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Casimir scaling and Yang-Mills glueballs

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Abstract

We conjecture that in Yang-Mills theories the ratio between the ground-state glueball mass squared and the string tension is proportional to the ratio of the eigenvalues of quadratic Casimir operators in the adjoint and the fundamental representations. The proportionality constant depends on the dimension of the space-time only, and is henceforth universal. We argue that this universality, which is supported by available lattice results, is a direct consequence of area-law confinement. In order to explain this universal behaviour, we provide three analytical arguments, based respectively on a Bethe-Salpeter analysis, on the saturation of the scale anomaly by the lightest scalar glueball and on QCD sum rules, commenting on the underlying assumptions that they entail and on their physical implications.

Keywords: Glueballs; Yang-Mills theories; Confinement; Casimir scaling

INTRODUCTION

Yang-Mills (YM) theories without matter fields are believed to exhibit a confining phase at low energies, in which all bound states (glueballs) are gapped and color-singlet. Confinement in YM theories is supported by lattice studies [1]. However, since glueballs are nonperturbative objects, we do not have yet good understanding of the properties of glueballs such as their mass spectrum or decay widths.

It has been suggested that color confinement can be described in terms of a dual Higgs mechanism or monopole condensation [2–4]. In this picture, monopoles, dual to color charges, condense in the color-confined phase, and 't Hooft operators develop a vacuum expectation value. The dynamical scale κ is set by the condensate, which should be responsible for all other dimensional quantities in the confined phase. Monopole condensation implies a linear potential between a pair of static color charges, or equivalently an area law for the Wilson loop.

In this letter we provide theoretical arguments and numerical evidence for the existence of a new universal law. The law states that the ratio of ground state glueball mass squared and the string tension is universally proportional to the ratio of the eigenvalues of quadratic Casimir operators for all confining gauge theories. The proportionality constant is independent of the gauge group and the strength of coupling as long as the area law arises. It depends only on the dimensionality of the space-time.

GLUEBALL MASS

Calculating the ground state glueball mass is tantamount to showing that there is a gap in the ground state of pure YM theory, which has never been proved analytically except in three dimensions [5]. Numerical calculations of glueball masses on the lattice show the existence of a gap in YM theories [6–13].

Asymptotically, for confining YM theories, the expectation value of rectangular Wilson loops \mathcal{C} can be written as

$$\langle W(\mathcal{C}) \rangle = \left\langle \frac{1}{N} e^{i \oint_{\mathcal{C}} A} \right\rangle = \exp[-\sigma LT + \dots], \quad (1)$$

where LT is the area of \mathcal{C} , σ is the string tension between a static quark-antiquark pair, and the ellipsis includes subleading corrections such as the Lüscher term. Following the area

law confinement, we write the string tension σ as to define κ via the proportionality to the quadratic Casimir operator on the fundamental representation

$$\sigma = \kappa^2 C_2(F), \quad (2)$$

which is consistent with lattice results [8, 14–18]. The glueball is a bound state of adjoint gluons. On dimensional grounds, its mass should be proportional to κ . For the ground state glueball we conjecture

$$m_{0^{++}}^2 = \eta \kappa^2 C_2(A), \quad (3)$$

where η is a universal ratio and $C_2(A)$ the quadratic Casimir for the adjoint representation. The existence of the universal ratio η is consistent with the large- N universality of YM theories, supported by Wilson loop calculations [19] and gauge-gravity dualities [20]. At finite N , the ratio of the eigenvalues of the relevant quadratic Casimir operators is [21]

$$\frac{C_2(A)}{C_2(F)} = \begin{cases} \frac{2N^2}{N^2-1} & \text{for SU}(N) \\ \frac{2(N-2)}{N-1} & \text{for SO}(N) \\ \frac{4(N+1)}{2N+1} & \text{for Sp}(2N) \end{cases}, \quad (4)$$

and approaches 2 in the large- N limit.

Glueball masses and string tensions have been calculated by various collaborations for YM theories in $3+1$ and $2+1$ dimensions [6–13]. From the continuum-extrapolated lattice results of glueball mass and string tension, taking the data from the most recent large- N calculations available in the literature [8, 11, 13] (Fig. 1), we find ¹

$$\eta(0^{++}) \equiv \frac{m_{0^{++}}^2}{\sigma} \cdot \frac{C_2(F)}{C_2(A)} = \begin{cases} 5.41(12), & (d = 3 + 1), \\ 8.440(14)(76), & (d = 2 + 1). \end{cases} \quad (5)$$

For $3+1$ dimensions Eq. (5) is the constant fit of $\text{SU}(N)$ results over $2 \leq N \leq 8$, with $\chi^2/\text{d.o.f.} \simeq 1$. For $2+1$ dimensions, lattice results are available for $\text{SU}(N)$, as well for $\text{SO}(N)$, with $2 \leq N \leq 16$, hence we performed a constant fit for the universal ratio η of

¹ Our conjecture for the universal ratio is also supported by the analytic calculation of the ground-state glueball mass in $2+1$ dimensional $\text{SU}(N)$ gauge theories [22], which finds $\eta(0^{++}) \simeq 8.41$, and suspected in the constituent gluon model in [23].

