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KESTEN'S BOUND FOR SUB-EXPONENTIAL DENSITIES ON THE REAL LINE AND ITS MULTI-DIMENSIONAL ANALOGUES

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Abstract

We study the tail asymptotic of sub-exponential probability densities on the real line. Namely, we show that the n-fold convolution of a sub-exponential probability density on the real line is asymptotically equivalent to this density times n. We prove Kesten's bound, which gives a uniform in n estimate of the n-fold convolution by the tail of the density. We also introduce a class of regular sub-exponential functions and use it to find an analogue of Kesten's bound for functions on $\mathbb{R}^d$. The results are applied for the study of the fundamental solution to a nonlocal heat-equation.

Keywords: sub-exponential densities; long-tail functions; heavy-tailed distributions; convolution tails; tail-equivalence; asymptotic behavior

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1. Introduction

Let F be a probability distribution on $\mathbb{R}$. Denote by $\overline{F}(s) := F\big((s,\infty)\big)$, $s \in \mathbb{R}$ its tail function. For probability distributions F_1, F_2 on $\mathbb{R}$, their convolution $F_1 * F_2$ has the tail function

$$\overline{F_1 * F_2}(s) = \int_{\mathbb{R}} \overline{F}_1(s-\tau)F_2(d\tau) = \int_{\mathbb{R}} \overline{F}_2(s-\tau)F_1(d\tau),$$

where $\overline{F}_1, \overline{F}_2$ are the corresponding tail functions of F_1, F_2.

If a probability distribution F is concentrated on $\mathbb{R}_+ := [0,\infty)$ and $\overline{F}(s) > 0$, $s \in \mathbb{R}$, then, see e.g. [7],

$$\liminf_{s\to\infty} \frac{\overline{F * F}(s)}{\overline{F}(s)} \geq 2. \tag{1.1}$$

If, additionally, F is heavy-tailed, i.e. $\int_{\mathbb{R}} e^{\lambda s} F(ds) = \infty$ for all $\lambda > 0$, then the equality holds in (1.1), see [13]. An important sub-class of heavy-tailed distributions concentrated on $\mathbb{R}_+$ constitute *sub-exponential* ones, for which

$$\lim_{s\to\infty} \frac{\overline{F * F}(s)}{\overline{F}(s)} = 2. \tag{1.2}$$

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Any sub-exponential distribution on $\mathbb{R}_+$ is (right-side) long-tailed on $\mathbb{R}$, see e.g. [7], i.e. (cf. Definition 2.1 below)

$$\lim_{s\to\infty} \frac{\overline{F}(s+t)}{\overline{F}(s)} = 1 \quad \text{for each } t>0. \tag{1.3}$$

If distributions F_1, F_2 on $\mathbb{R}$ have probability densities $f_1 \geq 0, f_2 \geq 0$, with $\int_{\mathbb{R}} f_1(s)\,ds = \int_{\mathbb{R}} f_2(s)\,ds = 1$, then $F_1 * F_2$ has the density

$$(f_1 * f_2)(s) := \int_{\mathbb{R}} f_1(s-t) f_2(t)\,dt, \quad s\in\mathbb{R}.$$

The density f of a sub-exponential distribution F concentrated on $\mathbb{R}_+$ (i.e. $f(s)=0$ for $s<0$) is said to be sub-exponential on $\mathbb{R}_+$ if f is long-tailed, i.e. (1.3) holds with $\overline{F}$ replaced by f (see also Definition 2.1 below), and, cf. (1.2),

$$\lim_{s\to\infty} \frac{(f*f)(s)}{f(s)} = 2. \tag{1.4}$$

It can be shown (see e.g. [2, 14, 15]) that, in this case, for any $n\in\mathbb{N}$,

$$\lim_{s\to\infty} \frac{f^{*n}(s)}{f(s)} = n, \tag{1.5}$$

where $f^{*n} := f * \ldots * f$ ($n-1$ times). Note that, in general, the density of a sub-exponential distribution concentrated on $\mathbb{R}_+$ even being long-tailed does not need to be a sub-exponential one; the corresponding characterisation can be found in [2, 14], see (2.13) below. The property (1.5) implies, in particular, that, for each $\delta>0$, $n\in\mathbb{N}$, there exists $s_n>0$, such that $f^{*n}(s) \leq (n+\delta) f(s)$ for $s>s_n$. In many situations, it is important to have similar inequalities 'uniformly' in n, i.e. on a set independent of n. A possible solution is given by the so-called Kesten's bound, see [2, 15]: for a bounded sub-exponential density f on $\mathbb{R}_+$ and for any $\delta>0$, there exist $c_\delta, s_\delta>0$, such that

$$f^{*n}(s) \leq c_\delta (1+\delta)^n f(s), \quad s>s_\delta,\ n\in\mathbb{N}. \tag{1.6}$$

For the corresponding results for distributions, see [3, 7, 8, 14]. Kesten's bounds were used to study series of convolutions of distributions on $\mathbb{R}_+$, $\sum_{n=1}^\infty \lambda_n F^{*n}$, and of the corresponding densities, $\sum_{n=1}^\infty \lambda_n f^{*n}$, appeared in different contexts: starting from the renewal theory (that was the motivation for the original paper [7]) to branching age dependent processes, random walks, queue theory, risk theory and ruin probabilities, compound Poisson processes, and the study of infinitely divisible laws, see e.g. [2, 4, 10, 14, 15, 22] and the references therein.

If F is a probability distribution on the whole $\mathbb{R}$, such that F^+, given by $F^+(B) := F(B\cap\mathbb{R}_+)$ for all Borel $B\subset\mathbb{R}$, is sub-exponential on $\mathbb{R}_+$, then (see e.g. [14, Lemma 3.4]) F is long-tailed on $\mathbb{R}$ and (1.2) holds. The distributions on $\mathbb{R}$ and their densities were considered by several authors, see [18, 20–22] and others. The reference [22], in particular, gives a review of difficulties appeared in the case of the whole $\mathbb{R}$ and closes several gaps in the preceding results. However, even some basic properties of sub-exponential densities on the whole $\mathbb{R}$ remained open.

Namely, in [14, Lemma 4.13], it was shown that if an integrable on $\mathbb{R}$ function f

- is right-side long-tail and, being restricted on $\mathbb{R}_+$ and normalized in $L^1(\mathbb{R}_+)$, satisfies (1.4) (we will say then that f is *weekly sub-exponential* on $\mathbb{R}$, cf. Definition 2.3 below), and if
- the condition
$$f(s+\tau) \leq K f(s), \quad s > \rho, \ \tau > 0 \tag{1.7}$$
holds, for some $K, \rho > 0$ (in particular, if f decays to 0 at ∞, cf. Definition 2.5),

then (1.4) holds for the original f on $\mathbb{R}$ as well. We generalize this to an analogue of (1.5) with a general $n \in \mathbb{N}$. In particular, we prove in Theorem 2.1 below that

Theorem 1.1. *Let f be an integrable weakly sub-exponential on $\mathbb{R}$ function, such that* (1.7) *holds. Then f, being normalized in $L^1(\mathbb{R})$, satisfies* (1.5) *for all $n \in \mathbb{N}$.*

Moreover, in Theorem 2.2, we prove that then (1.6) holds as well. Namely, one has the following result.

Theorem 1.2. *Let f be a bounded weakly sub-exponential probability density on $\mathbb{R}$, such that* (1.7) *holds. Then, for each $\delta > 0$, there exist $c_\delta, s_\delta > 0$, such that* (1.6) *holds.*

Note that the all 'classical' examples of sub-exponential functions satisfy assumptions of Theorems 1.1 and 1.2, see Subsection 3.2.

The multi-dimensional version of the constructions above is much more non-trivial. Currently, there exist at least three different definitions of sub-exponential distributions on $\mathbb{R}^d$ for $d > 1$, see [9, 17, 19]. The variety is mainly related to different possibilities to describe the zones in $\mathbb{R}^d$ where an analogue of the equivalence (1.2) takes place. However, any results about sub-exponential densities in $\mathbb{R}^d$, $d > 1$, seem to be absent at all. Note also that if, e.g. a is radially symmetric, i.e. $a(x) = b(|x|)$, $x \in \mathbb{R}^d$ (here $|x|$ denotes the Euclidean norm on $\mathbb{R}^d$) and b, being normalized, is a sub-exponential density on $\mathbb{R}_+$, then $(a * a)(x) := \int_{\mathbb{R}^d} a(x-y)a(y)\,dy = p(|x|)$, $x \in \mathbb{R}^d$, for some $p : \mathbb{R}_+ \to \mathbb{R}_+$ (i.e. $a * a$ is also radially symmetric), however, asymptotic behaviors of b and p at ∞ are hardly to be compared. Leaving this problem as on open, we focus in this paper on an analogue of Kesten's bound (1.6) in the multi-dimensional case.

To do this, we introduce a special class $\widetilde{\mathcal{S}}_{\mathrm{reg},d}$ of regular sub-exponential functions on $\mathbb{R}_+$ (see Definitions 3.1 and 4.2). Functions from this class are either inverse polynomials (i.e. (4.5) holds), or decay at ∞ faster than any polynomial (i.e. (4.15) holds), but slower than any exponential function, with the fastest allowed asymptotic $\exp\bigl(-s(\log s)^{-q}\bigr)$ with $q > 1$, cf. Remark 3.5. Then, in Corollary 4.1, we show the following result.

Theorem 1.3. *Let $a = a(x)$ be a probability density on $\mathbb{R}^d$, such that $a(x) = b(|x|)$, $x \in \mathbb{R}^d$, for some $b \in \widetilde{\mathcal{S}}_{\mathrm{reg},d}$. Then, for each $\delta > 0$ and for each $\alpha < 1$ close enough to* 1,
$$a^{*n}(x) \leq c_\alpha (1+\delta)^n a(x)^\alpha, \quad |x| > s_\alpha, \ n \in \mathbb{N} \tag{1.8}$$
for some $c_\alpha = c_\alpha(\delta) > 0$ and $s_\alpha = s_\alpha(\delta) > 0$.

Clearly, $a(x) = o\bigl(a(x)^\alpha\bigr)$, $|x| \to \infty$, for any $\alpha \in (0,1)$, hence the inequality (1.8) is weaker than (1.6) for the case $d = 1$.

The results of Corollary 4.1 is based on more general Theorem 4.1, which says that if, for some $b \in \widetilde{\mathcal{S}}_{\mathrm{reg},d}$ and decreasing on $\mathbb{R}_+$ function p,

$$a(x) \le p(|x|), \quad x \in \mathbb{R}^d, \qquad \log p(s) \sim \log b(s), \quad s \to \infty,$$

then (1.8) holds with $a(x)$ replaced by $b(|x|)$ in the right hand side.

The paper is organized as follows. In Section 2, we consider properties of general sub-exponential functions on the real line and prove the results which imply Theorems 1.1 and 1.2. In Section 3, we define and study properties of regular sub-exponential functions on the real line and consider the corresponding examples. In Section 4, we prove Theorem 1.3 and its generalizations. Finally, in Appendix, we apply the obtained results to the study of a non-local heat equation.

2. Sub-exponential functions and Kesten's bound on the real line

Definition 2.1. A function $b : \mathbb{R} \to \mathbb{R}_+$ is said to be *(right-side) long-tailed* if there exists $\rho \ge 0$, such that $b(s) > 0$, $s \ge \rho$; and, for any $\tau \ge 0$,

$$\lim_{s\to\infty} \frac{b(s+\tau)}{b(s)} = 1. \tag{2.1}$$

Remark 2.1. By [14, formula (2.18)], the convergence in (2.1) is equivalent to the locally uniform in τ convergence, namely, (2.1) can be replaced by the assumption that, for all $h > 0$,

$$\lim_{s\to\infty} \sup_{|\tau|\le h} \left| \frac{b(s+\tau)}{b(s)} - 1 \right| = 0. \tag{2.2}$$

A long-tailed function has to have a 'heavier' tail than any exponential function; namely, the following statement holds.

