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Deriving three-dimensional bosonization and the duality web

Horatiu Nastase [a], Carlos Núñez [b,*]

[a] Instituto de Física Teórica, UNESP-Universidade Estadual Paulista, R. Dr. Bento T. Ferraz 271, Bl. II, Sao Paulo 01140-070, SP, Brazil
[b] Department of Physics, Swansea University, Swansea SA2 8PP, United Kingdom

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ABSTRACT

Recently, a duality web for three dimensional theories with Chern–Simons terms was proposed. This can be derived from a single bosonization type duality, for which various supporting arguments (but not a proof) were given. Here we explicitly derive this bosonization, in the Abelian case and for a particular regime of parameters. To do this, we use the particle-vortex duality in combination with a Buscher-like duality (both considered in the regime of low energies). As a corollary, Son's conjectured duality is derived in a somewhat singular limit of vanishing mass.

1. Introduction

Duality symmetries are powerful tools that serve to constrain and understand non-perturbative physics. In $(2+1)$-dimensions, within the context of condensed matter systems, dualities have received less attention in comparison to their $(3+1)$-dimensional counterparts, that naturally appear in particle physics. Recently however, partly motivated by the desire to understand Son's conjecture [1] a web of dualities was proposed [2–4].

In fact, D.T. Son has proposed a relation between a massless Dirac 'fundamental' fermion and a 'composite' Dirac fermion coupled to a gauge field with BF-dynamics [1]. The 'fundamental' fermion is to be understood as a boundary mode in a topological insulator, while the 'composite' one should be thought of as an effective description for a half-filled lowest Landau level of a Fermi liquid [1], [5]. The whole idea is driven by the field theoretical descriptions of a (time reversal invariant) topological insulator and a topological superconductor.

The web of dualities mentioned above relates various bosonic theories (for scalars and gauge fields) with fermionic theories (coupled to a vector field), both with Chern–Simons terms. All fields transform under a $U(1)$ gauge symmetry. Extensions, including to non-Abelian cases have been considered in [6–10].

The web of dualities can be derived by assuming the validity of a basic correspondence between a bosonic theory and a fermionic one, which in the rest of this paper will be referred to as *three-dimensional bosonization*. These ideas were considered and extended to the context of supersymmetric theories by Aharony [11].

On the other hand, an explicit mapping between a bosonic and a fermionic theory was presented around twenty years ago in [12], [13,14]. It is based on the realisation that two-dimensional bosonization can be viewed as a Buscher-like duality and the extension of such procedure to three dimensions. We will refer to it as the *Burgess–Quevedo map* (or BQ-map). See the paper [15] for a careful account of the idea and technical details of the BQ-map.

Postulating the validity of the three-dimensional bosonization duality, one can derive the (bosonic) particle-vortex duality, or the fermionic duality conjectured by Son. Repeated application leads to a full duality web. While this basic three-dimensional bosonization was not proven, evidence indicative of its correctness was presented in [3,4] and subsequent papers, e.g. [16].

In this note, we will derive this basic three-dimensional bosonization conjecture as presented in [4,7]. To do this, we will assume the validity of the particle-vortex duality and combine it with the Buscher-like BQ-map [12], [13], [14]. In fact, a regime of sufficiently low energies, with special field configurations in the particle-vortex equivalence, together with a BQ-map improved by the presence of point like vortices in the system, are instrumental to our derivation.

Our approach will be phrased in the framework of the path integral formalism, as defined, for example, in [17], which is itself based on the earlier work [18] (see also [19,20] for an alternative viewpoint and [21] for the usual condensed matter formulation). The particle-vortex duality was discussed in works on supercon-

* Corresponding author.
E-mail addresses: nastase@ift.unesp.br (H. Nastase), c.nunez@swansea.ac.uk (C. Núñez).

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ductivity [22], [23,24], and also in the contexts of anyon superconductivity and the fractional quantum Hall effect [25].

This work is organized as follows. In Section 2, we summarize and streamline the background material needed for our purposes: the three-dimensional bosonization proposal, time reversal, Son's duality and the particle vortex duality. In Section 3 we derive the conjectured three dimensional bosonization, assuming the validity of the particle-vortex duality and the BQ-map. Section 4 closes the paper with final conclusions.

2. Three-dimensional bosonization and the duality web

As a warm-up, in this section we will review how (part of) the Abelian duality web is derived. We will also discuss the action of time reversal on the different dualities and go over the derivation in [3,4] of Son's conjectured relation [1]. Finally, the particle-vortex duality will be shown to arise from alternate integrations on a 'master' partition function that depends on both 'particle' and 'vortex' fields.

