Journal of Universal Computer Science, vol. 15, no. 6 (2009), 1280-1300
submitted: 1/11/08, accepted: 4/2/09, appeared: 28/3/09 © J.UCS

Representing Measurement Results

Arno Pauly
(University of Cambridge
United Kingdom
Arno.Pauly@cl.cam.ac.uk)

Abstract: To gain insight into the relationship between physical theories and computation, we examine the link between measurement devices and computers in the framework of TTE. Starting from a formal definition of a measurement procedure, different approaches to associate a representation with a measurement procedure are studied, and an equivalence class of representations suitable for representing the results of a measurement is defined for each measurement procedure.

Key Words: computable analysis, admissible representation, measurement, normal distribution, computation by physical devices
Category: F.0, F.1.1

1 Introduction

1.1 Computable Analysis and Physics

The search for a theory of computation capable of dealing with real numbers was driven, among other reasons, by the wish to study computational properties of physical systems. Due to the continuous nature of many physical models, computable analysis promises to be adequate for formalizing their computational aspects. However, a concise model of interaction between physical systems and computational devices has not been established yet. While recently some general steps for interpreting physical systems as computational devices, focusing on classical computability theory, were suggested in [Beggs and Tucker (2007)], a more detailed view seems to be necessary in the next step. A survey on the research in these directions can be found in [Beggs and Tucker (2004)]; a short introduction is given in [Yao (2003)].

The role of measurements has been neglected in a considerable part of the literature, and even when measurements are defined explicitly as in [Geroch and Hartle (1986)], the definition remains unprecise. The focus of the present work is a formalisation of measurements as an interface between physical reality and a computational device, rather than interpreting the outcome of a measurement as result of a computation. As noted e.g. in [Bosserhoff (2008)], measurements inevitably introduce stochastic concepts into a formal model.

An important distinction to be taken is between measurements in classical physics and measurements in quantum physics. The role of measurements in quantum physics is still subject of open philosophical debates, and in general of a different nature than the role of measurement in classical physics. Thus, in the present paper considerations will be limited to classical physics. For a mathematical presentation of classical mechanics we refer to [Arnold (1989)], a standard textbook is [Feyman et al. (1963)].

1.2 Stochastic Concepts in Computable Analysis

A straight-forward way to introduce stochastic concepts into computable analysis consists in creating representations for stochastic objects such as probability distributions. This approach was taken in e.g. [Weihrauch (1999)], and in a more general fashion in [Schröder and Simpson (2006)], [Schröder (2007)]. While representations of probability measures promise to be very useful for modeling the behaviour of quantum systems, they do not match the perspective on uncertainty in measurements widespread in classical physics, as the uncertainty is attributed to the measurement process rather than to the measured entity itself.

Another approach is presented by [Bosserhoff (2008)]; it consists in defining a notion of almost everywhere computable functions, as well as some related notions. The aspect of the measurement itself, however, is left out.

1.3 The Model

A measurement device is considered to be an interface between some kind of physical entity and a computer, which in particular is assumed to be digital. Real measurement devices usually put out finite decimal fractions, however, by choosing units appropriately, only natural numbers need to be permitted. For easier modeling, restrictions of output size, yielding a finite output range, are omitted, and every natural number is permitted as output.

To account for measuring errors[1], the output of the device is not completely determined by the state of the physical entity. Instead, the state of the physical entity only determines a probability density on the natural numbers, to be interpreted as the respective probabilities of a number occurring as output.

To be a valid source of insight about physical reality, an experiment is required to be reproducible[2]. Thus, in the limit of infinite repetitions, an infinite sequence of natural numbers arises. The probability density on the set of natural numbers determined by the state of the physical object induces naturally a probability measure on the set of sequences of natural numbers.

[1] If measurement errors were avoidable, the physical entity could be completely determined by the result of a single measurement. This, however, is a natural number, so the space of possible values would be countable. In our model, values with uncountable range can never be measured without errors.

[2] An interesting discussion of this postulate can be found in [Feyman et al. (1963), Part 1, Chapter 6, Section 1].

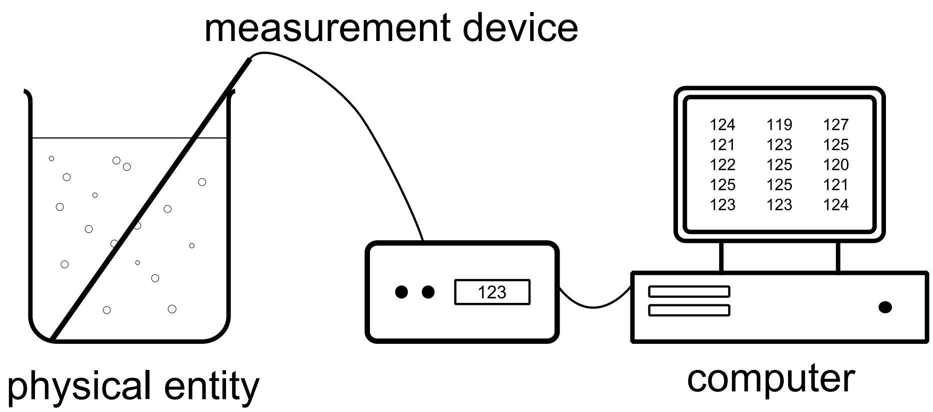

Figure 1: Modeling Measurements

The infinite sequences of natural numbers produced according to these probability measures now serve as input for a computer, modeled as a Type-2-Turing machine as defined in [Weihrauch (2000)]. Both δ-names (for a representation δ) and the results of a random process as described above are infinite sequences of natural numbers, however, the respective interpretation of these sequences are very distinct. Several approaches towards a link between these interpretations are presented here.

2 Preliminaries

2.1 General

The set of natural numbers is denoted by $\mathbb{N}$, 0 is not considered to be a natural number. $\mathbb{N}^*$ denotes the set of finite sequence of natural numbers, $\mathbb{N}^\omega$ denotes the set of infinite sequences of natural numbers. $\mathbb{Q}$ is the set of rational numbers and $\mathbb{R}$ is the set of real numbers.

For a finite sequence $s \in \mathbb{N}^*$, $|s|$ denotes the length of s. For $w \in \mathbb{N}^\omega$ the n-th number of w is referred to as w_n, the same holds for $s \in \mathbb{N}^*$, as long as $n \leq |s|$. $w_{\leq n} \in \mathbb{N}^*$ is the sequence consisting of the first n numbers in w for $w \in \mathbb{N}^\omega$, $s_{\leq n}$ is to be interpreted analogously. For a finite or infinite sequence w, $c_{k,n}(w)$ denotes the number of occurrences of the number k in $w_{\leq n}$.

For $S \subseteq \mathbb{N}^*$, $S\mathbb{N}^\omega$ denotes the set $\{w \in \mathbb{N}^\omega \mid \exists s \in S\ \exists n \in \mathbb{N}\ w_{\leq n} = s\}$, $s\mathbb{N}^\omega$ means the same as $\{s\}\mathbb{N}^\omega$. For $w \in \mathbb{N}^\omega$, let $R_w = \{n \in \mathbb{N} \mid \exists i \in \mathbb{N}\ w_i = n\}$.

$\langle\rangle$ denotes a certain bijective computable function from $\mathbb{N}^n$ to $\mathbb{N}$, so that the inverse function is again computable. The value of n will be clear from the context. $\nu_{\mathbb{Q}} : \mathbb{N} \to \mathbb{Q}$ is a fixed total numbering of $\mathbb{Q}$, so that all usual operations on $\mathbb{Q}$ are computable, $\nu_{\mathbb{Z}}$ is a fixed total numbering of $\mathbb{Z}$, again, all usual operations are assumed to be computable.

2.2 Topology

Though the relation to topology might not be obvious, topological concepts are needed for the following considerations. A topology $\mathcal{T}$ on a set X is a set of subsets of X containing $\emptyset$ and X, which is closed under finite intersections and arbitrary unions. The elements of a topology are called open sets, their complements are called closed sets. A topology $\mathcal{T}$ is T_0, if for all $x, y \in X$, $x \neq y$, a $U \in \mathcal{T}$ with either $x \in U$, $y \notin U$ or $x \notin U$, $y \in U$ exists.

