\|_T$, $u_1 \equiv 0$, $u_2 = \widetilde{u}$ (note that, evidently, $\mathcal{F}_r 0 = 0$). It yields $\widetilde{u} \geq 0$. Next, similarly, we can apply Theorem 4.2 for the case $r = \theta$, $u_1 = \widetilde{u}$, $u_2 \equiv \theta$ (since $\mathcal{F}_\theta\theta = 0$). It implies then that $\widetilde{u} \leq \theta$, that contradicts the assumption, therefore, $\|\widetilde{u}\|_T \leq \theta$. Apply once more Theorem 4.2 for the case $r = \theta$, $u_1 \equiv 0$, $u_2 = \widetilde{u}$, then $\widetilde{u} \geq 0$. As a result, the function $\widetilde{u} = 0 \vee \widetilde{u} \wedge \theta$ solves (1.1).

Choose an arbitrary extension of G on $\{0 \leq v \in E\}$ such that (3.5) holds. By Theorem 2.2, there exists a unique classical solution u to (1.1). Hence $0 \leq u = \widetilde{u} \leq \theta$. The proof is fulfilled. □

5 The hair-trigger effect: proofs of Theorems 2.5, 2.7

We are going to prove our main Theorems 2.5 and 2.7. The Section is organized as follows. First, in Propositions 5.1–5.2, we show some properties of solutions to (1.1) with continuous initial conditions. Note that, by existence and uniqueness Theorem 2.2, the solutions will be also continuous and, moreover, by comparison Theorem 2.3, any solution in $E = L^\infty(\mathbb{R}^d)$ can be estimated from above and below by continuous ones taking the corresponding estimates for the initial condition $u_0 \not\equiv 0$, cf. Remark 2.4.

Next, we describe general Weinberger's scheme [57] for a dynamical system in discrete time in the context of the equation (1.1) (Propositions 5.4 and 5.7, Lemma 5.8), and prove the corresponding result for continuous time (Proposition 5.11). The latter result is proved under additional assumptions inherited by general Weinberger's approach: a technical assumption (5.17) on the dynamical system and an assumption (5.18) on the initial condition u_0, which cannot be verified for particular examples of u_0, cf. Remark 5.9.

Then, in Proposition 5.13, by using Lemma 5.12, we prove that the technical assumption (5.17) holds. To get rid of restrictions on initial condition u_0, one needs more machinery. Namely, we find in Proposition 5.14 a useful sub-solution to the linearization of the equation (1.1) around the zero solution. Next, we show that (being multiplied on a small enough constant) it will be a sub-solution to the nonlinear equation (1.1) as well (Proposition 5.15) and, in Proposition 5.16, we show that a solution to (1.1) becomes larger than the sub-solution after a big enough time. As a result, one can show that Weinberger's assumption (5.18) on the initial condition is fulfilled (just starting from a moment of time $t_0 > 0$ rather than from 0). Finally, in the proof of Theorem 2.7, we show how to deal with the kernels without the first moment (where the assumption (A9) fails).

Proposition 5.1. *Let $0 \le u_0 \in C_{ub}(\mathbb{R}^d)$, and suppose that u is the corresponding classical solution to (1.1). Suppose also, that there exists $C > 0$, such that*

$$0 \le u(x,t) \le C, \quad x \in \mathbb{R}^d, \ t \ge 0,$$

and $g_C := \sup\limits_{v \in E_C^+} |Gv| < \infty$. Then $u \in C_{ub}(\mathbb{R}^d \times \mathbb{R}_+)$ and, moreover, $\|u(\cdot,t)\|_E \in C_{ub}(\mathbb{R}_+)$. In particular, these inclusions hold if we assume (A1)–(A4).

Proof. Being classical solution to (1.1), u satisfies the integral equation

$$u(x,t) = u_0(x) + \int_0^t \bigl(\varkappa(a*u)(x,s) - mu(x,s) - u(x,s)(Gu)(x,s)\bigr)\, ds.$$

Hence for any $x, y \in \mathbb{R}^d$, $0 \le \tau < t$, one has

$$\bigl|u(x,t) - u(y,\tau)\bigr| \le (2\varkappa C + 2mC + Cg_C)(t-\tau),$$

that fulfills the proof of the first inclusion. Then, the second one follows from the inequality $\bigl|\|u(\cdot,t)\|_E - \|u(\cdot,\tau)\|_E\bigr| \le \|u(\cdot,t) - u(\cdot,\tau)\|_E$, $t,\tau \in \mathbb{R}_+$. Finally, if the conditions (A1)–(A4) hold, then, by Proposition 4.3, one gets that the solution u exists and satisfies the conditions above if only $C := \theta$. Moreover, (A3) implies that, for any $v \in E_\theta^+$,

$$\|Gv\|_E \le \|G0\|_E + l_\theta \|v\|_E \le \|G0\|_E + \theta l_\theta < \infty,$$

that fulfills the proof. □

The maximum principle is a 'standard counterpart' of the comparison principle, see e.g. [17]. We will use in the sequel that, under some additional assumptions, the solutions to (1.1) are strictly positive; this is a quite common feature of linear parabolic equations, however, in general, it may fail for nonlinear ones. Consider the corresponding statement.

Proposition 5.2. *Let $E = C_b(\mathbb{R}^d)$. Let (A1)–(A5) hold with $G : E \to E$, such that $Gl \not\equiv \beta$, for $l \in (0,\theta)$. (In particular, the latter holds, if we assume, additionally, (A7)–(A8).) Let $u_0 \in E_\theta^+$, $u_0 \not\equiv \theta$, $u_0 \not\equiv 0$, be the initial condition to (1.1) and u be the corresponding solution. Then*

$$u(x,t) > \inf_{\substack{y \in \mathbb{R}^d \\ s > 0}} u(y,s) \ge 0, \qquad x \in \mathbb{R}^d, t > 0.$$

Proof. By Proposition 4.3, $0 \le u(x,t) \le \theta$, $x \in \mathbb{R}^d$, $t \ge 0$. We denote

$$(L_a u)(x,t) = \varkappa(a*u)(x,t) - \varkappa u(x,t). \tag{5.1}$$

Then, by (A2),

$$\frac{\partial u}{\partial t}(x,t) - (L_a u)(x,t) = u(x,t)(\beta - (Gu)(x,t)) \ge 0. \tag{5.2}$$

Prove that, under (5.2), u cannot attain its infimum on $\mathbb{R}^d \times (0,\infty)$ without being a constant. Indeed, suppose that, for some $x_0 \in \mathbb{R}^d$, $t_0 > 0$,

$$u(x_0,t_0) \le u(x,t), \quad x \in \mathbb{R}^d, t > 0. \tag{5.3}$$

Then, clearly,

$$\frac{\partial u}{\partial t}(x_0,t_0) = 0, \tag{5.4}$$

and (5.2) yields $(L_a u)(x_0,t_0) \le 0$. On the other hand, (5.1) and (5.3) imply $(L_a u)(x_0,t_0) \ge 0$. Therefore,

$$\int_{\mathbb{R}^d} a(x_0-y)(u(y,t_0)-u(x_0,t_0))\,dy = 0.$$

Then, by (A5), for all $y \in B_\varrho(x_0)$,

$$u(y,t_0) = u(x_0,t_0). \tag{5.5}$$

By the same arguments, for an arbitrary $x_1 \in \partial B_\varrho(x_0)$, we obtain (5.5), for all $y \in B_\varrho(x_1)$. Hence, (5.5) holds on $B_{2\varrho(x_0)}$, and so on. As a result, (5.5) holds, for all $y \in \mathbb{R}^d$, thus $u(\cdot,t_0)$ is a constant, i.e.

$$u(x,t_0) = u(x_0,t_0) =: l_0 \in [0,\theta], \quad x \in \mathbb{R}^d.$$

Then, considering (1.1) at (x_0,t_0), and taking into account (5.4), one gets

$$0 = u(x_0,t_0)\bigl(\beta - (Gu)(x_0,t_0)\bigr) = l_0(\beta - Gl_0).$$

By the assumption, the latter equality is possible if only $l_0 \in \{0,\theta\}$, i.e. either $u(\cdot,t_0) \equiv 0$ or $u(\cdot,t_0) \equiv \theta$. By (5.3), $u(x_0,t_0) = \theta \ge \sup_{y\in\mathbb{R}^d,s>0} u(y,s)$ implies $u \equiv \theta$, that contradicts $u_0 \not\equiv \theta$. Hence $u(x,t_0) = u(x_0,t_0) = 0$, $x \in \mathbb{R}^d$. Then, by Theorem 3.3, $u(x,t) = 0$, $x \in \mathbb{R}^d$, $t \ge t_0$. And now one can consider the reverse time in (1.1) starting from $t = t_0$. Namely, we set $w(x,t) := u(x,t_0-t)$, $t \in [0,t_0]$, $x \in \mathbb{R}^d$. Then $w(x,0) = u(x,t_0) = 0$, $x \in \mathbb{R}^d$, and

$$\frac{\partial w}{\partial t}(x,t) = w(x,t)(Gw)(x,t) - \varkappa^+(a^+ * w)(x,t) + mw(x,t). \tag{5.6}$$