A main basic ingredient in this work is the three dimensional bosonization that we now review, adopting the notation in [4]. The partition functions for a complex scalar $\phi=\phi_0 e^{i\theta}$ coupled to a vector A_μ (adding 'flux'), and that for a Dirac fermion ψ (both in the presence of a vectorial external source S_μ) are

$$\begin{aligned}\tilde{Z}_{\rm scalar+flux}[S] &\equiv \int \mathcal{D}\phi\mathcal{D}\phi^*\mathcal{D}A_\mu e^{iS_{\rm scalar}[\phi,A]+iS_{\rm CS}[A]+iS_{\rm BF}[A,S]}\\ &= \int \mathcal{D}\phi_0\mathcal{D}\theta\mathcal{D}A_\mu\\ &\qquad e^{iS_{\rm scalar}[\theta,A;\phi_0]-\frac{1}{2}\int d^3x(\partial_\mu\phi_0)^2+iS_{\rm CS}[A]+iS_{\rm BF}[A,S]}\\ Z_{\rm fermion}[S;m] &\equiv \int \mathcal{D}\psi\mathcal{D}\bar{\psi}e^{i\int\bar{\psi}(\partial\!\!\!/+m+S\!\!\!/)\psi}. \end{aligned} \tag{2.1}$$

We have denoted,

$$\begin{aligned} S_{\rm CS}[A] &\equiv \frac{1}{4\pi}\int d^3x\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho,\\ S_{\rm BF}[A,S] &\equiv \frac{1}{2\pi}\int d^3x\epsilon^{\mu\nu\rho}A_\mu\partial_\nu S_\rho. \end{aligned} \tag{2.2}$$

The action for the complex scalar ϕ is defined and can be rewritten according to,

$$\begin{aligned} S_{\rm scalar}[\phi,A] &\equiv -\frac{1}{2}\int d^3x|(\partial_\mu - iA_\mu)\phi|^2 \to S_{\rm scalar}[\theta,A;\phi_0]\\ &\equiv -\frac{1}{2}\int d^3x\phi_0^2(\partial_\mu\theta+A_\mu)^2. \end{aligned} \tag{2.3}$$

Note that the scalar action $S_{\rm scalar}[\theta,A;\phi_0]$ in the last expression of eq. (2.3) appears for the case in which the modulus ϕ_0 is constrained to be constant. Such an action is obtained from that of a complex scalar ϕ with a symmetry breaking Higgs-like potential,

$$\begin{aligned} S_{\rm scalar}[\theta,A;\phi_0] &= \lim_{\alpha\to\infty} S_{\rm scalar}[\theta,A;\phi_0]\\ &\quad -\int d^3x\left[\frac{1}{2}(\partial_\mu\phi_0)^2+\frac{\alpha}{2}(\phi_0^2-m)^2\right]\\ &= \lim_{\alpha\to\infty} S_{\rm scalar}[\phi,A] - \int d^3x\frac{\alpha}{2}(\phi_0^2-m)^2, \end{aligned} \tag{2.4}$$

with the coupling α taken to be very large, $\alpha\to\infty$. Equivalently, for low energies $E\ll\alpha$, the quantity ϕ_0 takes a constant value. In most of the analysis below we will consider ϕ_0 to be fixed, $\phi_0=\sqrt{m}$, and we will drop $\int\mathcal{D}\phi_0$ from the path integral.

Then, the basic three-dimensional bosonization duality, relates a fermion coupled to a background vectorial current with a complex scalar plus flux, considered in general with a fluctuating ϕ_0 (hence the tilde on $Z_{\rm scalar+flux}$). More explicitly,

$$Z_{\rm fermion}[S;m=0]e^{-\frac{i}{2}S_{\rm CS}[S]} = \tilde{Z}_{\rm scalar+flux}[S]\,. \tag{2.5}$$

In the paper [7], the authors proposed a more general duality for the bosonization of a massive fermion (of mass m). This extended relation that leads to a more general web of dualities reads

$$Z_{\rm fermion}[S;m]e^{-\frac{i}{2}S_{\rm CS}[S]} = Z_{\rm scalar+flux}[S]. \tag{2.6}$$

$$\begin{aligned} Z_{\rm scalar+flux}[S] &= \lim_{\alpha\to\infty,E\ll\alpha}\int\mathcal{D}A_\mu\mathcal{D}\phi_0\mathcal{D}\theta\mathcal{D}\sigma\\ &\qquad e^{iS_{\rm scalar}(\theta,A;\phi_0)+iS_{\rm CS}[A]+iS_{\rm BF}[A,S]-i\int d^3x\left[\frac{1}{2}(\partial_\mu\phi_0)^2+\sigma(\phi_0^2-m)+\frac{\sigma^2}{2\alpha}\right]}. \end{aligned}$$

