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Closed choice for finite and for convex sets*

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Abstract. We investigate choice principles in the Weihrauch lattice for finite sets on the one hand, and convex sets on the other hand. Increasing cardinality and increasing dimension both correspond to increasing Weihrauch degrees. Moreover, we demonstrate that the dimension of convex sets can be characterized by the cardinality of finite sets encodable into them. Precisely, choice from an $n+1$ point set is reducible to choice from a convex set of dimension n, but not reducible to choice from a convex set of dimension $n-1$.

1 Introduction

In the investigation of the computational content of mathematical theorems in the Weihrauch lattice, variations of closed choice principles have emerged as useful canonic characterizations [1,3,6]. Closed choice principles are multivalued functions taking as input a non-empty closed subset of some fixed space, and have to provide some element of the closed set as output. In [1,3] the influence of the space on the computational difficulty of (full) closed choice was investigated, whereas in [6] it turned out that the restriction of choice to connected closed subsets of the unit hypercube is equivalent to Brouwer's Fixed Point theorem for the same space.

Here the restrictions of closed choice to convex subsets (of the unit hypercube of dimension n), and to finite subsets (of a compact metric space) are the foci of our investigations. Via the connection between closed choice and non-deterministic computation [1, 7, 20, 25], in particular the latter problem is prototypic for those problems having only finitely many correct solutions where wrong solutions are identifiable. As such, some parts may be reminiscent of some ideas from [8, 15].

One of our main results shows that choice for finite sets of cardinality $n+1$ can be reduced to choice for convex sets of dimension n, but not to convex choice of dimension $n-1$. This demonstrates a computational aspect in which convex sets get more complicated with increasing dimension. As such, our work also

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continues the study of the structural complexity of various classes of subsets of the unit hypercubes done in [12, 14].

Some of the techniques used to establish our main results are promising with regards to further applicability to other classes of choice principles, or to even more general Weihrauch degrees. These techniques are presented in Section 2.

Due to lack of space, some of the proofs had to be omitted. A version including proofs and additional results is as available as [13].

1.1 Weihrauch reducibility

We briefly recall some basic results and definitions regarding the Weihrauch lattice. The original definition of Weihrauch reducibility is due to Weihrauch and has been studied for many years (see [10, 16, 17, 21–23]). Rather recently it has been noticed that a certain variant of this reducibility yields a lattice that is very suitable for the classification of mathematical theorems (see [1, 3–5, 9, 11, 18, 19]). A basic reference for notions from computable analysis is [24]. The Weihrauch lattice is a lattice of multi-valued functions on represented spaces. A represented space is a pair (X, δ_X) where $\delta_X :\subseteq \mathbb{N}^\mathbb{N} \to X$ is a partial surjection, called representation. In general we use the symbol "$\subseteq$" in order to indicate that a function is potentially partial. Using represented spaces we can define the concept of a realizer. We denote the composition of two (multi-valued) functions f and g either by $f \circ g$ or by fg.

Definition 1 (Realizer). *Let $f :\subseteq (X, \delta_X) \rightrightarrows (Y, \delta_Y)$ be a multi-valued function on represented spaces. A function $F :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ is called a* realizer *of f, in symbols $F \vdash f$, if $\delta_Y F(p) \in f\delta_X(p)$ for all $p \in \mathrm{dom}(f\delta_X)$.*

Realizers allow us to transfer the notions of computability and continuity and other notions available for Baire space to any represented space; a function between represented spaces will be called *computable*, if it has a computable realizer, etc. Now we can define Weihrauch reducibility.

Definition 2 (Weihrauch reducibility). *Let f, g be multi-valued functions on represented spaces. Then f is said to be* Weihrauch reducible *to g, in symbols $f \leq_W g$, if there are computable functions $K, H :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ such that $K\langle \mathrm{id}, GH\rangle \vdash f$ for all $G \vdash g$. Moreover, f is said to be* strongly Weihrauch reducible *to g, in symbols $f \leq_{sW} g$, if there are computable functions K, H such that $KGH \vdash f$ for all $G \vdash g$.*

Here $\langle , \rangle$ denotes some standard pairing on Baire space.

There are two operations defined on Weihrauch degrees that are used in the present paper, product $\times$ and composition $\star$. The former operation was originally introduced in [4], the latter in [5]. Informally, the product allows using both involved operations independently, whereas for the composition the call to the first operation may depend on the answer received from the second.

Definition 3. *Given $f :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$, $g :\subseteq \mathbf{U} \rightrightarrows \mathbf{V}$, define $f \times g :\subseteq (\mathbf{X} \times \mathbf{U}) \rightrightarrows (\mathbf{Y} \times \mathbf{V})$ via $(y, v) \in (f \times g)(x, u)$ iff $y \in f(x)$ and $v \in g(u)$.*

Definition 4. *Given* $f :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$, $g :\subseteq \mathbf{U} \rightrightarrows \mathbf{V}$, *let*

$$f \star g := \sup_{\leq_W}\{f' \circ g' \mid f' \leq_W f \wedge g' \leq_W g\}$$

where f', g' *are understood to range over all those multivalued functions where the composition is defined.*

Both $\times$ and $\star$ are associative, but only $\times$ is commutative. We point out that while it is not obvious that the supremum in the definition of $\star$ always exists, this is indeed the case, hence $\star$ is actually a total operation.