Lemma 2.1. ([14, Lemma 2.17].) *Let $b : \mathbb{R} \to \mathbb{R}_+$ be a long-tailed function. Then, for any $k > 0$, $\lim\limits_{s\to\infty} e^{ks} b(s) = \infty$.*

The constant h in (2.2) may be arbitrary big. It is quite natural to ask what will be if h increases to ∞ consistently with s.

Lemma 2.2. (Cf. [14, Lemma 2.19, Proposition 2.20].) *Let $b : \mathbb{R} \to \mathbb{R}_+$ be a long-tailed function. Then there exists a function $h : (0,\infty) \to (0,\infty)$, with $h(s) < \frac{s}{2}$ and $\lim\limits_{s\to\infty} h(s) = \infty$, such that, cf. (2.2),*

$$\lim_{s\to\infty} \sup_{|\tau|\le h(s)} \left| \frac{b(s+\tau)}{b(s)} - 1 \right| = 0. \tag{2.3}$$

Following [14], we will say then that b is h-insensitive. Of course, for a given long-tailed function b, the function h that fulfills (2.3) is not unique, see also [14, Proposition 2.20].

The convergence in (2.1) will not be, in general, monotone in s. To get this monotonicity, we consider the following class of functions.

Definition 2.2. A function $b : \mathbb{R} \to \mathbb{R}_+$ is said to be *(right-side) tail-log-convex*, if there exists $\rho > 0$, such that $b(s) > 0$, $s \ge \rho$, and the function $\log b$ is convex on $[\rho, \infty)$.

Remark 2.2. It is well-known that any function which is convex on an open interval is continuous there. Therefore, a tail-log-convex function $b = \exp(\log b)$ is continuous on (ρ_b, ∞) as well.

Lemma 2.3. *Let $b : \mathbb{R} \to \mathbb{R}_+$ be tail-log-convex, with $\rho = \rho_b$. Then, for any $\tau > 0$, the function $\frac{b(s+\tau)}{b(s)}$ is non-decreasing in $s \in [\rho, \infty)$.*

Proof. Take any $s_1 > s_2 \geq \rho$. Set $B(s) := \log b(s) \leq 0$, $s \in [\rho, \infty)$. Then the desired inequality $\frac{b(s_1+\tau)}{b(s_1)} \geq \frac{b(s_2+\tau)}{b(s_2)}$ is equivalent to $B(s_1 + \tau) + B(s_2) \geq B(s_2 + \tau) + B(s_1)$. Since B is convex, we have, for $\lambda = \frac{\tau}{s_1 - s_2 + \tau} \in (0, 1)$,

$$B(s_1) = B\bigl(\lambda s_2 + (1-\lambda)(s_1+\tau)\bigr) \leq \lambda B(s_2) + (1-\lambda)B(s_1+\tau),$$
$$B(s_2+\tau) = B\bigl((1-\lambda)s_2 + \lambda(s_1+\tau)\bigr) \leq (1-\lambda)B(s_2) + \lambda B(s_1+\tau),$$

that implies the needed inequality. □

Because of the terminology mentioned in the introduction, we will use the following definition.

Definition 2.3. We will say that a function $b : \mathbb{R} \to \mathbb{R}_+$ is *weakly (right-side) sub-exponential on* $\mathbb{R}$ if b is long-tailed, $b \in L^1(\mathbb{R}_+)$, and the function

$$b_+(s) := 1\!\!1_{\mathbb{R}_+}(s)\biggl(\int_{\mathbb{R}_+} b(\tau)d\tau\biggr)^{-1} b(s), \quad s \in \mathbb{R}, \tag{2.4}$$

satisfies the following asymptotic relation (as $s \to \infty$)

$$(b_+ * b_+)(s) = \int_{\mathbb{R}} b_+(s-\tau)b_+(\tau)\,d\tau = \int_0^s b_+(s-\tau)b_+(\tau)\,d\tau \sim 2b_+(s). \tag{2.5}$$

The next statement shows that a long-tailed tail-log-convex function is weakly sub-exponential on $\mathbb{R}$ provided that it decays at ∞ fast enough.

Lemma 2.4. (cf. [14, Theorem 4.15].) *Let $b : \mathbb{R} \to \mathbb{R}_+$ be a long-tailed tail-log-convex function such that $b \in L^1(\mathbb{R}_+)$. Suppose that, for a function $h : (0, \infty) \to (0, \infty)$, with $h(s) < \frac{s}{2}$ and $\lim\limits_{s\to\infty} h(s) = \infty$, the asymptotic* (2.3) *holds, and*

$$\lim_{s\to\infty} s\, b\bigl(h(s)\bigr) = 0. \tag{2.6}$$

Then b is weakly sub-exponential on $\mathbb{R}$.

Remark 2.3. Let $b : \mathbb{R} \to \mathbb{R}_+$ be a weakly sub-exponential function on $\mathbb{R}$. Then, by (2.4), (2.5), we have

$$\int_0^s b(s-\tau)b(\tau)\,d\tau \sim 2\biggl(\int_{\mathbb{R}_+} b(\tau)d\tau\biggr) b(s), \quad s \to \infty. \tag{2.7}$$

Definition 2.4. We will say that a function $b : \mathbb{R} \to \mathbb{R}_+$ is *(right-side) sub-exponential on* $\mathbb{R}$ if b is long-tailed, $b \in L^1(\mathbb{R})$, and the following asymptotic relation holds, cf. (2.5), (2.7),

$$(b * b)(s) = \int_{\mathbb{R}} b(s-\tau)b(\tau)\,d\tau \sim 2\biggl(\int_{\mathbb{R}} b(\tau)d\tau\biggr) b(s), \quad s \to \infty. \tag{2.8}$$

Remark 2.4. By [14, Lemma 4.12], a sub-exponential function on $\mathbb{R}$ is weakly sub-exponential there. The following lemma presents a sufficient condition to get the converse.

Lemma 2.5. (cf. [14, Lemma 4.13].) *Let $b \in L^1(\mathbb{R} \to \mathbb{R}_+)$ be a weakly sub-exponential function on $\mathbb{R}$. Suppose that there exists $\rho = \rho_b > 0$ and $K = K_b > 0$ such that*

$$b(s+\tau) \le K b(s), \quad s > \rho,\ \tau > 0. \tag{2.9}$$

Then (2.8) *holds, i.e. b is sub-exponential on $\mathbb{R}$.*

Remark 2.5. For $b \in L^1(\mathbb{R} \to \mathbb{R}_+)$, condition (2.9) yields that $\sup_{t\ge s} b(t) \to 0$, $s \to \infty$. In particular $b(s) \to 0$, as $s \to \infty$.

An evident sufficient condition which ensures (2.9) is that b is decreasing on $[\rho, \infty)$. Consider the corresponding definition.

Definition 2.5. A function $b : \mathbb{R} \to \mathbb{R}_+$ is said to be *(right-side) tail-decreasing* if there exists a number $\rho = \rho_b \ge 0$ such that $b = b(s)$ is strictly decreasing on $[\rho, \infty)$ to 0. In particular, $b(s) > 0$, $s \ge \rho$.

The proof of the following useful statement is straightforward.

Proposition 2.1. *Let $b : \mathbb{R} \to \mathbb{R}_+$ be a tail-decreasing function. Let $h : (0, \infty) \to (0, \infty)$, with $h(s) < \frac{s}{2}$ and $\lim\limits_{s\to\infty} h(s) = \infty$. Then* (2.3) *holds, if and only if*

$$\lim_{s\to\infty} \frac{b(s \pm h(s))}{b(s)} = 1. \tag{2.10}$$

The next statement and its proof follow ideas of [14, Lemma 4.13, Lemma 4.9].

Proposition 2.2. *Let $b : \mathbb{R} \to \mathbb{R}_+$ be a weakly sub-exponential function on $\mathbb{R}$, such that* (2.9) *holds. Let $b_1, b_2 \in L^1(\mathbb{R} \to \mathbb{R}_+)$ and there exist $c_1, c_2 \ge 0$, such that*

$$\lim_{s\to\infty} \frac{b_j(s)}{b(s)} = c_j, \quad j = 1, 2. \tag{2.11}$$

Then

$$\lim_{s\to\infty} \frac{(b_1 * b_2)(s)}{b(s)} = c_1 \int_{\mathbb{R}} b_2(\tau)\, d\tau + c_2 \int_{\mathbb{R}} b_1(\tau)\, d\tau. \tag{2.12}$$

Proof. Since b_+, given by (2.4), is long-tailed, and (2.5) holds, we have, by [14, Theorem 4.7], that there exists an increasing function $h : (0, \infty) \to (0, \infty)$, such that $h(s) < \frac{s}{2}$, $\lim\limits_{s\to\infty} h(s) = \infty$, and

$$\int_{h(s)}^{s-h(s)} b_+(s-\tau) b_+(\tau)\, d\tau = o(b_+(s)), \quad s \to \infty, \tag{2.13}$$

and, evidently, one can replace b_+ by b in (2.13).

Next, for any $b_1, b_2 \in L^1(\mathbb{R} \to \mathbb{R}_+)$, one can easily get that

$$(b_1 * b_2)(s) = \int_{-\infty}^{h(s)} \bigl(b_1(s-\tau) b_2(\tau) + b_1(\tau) b_2(s-\tau)\bigr)\, d\tau + \int_{h(s)}^{s-h(s)} b_1(s-\tau) b_2(\tau)\, d\tau.$$

Take an arbitrary $\delta \in (0,1)$. By (2.11), (2.3), and (2.13) (the latter, with b_+ replaced by b), there exist $K, \rho > 0$, such that (2.9) holds and, for all $s \geq h(\rho)$,

$$\bigl|b_j(s) - c_j b(s)\bigr| + \sup_{|\tau|\leq h(s)} \bigl|b(s+\tau) - b(s)\bigr| + \int_{h(s)}^{s-h(s)} b(s-\tau)b(\tau)\,d\tau \leq \delta b(s), \quad (2.14)$$

$$\int_{-\infty}^{-h(s)} b_j(\tau)d\tau + \int_{h(s)}^{\infty} b_j(\tau)d\tau \leq \delta, \quad j = 1,2. \quad (2.15)$$

Then, by (2.9), (2.14), (2.15), for $s \geq \rho > h(\rho)$,

$$\int_{-\infty}^{-h(s)} \bigl(b_1(s-\tau)b_2(\tau) + b_1(\tau)b_2(s-\tau)\bigr)\,d\tau \leq \delta K(c_1 + c_2 + 2\delta)b(s).$$

Set $B_j := \int_{\mathbb{R}} b_j(s)\,ds$, $j = 1,2$. By (2.14), for $s \geq \rho$,

$$\biggl|\int_{-h(s)}^{h(s)} b_1(s-\tau)b_2(\tau)\,d\tau - c_1 b(s) \int_{-h(s)}^{h(s)} b_2(\tau)\,d\tau\biggr| \leq \delta B_2(1 + c_1)b(s),$$

$$\int_{h(s)}^{s-h(s)} b_1(s-\tau)b_2(\tau)\,d\tau \leq \delta(c_1 + \delta)(c_2 + \delta)b(s).$$

From the obtained estimates, it is straightforward to get that, for some $M > 0$, $\bigl|(b_1 * b_2)(s) - (c_1 B_2 + c_2 B_1)b(s)\bigr| \leq \delta M b(s)$ for $s \geq \rho$. The latter implies (2.12). $\square$

Corollary 2.1. *The property of an integrable on $\mathbb{R}$ function to be weekly sub-exponential on $\mathbb{R}$ depends on its tail property only. Namely, for a weakly sub-exponential on $\mathbb{R}$ function $b \in L^1(\mathbb{R} \to \mathbb{R}_+)$ and for any $c \in L^1(\mathbb{R} \to \mathbb{R}_+)$ and $s_0 \in \mathbb{R}$, the function $\tilde{b}(s) = 1\!\!1_{(-\infty,s_0)}(s)c(s) + 1\!\!1_{[s_0,\infty)}(s)b(s)$ is weakly sub-exponential on $\mathbb{R}$, cf. also Theorem 3.1 below.*

Now one gets a generalization of Lemma 2.5.

