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Loop amplitudes in an extended gravity theory

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ABSTRACT

We extend the S-matrix of gravity by the addition of the minimal three-point amplitude or equivalently adding R^3 terms to the Lagrangian. We demonstrate how Unitarity can be used to simply examine the renormalisability of this theory and determine the R^4 counter-terms that arise at one-loop. We find that the combination of R^4 terms that arise in the extended theory is complementary to the R^4 counter-term associated with supersymmetric Lagrangians.

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1. Introduction

There are different approaches to defining a quantum field theory. One approach is to specify a local Lagrangian density containing fundamental fields and then use the associated Feynman diagrams to determine the scattering amplitudes that contribute to the S-matrix. Alternatively, one can use the on-shell amplitudes to define the theory [1]. In this approach singular structure and symmetries are crucial to constrain and ultimately compute the scattering amplitudes. With enough constraints, higher point amplitudes may be derived from a limited set of lower point functions [2]. In the minimal case all n-point amplitudes may be constructed from only the three-point amplitudes plus the requirement of factorisation. In [3] such theories were described as *fully constructible*. Yang–Mills and gravity both have fully constructible tree amplitudes. In [4] we showed that in a theory of gravity deformed by additional three point vertices the leading deformation to the tree amplitudes was also fully constructible.

In this article we examine the one-loop amplitudes for both extended Yang–Mills and gravity and in particular their ultra-violet structure. In agreement with long established results we find that the extended Yang–Mills theory is renormalised [5–8]. For gravity, we find that additional counter-terms/amplitudes are required. The use of four-dimensional unitarity techniques gives relatively easy access to the ultra-violet structures of these theories.

2. Structure of the amplitudes

A one-loop amplitude in a theory of massless particles can be expressed, after a Passarino–Veltman reduction [9], in the form

$$A_n^{\text{1-loop}} = \sum_{i \in \mathcal{C}} a_i I_4^i + \sum_{j \in \mathcal{D}} b_j I_3^j + \sum_{k \in \mathcal{E}} c_k I_2^k + R_n + O(\epsilon) \qquad (1)$$

where the I_m^i are m-point scalar integral functions and the a_i etc. are rational coefficients. $\mathcal{C}$ is the set of box integral functions with all allowed partitions of the external legs between the corners. For a colour-ordered Yang–Mills amplitude the allowed partitions respect the cyclic ordering of the legs. Similarly $\mathcal{D}$ and $\mathcal{E}$ are the sets of triangle and bubble integral functions. R_n is a purely rational term. The integral functions depend upon the number of "massive" legs (or more correctly legs with non-null momenta) so, for example, the triangle functions come in three types: with either one, two or three massive legs. This form is an expansion in terms of the integral functions appearing and so is useful when computing the coefficients from the cut singularities of the amplitude.

Alternately, we can re-express the one-loop amplitude for pure Yang–Mills or gravity in a form which highlights the singular structure of the amplitude,

$$A_n^{\text{1-loop}} = A_n^{\text{tree}} \mathcal{I}_n + \mathcal{G}_n + \mathcal{F}_n + \mathcal{R}_n \qquad (2)$$

where $\mathcal{I}_n$ contains the soft-singular Infra-Red (IR) [10] terms of the amplitude and is

$$\mathcal{I}_n = -\frac{c_\Gamma}{(4\pi)^2} \sum_{i=1}^{n} \frac{(-s_{ii+1}/\mu^2)^{-\epsilon}}{\epsilon^2} \qquad (3)$$

for a colour ordered gluon amplitude where $c_\Gamma = (4\pi)^\epsilon \Gamma^2(1-\epsilon)\Gamma(1+\epsilon)/\Gamma(1-2\epsilon)$. For a graviton scattering amplitude [11,12],

$$\mathcal{I}_n = -\frac{c_\Gamma}{(4\pi)^2} \sum_{i<j} s_{ij} \frac{(-s_{ij}/\mu^2)^{-\epsilon}}{\epsilon^2} . \qquad (4)$$

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Within the decomposition of (1) the contributions to $\mathcal{I}_n$ arise from the box integral functions and the one and two mass triangle integral functions. $\mathcal{G}_n$ is of the form

$$\mathcal{G}_n = \sum_i c_i \frac{(-s_{ii+1}/\mu^2)^{-\epsilon}}{\epsilon} \qquad (5)$$

for gluon scattering amplitudes and

$$\mathcal{G}_n = \sum_{i<j} c_{ij} \frac{(-s_{ij}/\mu^2)^{-\epsilon}}{\epsilon} \qquad (6)$$

for graviton scattering. Within the integral basis decomposition (1), the $\mathcal{G}_n$ arise from the bubble integral functions. The $\mathcal{G}_n$ terms contain both the collinear IR singular terms and the UV divergences. The function $\mathcal{F}_n$ contains the finite transcendental functions. These arise from both the box integral functions and from the three-mass triangle integral function. $\mathcal{R}_n$ is the remaining finite rational term.

The coefficients of the integral functions can be determined using four dimensional generalised unitarity techniques from the on-shell amplitudes. Computing $\mathcal{R}_n$ from unitarity requires using $d = 4 - 2\epsilon$ tree amplitudes.

We will find that the form of the IR singularities is not altered in the extended theories: as might be expected by naive power counting.