In the case of vanishing mass ($m=0$), integrating out the non-dynamical field σ we generate a potential $V=\phi_0^4/2\alpha$, which leads to the Wilson–Fisher fixed point at low energies. If $m>0$, integrating out σ we get the Higgs-like potential in eq. (2.4). At small enough energies $E\ll m=\phi_0^2$, $E\ll\alpha$, the dynamical field ϕ_0 freezes-out, leaving us simply with $Z_{\rm scalar+flux}[S]$ on the right hand side (notice that the integration in ϕ_0 is trivial, hence the absence of tilde in $Z_{\rm scalar+flux}[S]$). More explicitly, at low energies and after the constraint is imposed, we have

$$Z_{\rm scalar+flux}[S] = \int\mathcal{D}A_\mu\mathcal{D}\theta e^{iS_{\rm scalar}(\theta,A;\phi_0)+iS_{\rm CS}[A]+iS_{\rm BF}[A,S]}. \tag{2.7}$$

In the following we will consider the situation in which the constraint $\phi_0^2=m$ is enforced by the integration over the field σ, in the limit of low energies. More precisely, we will probe the dynamics with energies that are very small compared to those set by the two relevant scales, m and α.

2.1. Time-reversed relation

Another ingredient needed to prove different entries of the duality web comes from considering the effect of time reversal on the system. Time reversal invariance leads to relations, which change the sign of the Chern–Simons and BF terms. Indeed, we also have the duality,

$$\begin{aligned} Z_{\rm fermion}[S]e^{+\frac{i}{2}S_{\rm CS}[S]} &= \bar{\tilde{Z}}_{\rm scalar+flux}[S]\\ &\equiv \int\mathcal{D}\phi\mathcal{D}\phi^*\mathcal{D}A_\mu e^{iS_{\rm scalar}[\phi,A]-iS_{\rm CS}[A]-iS_{\rm BF}[A,S]}. \end{aligned} \tag{2.8}$$

The bosonic and fermionic particle-vortex dualities are obtained by applying and manipulating the three-dimensional bosonization relation in eq. (2.5), and using then the time-reversed bosonization relation above.

2.2. Son's duality from bosonization

As an example, we derive Son's conjectured duality between a massless Dirac fermion ψ coupled to an external field S_μ and a composite Dirac fermion χ, coupled to a dynamical field A_μ, which itself couples to the external S_μ through a BF coupling, denoted BF-QED. In what follows, we summarise a derivation in [3], [4]. Indeed, the dynamics of the composite fermion χ and the vector A_μ is described by

$$Z_{\rm BF-QED}[S;m] = \int\mathcal{D}A_\mu\mathcal{D}\chi\mathcal{D}\bar{\chi}e^{i\int\bar{\chi}(\partial\!\!\!/+m+A\!\!\!/)\chi+\frac{i}{2}S_{\rm BF}[A,S]}. \tag{2.9}$$

Son conjectured a duality between the composite, low energy, massless BF-QED theory (set $m=0$ in the above $Z_{\text{BF}-\text{QED}}$) and a massless Dirac fermion theory, both coupled to an external source S_μ,

$$Z_{\text{BF}-\text{QED}}[S] = Z_{\text{fermion}}[S]\,. \tag{2.10}$$

To derive eq. (2.10), one starts from eq. (2.5), changing the notation as $S_\mu \to \bar{A}_\mu$, adds a BF term $\frac{i}{2}S_{\text{BF}}[\bar{A},S]$ (where now S_μ is a new external field) on both sides. Takes the $e^{-\frac{i}{2}S_{\text{CS}}[S]}$ to the other side, and then integrates over $\bar{A}_\mu$ (formerly, the external field). Then the left hand side becomes $Z_{\text{BF}-\text{QED}}[S]$, while the right hand side turns into

$$\tilde{Z}_{\text{scalar+fluxes}}[S] = \int \mathcal{D}\phi\mathcal{D}\phi^*\mathcal{D}A_\mu\mathcal{D}\bar{A}_\mu \; e^{iS_{\text{scalar}}[\phi,A]+iS_{\text{CS}}[A]+iS_{\text{BF}}[A,\bar{A}]+\frac{i}{2}S_{\text{BF}}[\bar{A},S]+\frac{i}{2}S_{\text{CS}}[\bar{A}]}. \tag{2.11}$$

Performing the integration over $\bar{A}_\mu$ (which appears algebraically), we find the equation of motion $d\bar{A} = -(dS + 2dA)$. Finally replacing $\bar{A}_\mu = -(S_\mu + 2A_\mu)$ back in the scalar partition function of eq. (2.11), we find

$$Z_{\text{BF}-\text{QED}}[S] = \int \mathcal{D}\phi\mathcal{D}\phi^*\mathcal{D}A_\mu \; e^{iS_{\text{scalar}}[\phi,A]-iS_{\text{CS}}[A]-iS_{\text{BF}}[A,S]-\frac{i}{2}S_{\text{CS}}[S]}\,, \tag{2.12}$$

which because of eq. (2.8) (the *time-reversed* form of the basic bosonization duality) equals $Z_{\text{fermion}}[S]$. The final result is Son's relation in eq. (2.10).