Prove that the equation (5.6) with the initial condition $w(\cdot,0) \equiv 0$ has a unique classical solution $w \equiv 0$ in $C_b([0,t_0] \to E)$. Indeed, let $w \in C_b([0,t_0] \to E)$ solve (5.6). Suppose that the set

$$K := \bigl\{ t \in [0,t_0] \bigm| \|w(\cdot,t)\|_E > 0 \bigr\}$$

is not empty, i.e. $w \not\equiv 0$. We define then $T := \inf K$. In particular, $\|w(\cdot,t)\|_E = 0$ for $t \in [0,T)$ (note that the latter interval might be empty if $T = 0$). Since the function $\tau \mapsto \|w(\cdot,\tau)\|_E$ is continuous, we have that $\|w(\cdot,T)\|_E = 0$ as well. Therefore, $T = t_0$ would contradict the assumption $K \ne \emptyset$; hence $T < t_0$. Consider now the equation (5.6) for $t \in [T,t_0]$ with the initial value $w(\cdot,T) \equiv 0$. It is straightforward to check that the assumptions on G imply that, for any $r > 0$, there exists $\Delta T > 0$, such that $T + \Delta T < t_0$ and the mapping

$$\Psi(w)(x,t) = \int_T^{T+t} w(x,s)(Gw)(x,s) - \varkappa(a * w)(x,s) + mw(x,s)ds.$$

is a contraction on $C_b([0,\Delta T]\to E)$. Therefore, by the uniqueness arguments, $w(\cdot,t)\equiv 0$ for $t\in[T,T+\Delta T]$ that contradicts the choice of T. Therefore, $K=\emptyset$, i.e. $w(\cdot,t)\equiv 0$ for all $t\in[0,t_0]$, in particular, $u(\cdot,0)=w(\cdot,t_0)\equiv 0$, that contradicts $u_0\not\equiv 0$. Thus, the initial assumption was wrong, and (5.3) can not hold. The proof is fulfilled. □

In the sequel, it will be useful to consider the solution to (1.1) as a nonlinear transformation of the initial condition.

Definition 5.3. For a fixed $t>0$, define the mapping Q_t on $\{f\in E\mid f\geq 0\}$ by

$$(Q_t f)(x) := u(x,t), \quad x\in\mathbb{R}^d, \tag{5.7}$$

where $u(x,t)$ is the solution to (1.1) with the initial condition $u(x,0)=f(x)$.

Let us collect several properties of Q_t needed below.

Proposition 5.4. *Let* (A1)–(A8) *hold. Then, for any fixed $t>0$, the mapping $Q:=Q_t:\{f\in E\mid f\geq 0\}\to\{f\in E\mid f\geq 0\}$ satisfies the following properties*

(Q1) *$Q:E^+_\theta\to E^+_\theta$;*

(Q2) *let T_y, $y\in\mathbb{R}^d$, be a translation operator, given by* (2.2)*, then*

$$(QT_yf)(x)=(T_yQf)(x), \quad x,y\in\mathbb{R}^d,\ f\in E^+_\theta;$$

(Q3) *$Q0=0$, $Q\theta=\theta$, and $Qr>r$, for any constant $r\in(0,\theta)$;*

(Q4) *if $f,g\in E^+_\theta$ and $f\leq g$, then $Qf\leq Qg$;*

(Q5) *if $f_n\overset{\mathrm{loc}}{\Longrightarrow} f$, then $(Qf_n)(x)\to(Qf)(x)$ for (a.a.) $x\in\mathbb{R}^d$.*

Proof. The property (Q1) follows from Proposition 4.3. To prove (Q2) we note that, by (A7), $T_yG=GT_y$, and $T_y(a*u)=a*(T_yu)$, and then, by (3.10), $B(T_yv)=T_y(Bv)$. Using further the notations in the proof of Theorem 3.3, we will proceed by the induction in n. Namely, assume that $Q_tT_y=T_yQ_t$ for $t\in[0,\Upsilon_{n-1}]$. Denote $\Phi_\tau[u_\tau]:=\Phi_\tau$, given by (3.9) (to specify the dependence on the initial condition u_τ). Then $T_y(\Phi_\tau[u_\tau]v)=\Phi_\tau[T_yu_\tau](T_yv)$ for all $v\in\mathcal{X}^+_{\tau,\Upsilon}(r_n)$, where $[\tau,\Upsilon]:=[\Upsilon_{n-1},\Upsilon_n]$. Then, for $t\in(\tau,\Upsilon]$,

$$\begin{aligned}Q_tT_yf &= \lim_{N\to\infty}\bigl(\Phi_\tau[Q_\tau T_yf]\bigr)^N\bigl(T_yv(\cdot,t)\bigr)=\lim_{N\to\infty}\bigl(\Phi_\tau[T_yQ_\tau f]\bigr)^N\bigl(T_yv(\cdot,t)\bigr)\\ &= \lim_{N\to\infty}T_y(\Phi_\tau[Q_\tau f])^Nv(\cdot,t)=T_yQ_tf.\end{aligned}$$

By (Q2), $u_0(x)\equiv r\in(0,\theta)$ yields $u(\cdot,t)=\mathrm{const}$, $t\geq 0$. Then, by (A2) and (A8), for any $t\geq 0$, we have

$$Qr=u(t)=r+\int_0^t u(s)(\beta-(Gu)(s))ds>0.$$

Hence the property (Q3) holds. The property (Q4) holds by Theorem 4.2. The property (Q5) is a weaker version of Proposition 3.5. □

Remark 5.5. Take an arbitrary constant $r \in (0,\theta)$. One can treat then r as a constant function from E_θ^+. By (Q3) and (Q4), the sequence $\bigl(Q_t^n r\bigr)_{n\geq 1} \subset (0,\theta]$ is non-decreasing for an arbitrary $t > 0$. Hence there exists the limit $r_\infty := \lim\limits_{n\to\infty} Q_t^n r \in (0,\theta]$. Next, by (Q5),

$$Q_t r_\infty = Q_t \lim_{n\to\infty} Q_t^n r = \lim_{n\to\infty} Q_t^{n+1} r = r_\infty.$$

Hence, by (Q3), $r_\infty = \theta$. By Proposition 5.1, $Q_t r$ is uniformly continuous in $t > 0$. As a result, $\lim\limits_{t\to\infty} Q_t r = \theta$. Therefore, by (Q4), for any $u_0 \in E$ with $0 < r \leq u_0 \leq \theta$, we have

$$\theta = \lim_{t\to\infty} Q_t r \leq \lim_{t\to\infty} Q_t u_0 \leq \theta, \qquad \text{and hence } \lim_{t\to\infty} Q_t u_0 = \theta.$$

As a result, $u \equiv 0$ is unstable and $u \equiv \theta$ is asymptotically stable solutions to (1.1) in E_θ^+. For this reason, we refer to (1.1) as to a monostable-type equation.

Let S^{d-1} denote a unit sphere in $\mathbb{R}^d$ centered at the origin:

$$S^{d-1} = \bigl\{ x \in \mathbb{R}^d \bigm| |x| = 1 \bigr\};$$

in particular, $S^0 = \{-1, 1\}$.

Definition 5.6. A function $f \in E$ is said to be increasing (decreasing, constant) along the vector $\xi \in S^{d-1}$ if, for a.a. $x \in \mathbb{R}^d$, the function $f(x+s\xi) = (T_{-s\xi} f)(x)$ is increasing (decreasing, constant) in $s \in \mathbb{R}$, respectively.

Proposition 5.7. *Let* (A1)–(A8) *hold. Let $u_0 \in E_\theta^+$ be the initial condition for the equation* (1.1) *which is increasing (decreasing, constant) along a vector $\xi \in S^{d-1}$; and $u(\cdot,t) \in E_\theta^+$, $t \geq 0$, be the corresponding solution (cf. Proposition 4.3). Then, for any $t > 0$, $u(\cdot,t)$ is increasing (decreasing, constant, respectively) along the ξ.*

Proof. Let u_0 be decreasing along a $\xi \in S^{d-1}$. Take any $s_1 \leq s_2$ and consider two initial conditions to (1.1): $u_0^i(x) = u_0(x + s_i\xi) = (T_{-s_i\xi} u_0)(x)$, $i = 1, 2$ (cf. (2.2)). Since u_0 is decreasing, $u_0^1(x) \geq u_0^2(x)$, $x \in \mathbb{R}^d$. Then, by Proposition 5.4,

$$T_{-s_1\xi} Q_t u_0 = Q_t T_{-s_1\xi} u_0 = Q_t u_0^1 \geq Q_t u_0^2 = Q_t T_{-s_2\xi} u_0 = T_{-s_2\xi} Q_t u_0,$$

that proves the statement. The cases of a decreasing u_0 can be considered in the same way. The constant function along a vector is decreasing and decreasing simultaneously. $\square$

To prove the hair-trigger effect (Theorems 2.5, 2.7), we will follow the abstract scheme proposed in [57] for a dynamical system in discrete time. Note that all statements there were considered in the space $E = C_b(\mathbb{R}^d)$.