3. Yang–Mills case

Before looking at gravity theories, we consider the case of gluon scattering in pure Yang–Mills. We will consider a colour ordered formalism and examine the colour ordered partial amplitudes which have cyclic symmetry. The full amplitude can be reconstructed by multiplying by the colour factors and summing over permutations. We also restrict to the leading in N_c colour ordered partial amplitude for which the one-loop leading in colour amplitudes have a factor of N_c.

We define the theory from the fundamental three point amplitudes,

$$A(1^{h_1}, 2^{h_2}, 3^{h_3}) \qquad (7)$$

where h_i is the helicity of the i-th leg. For gluons the helicity is ± 1. Three point amplitudes vanish for real momenta but may be non-zero if we allow complex momenta. If we express the momenta in spinor variables $p^{\alpha\dot\alpha} = \lambda^\alpha \bar\lambda^{\dot\alpha}$ then amplitudes become functions of the bilinears $\langle a\,b \rangle = \epsilon_{\alpha\beta} \lambda_a^\alpha \lambda_b^\beta$ and $[a\,b] = -\epsilon_{\dot\alpha\dot\beta} \bar\lambda_a^{\dot\alpha} \bar\lambda_b^{\dot\beta}$. Since, for a three point amplitude of massless particles we must have $s_{ab} = \langle a\,b \rangle [b\,a] = 0$, there are two possibilities:

$$a) \quad \langle i\,j \rangle = 0, \ [i\,j] \neq 0$$

$$b) \quad [i\,j] = 0, \ \langle i\,j \rangle \neq 0 \qquad (8)$$

Consequently we can build three point amplitudes either from $\langle i\,j \rangle$ or $[i\,j]$ but not both. Under scaling $\lambda_i \longrightarrow t_i \lambda_i$, $\bar\lambda_i \longrightarrow t_i^{-1} \bar\lambda_i$ the amplitude must scale as t^{-2h_i}. Finally requiring that the amplitude vanishes for real momenta leads to the following unique non-zero three-point gluon amplitudes,

$$A_{3:\text{tree}}(1^-, 2^-, 3^+) = g \frac{\langle 1\,2 \rangle^3}{\langle 2\,3 \rangle \langle 3\,1 \rangle},$$

$$A_{3:\text{tree}}(1^+, 2^+, 3^-) = g \frac{[2\,1]^3}{[2\,3][3\,1]},$$

$$A_{3:\text{tree}}(1^+, 2^+, 3^+) = \alpha g\, [1\,2][2\,3][3\,1],$$

$$A_{3:\text{tree}}(1^-, 2^-, 3^-) = \alpha g\, \langle 1\,3 \rangle \langle 3\,2 \rangle \langle 2\,1 \rangle. \qquad (9)$$

The first two amplitudes are the well known MHV ("Maximally Helicity Violating") and $\overline{\text{MHV}}$ amplitudes. They form the top elements in a multiplet involving all the helicities of N = 4 super-Yang–Mills. The parameter α is dimensionful with mass dimension minus two, i.e. $\alpha = c/M^2$ where c is dimensionless and M is some mass scale[1] and the α expansion can be considered as an expansion in inverse powers of the mass M.

In this letter we consider expanding the theory by including the α-vertices in addition to the MHV vertices. The amplitudes in this theory can then be expanded as a power series in α,

$$A_n(1, \cdots, n) = \sum_{r=0} \alpha^r A_n^{\alpha^r}(1, \cdots, n) \qquad (10)$$

where $A_n^{\alpha^0}$ is the usual Yang–Mills amplitude.

From a Lagrangian field theory viewpoint we are extending the theory by

$$\mathcal{L}_{F^3} = \alpha' \text{Tr}(F_{\mu\nu} F^{\nu\rho} F_\rho{}^\mu), \qquad (11)$$

where $\alpha' = -\alpha g/3$ [13]. Having defined the three-point vertices, factorisation can be used [14,4] to obtain the leading four-point tree amplitudes:

$$A_{4:\text{tree}}^\alpha(1^+, 2^+, 3^+, 4^+) = 2\alpha \frac{stu}{\langle 1\,2 \rangle \langle 2\,3 \rangle \langle 3\,4 \rangle \langle 4\,1 \rangle}$$
$$= 2\alpha K_{++++} \times u,$$

$$A_{4:\text{tree}}^\alpha(1^-, 2^+, 3^+, 4^+) = -\alpha \frac{[2\,4]^2 st}{[1\,2] \langle 2\,3 \rangle \langle 3\,4 \rangle [4\,1]}$$
$$= \alpha K_{-+++} \times u. \qquad (12)$$

(These expressions match those obtained from Feynman diagram calculations [15], or colour-kinematics duality [13] or scattering equations [16].) The combinations K_{++++} and K_{-+++} carry all the necessary spinor weight of the amplitude with $|K_{++++}| = |K_{-+++}| = 1$ for real momenta. The factor K_{++++} has manifest cyclic symmetry but is also fully crossing symmetric since

$$\frac{st}{\langle 1\,2 \rangle \langle 2\,3 \rangle \langle 3\,4 \rangle \langle 4\,1 \rangle} = \frac{su}{\langle 1\,2 \rangle \langle 2\,4 \rangle \langle 4\,3 \rangle \langle 3\,1 \rangle}$$
$$= \frac{tu}{\langle 1\,4 \rangle \langle 4\,2 \rangle \langle 2\,3 \rangle \langle 3\,1 \rangle}. \qquad (13)$$

Similarly K_{-+++} has manifest flip-symmetry but is also invariant under exchange of the positive helicity legs, $2 \leftrightarrow 3$ etc. These factors can be written in many ways, e.g.

$$K_{++++} = -\frac{[1\,2]^2}{\langle 3\,4 \rangle^2} = -\frac{[1\,2]^2 [3\,4]^2}{s^2}. \qquad (14)$$

For the pure Yang–Mills theory, all tree amplitudes can be constructed using factorisation from the three-point trees [17]: i.e. the theory is "constructible" using the definition of [3]. The leading deformed amplitudes A_n^α also can also be constructed in this way, however amplitudes beyond this leading deformation are not constructible purely from factorisation. This will be pursued further in the context of gravity theories later.