A new result can be obtained if we start from the three-dimensional bosonization proposal in eq. (2.6), follow exactly the same procedure described above and derive a Son-like relation between a fundamental and a composite Dirac fermions, both with the same mass m,

$$Z_{\text{BF}-\text{QED}}[S;m] = Z_{\text{fermion}}[S;m]. \tag{2.13}$$

In Section 3, we will put this last correspondence on a firmer basis, by proving the equivalence in eqs. (2.6)–(2.7). Let us now revisit another important duality.

2.3. Review of the particle-vortex duality

Another ingredient needed in our derivation of Section 3, is a specific form of a particle-vortex duality. In the paper [17], a transformation was proposed that realizes a particle-vortex duality as an equivalence of partition functions. Getting rid of some unnecessary (for our purposes) extra ingredients, the two partition functions that are shown to be equivalent are

$$\begin{aligned} Z_{\text{particle}} &= \int \mathcal{D}\theta e^{iS} \\ &\equiv \int \mathcal{D}\theta \exp\Big[-i\int d^3x \frac{1}{2}\big[(\partial_\mu\phi_0)^2 + \phi_0^2(\partial_\mu\theta_{\text{smooth}} \\ &\quad + \partial_\mu\theta_{\text{vortex}} + A_\mu)^2\big]\Big]\,, \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} Z_{\text{vortex}} &= \int \mathcal{D}\lambda_\mu e^{iS_{\text{dual}}} = \int \mathcal{D}\lambda_\mu \exp\Big[-i\int d^3x\Big[\frac{1}{2}(\partial_\mu\phi_0)^2 \\ &\quad + \frac{1}{4(2\pi\phi_0)^2} f^{(\lambda)}_{\mu\nu} f^{(\lambda)\mu\nu} + \frac{1}{2\pi}\epsilon^{\mu\nu\rho}\lambda_\mu\partial_\nu A_\rho \\ &\quad + j^\mu_{\text{vortex}}\lambda_\mu\Big]\Big]\,. \end{aligned} \tag{2.15}$$

Let us clarify the different terms in these expressions. The expression in eq. (2.14) is written in terms of a dynamical field θ and two external ones A_μ and ϕ_0. We shall separate θ into a dynamical smooth part θ_{smooth} and a nondynamical vortex part θ_{vortex} that contains singularities (at $r=r_v$), i.e. $\int_0^{2\pi} d\alpha\,\partial_\alpha\theta_{\text{vortex}} = 2\pi N$, where α is the polar angle in 2 spatial dimensions, measured with respect to the positions $r=r_v$ of vortices. Thus the integral $\mathcal{D}\theta$ splits into $\int\mathcal{D}\theta_{\text{smooth}}$ times a sum over the nontrivial vortex numbers $\sum_N$ for the various θ_{vortex} sectors. More explicitly $\int\mathcal{D}\theta = \sum_N\int\mathcal{D}\theta_{\text{smooth}}$.

On the other hand, the partition function in eq. (2.15) is written in terms of a dynamical vector λ_μ, with external sources A_μ and ϕ_0. We have defined $f^{(\lambda)}_{\mu\nu} \equiv \partial_\mu\lambda_\nu - \partial_\nu\lambda_\mu$ and the vortex current,

$$j^\mu_{\text{vortex}} \equiv \frac{1}{2\pi}\epsilon^{\mu\nu\rho}\partial_\nu\tau_{\rho,\text{vortex}} = \frac{1}{2\pi}\epsilon^{\mu\nu\rho}\partial_\nu\partial_\rho\theta_{\text{vortex}}. \tag{2.16}$$

Let us now show the equivalence,

$$Z_{\text{particle}} = Z_{\text{vortex}}.$$

In order to do this, we use the usual trick of constructing a master partition function (dependent on two variables), that reduces either to Z_{particle}, or the dual vortex one Z_{vortex}, upon alternate integration-out of one or the other variable.