Theorem 2.1. *Let $b \in L^1(\mathbb{R} \to \mathbb{R}_+)$ be a weakly sub-exponential on $\mathbb{R}$ function, such that (2.9) holds (for example, let b be tail-decreasing). Then*

$$\lim_{s\to\infty} \frac{b^{*n}(s)}{b(s)} = n\Bigl(\int_{\mathbb{R}} b(\tau)\,d\tau\Bigr)^{n-1}, \quad n \geq 2, \quad (2.16)$$

where $b^{*n}(s) = (\underbrace{b * \ldots * b}_{n})(s)$, $s \in \mathbb{R}$.

Proof. Take in Proposition 2.2, $b_1 = b_2 = b$, i.e. $c_1 = c_2 = 1$. Then, for $B := \int_{\mathbb{R}} b(\tau)\,d\tau$, one gets $b^{*2}(s) \sim 2Bb(s)$, $s \to \infty$. Proving by induction, assume that $b^{*(n-1)}(s) \sim (n-1)B^{n-2}b(s)$, $s \to \infty$, $n \geq 3$. Take in Proposition 2.2, $b_1 = b$, $b_2 = b^{*(n-1)}$, $c_1 = 1$, $c_2 = (n-1)B^{n-2}$, then, since $\int_{\mathbb{R}} b^{*(n-1)}(\tau)\,d\tau = B^{n-1}$, one gets $\lim\limits_{s\to\infty} \frac{b^{*n}(s)}{b(s)} = B^{n-1} + B(n-1)B^{(n-2)} = nB^{(n-1)}$. $\square$

Consider now some general statements in the Euclidean space $\mathbb{R}^d$, $d \in \mathbb{N}$. Fix the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$ there. All functions on $\mathbb{R}^d$ in the sequel are supposed to be $\mathcal{B}(\mathbb{R}^d)$-measurable. Let $0 \leq a \in L^1(\mathbb{R}^d)$ be a fixed probability density on $\mathbb{R}^d$, i.e.

$$\int_{\mathbb{R}^d} a(x)\,dx = 1. \quad (2.17)$$

Let $f:\mathbb{R}\to\mathbb{R}$; we will say that the convolution

$$(a*f)(x):=\int_{\mathbb{R}^d} a(x-y)f(y)\,dy, \quad x\in\mathbb{R}^d$$

is well-defined if the function $y\mapsto a(x-y)f(y)$ belongs to $L^1(\mathbb{R}^d)$ for a.a. $x\in\mathbb{R}^d$. In particular, this holds if $f\in L^\infty(\mathbb{R}^d)$. Next, for a function $\phi:\mathbb{R}^d\to(0,+\infty)$, we define, for any $f:\mathbb{R}^d\to\mathbb{R}$,

$$\|f\|_\phi := \sup_{x\in\mathbb{R}^d}\frac{|f(x)|}{\phi(x)}\in[0,\infty].$$

Proposition 2.3. *Let a function $\phi:\mathbb{R}^d\to(0,+\infty)$ be such that $a*\phi$ is well-defined, $\|a\|_\phi<\infty$, and, for some $\gamma\in(0,\infty)$,*

$$\frac{(a*\phi)(x)}{\phi(x)}\leq\gamma, \quad x\in\mathbb{R}^d. \tag{2.18}$$

Then $a^{*n}(x)\leq\gamma^{n-1}\|a\|_\phi\phi(x)$, $x\in\mathbb{R}^d$.

Proof. For any $f:\mathbb{R}^d\to\mathbb{R}$, with $\|f\|_\phi<\infty$, we have, for $x\in\mathbb{R}^d$,

$$\left|\frac{(a*f)(x)}{\phi(x)}\right|\leq\int_{\mathbb{R}^d}\frac{a(y)\phi(x-y)}{\phi(x)}\frac{|f(x-y)|}{\phi(x-y)}dy\leq\frac{a*\phi(x)}{\phi(x)}\|f\|_\phi\leq\gamma\|f\|_\phi.$$

In particular, since $\|a\|_\phi<\infty$, one gets $\|a*a\|_\phi\leq\gamma\|a\|_\phi<\infty$. Proceeding inductively, one gets $\|a^{*n}\|_\phi\leq\gamma\|a*a^{n-1}\|_\phi\leq\gamma^{n-1}\|a\|_\phi<\infty$, that yields the statement. □

Proposition 2.4. *Let a function $\omega:\mathbb{R}^d\to(0,+\infty)$ be such that, for any $\lambda>0$,*

$$\Omega_\lambda:=\Omega_\lambda(\omega):=\bigl\{x\in\mathbb{R}^d:\omega(x)<\lambda\bigr\}\neq\emptyset. \tag{2.19}$$

Suppose further

$$\eta:=\limsup_{\lambda\to0+}\sup_{x\in\Omega_\lambda}\frac{(a*\omega)(x)}{\omega(x)}\in(0,\infty). \tag{2.20}$$

Then, for any $\delta\in(0,1)$, there exists $\lambda=\lambda(\delta,\omega)\in(0,1)$, such that (2.18) *holds, with*

$$\phi(x):=\omega_\lambda(x):=\min\bigl\{\lambda,\omega(x)\bigr\}, \quad x\in\mathbb{R}^d, \tag{2.21}$$

and $\gamma:=\max\{1,(1+\delta)\eta\}$.

Proof. By (2.21), for an arbitrary $\lambda>0$, we have $\omega_\lambda(x)\leq\lambda$, $x\in\mathbb{R}^d$; then $(a*\omega_\lambda)(x)\leq\lambda$, $x\in\mathbb{R}^d$, as well. In particular, cf. (2.21),

$$(a*\omega_\lambda)(x)\leq\omega_\lambda(x), \quad x\in\mathbb{R}^d\setminus\Omega_\lambda. \tag{2.22}$$

Next, by (2.20), for any $\delta>0$ there exists $\lambda=\lambda(\delta)\in(0,1)$ such that

$$\sup_{x\in\Omega_\lambda}\frac{(a*\omega)(x)}{\omega(x)}-\eta\leq\delta\eta,$$

in particular, $(a*\omega)(x)\leq(1+\delta)\eta\omega(x)=(1+\delta)\eta\omega_\lambda(x), \quad x\in\Omega_\lambda$. Therefore,

$$(a*\omega_\lambda)(x)=(a*\omega)(x)-\left(a*(\omega-\omega_\lambda)\right)(x)\leq(1+\delta)\eta\omega_\lambda(x), \tag{2.23}$$

for all $x\in\Omega_\lambda$; here we used that $\omega\geq\omega_\lambda$. Then (2.22)–(2.23) yield the statement. □

We are ready to prove now Kesten's bound on $\mathbb{R}$.

Theorem 2.2. *Let $b \in L^1(\mathbb{R} \to \mathbb{R}_+)$ be a bounded weakly sub-exponential on $\mathbb{R}$ function with $\int_{\mathbb{R}} b(s)\,ds = 1$, such that (2.9) holds. Then, for any $\delta \in (0,1)$, there exist $C_\delta, s_\delta > 0$, such that*

$$b^{*n}(s) \leq C_\delta (1+\delta)^n b(s), \qquad s > s_\delta, n \in \mathbb{N}. \tag{2.24}$$

Proof. Fix $\delta \in (0,1)$ and $\varepsilon \in \big(0, \delta]$ with $(1+\varepsilon)^3 \leq 1 + \frac{\delta}{2}$. Let $s_1, \lambda_1 > 0$ satisfy

$$\int_{-\infty}^{-s_1} b(s)ds + \int_{s_1}^{\infty} b(s)ds \leq \frac{\varepsilon}{2}, \qquad 4 s_1 \lambda_1 \leq \varepsilon. \tag{2.25}$$

Define the following functions, for $s \in \mathbb{R}$,

$$\begin{aligned} \tilde{b}(s) &:= 1\!\!1_{(-\infty,-s_1)}(s) b(-s) + 1\!\!1_{[-s_1,s_1]}(s) \max\{\lambda_1, b(s)\} + 1\!\!1_{(s_1,\infty)}(s) b(s), \\ b_1(s) &:= 1\!\!1_{(-\infty,-s_1)}(s) b(s), \qquad a(s) := \frac{\tilde{b}(s)}{\|\tilde{b}\|_1}. \end{aligned} \tag{2.26}$$

Here and below, $\|\cdot\|_1$ denotes the norm in $L^1(\mathbb{R})$. Then, by (2.25), (2.26), $\|b - \tilde{b}\|_1 \leq \varepsilon$, and hence

$$\|\tilde{b}\|_1 \leq 1 + \varepsilon. \tag{2.27}$$

By Corollary 2.1, both functions $s \mapsto \tilde{b}(s)$ and $s \mapsto \tilde{b}(-s)$ are weakly sub-exponential on $\mathbb{R}$. Hence by Theorem 2.1, $\lim\limits_{s\to\pm\infty} \frac{\tilde{b}^{*m}(s)}{\tilde{b}^{*k}(s)} = \frac{m}{k} \|\tilde{b}\|_1^{m-k}$ for any $k, m \in \mathbb{N}$. In particular, there exist $m_0 \in \mathbb{N}$ and $s_2 \geq s_1$, such that, for $\omega := \tilde{b}^{*m_0}$ and $|s| \geq s_2$,

$$\frac{\tilde{b} * \omega(s)}{\omega(s)} \leq (1+\varepsilon) \|\tilde{b}\|_1, \tag{2.28}$$

$$m_0 (1-\varepsilon) \|\tilde{b}\|_1^{m_0 - 1} \leq \frac{\omega(s)}{\tilde{b}(s)} \leq m_0 (1+\varepsilon) \|\tilde{b}\|_1^{m_0-1}. \tag{2.29}$$

For an $r > 0$, $\operatorname*{ess\,inf}\limits_{s \in [-r,r]} \tilde{b}(s) > 0$. It is straightforward to check then by induction, that

$$\operatorname*{ess\,inf}_{-r \leq s \leq r} \omega(s) = \operatorname*{ess\,inf}_{-r \leq s \leq r} \tilde{b}^{*m_0}(s) > 0, \qquad r > 0. \tag{2.30}$$

On the other hand, by Remark 2.5 and (2.29), we have

$$\lim_{s \to \pm\infty} \omega(s) = \lim_{s \to \pm\infty} \tilde{b}(s) = 0. \tag{2.31}$$

Hence there exists $\lambda_2 \in (0, \lambda_1]$, such that for any $\lambda \in (0, \lambda_2)$, the set Ω_λ defined by (2.19) will be a non-empty subset of $(-\infty, -s_\lambda) \cup (s_\lambda, \infty)$, where $s_\lambda \to \infty$, as $\lambda \searrow 0$. Therefore, by (2.28), the condition (2.20) holds with a given by (2.26) and $\eta \leq 1 + \varepsilon$. Then, by Proposition 2.4, there exists $\lambda_3 \in (0, \min\{\lambda_2, 1\})$, such that (2.18) holds with

$$\phi(s) := \min\{\lambda_3, \omega(s)\}, \quad s \in \mathbb{R}, \qquad \gamma := (1+\varepsilon)^2.$$

By (2.29), (2.30) and since $\tilde{b}$ is bounded, we have $\|\tilde{b}\|_\phi < \infty$. Hence, by Proposition 2.3 and using that $\gamma\|\tilde{b}\|_1 \leq 1 + \frac{\delta}{2}$ because of (2.27) and the choice of ε, one easily gets

$$\tilde{b}^{*n}(s) \leq \Bigl(1 + \frac{\delta}{2}\Bigr)^{n-1} \|\tilde{b}\|_\phi \min\{\lambda_3, \omega(s)\}, \qquad s \in \mathbb{R},\ n \in \mathbb{N}.$$

By (2.31), (2.29), $\|\tilde{b}\|_\phi \min\{\lambda_3, \omega(s)\} \leq C_\delta \tilde{b}(s) = C_\delta b(s)$ for $s > s_3$, with some $s_3 \geq s_2$ and $C_\delta = C_\delta(m_0) > 0$. As a result,

$$\tilde{b}^{*n}(s) \leq C_\delta \Bigl(1 + \frac{\delta}{2}\Bigr)^{n} b(s), \qquad s > s_3,\ n \in \mathbb{N}. \tag{2.32}$$

