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Multi-valued functions in Computability Theory*

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Abstract. Multi-valued functions are common in computable analysis (built upon the Type 2 Theory of Effectivity), and have made an appearance in complexity theory under the moniker *search problems* leading to complexity classes such as PPAD and PLS being studied. However, a systematic investigation of the resulting degree structures has only been initiated in the former situation so far (the Weihrauch-degrees).
A more general understanding is possible, if the category-theoretic properties of multi-valued functions are taken into account. In the present paper, the category-theoretic framework is established, and it is demonstrated that many-one degrees of multi-valued functions form a distributive lattice under very general conditions, regardless of the actual reducibility notions used (e.g. Cook, Karp, Weihrauch).
Beyond this, an abundance of open questions arises. Some classic results for reductions between functions carry over to multi-valued functions, but others do not. The basic theme here again depends on category-theoretic differences between functions and multi-valued functions.

Keywords: Multi-valued functions, many-one reduction, Weihrauch reducibility, category theory, degree structure

1 Introduction

What are multi-valued functions? A (partial) multi-valued function $f :\subseteq A \rightrightarrows B$ is just a set $f \subseteq A \times B$ – i.e. a relation. However, the category of multi-valued functions is not the category of relations! We write $f(a)$ for $\{b \in B \mid (a, b) \in f\}$ and $\operatorname{dom}(f) = \{a \in A \mid \exists b \in f(a)\}$. Then the composition of multi-valued functions $f :\subseteq A \rightrightarrows B$, $g :\subseteq B \rightrightarrows C$ is defined via $c \in (g \circ f)(a)$ iff $f(a) \subseteq \operatorname{dom}(g)$ and $\exists b \in f(a)$ s.t. $c \in g(a)$. In the usual definition of the composition for relations, the former condition is absent!

The intended interpretation of a multi-valued function $f :\subseteq A \rightrightarrows B$ is that it links problem instances to solutions. This draws interest to the following partial order:

$$f \preceq g \ \Leftrightarrow\ \operatorname{dom}(f) \subseteq \operatorname{dom}(g) \wedge g_{|\operatorname{dom}(f)} \subseteq f$$

* This paper is based on the second chapter of the authors thesis [18]. An extended version including proofs is available at the arXiv, see [17].

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We can read $f \preceq g$ as f *is easier as* g: There may be fewer instances for f than for g, and a solution to a problem instance in g is a solution for it in f, too, where applicable. This has the consequence that any procedure solving g also solves f.

For any two multi-valued functions $f, g :\subseteq A \rightrightarrows B$ there exists a hardest multi-valued function easier than both, i.e. there are binary infima w.r.t. $\preceq$. These are given by $f \wedge g = (f \cup g)_{|\operatorname{dom}(f) \cap \operatorname{dom}(g)}$.

Why use them? First, multi-valued functions are natural: From elimination orders on graphs over Nash equilibria in games to fixed points of continuous mappings, there are plenty of problems without a natural way to specify the desired solution uniquely. In fact, if one accepts their formulation as multi-valued functions, one can even prove that the latter two are non-equivalent to any function!

Then, they are well-behaved under realizability: It is a common situation in computability and complexity theory that we have an algorithmic notion for some functions on some special sets X, Y which we intend to lift to more general sets A, B. We do this by fixing surjective encodings $\delta_A :\subseteq X \to A$, $\delta_B :\subseteq Y \to B$, and then calling e.g. a function $f : A \to B$ computable, iff there is a computable function $F :\subseteq X \to Y$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{F} & Y \\ \downarrow{\scriptstyle \delta_A} & & \downarrow{\scriptstyle \delta_B} \\ A & \xrightarrow{f} & B \end{array}$$

In general (depending on δ_A, δ_B), there will be algorithms (i.e. functions $F :\subseteq X \to Y$) which do not compute any function $f : A \to B$, which leads to the canonization problem: The desire to find an algorithm $C_A :\subseteq X \to X$ with the properties $C_A(x) = C_A(y)$ whenever $\delta_A(x) = \delta_A(y)$, and $\delta_A(C_A(x)) = \delta_A(x)$.

On the other hand, every algorithm computes a multi-valued function, hence, the canonization problem is relegated to a far less fundamental position.