1.2 Closed Choice and variations thereof

The space of continuous functions from a represented space $\mathbf{X}$ to $\mathbf{Y}$ has a natural representation itself, as a consequence of the UTM-theorem. This represented space is denoted by $\mathcal{C}(\mathbf{X}, \mathbf{Y})$. A special represented space of utmost importance is Sierpiński space $\mathbb{S}$ containing two elements $\{\top, \bot\}$ represented by $\delta_\mathbb{S} : \mathbb{N}^\mathbb{N} \to \mathbb{S}$ where $\delta_\mathbb{S}(0^\mathbb{N}) = \bot$ and $\delta_\mathbb{S}(p) = \top$, iff $p \neq 0^\mathbb{N}$. The space $\mathcal{A}(\mathbf{X})$ of closed subsets of $\mathbf{X}$ is obtained from $\mathcal{C}(\mathbf{X}, \mathbb{S})$ by identifying a set $A \subseteq \mathbf{X}$ with the characteristic function $\chi_{X \setminus A} : \mathbf{X} \to \mathbb{S}$ of its complement.

For a computable metric space $\mathbf{X}$, an equivalent representation $\psi_- : \mathbb{N}^\mathbb{N} \to \mathcal{A}(\mathbf{X})$, can be defined by $\psi_-(p) := X \setminus \bigcup_{i=0}^{\infty} B_{p(i)}$, where B_n is some standard enumeration of the open balls of X with center in the dense subset and rational radius (possibly 0). The computable points in $\mathcal{A}(X)$ are called *co-c.e. closed sets*. We are primarily interested in closed choice on computable metric spaces; additionally, most of our considerations pertain to compact spaces.

Definition 5 (Closed Choice, [3]). *Let* $\mathbf{X}$ *be a represented space. Then the* closed choice *operation of this space is defined by* $C_\mathbf{X} :\subseteq \mathcal{A}(\mathbf{X}) \rightrightarrows \mathbf{X}, A \mapsto A$ *with* $\mathrm{dom}(C_X) := \{A \in \mathcal{A}(\mathbf{X}) : A \neq \emptyset\}$.

Intuitively, $\mathrm{C}_\mathbf{X}$ takes as input a non-empty closed set and it produces an arbitrary point of this set as output. Hence, $A \mapsto A$ means that the multi-valued map $\mathrm{C}_\mathbf{X}$ maps the input set $A \in \mathcal{A}(\mathbf{X})$ to the points in $A \subseteq \mathbf{X}$ as possible outputs.

Definition 6. *For a represented space* $\mathbf{X}$ *and* $1 \leq n \in \mathbb{N}$,
let $C_{\mathbf{X},\sharp=n} := C_\mathbf{X}|_{\{A \in \mathcal{A}(\mathbf{X}) || A|=n\}}$ *and* $C_{\mathbf{X},\sharp\leq n} := C_\mathbf{X}|_{\{A \in \mathcal{A}(\mathbf{X}) | 1 \leq |A| \leq n\}}$.

More generally, for any choice principle the subscript $\sharp = n$ denotes the restriction to sets of cardinality n, and the subscript $\sharp \leq n$ to sets of cardinality less or equal than n. In the same spirit, the subscript $\lambda > \epsilon$ denotes the restriction to sets of outer diameter greater than ϵ, and $\mu > \epsilon$ the restriction to those sets where some value μ is greater than ε.

Definition 7. *Let* $XC_n := C_{[0,1]^n}|_{\{A \in \mathcal{A}([0,1]^n) | A \text{ is convex }\}}$.

The proof of the following proposition has been inspired by the proof of [15, Theorem 3.1] by LONGPRÉ et al. , which the proposition generalizes in some sort. In fact, the study of $\mathrm{C}_{[0,1],\sharp=m}$ is quite closely related to the theme of [15].

Proposition 1. *Let* $\mathbf{X}$ *be a computably compact computable metric space. Then* $C_{\mathbf{X},\sharp=n} \leq_{sW} C_{\{0,1\}^{\mathbb{N}},\sharp=n}$ *and* $C_{\mathbf{X},\sharp\leq n} \leq_{sW} C_{\{0,1\}^{\mathbb{N}},\sharp\leq n}$.