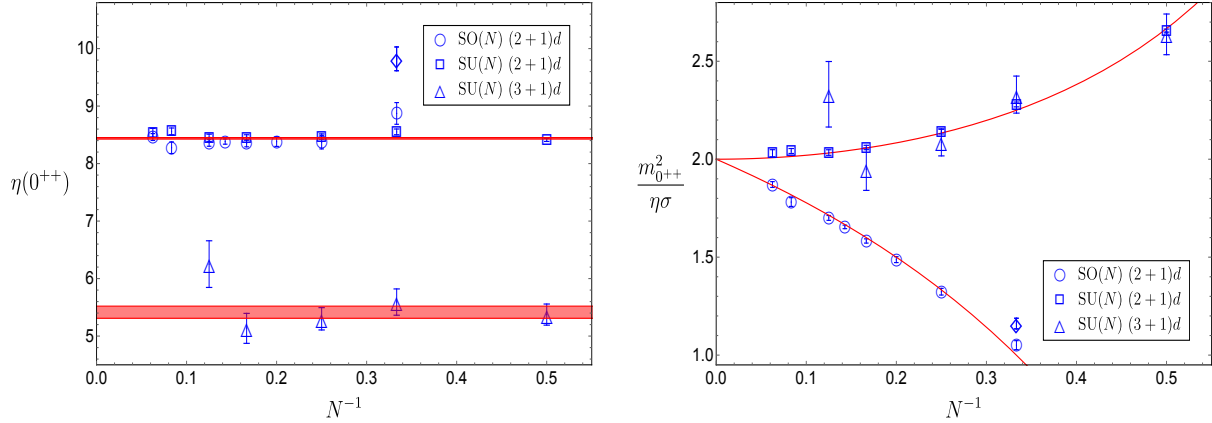


FIG. 1. The universal ratio η (left panel), and glueball masses squared in units of the string tension (right panel), for various YM theories as a function of $1/N$. The solid curves are the Casimir ratio $C_2(A)/C_2(F)$ for $SU(N)$ (upper curve) and $SO(N)$ (lower curve), respectively. The value of η from the tension of the $SO(3)$ fundamental string is marked as \diamond .

both data sets.² The resulting statistical error is quoted in the first parentheses in Eq. (5), with somewhat larger value of $\chi^2/\text{d.o.f.} \simeq 1.9$.³

Deviations from universality in 2+1 dimensions between two classes of gauge groups are assessed by calculating η separately. We find $\eta = 8.386(25)$ ($\chi^2/\text{d.o.f.} \simeq 1.3$) for $SO(N)$ and $\eta = 8.462(16)$ ($\chi^2/\text{d.o.f.} \simeq 1.9$) for $SU(N)$. Given the expectation that the large- N limit of the two sets should coincide, this difference of 3σ level is probably due to the systematic errors in the lattice data. We account for the discrepancy with a systematic error reported in the second parenthesis in Eq. (5). We also studied two heavier states, the 2^{++} glueball and the first excited scalar glueball, 0^{*++} . The excited states start to see the deviation from the area-law confinement, hence it is not surprising that the 0^{*++} does not show universal behavior. (See Fig. 2). For the 2^{++} , however, it is inclusive, because the constant fit gives a poor $\chi^2/\text{d.o.f.} \simeq 19$ for the 2^{++} tensor glueballs in 2+1 dimensions, while it fits much better in 3+1 dimensions with $\chi^2/\text{d.o.f.} \simeq 1.1$.

² The string tension can be defined also for $SO(3)$ by considering distances of the order of the confinement scale. Yet, it is affected by large systematic uncertainties due to its instability [11, 13]. To mitigate the systematics, instead of this quantity, we use the string tension obtained from the fundamental of $SU(2)$, assuming Casimir scaling for the string tension. We checked that by using the measured value of the string tension of $SO(3)$, the value of η does not change but yields a poor $\chi^2/\text{d.o.f} \simeq 4.8$.