We now wish to determine the one-loop amplitudes in the extended theory to leading order in α. Unitarity methods have proven very efficient in determining one-loop amplitudes using the on-shell tree amplitudes.

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[1] With the normalisation of (9) each n-point, L-loop amplitude contains a factor of g^{n-2+2L} which we suppress.
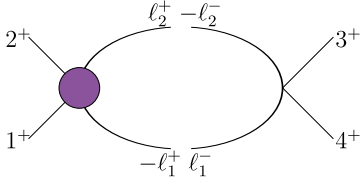
Fig. 1. The two-particle cut of the all-plus amplitude.

Working in four dimensions the simplicity of the four-dimensional trees greatly simplifies the calculation of the coefficients of the integral functions, but does not allow us to compute the finite rational terms $\mathcal{R}$. Since we are mainly interested in the UV singularities, which come with an accompanying $\ln(\mu^2/s)$, four dimensional two-particle cuts suffice [18].

Noting that

$$A^\alpha_{4\text{:1-loop}}(1^-, 2^-, 3^+, 4^+)|_{cut\ part} = 0 \tag{15}$$

since there are no non-vanishing four dimensional cuts for these amplitudes, the non-vanishing α^1 amplitudes are the all-plus and the single minus.

Calculating the s-channel two-particle cut for the all-plus amplitude, as shown in Fig. 1, gives

$$
\begin{aligned}
C_2 &= \int dLIPS\, A^\alpha_{4\text{:tree}}(1^+, 2^+, \ell_2^+, -\ell_1^+) \\
&\quad \times A^{\alpha^0}_{4\text{:tree}}(\ell_1^-, -\ell_2^-, 3^+, 4^+) \\
&= -2\alpha \int dLIPS\, \frac{s_{12}s_{1\ell_1}s_{1\ell_2}}{\langle 1\,2\rangle \langle 2\,\ell_2\rangle \langle \ell_2\,\ell_1\rangle \langle \ell_1\,1\rangle} \\
&\quad \times \frac{\langle \ell_1\,\ell_2\rangle^4}{\langle 3\,4\rangle \langle 4\,\ell_1\rangle \langle \ell_1\,\ell_2\rangle \langle \ell_2\,3\rangle} \\
&= -2\alpha \int dLIPS\, \frac{s_{12}}{\langle 1\,2\rangle \langle 3\,4\rangle} \frac{s_{1\ell_2}[\ell_1\,1]}{\langle 2\,\ell_2\rangle} \times \frac{\langle \ell_1\,\ell_2\rangle^2}{\langle 4\,\ell_1\rangle \langle \ell_2\,3\rangle} \tag{16}
\end{aligned}
$$

where we also have a configuration where the α vertex is on the right-hand side. Manipulating eq. (16) using

$$\frac{\langle \ell_2\,\ell_1\rangle [\ell_1\,1]}{\langle 2\,\ell_2\rangle} = \frac{\langle \ell_2\,2\rangle [2\,1]}{\langle 2\,\ell_2\rangle} = [1\,2] , \tag{17}$$

we have

$$C_2 = -2\alpha \int dLIPS\, \frac{[1\,2]^2}{\langle 3\,4\rangle} \times \frac{s_{1\ell_2}\langle \ell_1\,\ell_2\rangle}{\langle 4\,\ell_1\rangle \langle \ell_2\,3\rangle} . \tag{18}$$

Now

$$
\begin{aligned}
\langle 4\,\ell_1\rangle \langle \ell_2\,3\rangle \times [\ell_1\,\ell_2] &= \langle 4|\ell_1\ell_2|3\rangle = -\langle 4\,3\rangle [3|\ell_2|3\rangle \\
&= -\langle 3\,4\rangle (\ell_2 - k_3)^2 , \tag{19}
\end{aligned}
$$

so

$$C_2 = 2\alpha \int dLIPS\, \frac{[1\,2]^2 s_{12}}{\langle 3\,4\rangle^2} \times \frac{[1|\ell_2|1\rangle}{(\ell_2 - k_3)^2} . \tag{20}$$