Consider the set N_θ of all non-increasing functions $\varphi \in C(\mathbb{R})$, such that $\varphi(s) = 0$, $s \geq 0$, and

$$\varphi(-\infty) := \lim_{s\to-\infty} \varphi(s) \in (0,\theta).$$

For arbitrary $s \in \mathbb{R}$, $c \in \mathbb{R}$, $\xi \in S^{d-1}$, define the following mapping $V_{s,c,\xi} : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R}^d)$

$$(V_{s,c,\xi} g)(x) = g(x\cdot\xi + s + c), \quad x \in \mathbb{R}^d. \tag{5.8}$$

Fix an arbitrary $\varphi \in N_\theta$. For $t > 0$, $c \in \mathbb{R}$, $\xi \in S^{d-1}$, consider the mapping $R_{t,c,\xi} : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$, given by

$$(R_{t,c,\xi}g)(s) = \max\bigl\{\varphi(s), (Q_t(V_{s,c,\xi}g))(0)\bigr\}, \quad s \in \mathbb{R}, \tag{5.9}$$

where $Q_t : E \to E$ is a mapping which satisfies the conditions (Q1)–(Q5) in Proposition 5.4 (in particular, one can consider Q_t given by (5.7) provided that (A1)–(A8) hold). Consider now the following sequence of functions

$$f_{n+1}(s) = (R_{t,c,\xi}f_n)(s), \quad f_0(s) = \varphi(s), \qquad s \in \mathbb{R}, n \in \mathbb{N} \cup \{0\}. \tag{5.10}$$

By Proposition 5.4 and [57, Lemma 5.1], $0 \le \phi(s) \le \theta$, $s \in \mathbb{R}$, implies $0 \le f_n(s) \le f_{n+1}(s) \le \theta$, $s \in \mathbb{R}$, $n \in \mathbb{N}$; hence one can define the following limit

$$f_{t,c,\xi}(s) := \lim_{n\to\infty} f_n(s), \quad s \in \mathbb{R}. \tag{5.11}$$

Also, by [57, Lemma 5.1], for fixed $\xi \in S^{d-1}$, $t > 0$, $n \in \mathbb{N}$, the functions $f_n(s)$ and $f_{t,c,\xi}(s)$ are non-increasing in s and in c; moreover, $f_{t,c,\xi}(s)$ is a lower semi-continuous function of s, c, ξ, as a result, this function is continuous from the right in s and in c. Note also, that $0 \le f_{t,c,\xi} \le \theta$. Then, for any c, ξ, one can define the limiting value

$$f_{t,c,\xi}(\infty) := \lim_{s\to\infty} f_{t,c,\xi}(s).$$

Next, for any $t > 0$, $\xi \in S^{d-1}$, we define

$$c_t^*(\xi) = \sup\{c \mid f_{t,c,\xi}(\infty) = \theta\} \in \mathbb{R} \cup \{-\infty, \infty\},$$

where, as usual, $\sup\emptyset := -\infty$. By [57, Propositions 5.1, 5.2], one has

$$f_{t,c,\xi}(\infty) = \begin{cases} \theta, & c < c_t^*(\xi), \\ 0, & c \ge c_t^*(\xi), \end{cases} \tag{5.12}$$

cf. also [57, Lemma 5.5]; moreover, $c_t^*(\xi)$ is a lower semi-continuous function of ξ. It is crucial that, by [57, Lemma 5.4], neither $f_{t,c,\xi}(\infty)$ nor $c_t^*(\xi)$ depends on the choice of $\varphi \in N_\theta$. Note that the monotonicity of $f_{t,c,\xi}(s)$ in s and (5.12) imply that, for $c < c_t^*(\xi)$, $f_{t,c,\xi}(s) = \theta$, $s \in \mathbb{R}$.

Define

$$\Upsilon_t := \bigl\{x \in \mathbb{R}^d \bigm| x \cdot \xi \le c_t^*(\xi), \xi \in S^{d-1}\bigr\}, \quad t > 0. \tag{5.13}$$

For $A \subset \mathbb{R}^d$, $x \in \mathbb{R}^d$, $s \in \mathbb{R}$, we denote also

$$x + A := \{x + y \mid y \in A\} \subset \mathbb{R}^d, \quad sA := \{sy \mid y \in A\} \subset \mathbb{R}^d.$$

We will need the following Weinberger's result:

Lemma 5.8 (cf. [57, Theorem 6.2]). *Let $E = C_b(\mathbb{R}^d)$ and $v_0 \in E_\theta^+$. Let, for some fixed $t > 0$, $Q = Q_t : E \to E$ be a mapping which satisfies the conditions (Q1)–(Q5) in Proposition 5.4, and Υ_t be defined by* (5.13). *Suppose that*

$$\operatorname{int}(\Upsilon_t) \ne \emptyset. \tag{5.14}$$

Then, for any compact set $\mathscr{C}_t \subset \operatorname{int}(\Upsilon_t)$ and for any $\sigma \in (0,\theta)$, one can choose a radius $r_\sigma = r_\sigma(Q_t, \mathscr{C}_t) > 0$, such that, for any fixed $x_0 \in \mathbb{R}^d$,

$$v_0(x) \geq \sigma, \quad x \in B_{r_\sigma}(x_0), \tag{5.15}$$

implies

$$\lim_{n\to\infty} \min_{x\in n\mathscr{C}_t} Q_t^n v_0(x) = \theta. \tag{5.16}$$

Remark 5.9. Note that, in [57, Theorem 6.2], the existence of r_σ is proved only; there are not any estimates on it. As a result, for a given $v_0 \in E_\theta^+$, the condition (5.15) cannot be checked directly.

Remark 5.10. There is no loss of generality if we assume that (5.15) holds for $x_0=0$ only. Indeed, for any $x_0 \in \mathbb{R}^d$, $\mathscr{C}_t \subset \operatorname{int}(\Upsilon_t)$, there exist $N = N(x_0, \mathscr{C}_t)$, $\widetilde{\mathscr{C}_t} \subset \operatorname{int}(\Upsilon_t)$, such that, for all $n \geq N$, one gets $x_0 + n\mathscr{C}_t \subset n\widetilde{\mathscr{C}_t}$. Therefore, we have

$$\begin{aligned}\theta &\geq \lim_{n\to\infty} \min_{x\in n\mathscr{C}_t} (Q_t^n T_{-x_0} u_0)(x) = \lim_{n\to\infty} \min_{x\in n\mathscr{C}_t} (T_{-x_0} Q_t^n u_0)(x)\\ &= \lim_{n\to\infty} \min_{x\in x_0+n\mathscr{C}_t} (Q_t^n u_0)(x) \geq \lim_{n\to\infty} \min_{x\in n\widetilde{\mathscr{C}_t}} (Q_t^n u_0)(x) = \theta.\end{aligned}$$

The following statement presents a counterpart of Lemma 5.8 for continuous time provided that the mapping Q_t is given by the solution to (1.1) as in (5.7).

Proposition 5.11. *Let* (A1)–(A8) *hold and $u_0 \in C_{ub}(\mathbb{R}^d)$. Let Q_t, $t > 0$, be given by* (5.7)*, and let the corresponding Υ_t, $t > 0$, be given by* (5.13)*. Suppose that, for some compact $\mathscr{C} \subset \operatorname{int}(\Upsilon_1)$, there exists $\mathfrak{n} \in \operatorname{int}(\mathscr{C})$, such that*

$$\frac{1}{j}\mathfrak{n} \in \operatorname{int}(\Upsilon_{\frac{1}{j}}), \qquad j \in \mathbb{N}. \tag{5.17}$$

Let $\sigma \in (0,\theta)$ and $r_\sigma = r_\sigma(Q_1, \mathscr{C})$ be chosen according to Lemma 5.8. Suppose that

$$u_0(x) \geq \sigma, \qquad x \in B_{r_\sigma(Q_1,\mathscr{C})}. \tag{5.18}$$

Then, for the corresponding solution u to (1.1) *and for any compact $K \subset \mathbb{R}^d$, the following limit holds*

$$\min_{x\in K} u(x + t\mathfrak{n}, t) \to \theta, \qquad t \to \infty. \tag{5.19}$$

Proof. First, we note that, by Proposition 5.4, the conditions (Q1)–(Q5) hold for all $Q = Q_t$, $t > 0$. We denote $K_1 := -\mathfrak{n} + \mathscr{C}$. Because of (5.18), one can apply Lemma 5.8 for $t = 1$ and $v_0(x) := u_0(x)$, $x \in \mathbb{R}^d$. Namely, since $Q_1^n v_0(y) = Q_1^n u_0(y) = u(y, n)$, $y \in \mathbb{R}^d$, one gets, by (5.16),

$$\min_{x\in nK_1} u(x + n\mathfrak{n}, n) = \min_{y\in n\mathscr{C}} u(y, n) \to \theta, \quad n \to \infty. \tag{5.20}$$