Proof. We associate a labeled finitely-branching infinite tree (with given bounds) with the space $\mathbf{X}$, where each vertex is labeled by an open subset of $\mathbf{X}$. The root is labeled by $\mathbf{X}$. Then we find a finite open cover of $\mathbf{X}$ by open balls $B(x_1, 2^{-1}), \ldots, B(x_n, 2^{-1})$ using the computable dense sequence and the computable compactness provided by $\mathbf{X}$. The $B(x_i, 2^{-1})$ form the second layer of the tree. For the third layer, each $B(x_i, 2^{-1})$ (whose closure is computably compact) is covered by finitely many $B(x_{i,j}, 2^{-2})$, and we then use $B(x_i, 2^{-1}) \cap B(x_{i,j}, 2^{-2})$ as labels. This process is iterated indefinitely, yielding finer and finer coverings of the space at each layer.

Any closed subset of a computably compact space is compact (in a uniform way), so we can assume the input to $\mathrm{C}_{\mathbf{X},\sharp=n}$ ($\mathrm{C}_{\mathbf{X},\sharp\leq n}$) to be a compact set A of cardinality n (less-or-equal n). On any layer of the tree, there are n vertices such that the union of their labels covers A. It is recognizable when an open set includes a compact set, so we will find suitable n vertices eventually. Also, we can require that the vertices chosen on one level are actually below those chosen on the previous level.

Any finitely-branching tree with at most n nodes per layer can be encoded in a binary tree with at most n nodes per layer, and this just represents a closed subset of Cantor space with cardinality less-or-equal n. If the initial set A has exactly n elements, at some finite stage in the process any of the n open sets used to cover it will actually contain a point of A. Hence, from that stage onwards no path through the finitely-branching tree dies out, which translates to no path through the binary tree dying out. But then, the closed subset of Cantor space has cardinality exactly n.

Any point from the subset of Cantor space is an infinite path through the two trees constructed, hence, gives us a sequence of rational balls with diameter shrinking to 0 and non-empty intersection. This provides us with a name of a point in the original set, completing the reduction.

It is rather obvious that if $\mathbf{X}$ is a co-c.e. closed subspace of $\mathbf{Y}$, then $\mathrm{C}_{\mathbf{X},\sharp=n} \leq_{sW} \mathrm{C}_{\mathbf{Y},\sharp=n}$ and $\mathrm{C}_{\mathbf{X},\sharp\leq n} \leq_{sW} \mathrm{C}_{\mathbf{Y},\sharp\leq n}$ (compare [1, Section 4]). We recall that a computable metric space $\mathbf{X}$ is called *rich*, if it has Cantor space as computably isomorphic to a subspace (then this subspace automatically is co-c.e. closed). [2, Proposition 6.2] states that any non-empty computable metric space without isolated points is rich.

Corollary 1. *Let* $\mathbf{X}$ *be a rich computably compact computable metric space. Then* $C_{\mathbf{X},\sharp=n} \equiv_{sW} C_{\{0,1\}^{\mathbb{N}},\sharp=n}$ *and* $C_{\mathbf{X},\sharp\leq n} \equiv_{sW} C_{\{0,1\}^{\mathbb{N}},\sharp\leq n}$.

By inspection of the proof of Proposition 1, we notice that the names produced there as inputs to $\mathrm{C}_{\{0,1\}^{\mathbb{N}},\sharp=n}$ or $\mathrm{C}_{\{0,1\}^{\mathbb{N}},\sharp\leq n}$ have a specific form: If we consider the closed subsets of Cantor space to be represented as the sets of paths of infinite binary trees, the trees involved will have exactly n vertices on all layers admitting at least n vertices in a complete binary tree. The names used for $\mathrm{C}_{\{0,1\}^{\mathbb{N}},\sharp=n}$ moreover have the property that from some finite depths onwards, all vertices have exactly one child. The restrictions of $\mathrm{C}_{\{0,1\}^{\mathbb{N}},\sharp=n}$ and $\mathrm{C}_{\{0,1\}^{\mathbb{N}},\sharp\leq n}$ to inputs of the described type shall be denoted by $\mathrm{C}_{\sharp=n}$ and $\mathrm{C}_{\sharp\leq n}$. We directly conclude $\mathrm{C}_{\sharp=n} \equiv_{sW} \mathrm{C}_{\{0,1\}^{\mathbb{N}},\sharp=n} \equiv_{sW} \mathrm{C}_{[0,1],\sharp=n}$ and $\mathrm{C}_{\sharp\leq n} \equiv_{sW} \mathrm{C}_{\{0,1\}^{\mathbb{N}},\sharp\leq n} \equiv_{sW} \mathrm{C}_{[0,1]^k,\sharp\leq n}$.