Now, rather than perform the integration over the cut momenta we recognise this as the cut of a covariant integral which we can evaluate:

$$\int dLIPS\, f(\ell_1, \ell_2) \longrightarrow \int d^D\ell\, \frac{f(\ell_1, \ell_2)}{\ell_1^2 \ell_2^2} \tag{21}$$

where we only keep terms with a s-channel cut in the resultant integral [19,20]. Consequently we replace C_2 by a triangle integral with linear numerator $I_3[\ell_2^\mu]$. Using

$$I_3[\ell_2^\mu] = \frac{(k_4 - k_3)^\mu}{s} I_2(s) + k_3^\mu I_3(s) , \tag{22}$$

where $I_2(s)$ is the scalar bubble integral function and $I_3(s)$ is the one-mass scalar triangle integral function which only depend on the kinematic variable s, we obtain

$$
\begin{aligned}
&-2\alpha \frac{[1\,2]^2}{\langle 3\,4\rangle^2}[1|(4-3)|1\rangle I_2(s) - 2\alpha \frac{[1\,2]^2 s_{12}}{\langle 3\,4\rangle^2}[1|3|1\rangle I_3(s) \\
&= -2\alpha \frac{[1\,2]^2}{\langle 3\,4\rangle^2}(t-u) I_2(s) - 2\alpha \frac{[1\,2]^2 s_{12}s_{13}}{\langle 3\,4\rangle^2} I_3(s) \\
&= 2\alpha K_{++++}(t-u) I_2(s) + 2\alpha K_{++++}u \times s I_3(s) . \tag{23}
\end{aligned}
$$

Doubling this to account for inserting the F^3 operator on the opposite side of the cut and combining with the t-channel cut leads to the amplitude

$$
\begin{aligned}
A^\alpha_{1\text{-loop}}(1^+, 2^+, 3^+, 4^+) &= A^\alpha_{\text{tree}}\mathcal{I}_4 \\
&\quad + 4\alpha K_{++++}\Big[(t-u) I_2(s) + (s-u) I_2(t)\Big] + \mathcal{R}_4 \tag{24}
\end{aligned}
$$

where we have used

$$2s I_3(s) + 2t I_3(t) = \mathcal{I}_4 . \tag{25}$$

Note that there are no box functions in this amplitude and the triangle functions generate the soft part of the IR term. The overall coefficient of ϵ^{-1} is

$$
\begin{aligned}
4\frac{\alpha}{(4\pi)^2} ((t-u) + (s-u)) K_{++++} &= -\frac{12}{(4\pi)^2} u\alpha K_{++++} \\
&= -\frac{6}{(4\pi)^2} \times A^\alpha_{\text{tree}} \tag{26}
\end{aligned}
$$

which is proportional to the tree and so the one-loop UV infinity leads to a renormalisation of the αF^3 term.

We also compute the one-loop contribution to the single minus amplitude $A^{(1)}(1^-, 2^+, 3^+, 4^+)$. There are three non-zero configurations as shown in Fig. 2. The first of these is

$$
\begin{aligned}
C_2^A &= A^\alpha_{4\text{:tree}}(1^-, 2^+, l_2^+, -l_1^+) \times A^{\alpha^0}_{4\text{:tree}}(l_1^-, -l_2^-, 3^+, 4^+) \\
&= -\alpha \frac{[2-l_1]^2 s_{12}s_{2l_2}}{[1\,2]\langle 2\,l_2\rangle \langle l_2 - l_1\rangle [-l_1\,1]} \times \frac{\langle l_1 - l_2\rangle^3}{\langle -l_2\,3\rangle \langle 3\,4\rangle \langle 4\,l_1\rangle} \\
&= -\alpha \frac{\langle 1\,2\rangle}{\langle 3\,4\rangle} \frac{[2\,l_1]^2 [2\,l_2]}{[l_1\,1]} \times \frac{\langle l_1\,l_2\rangle^2}{\langle l_2\,3\rangle \langle 4\,l_1\rangle} . \tag{27}
\end{aligned}
$$

This can be rearranged to

$$C_2^A = -\alpha \frac{s_{12}^2}{\langle 3\,4\rangle^2} \times \frac{[2\,l_1][2\,l_2]\langle l_1\,1\rangle \langle l_2\,1\rangle}{(l_1 - k_1)^2 (l_1 + k_4)^2} \tag{28}$$

which is the two particle cut of a quadratic box. Replacing this by a covariant integral and only keeping terms with a s-channel cut we obtain

$$C_2^A = -\alpha K_{-+++}\left(\frac{s^3 t}{2u} I_4 + \frac{s^3}{u} I_3(s) - su I_3(s) - (2t+s) I_2(s)\right) . \tag{29}$$

The other terms are a little more complex (requiring a quartic box integral) and yield

$$
\begin{aligned}
C_2^B &= -\alpha K_{-+++}\left(\frac{s t^3}{2u} I_4(s,t) + \frac{s t^2}{u} I_3(s) - t I_2(s)\right) , \\
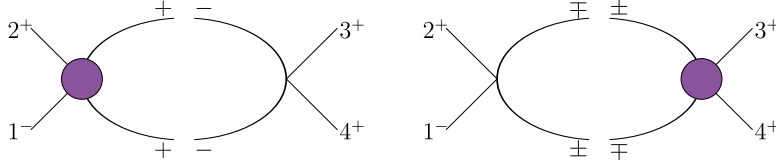C_2^C &= -\alpha K_{-+++}\left(\frac{s t u}{2} I_4(s,t) + u I_2(s)\right) . \tag{30}
\end{aligned}
$$

Fig. 2. The bubbles for single minus.