³ The χ^2 distribution does not improve significantly, even if the data for the lowest N is excluded.

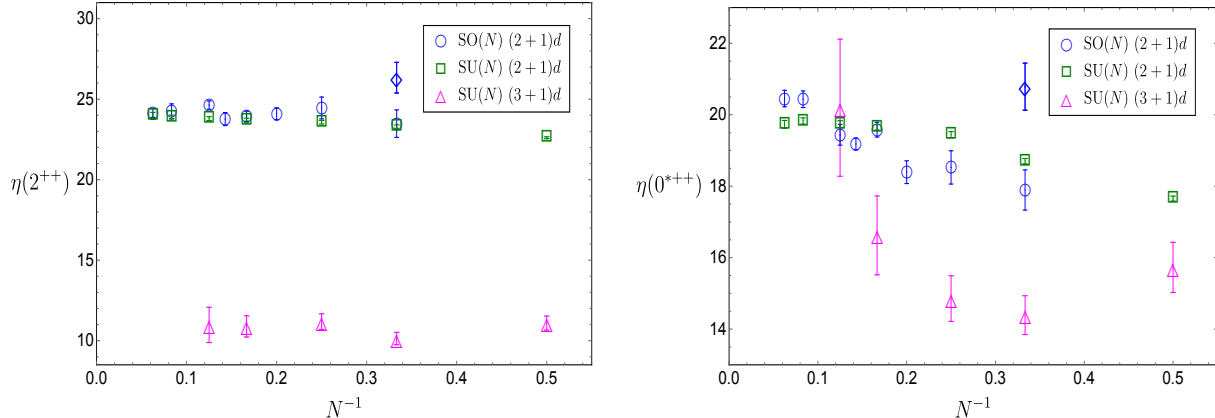


FIG. 2. The ratio η for the lowest-lying 2^{++} and 0^{*++} (first excitation in the scalar channel) as a function of $1/N$. The value of η from the tension of the $\text{SO}(3)$ fundamental string is marked as \diamond .

GLUEBALL MASS AND CASIMIR SCALING

Motivated by the strong numerical evidence for Casimir scaling, we provide three analytical arguments to explain its origin. None of the arguments is fully conclusive, as they all rely on specific dynamical assumptions that we highlight explicitly, yet the picture that emerges is that Casimir scaling of ground state mass should capture much of the essence of the confinement properties of YM theories.

Bethe-Salpeter equation

The amplitude for creating two gluons out of vacuum to form a color-singlet bound state of momentum P with a polarization λ can be defined as

$$\Gamma_R^{\mu\nu}(x_1, x_2; P, \lambda) = \langle 0 | T A^{\mu a}(x_1) A^{\nu a}(x_2) | R(P, \lambda) \rangle, \quad (6)$$

where T denotes the time-ordered product and $\langle 0 |$ is the vacuum. Summation over color indices a is understood.

The bound state amplitude satisfies the Bethe-Salpeter (BS) equations, obtained from the gluon four-point scattering amplitude near the pole, which are diagrammatically shown for the amputated BS amplitude in Fig 3.

From the BS equation, the scalar (amputated) amplitude χ_P obeys, in Euclidean space,

$$[\partial^2 - P^2] \chi_P(x) = \int d^4 y V(x - y) \chi_P(y), \quad (7)$$

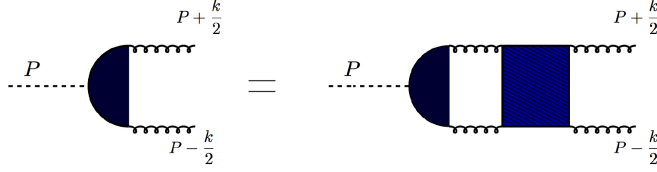


FIG. 3. The BS equation for the glueballs. The half disk denotes the BS amplitude of momentum P and the relative momentum k of two gluons. The box denotes the BS kernel.

with $x = x_1 - x_2$ the displacement of two external gluons.