Next, by (5.17), $0 \in -\frac{1}{2}\mathfrak{n} + \operatorname{int}(\Upsilon_{\frac{1}{2}})$. Choose now any compact $K_2 \subset -\frac{1}{2}\mathfrak{n} + \operatorname{int}(\Upsilon_{\frac{1}{2}})$, such that $0 \in \operatorname{int}(K_2)$. By Lemma 5.8 for $t = \frac{1}{2}$ and $\mathscr{C}_{\frac{1}{2}} := K_2 + \frac{1}{2}\mathfrak{n} \subset$

$\mathrm{int}(\Upsilon_{\frac12})$ there exists a radius $r_\sigma(Q_{\frac12},\mathscr{C}_{\frac12})>0$. By (5.20), there exists $N_1\geq 1$, such that, for all $n\geq N_1$,

$$B_{r_\sigma(Q_{\frac12},\mathscr{C}_{\frac12})}(0)\cup K\subset nK_1,\qquad u(x+n\mathfrak{n},n)\geq\sigma,\quad x\in nK_1. \tag{5.21}$$

Set $S_1:=N_1$; by the latter inclusion and (5.21), one can apply Lemma 5.8 for $v_0(x):=u(x+S_1\mathfrak{n},S_1)$, $x\in\mathbb{R}^d$. Then

$$Q^n_{\frac12}v_0(y)=u\Bigl(y+S_1\mathfrak{n},S_1+\frac{n}{2}\Bigr),\quad y\in\mathbb{R}^d,$$

and hence

$$\begin{aligned}&\min_{x\in nK_2}u\Bigl(x+\Bigl(S_1+\frac{n}{2}\Bigr)\mathfrak{n},S_1+\frac{n}{2}\Bigr)\\&=\min_{y\in n\mathscr{C}_{\frac12}}u\Bigl(y+S_1\mathfrak{n},S_1+\frac{n}{2}\Bigr)\to\theta,\quad n\to\infty.\end{aligned}\tag{5.22}$$

Similarly, choose a compact $K_3\subset-\frac13\mathfrak{n}+\mathrm{int}(\Upsilon_{\frac13})$ with $0\in\mathrm{int}(K_3)$, and consider Lemma 5.8 with $t=\frac13$ and $\mathscr{C}_{\frac13}:=K_3+\frac13\mathfrak{n}\subset\mathrm{int}(\Upsilon_{\frac13})$. Then, there exists a radius $r_\sigma(Q_{\frac13},\mathscr{C}_{\frac13})>0$, and, by (5.22), there exists $N_2\geq 2$ such that for all $n\geq N_2$,

$$B_{r_\sigma(Q_{\frac13},\mathscr{C}_{\frac13})}\cup K\subset nK_2$$

and

$$u\Bigl(x+\Bigl(S_1+\frac{n}{2}\Bigr)\mathfrak{n},S_1+\frac{n}{2}\Bigr)\geq\sigma,\quad x\in nK_2.$$

Set $S_2:=S_1+\frac{N_2}{2}=N_1+\frac{N_2}{2}\geq 2$ and apply Lemma 5.8 with $v_0(x):=u(x+S_2\mathfrak{n},S_2)$, $x\in\mathbb{R}^d$. We have

$$\min_{x\in nK_3}u\Bigl(x+\Bigl(S_2+\frac{n}{3}\Bigr)\mathfrak{n},S_2+\frac{n}{3}\Bigr)=\min_{x\in n\mathscr{C}_{\frac13}}u\Bigl(x+S_2\mathfrak{n},S_2+\frac{n}{3}\Bigr)\to\theta,\quad n\to\infty.$$

By induction, for any $K_j\subset-\frac1j\mathfrak{n}+\mathrm{int}(\Upsilon_{\frac1j})$, $j\geq 3$, with $0\in\mathrm{int}(K_j)$, one can set $\mathscr{C}_{\frac1j}:=K_j+\frac1j\mathfrak{n}\subset\mathrm{int}(\Upsilon_{\frac1j})$ and choose $N_{j-1}\geq j-1$ such that for all $n\geq N_{j-1}$,

$$\begin{gathered}B_{r_\sigma(Q_{\frac1j},\mathscr{C}_{\frac1j})}\cup K\subset nK_{j-1},\\ u\Bigl(x+\Bigl(S_{j-2}+\frac{n}{j-1}\Bigr)\mathfrak{n},S_{j-2}+\frac{n}{j-1}\Bigr)\geq\sigma,\quad x\in nK_{j-1}.\end{gathered}$$

Set

$$S_{j-1}:=S_{j-2}+\frac{N_{j-1}}{j-1}=N_1+\frac{N_2}{2}+\ldots+\frac{N_{j-1}}{j-1}\geq j-1.$$

Then, by Lemma 5.8, similarly to the above,

$$\min_{x\in nK_j}u\Bigl(x+\Bigl(S_{j-1}+\frac{n}{j}\Bigr)\mathfrak{n},S_{j-1}+\frac{n}{j}\Bigr)\to\theta,\quad n\to\infty. \tag{5.23}$$

Suppose that (5.19) does not hold. Then, for some $\varepsilon>0$, there exist sequences $x_m\in K$, $m\in\mathbb{N}$, and $t_m\to\infty$ as $m\to\infty$, such that

$$u(x_m+t_m\mathfrak{n},t_m)\leq\theta-\varepsilon. \tag{5.24}$$

Since $u_0 \in C_{ub}(\mathbb{R}^d)$, then by Proposition 5.1, $u \in C_{ub}(\mathbb{R}^d \times \mathbb{R}_+)$. Thus there exists $\delta = \delta(\varepsilon)$, such that

$$|u(x,t) - u(y,s)| < \frac{\varepsilon}{2}, \qquad |x-y| < \delta,\ |t-s| < \delta.$$

We choose $j \in \mathbb{N}$ such that $\max\{1, |\mathfrak{n}|\} < \delta j$. By (5.23), there exists $N_j' > N_{j-1}$, such that, for all $n \geq N_j'$, we have that $K \subset nK_j$ and

$$\min_{x\in K} u\Big(x + \Big(S_{j-1} + \frac{n}{j}\Big)\mathfrak{n}, S_{j-1} + \frac{n}{j}\Big) > \theta - \frac{\varepsilon}{4}.$$

Choose m, such that $t_m \geq S_{j-1} + \dfrac{N_j'}{j}$. Let n_m be the entire part of $j(t_m - S_{j-1})$. Then $n_m \geq N_j'$ and, for $q_m := S_{j-1} + \dfrac{n_m}{j}$, we easily get that

$$\max\{1, |\mathfrak{n}|\}\, |t_m - q_m| < \delta.$$

Therefore,

$$\begin{aligned} u(x_m + t_m\mathfrak{n}, t_m) &\geq \min_{x\in K} u(x + t_m\mathfrak{n}, t_m) \\ &\geq \min_{x\in K} u(x + q_m\mathfrak{n}, q_m) - \max_{x\in K} \big|u(x + q_m\mathfrak{n}, q_m) - u(x + t_m\mathfrak{n}, t_m)\big| \\ &\geq \min_{x\in K} u(x + q_m\mathfrak{n}, q_m) - \frac{\varepsilon}{2} > \theta - \frac{3}{4}\varepsilon, \end{aligned}$$

that contradicts (5.24). Therefore (5.19) holds and the proof is fulfilled. □

We are going now to get rid of the assumptions (5.17) and (5.18) in Proposition 5.11. We start with the following lemma.