Combining these gives

$$A^\alpha_{1\text{-loop}}(1^-, 2^+, 3^+, 4^+)$$

$$= -\alpha K_{-+++}\left(\frac{st}{2u}(s^2 + t^2 + u^2)I_4(s,t)\right.$$

$$+ \frac{(s^2 + t^2 - u^2)}{u}(sI_3(s) + tI_3(t))$$

$$\left. - (2s + 4t)I_2(s) - (4s + 2t)I_2(t)\right) + \mathcal{R}_4 . \tag{31}$$

The box functions combine with the triangle functions to generate the IR singular terms and the finite transcendental function,

$$aI_4 + \sum_{s,t} b_j I_3^j = A_4^{\text{tree}}\mathcal{I}_4 + \mathcal{F}_4 \tag{32}$$

where

$$\mathcal{F}_4 = -\frac{\alpha}{(4\pi)^2}K_{-+++} \times \frac{(s^2 + u^2 + t^2)}{u}\ln^2(s/t) \tag{33}$$

and we find no corrections to the IR structure.

We also have the UV terms: the coefficient of ϵ^{-1} in these terms is

$$\frac{\alpha}{(4\pi)^2}K_{-+++}(2s + 4t) + (4s + 2t)) = -6\frac{\alpha}{(4\pi)^2}uK_{-+++}$$

$$= -\frac{6}{(4\pi)^2}A^\alpha_{\text{tree}}(1^-, 2^+, 3^+, 4^+) \tag{34}$$

which matches the singularity found for the $A(1^+2^+3^+4^+)$ case.

Equations (26) and (34) are the singularities in the bare amplitudes. The amplitudes contain universal collinear IR singularities [10]:

$$-\frac{n\beta_0 g^2}{2(4\pi)^2}A^\alpha_{\text{tree}} = -\frac{22N_c g^2}{3(4\pi)^2}A^\alpha_{\text{tree}} \tag{35}$$

where $\beta_0 = 11N_c/3$. We determine the UV divergence by first subtracting these from the bare singularity.

When renormalising the theory there must be a simultaneous renormalisation of g^2 and α. The renormalisation of g^2 is unaltered since it is determined by the α^0 amplitudes.

$$g^2\alpha \longrightarrow g^2\alpha - g^4\beta_0\alpha + g^2\delta\alpha \tag{36}$$

so that (reinserting g and N_c)

$$g^2\delta\alpha - \frac{g^4\beta_0\alpha}{\epsilon} - \frac{6N_c\alpha}{(4\pi)^2\epsilon} = -\frac{22N_c g^2}{3(4\pi)^2}\alpha \tag{37}$$

so that

$$\delta\alpha = \frac{g^2 N_c\alpha}{(4\pi)^2\epsilon}(6 - \frac{11}{3}) = \frac{7N_c\alpha}{3}\frac{g^2}{(4\pi)^2} \tag{38}$$

This value of $7N_c/3$ matches previous calculations of the anomalous dimension of Yang–Mills extended by the F^3 operator [5–8].

4. Extended gravity amplitudes

We now consider extending gravity by additional three point vertices. For gravitons with $h = \pm 2$ the possible three point amplitudes are[2]

$$M_{3:\text{tree}}(1^-, 2^-, 3^+) = \frac{\langle 12\rangle^6}{\langle 23\rangle^2 \langle 31\rangle^2} ,$$

$$M_{3:\text{tree}}(1^+, 2^+, 3^-) = \frac{[12]^6}{[23]^2 [31]^2} ,$$

$$M_{3:\text{tree}}(1^+, 2^+, 3^+) = \alpha [12]^2 [23]^2 [31]^2 ,$$

$$M_{3:\text{tree}}(1^-, 2^-, 3^-) = \alpha \langle 12\rangle^2 \langle 23\rangle^2 \langle 31\rangle^2 . \tag{39}$$

From a Lagrangian viewpoint we are considering the theory

$$L = \int d^D x\sqrt{-g}(R + \frac{\alpha}{60}R_{abcd}R^{cdef}R_{ef}{}^{ab}) . \tag{40}$$

The non-vanishing four-point α^0 tree is

$$M^{\alpha^0}_{4:\text{tree}}(1^-, 2^-, 3^+, 4^+) = -\frac{[34]\langle 12\rangle^6}{\langle 34\rangle \langle 13\rangle \langle 14\rangle \langle 23\rangle \langle 24\rangle} \tag{41}$$

and the non-zero α^1 trees are

$$M^\alpha_{4:\text{tree}}(1^+, 2^+, 3^+, 4^+) = -10\left(\frac{st}{\langle 12\rangle \langle 23\rangle \langle 34\rangle \langle 41\rangle}\right)^2 stu$$

$$= -10K^2_{++++} \times stu ,$$

$$M^\alpha_{4:\text{tree}}(1^-, 2^+, 3^+, 4^+) = \left(\frac{[24]^2}{[12]\langle 23\rangle \langle 34\rangle [41]}\right)^2 \frac{-s^3 t^3}{u}$$

$$= -K^2_{-+++} \times stu . \tag{42}$$

These four-point amplitudes due to a R^3 term have been computed using field theory methods long ago [21] but can also be obtained from the three-point amplitudes by factorisation [14,4]. These expressions also appear as the UV infinite pieces of both two-loop gravity in four dimensions [22,23] and one-loop gravity in six dimensions [24].