A topological space is a pair $(X, \mathcal{T})$ consisting of a set X and a topology $\mathcal{T}$ on X. A function $f : X \to Y$ is continuous with respect to topologies $\mathcal{T}_X$ on X and $\mathcal{T}_Y$ on Y, if for all $O \in \mathcal{T}_Y$, $f^{-1}(O) \in \mathcal{T}_X$ holds. A partial function is continuous, if its restriction to its domain is continuous. A sequence $(y_n)_{n\in\mathbb{N}}$ of elements of X converges to $x \in X$ in the space $(X, \mathcal{T})$, if for every $U \in \mathcal{T}$ with $x \in U$ there is a $n_0 \in \mathbb{N}$ so that for $n \geq n_0$, $y_n \in U$ follows.

Following [Schröder (2002)], a set $\mathcal{B}$ of subsets of X is called a pseudobase of a topological space $(X, \mathcal{T})$, if for every open set $U \in \mathcal{T}$, every $x \in U$ and every sequence $(y_n)_{n\in\mathbb{N}}$ converging to x, there is a $B \in \mathcal{B}$ and a $n_0 \in \mathbb{N}$ satisfying $x \in B$ and $y_n \in B$ for $n \geq n_0$. A pseudosubbase of $(X, \mathcal{T})$ is a set $\mathcal{B}$ of subsets of X, so that the set of finite intersections of elements of $\mathcal{B}$ is a pseudobase of $(X, \mathcal{T})$.

On $\mathbb{N}^\omega$, a standard topology $\mathcal{S}$ is given by $\mathcal{S} = \{W\mathbb{N}^\omega \mid W \subseteq \mathbb{N}^*\} \cup \{\mathbb{N}^\omega\}$. On $\mathbb{N}$ and $\mathbb{Q}$ the discrete topologies $2^{\mathbb{N}}$ and $2^{\mathbb{Q}}$ are used, on $\mathbb{R}$ we assume the usual Euclidean topology. For a comprehensive presentation of topology, we refer the reader to [Dugundji (1970)].

2.3 Computable Analysis

A notion of computability for partial functions on $\mathbb{N}^\omega$ is constructed by Type-2-Turing machines; each computable function is also continuous. For definitions we refer to [Weihrauch (2000)]. A representation is a surjective partial function from $\mathbb{N}^\omega$ to the represented set X. Using these, a relativised concept of computability for functions $f : X \to Y$ can be derived as following: f is called (δ, ρ)-computable for a representation δ of X and a representation ρ of Y, if there is a computable partial function F satisfying $\rho \circ F(w) = f \circ \delta(w)$ for all $w \in \mathrm{dom}(\delta)$.

A representation ρ is (computably) reducible to δ, $\rho \preceq \delta$ ($\rho \preceq_c \delta$), if there exists a continuous (computable) partial function f on $\mathbb{N}^\omega$ with $\rho(w) = \delta(f(w))$ for all $w \in \mathrm{dom}(\rho)$. ρ and δ are (computably) equivalent, $\rho \equiv \delta$ ($\rho \equiv_c \delta$), if $\delta \preceq \rho$ and $\rho \preceq \delta$ ($\delta \preceq_c \rho$ and $\rho \preceq_c \delta$) hold.

If the represented set X is equipped with a topology $\mathcal{T}$, a representation δ of X is continuous, if it is continuous as a partial function. For a T_0-space $(X, \mathcal{T})$ with a countable pseudosubbase $\{B_n \mid n \in \mathbb{N}\}$, a standard representation δ is introduced, generalising a notion from [Schröder (2002)][3], defined through $\delta(w) = x$ if $x \in B_{w_n}$ for all $n \in \mathbb{N}$ and for each $U \in \mathcal{T}$ with $x \in U$, there is an $i \in \mathbb{N}$ so that $\bigcap_{j<i} B_{w_j} \subseteq U$ holds. A representation ρ of X is admissible with respect to a topology $\mathcal{T}$ on X, if ρ is equivalent to a standard representation δ of $(X, \mathcal{T})$. Especially, only those topological spaces possess admissible representations, that are T_0 and have a countable pseudobase. Admissible representations employ a range of useful properties, for details we refer to [Weihrauch (2000)] or [Schröder (2002)].

2.4 Probability Theory

A σ-algebra $\mathcal{A}$ on a set X is a set of subsets closed under countable intersections, countable unions, and the formation of complements, and includes the set X. The set of subsets of $\mathbb{N}$ forms the standard σ-algebra on $\mathbb{N}$. The standard σ-algebra $\sigma_{\mathbb{N}^\omega}$ on $\mathbb{N}^\omega$ is generated by the standard topology on $\mathbb{N}^\omega$.

A probability density on $\mathbb{N}$ is a function $p : \mathbb{N} \to \mathbb{I}$, satisfying $\sum_{n=1}^{\infty} p(n) = 1$. The set of probability densities on $\mathbb{N}$ is denoted by $\mathbb{P}$. Each probability density p on $\mathbb{N}$ uniquely determines a probability measure $\bar{p}$ on $\mathbb{N}$ through $\bar{p}(A) = \sum_{n \in A} p(n)$.

A probability measure on $\mathbb{N}^\omega$ is a function $P : \mathcal{A} \to \mathbb{R}$, where $\mathcal{A}$ is the standard σ-algebra on $\mathbb{N}^\omega$, satisfying $P(\mathbb{N}^\omega) = 1$ and $P(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)$ for all sequences of pairwise disjunct sets A_n in $\mathcal{A}$. Each probability density p on $\mathbb{N}$ uniquely defines a probability measure $\hat{p}$ on $\mathbb{N}^\omega$ as extension of $\hat{p}(a_1 a_2 ... a_n \mathbb{N}^\omega) = \prod_{n=1}^{n} p(a_n)$.

A probability measure P on $\mathbb{N}^\omega$ can be extended to an outer measure $\mu_P : 2^{\mathbb{N}^\omega} \to \mathbb{R}$ by defining $\mu(A) = \inf_{M \in \mathcal{A}, A \subseteq M} P(M)$. We will identify a probability measure with the induced outer measure, allowing us to disregard issues of measurability.

We refer the reader to [Shiryaev (1996)] for further elaboration.

[3] In [Schröder (2002)], standard representations where defined only for pseudobases; the standard representation for a pseudosubbase defined here is equivalent to the standard representation for the pseudobase obtained by forming all finite intersections.

3 Measurement procedures and representations

3.1 Measurement procedures

To formalise the model set up in [Subsection 1.3], the notion of a measurement procedure shall be introduced:

Definition 1. A measurement procedure[4] (for a set X) is an injective function $\mathcal{M} : X \to \mathbb{P}$. If $\mathcal{M}(x) = p$, then $\hat{\mathcal{M}}(x)$ is defined as $\hat{\mathcal{M}}(x) = \hat{p}$.

To use data resulting from experiments as input for calculations on Type-2-Turing machines, representations have to be found that correspond suitably to the measurement procedures at hand. While the rest of this section will be dealing with ways to associate representations to measurement procedures, first a certain property of measurement procedures shall be introduced.

Definition 2. A measurement procedure $\mathcal{M}$ is degenerate, if

$$\{n \in \mathbb{N} \mid \mathcal{M}(x)(n) > 0\} = \{n \in \mathbb{N} \mid \mathcal{M}(y)(n) > 0\} \Leftrightarrow x = y$$

holds for all $x, y \in X$.

While degenerate measurement procedures will be shown to employ some interesting theoretical properties, they do not contain natural examples of measurement procedures. Thus, the property of non-degenerateness seems to be desirable.

3.2 Almost surely associated representations

Definition 3. A representation ρ of X is almost surely associated with a measurement procedure $\mathcal{M}$, if $\hat{\mathcal{M}}(x)(\rho^{-1}(\{x\})) = 1$ holds[5] for all $x \in X$.

If ρ is almost surely associated with a measurement procedure $\mathcal{M}$, then the probability of retrieving a ρ-name for the actual state of the measured physical entity when repeating the measurement infinitely often is 1. Considering the stochastic nature of measurements, a stronger link between measurement procedures and representations cannot expected to be feasible.