Lemma 5.12. *Let $b \in L^1(\mathbb{R} \to \mathbb{R}_+)$ be such that*

$$\int_{\mathbb{R}} b(s)\, ds = 1, \qquad \int_{\mathbb{R}} |s| b(s) ds < \infty,$$

and let $v \in L^\infty(\mathbb{R} \to \mathbb{R}_+)$ be a non-increasing function. Then the following limit holds

$$\lim_{r\to\infty} \int_{-r}^{r} \big((b * v)(s) - v(s)\big) ds = \big(v(-\infty) - v(\infty)\big) \int_{\mathbb{R}} s\, b(s)\, ds. \tag{5.25}$$

Proof. For $r > 0$ and $\varrho := \dfrac{r}{2}$, we have, by Fubini's theorem,

$$\begin{aligned} &\int_{-r}^{r} (b * v)(s) ds - \int_{-r}^{r} v(s) ds = \int_{-\infty}^{\infty} b(y) \int_{-r}^{r} v(s-y) ds dy - \int_{-r}^{r} v(s) ds \\ &= \int_{-\infty}^{\infty} b(y) W_r(y)\, dy = I_1(r) + I_2(r), \end{aligned}$$

where

$$W_r(y) := \int_{-r-y}^{r-y} v(s) ds - \int_{-r}^{r} v(s) ds, \quad y \in \mathbb{R},$$

$$I_1(r) := \int_{|y|\leq\varrho} b(y) W_r(y)\, dy, \qquad I_2(r) := \int_{|y|>\varrho} b(y) W_r(y)\, dy.$$

Clearly,

$$\sup_{|y|\leq\varrho} b(y)|W_r(y)| \leq 2\|v\|_E |y| b(y) \in L^1(\mathbb{R}).$$

Next, because of the monotonicity of v, we have,

$$yv(-r) \leq \int_{-r-y}^{-r} v(s)ds \leq yv(-r-y). \tag{5.26}$$

Since $-r-y < -\frac{r}{2}$ for $|y| \leq \varrho = \frac{r}{2}$, we have that

$$1\!\!1_{|y|\leq\varrho} \int_{-r-y}^{-r} v(s)ds \to v(-\infty)y, \qquad r \to \infty;$$

and, similarly, $1\!\!1_{|y|\leq\varrho} \int_{r-y}^{r} v(s)ds \to v(\infty)y$. Therefore, by the dominated convergence theorem,

$$\lim_{r\to\infty} I_1(r) = \bigl(v(-\infty) - v(\infty)\bigr) \int_{|y|\leq\varrho} yb(y)dy. \tag{5.27}$$

On the other hand,

$$|I_2(r)| \leq 2v(-\infty)\, r \int_{|y|>\varrho} b(y)dy \leq 4v(-\infty)\, \frac{r}{\varrho} \int_{|y|>\varrho} b(y)|y|dy \to 0, \tag{5.28}$$

as $r \to \infty$. Combining (5.27) and (5.28), one gets the statement. $\square$

The following statement yields sufficient conditions for (5.17).

Proposition 5.13. *Let* (A1)–(A9) *hold. Let* Υ_t*,* $t > 0$*, be defined by* (5.13)*, and* $\mathfrak{m}$ *be defined by* (2.4)*. Then*

$$t\,\mathfrak{m} \in \operatorname{int}(\Upsilon_t). \tag{5.29}$$

Proof. Fix $t > 0$. For a $\xi \in S^{d-1}$, we set

$$c := \varkappa t \int_{\mathbb{R}^d} y{\cdot}\xi a(y)dy = t\,\mathfrak{m}{\cdot}\xi \in \mathbb{R}. \tag{5.30}$$

Let $f_{t,c,\xi}$ be defined by (5.11). By the definition of Υ_t and (5.12), we have that if $f_{t,c,\xi}(\infty) = \theta$ for all $\xi \in S^{d-1}$, then (5.29) holds. Suppose, in contrast, that, for some $\xi \in S^{d-1}$, $f_{t,c,\xi}(\infty) = 0$. Fix such a ξ, consider the corresponding c according to (5.30), and denote $f := f_{t,c,\xi}$. Note that, by [57, Lemma 5.2] and the discussion thereafter, $f(-\infty) = \theta$.

We set $u_0(x) := f(x \cdot \xi)$, $x \in \mathbb{R}^d$, and consider the corresponding solution u to (1.1). Then, by (5.8), we evidently have

$$(V_{s,c,\xi} f)(x) = (T_{-(c+s)\xi} u_0)(x), \quad x \in \mathbb{R}^d.$$

Next, as it was mentioned above, the functions f_n and $f = f_{t,c,\xi}$ in (5.11) are monotone, hence the limit in (5.11) is locally uniform. Therefore, passing n to ∞ in (5.10), we will get from (5.9) and Proposition 5.4, that

$$\begin{aligned} f(s) &= \max\bigl\{\varphi(s), (Q_t(V_{s,c,\xi} f))(0)\bigr\} = \max\bigl\{\varphi(s), (T_{-(c+s)\xi} Q_t u_0)(0)\bigr\} \\ &= \max\bigl\{\varphi(s), u\bigl((c+s)\xi, t\bigr)\bigr\}. \end{aligned} \tag{5.31}$$

Since f is non-increasing on $\mathbb{R}$, u_0 is non-increasing along ξ, cf. Definition 5.6; then, by Proposition 5.7, u also has the same property. As a result, the function

$$v(s) := u\big((c+s)\xi, t\big), \quad s \in \mathbb{R} \tag{5.32}$$

is non-increasing on $\mathbb{R}$. Next, by our assumptions, $f(-\infty) = \theta > \varphi(-\infty)$ and $f(\infty) = 0$; therefore, we get from (5.31), that

$$\lim_{s\to\infty} v(s) = 0, \qquad \lim_{s\to-\infty} v(s) = \theta. \tag{5.33}$$

Next, (5.31) implies that, for each $s \in \mathbb{R}$, cf. (1.4),

$$\begin{aligned} u_0(s\xi) &\geq u\big((c+s)\xi, t\big) \\ &= u_0\big((c+s)\xi\big) + \int_0^t \varkappa\Big((a*u)\big((c+s)\xi,\tau\big) - u\big((c+s)\xi,\tau\big)\Big)d\tau \\ &\quad + \int_0^t u\big((c+s)\xi,\tau\big)\Big(\beta - (Gu)\big((c+s)\xi,\tau\big)\Big)d\tau. \end{aligned}$$

Therefore, for $r > c$,

$$\begin{aligned} 0 &\geq \varkappa \int_{-r}^{r}\int_0^t \Big((a*u)\big((c+s)\xi,\tau\big) - u\big((c+s)\xi,\tau\big)\Big)d\tau\, ds \\ &\quad + \int_{-r}^{r}\int_0^t u\big((c+s)\xi,\tau\big)\Big(\beta - (Gu)\big((c+s)\xi,\tau\big)\Big)d\tau\, ds \\ &\quad + \int_{-r}^{r}\Big(u_0\big((c+s)\xi\big) - u_0(s\xi)\Big)\, ds =: S_1(r) + S_2(r) + S_3(r). \end{aligned} \tag{5.34}$$

Note that u_0 is constant along any $\eta \in S^{d-1}$ orthogonal to ξ, cf. Definition 5.6; and, by Proposition 5.7, u has the same property. Namely, for each $s \in \mathbb{R}$ and $\eta \in S^{d-1}$ orthogonal to ξ,

$$u(x,t) = u(x + s\eta, t), \qquad t \geq 0,\ x \in \mathbb{R}^d. \tag{5.35}$$

For $d \geq 2$, choose any $\{\eta_1,\ \eta_2,\ \ldots,\ \eta_{d-1}\} \subset S^{d-1}$ which form a complement of $\xi \in S^{d-1}$ to an orthonormal basis in $\mathbb{R}^d$. Then

$$\begin{aligned} (a*u)(s\xi,t) &= \int_{\mathbb{R}^d} a(y)u(s\xi - y, t)dy \\ &= \int_{\mathbb{R}^d} a\Big(\sum_{j=1}^{d-1} y_j\eta_j + y_d\xi\Big)\, u\Big(-\sum_{j=1}^{d-1} y_j\eta_j + (s-y_d)\xi, t\Big)\, dy_1\ldots dy_d \\ &= \int_{\mathbb{R}}\bigg(\int_{\mathbb{R}^{d-1}} a\Big(\sum_{j=1}^{d-1} y_j\eta_j + y_d\xi\Big)\, dy_1\ldots dy_{d-1}\bigg)u\big((s-y_d)\xi,t\big)\, dy_d, \end{aligned} \tag{5.36}$$

where we used (5.35) with $\eta = -\sum\limits_{j=1}^{d-1} y_j\eta_j$, which is orthogonal to the ξ. Therefore, one can set

$$\check{a}(s) := \begin{cases} \displaystyle\int_{\mathbb{R}^{d-1}} a\Big(\sum_{j=1}^{d-1} y_j\eta_j + s\xi\Big)\, dy_1\ldots dy_{d-1}, & d \geq 2, \\ a^{\pm}(s\xi), & d = 1 \end{cases}$$

for $s\in\mathbb{R}$. We also denote $\check{u}(s,t):=u(s\xi,t)$, $s\in\mathbb{R}$. Then one can continue (5.36), as follows: $(a*u)(s\xi,t)=(\check{a}*\check{u})(s,t)$, where the convolution in the right-hand side is in $s\in\mathbb{R}$. Since $\int_{\mathbb{R}}\check{a}(s)ds=\int_{\mathbb{R}^d}a(y)dy=1$ and (A9) yields

$$\int_{\mathbb{R}}|s|\check{a}(s)ds=\int_{\mathbb{R}^d}|y\cdot\xi|a(y)dy<\infty,$$

we may apply Lemma 5.12 with $b=\check{a}$ and v given by (5.32). Then, by (5.25), (5.33) and the dominated convergence theorem, we have

$$\begin{aligned}S_1(r)&=\varkappa\int_0^t\int_{-r}^r\bigl((\check{a}*\check{u})(s+c,\tau)-\check{u}(s+c,\tau)\bigr)dsd\tau\\&\to\varkappa t\theta\int_{\mathbb{R}}s\check{a}(s)ds=\varkappa\theta t\int_{\mathbb{R}^d}y\cdot\xi a(y)dy=\varkappa\theta t\,\xi\cdot\mathfrak{m},\end{aligned}\tag{5.37}$$

as $r\to\infty$. Next, by (5.33), (5.30), we have, cf. (5.26),

$$S_3(r)=\int_r^{r+c}u_0(s\xi)ds-\int_{-r}^{-r+c}u_0(s\xi)ds\to-\theta c=-\theta\varkappa t\,\xi\cdot\mathfrak{m},\tag{5.38}$$

as $r\to\infty$. Therefore, combining (5.34), (5.37), (5.38) with the inequality $u(\beta-Gu)\geq 0$, we deduce that

$$\int_{-\infty}^{\infty}\int_0^t u\bigl((c+s)\xi,\tau\bigr)\Bigl(\beta-(Gu)\bigl((c+s)\xi,\tau\bigr)\Bigr)d\tau\,ds=\lim_{r\to\infty}S_2(r)=0.\tag{5.39}$$