We evaluate the one-loop amplitude for the all-plus amplitude via the two particle cuts

$$C_2 = \int dLIPS\, M^\alpha_{4:\text{tree}}(1^+, 2^+, \ell_2^+, -\ell_1^+)$$

$$\times M^{\alpha^0}_{4:\text{tree}}(\ell_1^-, -\ell_2^-, 3^+, 4^+) \tag{43}$$

Arranging the trees carefully we obtain

$$C_2 = 10\int dLIPS\, \frac{[12]^2 [\ell_1\ell_2]}{\langle \ell_1\ell_2\rangle}[2\ell_1][1\ell_2][2\ell_2][\ell_1 1]$$

$$\times \frac{s\langle \ell_1\ell_2\rangle^4 [34]}{\langle 34\rangle^3}\left(\frac{1}{s_{\ell 14}} + \frac{1}{s_{\ell 13}}\right)$$

---

[2] We remove a factor of $i(\kappa/2)^{n-2+2L}(4\pi)^{-(2-\epsilon)L}$ from the n-point L-loop amplitude.

$$= 10 \int dLIPS \, \frac{s^2 \, [1\,2]^2 \, [3\,4]}{\langle 3\,4 \rangle^3} [2\,\ell_1][1\,\ell_2][2\,\ell_2][\ell_1\,1]$$

$$\times \, \langle \ell_1 \, \ell_2 \rangle^2 \left(\frac{1}{s_{\ell_1 4}} + \frac{1}{s_{\ell_1 3}} \right)$$

$$= 10 \int dLIPS \, \frac{s^2 \, [1\,2]^4 \, [4\,3]}{\langle 3\,4 \rangle^3} [1|\ell_1|1\rangle [1|\ell_2|1\rangle$$

$$\times \left(\frac{1}{s_{\ell_1 4}} + \frac{1}{s_{\ell_1 3}} \right) \tag{44}$$

which is the cut of a pair of quadratic triangle integrals. Replacing this by the covariant integral and evaluating yields

$$10 \frac{tu}{s^3} [1\,2]^4 [3\,4]^4 \times s^2 I_3(s)$$

$$+ 10 \frac{[1\,2]^4 [3\,4]^4}{s^2} \times \left(t^2 + u^2 - 4tu \right) I_2(s) , \tag{45}$$

where both integrals yield the same resultant integral function. After doubling this to account for inserting the R^3 operator on the opposite side of the cut, this can be rewritten as

$$20 K_{++++}^2 stu \times s^2 I_3(s) + 20 K_{++++}^2 \times s^2 \left(t^2 + u^2 - 4tu \right) I_2(s) . \tag{46}$$

The triangle functions correctly generate the IR terms and so the amplitude can be expressed as

$$M_{4:\text{1-loop}}^\alpha = M_{4:\text{tree}}^\alpha \mathcal{I}_4 + \mathcal{F}_4 + \mathcal{G}_4 + \mathcal{R}_4 \tag{47}$$

with $\mathcal{F}_4 = 0$ and

$$\mathcal{G}_4 = 20 K_{++++}^2 \times \left[s^2 \left(t^2 + u^2 - 4tu \right) I_2(s) \right.$$

$$+ t^2 \left(s^2 + u^2 - 4su \right) I_2(t)$$

$$\left. + u^2 \left(t^2 + s^2 - 4ts \right) I_2(u) \right] . \tag{48}$$

The coefficient of ϵ^{-1} is

$$20 K_{++++}^2 \times \left(s^2 \left(t^2 + u^2 - 4tu \right) + \{s \leftrightarrow t\} + \{s \leftrightarrow u\} \right)$$

$$= 10 K_{++++}^2 \times (s^2 + t^2 + u^2)^2 . \tag{49}$$

In Einstein gravity the IR singularity is of the form $\sum s \ln(s)/\epsilon$ [11, 12] and the additional vertex will not affect this by power counting. Therefore the rational ϵ^{-1} singularities in eq. (49) represent the ultra-violet divergence. Unlike the Yang–Mills case this is not a renormalisation of the cubic vertex but must be cancelled by the addition of a four-point vertex produced by a higher-dimension local operator.

As a consistency check we also consider the single minus amplitude. The s-channel bubble in this case has three configurations:

$$M_{4:\text{tree}}^\alpha(1^-, 2^+, \ell_2^+, -\ell_1^+) \times M_{4:\text{tree}}^{\alpha 0}(\ell_1^-, -\ell_2^-, 3^+, 4^+) , \tag{50}$$

$$M_{4:\text{tree}}^{\alpha 0}(1^-, 2^+, \ell_2^\mp, -\ell_1^\pm) \times M_{4:\text{tree}}^\alpha(\ell_1^\pm, -\ell_2^\mp, 3^+, 4^+) . \tag{51}$$

These give contributions to the coefficients of the bubble integral functions. We find

$$c_A = K_{-+++}^2 \times \frac{s^2}{tu} \left(2(t^4 + u^4) + 5ut(t^2 + u^2) \right) ,$$

$$c_B = K_{-+++}^2 \times \frac{s^2}{tu} \left(2(t^4 + u^4) - 3ut(t^2 + u^2) \right) \tag{52}$$

giving the overall coefficient of $I_2(s)$ to be

$$c = K_{-+++}^2 \times \frac{s^2}{tu} \left(4(t^4 + u^4) + 2ut(t^2 + u^2) \right) . \tag{53}$$

Extracting the UV divergence we find

$$\frac{1}{\epsilon} \times \left(\frac{s^2}{tu} \left(4(t^4 + u^4) + 2ut(t^2 + u^2) \right) + \{s \leftrightarrow u\} + \{s \leftrightarrow t\} \right)$$

$$= 0 \tag{54}$$

and so this amplitude has no UV divergence.