By (2.26), $b \leq b_1 + \tilde{b}$ and hence $b^{*n} \leq \sum_{k=0}^{n} \binom{n}{k} b_1^{*k} * \tilde{b}^{*(n-k)}$, $n \in \mathbb{N}$, pointwise. Since $b_1^{*k}(s) = 0$ for all $s \geq -s_1$, $k \in \mathbb{N}$, then, by (2.32), one gets, for $s \geq s_3$,

$$\begin{aligned} &b_1^{*k} * \tilde{b}^{*(n-k)}(s) = \int_{s+s_1}^{\infty} b_1^{*k}(s-y)\tilde{b}^{*(n-k)}(y)dy \\ &\leq C_\delta \Bigl(1 + \frac{\delta}{2}\Bigr)^{n-k} \int_{s+s_1}^{\infty} b_1^{*k}(s-y)\tilde{b}(y)dy = C_\delta \Bigl(1 + \frac{\delta}{2}\Bigr)^{n-k} \int_{-\infty}^{-s_1} b_1^{*k}(y)\tilde{b}(s-y)dy, \end{aligned}$$

and by (2.9), (2.25), and (2.27), one can continue, for $s \geq s_\delta := \max\{s_3, \rho\}$,

$$\leq C_\delta \Bigl(1 + \frac{\delta}{2}\Bigr)^{n-k} \Bigl(\frac{\varepsilon}{2}\Bigr)^{k} K\tilde{b}(s) \leq C_\delta \Bigl(1 + \frac{\delta}{2}\Bigr)^{n-k} \Bigl(\frac{\delta}{2}\Bigr)^{k} b(s),$$

that yields (2.24). □

Remark 2.6. Following the scheme of the proof for [2, Proposition 8], it may be shown that Kesten's bound (2.24) holds under a weaker assummption that $b \in L^1(\mathbb{R} \to \mathbb{R}_+)$ is a bounded on $\mathbb{R}$ function with $\int_{\mathbb{R}} b(s)\,ds = 1$ such that (2.16) holds, i.e.

$$b^{*n}(x) \sim nb(x), \quad x \to \infty,\ n \geq 2$$

(recall that, in contrast to (1.5) b is not necessary concentrated on $\mathbb{R}_+$). By Theorem 2.1, a sufficient condition for the latter asymptotic relation is that b is weakly subexponential (i.e., cf. (2.4)–(2.5), its normalised restriction b_+ on $\mathbb{R}_+$ is subexponential) and the inequality (2.9) holds.

3. Regular sub-exponential densities on $\mathbb{R}$

We are going to obtain an analogue of Kesten's bound on $\mathbb{R}^d$ with $d > 1$, at least for radially symmetric functions. Our technique will require to deal with functions $b(|x|)^\alpha$, $x \in \mathbb{R}^d$, where b is a sub-exponential function on $\mathbb{R}_+$ and $\alpha < 1$ is close enough to 1; in particular, we have to be sure that b^α is also sub-exponential on $\mathbb{R}_+$. Moreover, to weaken the condition of radial symmetry, we will allow double-side estimates by functions of the form $p(|x|)b(|x|)$ for appropriate p on $\mathbb{R}_+$ (say, polynomial). Again, we will need have to check whether the functions pb is also sub-exponential on $\mathbb{R}_+$. To check such a stability of the class of sub-exponential on $\mathbb{R}_+$ functions with respect to power and multiplicative perturbations, we have to reduce the class to appropriately regular sub-exponential functions. Then the mentioned stability takes place, see Theorem 3.1 and Proposition 3.3. The examples of regular sub-exponential functions are given in Subsection 3.2. The analogues of Kesten's bound on $\mathbb{R}^d$ are considered in Section 4.

3.1. Main properties

Definition 3.1. Let $\mathcal{S}_{\mathrm{reg}}$ be the set of all functions $b:\mathbb{R}\to\mathbb{R}_+$ such that 1) $b\in L^1(\mathbb{R}_+)$ and b is bounded on $\mathbb{R}$; 2) there exists $\rho=\rho_b>1$, such that b is log-convex and strictly decreasing to 0 on $[\rho,\infty)$ (i.e. b is simultaneously tail-decreasing and tail-log-convex), and $b(\rho)\leq 1$ (without loss of generality); 3) there exist $\delta=\delta_b\in(0,1)$ and an increasing function $h=h_b:(0,\infty)\to(0,\infty)$, with $h(s)<\frac{s}{2}$ and $\lim\limits_{s\to\infty}h(s)=\infty$, such that the asymptotic (2.10) holds, and, cf. (2.6),

$$\lim_{s\to\infty} b\bigl(h(s)\bigr)s^{1+\delta}=0. \tag{3.1}$$

For any $n\in\mathbb{N}$, we denote by $\mathcal{S}_{\mathrm{reg},n}$ the subclass of functions b from $\mathcal{S}_{\mathrm{reg}}$ such that

$$\int_{-\infty}^{\rho} b(s)\,ds+\int_{\rho}^{\infty} b(s)s^{n-1}\,ds<\infty. \tag{3.2}$$

Remark 3.1. It is worth noting again that, for a tail-decreasing function, (2.10) implies that b is long-tailed.

Remark 3.2. By Lemma 2.4, any function $b\in\mathcal{S}_{\mathrm{reg}}$ is weakly sub-exponential on $\mathbb{R}$. Moreover, by Lemma 2.5, any function $b\in\mathcal{S}_{\mathrm{reg},1}$ is sub-exponential on $\mathbb{R}$.

Remark 3.3. Let $b\in\mathcal{S}_{\mathrm{reg}}$, and $s_0>0$ be such that $h(2s_0)>\rho$. Then the monotonicity of b and h implies $b(s)\leq b(h(2s))$, $s>s_0$; and hence, because of (3.1), for $B:=2^{-1-\delta}$, there exists $s_1\geq s_0$, such that

$$b(s)\leq Bs^{-1-\delta},\quad s\geq s_1. \tag{3.3}$$

Below we will show that $\mathcal{S}_{\mathrm{reg}}$ and $\mathcal{S}_{\mathrm{reg},n}$, $n\in\mathbb{N}$ are closed under some simple transformations of functions. For an arbitrary function $b\in\mathcal{S}_{\mathrm{reg}}$, we consider the following transformed functions: 1) for fixed $p>0$, $q>0$, $r\in\mathbb{R}$, we set

$$\widetilde{b}(s):=pb(qs+r),\quad s\in\mathbb{R}; \tag{3.4}$$

2) for a fixed $s_0>0$ and a fixed bounded function $c:\mathbb{R}\to\mathbb{R}_+$, we set

$$\breve{b}(s):=1\!\!1_{(-\infty,s_0)}(s)c(s)+1\!\!1_{[s_0,\infty)}(s)b(s),\quad s\in\mathbb{R}; \tag{3.5}$$

3) for any $\alpha\in(0,1]$, we denote

$$b_\alpha(s):=\bigl(b(s)\bigr)^\alpha,\quad s\in\mathbb{R}.$$

Theorem 3.1. *1. Let $b\in\mathcal{S}_{\mathrm{reg}}$. Then the functions $\widetilde{b}$ and $\breve{b}$ defined in* (3.4) *and* (3.5)*, correspondingly, also belong to $\mathcal{S}_{\mathrm{reg}}$ for all admissible values of their parameters. If, additionally, there exists $\alpha'\in(0,1)$ such that $b_{\alpha'}\in L^1(\mathbb{R}_+)$, then there exists $\alpha_0\in(\alpha',1)$, such that $b_\alpha\in\mathcal{S}_{\mathrm{reg}}$ for all $\alpha\in[\alpha_0,1]$.*

2. Let $b\in\mathcal{S}_{\mathrm{reg},n}$ for some $n\in\mathbb{N}$. Then $\widetilde{b}\in\mathcal{S}_{\mathrm{reg},n}$. If, additionally, the function c in (3.5) *is integrable on $(-\infty,s_0)$, then $\breve{b}\in\mathcal{S}_{\mathrm{reg},n}$. Finally, if there exists $\alpha'\in(0,1)$ such that* (3.2) *holds for $b=b_{\alpha'}$, then there exists $\alpha_0\in(\alpha',1)$, such that $b_\alpha\in\mathcal{S}_{\mathrm{reg},n}$ for all $\alpha\in[\alpha_0,1]$. Moreover, in the latter case, there exist $B_0>0$ and $\rho_0>0$, such that, for all $\alpha\in(\alpha_0,1]$,*

$$\int_{\mathbb{R}}\bigl(b(s-\tau)\bigr)^\alpha\bigl(b(\tau)\bigr)^\alpha\,d\tau\leq B_0\bigl(b(s)\bigr)^\alpha,\quad s\geq\rho_0, \tag{3.6}$$

Proof. It is very straightforward to check that if b is long-tailed, tail-decreasing and tail-log-convex, then $\widetilde{b}, \breve{b}, b_\alpha$ also have these properties for all admissible values of their parameters. Let $h:(0,\infty)\to(0,\infty)$ be such that $h(s)<\frac{s}{2}$, $\lim\limits_{s\to\infty} h(s)=\infty$, and (2.10) hold. Let also (3.1) hold for some $\delta>0$.

(i) Evidently, both (2.10) and (3.1) hold, with b replaced by $\breve{b}$. Next, $\breve{b}\in L^1(\mathbb{R}_+)$ and $\breve{b}$ is bounded. Hence $\breve{b}\in\mathcal{S}_{\mathrm{reg}}$. If $b\in\mathcal{S}_{\mathrm{reg},n}$ and c is integrable on $(-\infty,s_0)$, then (3.2) holds for b replaced by $\breve{b}$. Cf. also Corollary 2.1.

(ii) Set, for the given $q>0$, $r\in\mathbb{R}$, $\widetilde{h}(s):=\frac{1}{q}h(qs+r)-\frac{r}{2q}1\!\!1_{\mathbb{R}_+}(r)$, $s\in[s_1,\infty)$, where $s_1>0$ is such that $qs_1+r>0$ and $h(qs+r)>\frac{r}{2q}$ for all $s\geq s_1$, and $\widetilde{h}$ is increasing on $(0,s_1)$, such that $\widetilde{h}(s)<\min\bigl\{\frac{s}{2},\widetilde{h}(s_1)\bigr\}$, $s\in(0,s_1)$. By Proposition 2.1, (2.10) is equivalent to (2.3). Then, by (3.4), we have, $\sup_{|\tau|\leq\widetilde{h}(s)}\Bigl|\frac{\widetilde{b}(s+\tau)}{\widetilde{b}(s)}-1\Bigr|\to0$, as $s\to\infty$. Therefore, again by Proposition 2.1, (2.10) holds for b replaced by $\widetilde{b}$. Next, it is straightforward to get from (2.1), (3.1), that $\widetilde{b}\bigl(\widetilde{h}(s)\bigr)s^{1+\delta}\to0$, as $s\to\infty$. Therefore, $\widetilde{b}\in\mathcal{S}_{\mathrm{reg}}$. Finally, $b\in\mathcal{S}_{\mathrm{reg},n}$ for some $n\in\mathbb{N}$, trivially implies $\widetilde{b}\in\mathcal{S}_{\mathrm{reg},n}$.