Algorithms lacking semantics as a function can nonetheless be very meaningful. A common example for this is the multi-valued function $\chi : \mathbb{R} \to \{0, 1\}$ with $0 \in \chi(x)$ iff $x \leq 1$ and $1 \in \chi(x)$ iff $x \geq 0$. χ is computable – but the only computable functions from $\mathbb{R}$ to $\{0, 1\}$ are the constant ones! Hence, when working with real numbers, tests will have to be non-deterministic, i.e. multi-valued functions.

Finally, as will be demonstrated in this paper, the properties of multi-valued functions have a nice impact on the degree-structure of many-one reductions: One always obtains a distributive lattice here.

Due to lack of space, proofs are omitted here. They can be found in [17, 18].

2 Background

Many-one reductions between multi-valued functions have been studied in complexity theory for several decades now, with the complexity classes PPAD [14] and PLS [12] garnering a lot of attention. Both have a number of very interesting complete problems, we just mention finding Nash equilibria in finite two player games with integer payoffs as a complete problem for PPAD [7].

There also are a several problems which are known to be in both PPAD and PLS, but where this is the best classification available. Deciding the winner in parity or discounted payoff games is a typical example here. Despite this strong motivation to study PPAD $\cap$ PLS, only recently it was noticed (in a publication) that this class actually has complete problems [8] - a fact that is an obvious consequence of the degree structure being a distributive lattice (which we show here). A systematic investigation of the degree structure seems to be missing so far.

In another setting for many-one reductions between multi-valued functions is the programme to classify the computational content of mathematical theorems initiated in [3], [10]. Here a mathematical theorem of the form

$$\forall x \in X \, (x \in D \Rightarrow \exists y \in Y \ T(x, y))$$

is read as a multi-valued function $T :\subseteq X \rightrightarrows Y$ with $\mathrm{dom}(T) = D$ which has to find a witness $y \in Y$ given some $x \in X$. The tool for classification is Weihrauch-reducibility, a form of many-one reducibility introduced originally in [20], [21].

Various theorems been classified in this framework: e.g. the Hahn-Banach theorem [10], Weak König's Lemma, the Intermediate Value theorem [4], Nash's theorem on the existence of equilibria [15], Bolzano-Weierstrass [5], Brouwer's Fixed Point theorem [6].

Accompanying the investigation of specific degrees, also the overall degree structure has been studied. The Weihrauch degrees form a distributive lattice [4], [16], and can be turned into a Kleene algebra when equipped with additional natural operations $\times, ^*$ [11]. While some additional results in this area do depend on specific properties of Weihrauch reducibility, the fundamental ones only use generic properties of many-one reductions and multi-valued functions - and as such would also apply to the study of PPAD, PLS, etc.!

In the present paper we outline how the notion of *generic properties of many-one reductions between multi-valued functions* can be formalized, and show the fundamental structural results derivable from them. Then we introduce some properties that do depend on specific reducibilities, and both present some results and open questions.

3 The category of multi-valued functions

It is easy to see that composition of multi-valued functions is associative, so they form a category *Mult*. One can lift disjoint unions and cartesian products from sets to multi-valued functions in the straight-forward way, we will denote the results by $f + g$ and $f \times g$. The disjoint union retains its rôle as the coproduct, however, the cartesian product is **not** the categorical product!

This situation is reminiscent of categories of partial functions, and indeed we can borrow the following:

Definition 1 (Robinson and Rosolini [19]). *A p-category is a category $\mathcal{C}$ together with a naturally associative and naturally commutative bifunctor $\times : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (the (cartesian) product), a natural transformation Δ (the diagonal) between the identity functor and the derived functor $X \mapsto X \times X$, and two families of natural transformations $(\pi_1^A)_{A \in Ob(\mathcal{C})}$ and $(\pi_2^B)_{B \in Ob(\mathcal{C})}$ (the projections) where π_1^A is between the derived functor $X \mapsto X \times A$ and the identity, while π_2^B is between the derived functor $X \mapsto B \times X$ and the identity, such that the following properties are given:*