In summary, the UV infinities for the four-point one-loop amplitudes are, (re-inserting the appropriate factors)

$$M_{4:\text{1-loop}}^\alpha(1^-, 2^-, 3^+, 4^+) \Big|_{1/\epsilon} = 0 ,$$

$$M_{4:\text{1-loop}}^\alpha(1^+, 2^+, 3^+, 4^+) \Big|_{1/\epsilon} = \alpha \frac{i}{(4\pi)^2} \left(\frac{\kappa}{2} \right)^4 \frac{1}{\epsilon} \times 10 K_{++++}^2$$

$$\times (s^2 + t^2 + u^2)^2 ,$$

$$M_{4:\text{1-loop}}^\alpha(1^-, 2^+, 3^+, 4^+) \Big|_{1/\epsilon} = 0 . \tag{55}$$

Unlike the Yang–Mills case, the UV infinity is not removed by a renormalisation of the three-point vertex but requires the addition of a four-point vertex which acts as a counterterm. In the following section we examine the counterterm that is required.

5. R^4 operators

From [25] the general R^4 counterterm in arbitrary dimension is

$$\frac{i}{(4\pi)^4 \epsilon} \left[a_1 T_1 + a_2 T_2 + a_3 T_3 + a_4 T_4 + a_5 T_5 + a_6 T_6 + a_7 T_7 \right] \tag{56}$$

where

$$T_1 = (R_{abcd} R^{abcd})^2 ,$$

$$T_2 = R_{abcd} R^{abc}{}_e R_{fgh}{}^d R^{fghe} ,$$

$$T_3 = R^{ab}{}_{cd} R^{cd}{}_{ef} R^{ef}{}_{gh} R^{gh}{}_{ab} ,$$

$$T_4 = R_{abcd} R^{ab}{}_{ef} R^{ce}{}_{gh} R^{dfgh} ,$$

$$T_5 = R_{abcd} R^{ab}{}_{ef} R^c{}_g{}^e{}_h R^{dgfh} ,$$

$$T_6 = R_{abcd} R^a{}_e{}^c{}_f R^e{}_g{}^f{}_h R^{bgdh} ,$$

$$T_7 = R_{abcd} R^a{}_e{}^c{}_f R^e{}_g{}^b{}_h R^{fgdh} \tag{57}$$

and the combination

$$-\frac{T_1}{16} + T_2 - \frac{T_3}{8} - T_4 + 2T_5 - T_6 + 2T_7 \tag{58}$$

vanishes on-shell in any dimension due to it being proportional to the Euler form

$$\epsilon_{a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8} \epsilon^{b_1 b_2 b_3 b_4 b_5 b_6 b_7 b_8} R^{a_1 a_2}{}_{b_1 b_2} R^{a_3 a_4}{}_{b_3 b_4} R^{a_5 a_6}{}_{b_5 b_6} R^{a_7 a_8}{}_{b_7 b_8} . \tag{59}$$

In $D = 4$ these R^4 tensors reduce to two independent tensors. One of these is the square of the "Bel–Robinson" tensor [26] which was shown to be consistent with supersymmetry and thus became a candidate counterterm for supergravity theories [27]. In higher dimensions this tensor extends to a two-parameter set [28].
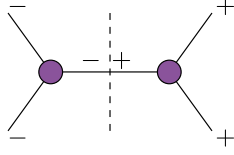
Fig. 3. Factorisations of the four-point MHV amplitude at α^2.

Computing with the general counterterm [24,29][3]

$$M^c(1^+, 2^+, 3^+, 4^+) = 8\,(8a_1 + 2a_2 + 4a_3 + a_6)$$
$$\times (s^2 + t^2 + u^2)^2 K^2_{++++}\,,$$

$$M^c(1^-, 2^+, 3^+, 4^+) = 0\,,$$

$$M^c(1^-, 2^-, 3^+, 4^+) = 8\,(16a_1 + 4a_2 + 4a_4 + 3a_6 + 2a_7)$$
$$\times \langle 1\,2 \rangle^8 K^2_{++++} \qquad (60)$$

where we see explicitly the two independent choices of tensors expressed in amplitudes. In general a tensor could be expressed as

$$c_+ R^4_+ + c_- R^4_- \qquad (61)$$

where R^4_+ is the counterterm consistent with supersymmetry and R^4_- is *orthogonal* to it, in the sense that it yields $M^c(1^-, 2^-, 3^+, 4^+) = 0$. A general counterterm would be a combination of the two.

Clearly the order α^1 theory is made UV finite by the addition of the orthogonal counterterm, R^4_-. This could be realised, for example, by choosing $R^4_- = T_3$ with

$$R^4_{counter} = \frac{i}{(4\pi)^4 \epsilon} T_3 \qquad (62)$$

6. Beyond cubic vertices

The non-extended theory of graviton scattering (and of gluons) is *constructible*: that is the entire S-matrix can be generated by demanding that the amplitudes are factorisable [30]. In practice the factorisation can be excited by the BCFW shift. In the extended theory the leading deformation of the S-matrix is also constructible [4] albeit by using alternative shifts [31,32]. However at order α^2, if we consider $M^{\alpha^2}_{4:\text{tree}}(1^-, 2^-, 3^+, 4^+)$ there is a single factorisation as shown in Fig. 3. The amplitude

$$M^{\alpha^2}_{4:\text{tree}}(1^-, 2^-, 3^+, 4^+) = \alpha^2 \langle 1\,2 \rangle^4 [3\,4]^4 \left(\frac{tu + \beta s^2}{s} \right) \qquad (63)$$

has the correct factorisation for any choice of β. This ambiguity means we also have to specify the four-point amplitude to determine the S-matrix. In the diagrammar approach this ambiguity arises due to the existence of a polynomial function with the correct symmetries and spinor and momentum weight. From a field theory perspective, additional counterterms can contribute to this amplitude. Specifically, we could deform the theory via

$$R \longrightarrow R + C_\alpha R^3 + C_\beta D^2 R^4 \qquad (64)$$

and the four-point amplitude is only specified once C_α and C_β are determined.