Let $w_0\in C_b(\mathbb{R}^d)$ be such that $0\leq w_0\leq u_0$ and $w_0\not\equiv 0$. The by Theorem 4.2 and Proposition 5.2, we have

$$u(x,\tau)\geq w(x,\tau)>0,\quad x\in\mathbb{R}^d,\ \tau>0.$$

Hence (5.39) is possible if and only if $(Gu)\bigl(s\xi,\tau\bigr)=\beta$ for (a.a.) $s\in\mathbb{R}$ and all $\tau\in[0,t]$; note that $u(\cdot,\tau)$ is continuous in $\tau\geq 0$ and G is continuous on E_θ^+ because of (A3). In particular, $(Gu_0)\bigl(s\xi\bigr)=\beta$, $s\in\mathbb{R}$. Then we have by (A7) that, for any $p>0$,

$$(GT_{-p\xi}u_0)(s\xi)=(T_{-p\xi}Gu_0)(s\xi)=(Gu_0)((s+p)\xi)=\beta,\quad s\in\mathbb{R}.\tag{5.40}$$

Since, $(T_{-p\xi}u_0)(x)=f(x\cdot\xi+p)$, $x\in\mathbb{R}^d$, and $f(\infty)=0$, we have that $T_{-p\xi}u_0\overset{\text{loc}}{\Longrightarrow}0$, as $p\to\infty$. Then, by (A6), (A2) we get that $GT_{-p\xi}u_0\overset{\text{loc}}{\Longrightarrow}G0=0$, as $p\to\infty$, that contradicts (5.40). The proof is fulfilled. $\square$

Therefore, under assumptions (A1)–(A9), one has that (5.14) holds for all $T>0$ and, moreover, (5.17) holds for $\mathfrak{n}=\mathfrak{m}$ given by (2.4). Now, we are going to get rid of the condition (5.18).

We find first a useful sub-solution to the linearization of (1.1) around the zero solution, namely

$$\frac{\partial v}{\partial t}(x,t)=\varkappa(a*v)(x,t)-mv(x,t).\tag{5.41}$$

Proposition 5.14. *Let* (A1), (A5), (A9) *hold and* $\mathfrak{m}$ *be given by* (2.4). *Then there exists* $\alpha_0 > 0$, *such that, for all* $\alpha \in (0, \alpha_0)$, *there exists* $T = T(\alpha) > 0$, *such that, for all* $q > 0$, *the function*

$$w(x,t) = q \exp\biggl(-\frac{|x - t\mathfrak{m}|^2}{\alpha t}\biggr), \quad x \in \mathbb{R}^d, \ t > T, \tag{5.42}$$

is a sub-solution to (5.41) *on* $t > T$; *i.e., cf.* (4.1),

$$(\widetilde{\mathcal{F}}w)(x,t) := \frac{\partial w(x,t)}{\partial t} - \varkappa(a * w)(x,t) + mw(x,t) \le 0 \tag{5.43}$$

for all $x \in \mathbb{R}^d$, $t > T$.

The proof is very similar to that in [30, Proposition 5.19]. For reader convenience, we provide the proof in the Appendix.

Now, we will show that (5.42) is a sub-solution to (1.1) provided that q is small enough.

Proposition 5.15. *Let* (A1)–(A9) *hold and* $\mathfrak{m}$ *be given by* (2.4). *Then there exists* $q_0 \in (0, \theta)$ *and* $\alpha_0 > 0$, *such that, for all* $\alpha \in (0, \alpha_0)$, *there exists* $T = T(\alpha) > 0$, *such that, for all* $q \in (0, q_0)$, *the function* (5.42) *is a sub-solution to* (1.1) *on* $t > T$; *i.e., cf.* (4.1) *and* (5.43),

$$(\mathcal{F}_\theta w)(x,t) := \frac{\partial w(x,t)}{\partial t} - \varkappa(a * w)(x,t) + mw(x,t) + w(x,t)(Gw)(x,t) \le 0$$

for all $x \in \mathbb{R}^d$, $t > T$.

Proof. By (A2), (A3), for each $0 < q_0 < \min\bigl\{\theta, \frac{\beta}{2l_\theta}\bigr\}$ (where, recall, $\beta = \varkappa - m$), we have that $v \in E^+_{q_0}$ yields $0 \le Gv \le \frac{\beta}{2}$. Then, for each $q \in (0, q_0)$,

$$\mathcal{F}_\theta w \le \frac{\partial w}{\partial t} - \varkappa a * w + \Bigl(m + \frac{\beta}{2}\Bigr) w.$$

Since (A1) yields $m + \frac{\beta}{2} < \varkappa$, the statement follows from Proposition 5.14 applied for (5.43) with m replaced by $m + \frac{\beta}{2}$. ◻

The next statement shows that a solution to (1.1) becomes larger than the sub-solution (5.42) after a big enough time.

Proposition 5.16. *Let* (A1)–(A10) *hold. Then, there exists* $t_1 > 0$, *such that, for any* $t > t_1$ *and for any* $\tau > 0$, *there exists* $q_1 = q_1(t, \tau) > 0$, *such that the following holds. If* $u_0 \in E^+_\theta$ *is such that there exist* $\eta > 0$, $r > 0$, $x_0 \in \mathbb{R}^d$ *with* $u_0(x) \ge \eta$, $x \in B_r(x_0)$ *and* u *is the corresponding solution to* (1.1), *then*

$$u(x,t) \ge q_1 \exp\Bigl(-\frac{|x - x_0|^2}{\tau}\Bigr), \quad x \in \mathbb{R}^d.$$

The proof is, as a matter of fact, the same as that in [30, Proposition 5.20]. Again, for reader convenience, we provide the proof in the Appendix.

Now we are finally ready to proof Theorems 2.5, 2.7.

Proof of Theorem 2.5. As it was mentioned above, one can get the statement, combining Propositions 5.11 and 5.13, provided that (5.18) holds. To get rid of the latter assumption, one can literally follow the proof of [30, Theorem 5.10] using the results of Propositions 5.15 and 5.16. □

Proof of Theorem 2.7. Without loss of generality we can assume that $\theta - \theta_n \le \frac{\theta}{2}$, $n \in \mathbb{N}$. Consider $v_0 \in E^+_{\theta/2} \cap C^\infty(\mathbb{R}^d)$, such that for some $x_0 \in \mathbb{R}^d$, $\delta \in (0, \frac{\theta}{2})$,

$$\delta 1\!\!1_{B_\delta(x_0)}(x) \le v_0(x) \le u_0(x), \qquad x \in \mathbb{R}^d.$$

Let $u_n(x,0) = v_0(x)$ and u_n solves the following equation

$$\mathcal{F}^{(n)}_{\theta_n} u_n := \frac{\partial u_n}{\partial t} - \varkappa_n a_n * u_n + u_n G_n u_n + m u_n = 0. \tag{5.44}$$

Therefore by (A11) we obtain,

$$\mathcal{F}^{(n)}_{\theta_n} u_n = 0 = \mathcal{F}_\theta u \le \mathcal{F}^{(n)}_{\theta_n} u.$$

Hence by Theorem 4.2 applied to $\mathcal{F}^{(n)}_{\theta_n}$, we obtain

$$0 \le u_n(x,t) \le u(x,t) \le \theta.$$

Applying Theorem 2.5 to the equation (5.44), we have

$$\begin{aligned} \theta - \frac{1}{n} \le \theta_n = \lim_{t\to\infty} \operatorname*{essinf}_{x\in K} u_n(x + t\mathfrak{m}_n, t) &\le \liminf_{t\to\infty} \operatorname*{essinf}_{x\in K} u(x + t\mathfrak{m}_n, t) \\ &\le \limsup_{t\to\infty} \operatorname*{essinf}_{x\in K} u(x + t\mathfrak{m}_n, t) \le \theta, \end{aligned}$$

that fulfills the proof. □

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Appendix

Proof of Proposition 5.14. For $q > 0$, consider the function (5.42). By (5.43), one gets

$$(\widetilde{\mathcal{F}} w)(x,t) = w(x,t)\left(\frac{|x|^2}{\alpha t^2} - \frac{|\mathfrak{m}|^2}{\alpha}\right) - \varkappa (a * w)(x,t) + m w(x,t).$$