$$\pi_1^X(X) \circ \Delta(X) = \pi_2^X(X) \circ \Delta(X) = \mathrm{id}_X \quad (\pi_1^Y(X) \times \pi_2^X(Y)) \circ \Delta(X \times Y) = \mathrm{id}_{X \times Y}$$
$$\pi_1^Y(X) \circ (\mathrm{id}_X \times \pi_1^Z(Y)) = \pi_1^{(Y \times Z)}(X) \quad \pi_1^Z(X) \circ (\mathrm{id}_X \times \pi_2^Y(Z)) = \pi_1^{(Y \times Z)}(X)$$
$$\pi_2^X(Y) \circ (\pi_1^Y(X) \times \mathrm{id}_Z) = \pi_2^{(X \times Y)}(Z) \quad \pi_2^X(Z) \circ (\pi_2^X(Y) \times \mathrm{id}_Z) = \pi_2^{(X \times Y)}(Z)$$

For easier reading, we shall write $\pi_1^{X,Y}$ instead of $\pi_1^Y(X)$, $\pi_2^{X,Y}$ for $\pi_2^X(Y)$ and finally Δ_X in place of $\Delta(X)$. If the superscripts are obvious from the context, they may be dropped.

The treatment of partial maps in a categorical framework causes the concept of the domain of a map to split into two separate ones. With $\mathrm{Dom}(f)$ we denote the object A, if $f : A \to B$ is a morphism (and likewise CDom denotes B here). Following DiPaola and Heller [9], we write $\mathrm{dom}(f)$ for the morphism $\pi_1^{A,B} \circ (\mathrm{id}_A \times f) \circ \Delta_A$, where π_1 is the first projection of the product $X \times Y$. One can interpret $\mathrm{dom}(f) : A \to A$ as the partial identity on that part of A where the partial map f is actually defined. If $\mathrm{dom}(f) = \mathrm{id}_{\mathrm{CDom}(f)}$, we call f *total*.

Additionally we will assume that the categories underlying our p-categories have coproducts, and that the functor $\times$ distributes over the coproducts.

We already mentioned the fundamental partial order $\preceq$. As it is compatible with the composition of multi-valued functions, as well as with the cartesian product and the coproduct, we introduce the notion of a *poset enriched p-category* for such structures. If also binary infima exists, and are compatible with composition, cartesian products and coproducts, we have a *meet-semilattice enriched p-category*. These concepts come with a natural concept of a substructure, which we will use.

Moreover, we need two minor properties: A sub poset enriched p-category is called *wide*, if it contains all objects of the superstructure. A poset-enriched p-category is totally connected, if for any two objects A, B there is a morphisms $c_{A,B} : A \to B$. With these notions available, we can now provide the setting we need to introduce many-one reductions:

Definition 2. *A* many-one category extension *(moce) shall be a meet-semilattice enriched p-category $\mathcal{P}$ with coproducts, together with a wide and totally connected sub-poset enriched p-category $\mathcal{S}$, where $\times$ distributes over the coproducts.*

The intuition behind the preceding definition is that $\mathcal{P}$ is the category of problems one wishes to structure by reductions, whereas $\mathcal{S}$ is the subcategory of *simple* multi-valued functions that serve as reduction witnesses. Typical choices for $\mathcal{S}$ would be computable or polynomial-time computable functions (or multi-valued functions).

4 The lattice of many-one degrees

There are two definitions of many-one reductions commonly found in the literature on (multi-valued) functions, which differ in the question whether the post-processing of the oracle answer still has access to the input. Forgetting the input leads to a simpler definition, and may make proofs of non-reducibility easier, while retaining it yields the nicer degree structure and allows to formulate stronger and more meaningful separation statements. We shall speak of strong many-one reductions if the original input is forgotten, and of many-one reductions otherwise.

Throughout this subsection, we assume that some moce $(\mathcal{P}, \mathcal{S}, \times, \preceq)$ is given, and refrain from mentioning it explicitly where this is unnecessary.