Definition 4. For a measurement procedure $\mathcal{M}$ for X, define a representation α of X by $\alpha(w) = x$, if $\lim\limits_{n\to\infty} \frac{c_{k,n}(w)}{n} = \mathcal{M}(x)(k)$ holds for each $k \in \mathbb{N}$.

[4] In an earlier version of this paper, [Pauly (2008)], the notion of stochastic representations was used instead. Measurement procedures and stochastic representations are the inverse functions of each other, so the necessary changes in the following results and notions are rather small.

[5] This means that every measurable set containing $\rho^{-1}(\{x\})$ must have measure 1. If there is a T_1 topology τ on X, so that ρ is continuous with respect to τ, then $\rho^{-1}(\{x\})$ is closed and hence measurable itself.

Theorem 5. *The representation α is well-defined and almost surely associated with $\mathcal{M}$.*

Proof. For $x, y \in X$ with $x \neq y$, also $\mathcal{M}(x) \neq \mathcal{M}(y)$ holds, since $\mathcal{M}$ is required to be injective. Thus, there exists a $k \in \mathbb{N}$ with $\mathcal{M}(x)(k) \neq \mathcal{M}(y)(k)$. So for $x \neq y$, $\forall k \in \mathbb{N} \lim\limits_{n\to\infty} \frac{c_{k,n}(w)}{n} = \mathcal{M}(x)(k)$ and $\forall k \in \mathbb{N} \lim\limits_{n\to\infty} \frac{c_{k,n}(w)}{n} = \mathcal{M}(y)(k)$ are mutually exclusive for each $w \in \mathbb{N}^\omega$. It follows that α is well-defined as a partial function.

Since $c_{k,n}$ can be considered as a random variable expressible as a sum of n independent identically distributed random variables with mean $\mathcal{M}(x)(k)$, the Strong Law of Large Numbers yields the convergence of $\frac{c_{k,n}(w)}{n}$ to $\mathcal{M}(x)(k)$ almost surely. Rephrased, for a fixed $k \in \mathbb{N}$,

$$\hat{\mathcal{M}}(x)(\{w \in \mathbb{N}^\omega \mid \lim_{n\to\infty} \frac{c_{k,n}(w)}{n} = \mathcal{M}(x)(k)\}) = 1$$

was obtained. Since a union of countably many null sets is a null set, the intersection of countably many sets with probability 1 yields a set with probability 1, so

$$\hat{\mathcal{M}}(x)(\{w \in \mathbb{N}^\omega \mid (\forall k \in \mathbb{N}) \lim_{n\to\infty} \frac{c_{k,n}(w)}{n} = \mathcal{M}(x)(k)\}) = 1$$

follows. As the preimage $\alpha^{-1}(\{x\})$ has measure 1 for each x, it is non-empty. Thus, α is a representation, and indeed almost surely associated with $\mathcal{M}$.

While a direct consequence of Theorem 5 is that for each measurement procedure $\mathcal{M}$ there is a representation ρ, so that ρ is almost surely associated with $\mathcal{M}$, the reverse statement does not hold in general[6]. However, through restricting the scope of consideration to standard representations as introduced in [Subsection 2.3], a positive result can be obtained.

Theorem 6. *For a standard representation δ_S of a space X there is a measurement procedure $\mathcal{M}$, so that δ_S is almost surely associated with $\mathcal{M}$.*

Proof. It follows directly from the definition of a standard representation that $\delta_S(w)$, as well as the membership of w in $dom(\delta_S)$ only depends on R_w. This allows to define $R_x := R_{w(x)}$, where $w(x)$ is a choice function satisfying $w(x) \in \delta_S^{-1}(x)$. Let $N_x = \sum\limits_{n \in R_x} 2^{-n}$. Now for $n \in R_x$ one sets $p_x(n) = \frac{2^{-n}}{N_x}$, and for $n \notin R_x$ one sets $p_x(n) = 0$.

It is easy to see that p_x is a probability density on $\mathbb{N}$, and as R_x determines x, for different $x \neq y$ there has to be an $m \in (R_x \setminus R_y)$ or an $n \in (R_y \setminus R_x)$. In

[6] It is trivial to prove that for each representation ρ of a set X there is a computationally equivalent representation δ of X, so that δ is not almost surely associated with any measurement procedure. Define $\delta(01w) = \rho(w)$, and let δ be undefined elsewhere. Then $\hat{p}(\mathbb{N}^\omega \setminus \mathrm{dom}(\delta)) > 0$ for every $p \in \mathbb{P}$.

the former case, $p_y(m) = 0 \neq p_x(m)$, the latter is analogously. This shows that a measurement procedure $\mathcal{M}$ can be defined by $\mathcal{M}(x) = p_x$.

For a fixed $x \in X$, let n be given with $n \notin R_x$. For $a_1, .., a_k \in \mathbb{N}$, one gets $\hat{\mathcal{M}}(x)(a_1..a_k n\mathbb{N}^\omega) = p_x(n) * \prod_{i=1}^{k} p_x(a_i) = 0$, since $p_x(n) = 0$. So in

$$\{w \in \mathbb{N}^\omega \mid n \in R_w\} = \bigcup_{k\in\mathbb{N}} \bigcup_{(a_1,..,a_k)\in\mathbb{N}^k} a_1..a_k n\mathbb{N}^\omega$$

the right side is a union of countably many null sets, proving the left side to be a null set, too. The argument proves that $\hat{\mathcal{M}}(x)(\{w \in \mathbb{N}^\omega \mid R_w \not\subseteq R_x\}) = 0$ holds.

Now, let $n \in R_x$, so $p_x(n) > 0$. For $k \in \mathbb{N}$, consider the set

$$W^k_{x,n} := \{w \in \mathbb{N}^\omega \mid \forall i \leq k \; w(i) \neq n\}$$

One gets $\hat{\mathcal{M}}(x)(W^k_{x,n}) = [1 - p_x(n)]^k$. Since

$$W_{x,n} := \{w \in \mathbb{N}^\omega \mid \forall i \in \mathbb{N} \; w_i \neq n\} \subseteq W^k_{x,n}$$

directly $\hat{\mathcal{M}}(x)(W_{x,n}) \leq [1 - p_x(n)]^k$ follows for all $k \in \mathbb{N}$. Since $p_x(n) > 0$, this means $\hat{\mathcal{M}}(x)(W_{x,n}) = 0$.

Finally, notice $\mathbb{N}^\omega \setminus \delta_S^{-1}(x) \subseteq \{w \in \mathbb{N}^\omega \mid R_w \not\subseteq R_x\} \cup (\bigcup_{n\in R_x} W_{x,n})$. The right side is a union of countably many null sets w.r.t. $\hat{\mathcal{M}}(x)$, so the left side is also a null set, yielding $\hat{\mathcal{M}}(x)(\delta_S^{-1}(x)) = 1$, which completes the proof.

The measurement procedure constructed in the proof of Theorem 6 is degenerate. A following result shows that this is not an artefact of the proof, but rather a necessary condition for a measurement procedure to have an almost surely associated standard representation. On the other hand, for a degenerate measurement procedure it is always possible to construct a standard representation almost surely associated with it.

Theorem 7. *For each degenerate measurement procedure $\mathcal{M}$ for X there is a T_0 topology $\mathcal{T}$ with a countable subbase on X, so that the corresponding standard representation δ_S is almost surely associated with $\mathcal{M}$.*

Proof. Define $B_n = \{x \in X \mid \mathcal{M}(x)(n) > 0\}$, and let $\mathcal{T}$ be the topology induced by the sets B_n serving as subbase. It follows directly that the B_n form also a pseudosubbase.

For abbreviation, let $R_x = \{n \in \mathbb{N} \mid \mathcal{M}(x)(n) > 0\}$. Since $\mathcal{M}$ is degenerate, $x \neq y$ implies $R_x \neq R_y$. Thus, there is either an $n \in R_x \setminus R_y$ or an $m \in R_y \setminus R_x$. So either B_n is an open set with $x \in B_n$, $y \notin B_n$ or B_m is an open set with $x \notin B_m$, $y \in B_m$, so $\mathcal{T}$ is a T_0 topology.