Therefore, to have $\widetilde{\mathcal{F}} w \le 0$, it is enough to claim that, for all $x \in \mathbb{R}^d$,

$$m + \frac{|x|^2}{\alpha t^2} - \frac{|\mathfrak{m}|^2}{\alpha} \le \varkappa \exp\left(\frac{|x - t\mathfrak{m}|^2}{\alpha t}\right) \int_{\mathbb{R}^d} a(y) \exp\left(-\frac{|x - y - t\mathfrak{m}|^2}{\alpha t}\right) dy.$$

By changing x onto $x+t\mathfrak{m}$ and a simplification, one gets an equivalent inequality

$$m + \frac{|x|^2}{\alpha t^2} + \frac{2\, x\cdot \mathfrak{m}}{\alpha t} \leq \varkappa \int_{\mathbb{R}^d} a(y) \exp\Bigl(\frac{2x\cdot y}{\alpha t}\Bigr) \exp\Bigl(-\frac{|y|^2}{\alpha t}\Bigr)\, dy =: I(t). \quad (5.45)$$

One can rewrite $I(t) = I_0(t) + I^+(t) + I^-(t)$, where

$$I_0(t) := \varkappa \int_{\mathbb{R}^d} a(y) e^{-\frac{|y|^2}{\alpha t}}\, dy; \qquad I^+(t) := \varkappa \int_{x\cdot y\geq 0} a(y) e^{-\frac{|y|^2}{\alpha t}} \Bigl(e^{\frac{2x\cdot y}{\alpha t}} - 1\Bigr) dy;$$

$$I^-(t) := \varkappa \int_{x\cdot y< 0} a(y) e^{-\frac{|y|^2}{\alpha t}} \Bigl(e^{\frac{2x\cdot y}{\alpha t}} - 1\Bigr) dy.$$

Using that $e^s - 1 \geq s$, for all $s\in\mathbb{R}$, and $e^s - 1 \geq s + \frac{s^2}{2}$, for all $s\geq 0$, one gets the following estimates

$$\begin{aligned} I^+(t) &\geq \frac{2\varkappa}{\alpha t} \int_{x\cdot y\geq 0} a(y) e^{-\frac{|y|^2}{\alpha t}} (x\cdot y) dy + \frac{2\varkappa}{\alpha^2 t^2} \int_{x\cdot y\geq 0} a(y) e^{-\frac{|y|^2}{\alpha t}} (x\cdot y)^2 dy,\\ I^-(t) &\geq \frac{2\varkappa}{\alpha t} \int_{x\cdot y< 0} a(y) e^{-\frac{|y|^2}{\alpha t}} (x\cdot y) dy. \end{aligned}$$

Therefore,

$$I(t) \geq I_0(t) + \frac{2}{\alpha t}\, x\cdot I_1(t) + \frac{2}{\alpha^2 t^2} I_2(t), \quad (5.46)$$

where

$$\begin{aligned} I_1(t) &:= \varkappa \int_{\mathbb{R}^d} a(y) e^{-\frac{|y|^2}{\alpha t}}\, y\, dy \in \mathbb{R}^d,\\ I_2(t) &:= \varkappa \int_{x\cdot y\geq 0} a(y) e^{-\frac{|y|^2}{\alpha t}} (x\cdot y)^2 dy \in \mathbb{R}. \end{aligned}$$

By (A9), (2.4), and the dominated convergence theorem, we will get that $I_0(t) \nearrow \varkappa > m$ and $I_1(t) \to \mathfrak{m}\in\mathbb{R}^d$ as $t\to\infty$. Therefore, for any $\varepsilon>0$ with $m+2\varepsilon<\varkappa$, there exists $T_1 = T_1(\varepsilon)>0$, such that, for all $\alpha>0$ and $t>0$ with $\alpha t > T_1$, one has

$$\varkappa \geq I_0(t) > m+\varepsilon, \qquad |I_1(t) - \mathfrak{m}| < \varepsilon. \quad (5.47)$$

Let $T > \frac{T_1}{\alpha}$ be chosen later. The function $I_2(t)$ is also increasing in $t>0$. Therefore, by (5.46) and (5.47), one gets, for $t > T > \frac{T_1}{\alpha}$,

$$\begin{aligned} I(t) &> m + \varepsilon + \frac{2}{\alpha t} x\cdot (I_1(t) - \mathfrak{m}) + \frac{2}{\alpha t} x\cdot \mathfrak{m} + \frac{2}{\alpha^2 t^2} I_2(t)\\ &\geq m + \varepsilon - \frac{2\varepsilon}{\alpha t}|x| + \frac{2}{\alpha t} x\cdot\mathfrak{m} + \frac{2}{\alpha^2 t^2} I_2(T). \end{aligned} \quad (5.48)$$

Let $\varrho>0$ be as in (A5). For an arbitrary $x\in\mathbb{R}^d$, consider the set

$$B_x = \Bigl\{ y\in\mathbb{R}^d \Bigm| |y|\leq \varrho, \frac{1}{2} \leq \frac{x\cdot y}{|x||y|} \leq 1 \Bigr\}.$$

Then

$$I_2(T) \geq \frac{\varkappa\varrho}{4} |x|^2 \int_{B_x} |y|^2 e^{-\frac{|y|^2}{\alpha T}}\, dy. \quad (5.49)$$

The set B_x is a cone inside the ball $B_\varrho(0)$, with the apex at the origin, the height which lies along x, and the apex angle $2\pi/3$. Since the function inside the integral in the right-hand side of (5.49) is radially symmetric, the integral does not depend on x. Fix an arbitrary $\bar{x} \in \mathbb{R}^d$ and denote

$$A(\tau) = A(\tau, \varrho) = \int_{B_{\bar{x}}} |y|^2 e^{-\frac{|y|^2}{\tau}} dy \nearrow \int_{B_{\bar{x}}} |y|^2 dy =: \bar{B}_\varrho, \quad \tau \to \infty. \tag{5.50}$$

Then, by (5.48) and (5.49), one has, for $t > T$,

$$I(t) > m + \varepsilon - \frac{2\varepsilon}{\alpha t}|x| + \frac{2}{\alpha t} x \cdot \mathfrak{m} + \frac{\varkappa \varrho A(\alpha T)}{2\alpha^2 t^2}|x|^2. \tag{5.51}$$

By (5.51), to prove (5.45), it is enough to show that

$$\varepsilon - \frac{2\varepsilon}{\alpha t}|x| + \frac{\varkappa \varrho A(\alpha T)}{2\alpha^2 t^2}|x|^2 \geq \frac{|x|^2}{\alpha t^2}, \qquad t > T, \ x \in \mathbb{R}^d,$$

or, equivalently, for $2\alpha < \varkappa \varrho A(\alpha T)$,

$$\begin{aligned} &\left(\sqrt{\frac{\varkappa \varrho A(\alpha T) - 2\alpha}{2}} \frac{|x|}{\alpha t} - \varepsilon \sqrt{\frac{2}{\varkappa \varrho A(\alpha T) - 2\alpha}} \right)^2 \\ &\qquad\qquad\qquad\qquad + \varepsilon - \varepsilon^2 \frac{2}{\varkappa \varrho A(\alpha T) - 2\alpha} \geq 0. \end{aligned} \tag{5.52}$$

To get (5.52), we proceed as follows. For a given $\varrho > 0$ which provides (A5), we set $\alpha_0 := \frac{1}{2}\varkappa \varrho \bar{B}_\varrho$, cf. (5.50). Then, for any $\alpha \in (0, \alpha_0)$, there exists $T_2 = T_2(\alpha) > 0$, such that

$$2\alpha < \varkappa \varrho A(\alpha T_2) < \varkappa \varrho \bar{B}_\delta.$$

Choose now $\varepsilon = \varepsilon(\alpha) > 0$, such that $m + 2\varepsilon < \varkappa$ and

$$\varepsilon < \frac{1}{2}(\varkappa \varrho A(\alpha T_2) - 2\alpha) < \frac{1}{2}(\varkappa \varrho A(\alpha T) - 2\alpha), \quad T > T_2. \tag{5.53}$$

Then, find $T_1 = T_1(\alpha) > 0$ which gives (5.47) for $\alpha t > T_1$; and, finally, take $T = T(\alpha) > T_2$ such that $\alpha T > T_1$. As a result, for $t > T$, one has $\alpha t > \alpha T > T_1$, thus (5.47) holds, whereas (5.53) yields (5.52). The latter inequality gives (5.45), and hence, for all $q > 0$, $\mathcal{F} w \leq 0$, for w given by (5.42). The statement is proved. □