Definition 3 (Strong many-one reductions). *Let $f \leq_{sm} g$ hold for $f, g \in \mathcal{P}$, if there are $H, K \in \mathcal{S}$ with $f \preceq H \circ g \circ K$.*

Definition 4 (Many-one reductions). *Let $f \leq_m g$ hold for $f, g \in \mathcal{P}$, if there are $H, K \in \mathcal{S}$ with $f \preceq H \circ (\mathrm{id}_{\mathrm{Dom}(f)} \times (g \circ K)) \circ \Delta_{\mathrm{Dom}(f)}$.*

Proposition 1. *Both $\leq_{sm}$ and $\leq_m$ define preorders on $\mathcal{P}$. For any, $f, g \in \mathcal{P}$, $f \leq_{sm} g$ implies $f \leq_m g$.*

By $\mathfrak{D}$ we shall denote the partially ordered class of $\leq_m$-equivalence classes in $\mathcal{P}$. Both the coproduct $+$ and the cartesian product $\times$ in $\mathcal{P}$ can be lifted to operations on $\mathfrak{D}$, which we shall denote by $+$, $\times$ again. We need a third operation to be lifted from $\mathcal{P}$ to $\mathfrak{D}$. The coproduct injections shall be $\iota_1^{X,Y} : X \to X + Y$ and $\iota_2^{X,Y} : Y \to X + Y$, and we denote the infimum regarding $\preceq$ via $\wedge$. Now for $f : X \to Y$, $g : A \to B$ define $f \oplus g : X \times A \to Y + B$ via:

$$f \oplus g = [(\iota_1^{Y,B} \circ \pi_1^{Y,B}) \wedge (\iota_2^{Y,B} \circ \pi_2^{Y,B})] \circ (f \times g)$$

Informally, $f \oplus g$ receives a problem instance to each of f and g, and has to produce a solution to one of them. Unlike $+, \times$, it is clear that $\oplus$ lacks a counterpart for functions. Its degree-theoretic relevance follows from the following main result:

Theorem 1. *$\mathfrak{D}$ is a distributive lattice, with $\oplus$ as infimum and $+$ as supremum.*

The presence of certain distinguished objects in $\mathcal{P}$ translates into the existence of special degrees in $\mathfrak{D}$. As usual, we call an object $I \in Ob(\mathcal{P})$ *initial*, iff for any object $A \in Ob(\mathcal{P})$ there is exactly one morphism $f : I \to A$. The concept is extended to domains in p-categories by calling $\operatorname{dom} i$ initial, iff for any $A \in Ob(\mathcal{P})$ there is exactly one morphism f with $\operatorname{CDom}(f) = A$ and $f = f \circ \operatorname{dom} i$.

Our notion of emptiness for objects in p-categories does not amount to emptiness in the underlying category. Instead, we call $E \in Ob(\mathcal{P})$ *empty*, iff for any total morphism $g : A \to E$ we find A to be initial. Likewise, a domain $\operatorname{dom} e$ is empty, iff for any total morphism g with $(\operatorname{dom} e) \circ g = g$, we find $\operatorname{Dom}(g)$ to be initial. Note that empty implies initial. Objects are initial (empty), iff the corresponding identity is initial (empty) as a domain.

An object $F \in Ob(\mathcal{P})$ is called final, iff for any object $A \in Ob(\mathcal{P})$ there is exactly one total morphism $g : A \to F$. Likewise, a domain $\operatorname{dom} f$ is final, iff for any $A \in Ob(\mathcal{P})$ there is exactly one total morphism g with $\operatorname{Dom}(g) = A$ and $\operatorname{dom} f \circ g = g$. Objects are initial (empty, final), iff the corresponding identity is initial (empty, final) as a domain.

Proposition 2. *Let* $\operatorname{dom} i$ *be initial in both* $\mathcal{S}$ *and* $\mathcal{P}$*. Then its degree (denoted by* 0*) is the bottom element in* $\mathfrak{D}$*.*

In particular, this shows that all initial domains are equivalent w.r.t. $\leq_m$. The same holds for final domains, whose degree (if present) we denote by 1. Then we find:

Theorem 2. *If* $\mathcal{P}$ *and* $\mathcal{S}$ *share an empty domain and a final domain, then* $\mathfrak{D}$ *with the operations* $\times$*,* $+$ *and the constants* 0*,* 1 *is an idempotent commutative semiring, i.e. the following equations hold for all* $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathfrak{D}$*:*

1. $\mathbf{a} + \mathbf{a} = \mathbf{a}$, $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$, $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
2. $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{a}$, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$
4. $0 + \mathbf{a} = \mathbf{a}$, $0 \times \mathbf{a} = 0$, $1 \times \mathbf{a} = \mathbf{a}$

We remark that in a similar fashion, an operation * can be introduced (albeit requiring slightly more assumption) turning $\mathfrak{D}$ into a Kleene-algebra. As a consequence, a definition of wtt-reductions can be derived from our definition of many-one reductions.