Now consider the standard representation δ_S to the subbase B_n. Since $\{w \in \mathbb{N}^\omega \mid R_w = R_x\} \subseteq \delta_S^{-1}(\{x\})$, it suffices to prove

$$\hat{\mathcal{M}}(\{w \in \mathbb{N}^\omega \mid R_w = R_x\}) = 1$$

to show that δ_S is almost surely associated with $\mathcal{M}$. The remaining part of the proof is completely analogously to the corresponding part in the proof of Theorem 6.

Non-degenerate measurement procedures, however, are supposed to include the natural examples from physics, while admissibility is a desirable property for representations. The next result shows that these properties cannot be linked via almost sure association.

Theorem 8. *Let $\mathcal{M}$ be a non-degenerate measurement procedure for a set X with $|X| > 1$, and let ρ be a representation of X, so that ρ is almost surely associated to $\mathcal{M}$. Further, let τ be a T_0-topology on X. Then ρ is not continuous w.r.t. τ.*

Proof. As $\mathcal{M}$ is non-degenerate, there are two different elements $x \neq y$ of X with $\{n \in \mathbb{N} \mid \mathcal{M}(x)(n) > 0\} = \{n \in \mathbb{N} \mid \mathcal{M}(y)(n) > 0\}$; these shall be fixed for the following considerations. Since τ is T_0, there is an open $O \in \tau$ with either $x \in O$, $y \notin O$ or $x \notin O$, $y \in O$, w.l.o.g. the former shall be assumed.

Now assume ρ to be continuous, so $U := \rho^{-1}(O)$ is open in $\mathrm{dom}(\rho)$ and satisfies $\rho^{-1}(\{x\}) \subseteq U$ and $\rho^{-1}(\{y\}) \subseteq \mathbb{N}^\omega \setminus U$. U has the form $U = W\mathbb{N}^\omega$ with $W \subseteq \mathbb{N}^*$. Next, we assume ρ to be almost surely associated to $\mathcal{M}$. This implies $\hat{\mathcal{M}}(x)(U) = 1$ and $\hat{\mathcal{M}}(y)(U) = 0$.

Since W is countable and all occurring sets are measurable, one gets:

$$\sum_{a \in W} \hat{\mathcal{M}}(x)(a\mathbb{N}^\omega) \geq 1 \quad \text{and} \quad \forall a \in W\ \hat{\mathcal{M}}(y)(a\mathbb{N}^\omega) = 0$$

So there is at least one $a \in W$ with $\hat{\mathcal{M}}(x)(a\mathbb{N}^\omega) > 0$ and $\hat{\mathcal{M}}(y)(a\mathbb{N}^\omega) = 0$. Choose such an a, which shall have the form $a := a_1 ... a_k$. From the definition of $\hat{\mathcal{M}}(x)$, this leads to $\prod_{i=1}^{k} \mathcal{M}(x)(a_i) > 0$ and $\prod_{i=1}^{k} \mathcal{M}(y)(a_i) = 0$. The second equation proves the existence of an a_i with $\mathcal{M}(y)(a_i) = 0$. However, x and y were chosen in a fashion that $\mathcal{M}(y)(a_i) = 0$ implies $\mathcal{M}(x)(a_i) = 0$. This contradicts $\prod_{i=1}^{k} \mathcal{M}(x)(a_i) > 0$, so the assumption of ρ being continuous w.r.t. τ is refuted.

Corollary 9. *Let $\mathcal{M}$ be a non-degenerate measurement procedure for a set X with $|X| > 1$, and let ρ be a representation of X, so that ρ is almost surely associated with $\mathcal{M}$. Then ρ is not admissible w.r.t. any topology τ on X.*

Proof. We recall that ρ being admissible w.r.t. τ implies ρ to be continuous w.r.t. τ and τ to be a T_0-topology, as proven in [Schröder (2002), Theorem 13]. The corollary now follows from Theorem 8.

Combining Theorems 5, 6 and 8, we see that every non-degenerate measurement procedure has an almost surely associated representation, however, not an admissible one. Similarly, every admissible representation has a measurement procedure an equivalent representation is almost surely associated with, however, not a non-degenerate one.

Faced with the problem to model the results of a measurement by a representation, one cannot achieve complete accuracy and desirable properties of the representation simultaneously. It was already shown that complete accuracy can be ensured, if the resulting representation is accepted not to be admissible. Whether an admissible representation could be associated with a non-degenerate measurement procedure in a less strict way, will be the focus of the next section.

3.3 Associated Representations

Definition 10. A representation ρ is called associated with probability ε with a measurement procedure $\mathcal{M}$, if $\hat{\mathcal{M}}(x)(\rho^{-1}(\{x\})) \geq \varepsilon$ holds for all $x \in X$ and $\hat{\mathcal{M}}(x)(\rho^{-1}(\{y\})) = 0$ holds for all $x, y \in X$, $x \neq y$.

Association with probability ε generalises almost sure association because association with probability 1 coincides with almost sure association. However, for obtaining the following results, $0 < \varepsilon < 1$ shall always be assumed.

Note that association with probability ε allows an infinitely repeated measurement to yield no valid ρ-name at all with probability $1 - \varepsilon$, while a wrong ρ-name has zero probability.

The goal of this section is to prove that for every measurement procedure $\mathcal{M}$ for X and every $\varepsilon < 1$ there is an admissible[7] representation associated with $\mathcal{M}$ with probability ε. Therefore, for each $\mathcal{M}$ and each ε, two equivalent representations are defined. One will be shown to be associated with $\mathcal{M}$ with probability ε, the other one is an admissible representation w.r.t. a certain topology on X.

Definition 11. For a measurement procedure $\mathcal{M}$ for a set X and an ε, define the representation α_ε of X through $\alpha_\varepsilon(w) = x$, if $\lim\limits_{k \to \infty} \frac{c_{i,k}(w)}{k} = \mathcal{M}(x)(i)$ and $\frac{c_{i,n}(w)}{n} < \mathcal{M}(x)(i) + f_n$ hold for all $i, n \in \mathbb{N}$, $n > 1$. The number f_n is defined through $f_n = \frac{1}{\sqrt[4]{4n(1-\varepsilon)}}$.

[7] As there is no fixed topology on X, the announced result just states that a suitable topology can be constructed. Actually, as will be argued later, a natural topology on a set of values for a physical entity is derived from the properties of the possible measurements, as points are assumed to be nearer, if they are more likely to be confused by the measurement.

Theorem 12. *The representation α_ε from Definition 11 is associated with $\mathcal{M}$ with probability ε.*

Proof. From Definitions 4, 11 follows directly $\alpha_\varepsilon^{-1}(\{x\}) \subseteq \alpha^{-1}(\{x\})$. So from $\hat{\mathcal{M}}(y)(\alpha^{-1}(\{x\})) = 0$ for $x \neq y$, as shown in the proof of Theorem 5, also $\hat{\mathcal{M}}(y)(\alpha_\varepsilon^{-1}(\{x\})) = 0$ follows.

Now assume that for a given sequence w, $\frac{c_{i,n}(w)}{n} < \mathcal{M}(x)(i) + f_n$ holds for all $i \in \mathbb{N}$ and all $n < n_0$. Then $\frac{c_{i,n_0}(w)}{n_0} < \mathcal{M}(x)(i) + f_{n_0}$ follows for $i \neq w(n_0)$. This results from $c_{i,n_0}(w) = c_{i,n_0-1}(w)$ for $i \neq w(n_0)$ and the fact that f_n is decreasing more slowly than $\frac{1}{n}$. So $\alpha_\varepsilon^{-1}(\{x\})$ can be written as:

$$\alpha_\varepsilon^{-1}(\{x\}) = \alpha^{-1}(\{x\}) \cap \{w \in \mathbb{N}^\omega \mid \forall n > 1 \ \frac{c_{w(n),n}(w)}{n} < \mathcal{M}(x)(w(n)) + f_n\}$$

Using [Shiryaev (1996), equation 39, page 69], for a given $n \in \mathbb{N}$, one gets:

$$\hat{\mathcal{M}}(x)(\{w \in \mathbb{N}^\omega \mid \frac{c_{w(n),n}(w)}{n} > \mathcal{M}(x)(w(n)) + f_n\}) \leq e^{-2nf_n^2}$$

By summing up over countably many probabilities,

$$P := \hat{\mathcal{M}}(x)(\mathbb{N}^\omega \setminus \alpha_\varepsilon^{-1}(\{x\})) \leq \sum_{n=1}^{\infty} e^{-2nf_n^2}$$

follows. Cauchy's integral criterion leads to $P \leq \int_0^\infty e^{-2nf_n^2} dn$. This integral can be evaluated to $1 - \varepsilon$, what completes the proof.