Proof of Proposition 5.16. By (Q2) in Proposition 5.4, it is enough to prove the statement for $x_0 = 0$. Consider arbitrary functions $j, v_0 \in C^\infty(\mathbb{R}^d)$, such that

$$\begin{aligned} \operatorname{supp} j = B_\delta(0), \qquad & 0 < j(x) = j(|x|) \leq \delta, \quad && x \in \operatorname{int}(B_\delta(0)); \\ \operatorname{supp} v_0 = B_r(0), \qquad & 0 < v_0(x) \leq \eta, && x \in \operatorname{int}(B_r(0)); \\ \exists\, 0 < p < \min\{r, 1\},\ 0 < \nu < \eta, \qquad & \text{such that } v_0(x) \geq \nu, && x \in B_p(0), \end{aligned}$$

where δ is the same as in (A10). We choose p and b as in (A10). Then one can rewrite (1.1) as follows

$$\frac{\partial}{\partial t} u(x,t) = \varkappa (j * u)(x,t) - (m+q)u(x,t) + f(x,t),$$

where, for all $x \in \mathbb{R}^d$ and $t \geq 0$,

$$f(x,t) := \varkappa((a-j)*u)(x,t) - u(x,t)(Gu)(x,t) + qu(x,t) \geq 0,$$

because of (A10). Since $j \geq 0$ and $Ju = j * u$ defines a bounded operator on $L^\infty(\mathbb{R}^d)$, one has that $e^{tJ} f(x,s) \geq 0$, for all $t, s \geq 0$, $x \in \mathbb{R}^d$. By the same argument, $u_0(x) \geq \eta \mathbb{1}_{B_r(0)}(x) \geq v_0(x) \geq 0$ implies $(e^{tJ}u_0)(x) \geq (e^{tJ}v_0)(x)$. Therefore,

$$\begin{aligned} u(x,t) &= e^{-t(m+q)}(e^{tJ}u_0)(x) + \int_0^t e^{-(t-s)(m+q)}(e^{(t-s)J}f)(x,s)ds \\ &\geq e^{-t(m+q)}(e^{tJ}u_0)(x) \geq e^{-(m+q-\langle j\rangle)t}(e^{tL_j}v_0)(x), \quad x \in \mathbb{R}^d. \end{aligned} \tag{5.54}$$

where $\langle j \rangle := \int_{\mathbb{R}^d} j(x)\,dx > 0$ and $L_j u = Ju - \langle j \rangle u$.

We are going to apply now the results of [9]. To do this, set $\alpha := \langle j \rangle^{-1}$. Then,

$$(e^{tL_j}v_0)(x) = (e^{\langle j\rangle t(\alpha L_j)}v_0)(x) = v(x, \langle j \rangle t), \tag{5.55}$$

where v solves the differential equation $\frac{d}{dt}v = \alpha L_j$. Since $\int_{\mathbb{R}^d} \alpha j(x)\,dx = 1$, then, by [4, Theorem 1.4, Lemma 1.6],

$$v(x,t) = e^{-t}v_0(x) + (w * v_0)(x,t), \tag{5.56}$$

where $w(x,t)$ is a smooth function. Moreover, by [9, Proposition 5.1], for any $\omega \in (0, \delta)$ there exist $c_1 = c_1(\omega) > 0$ and $c_2 = c_2(\omega) \in \mathbb{R}$, such that

$$\begin{aligned} &w(x,t) \geq h(x,t), \quad x \in \mathbb{R}^d, t \geq 0, \\ &h(x,t) := c_1 t \exp\Bigl(-t - \frac{1}{\omega}|x|\log|x| + (\log t - c_2)\Bigl[\frac{|x|}{\omega}\Bigr]\Bigr). \end{aligned} \tag{5.57}$$

Here $[\alpha]$ means the entire part of an $\alpha \in \mathbb{R}$, and $0 \log 0 := 1$, $\log 0 := -\infty$.

Set $t_1 = e^{c_2} > 0$. Since $[\alpha] > \alpha - 1$, $\alpha \in \mathbb{R}$, one has, for $t > t_1$,

$$h(x,t) \geq c_1 e^{c_2} \exp\Bigl(-t - \frac{1}{\omega}|x|\log|x| + (\log t - c_2)\frac{|x|}{\omega}\Bigr) \geq c_3 g(x,t),$$

where $c_3 = c_1 e^{c_2} > 0$ and

$$g(x,t) := \exp\Bigl(-t - \frac{1}{\omega}|x|\log|x|\Bigr), \quad x \in \mathbb{R}^d, t > t_1.$$

Since $v_0 \geq \nu \mathbb{1}_{B_p(0)}$, one gets from (5.56) and (5.57), that

$$v(x,t) \geq \nu e^{-t}\mathbb{1}_{B_p(0)}(x) + \nu c_3 \int_{B_p(x)} g(y,t)\,dy \tag{5.58}$$

Set $V_p := \int_{B_p(0)} dx$. For any fixed $t > t_1$, since $g(\cdot, t) \in C(B_p(x))$, there exists $y_0, y_1 \in B_p(x)$, such that $g(y,t)$ attains its minimal and maximal values on $B_p(x)$ at these points, respectively. Since $B_p(x)$ is a convex set, one gets that, for any $\gamma \in (0,1)$, $y_\gamma := \gamma y_1 + (1-\gamma)y_0 \in B_p(x)$. Then

$$V_p g(y_0,t) \leq \int_{B_p(x)} g(y_\gamma, t)\,dy \leq V_p g(y_1,t).$$

Therefore, by the intermediate value theorem there exists, $\tilde{y}_t = \tilde{y}(x,t) \in B_p(x)$, $t > t_1$, $x \in \mathbb{R}^d$, such that $\int_{B_p(x)} g(y,t)\,dy = V_p g(\tilde{y}_t, t)$. Hence one gets from (5.54), (5.55), (5.58), that

$$\begin{aligned} u(x,t) &\geq c_4 e^{-(m+q-\langle j\rangle)t} g\big(\tilde{y}_t, \langle j\rangle t\big) \\ &= c_4 \exp\Big(-(m+q)t - \frac{1}{\omega}|\tilde{y}_t| \log|\tilde{y}_t|\Big), \end{aligned} \tag{5.59}$$

for $\tilde{y}_t = \tilde{y}(x,t) \in B_p(x)$, $t > t_1$; here $c_4 = c_3 \nu V_p > 0$.

As a result, to get the statement, it is enough to show that, for any $t > t_1$ and for any $\tau > 0$, there exists $q_1 = q_1(t,\tau) > 0$, such that the r.h.s. of (5.59) is estimated from below by $q_1 e^{-\frac{|x|^2}{\tau}}$, i.e. that

$$(m+q)t + \frac{1}{\omega}|\tilde{y}_t| \log|\tilde{y}_t| - \log c_4 \leq \frac{|x|^2}{\tau} - \log q_1, \quad x \in \mathbb{R}^d, \tag{5.60}$$

Note that $\tilde{y}_t \in B_p(x)$ implies $|\tilde{y}_t| \leq p + |x|$, $x \in \mathbb{R}^d$.

Let $p + |x| \leq 1$. Then $\log|\tilde{y}_t| \leq 0$, and the l.h.s. of (5.60) is majorized by $(m+q)t - \log c_4$. Therefore, to get (5.60), it is enough to have $q_1 < c_4 e^{-(m+q)t}$, regardless of τ.

Let now $|x| + p > 1$. Recall that we chose $p < 1$. The function $s \log s$ is increasing on $s > 1$. Hence to get (5.60), we claim

$$(|x|+1)\log(|x|+1) \leq \frac{\omega}{\tau}|x|^2 - \omega(m+q)t + \omega \log c_4 - \omega \log q_1. \tag{5.61}$$

Consider now the function $f(s) = as^2 - (s+1)\log(s+1)$, $s \geq 0$, $a = \frac{\omega}{\tau} > 0$. Then $f(0) = 0$, $f'(s) = 2as - \log(s+1) - 1$, $f'(0) = -1$, $f''(s) = 2a - \frac{1}{s+1}$. Since $f''(s) \nearrow 2a > 0$, $s \to \infty$, there exists $s_0 > 0$, such that $f''(s) > 0$, for all $s > s_0$, i.e. $f'(s)$ increases on $s > s_0$. Since $f'(s) \to \infty$, $s \to \infty$, there exists $s_1 > s_0$, such that $f'(s) > 0$, for all $s > s_1$, i.e. f is increasing on $s > s_1$. Finally, for any $t > t_1$, one can choose $q_1 = q_1(t,\tau) > 0$ small enough, to get

$$\min_{s \in [0,s_1]} f(s) - \omega(m+q)t + \omega \log c_4 - \omega \log q_1 > 0$$

and to fulfill (5.61), for all $x \in \mathbb{R}^d$. The statement is proved. □

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