5 Some examples

In this section, we will exhibit two basic examples for our framework, namely the adapted versions of Karp and Cook reducibilities to multi-valued functions. Another prime example is Weihrauch reducibility. The structural investigation of Weihrauch degrees served as inspiration for the present work, and we refer to the original literature for the details [4], [16], [2], [11].

These examples do not exhaust the range of applicability, though: Medvedev-reducibility and many-one reductions between parameterized search problems are omitted due to limited space; resource-bounded variants of Weihrauch reducibility also satisfy the requirements. Beyond computability, also continuity may be used as the decisive property of reduction witnesses.

5.1 Computable many-one reductions

Here we consider the category of multi-valued functions from $\{0,1\}^*$ to $\{0,1\}^*$ in the rôle of $\mathcal{P}$, while the category of reduction witnesses $\mathcal{S}$ is given by the category $\mathcal{C}_1$ of restrictions of partial computable functions. These categories satisfy our conditions, with $f+g$, $f \times g$ defined via $(f+g)(0x) = 0f(x)$, $(f+g)(1x) = 1g(x)$, $(f \times g)(\langle x, y\rangle) = \langle f(x), g(x)\rangle$.

Definition 5 (special case of Definition 4). *For two multi-valued functions $f, g :\subseteq \{0,1\}^* \rightrightarrows \{0,1\}^*$, define $f \leq_m g$, if there are computable functions $H, K :\subseteq \{0,1\}^* \to \{0,1\}^*$ with $H\langle x, y\rangle \in f(x)$ whenever $y \in g(K(x))$.*

We use $\mathfrak{C}_1$ to denote the set of degrees in this setting.

Corollary 1 (of Theorem 1). *$(\mathfrak{C}_1, \oplus, +)$ is a distributive lattice.*

In $\mathcal{C}_1$, there exists both an empty domain and final domains, namely the no-where defined multi-valued function $\emptyset \subset \{0,1\}^* \times \{0,1\}^*$ and any $\{(x,x)\} \subseteq \{0,1\}^* \times \{0,1\}^*$. The corresponding degrees shall be denoted by $0, 1 \in \mathfrak{C}_1$.

Proposition 3. *1 is the least element in $\mathfrak{C}_1 \setminus \{0\}$ and contains exactly those multi-valued functions admitting a computable choice function.*

We do point out that decision problems cannot be considered as a special case of multi-valued functions in the straight-forward way, as our definition of many-one reductions allows modifications of the output; in particular, the characteristic function of a set is trivially equivalent to the characteristic function of its complement. However, many results proven for many-one reductions between search problems hold - with identical proofs - also for Turing reductions with the number of oracle queries limited to 1, which corresponds to the notion employed here.

For example, Yates' result [22] regarding the existence of minimal pairs applies here as follows:

Proposition 4 (Yates [22]). *There are $\mathbf{a}, \mathbf{b} \in \mathfrak{C}_1 \setminus \{0,1\}$ with total representatives such that for any $\mathbf{c} \leq_m (\mathbf{a} \oplus \mathbf{b})$ that has a representative $f \in \mathbf{c}$ of the type $f : \{0,1\}^* \to \{0,1\}$, we find $\mathbf{c} = 1$.*

However, the cumbersome restriction to degrees admitting a function representative is necessary, as minimal pairs for multi-valued functions do not exist in the computable case:

Proposition 5. *If $\mathbf{a}, \mathbf{b} \in \mathfrak{C}_1$ have total representatives, then $\mathbf{a} \oplus \mathbf{b} = 1$ implies $\mathbf{a} = 1$ or $\mathbf{b} = 1$.*

The proof of the preceding proposition is based on the following technical lemma:

Lemma 1. *There are Turing functionals Ψ, Φ, such that for all total multi-valued functions $f, g : \{0,1\}^* \rightrightarrows \{0,1\}^*$ and for any choice function I of $(f \oplus g)$, either Ψ^I is a choice function of f or Φ^I is a choice function of g.*

5.2 Polynomial-time many reductions

Proceeding as above, but taking the category of polynomial-time computable functions as the category $\mathcal{S}$ of reduction witnesses, we again obtain a degree structure w.r.t. many-one reductions in the generic way, which we shall denote by $\mathfrak{P}_1$, and the reducibility by $\leq_m^p$.