2 Relative separation techniques

The relative separation techniques to be developed in this section do not enable us to prove separation results just on their own; instead they constitute statements that some reduction $f \leq_W g$ implies some reduction $f' \leq_W g'$, so by contraposition $f' \nleq_W g'$ (which may be easier to prove) implies $f \nleq_W g$. A particular form of these implications are absorption theorems. These show that for special degrees h, whenever f has a certain property, then $f \leq_W g \star h$ (or $f \leq_W h \star g$) implies $f \leq_W g$. A known result of this form is the following:

Theorem 1 (Brattka, de Brecht & Pauly [1, Theorem 5.1][3]). *Let $f : \mathbf{X} \to \mathbf{Y}$ be single-valued and $\mathbf{Y}$ admissible. Then $f \leq_W C_{\{0,1\}^{\mathbb{N}}} \star g$ implies $f \leq_W g$.*

We call a Weihrauch-degree a *fractal*, if each of its parts is again the whole. The concept was introduced by Brattka, de Brecht and Pauly in [1] as a criterion for a degree to be join-irreducible (all fractals are join-irreducible, cf. Lemma 1). The formalization uses the operation $f \mapsto f_A$ introduced next.

For some represented space $\mathbf{X} = (X, \delta_X)$ and $A \subseteq \mathbb{N}^{\mathbb{N}}$, we use the notation $\mathbf{X}_A$ for the represented space $(\delta_X[A], (\delta_X)|_A)$. This is a proper generalization of the notion of a subspace. Given $f :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$ and $A \subseteq \mathbb{N}^{\mathbb{N}}$, then f_A is the induced map $f_A :\subseteq \mathbf{X}_A \rightrightarrows \mathbf{Y}$.

Definition 8. *We call f a* fractal *iff there is some $g : \mathbf{U} \rightrightarrows \mathbf{V}$, $\mathbf{U} \neq \emptyset$ such that for any clopen $A \subseteq \mathbb{N}^{\mathbb{N}}$, either $g_A \equiv_W f$ or $g_A \equiv_W 0$. If we can choose $\mathbf{U}$ to be represented by a total representation $\delta_{\mathbf{U}} : \mathbb{N}^{\mathbb{N}} \to \mathbf{U}$, we call f a* closed fractal.

We will prove two absorption theorems, one for fractals and one for closed fractals. These essentially state that certain Weihrauch degrees are useless in solving a (closed) fractal.

Theorem 2 (Fractal absorption). *For fractal f, $f \leq_W g \star C_{\{1,\dots,n\}}$ implies $f \leq_W g$.*

[3] The precise statement of [1, Theorem 5.1] is weaker than the one given here, but a small modification of the proof suffices to obtain the present form.

2.1 Baire Category Theorem as separation technique

The absorption theorem for closed fractals is a consequence of the Baire Category Theorem, and was first employed as a special case in [3, Proposition 4.9] by Brattka and Gherardi.

Theorem 3 (Closed fractal absorption). *For a closed fractal f, $f \leq_W g \star C_{\mathbb{N}}$ implies $f \leq_W g$.*

Proof. The closed sets $A_n = \{p \mid n \in \psi_-^{\mathbb{N}}(p)\}$ cover $\operatorname{dom}(\mathrm{C}_{\mathbb{N}} \circ \psi_-^{\mathbb{N}}) \subseteq \mathbb{N}^{\mathbb{N}}$, and the corresponding restrictions $(\mathrm{C}_{\mathbb{N}})_{A_n}$ is computable for each $n \in \mathbb{N}$. Let δ be the representation used on the domain of $g \star \mathrm{C}_{\mathbb{N}}$, and $B_n \subseteq \operatorname{dom}((g \star \mathrm{C}_{\mathbb{N}}) \circ \delta) \subseteq \mathbb{N}^{\mathbb{N}}$ be the closed set of those names of inputs to $g \star \mathrm{C}_{\mathbb{N}}$ such that the call to $\mathrm{C}_{\mathbb{N}}$ involved is an element of A_n. The sets $(B_n)_{n \in \mathbb{N}}$ cover $\operatorname{dom}((g \star \mathrm{C}_{\mathbb{N}}) \circ \delta)$, and we find $(g \star \mathrm{C}_{\mathbb{N}})_{B_n} \leq_W g$.

W.l.o.g. assume that f witnesses its own fractality. Now let $f \leq_W g \star \mathrm{C}_{\mathbb{N}}$ be witnessed by computable K, H, and let ρ be the representation on the domain of f. Then the closed sets $H^{-1}(B_n)$ cover $\operatorname{dom}(f \circ \rho) = \mathbb{N}^{\mathbb{N}}$. We can apply the Baire Category Theorem, and find that there exists some n_0 such that $H^{-1}(B_{n_0})$ contains some non-empty clopen ball. As f is a fractal, we know:

$$f \leq_W f_{H^{-1}(B_{n_0})} \leq_W (g \star \mathrm{C}_{\mathbb{N}})_{B_{n_0}} \leq_W g$$

The preceding result occasionally is more useful in a variant adapted directly to choice principles in the rôle of g. For this, we recall the represented space $\mathbb{R}_>$, in which decreasing sequences of rational numbers are used to represent their limits as real numbers. Note that $\mathrm{id} : \mathbb{R} \to \mathbb{R}_>$ is computable but lacks a computable inverse. A generalized measure on some space $\mathbf{X}$ is a continuous function $\mu : \mathcal{A}(\mathbf{X}) \to \mathbb{R}_>$ taking only non-negative values. The two variants are connected by the following result:

Proposition 2. *Define* $\textsc{Lb} : \{x \in \mathbb{R}_> \mid x > 0\} \to \mathbb{N}$ *via* $\textsc{Lb}(x) = \min\{n \in \mathbb{N} \mid n^{-1} \leq x\}$. *Then* $\textsc{Lb} \equiv_{sW} C_{\mathbb{N}}$.