From a constructibility viewpoint, defining the theory from its four-point amplitudes is much less constrained than using the

three-point amplitudes because momenta constraints do not limit the vertex to be constructed only from λ_i or $\bar\lambda_i$ but can involve both (or more likely momenta invariants). Specifically we could introduce a fundamental amplitude

$$M_{4:\text{tree}}(1^-, 2^-, 3^+, 4^+) = \beta' s \langle 1\,2 \rangle^4 [3\,4]^4 \qquad (65)$$

From a Lagrangian perspective this would be implemented by a $D^2 R^4$ operator giving non-vanishing $M(1^-, 2^-, 3^+, 4^+)$ but vanishing $M(1^+, 2^+, 3^+, 4^+)$ and $M(1^-, 2^+, 3^+, 4^+)$. As the all-plus and single-minus amplitudes vanish in a supersymmetric theory, any operator compatible with supersymmetry that generates a non-vanishing four-point MHV amplitude will suffice. A specific choice of this is [33]

$$t_{10} t_8 \partial^2 R^4 = t_{10}^{a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9 a_{10}} t_{8 b_1 b_2 b_3 b_4 b_5 b_6 b_7 b_8} \partial_{a_1} R_{a_2 a_3}{}^{b_1 b_2}$$
$$\times \partial_{a_4} R_{a_5 a_6}{}^{b_3 b_4} R_{a_7 a_8}{}^{b_5 b_6} R_{a_9 a_{10}}{}^{b_7 b_8} \qquad (66)$$

where

$$t_{10}^{a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9 a_{10}} = \delta^{a_1 a_4} \delta^{a_2 a_5} \delta^{a_3 a_6} \delta^{a_7 a_9} \delta^{a_8 a_{10}}$$
$$- 4 \delta^{a_1 a_4} \delta^{a_2 a_{10}} \delta^{a_3 a_5} \delta^{a_6 a_7} \delta^{a_8 a_9} \qquad (67)$$

and, where possible, this should be anti-symmetrised with respect to the pairs of indices $a_2 \leftrightarrow a_3$ etc. and symmetrised with respect to pairs of couples of indices $(a_2 a_3) \leftrightarrow (b_1 b_2)$ [33].

Also, $t_8 = \frac{1}{2}\left(t_{(12)} + t_{(48)}\right)$ with

$$t_{(12)}^{ijklmnpq} = -\Big((\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk})(\delta^{mp}\delta^{nq} - \delta^{mq}\delta^{np})$$
$$+ (\delta^{km}\delta^{ln} - \delta^{kn}\delta^{lm})(\delta^{pi}\delta^{qj} - \delta^{pj}\delta^{qi})$$
$$+ (\delta^{im}\delta^{jn} - \delta^{in}\delta^{jm})(\delta^{kp}\delta^{lq} - \delta^{kq}\delta^{lp}) \Big) \qquad (68)$$

$$t_{(48)}^{ijklmnpq} = \Big(\delta^{jk}\delta^{lm}\delta^{np}\delta^{qi} + \delta^{jm}\delta^{nk}\delta^{lp}\delta^{qi} + \delta^{jm}\delta^{np}\delta^{qk}\delta^{li}$$
$$+ [i \leftrightarrow j] + [k \leftrightarrow l] + [m \leftrightarrow n] \Big)$$

where $[i \leftrightarrow j]$ denotes antisymmetrisation with respect to i and j. The tensor t_8 appears in many situations involving maximal supergravity [34] and t_{10} appears in loop amplitudes of less than maximal supergravity [33].

Although constructability from three-point vertices is an attractive concept, unfortunately we find the theory is not completely specified by the three-point vertex.

7. Conclusions

We have studied the S-matrix of extended Yang–Mills and gravity using a diagrammar approach in which the theory is defined by its on-shell amplitudes. If we wish to extend either pure Yang–Mills or gravity by the addition of a three-point interaction there is an essentially unique choice. This choice leads to a theory in which the leading deformation is constructible from three-point amplitudes although higher order deformations require further information to fix the amplitudes. In this letter we have studied the one-loop corrections to these theories and demonstrated how Unitarity can be used to simply examine the renormalisability. For Yang–Mills the one-loop UV infinities renormalise the three-point vertex at leading order. For gravity however the UV infinities must be cancelled by four-point amplitudes arising from a different source. Extending the S-matrix of gravity by the addition of the minimal three-point amplitude is equivalent to adding R^3

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[3] This is in arbitrary dimension D where the momenta of the four particles define a four dimensional plane and we can choose two of the helicities to lie in this plane.

terms to the Lagrangian. From a Lagrangian view point this is then renormalised at one-loop by R^4 counterterms. For the leading deformations we find that these counterterms are the combination that is orthogonal to the R^4 counterterm associated with supersymmetric Lagrangians.

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