To construct such master path integral, first replace $\partial_\mu\theta = \partial_\mu\theta_{\text{smooth}} + \partial_\mu\theta_{\text{vortex}}$ with an independent variable $\tau_\mu = \tau_{\mu,\text{smooth}} + \tau_{\mu,\text{vortex}}$, and then impose the flatness of the smooth part's curvature by $\epsilon^{\mu\nu\rho}\partial_\nu\tau_{\rho,\text{smooth}} = 0$, with Lagrange multiplier λ_μ. We then obtain the master partition function,

$$\begin{aligned} Z_{\text{master}} &= \int \mathcal{D}\tau_\mu\mathcal{D}\lambda_\mu e^{iS_{\text{master}}} \\ &\equiv \int \mathcal{D}\tau_\mu\mathcal{D}\lambda_\mu \exp\Big[i\int d^3x\Big\{-\frac{1}{2}(\partial_\mu\phi_0)^2 \\ &\quad -\frac{1}{2}\phi_0^2(\tau_{\mu,\text{smooth}} + \tau_{\mu,\text{vortex}} + A_\mu)^2 \\ &\quad + \frac{1}{2\pi}\epsilon^{\mu\nu\rho}\lambda_\mu\partial_\nu\tau_{\rho,\text{smooth}}\Big\}\Big]\,, \end{aligned} \tag{2.17}$$

where again $\int\mathcal{D}\tau$ is understood as $\sum_N\int\mathcal{D}\tau_{\mu,\text{smooth}}$.

If we solve for the Lagrange multiplier λ_μ (and integrate it out), we obtain $\tau_{\mu,\text{smooth}} = \partial_\mu\theta_{\text{smooth}}$. Substituting it back into eq. (2.17), we get back to the original particle path integral Z_{particle} in eq. (2.14). On the other hand, if we integrate out the field τ_μ, we find the equation of motion,

$$(\tau_\mu + A_\mu)\phi_0^2 = \frac{1}{2\pi}\epsilon^{\mu\nu\rho}\partial_\nu\lambda_\rho\,. \tag{2.18}$$

By replacing this in eq. (2.17), we obtain the dual path integral, for the Lagrange multipliers λ_μ, as in eq. (2.15).

We have then proven, at the level of partition functions, the particle-vortex duality or equivalence between eqs. (2.14) and (2.15). We will now use the results summarized in this section to prove the three-dimensional bosonization in eqs. (2.5)–(2.7).

3. Proof of the three-dimensional bosonization duality

In this section we provide a proof of the basic three-dimensional bosonization duality, in its mass deformed version, as written in eqs. (2.6)–(2.7).

We first discuss the Burgess–Quevedo map (BQ-map). This should be thought of as a bosonization relation that can be derived in a self-consistent manner [12–14], see [15] for a careful account of the BQ-map. We are interested in the formulation presented in the paper [13], that proceeds by explicitly integrating out *massive* fermions *at low energies* (the energies are E much smaller than the mass m) in the presence of a vector field background. In fact, we approximate the fermionic determinant by calculating a fermion-loop with only two external vector insertions. On top of this, we approximate this result for the case of large masses (see the paper [26] for details). Both approximations are well-justified in a $\frac{k}{m}$-expansion. After various algebraic manipulations described in [15], one obtains the approximate relation,

$$Z_{\text{fermion}}[S;m] = Z_{\text{gauge}}[S] = \int \mathcal{D}\lambda_\mu e^{-i\left[\frac{1}{2k_3}\epsilon^{\mu\nu\rho}\lambda_\mu\partial_\nu\lambda_\rho + \epsilon^{\mu\nu\rho}\lambda_\mu\partial_\nu S_\rho\right]}\,, \tag{3.1}$$

where $k_3 = sign(m)/(4\pi)$.[1] Defining $\tilde{\lambda}_\mu = 2\pi\lambda_\mu$, for $m > 0$, this becomes[2]

$$Z_{\text{fermion}}[S;m] = Z_{\text{gauge}}[S] = \int \mathcal{D}\tilde{\lambda}_\mu e^{-2iS_{\text{CS}}[\tilde{\lambda}] - iS_{\text{BF}}[\tilde{\lambda},S]}\,. \tag{3.2}$$

It is nice to notice that we can supplement the BQ-map in eq. (3.2), extending it to the situation in which the system is in the presence of topological objects, like the singular vortices of the previous section. Indeed, representing these vortices by a multiple-valued angle θ_{vortex}, the associated current j^μ_{vortex} as defined in eq. (2.16), and replacing $S_\mu \to S_\mu + \partial_\mu\theta_{\text{vortex}}$ in the BQ-map of eq. (3.2), we find

$$\begin{aligned}
Z_{\text{fermion+vortex}}[S;m] &= \int \mathcal{D}\psi\mathcal{D}\bar{\psi} e^{i\int\bar{\psi}(\partial\!\!\!/ + m + S\!\!\!/ + \partial\!\!\!/\theta_{\text{vortex}})\psi}\,,\\
Z_{\text{gauge+vortex}}[S] &= \int \mathcal{D}\tilde{\lambda}_\mu e^{-2iS_{\text{CS}}[\tilde{\lambda}] - iS_{\text{BF}}[\tilde{\lambda},S] - i\int d^3x j^\mu_{\text{vortex}}\tilde{\lambda}_\mu}\,,\\
Z_{\text{fermion+vortex}}[S;m] &= Z_{\text{gauge+vortex}}[S].
\end{aligned} \tag{3.3}$$

The last equality in eq. (3.3) is valid, as discussed above, in the regime of low energies (or large mass, in a $\frac{k}{m}$-expansion).