To obtain an admissible representation on X, we start with an admissible representation on $\mathbb{P}$, which will be lifted according to the following theorem, which is a special case of [Zhong and Weihrauch (2003), Lemma 2.10].

Theorem 13. *Let $f : X \to Y$ be an injective function, and δ be a representation for Y that is admissible w.r.t. a topology $\mathcal{S}$ on Y. Let $\mathcal{T} := \{f^{-1}(U) \mid U \in \mathcal{S}\}$ be the initial topology regarding f. Then $f^{-1} \circ \delta$ is a representation of X that is admissible w.r.t. $\mathcal{T}$.*

Representations for probability measures on represented metrisable spaces have been investigated in a general fashion in [Schröder (2007)], yielding an equivalence class of representations bearing several interesting characterizations including admissibility w.r.t. the weak topology on the space of probability measures. The following standard representation of $\mathbb{P}$ can also be considered as restriction of the standard representation of sequences of real numbers to $\mathbb{P}$.

Definition 14. The standard representation θ of $\mathbb{P}$ is defined through $\theta(w) = p$, if $\lim_{n\to\infty} \nu_{\mathbb{Q}}(w(\langle i, n\rangle)) = p(i)$ is satisfied for all $i \in \mathbb{N}$, and $\nu_{\mathbb{Q}}(w(\langle i, n\rangle)) \leq p(i)$ holds for all $n, i \in \mathbb{N}$.

In the following, we will prove α_ε to be computationally equivalent to $\mathcal{M}^{-1}\circ\theta$ for $0 < \varepsilon < 1$. Since $\mathcal{M}^{-1}\circ\theta$ does not depend on ε, this implies that the superficial arbitrariness in the choice of ε is not problematic. $\mathcal{M}^{-1}\circ\theta$ will be abbreviated with δ.

Theorem 15. *For a measurement procedure $\mathcal{M}$ and an ε, $\alpha_\varepsilon \preceq_c \delta$.*

Proof. Let λ_n be a computable sequence of rational numbers satisfying $f_n \leq \lambda_n$ and $\lim\limits_{n\to\infty} \lambda_n = 0$. Then define the function $G : \mathbb{N}^\omega \to \mathbb{N}^\omega$ through:

$$G(w)(\langle i,n\rangle) = \nu_{\mathbb{Q}}^{-1}(\frac{c_{i,n}(w)}{n} - \lambda_n)$$

Clearly, G is computable. Now, $\alpha_\varepsilon(w) = \delta \circ G(w)$ for all $w \in \mathrm{dom}(\alpha_\varepsilon)$ follows from the relevant definitions.

Theorem 16. *For a measurement procedure $\mathcal{M}$ and an ε, $\delta \preceq_c \alpha_\varepsilon$.*

Proof. We define the partial function $H :\subseteq \mathbb{N}^\omega \to \mathbb{N}^\omega$ through the following informal description of a Type-2 machine M which works on a δ-name w. In the nth step with current output σ, M computes the numbers $k_{i,n}$, $1 \leq i \leq n$ according to $k_{i,n} = \max\{k \in \mathbb{N} \mid \exists m \leq n,\ \frac{k}{n} \leq \nu_{\mathbb{Q}}(w(\langle i,m\rangle))\}$. Then M prints the smallest $i \leq n$ with $k_{i,n} > c_{i,|\sigma|}(\sigma)$, if such an n exists, or otherwise the smallest i with $c_{i,|\sigma|}(\sigma) = 0$. After that, M proceeds to the next step.

Assume $c_{i,n}(H(w)) > 1$ for any n, i. Then there is an $m \leq n$, so that the last i in $H(w)_{\leq n}$ was printed by M in the mth step. Therefore, we have $c_{i,n}(H(w)) = c_{i,m}(H(w)) \leq k_{i,m}$, so also:

$$\frac{c_{i,n}(H(w))}{n} \leq \frac{k_{i,m}}{n} \leq \nu_{\mathbb{Q}}(w(\langle i,n\rangle)) \leq \mathcal{M}(\delta(w))(i)$$

Consequently, the first condition in Definition 11 is fulfilled. In the case $c_{i,n}(H(w)) \leq 1$, it is trivially fulfilled.

It remains to prove $\lim\limits_{n\to\infty} \frac{c_{i,n}(H(w))}{n} = \mathcal{M}(\delta(w))(i)$ for all $i \in \mathbb{N}$. Since we know the limit is an upper bound for the sequence, we just have to prove that for every $r < \mathcal{M}(\delta(w))(i)$ there is an n_r, so that $r \leq \frac{c_{i,n}(H(w))}{n}$ holds for all $n \geq n_r$. From $\lim\limits_{n\to\infty} \nu_{\mathbb{Q}}(w(\langle i,n\rangle)) = \mathcal{M}(\delta(w))(i)$ we obtain an n_0 with $r < \nu_{\mathbb{Q}}(w(\langle i,n_0\rangle))$. Thus, there also must be an n_1 with $r < \frac{k_{i,n_1}}{n_1}$.

Define $g_n = |\{i \leq n \mid c_{i,n}(H(w)) > k_{i,n}\}|$. If there is an $i \leq n+1$ with $k_{i,(n+1)} > c_{i,(n+1)}(H(w))$, then $g_{n+1} \leq g_n$ holds. Furthermore, we have $\frac{k_{i,m}}{m} \leq \frac{c_{i,n}(H(w))+g_m}{n}$ for all $m \in \mathbb{N}$, $i \leq m$, $m \leq n$. Especially, $r < \frac{c_{i,n}(H(w))+g_{n_1}}{n}$ for all $n \geq n_1$. Since g_{n_1} does not depend on n and therefore $\lim\limits_{n\to\infty} \frac{g_{n_1}}{n} = 0$, there must be an n_2 with $r \leq \frac{c_{i,n}(H(w))}{n}$ for all $n \geq n_2$, which completes the proof.

3.4 Common measurement procedures in physics

In most parts of classical physics, measurement errors of real-valued physical entities are usually assumed to follow a normal distribution. The reasons for this model are illustrated in [Parratt (1961), Chapter 4]. [Patel and Read (1996)] provide a historic excursion on the normal distribution. A central argument for the use of the normal distribution is the Central Limit Theorem [van Kampen (1992), Chapter 1.7], stating roughly that measurement errors induced by an infinite number of insignificantly small errors follow a normal distribution.

Taking into account the inherent discrete nature of digital measurement devices, and assuming a uniform variance σ^2, a measurement procedure can be obtained from the normal distribution:

Definition 17. The measurement procedure $\mathcal{N}$ for the set $\mathbb{R}$ is defined through:

$$\mathcal{N}(x)(n) = \int_{\nu_{\mathbb{Z}}(n)-\frac{1}{2}}^{\nu_{\mathbb{Z}}(n)+\frac{1}{2}} \frac{1}{\sigma\sqrt{2\pi}} \exp(-\frac{1}{2}(\frac{y-x}{\sigma})^2)dy$$

Theorem 18. *$\mathcal{N}$ is a measurement procedure.*

Proof. It is clear that each $\mathcal{N}(x)(n)$ is a non-negative real number. As

$$\sum_{n\in\mathbb{N}} \mathcal{N}(x)(n) = \int_{-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp(-\frac{1}{2}(\frac{y-x}{\sigma})^2)dy = 1$$

holds, directly $\mathcal{N}(x) \in \mathbb{P}$ follows for all $x \in \mathbb{R}$. It remains to show that $\mathcal{N}$ is injective. Assume $x < y$ and $\mathcal{N}(x) = \mathcal{N}(y)$. For $z \geq y$, $\exp(-\frac{1}{2}(\frac{z-x}{\sigma})^2) < \exp(-\frac{1}{2}(\frac{z-y}{\sigma})^2)$ holds. Consider an n with $\nu_{\mathbb{Z}}(n) - \frac{1}{2} \geq z$. From $\mathcal{N}(x)(n) = \mathcal{N}(y)(n)$ follows:

$$\int_{\nu_{\mathbb{Z}}(n)-\frac{1}{2}}^{\nu_{\mathbb{Z}}(n)+\frac{1}{2}} \exp(-\frac{1}{2}(\frac{z-y}{\sigma})^2) - \exp(-\frac{1}{2}(\frac{z-x}{\sigma})^2)dz = 0$$

The integrand is non-negative, as explained above, so it has to be 0 everywhere. This contradicts $\exp(-\frac{1}{2}(\frac{z-x}{\sigma})^2) < \exp(-\frac{1}{2}(\frac{z-y}{\sigma})^2)$, so $\mathcal{N}$ has to be injective.