(iii) Evidently, the convergence (2.10) implies the same one with b replaced by b_α, with the same h and for any $\alpha\in(0,1)$. Next, let $\alpha'\in(0,1)$ be such that $b_{\alpha'}\in L^1(\mathbb{R}_+)$. By the well-known log-convexity of L^p-norms (for $p>0$), for any $\alpha\in(\alpha',1)$ and for $\beta:=\frac{\alpha-\alpha'}{\alpha(1-\alpha')}\in(0,1)$, we have $\frac{1}{\alpha}=\frac{1-\beta}{\alpha'}+\beta$ and

$$\|b\|_{L^\alpha(\mathbb{R}_+)}\leq\|b\|^{1-\beta}_{L^{\alpha'}(\mathbb{R}_+)}\,\|b\|^{\beta}_{L^1(\mathbb{R}_+)}<\infty, \tag{3.7}$$

i.e. $b_\alpha\in L^1(\mathbb{R}_+)$ for all $\alpha\in(\alpha',1)$. Take and fix an $\alpha_0\in\Bigl(\max\Bigl\{\alpha',\frac{1}{1+\delta}\Bigr\},1\Bigr)$. Then, for any $\alpha\in[\alpha_0,1]$, we have that $\delta':=\alpha(1+\delta)-1\in(0,\delta]$, and hence, by (3.1),

$$\lim_{s\to\infty} b_\alpha\bigl(h(s)\bigr)s^{1+\delta'}=\lim_{s\to\infty}\Bigl(b\bigl(h(s)\bigr)s^{1+\delta}\Bigr)^\alpha=0.$$

Therefore, $b_\alpha\in\mathcal{S}_{\mathrm{reg}}$, $\alpha\in[\alpha_0,1]$.

Let, additionally, (3.2) hold for both b and $b_{\alpha'}$ (i.e., in particular, $b\in\mathcal{S}_{\mathrm{reg},n}$) and for some $n\in\mathbb{N}$. Then one can use again the log-convexity of L^p-norms, now for $L^p\bigl((\rho,\infty),s^n\,ds\bigr)$ spaces, to deduce that $b_\alpha\in\mathcal{S}_{\mathrm{reg},n}$, $\alpha\in[\alpha_0,1]$.

Finally, $b,b_{\alpha_0}\in\mathcal{S}_{\mathrm{reg},n}$, $n\in\mathbb{N}$, implies $b,b_{\alpha_0}\in\mathcal{S}_{\mathrm{reg},1}$, and hence, cf. Remark 3.2, b and b_{α_0} are sub-exponential on $\mathbb{R}$, i.e. (2.8) holds for both b and b_{α_0}. Therefore, for an arbitrary $\varepsilon\in(0,1)$, there exists $\rho_0=\rho_0(\varepsilon,b,b_{\alpha_0})>\rho$ (where ρ is from Definition 3.1) and $B_0:=2(1+\varepsilon)\max\bigl\{\int_{\mathbb{R}}b(s)\,ds,\int_{\mathbb{R}}b_{\alpha_0}(s)\,ds\bigr\}>0$, such that, for all $s\geq\rho_0$,

$$\int_{\mathbb{R}}b(s-\tau)b(\tau)\,d\tau\leq B_0b(s),\qquad \int_{\mathbb{R}}b_{\alpha_0}(s-\tau)b_{\alpha_0}(\tau)\,d\tau\leq B_0b_{\alpha_0}(s). \tag{3.8}$$

Then, applying again the norm log-convexity arguments, cf. (3.7), one gets, for any fixed $s\geq\rho_0$ and for all $\alpha\in(\alpha_0,1)$

$$\int_{\mathbb{R}}\bigl(b(s-\tau)b(\tau)\bigr)^\alpha d\tau\leq\biggl(\int_{\mathbb{R}}\bigl(b(s-\tau)b(\tau)\bigr)^{\alpha_0}d\tau\biggr)^{\frac{1}{\alpha_0}(1-\beta)\alpha}\biggl(\int_{\mathbb{R}}b(s-\tau)b(\tau)d\tau\biggr)^{\beta\alpha},$$

where $\beta = \frac{\alpha-\alpha_0}{\alpha(1-\alpha_0)} \in (0,1)$. Combining the latter inequality with (3.8), one gets

$$\int_{\mathbb{R}} \bigl(b(s-\tau)b(\tau)\bigr)^{\alpha} d\tau \le \bigl(B_0(b(s))^{\alpha_0}\bigr)^{\frac{1}{\alpha_0}(1-\beta)\alpha} \bigl(B_0 b(s)\bigr)^{\beta\alpha} = B_0\bigl(b(s)\bigr)^{\alpha}.$$

The theorem is fully proved now. □

It is naturally to expect that asymptotically small changes of tail properties preserve the sub-exponential property of a function. Namely, consider the following definition.

Definition 3.2. Two functions $b_1, b_2 : \mathbb{R} \to \mathbb{R}_+$ are said to be *weakly tail-equivalent* if

$$0 < \liminf_{s\to\infty} \frac{b_1(s)}{b_2(s)} \le \limsup_{s\to\infty} \frac{b_1(s)}{b_2(s)} < \infty,$$

or, in other words, if there exist $\rho > 0$ and $C_2 \ge C_1 > 0$, such that,

$$C_1 b_1(s) \le b_2(s) \le C_2 b_1(s), \quad s \ge \rho. \tag{3.9}$$

The proof of the following statement is a straightforward consequence of [14, Theorem 4.8].

Proposition 3.1. *Let $b_1 : \mathbb{R} \to \mathbb{R}_+$ be a weakly sub-exponential on $\mathbb{R}$ function. Let $b_2 : \mathbb{R} \to \mathbb{R}_+$ be a long-tailed function which is weakly tail-equivalent to b_1. Then b_2 is weakly sub-exponential on $\mathbb{R}$ as well. If, additionally,* (2.9) *holds for $b = b_1$, then b_2 is sub-exponential on $\mathbb{R}$.*

Proposition 3.2. *Let $b_1 \in \mathcal{S}_{\mathrm{reg}}$ and let $b_2 : \mathbb{R} \to \mathbb{R}_+$ be a bounded tail-decreasing and tail-log-convex function, such that*

$$\lim_{s\to\infty} \frac{b_2(s)}{b_1(s)} = C \in (0,\infty). \tag{3.10}$$

Then $b_2 \in \mathcal{S}_{\mathrm{reg}}$.

Proof. Clearly, b_2 is long-tailed as b_1 is such. Let $\delta \in (0,1)$ and h be an increasing function, with $h(s) < \frac{s}{2}$, $\lim\limits_{s\to\infty} h(s) = \infty$, such that (2.10) and (3.1) hold for $b = b_1$. Let $\varepsilon \in \bigl(0, \min\{1, C\}\bigr)$, and choose $\rho > 1$, such that b_2 is decreasing and log-convex on $[\rho,\infty)$, and $b_2(\rho) \le 1$. By (3.10) and (2.10) (for $b = b_1$), there exists $\rho_1 \ge \rho$, such that

$$0 < (C-\varepsilon)b_1(s) \le b_2(s) \le (C+\varepsilon)b_1(s), \qquad \left| \frac{b_1(s \pm h(s))}{b_1(s)} - 1 \right| < \varepsilon \tag{3.11}$$

for all $s \ge \rho_1$. Since b_2 is bounded and $b_1 \in L^1(\mathbb{R}_+)$, we have from (3.11) that $b_2 \in L^1(\mathbb{R}_+)$. By (3.11), for any $s \ge 2\rho_1$,

$$\left| \frac{b_2(s \pm h(s))}{b_2(s)} - 1 \right| < \max\left\{ \varepsilon \frac{C+\varepsilon}{C-\varepsilon} + \frac{C+\varepsilon}{C-\varepsilon} - 1,\ \varepsilon \frac{C-\varepsilon}{C+\varepsilon} + 1 - \frac{C-\varepsilon}{C+\varepsilon} \right\}.$$

Since the latter expression may be arbitrary small, by an appropriate choice of ε, one gets that (2.10) holds for $b = b_2$. Finally, (3.1) with $b = b_1$ and (3.10) imply that (3.1) holds with $b = b_2$ and the same δ and h. □

Remark 3.4. In the assumptions of the previous theorem, if, additionally, $b_1 \in \mathcal{S}_{\mathrm{reg},n}$ for some $n \in \mathbb{N}$, and b_2 is integrable on $(-\infty, -\rho_2)$ for some $\rho_2 > 0$, then $b_2 \in \mathcal{S}_{\mathrm{reg},n}$ (because of (3.11) and the boundedness of b_2).

On the other hand, if one can check that both functions b_1 and b_2 satisfy (2.10) with the same function $h(s)$, then the sufficient condition to verify (3.1) for $b = b_2$, provided that it holds for $b = b_1$, is much weaker than (3.10). To present the corresponding statement, consider the following definition.

Definition 3.3. Let $b_1, b_2 : \mathbb{R} \to \mathbb{R}_+$ and, for some $\rho \geq 0$, $b_i(s) > 0$ for all $s \in [\rho, \infty)$, $i = 1, 2$. The functions b_1 and b_2 are said to be *(asymptotically) log-equivalent*, if

$$\log b_1(s) \sim \log b_2(s), \quad s \to \infty. \tag{3.12}$$

Proposition 3.3. *Let $b_1 \in \mathcal{S}_{\mathrm{reg}}$ and let h be the function corresponding to Definition 3.1 with $b = b_1$. Let $b_2 : \mathbb{R} \to \mathbb{R}_+$ be a bounded tail-decreasing and tail-log-convex function, such that* (2.10) *holds with $b = b_2$ and the same h. Suppose that b_1 and b_2 are log-equivalent. Then $b_2 \in \mathcal{S}_{\mathrm{reg}}$. If, additionally, there exists $\alpha' \in (0, 1)$, such that* (3.2) *holds with $b = (b_1)^{\alpha'}$ and b_2 is integrable on $(-\infty, \rho)$, then $b_2 \in \mathcal{S}_{\mathrm{reg},n}$.*

Proof. Let $\delta \in (0, 1)$ be such that (3.1) holds for b replaced by b_1. Take an arbitrary $\varepsilon \in \bigl(0, \frac{\delta}{1+\delta}\bigr)$. By (3.12), there exists $\rho_\varepsilon > 0$, such that $b_i(s) < 1$, $s > \rho_\varepsilon$, $i = 1, 2$, and

$$\begin{aligned} -(1-\varepsilon) \log b_1(s) \leq - \log b_2(s) \leq -(1+\varepsilon) \log b_1(s), \quad s > \rho_\varepsilon, \\ b_1(s)^{1+\varepsilon} \leq b_2(s) \leq b_1(s)^{1-\varepsilon}, \quad s > \rho_\varepsilon. \end{aligned} \tag{3.13}$$

Since $h(s) \to \infty$, $s \to \infty$, there exists $\rho_0 > \rho_\varepsilon$, such that $h(s) > \rho_\varepsilon$ for any $s > \rho_0$. Then, by (3.13), we have, for all $s > \rho_0$,

$$b_2(h(s)) s^{(1+\delta)(1-\varepsilon)} < b_1(h(s))^{1-\varepsilon} s^{(1+\delta)(1-\varepsilon)} = \bigl(b_1(h(s)) s^{1+\delta}\bigr)^{1-\varepsilon},$$

and therefore, (3.1) holds with $b = b_2$ and δ replaced by $(1+\delta)(1-\varepsilon)-1 = \delta - \varepsilon(1+\delta) \in (0, 1)$, that proves the first statement. To prove the second one, assume, additionally, that $\varepsilon < 1 - \alpha'$. Then, by (3.13), we have $b_2(s) s^{n-1} \leq b_1(s)^{1-\varepsilon} s^{n-1} < b_1(s)^{\alpha'} s^{n-1}$ for all $s > \rho_\varepsilon$, as $b_1(s) < 1$ here. □

3.2. Examples

We consider now examples of functions $b \in \mathcal{S}_{\mathrm{reg}}$. Because of Proposition 3.3, we will classify these functions 'up to log-equivalence', i.e. by tail properties of the function

$$l(s) := -\log b(s).$$

Taking into account the result of Theorem 3.1 concerning the function $\check{b}$, it will be enough to define b on some (s_0, ∞), $s_0 > 0$ only. Next, by Lemma 2.4, the function b_+ defined by (2.4) is a sub-exponential density on $\mathbb{R}_+$. Therefore, one can use the classical examples of such densities, see e.g. [14]. However, using the result of Theorem 3.1 concerning the function $\widetilde{b}$, one can consider that examples in their 'simplest' forms (ignoring any shifts of the argument or scales of the argument or the function itself).

Now we consider different asymptotics of the function $l(s) = -\log b(s)$. In all particular examples below, it is straightforward to check that each particular bounded functions b is such that $b'(s) < 0$ and $(\log b(s))'' > 0$ for all big enough values of s, i.e. b is tail-decreasing and tail-log-convex.