The preceding result indirectly shows how a closed choice principle for some class $\mathfrak{A} \subseteq \mathcal{A}(\mathbf{X})$ of closed sets with positive generalized measure μ can be decomposed into the slices with fixed lower bounds $\mu > n^{-1}$. For this, we recall the infinitary coproduct (i.e. disjoint union) $\coprod_{n \in \mathbb{N}}$ defined both for represented spaces and multivalued functions between them via $\left(\coprod_{n \in \mathbb{N}} f_n\right)(i, x) = (i, f_i(x))$.

Corollary 2. $\mathrm{C}_{\mathbf{X}}|_{\mathfrak{A},\mu>0} \leq_W \left(\coprod_{n \in \mathbb{N}} \mathrm{C}_{\mathbf{X}}|_{\mathfrak{A},\mu>n^{-1}}\right) \star C_{\mathbb{N}}$

Lemma 1 (σ-join irreducibility of fractals [1, Lemma 5.5]). *Let f be a fractal and satisfy $f \leq_W \coprod_{n \in \mathbb{N}} g_n$. Then there is some $n_0 \in \mathbb{N}$ such that $f \leq_W g_{n_0}$.*

Theorem 4. *Let f be a closed fractal such that $f \leq_W C_{\mathbf{X}}|_{\mathfrak{A},\mu>0}$. Then there is some $n \in \mathbb{N}$ such that $f \leq_W C_{\mathbf{X}}|_{\mathfrak{A},\mu>n^{-1}}$.*

Proof. By Corollary 2 we find $f \leq_W \left(\coprod_{n \in \mathbb{N}} \mathrm{C}_{\mathbf{X}}|_{\mathfrak{A},\mu>n^{-1}}\right) \star \mathrm{C}_{\mathbb{N}}$. Then Theorem 3 implies $f \leq_W \left(\coprod_{n \in \mathbb{N}} \mathrm{C}_{\mathbf{X}}|_{\mathfrak{A},\mu>n^{-1}}\right)$. By Lemma 1 there has to be some n_0 with $f \leq_W \mathrm{C}_{\mathbf{X}}|_{\mathfrak{A},\mu>n_0^{-1}}$.

2.2 Large diameter technique

Whereas Theorem 4 allows us to bound any positive generalized measure on the closed sets used to compute a function f away from 0, provided f is a closed fractal, the separation technique to be developed next bounds away only a specific generalized measure - the outer diameter - yet needs neither positivity nor the closed fractal property.

For $\varepsilon > 0$ and some class $\mathfrak{A} \subseteq \mathcal{A}(\mathbf{X})$, we introduce:

$$X_\varepsilon(\mathfrak{A}) = \overline{\psi_-^{-1}\left(\{A \in \mathfrak{A} \mid \forall x \in \mathbf{X} \exists B \in \mathfrak{A}\ B \subseteq A \setminus B(x, \varepsilon)\}\right)} \subseteq \mathbb{N}^\mathbb{N}$$

This means that the names in $X_\varepsilon(\mathfrak{A})$ are for sets large enough such that arbitrarily late an arbitrary ball of radius ε can be removed from them, and still a closed set in the class $\mathfrak{A}$ remains as a subset.

We proceed to show that a reduction between choice principles has to map sets large in this sense to sets with large outer diameter (denoted by λ).

Lemma 2 (Large Diameter Principle). *Let H and K witness a reduction $C_\mathbf{X}|_\mathfrak{A} \leq_W C_\mathbf{Y}|_\mathfrak{B}$, where $\mathbf{Y}$ is compact and $\mathfrak{A} \subseteq \mathcal{A}(\mathbf{X})$, $\mathfrak{B} \subseteq \mathcal{A}(\mathbf{Y})$. Then*

$$\forall p \in \operatorname{dom}(C_\mathbf{X}|_\mathfrak{A}\psi)\ \forall \varepsilon > 0\ \exists n \in \mathbb{N}\ \exists \delta > 0,\ q \in X_\varepsilon(\mathfrak{A}) \cap B(p, 2^{-n}) \Rightarrow \lambda\psi K(q) > \delta$$

Proof. Assume the claim were false, and let $p \in \operatorname{dom}(\mathrm{C}_\mathbf{X}|_\mathfrak{A}\psi)$ and $\varepsilon > 0$ be witness for the negation. There has to be a sequence $(p_n)_{n\in\mathbb{N}}$ such that $p_n \in X_\varepsilon(\mathfrak{A})$, $d(p, p_n) < 2^{-n}$ and $\lambda\psi H(p_n) < 2^{-n}$. As the p_n converge to p and H is continuous, we find that $\lim_{n\to\infty} H(p_n) = H(p)$. For the closed sets represented by these sequences, this implies $\left(\bigcap_{n\in\mathbb{N}} \overline{\bigcup_{i\geq n} \psi H(p_i)}\right) \subseteq \psi H(p)$. As $\mathbf{Y}$ is compact, the left hand side contains some point x.