It was already established that the equivalence class of representations containing δ and α_ε is a good candidate for representing the results of such kind of measurements. However, there is already a standard representation of the real numbers[8], which is admissible w.r.t. the Euclidean topology on $\mathbb{R}$. The following

[8] A discussion of various representations of $\mathbb{R}$ are their respective relations can be found in [Weihrauch (1992)].

theorem establishes computational equivalence between these representations, provided the variance σ^2 is computable. For non-computable σ^2, continuous equivalence follows.

Theorem 19. *Let δ be the representation obtained as $\delta := \mathcal{N}^{-1} \circ \theta$, where σ^2 is computable, and let ρ be the standard representation of $\mathbb{R}$. Then $\delta \equiv_c \rho$ follows.*

Proof. The function $x \mapsto \mathcal{N}(x)(n)$ is (ρ, ρ)-computable[9] for each n, so the function $x \mapsto \mathcal{N}(x)$ is $(\rho, [\rho]^\omega)$ and thus (ρ, θ) computable. This implies that $x \mapsto x$ is (ρ, δ)-computable, establishing $\rho \preceq_c \delta$.

For the other direction, consider the function R defined through:

$$R(x) = \int_{\frac{1}{2}}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp(-\frac{1}{2}(\frac{y-x}{\sigma})^2)dy$$

Assuming that $\{n \mid \nu_{\mathbb{Z}}(n) \geq 1\}$ is decidable, R is $(\delta, \rho_<)$-computable, the same holds for $x \mapsto 1 - R(x)$, so R is even (δ, ρ)-computable. By the same reasoning as above R is (ρ, ρ)-computable. Since R is strictly increasing, according to [Weihrauch (2000), Theorem 6.3.11], R^{-1} is also (ρ, ρ)-computable. Thus $x \mapsto x$ is (δ, ρ)-computable, establishing $\delta \preceq_c \rho$.

While Theorem 19 directly yields an independent argument for the use of the standard representation when dealing with real numbers, it can also be interpreted as stating that the Euclidean topology is indeed the most suitable topology on the real numbers regarding measurements.

More generally, assume a probability measure on $\mathbb{R}$ derived from a differentiable probability density f. Through linear translation, a measurement procedure $\mathcal{M}$ is given by $\mathcal{M}(x)(n) = \int_{\nu_{\mathbb{Z}}(n)-\frac{1}{2}}^{\nu_{\mathbb{Z}}(n)+\frac{1}{2}} f(y-x)dy$. The representations α_ε for $\mathcal{M}$ are equivalent to the standard representation of $\mathbb{R}$, the proofs are analogously to the respective proofs for the normal distribution. Examples for such measures include the Cauchy (or Lorentz) distribution, the logistic distribution, the Fisher-Tippett distribution and the Laplace distribution.

4 Reductions between measurement procedures

4.1 Definitions

To be able to compare measurement procedures regarding the amount of information they supply, a notion of reducibility between measurement procedures

[9] To prove this, a computable version of [Lang (1968), Chapter X, Theorem 11] is useful, which can easily be obtained from the results presented in [Weihrauch (2000), Chapter 6].

is desirable. Depending on the placement of the reduction in the model different definitions seem to be natural, which will be introduced and compared in the following. However, further results are needed to make a convincing case for choosing one of them.

The first definition assumes that the reduction works directly and memoryless on the output of our measurement device. The measurement procedure $\mathcal{M}$ is simulated by $\mathcal{N}$, in a way that each natural number n obtained from the measurement according to $\mathcal{N}$ is simply replaced by $f(n)$.

Definition 20. $\mathcal{M} \leq_1 (\leq_1^c) \mathcal{N}$, if there is a (computable) function $f : \mathbb{N} \to \mathbb{N}$ with $\mathcal{M}(x)(n) = \sum_{k \in f^{-1}(\{n\})} \mathcal{N}(x)(k)$ for all $x \in X$, $n \in \mathbb{N}$.

A more general reduction would still modify the output of the measurement device, but include a record of earlier measurements. This replaces the function $f : \mathbb{N} \to \mathbb{N}$ by a continuous partial function $F :\subseteq \mathbb{N}^\omega \to \mathbb{N}^\omega$. The move to partial functions is debatable, in fact, this allows the alleged simulation to fail sometimes, however, only with probability 0. We agree with [Parker (2003)] and [Bosserhoff (2008)] that if wrong results are accepted with probability zero, no results should also be accepted with probability 0. Another argument in favour of choosing partial functions in the following definition will be given by Theorem 27.

Definition 21. $\mathcal{M} \leq_2 (\leq_2^c) \mathcal{N}$, if there is a continuous (computable) partial function $F :\subseteq \mathbb{N}^\omega \to \mathbb{N}^\omega$ with $\hat{\mathcal{M}}(x)(A) = \hat{\mathcal{N}}(x)(F^{-1}(A))$ for all $x \in X$, $A \subseteq \mathbb{N}^\omega$.

Instead of caring about simulating the complete measurement, for the next definitions its information content will be expressed in terms of the associated representations. The set of representations almost surely associated with a measurement procedure (or associated with a measurement procedure with probability ε) uniquely determines the measurement procedure. As both sets contain the representation α from Definition 4, it suffices to realise that this representation can only be associated to a single measurement procedure. We suggest a reduction that is uniform in the associated representations to account for the arbitrariness in the choice of associated representations.

Definition 22. $\mathcal{M} \leq_3 (\leq_3^c) \mathcal{N}$, if there is a continuous (computable) partial function $F :\subseteq \mathbb{N}^\omega \to \mathbb{N}^\omega$, so that for each $0 < \varepsilon < 1$, for each representation ρ of X that is associated with $\mathcal{M}$ with probability ε, there is a representation δ of X with $\delta(w) = \rho(F(w))$ for all $w \in \mathrm{dom}(\delta)$, which is associated with $\mathcal{N}$ with probability ε.

Definition 23. $\mathcal{M} \leq_4 (\leq_4^c) \mathcal{N}$, if there is a continuous (computable) partial function $F :\subseteq \mathbb{N}^\omega \to \mathbb{N}^\omega$, so that for each representation ρ of x that is almost surely associated with $\mathcal{M}$, there is a representation δ of X with $\delta(w) = \rho(F(w))$ for all $w \in \mathrm{dom}(\delta)$, which is almost surely associated with $\mathcal{N}$.

The reductions from Definitions 22 and 23 can be expressed in terms of measure theory as well, this will be demonstrated in the next subsection.

4.2 Implications between the reducibility relations

It is clear that for $i \in \{1, 2, 3, 4\}$, $\mathcal{M} \leq_i^c \mathcal{N}$ always implies $\mathcal{M} \leq_i \mathcal{N}$, as each computable (partial) function is continuous w.r.t. the relevant topologies.

Theorem 24. *$\mathcal{M} \leq_1 \mathcal{N}$ does not imply $\mathcal{M} \leq_1^c \mathcal{N}$.*

Proof. Consider $X = \{x\}$, and $\mathcal{N}(x)(n) = 2^{-n}$. Let A be any subset of $\mathbb{N}$, and $\mathcal{M}(x)(1) = \sum_{n \in A} 2^{-1}$, $\mathcal{M}(x)(2) = 1 - \mathcal{M}(x)(1)$. Then there is exactly one function $f : \mathbb{N} \to \mathbb{N}$ with $\mathcal{M}(x)(n) = \sum_{k \in f^{-1}(\{n\})} \mathcal{N}(x)(k)$, which is given by $f(n) = 1$ for $n \in A$ and $f(n) = 2$ for $n \in \mathbb{N} \setminus A$. For a non-decidable A, F is non-computable, so $\mathcal{M} \leq_1 \mathcal{N}$ holds, while $\mathcal{M} \leq_1^c \mathcal{N}$ is false.