Class 1: $l(s) \sim D \log s$, $s \to \infty$, $D > 1$

Polynomial decay. Let $b : \mathbb{R} \to \mathbb{R}_+$ be a bounded tail-decreasing tail-log-convex function, such that $b(s) \sim qs^{-D}$, $s \to \infty$, $D > 1$, $q > 0$. By Proposition 3.2, to show that $b \in \mathcal{S}_{\mathrm{reg}}$, it is enough to prove this for

$$b(s) = 1\!\!1_{\mathbb{R}_+}(s)(1+s)^{-D}, \quad s \in \mathbb{R}.$$

For an arbitrary $\gamma \in \bigl(\frac{1}{D}, 1\bigr)$, consider $h(s) = s^\gamma$, $s > 0$. Then it is straightforward to check that (2.10) and (3.1) hold, provided that $\delta \in (0, \gamma D - 1) \subset (0,1)$. As a result, $b \in \mathcal{S}_{\mathrm{reg}}$. Clearly, $b \in \mathcal{S}_{\mathrm{reg},n}$ for $D > n$, cf. Remark 3.4.

Classical examples of the polynomially decaying probability densities in [14] can be described by the following functions:

1. *Student's t-function*: $\mathscr{T}(s) = (1+s^2)^{-p}$, $s > 0$, $p > \frac{1}{2}$. Note that $\mathscr{T} \in \mathcal{S}_{\mathrm{reg},n}$, $n \in \mathbb{N}$, if only $p > \frac{n}{2}$. The case $p = 1$ is referred to the Cauchy distribution, the corresponding function belongs to $\mathcal{S}_{\mathrm{reg},n}$ for $n = 1$ only.
2. *The Lévy function*: $\mathscr{L}(s) = s^{-\frac{3}{2}} \exp\bigl(-\frac{c}{s}\bigr)$, $s > 0$, $c > 0$.
3. *The Burr function*: $\mathscr{B}(s) = s^{c-1}(1+s^c)^{-k-1}$, $s > 0$, $c > 0$, $k > 0$. Note that the case $c = 1$ is related to the Pareto distribution; the latter has the density $kp^k \mathscr{B}(s-1) 1\!\!1_{[p,\infty)}(s)$ for any $p > 0$.

Logarithmic perturbation of the polynomial decay. Let $D > 1$, $\nu \in \mathbb{R}$, and

$$b(s) = 1\!\!1_{(1,\infty)}(s)(\log s)^\nu s^{-D}, \quad s \in \mathbb{R}.$$

We are going to apply Proposition 3.3 now, with $b_1(s) = s^{-D}$ and $b_2(s) = (\log s)^\nu s^{-D}$. Indeed, then (3.12) evidently holds. It remains to check that (2.10) holds for both b_1 and b_2 with the same $h(s) = s^\gamma$, $\gamma \in (0,1)$. One has then $\frac{\log(s \pm s^\gamma)}{\log s} \to 1$ as $s \to \infty$, that yields the needed.

Class 2: $l(s) \sim D(\log s)^q$, $s \to \infty$, $q > 1$, $D > 0$

Consider the function

$$N(s) := 1\!\!1_{\mathbb{R}_+}(s) \exp\bigl(-D(\log s)^q\bigr), \quad s \in \mathbb{R}.$$

Take $h(s) = 1\!\!1_{[\rho,\infty)}(s) s^{\frac{1}{q}}$, where $\rho > 1$ is chosen such that $h(s) < \frac{s}{2}$ for $s \geq \rho$. Since $q > 1$, we have

$$\begin{aligned}\frac{N(s \pm h(s))}{N(s)} &= \exp\biggl\{ D(\log s)^q \biggl(1 - \Bigl(1 + \frac{\log(1 \pm s^{\frac{1}{q}-1})}{\log s}\Bigr)^q \biggr) \biggr\} \\ &\sim \exp\biggl\{ D(\log s)^q \biggl(\mp q \frac{s^{\frac{1}{q}-1}}{\log s} \biggr) \biggr\} \sim 1, \quad s \to \infty,\end{aligned}$$

that proves (2.10). Next, for any $\delta \in \mathbb{R}$,

$$N\bigl(s^{\frac{1}{q}}\bigr) s^{1+\delta} = \exp\bigl(-Dq^{-q}(\log s)^q + (1+\delta)\log s\bigr) \to 0, \quad s \to \infty.$$

As a result, $N\in\mathcal{S}_{\mathrm{reg}}$. Moreover, evidently, $N\in\mathcal{S}_{\mathrm{reg},n}$, for any $n\in\mathbb{N}$.

We may also consider Proposition 3.3 for $b_1=b$ and $b_2=pb$, where b_2 is tail-decreasing and tail-log-convex function, such that $\log p=o(\log b)$ (that is equivalent to $\log b_1\sim\log b_2$) and p satisfies (2.10) with $h(s)=s^{\frac{1}{q}}$. According to the results above, a natural example of such $p(s)$ might be s^D, $D\in\mathbb{R}$. It is straightforward to verify that, for any $D\in\mathbb{R}$, $b_2=pb_1$ is tail-decreasing and tail-log-convex. As a result, then $b_2\in\mathcal{S}_{\mathrm{reg},n}$, $n\in\mathbb{N}$.

The classical log-normal distribution has the density described by the function $\mathscr{N}(s)=\frac{1}{s}\exp\Bigl(-\frac{(\log s)^2}{2\gamma^2}\Bigr)$, $s>0$, $\gamma>0$, that can be an example of the function b_2 above.

Class 3: $l(s)\sim s^\alpha$, $\alpha\in(0,1)$

Consider, for any $\alpha\in(0,1)$, the so-called *fractional exponent*

$$w(s)=1\!\!1_{\mathbb{R}_+}(s)\exp(-s^\alpha),\qquad s\in\mathbb{R}. \tag{3.14}$$

Set $h(s)=1\!\!1_{[\rho,\infty)}(s)(\log s)^{\frac{2}{\alpha}}$, where $\rho>0$ is chosen such that $h(s)<\frac{s}{2}$ for $s\geq\rho$. Then $h(s)=o(s)$, $s\to\infty$, and hence

$$\frac{w\bigl(s\pm h(s)\bigr)}{w(s)}=\exp\biggl\{-s^\alpha\biggl(\Bigl(1\pm\frac{h(s)}{s}\Bigr)^\alpha-1\biggr)\biggr\}\sim\exp\biggl\{-s^\alpha\biggl(\pm\alpha\frac{h(s)}{s}\biggr)\biggr\}\sim 1,$$

as $s\to\infty$, that proves (2.10). Next, for any $\delta\in\mathbb{R}$,

$$w\bigl(h(s)\bigr)s^{1+\delta}=\exp\bigl(-(\log s)^2+(1+\delta)\log s\bigr)\to 0,\qquad s\to\infty.$$

As a result, $w\in\mathcal{S}_{\mathrm{reg}}$. It is clear also that $w\in\mathcal{S}_{\mathrm{reg},n}$ for all $n\in\mathbb{N}$.

Similarly to the above, one can show that $pw\in\mathcal{S}_{\mathrm{reg}}$, provided that, in particular, $\log p=o(\log w)$ and (2.10) holds for $b=p$ and $h(s)=(\log s)^{\frac{2}{\alpha}}$. Again, one can consider $p(s)=s^D$, $D\in\mathbb{R}$, since it satisfies (2.10) with $h(s)=s^\gamma>(\log s)^{\frac{2}{\alpha}}$, $\alpha,\gamma\in(0,1)$, and big enough s. As before, the verification that, for any $D\in\mathbb{R}$, $b_2=pb_1$ is tail-decreasing and tail-log-convex is straightforward.

The probability density of the classical Weibull distribution is described by the function $\mathscr{W}(s)=s^{\alpha-1}\exp(-s^\alpha)$, $s\geq\rho>0$, $\alpha\in(0,1)$. By the above, $\mathscr{W}\in\mathcal{S}_{\mathrm{reg},n}$, $n\in\mathbb{N}$. Note that $\int_s^\infty\mathscr{W}(\tau)\,d\tau=\frac{1}{\alpha}w(s)$, where w is given by (3.14).

Class 4: $l(s)\sim\frac{s}{(\log s)^q}$, $q>1$

Consider also a function which decays 'slightly' slowly than an exponential function. Namely, let, for an arbitrary fixed $q>1$,

$$g(s)=1\!\!1_{\mathbb{R}_+}(s)\exp\Bigl(-\frac{s}{(\log s)^q}\Bigr),\qquad s\in\mathbb{R}.$$

Take, for an arbitrary $\gamma\in(1,q)$, $h(s)=(\log s)^\gamma$, $s>0$; and denote, for a brevity, $p(s):=\frac{h(s)}{s}\to 0$, $s\to\infty$. Then, $\log(s+h(s))=\log s+\log(1+p(s))$. Set also

$r(s) := \frac{\log(1+p(s))}{\log s} \to 0$, $s \to \infty$. Then, for any $s > e^{q+1}$, we have

$$\log \frac{g(s+h(s))}{g(s)} = \frac{s}{(\log s)^q}\Bigl(1 - \frac{1+p(s)}{\bigl(1+r(s)\bigr)^q}\Bigr)$$
$$= \frac{1}{\bigl(1+r(s)\bigr)^q}\biggl(q\frac{\bigl(1+r(s)\bigr)^q - 1}{qr(s)}\frac{\log(1+p(s))}{p(s)}(\log s)^{\gamma-q-1} - (\log s)^{\gamma-q}\biggr) \to 0,$$

as $s \to \infty$, since $\gamma < q$; and similarly $\log \frac{g(s-h(s))}{g(s)} \to 0$, $s \to \infty$. Therefore, (2.10) holds for $b = g$. Next,

$$\log\bigl(g(h(s))s^{1+\delta}\bigr) = -(\log s)\Bigl(\frac{(\log s)^{\gamma-1}}{\gamma^q(\log\log s)^q} - (1+\delta)\Bigr) \to -\infty, \quad s \to \infty,$$

that yields (3.1) for $b = g$. As a result, $g \in \mathcal{S}_{\mathrm{reg}}$. Again, evidently, $g \in \mathcal{S}_{\mathrm{reg},n}$, $n \in \mathbb{N}$. The same arguments as before show that, for any $D \in \mathbb{R}$, the function $s^D g(s)$ belongs to $\mathcal{S}_{\mathrm{reg},n}$ as well.

Remark 3.5. Naturally, $q \in (0,1]$ gives behavior of $g(s)$ more 'close' to the exponential function. Unfortunately, our approach does not cover this case: the analysis above shows that $h(s)$, to fulfill even (2.6), must grow faster than $\log s$, whereas so 'big' $h(s)$ would not fulfill (2.10). In general, Lemma 2.4 gives a sufficient condition only, to get a sub-exponential density on $\mathbb{R}_+$. It can be shown, see e.g. [10, Example 1.4.3], that a probability distribution, whose density b on $\mathbb{R}_+$ is such that $\int_s^\infty b(\tau)\,d\tau \sim g(s)$, $s \to \infty$, with $q > 0$, is a sub-exponential distribution (for the latter definition, see e.g. [14, Definition 3.1]). Then we expect that $b(s) \sim -g'(s)$, $s \to \infty$, and it is easy to see that $\log(-g'(s)) \sim \log g(s)$, $s \to \infty$. It should be stressed though that, in general, sub-exponential property of a distribution does not imply the corresponding property of its density, cf. [14, Section 4.2]. Therefore, we can not state that the function b above is a sub-exponential one for $q \in (0,1]$.

Combining the results above, one gets the following statement.