The area law for confinement is associated with the Regge behavior of the spectrum: $M_n^2 \sim n$, where $n = 1, 2, \dots$ are the radial quantum numbers, reproduced by the approximate BS kernel

$$V(x - y) \approx \frac{1}{2} \omega^2 x^2 \delta^4(x - y). \quad (8)$$

The BS kernel is nothing but the four-point function of gluons, properly projected for the spin-0 state. If the string flux picture holds for the glueball states, ω should be the string tension of the Nambu-Goto action for the closed string that describes glueballs, $\omega \sim \sigma(A) = C_2(A)\kappa^2$ (see for example Eq. 2.26 in [24]). The radially excited scalar glueball mass is then (for $n = 1, 2, \dots$)

$$M_n^2 \sim C_2(A)\kappa^2 (n + 1). \quad (9)$$

Since the string tension is $\sigma = \kappa^2 C_2(F)$, for the mass of ground state ($n = 1$) glueball we find

$$\frac{m_{0++}^2}{\sigma} = \eta \frac{C_2(A)}{C_2(F)}. \quad (10)$$

There are corrections to Casimir scaling, coming from the corrections to the area law in Eq. (1). But such corrections are suppressed, arising at the next-to-next-to-leading order. Namely, the Lüscher term in the expectation value of the Wilson loop in (1) does not modify Casimir scaling, Eq. (10), since the Lüscher term is a universal number [25] correcting the BS kernel by a shift itself proportional to the Casimir; for $|x| \gg \kappa^{-1}$

$$\frac{1}{2} \left(C_2(A)\kappa^2 x - \frac{\alpha}{x} \right)^2 \approx \frac{1}{2} C_2(A)^2 \kappa^4 x^2 - \alpha C_2(A)\kappa^2, \quad (11)$$

where $\alpha = (D - 2)\pi/24$ is the universal coefficient of the Lüscher term in D dimensions. The ground state glueball mass then is corrected as

$$m_{0++}^2 \sim C_2(A)\kappa^2 (2 - \alpha), \quad (12)$$

which does not change the universal scaling law.

The corrections become more important at high energies (short distances), in particular for the excited states, for which we expect violations of the Casimir scaling to show. As discussed in following, the characteristic behavior of the 0^{++} ground state may be understood in terms of its special role with respect to the scale symmetry of the system.

Scale anomaly

Pure Yang-Mills theories in four dimensions are classically scale invariant, but scale symmetry is anomalous, broken by quantum effects. Furthermore, the (anomalous) scale symmetry in YM theory is spontaneously broken as well, since the YM vacuum develops a non-vanishing expectation value for the order parameter for confinement. If the scale anomaly is parametrically small, compared to such vacuum expectation value of the order parameter, then there should be a (pseudo) Nambu-Goldstone boson in YM theory, associated with spontaneous breaking of the scale symmetry. Namely, by Goldstone's theorem, the dilatation current, associated with the scale symmetry $x^\rho \rightarrow e^\lambda x^\rho$, creates a state, called dilaton:

$$\langle 0 | D^\mu(x) | D(p) \rangle = -i f_D p^\mu e^{-ip \cdot x}, \quad (13)$$

where the dilatation current $D_\mu = x^\nu \theta_{\mu\nu}$ with the improved energy-momentum tensor $\theta_{\mu\nu}$ [26] and f_D is the dilaton decay constant.

The scale anomaly in pure YM theory is given as

$$\partial^\mu D_\mu = -\frac{\beta(g)}{2g} F_{\mu\nu}^a F^{a\mu\nu}, \quad (14)$$

where $\beta(g)$ is the beta function and $F_{\mu\nu}^a$ the field-strength tensor. Since the divergence of the dilatation current can be written in terms of the trace of the energy-momentum tensor as $\partial_\mu D^\mu = \theta^\mu_\mu$, the anomalous Ward identity (14) relates the two-point function of the trace of the energy-momentum tensor to its one-point function or the scale anomaly:

$$\int_x \langle 0 | T \theta^\mu_\mu(x) \theta^\nu_\nu(y) | 0 \rangle = -4 \langle \theta^\mu_\mu(y) \rangle. \quad (15)$$

As all the gluons equally and additively contribute to the vacuum energy, the scale anomaly should be given on dimensional ground as $\langle \theta^\mu_\mu \rangle = -\tilde{\beta} C_2(A) \kappa^4$, after subtracting out the part that is independent of the condensate. If the scale anomaly is parametrically small or

$|\tilde{\beta}| \ll 1$, there should be a light dilaton, defined as in Eq. (13), that saturates the two-point function in (15):

$$\int_x \langle 0 | T \theta_\mu^\mu(x) \theta_\nu^\nu(y) | 0 \rangle \approx f_D^2 m_D^2. \quad (16)$$

We then have a so-called partially conserved dilatation current (PCDC) relation,

$$f_D^2 m_D^2 = -4 \langle \partial_\mu D^\mu \rangle = -16 \mathcal{E}_{\text{vac}}, \quad (17)$$

where $\mathcal{E}_{\text{vac}} = -\tilde{\beta} C_2(A) \kappa^4 / 4$ is the vacuum energy density of YM theories in the confined phase. The vacuum energy (density) scales as $C_2(A)$. On the other hand the dilaton decay constant, f_D , measures the strength of the amplitude that creates the dilaton out of vacuum, which should not depend on the number of gluon fields, but only on the characteristic scale κ that defines the scale of spontaneous scale-symmetry breaking. We hence find the ground state glueball mass $m_{0^{++}}^2 \propto \tilde{\beta} C_2(A) \kappa^2$, if identified as dilaton, and it becomes parametrically small if $\tilde{\beta} \ll 1$.