We will now use the BQ-map in the versions discussed above, together with the particle-vortex duality derived in the previous section, to prove the three dimensional bosonization duality in eqs. (2.5)–(2.7).

We proceed as follows: first we set $\phi_0 =$ constant on both sides of particle-vortex duality, eqs. (2.14)–(2.15). This implies that the corresponding vortices are point-like and singular. Then, we consider the situation in which we probe the system with very small energies, specifically $E \ll \phi_0^2$, so that we can neglect the Maxwell kinetic term in comparison with the BF kinetic term in eq. (2.15). In this situation, the dynamics consists of point-like vortices coupled to a Chern–Simons gauge field.

Now, we add $S_{\text{CS}}[A] + S_{\text{BF}}[A;S]$ to the actions in both path integrals (which adds 'flux' to both sides) and integrate over A_μ as well. We obtain the equality of the modified particle path integrals for the two systems, one with particles and flux and the other with vortices and flux. On the particle with flux side we have (note that we change $\lambda_\mu \to -\lambda_\mu$ in the path integral),

$$\begin{aligned}
Z'_{\text{particle+flux}}[S] &= \int \mathcal{D}A_\mu\mathcal{D}\theta e^{[iS_{\text{scalar}}[\theta,A;\phi_0] + iS_{\text{CS}}[A] + iS_{\text{BF}}[A;S]]}\\
&= Z_{\text{scalar+flux}}[S]\,,
\end{aligned} \tag{3.4}$$

which as we can see is equal to the scalar+flux path integral in the bosonization relation of eqs. (2.5)–(2.7). Notice that although ϕ_0 appears here, it is not a true parameter. Indeed, by rescaling the dimensionless θ by ϕ_0 we simply construct a scalar with the canonical dimension. In other words, ϕ_0 simply defines units.

On the other hand, on the vortex side of the duality, we are left with a modified vortex path integral,

$$Z'_{\text{vortex+flux}}[S] = \int \mathcal{D}A_\mu\mathcal{D}\lambda_\mu\, e^{iS_{\text{BF}}[\lambda;A] + iS_{\text{BF}}[A;S] + iS_{\text{CS}}[A] + i\int d^3x j^\mu_{\text{vortex}}\lambda_\mu}\,. \tag{3.5}$$

Evaluating the integral over A_μ, we obtain the equation of motion,

$$dA = -dS - d\lambda\,, \tag{3.6}$$

which when substituted back into the path integral of eq. (3.5) gives,

$$Z'_{\text{vortex+flux}}[S] = \int \mathcal{D}\lambda_\mu e^{-iS_{\text{CS}}[S] - iS_{\text{CS}}[\lambda] - iS_{\text{BF}}[S,\lambda] + i\int d^3x j^\mu_{\text{vortex}}\lambda_\mu}\,. \tag{3.7}$$

Now, we redefine the path integral variable (with trivial Jacobian) by

$$\lambda_\mu = \sqrt{2}\tilde{\lambda}_\mu + S_\mu\left(-1 + \frac{1}{\sqrt{2}}\right)\,, \tag{3.8}$$

to finally obtain

$$Z'_{\text{vortex+flux}}[S] = Z'_{\text{gauge+vortex}}[S] e^{-\frac{i}{2}S_{\text{CS}}[S] - i\int d^3x j^\mu_{\text{vortex}}S_\mu\left(-1+\frac{1}{\sqrt{2}}\right)}\,. \tag{3.9}$$

Two comments are in order. First, we have identified the gauge path integral from the BQ-map [13] in the presence of non-trivial topology, with the explicit insertion of the vortex current, as in eq. (3.3). Note however that, since the vortex current multiplies λ_μ, and not $\tilde{\lambda}_\mu$, we obtain an extra term coupling it to S_μ, and also we get a $\sqrt{2}$ factor in the $j^\nu_\mu\tilde{\lambda}^\mu$ term of eq. (3.3), which is why we have put a prime on $Z_{\text{gauge+vortex}}$. This will translate into the same factors on the fermion-vortex side. Second, related to eq. (3.8), we see that if the field λ_μ has quantized flux across a two sphere, the flux of $\tilde{\lambda}_\mu$ will not be quantized. This is a shortcoming of the change in eq. (3.8).