Theorem 25. *$\mathcal{M} \leq_1 (\leq_1^c)\ \mathcal{N}$ implies $\mathcal{M} \leq_2 (\leq_2^c)\ \mathcal{N}$.*

Proof. Let F be the (computable) function used in Definition 20. Define $\hat{F} : \mathbb{N}^\omega \to \mathbb{N}^\omega$ through $(\hat{F}(w))_n = F(w_n)$. Clearly, $\hat{F}$ is continuous, and if F is computable, so is $\hat{F}$. As $\hat{F}$ satisfies the condition from Definition 21, the claim follows.

Theorem 26. *$\mathcal{M} \leq_2^c \mathcal{N}$ does not imply $\mathcal{M} \leq_1 \mathcal{N}$.*

Proof. Consider the set $X = \{x\}$, the function $F : \mathbb{N}^\omega \to \mathbb{N}^\omega$ defined through $(F(w))_n = \langle w_{2n-1}, w_{2n} \rangle$, and measurement procedures $\mathcal{N}$, $\mathcal{M}$ given by:

$$\mathcal{N}(x)(1) = \mathcal{N}(x)(2) = \frac{1}{2}$$

$$\mathcal{M}(x)(\langle 1, 1 \rangle) = \mathcal{M}(x)(\langle 1, 2 \rangle) = \mathcal{M}(x)(\langle 2, 1 \rangle) = \mathcal{M}(x)(\langle 2, 2 \rangle) = \frac{1}{4}$$

Then $\hat{\mathcal{M}}(x)(A) = \hat{\mathcal{N}}(x)(F^{-1}(A))$ holds, but there is no $f : \mathbb{N} \to \mathbb{N}$ with $\mathcal{M}(x)(n) = \sum_{k \in f^{-1}(\{n\})} \mathcal{N}(x)(k)$.

Although the Definitions of $\leq_2$ and $\leq_3$ might appear quite different on the first glance, the following theorem establishes that the two reducibilities are equivalent. This provides another link between measurement procedures and their associated representations.

Theorem 27. *$\mathcal{M} \leq_2 (\leq_2^c)\ \mathcal{N}$ is equivalent to $\mathcal{M} \leq_3 (\leq_3^c)\ \mathcal{N}$.*

Proof. The first implication from $\leq_2$ to $\leq_3$ can easily be proven. Let F be the function used in Definition 21. Then

$$\hat{\mathcal{M}}(x)(\rho^{-1}(\{x\}) = \hat{\mathcal{N}}(x)(F^{-1}(\rho^{-1}(\{x\}))) = \hat{\mathcal{N}}(x)((\rho \circ F)^{-1}(\{x\}))$$

so $\rho \circ F$ is the representation δ needed for Definition 22. That $\rho \circ F$ actually is a representation follows from $\hat{\mathcal{N}}(x)((\rho \circ F)^{-1}(\{x\})) > 0$ for all $x \in X$.

Now assume that F is a partial function witnessing $\mathcal{M} \leq_3 (\leq_3^c) \mathcal{N}$. We have to show $\hat{\mathcal{M}}(x)(A) = \hat{\mathcal{N}}(x)(F^{-1}(A))$ for all $x \in X$, $A \subseteq \mathbb{N}^\omega$. Since $\hat{\mathcal{M}}(x)(A) = \hat{\mathcal{M}}(x)(A \cap \alpha^{-1}(\{x\}))$, where α is the representation introduced in Definition 4 for $\mathcal{M}$, we can restrict considerations to the case $A \subseteq \alpha^{-1}(\{x\})$ for given $x \in X$. We further assume $\varepsilon := \hat{\mathcal{M}}(x)(A) > 0$ for now.

In the next step, we define a representation ρ of X by $\rho(w) = \alpha(w)$ for all $w \in \alpha^{-1}(X \setminus \{x\}) \cup A$. ρ shall be undefined elsewhere. Then ρ is associated with $\mathcal{M}$ with probability ε. Thus $\rho \circ F$ extends a representation δ which is associated with $\mathcal{N}$ with probability ε, yielding $\hat{\mathcal{N}}(x)(F^{-1}(A)) \geq \varepsilon = \hat{\mathcal{M}}(x)(A)$. Since this holds for all $A \subseteq \mathbb{N}^\omega$, even the equality must hold. The null sets disregarded earlier are taken care of by considering their complements.

It is possible to weaken Definition 21 to obtain a definition equivalent to Definition 23, and thereby establishing an implication between $\leq_2$ (or $\leq_3$) and $\leq_4$.

Theorem 28. $\mathcal{M} \leq_4 (\leq_4^c) \mathcal{N}$ *is true, if and only if there is a continuous (computable) partial function* $F :\subseteq \mathbb{N}^\omega \to \mathbb{N}^\omega$*, so that* $\hat{\mathcal{M}}(x)(A) = 1$ *implies* $\hat{\mathcal{N}}(x)(F^{-1}(A)) = 1$ *for any* $x \in X$*,* $A \subseteq \mathbb{N}^\omega$*.*

Proof. If the second condition holds, then for any representation ρ that is almost surely associated to $\mathcal{M}$, $\rho \circ F$ already is almost surely associated to $\mathcal{N}$, thus fulfilling Definition 23.

For the other direction, let F witness $\mathcal{M} \leq_4 (\leq_4^c) \mathcal{N}$. Given A and x with $\hat{\mathcal{M}}(x)(A) = 1$, we define a representation ρ of X by $\rho(w) = \alpha(w)$ for $w \in \alpha^{-1}(X \setminus \{x\})$, where α is the representation from Definition 4 applied to $\mathcal{M}$, and $\rho(w) = x$ for $w \in A$. Then ρ is almost surely associated with M, so $\rho \circ F$ extends a representation δ which is almost surely associated with $\mathcal{N}$. This implies $\hat{\mathcal{N}}(x)(F^{-1}(A)) = 1$.

Theorem 29. *$\mathcal{M} \leq_4^c \mathcal{N}$ does not imply $\mathcal{M} \leq_3 \mathcal{N}$.*

Proof. We consider a singleton set $X = \{x\}$, and the measurement procedures $\mathcal{N}$ and $\mathcal{M}$ defined by $\mathcal{M}(x)(1) = \mathcal{M}(x)(2) = \frac{1}{2}$, $\mathcal{N}(x)(1) = \frac{2}{3}$ and $\mathcal{N}(x)(2) = \frac{1}{3}$. To establish $\mathcal{M} \leq_4^c \mathcal{N}$, we use a method introduced by von Neumann to simulate a perfect source of random bits by a distorted source of random bits. More precise, we consider a machine that reads two input symbols in each step, and writes 1 if it read 12, and 2 if it read 21. All other input is ignored. The partial function F computed by this machine satisfies $\hat{\mathcal{N}}(x)(F^{-1}(A)) = 1$ for all A with $\hat{\mathcal{M}}(x)(A) = 1$.

Now consider any continuous partial function G, and observe

$$\hat{\mathcal{N}}(x)(G^{-1}(1\mathbb{N}^\omega)) = \frac{n-1}{3^m}$$

with $n, m \in \mathbb{N}$, so $\hat{\mathcal{N}}(x)(G^{-1}(1\mathbb{N}^\omega)) \neq \hat{\mathcal{M}}(x)(1\mathbb{N}^\omega) = \frac{1}{2}$ follows. This proves $\mathcal{M} \not\leq_3 \mathcal{N}$.

4.3 Reductions and null sets

It seems useful to investigate the structure of the set of representations almost surely associated with a certain measurement procedure, both for its own sake, and to gain further insight into the reduction $\leq_4$. The set of representations associated with a measurement procedure with probability ε is equally important, among other reasons to study $\leq_3$.