Corollary 3.1. *Let $b : \mathbb{R} \to \mathbb{R}_+$ be a bounded tail-decreasing and tail-log-convex function, such that, for some $C > 0$, the function $Cb(s)$ has either of the following asymptotics as $s \to \infty$*

$$(\log s)^\mu s^{-(n+\delta)}, \qquad (\log s)^\mu s^\nu \exp\bigl(-D(\log s)^q\bigr),$$
$$(\log s)^\mu s^\nu \exp\bigl(-s^\alpha\bigr), \qquad (\log s)^\mu s^\nu \exp\Bigl(-\frac{s}{(\log s)^q}\Bigr),$$

where $D, \delta > 0$, $q > 1$, $\alpha \in (0,1)$, $\nu, \mu \in \mathbb{R}$. Then $b \in \mathcal{S}_{\mathrm{reg},n}$, $n \in \mathbb{N}$.

4. Analogues of Kesten's bound on $\mathbb{R}^d$

We start with a simple corollary of Propositions 2.3 and 2.4.

Proposition 4.1. *Let a function $\omega : \mathbb{R}^d \to (0,+\infty)$ be such that (2.19) holds, and*

$$\limsup_{\lambda \to 0+} \sup_{x \in \Omega_\lambda} \frac{(a * \omega)(x)}{\omega(x)} \le 1. \tag{4.1}$$

Let also $a \in L^\infty(\mathbb{R}^d)$ and $\|a\|_\omega < \infty$. Then, for any $\delta \in (0,1)$, there exist $c_\delta > 0$ and $\lambda = \lambda(\delta) \in (0,1)$, such that

$$a^{*n}(x) \leq c_\delta (1+\delta)^{n-1} \min\bigl\{\lambda, \omega(x)\bigr\}, \quad x \in \mathbb{R}^d. \tag{4.2}$$

Proof. Take any $\delta \in (0,1)$. By Proposition 2.4, there exists $\lambda = \lambda(\delta, \omega) \in (0,1)$, such that (2.18) holds, with ϕ given by (2.21) and $\gamma = 1 + \delta$. Denote $\|a\|_\infty := \|a\|_{L^\infty(\mathbb{R}^d)}$. We have

$$\frac{a(x)}{\omega_\lambda(x)} \leq \frac{\|a\|_\infty}{\lambda} 1\!\!1_{\mathbb{R}^d \setminus \Omega_\lambda}(x) + \frac{a(x)}{\omega(x)} 1\!\!1_{\Omega_\lambda}(x) \leq \frac{\|a\|_\infty}{\lambda} + \|a\|_\omega =: c_\delta < \infty, \tag{4.3}$$

and one can apply Proposition 2.3 that yields the statement. □

Remark 4.1. Note that, for $d = 1$, Proposition 2.2 implies that, if only $\omega \in \mathcal{S}_{\mathrm{reg},1}$, $a(s) = o(\omega(s))$, $s \to \infty$, and (2.17) holds, then $(a * \omega)(s) \sim \omega(s)$, $s \to \infty$; and then, in particular, (4.1) holds. Next, in the course of the proof of Theorem 2.2 (still for $d = 1$), we slightly weakened the restriction $a = o(\omega)$ for the case where a itself is sub-exponential, by setting $\omega := a^{*m}$ with large enough $m \in \mathbb{N}$, since then $\frac{a}{\omega} \sim \frac{1}{m}$ may be chosen arbitrary small. As it was mentioned in the Introduction, for the multi-dimensional case, we do not have a theory of sub-exponential densities. Therefore, we consider more 'rough' candidate for ω to ensure (4.1), namely, $\omega(x) = a(x)^\alpha$ for $\alpha \in (0,1)$; then, in particular, $a(x) = o(\omega(x))$, $|x| \to \infty$ if, for example, $a(x) = b(|x|)$, $x \in \mathbb{R}^d$, with a tail-decreasing function b, cf. Definition 4.1 below. In particular, the results of this Section, following from (4.2), yield upper bounds for $a^{*n}(x)$ with the right-hand side heavier than $a(x)$ at infinity.

Definition 4.1. Let $\mathcal{D}_d$ be the set of all bounded functions $b : \mathbb{R} \to (0,\infty)$, such that b is tail-decreasing (cf. Definition 2.5) and $\int_0^\infty b(s) s^{d-1}\, ds < \infty$. Let $\widetilde{\mathcal{D}}_d \subset \mathcal{D}_d$ denote the subset of all functions from $\mathcal{D}_d$ which are (strictly) decreasing to 0 on $\mathbb{R}_+$.

Remark 4.2. It is easy to see that, if $b^{\alpha_0} \in \widetilde{\mathcal{D}}_d$ for some $\alpha_0 \in (0,1)$, then $b^\alpha \in \widetilde{\mathcal{D}}_d$ for all $\alpha \in [\alpha_0, 1]$.

Consider an analogue of long-tailed functions on $\mathbb{R}^d$.

Lemma 4.1. *Let $b : \mathbb{R}_+ \to \mathbb{R}_+$ be a long-tailed function (cf. Definition 2.1), and $c(x) = b(|x|)$, $x \in \mathbb{R}^d$. Then, for any $r > 0$,*

$$\lim_{|x|\to\infty} \sup_{|y|\leq r} \left| \frac{c(x+y)}{c(x)} - 1 \right| = 0. \tag{4.4}$$

Proof. Evidently, $|y| \leq r$ implies that $h := |x+y| - |x| \in [-r, r]$ for each $x \in \mathbb{R}^d$. Therefore,

$$\sup_{|y|\leq r} \left| \frac{b(|x+y|)}{b(|x|)} - 1 \right| \leq \sup_{h \in [-r,r]} \left| \frac{b(|x|+h)}{b(|x|)} - 1 \right| \to 0, \quad |x| \to \infty,$$

because of (2.2). □

We will assume in the sequel, that a is bounded by a radially symmetric function:

$$\text{There exists } b^+ \in \widetilde{\mathcal{D}}_d, \text{ such that } a(x) \le b^+(|x|), \text{ for a.a. } x \in \mathbb{R}^d. \tag{4.5}$$

We start with the following sufficient condition.

Proposition 4.2. *Let* (4.5) *hold with* $b^+ \in \widetilde{\mathcal{D}}_d$ *which is log-equivalent, cf. Definition 3.3, to the function* b*, given by*

$$b(s) := 1\!\!1_{\mathbb{R}_+}(s)\frac{M}{(1+s)^{d+\mu}}, \quad s \in \mathbb{R}, \tag{4.6}$$

for some $\mu, M > 0$*. Then there exists* $\alpha_0 \in (0,1)$*, such that, for all* $\alpha \in (\alpha_0, 1)$*, the function* $\omega(x) = b(|x|)^\alpha$*,* $x \in \mathbb{R}^d$*, satisfies* (4.1)*.*

Proof. Set $\alpha_0 := \frac{d+\frac{\mu}{2}}{d+\mu} \in (0,1)$. Take arbitrary $\alpha \in (\alpha_0, 1)$ and $\epsilon \in (0, 1-\alpha)$. Take also an arbitrary $\delta \in (0,1)$, and define $h(s) = s^\delta$, $s > 0$. By (3.13), applied to $b_1 = b$ and $b_2 = b^+$, there exists $s_\delta > 2r$ such that, for all $s > s_\delta$,

$$h(s) < \frac{s}{2}, \qquad b^+(s) \le \bigl(b(s)\bigr)^{1-\epsilon}. \tag{4.7}$$

For an arbitrary $x \in \mathbb{R}^d$ with $|x| > s_\delta$, we have a disjoint expansion $\mathbb{R}^d = D_1(x) \sqcup D_2(x) \sqcup D_3(x)$, where

$$D_1(x) := \bigl\{|y| \le h(|x|)\bigr\}, \quad D_2(x) := \Bigl\{h(|x|) < |y| \le \frac{|x|}{2}\Bigr\}, \quad D_3(x) = \Bigl\{|y| \ge \frac{|x|}{2}\Bigr\}.$$

Then, $\frac{(a*\omega)(x)}{\omega(x)} = I_1(x) + I_2(x) + I_3(x)$, where

$$I_j(x) := \int_{D_j(x)} a(y)\frac{(1+|x|)^{(d+\mu)\alpha}}{(1+|x-y|)^{(d+\mu)\alpha}}dy, \quad j = 1,2,3.$$

Using the inequality $|x-y| \ge \bigl||x|-|y|\bigr|$, $x, y \in \mathbb{R}^d$, one has that $|x-y| \ge |x|-|y| \ge |x|-|x|^\delta$ for $y \in D_1(x)$, $|x| > s_\delta$. Then

$$I_1(x) \le \biggl(\frac{1+|x|}{1+|x|-|x|^\delta}\biggr)^{(d+\mu)\alpha}\int_{D_1(x)} a(y)dy \to 1, \quad |x| \to \infty.$$

Next, we have, for any $|y| < \frac{|x|}{2}$, that $1+|x-y| \ge 1+|x|-|y| \ge \frac{1}{2}(1+|x|)$; therefore,

$$I_2(x) \le 2^{(d+\mu)\alpha}\int_{\{|y|>|x|^\delta\}} a(y)dy \to 0, \quad |x| \to \infty.$$

Finally, by (4.5) and (4.7), the inclusions $y \in D_3(x)$ and $|x| > s_\delta$ imply $a(y) \le b^+(|y|) \le b(|y|)^{1-\epsilon} \le b\bigl(\frac{|x|}{2}\bigr)^{1-\epsilon}$, and, therefore,

$$\begin{aligned} I_3(x) &\le M\frac{(1+|x|)^{(d+\mu)\alpha}}{\bigl(1+\frac{|x|}{2}\bigr)^{(d+\mu)(1-\epsilon)}}\int_{D_3(x)}\frac{1}{(1+|x-y|)^{(d+\mu)\alpha_0}}dy \\ &\le M\frac{(1+|x|)^{(d+\mu)\alpha}}{\bigl(1+\frac{|x|}{2}\bigr)^{(d+\mu)(1-\epsilon)}}\int_{\mathbb{R}^d}\frac{1}{(1+|y|)^{d+\frac{\mu}{2}}}dy \to 0, \quad |x| \to \infty, \end{aligned}$$

as $1-\epsilon > \alpha$. Since b is decreasing on $\mathbb{R}_+$, we have, by (2.19), that, for any $\lambda > 0$, there exists $\rho_\lambda > 0$, such that $\Omega_\lambda = \{x \in \mathbb{R}^d : |x| > \rho_\lambda\}$. This yields (4.1). $\square$

Lemma 4.2. *Let $b \in L^1(\mathbb{R})$ be even, positive, decreasing to 0 on the whole $\mathbb{R}_+$, and long-tailed function. Suppose that there exist $B, r_b, \rho_b > 0$, such that*

$$\int_{r_b}^{\infty} b(s-\tau)b(\tau)\, d\tau \leq Bb(s), \quad s > \rho_b. \tag{4.8}$$

Suppose also that

$$\lim_{|x|\to\infty} \frac{a(x)|x|^{d-1}}{b(|x|)} = 0. \tag{4.9}$$

Then the inequality (4.1) *holds for $\omega(x) := b(|x|)$, $x \in \mathbb{R}^d$.*

Proof. The assumption (4.9) implies that

$$g(r) := \sup_{|x|\geq r} \frac{a(x)|x|^{d-1}}{\omega(x)} \to 0, \quad r \to \infty. \tag{4.10}$$

Take an arbitrary $\delta \in (0,1)$. By (4.10), one can take then $r = r(\delta) > r_b$ such that $g(r) < \delta$. Next, by Lemma 4.1, the inequality (4.4) holds for $c = \omega$. Therefore, there exists $\rho = \rho(\delta, r) = \rho(\delta) > \max\{r, \rho_b\}$, such that

$$\sup_{|y|\leq r} \frac{\omega(x-y)}{\omega(x)} < 1 + \delta, \quad |x| \geq \rho. \tag{4.11}$$