The assumption about saturation of the Ward identity by the lightest 0^{++} state, Eq. (16), is equivalent to assuming the existence of a weakly-coupled low-energy effective field theory for the 0^{++} state in terms of the dilaton field, in spite of its mass not being particularly small, compared to other excited states like 0^{*++} glueball states, which implies $\tilde{\beta} \sim 1$. The fact that Casimir scaling holds for the ground state glueball but not for excited states (as hinted also by lattice calculations) is therefore quite intriguing, and very distinctive from analysis based on other approaches that do not differentiate the lightest state.

Sum rules

The glueball mass can be extracted from the correlators of interpolating operators made of gluons. For scalar glueballs one considers the correlator of the gluonic field strength tensor $\mathcal{O}_S(x) \equiv \alpha_s F_{\mu\nu}^a F^{a\mu\nu}$:

$$\Pi_S(x) = \langle 0 | T [\mathcal{O}_S(x) \mathcal{O}_S(0)] | 0 \rangle = \sum_n c_n e^{-m_n |x|}, \quad (18)$$

where T is the time-ordering operation and the smallest m_n will be the mass of ground state glueball 0^{++} .

The sum rules, associated with the moments of the correlators, exploit the operator production expansion. For the zero moment, one finds [27]

$$\int d^4x \Pi_S(x) = \frac{32\pi^2}{b} \langle 0 | \frac{\alpha_s}{\pi} F_{\mu\nu}^a F^{a\mu\nu} | 0 \rangle , \quad (19)$$

where b is the first coefficient of the beta function and the integral is regularized by subtracting out the perturbative contributions. Assuming single-particle states to be stable and inserting a complete set between the interpolating operators in (18), we have

$$\sum_{n=0}^{\infty} f_n^2 m_n^2 = \frac{32\pi^2}{b} \langle 0 | \frac{\alpha_s}{\pi} F_{\mu\nu}^a F^{a\mu\nu} | 0 \rangle , \quad (20)$$

where the decay constants f_n are normalized by

$$f_n m_n^2 \equiv \langle 0 | \mathcal{O}_S(p) | n \rangle_{p^2=0} . \quad (21)$$

Because of the summation over gluons in the condensate in (20), we expect the scalar glueball mass squared to be proportional to $C_2(A)$. We note the similarity with the low energy theorem (17), if the sum rule (20) is saturated by the ground state or, equivalently, for the excited states $f_n m_n^2 \ll f_0 m_0^2$, which suggests that the excited states have very little overlap with the operator \mathcal{O}_S . The numerical analysis we report in this letter seems to suggest that this is the case, as we do not see evidence of Casimir scaling in the excited states, but only in the ground state.

DISCUSSIONS AND CONCLUSION

For Yang-Mills theories, we conjectured that the ground state glueball mass squared, measured in units of string tension, is universally proportional to the ratio of the eigenvalues of the quadratic Casimir operator of the adjoint over that of the fundamental representation. The conjecture relies on the area law for confinement, and the specific coefficient should depend only on the dimensionality of the space-time, but not on the specific group.

We provided three analytical arguments to justify Casimir scaling, based respectively on the Bethe-Salpeter equation, scale anomaly, and sum rules. We tested this law on existing numerical lattice results in pure $SU(N)$ and $SO(N)$ Yang-Mills theories in $3+1$ and $2+1$ dimensions. The data strongly support Casimir scaling for the ground state. The values of the universal constant extracted from lattice data are $\eta(0^{++}) = 5.41(12)$ for $3+1$ dimensions

and $\eta(0^{++}) = 8.444(15)(85)$ for 2+1 dimensions. Numerical results are inconclusive for the 2^{++} state, while showing that universality does not hold for the first excitation in the 0^{++} channel.

If the conjectured universal scaling is confirmed, it would shed light on the mechanism yielding confinement in YM theories. It would be therefore quite interesting to test further numerically our conjecture for other gauge groups such as $Sp(2N)$ and $SO(N)$ in $3 + 1$ dimensions, and to extend the arguments discussed here to provide systematic control over sub-leading corrections (if they exist) to exact Casimir scaling.

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