Theorem 30. *Let ρ be a representation of X almost surely associated with the measurement procedure $\mathcal{M}$, and δ be another representation of X. Let D_x denote the set $(\rho^{-1}(\{x\}) \setminus \delta^{-1}(\{x\})) \cup (\delta^{-1}(\{x\}) \setminus \rho^{-1}(\{x\}))$ for $x \in X$. Then δ is almost surely associated with $\mathcal{M}$, if and only if $\hat{\mathcal{M}}(x)(D_x) = 0$ for all $x \in X$.*

Theorem 31. *Let ρ be a representation of X with the measurement procedure $\mathcal{M}$ with probability ε, and δ be another representation of X. Let D_x denote the set $(\rho^{-1}(\{x\}) \setminus \delta^{-1}(\{x\})) \cup (\delta^{-1}(\{x\}) \setminus \rho^{-1}(\{x\}))$ for $x \in X$. If $\hat{\mathcal{M}}(x)(D_x) = 0$ for all $x \in X$, then δ is associated with $\mathcal{M}$ with probability ε.*

A direct consequence of Theorem 30 will be useful:

Theorem 32. *Let ρ be a representation of X almost surely associated with the measurement procedure $\mathcal{M}$, and ρ be a restriction of the representation δ, then δ is almost surely associated with $\mathcal{M}$.*

As the values of a representation ρ on a null set are not relevant for association with a measurement procedure, it seems straightforward to include this in the definitions of the relevant reductions. Following [Bosserhoff (2008)], we call a function f μ-almost everywhere continuous (computable), if there is a μ-null set N, so that $f_{|\operatorname{dom}(f)\setminus N}$ is continuous (computable). A function is $\mathcal{M}$-almost everywhere continuous (computable), if it is $\hat{\mathcal{M}}(x)$-almost everywhere continuous (computable) for each $x \in X$.

Theorem 33. *$\mathcal{M} \leq_3$ ($\leq_3^c$) $\mathcal{N}$, if and only if there is a $\mathcal{N}$-almost everywhere continuous (computable) function $F :\subseteq \mathbb{N}^\omega \to \mathbb{N}^\omega$, so that for each $\varepsilon > 0$ and each representation ρ of x that is associated with $\mathcal{M}$ with probability ε, there is a representation δ of X with $\delta(w) = \rho(F(w))$ for all $w \in \operatorname{dom}(\delta)$, which is associated with $\mathcal{N}$ with probability ε.*

Theorem 34. *$\mathcal{M} \leq_4$ ($\leq_4^c$) $\mathcal{N}$, if and only if there is a $\mathcal{N}$-almost everywhere continuous (computable) function $F :\subseteq \mathbb{N}^\omega \to \mathbb{N}^\omega$, so that for each representation ρ of x that is almost surely associated with $\mathcal{M}$, there is a representation δ of X with $\delta(w) = \rho(F(w))$ for all $w \in \operatorname{dom}(\delta)$, which is almost surely associated with $\mathcal{N}$.*

5 Conclusions and Open Questions

In this section, we will try to illuminate the relevance of the definitions and results presented so far, and point out possible starting points for further inquiry. We have provided a framework for studying computability aspects of physical theories. In the first step, a physical theory has to be effectivized by stating the measurement procedure as introduced in Definition 1 used to conduct measurements. The measurement procedure as defined here can be inferred from the distribution of measurement errors assumed for the physical theory.

To obtain a representation of the set of possible measurement results from the used measurement procedure, three different approaches are presented here. All three have in common that the sequence of repeated measurement results is regarded as a name for the state the measured entity is in. If the measurement procedure is degenerate (Definition 2), Theorem 7 provides a topology on the space of measurement results, so the corresponding standard representation can be used with zero probability of errors.

If the measurement procedure at hand is non-degenerate, either the representation α from Definition 4 can be used, which has the advantage that α-names can be produced by repeated measurement with zero probability of errors, but is not continuous w.r.t. any T_0 topology on the set of possible measurement results; or the representation α_ε for any $0 < \varepsilon < 1$ presented in Definition 11, which is admissible w.r.t. a certain topology, but comes with an error probability of up to $1-\varepsilon$. While the dilemma presented here is unavoidable, we do not know whether there is another general family of representations with similar properties as the α_ε, but admissible w.r.t. another topology.

For a number of common possible distributions of measurement errors the analysis outlined above was conducted in Subsection 3.4, showing that the standard representation ρ of the real numbers is suitable to represent measurement results in most cases, however, with an arbitrarily small positive probability of errors occurring.

A field of questions mainly left open by this paper is how to compare the information content of different measurement procedures. A couple of possible definitions, and their relations to each other, are given in Section 4; but further results are needed to settle the issue.

Acknowledgements

I am deeply grateful to the participants and referees of the CCA 2008, for the large number of comments, questions and suggestions on earlier versions of this paper ([Pauly (2008)] in particular), and to Klaus Weihrauch for introducing me to computable analysis. Furthermore, I would like to thank Michael Wenzel for drawing [Figure 1]; and my supervisor, Anuj Dawar, for his support.

References

[Arnold (1989)] Arnold, V.I.: "Mathematical Methods of Classical Mechanics"; Springer (1989)

[Beggs and Tucker (2004)] Beggs, E., Tucker, J.: "Computation via experiments with kinematic systems"; Technical Report 5-2004, Department of Computer Science, University of Wales Swansea (2004)

[Beggs and Tucker (2007)] Beggs, E., Tucker, J.: "Can newtonian systems, bounded in space, time, mass and energy compute all functions?"; "Theoretical Computer Science"; 371 (2007), 4-19.

[Bosserhoff (2008)] Bosserhoff, V.: "Notions of probabilistic computability on represented spaces"; J.UCS (Journal of Universal Computer Science), 14, 6 (2008), 956-995.

[Dugundji (1970)] Dugundji, J.: "Topology"; Allyn and Bacon, Boston, 6 edition, (1970)

[Feyman et al. (1963)] Feynman, R., Leighton, R., Sands, M.: "The Feynman Lectures on Physics"; Addison-Wesley, (1963)

[Geroch and Hartle (1986)] Geroch, R., Hartle, J.: "Computability and Physical Theories"; Foundations Phys., 16, 6 (1986), 533 - 550.

[Lang (1968)] Lang, S.: "Analysis 1"; Addison-Wesley, (1968)

[Parker (2003)] Parker, M.W.: "Undecidability in $\mathbb{R}^n$: Riddled basins, the KAM tori, and the stability of the solar system"; Philos. Sci., 70, 2 (2003), 359 - 382.

[Parratt (1961)] Parratt, L.: "Probability and Experimental Errors in Science"; John Wiley and Sons, New York, (1961)

[Patel and Read (1996)] Patel, J., Read, C.: "Handbook of the Normal Distribution" Marcel Dekker, (1996)

[Pauly (2008)] Pauly, A.: "Representing measurement results" In Brattka, V. et al. (eds) "Conference on Computability and Complexity in Analysis", Informatik Berichte 348, FernUniversität Hagen (2008), 187 - 198.

[Schröder (2002)] Schröder, M.: "Extended admissibility"; Theoretical Computer Science, 284, 2 (2002), 519-538.

[Schröder (2007)] Schröder, M.: "Admissible representations for probability measures"; Mathematical Logic Quaterly, 53, 4 (2007), 431-445.

[Schröder and Simpson (2006)] Schröder, M., Simpson, A.: "Representing probability measures using probabilistic processes"; Journal of Complexity, 22, 6 (2006), 768-782.

[Shiryaev (1996)] Shiryaev, A.N.: "Probability"; Springer, 2 edition, (1996)

[van Kampen (1992)] van Kampen, N.G.: "Stochastic Processes in Physics and Chemistry"; Elsevier Science, Amsterdam, 2 edition, (1992)

[Weihrauch (1992)] Weihrauch, K.: "The degrees of discontinuity of some translators between representations of the real numbers"; Informatik Berichte 129, FernUniversität Hagen (1992)

[Weihrauch (1999)] Weihrauch, K.: "Computability on the probability measures on the borel sets of the unit interval"; Theoretical Computer Science, 219 (1999), 421-437.

[Weihrauch (2000)] Weihrauch, K.: "Computable Analysis"; Springer-Verlag (2000)

[Yao (2003)] Yao, A.C.: "Classical physics and the Church-Turing thesis". J. ACM, 50, 1 (2003), 100-105.

[Zhong and Weihrauch (2003)] Zhong, N., Weihrauch, K.: "Computability theory of generalized functions" J.ACM, 50, 4 (2003), 469 - 505