Corollary 2 (of Theorem 1). *$(\mathfrak{P}_1, \oplus, +)$ is a distributive lattice.*

Again, 0 and 1 exist and are the bottom and second-least element respectively. $0 \in \mathfrak{P}_1$ contains only the no-where defined multi-valued function, whereas 1 contains exactly those multi-valued functions with non-empty domain admitting a polynomial-time computable choice function.

Again, some results for functions or decision problems can be transferred. As a demonstration, we extend LADNER's density result [13, Theorem 2] to multi-valued functions. For this, note that two notions coinciding for single-valued functions differ for multi-valued functions, namely the existence of a computable choice function and the decidability of the graph. We call those multi-valued functions satisfying the former condition *computable*. The latter condition has the disadvantage of not being preserved downwards by many-one reductions. However, a decidable graph is the condition needed for the following theorem. Its proof closely resembles the one of [13, Theorem 2].

Theorem 3. *Let $\mathbf{a}, \mathbf{b} \in \mathfrak{P}_1$ admit representatives with decidable graphs and satisfy $\mathbf{b} \not\leq_m^p \mathbf{a}$. Then there are $\mathbf{b}_0, \mathbf{b}_1 \in \mathfrak{P}_1$ with $\mathbf{b} = \mathbf{b}_0 + \mathbf{b}_1$, $\mathbf{b}_i \not\leq_m^p \mathbf{a}$ and $\mathbf{b} \not\leq_m^p \mathbf{a} + \mathbf{b}_i$ for both $i \in \{0, 1\}$.*

Corollary 3. *The degrees in $\mathcal{P}_1$ admitting decidable graphs are dense (in themselves).*

A question that has received a lot of attention regarding (polynomial-time) many-one reductions between decision problems is about the existence and nature of minimal pairs. In terms of lattice theory, this asks whether the degree 1 is meet-irreducible, and if not, what kind of pairs can satisfy $\mathbf{a} \oplus \mathbf{b} = 1$. Following the initial result by LADNER that minimal pairs for polynomial-time many-one reductions between decision problems exist [13], AMBOS-SPIES could prove that every computable super-polynomial degree is part of a minimal pair [1].

For search problems, however, the question remains open:

Question 1. Is $1 \in \mathcal{P}_1$ meet-irreducible?

The techniques used to construct a minimal pair in [13], [1] diagonalize against pairs of reductions R_e, R_f trying to prevent $R_e(a) = R_f(b)$ for the constructed representatives a, b. If the equality cannot be prevented, then one can prove that the resulting set is already polynomial-time decidable using a constant prefix of b, hence, polynomial-time decidable. However, for search problems any pair of reductions to a pair of search problems produces a search problem, namely $R_e(a) \cup R_f(b)$.

A non-computable minimal pair for Type-2 search problems was constructed in [11] by Higuchi and Pauly. There, the crucial part is the identifiability of hard and easy instances, which is not available in a Type-1 setting. The negative answer we obtained for computable many-one reductions in Subsection 5.1 relied on Lemma 1, which again cannot be transferred to the time-bounded case: There are polynomial-time decidable relations R such that neither R nor its inverse $\neg R^\dagger$ admit a polynomial-time choice function, even if $\mathrm{P} = \mathrm{NP}$ should hold.

6 Outlook

Hopefully we have made the case that investigating the degree structures of many-one reductions between multi-valued functions is both intrinsically and extrinsically interesting. The basic results follow from our generic results in Section 4, but beyond that the various kinds require specific attention. To some extent proof concepts can be extended from the traditional setting of many-one reductions between functions, but beyond that novel techniques are called for.

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