Then combining the various expressions in eqs. (3.3), (3.4) and (3.9), we obtain a chain of equalities,

$$\begin{aligned}
Z_{\text{scalar+flux}}[S] &= Z'_{\text{particle+flux}}[S] = Z'_{\text{vortex+flux}}[S]\\
&= Z'_{\text{gauge+vortex}}[S] e^{-\frac{i}{2}S_{\text{CS}}[S] - i\int d^3x j^\mu_{\text{vortex}}S_\mu\left(-1+\frac{1}{\sqrt{2}}\right)}\\
&= Z'_{\text{fermion+vortex}}[S] e^{-\frac{i}{2}S_{\text{CS}}[S] - i\int d^3x j^\mu_{\text{vortex}}S_\mu\left(-1+\frac{1}{\sqrt{2}}\right)}\,.
\end{aligned} \tag{3.10}$$

[1] Note that [13] has a different coefficient, but the original article [27] quoted there implies this value. Indeed, integrating over λ_μ we get $S_{\text{eff}} = \frac{k_3}{2}\int d^3x\epsilon^{\mu\nu\rho}S_\mu\partial_\nu S_\rho$, and eq. (4.26) of the paper [27] gives the coefficient as $k_3/2 = sign(m)e^2/(8\pi)$. In this sense, the parameter m is also present in $Z_{\text{gauge}}[S]$.

[2] Note that in the work [26] a generic expression for the fermionic determinant and the BQ-map was obtained. The limit $m \to \infty$ of their expression, gives the same path integral quoted in eq. (3.2), after a suitable rescaling in eq. (22) of [26].

Focusing our attention on the first and last terms of the chain of equalities in eq. (3.10), we find that this is the three-dimensional bosonization duality (2.6)–(2.7) that we wanted to prove (in the presence of non-trivial topology).

Our derivation needs the addition of the vortex coupling $-\sqrt{2}\bar{\psi}\gamma^\mu\psi\partial_\mu\theta_{\text{vortex}}$ on the fermionic side—the term $\not{\partial}\theta_{\text{vortex}}$ eq. (3.3), multiplied by $\sqrt{2}$. On the scalar side, we have an integration over the full θ variable, which can be split into an integral over θ_{smooth} and a sum over θ_{vortex} sectors. Hence the presence of topology (nontrivial θ_{vortex}). Both the fermionic and bosonic sides of the three-dimensional bosonization duality are more general than initially assumed and were both supplemented by the presence of non-trivial topology. One may consider (after the derivation is complete) the limit $\theta_{\text{vortex}} = 0$, to get back to the situation with no topology in eqs. (2.6)–(2.7). This completes our derivation. Once again, we emphasize that the result has only been obtained for small energies $E \ll m = \phi_0^2$, $E \ll \alpha$.

4. Discussion and conclusions

We set out to prove the basic three-dimensional bosonization relation in eq. (2.5), or its generalization by a mass in eqs. (2.6)–(2.7), which are at the basis of the derivation of the original web of dualities. Indeed, once the validity of eq. (2.5) is assumed, it can be used to prove Son's conjecture [1]. We have extended Son's relation to the case in which both fundamental and composite fermions are massive (with the same mass). This version of the conjecture has been put on a firmer basis in our work.

We used a combination of the Burgess–Quevedo map, which is a Buscher-like correspondence between bosonic and fermionic theories [13] and assumed the validity of the particle-vortex duality as defined in [17]. With this, we have shown that eqs. (2.5)–(2.7) hold *at low energies* $E \ll m = \phi_0^2$, $E \ll \alpha$, with the addition of a vortex current term.

The vortex current term would not be relevant for dualities between two bosonic theories, or between two fermionic ones, for these would need to apply twice (in opposite directions) the basic bosonization duality of eq. (2.5). But it would influence other Bose to Fermi dualities, for which we would apply it an odd number of times.

It is significant the fact that the derivation was only valid for energies $E \ll m = \phi_0^2$, $E \ll \alpha$. It was already understood that the existence of dual pairs were valid for the low energy theories. Following our procedure, we can only obtain eq. (2.5) for energies below m, which makes the $m \to 0$ limit singular. That means that results like the original Son's conjecture [1], are harder to understand, and would need extra arguments, to ensure their validity for the $m \to 0$ limit.