As $x \in \psi H(p)$, for any $q \in \delta_\mathbf{Y}^{-1}(\{x\})$ we find $\langle p, q\rangle \in \operatorname{dom}(K)$. We fix such a q and $y = \delta_\mathbf{X}(K(\langle p, q\rangle))$. By continuity, there is some $N \in \mathbb{N}$ such that for any $\langle p', q'\rangle \in (B(p, 2^{-N}) \times B(q, 2^{-N})) \cap \operatorname{dom}(\delta_\mathbf{X} K)$ we find $\delta_\mathbf{X} K(\langle p', q'\rangle) \in B(y, \varepsilon)$.

By choice of x, for any $i \in \mathbb{N}$ there is some $k_i \geq i$ such that $d(x, \psi H(p_{k_i})) < 2^{-i}$. By choice of the p_n, this in turn implies $\psi H(p_{k_i}) \subseteq B(x, 2^{-i} + 2^{-k_i})$. Let $I \in \mathbb{N}$ be large enough, such that for any $x' \in B(x, 2^{-I} + 2^{-k_I})$ we find $\delta_\mathbf{Y}^{-1}(x') \cap B(q, 2^{-N}) \neq \emptyset$. The inclusion $\psi H(p_{k_I}) \subseteq B(x, 2^{-I} + 2^{-k_I})$ of a compact set in an open set implies that there is some $L > k_I$ such that for all $p' \in B(p_{k_I}, 2^{-L}) \cap \operatorname{dom}(\mathrm{C}_\mathbf{X}|_\mathfrak{A}\psi)$ we find $\psi H p' \subseteq B(x, 2^{-I} + 2^{-k_I})$.

The choice of p_{k_I}, L and the point $y \in \mathbf{X}$ ensures that our reduction may answer any valid input to $\mathrm{C}_\mathbf{X}|_\mathfrak{A}$ sharing a prefix of length L with p_{k_I} with a name of some point $y' \in B(y, \varepsilon)$. However, as we have $p_{k_I} \in X_\varepsilon(\mathfrak{A})$, we can extend any long prefix of p_{k_I} to a name of a set not intersecting the ball $B(y, \varepsilon)$ – this means, our reduction would answer wrong, and we have found the desired contradiction.

Corollary 3 (Large Diameter Principle for fractals). *Let $C_\mathbf{X}|_\mathfrak{A}$ be a fractal, $\mathbf{Y}$ be compact and $C_\mathbf{X}|_\mathfrak{A} \leq_W C_\mathbf{Y}|_\mathfrak{B}$. Then for any $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$(C_\mathbf{X})_{X_\varepsilon(\mathfrak{A})} \leq_W C_\mathbf{Y}|_{\mathfrak{B},\lambda>\delta}$$

3 Separation results for finite and convex choice

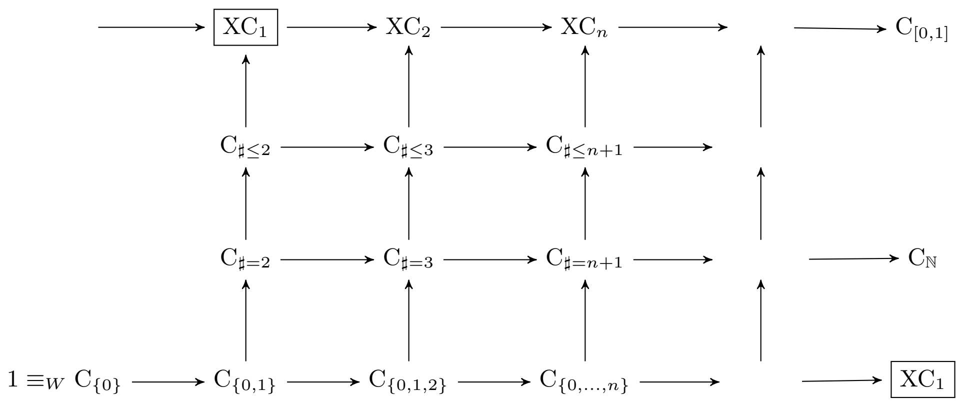

Fig. 1. The reducibilities

We now have the tools available to completely characterize the valid reductions between $\mathrm{C}_{\{0,\ldots,n\}}$, XC_m, $\mathrm{C}_{\sharp\leq i}$ and $\mathrm{C}_{\sharp=j}$. Figure 1 provides an overview – the absence of an arrow (up to transitivity) indicates a proof of irreducibility. Besides an application of the general techniques of the preceding section, more specialized proof methods are employed, some with a rather combinatorial character, others based on the properties of simplices. We also exhibit a technique suitable to transfer results from the compact case to the locally compact case.