Then, by (4.10) and (4.11), we have

$$\begin{aligned}(a * \omega)(x) &= \omega(x) \int_{\{|y|\leq r\}} a(y) \frac{\omega(x-y)}{\omega(x)} dy + \omega(x) \int_{\{|y|\geq r\}} \frac{a(y)|y|^{d-1}}{\omega(y)} \frac{\omega(x-y)\omega(y)}{\omega(x)|y|^{d-1}} dy \\ &\leq \omega(x)(1+\delta) \int_{|y|\leq r} a(y) dy + g(r)\omega(x) \int_{\{|y|\geq r\}} \frac{b(|x-y|)b(|y|)}{b(|x|)|y|^{d-1}} dy,\end{aligned}$$

and using that b is decreasing on $\mathbb{R}_+$ and the inequality $|x-y| \geq \bigl||x|-|y|\bigr|$, one gets, cf. (2.17) and recall that $g(r) < \delta$,

$$\begin{aligned}&\leq \omega(x)(1+\delta) + \delta\omega(x) \int_{\{|y|\geq r\}} \frac{b\bigl(\bigl||x|-|y|\bigr|\bigr)b(|y|)}{b(|x|)|y|^{d-1}} dy, \\ &\leq \omega(x)(1+\delta) + \delta\omega(x)\sigma_d \int_r^{\infty} \frac{b\bigl(|x|-p\bigr)b(p)}{b(|x|)} dp, \end{aligned} \tag{4.12}$$

where σ_d is the hyper-surface area of a unit sphere in $\mathbb{R}^d$ (note that we have omitted an absolute value, as b is even). Finally, using that $r > r_b$ and $\rho > \rho_b$, we obtain from (4.8) and (4.12) that, for any $\delta \in (0,1)$,

$$(a * \omega)(x) \leq \omega(x)\bigl(1 + \delta(1 + \sigma_d B)\bigr), \quad |x| > \rho(\delta),$$

that implies the statement. □

Lemma 4.3. *Let $b \in \mathcal{S}_{\mathrm{reg},1}$ be an even function. Suppose that there exists $\alpha' \in (0,1)$ such that $b^{\alpha'} \in \widetilde{\mathcal{D}}_d$, i.e.*

$$\int_0^\infty b(s)^{\alpha'} s^{d-1}\, ds < \infty, \tag{4.13}$$

and, for any $\alpha \in (\alpha', 1)$,

$$\lim_{|x|\to\infty} \frac{a(x)}{b(|x|)^\alpha} |x|^{d-1} = 0. \tag{4.14}$$

Then there exists $\alpha_0 \in (\alpha', 1)$ such that the inequality (4.1) holds for $\omega(x) = b(|x|)^\alpha$, $x \in \mathbb{R}^d$ for all $\alpha \in (\alpha_0, 1)$.

Proof. We apply the second part of Theorem 3.1, for $n = 1$; note that then (4.13) implies (3.2). As a result, for any $\alpha \in (\alpha_0, 1)$, the inequality (3.6) holds; in particular, then (4.8) holds with b replaced by b^α. The latter together with (4.14) allows to apply Lemma 4.2 for b replaced by b^α, that fulfils the statement. □

Remark 4.3. Note that, by Remark 4.2, (4.13) implies that $b \in \widetilde{\mathcal{D}}_d$ and hence, cf. Definition 3.1, $b \in \mathcal{S}_{\mathrm{reg},d}$.

As a result, one gets a counterpart of Proposition 4.2, for the case when the function b^+ in (4.5) decays faster than polynomial and $d > 1$.

Proposition 4.3. *Let (4.5) hold for a function $b^+ \in \widetilde{\mathcal{D}}_d$ which is loq-equivalent to a function $b \in \mathcal{S}_{\mathrm{reg},1}$. For $d > 1$, we suppose, additionally, that*

$$\lim_{s\to\infty} b(s) s^\nu = 0 \qquad \textit{for all } \nu \geq 1. \tag{4.15}$$

Then there exists $\alpha_0 \in (0,1)$, such that, for all $\alpha \in (\alpha_0, 1)$, the function $\omega(x) = b(|x|)^\alpha$, $x \in \mathbb{R}^d$ satisfies (4.1).

Proof. We will use Lemma 4.3. For $d > 1$, one gets from (4.15) that, for any $\nu > 0$, there exists $\rho_\nu \geq 1$, such that $b(s) \leq s^{-\nu}$, $s > \rho_\nu$. In particular, for any $\alpha' \in (0,1)$, one has (4.13). For $d = 1$, $\sigma = 0$, we use instead that $b \in \mathcal{S}_{\mathrm{reg},1}$ implies (3.3), and hence we get (4.13), if only $\alpha' \in \bigl(\frac{1}{1+\delta}, 1\bigr)$.

Next, for any $d \in \mathbb{N}$, choose an arbitrary $\alpha \in (\alpha', 1)$. Then, by (4.5) and (3.13) applied for $b_1 = b$ and $b_2 = b^+$, we have that, for any $\epsilon \in (0, 1-\alpha)$, there exists $\rho_\epsilon > 0$, such that, for all $|x| > \rho_\varepsilon$,

$$\frac{a(x)}{b(|x|)^\alpha} |x|^{d-1} \leq b(|x|)^{1-\epsilon-\alpha} |x|^{d-1} = \Bigl(b(|x|)|x|^\nu\Bigr)^{1-\epsilon-\alpha}, \tag{4.16}$$

where $\nu = \frac{d-1}{1-\epsilon-\alpha} \geq 0$, as $\alpha < 1 - \epsilon$. Clearly, (4.16) together with (4.15), in the case $d > 1$, imply (4.14), that fulfills the statement. □

Remark 4.4. Note that, in Proposition 4.2, for the function b given by (4.6), one can choose $\alpha' \in (0,1)$ such that (4.13) holds. The same property we have checked above for the function b which satisfies assumptions of Proposition 4.3. As a result, by Remark 4.2, the functions $\omega(x) = b(|x|)^\alpha$, $x \in \mathbb{R}^d$ in these Propositions are integrable for all $\alpha \in (\alpha_0, 1)$.

Definition 4.2. Let the set $\widetilde{\mathcal{S}}_{\mathrm{reg},d} \subset \mathcal{S}_{\mathrm{reg},d}$, $d \in \mathbb{N}$ be defined as follows. Let $\widetilde{\mathcal{S}}_{\mathrm{reg},1}$ be just the set $\mathcal{S}_{\mathrm{reg},1}$, whereas, for $d > 1$, let $\widetilde{\mathcal{S}}_{\mathrm{reg},d}$ be the set of all functions $b \in \mathcal{S}_{\mathrm{reg},d}$, such that b is either given by (4.6) for some $M, \mu > 0$ or b satisfies (4.15).

Remark 4.5. All functions in Classes 2–4 in Subsection 3.2 evidently satisfy (4.15) and hence belong to $\widetilde{\mathcal{S}}_{\mathrm{reg},d}$.

Theorem 4.1. *Let* (4.5) *hold with* $b^+ \in \widetilde{\mathcal{D}}_d$ *which is log-equivalent to a function* $b \in \widetilde{\mathcal{S}}_{\mathrm{reg},d}$. *Then there exists* $\alpha_0 \in (0,1)$, *such that, for any* $\delta \in (0,1)$ *and* $\alpha \in (\alpha_0, 1)$, *there exist* $c_1 = c_1(\delta,\alpha) > 0$ *and* $\lambda = \lambda(\delta,\alpha) \in (0,1)$, *such that*

$$a^{*n}(x) \leq c_1(1+\delta)^n \min\bigl\{\lambda, b(|x|)^\alpha\bigr\}, \quad x \in \mathbb{R}^d.$$

In particular, for some $c_2 = c_2(\delta,\alpha) > 0$ *and* $s_\alpha = s_\alpha(\delta) > 0$,

$$a^{*n}(x) \leq c_2(1+\delta)^n b(|x|)^\alpha, \quad |x| > s_\alpha, n \in \mathbb{N}. \tag{4.17}$$

Proof. Combining Proposition 4.2 and 4.3 with Proposition 4.1, we get by (4.2) and (4.3) that there exist $\tilde{c_\delta} = \tilde{c_\delta}(\omega)$, $\lambda = \lambda(\delta,\alpha) \in (0,1)$, where $\omega(x) = b(|x|)^\alpha$, $x \in \mathbb{R}^d$, such that

$$a^{*n}(x) \leq \tilde{c}_\delta(1+\delta)^{n-1} \min\bigl\{\lambda, b(|x|)^\alpha\bigr\}, \quad x \in \mathbb{R}^d, \tag{4.18}$$

that evidently yields (4.18). Since b is tail-decreasing, we have that, for some $s_\alpha > 0$, $b(|x|)^\alpha < \lambda$ for $|x| > s_\alpha$. This implies (4.17). $\square$

Corollary 4.1. *Let* $a(x) = b(|x|)$, $x \in \mathbb{R}$ *for some* $b \in \widetilde{\mathcal{S}}_{\mathrm{reg},d}$. *Then there exists* $\alpha_0 \in (0,1)$, *such that, for any* $\delta \in (0,1)$ *and* $\alpha \in (\alpha_0,1)$, *there exist* $c_{\delta,\alpha} > 0$ *and* $s_\alpha = s_\alpha(\delta) > 0$, *such that, for all ,*

$$a^{*n}(x) \leq c_{\delta,\alpha}(1+\delta)^n a(x)^\alpha, \quad |x| > s_\alpha, n \in \mathbb{N}.$$

Proof. Since, for some $\rho > 0$, b is decreasing on (ρ,∞) and (3.2) holds, there exists $b^+ \in \widetilde{\mathcal{D}}_d$, such that $b^+(s) = b(s)$, $s > \rho$ and $b^+(s) \geq b(s)$, $s \in [0,\rho]$. Then one can apply Theorem 4.1. $\square$

Appendix: Non-local heat equation

We can apply the obtained results to the study of the regular part of the fundamental solution to the non-local heat equation

$$\frac{\partial}{\partial t} u(x,t) = \varkappa \int_{\mathbb{R}^d} a(x-y)\bigl(u(y,t) - u(x,t)\bigr)\, dy, \quad x \in \mathbb{R}^d, \tag{A.1}$$

where $\varkappa > 0$ and $0 \leq a \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ is normalized, i.e. $\int_{\mathbb{R}^d} a(x)\, dx = 1$; see e.g. [1, 5, 16]. Consider an initial condition $u(x,0) = u_0(x)$, $x \in \mathbb{R}^d$ to (A.1) with u_0 from a space E of bounded on $\mathbb{R}^d$ functions. Since the operator $Au = \varkappa a * u - \varkappa u$ in the right hand side of (A.1) is bounded on E, the unique solution to (A.1) is given by

$$u(x,t) = e^{-\varkappa t}\bigl((\delta_0 + \phi_\varkappa(t)) * u_0\bigr)(x), \tag{A.2}$$

where δ_0 is the Dirac delta at $0 \in \mathbb{R}^d$ and

$$\phi_\varkappa(x,t) := \sum_{n=1}^\infty \frac{\varkappa^n t^n}{n!} a^{*n}(x), \quad x \in \mathbb{R}^d, t \geq 0. \tag{A.3}$$

Note that it was shown in [6, Lemma 2.2], that if a is a rapidly decreasing smooth function, then ϕ_k is indeed the solution to (A.1) with $u_0 = \delta_0$.

Now, for $d = 1$, suppose that $a(x) = b(x)$, $x \in \mathbb{R}^1$, and b satisfies the conditions of Theorem 2.2. Then, by (2.24), the series in (A.3) converges uniformly on finite time intervals for each $x > s_0$, and therefore, by (2.16),

$$\phi_\varkappa(x,t) \sim kte^{\varkappa t}a(x), \quad x \to \infty, \ t > 0.$$

For $d > 1$, let a and b satisfy the conditions of Theorem 4.1. Then, for each $\delta > 0$ and for each $\alpha < 1$ close enough to 1,

$$\phi_\varkappa(x,t) \leq c_{\delta,\alpha}\big(e^{\varkappa t(1+\delta)} - 1\big)b(|x|)^\alpha, \quad |x| > s_\alpha, \ t > 0 \tag{A.4}$$

for some $c_{\delta,\alpha} > 0$ and $s_\alpha = s_\alpha(\delta) > 0$. In particular, if a is radially symmetric and the conditions of Corollary 4.1 hold, then one can replace $b(|x|)$ on $a(x)$ in (A.4).

Moreover, combining (4.18) with (A.2), one can get an estimate for the solution u to (A.1) as well. The further analysis of solutions to (A.1) can be found in [11, 12].

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