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References

[1] D.T. Son, Is the composite fermion a Dirac particle?, Phys. Rev. X 5 (3) (2015) 031027, arXiv:1502.03446 [cond-mat.mes-hall].
[2] J. Murugan, H. Nastase, Particle-vortex duality in topological insulators and superconductors, arXiv:1606.01912 [hep-th].
[3] N. Seiberg, T. Senthil, C. Wang, E. Witten, A duality web in $2+1$ dimensions and condensed matter physics, Ann. Phys. 374 (2016) 395–433, arXiv:1606.01989 [hep-th].
[4] A. Karch, D. Tong, Particle-vortex duality from 3d bosonization, Phys. Rev. X 6 (3) (2016) 031043, arXiv:1606.01893 [hep-th].
[5] M.A. Metlitski, A. Vishwanath, Particle-vortex duality of two-dimensional Dirac fermion from electric-magnetic duality of three-dimensional topological insulators, Phys. Rev. B 93 (24) (2016) 245151, arXiv:1505.05142 [cond-mat.str-el].
[6] P.-S. Hsin, N. Seiberg, Level/rank duality and Chern–Simons-matter theories, J. High Energy Phys. 09 (2016) 095, arXiv:1607.07457 [hep-th].
[7] A. Karch, B. Robinson, D. Tong, More Abelian dualities in 2+1 dimensions, J. High Energy Phys. 01 (2017) 017, arXiv:1609.04012 [hep-th].
[8] O. Aharony, F. Benini, P.-S. Hsin, N. Seiberg, Chern–Simons-matter dualities with SO and USp gauge groups, arXiv:1611.07874 [cond-mat.str-el].
[9] S. Kachru, M. Mulligan, G. Torroba, H. Wang, Nonsupersymmetric dualities from mirror symmetry, Phys. Rev. Lett. 118 (1) (2017) 011602, arXiv:1609.02149 [hep-th].
[10] D. Radicevic, D. Tong, C. Turner, Non-abelian 3D bosonization and quantum Hall states, J. High Energy Phys. 12 (2016) 067, arXiv:1608.04732 [hep-th].
[11] O. Aharony, Baryons, monopoles and dualities in Chern–Simons-matter theories, J. High Energy Phys. 02 (2016) 093, arXiv:1512.00161 [hep-th].
[12] C.P. Burgess, F. Quevedo, Bosonization as duality, Nucl. Phys. B 421 (1994) 373–390, arXiv:hep-th/9401105.
[13] C.P. Burgess, C.A. Lutken, F. Quevedo, Bosonization in higher dimensions, Phys. Lett. B 336 (1994) 18–24, arXiv:hep-th/9407078.
[14] E.H. Fradkin, F.A. Schaposnik, The fermion-boson mapping in three-dimensional quantum field theory, Phys. Lett. B 338 (1994) 253–258, arXiv:hep-th/9407182.
[15] F.A. Schaposnik, A comment on bosonization in d $\geq$ two-dimensions, Phys. Lett. B 356 (1995) 39–44, arXiv:hep-th/9505049.
[16] E.G. Filothodoros, A.C. Petkou, N.D. Vlachos, 3*d* fermion-boson map with imaginary chemical potential, arXiv:1608.07795 [hep-th].
[17] J. Murugan, H. Nastase, N. Rughoonauth, J.P. Shock, Particle-vortex and Maxwell duality in the $AdS_4 \times \mathbb{CP}^3$/ABJM correspondence, J. High Energy Phys. 10 (2014) 51, arXiv:1404.5926 [hep-th].
[18] C.P. Burgess, B.P. Dolan, Particle vortex duality and the modular group: applications to the quantum Hall effect and other 2-D systems, Phys. Rev. B 63 (2001) 155309, arXiv:hep-th/0010246.
[19] R.O. Ramos, J.F. Medeiros Neto, D.G. Barci, C.A. Linhares, Abelian Higgs model effective potential in the presence of vortices, Phys. Rev. D 72 (2005) 103524, arXiv:hep-th/0506052.
[20] R.O. Ramos, J.F. Medeiros Neto, Transition point for vortex condensation in the Chern–Simons Higgs model, Phys. Lett. B 666 (2008) 496–501, arXiv:0711.0798 [hep-th].
[21] A. Zee, Quantum Field Theory in a Nutshell, Princeton University Press, 2003.
[22] C. Dasgupta, B.I. Halperin, Phase transition in a lattice model of superconductivity, Phys. Rev. Lett. 47 (1981) 1556–1560.
[23] E.C. Marino, Quantum theory of nonlocal vortex fields, Phys. Rev. D 38 (1988) 3194.
[24] E.C. Marino, Duality, quantum vortices and anyons in Maxwell–Chern–Simons–Higgs theories, Ann. Phys. 224 (1993) 225–274, arXiv:hep-th/9208062.
[25] D.H. Lee, M.P.A. Fisher, Anyon superconductivity and the fractional quantum Hall effect, Phys. Rev. Lett. 63 (1989) 903–906.
[26] D.G. Barci, C.D. Fosco, L.E. Oxman, On bosonization in three-dimensions, Phys. Lett. B 375 (1996) 267–272, arXiv:hep-th/9508075.
[27] A.N. Redlich, Parity violation and gauge noninvariance of the effective gauge field action in three-dimensions, Phys. Rev. D 29 (1984) 2366–2374.