Observation 5 *$C_{\sharp=n}$ is a fractal. XC_n and $C_{\sharp\leq n}$ are even closed fractals.*

Corollary 4. *$C_{\sharp=n} \not\leq_W C_{\{0,\ldots,m\}}$ for all $n > 1, m \in \mathbb{N}$.*

Proof. Assume the reduction would hold for some $n, m \in \mathbb{N}$. Observation 5 allows us to use Theorem 2 to conclude $\mathrm{C}_{\sharp=n}$ to be computable - a contradiction for $n > 1$.

Proposition 3. *$C_{\sharp=n} \leq_W C_{\mathbb{N}}$*

Corollary 5. *$C_{\sharp\leq 2} \not\leq_W C_{\sharp=n}$*

Proof. Assume $\mathrm{C}_{\sharp\leq 2} \leq_W \mathrm{C}_{\sharp=n}$ for some $n \in \mathbb{N}$. By Proposition 3, this implies $\mathrm{C}_{\sharp\leq 2} \leq_W \mathrm{C}_{\mathbb{N}}$. Observation 5 together with Theorem 3 would show $\mathrm{C}_{\sharp\leq 2}$ to be computable, contradiction.

3.1 Combinatorial arguments

Proposition 4. $C_{\{0,\dots,n\}} <_{sW} C_{\sharp=n+1}$.

Proposition 5 (Pigeonhole principle). $C_{\{0,\dots,n\}} \nleq_W C_{\sharp\leq n}$

Proposition 6. $C_{\sharp=n+1} \leq_W C_{\sharp=2}^n$ *and* $C_{\sharp\leq n+1} \leq_W C_{\sharp\leq 2}^n$

Proof (Sketch). When trying to find a path through an infinite tree with exactly $n+1$ vertices per level, there are at any moment n vertices where both the left and the right successor could potentially lead to an infinite path. The difficulty solely lies in picking a suitable successor at each of these vertices in each stage. It is possible to disentangle these decisions, yielding n trees with just two vertices per level such that any problematic vertex in the original tree is mapped to a problematic vertex in one of the new trees in a way that preserves correct choices of successors.

As a consequence from the independent choice theorem in [1] together with Proposition 1 we obtain the following, showing ultimately that picking an element from a finite number of 2-element sets in parallel is just as hard as picking finitely many times from finite sets, with the later questions depending on the answers given so far:

Observation 6 $C_{\sharp=n} \times C_{\sharp=m} \leq_W C_{\sharp=n} \star C_{\sharp=m} \leq_W C_{\sharp=(nm)}$ *and* $C_{\sharp\leq n} \times C_{\sharp\leq m} \leq_W C_{\sharp\leq n} \star C_{\sharp\leq m} \leq_W C_{\sharp\leq(nm)}$

Corollary 6. $C_{\sharp=2}^* \equiv_W \left(\coprod_{n\in\mathbb{N}} C_{\sharp=n}\right) \equiv_W \left(\coprod_{n,k\in\mathbb{N}} C_{\sharp=n}^{(k)}\right)$

Corollary 7. $C_{\sharp\leq 2}^* \equiv_W \left(\coprod_{n\in\mathbb{N}} C_{\sharp\leq n}\right) \equiv_W \left(\coprod_{n,k\in\mathbb{N}} C_{\sharp\leq n}^{(k)}\right)$

Whether this property (that sequential uses of some closed choice principle are equivalent to parallel uses) also applies to convex choice XC_1 remains open at this stage. As we do have $\mathrm{XC}_1^k \leq_W \mathrm{XC}_k \leq_W \mathrm{XC}_1^{(k)}$, a positive answer would also imply that increasing the dimension for convex choice means climbing the same hierarchy, again. We point out that the question is related to the open question in [6] whether connected choice in two dimensions is equivalent to connected choice in three dimensions.

Question 1. Is there some $k \in \mathbb{N}$ such that $\mathrm{XC}_1 \star \mathrm{XC}_1 \leq_W \mathrm{XC}_1^k$?

3.2 Simplex choice

Proposition 7. *Given a closed set $A \subseteq [0,1]$ with $|A| \leq n$, we can compute a closed set $B \subseteq [0,1]^{n-1}$ with $|A| = |B|$, $\pi_1(B) = A$, and such that the points in B are affinely independent.*

Proposition 8. *Given a closed set $A \subseteq [0,1]^n$ with $|A| = n+1$ such that the points in A are affinely independent, we can compute a set $A \cup \{c\}$, where c is a point in the interior of the convex hull of A.*

Proposition 9. *Given a finite closed set $A \subseteq [0,1]^n$, such that the points in A are affinely independent, as well as a point x in the convex hull of A, we can compute a point in A.*

Proof (Sketch). We can obtain positive information about A by noticing that removing some area from A makes x fall out of the convex hull of the remaining set. But then this area must actually contain a point. Iteratively refining this information yields some point in A.

Corollary 8. $C_{\sharp \leq n} \leq_W XC_{n-1}$

Theorem 7. $C_{\sharp = n} <_W C_{\sharp = n+1}$

Proof. By Corollary 1, we can freely change the space we are working in among any rich computably compact computable metric space. We start with an n-point subset of $[0,1]$ and apply Proposition 7 to obtain n affinely independent points. Then we use Proposition 8 to obtain a set of cardinality $n+1$ containing the n previous points and some additional point in the interior of their convex hull. This is a valid input to $\mathrm{C}_{\sharp = n+1}$ (using Corollary 1 again), and we obtain one of the points, which certainly is contained in the convex hull. Hence, Proposition 9 allows us to find one of the vertices, which by Proposition 7 is sufficient to compute one of the points in the original set.

That the reduction is strict follows from Propositions 4 and 5.

Note that while $\mathrm{C}_{\sharp \leq n} \leq_W \mathrm{C}_{\sharp \leq n+1}$ is trivially true, the positive part of the preceding result is not obvious.

3.3 Application of the large diameter technique

The usefulness of the large diameter technique for disproving reducibility to convex choice lies in the observation that convex sets with large outer diameter are simpler, as we can then cut by a hyperplane and obtain another convex set of smaller dimension:

Proposition 10 (Cutting). $XC_{n,\lambda > m^{-1}} \leq_W XC_{n-1} \star C_{\{1,\ldots,(m-1)n\}}$

Proof (Sketch). Because the input set has a large outer diameter, we can find $(m-1)n$ hyperplanes such that at least one of them intersects the input set. Picking a suitable hyperplane is done with $\mathrm{C}_{\{1,\ldots,(m-1)n\}}$, and the intersection then is a convex set of lower dimension, hence a valid input for XC_{n-1}.

Corollary 9. *Let $C_{\mathbf{X}}|_{\mathfrak{A}}$ and $(C_{\mathbf{X}})_{X_\varepsilon(\mathfrak{A})}$ be fractals and $C_{\mathbf{X}}|_{\mathfrak{A}} \leq_W XC_{n+1}$. Then we find $(C_{\mathbf{X}})_{X_\varepsilon(\mathfrak{A})} \leq_W XC_n$.*

Proof. Corollary 3 gives us $(\mathrm{C}_{\mathbf{X}})_{X_\varepsilon(\mathfrak{A})} \leq_W \mathrm{XC}_{n+1,\lambda > m^{-1}}$ for some $m \in \mathbb{N}$, then Proposition 10 implies $(\mathrm{C}_{\mathbf{X}})_{X_\varepsilon(\mathfrak{A})} \leq_W \mathrm{XC}_n \star \mathrm{C}_{\{1,\ldots,(m-1)(n+1)\}}$ and finally Theorem 2 fills the gap to $(\mathrm{C}_{\mathbf{X}})_{X_\varepsilon(\mathfrak{A})} \leq_W \mathrm{XC}_n$.

For $n \geq k \geq 1$ let $\mathrm{C}_{\sharp=n \rhd k} := \mathrm{C}_{[0,1]}|_{\{A \in \mathcal{A}([0,1]) \mid |A|=n \wedge |\{i<n \mid [\frac{2i}{2n}, \frac{2i+1}{2n}] \cap A \neq \emptyset\}| \geq k\}}$. So $\mathrm{C}_{\sharp=n \rhd k}$ is choice for n element sets, where we know that our set intersects at least k of a collection of fixed distinct regions. We shall need three properties of these choice principles:

Proposition 11. *1.* $C_{\sharp=n \rhd (k+1)} \leq_W (C_{[0,1]})_{X_{(5n)}^{-1}(\mathrm{dom}(C_{\sharp=n \rhd k}))}$.
2. $C_{\sharp=n+1 \rhd n}$ *is not computable.*
3. Any $C_{\sharp=n \rhd k}$ *is a fractal.*

Corollary 10. $C_{\sharp=n \rhd k} \leq_W XC_{m+1}$ *implies* $C_{\sharp=n \rhd (k+1)} \leq_W XC_m$.

Theorem 8. $C_{\sharp n+2} \not\leq_W XC_n$.

Proof. Assume $\mathrm{C}_{\sharp=n+2} = \mathrm{C}_{\sharp=(n+2) \rhd 1} \leq_W \mathrm{XC}_n$. Iterated use of Corollary 10 allows us to conclude that $\mathrm{C}_{\sharp=(n+2) \rhd (n+1)}$ is computable, which contradicts Proposition 11 (2).

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