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Paper:
Pauly, A. (2010). On the (semi)lattices induced by continuous reducibilities. *Mathematical Logic Quarterly, 56*(5), 488-502.
http://dx.doi.org/10.1002/malq.200910104

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Math. Log. Quart. **56**, No. 5, 488 – 502 (2010) / **DOI** 10.1002/malq.200910104

On the (semi)lattices induced by continuous reducibilities

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Received 11 March 2009, revised 1 December 2009, accepted 14 December 2009
Published online 17 September 2010

Key words Computable analysis, continuous reducibility, Weihrauch-reducibility, category theory.
MSC (2010) 03D30

Continuous reducibilities are a proven tool in Computable Analysis, and have applications in other fields such as Constructive Mathematics or Reverse Mathematics. We study the order-theoretic properties of several variants of the two most important definitions, and especially introduce suprema for them. The suprema are shown to commutate with several characteristic numbers.

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1 Introduction

Studying discontinuity of functions is an interesting topic on its own, an observation that is fortified by noting that continuity behaves similarly to computability in the framework of Computable Analysis. This suggests to compare the discontinuity of functions through continuous reducibilities. In the present paper, a continuous version of bounded Turing reducibility ($\leq_2$) and a continuous version of many-one reducibility ($\leq_0$) are studied.

While the implicitly involved concept of an Oracle-Type-2-Machine as defined in [16] is significantly more complicated than its classical counterpart, the resulting properties are basically the same. As we are primarily interested in the topological variants of the relations of relative computability in the present paper, we do not need to consider any details.

Another motivation for the study of these relations stems from parallels between Computable Analysis and Bishop's Constructive Mathematics ([1]) for $\leq_0$ and $\leq_2$ and between Computable Analysis and Reverse Mathematics ([18]) for $\leq_2$. As spelled out in [20], statements of the form $f \leq_0 g$ often correspond to set inclusions in Constructive Mathematics. The relationship between discontinuity and inconstructibility was studied in [21]. In Reverse Mathematics, $f \leq_2 g$ can correspond to the observation that a statement A can be proven with no more axioms than needed for proving B, as demonstrated in [8]. Neither of the two parallels is strict, but both were successfully used to derive new insight in one of the respective fields.

An even stronger link between continuous reducibilities and the foundations of mathematics was suggested recently in [4]. Through careful identification of mathematical theorems with sets of (discontinuous) functions, the reducibilities discussed here can be used to compare the effective content of these theorems. For a further presentation of this approach, we refer to [4].

2 Preliminaries

2.1 Topology

Given a set X, a *topology* $\mathcal{T}$ on X is a set of subsets of X including $\emptyset$ and X, which is closed under formation of arbitrary unions and finite intersections. The elements of a topology are called *open sets*, their complements are called *closed sets*. Since any union of open sets returns an open set, any intersection of closed sets is closed, enabling the definition of $\operatorname{cl} U$ as the smallest closed set containing $U \subseteq X$. For each set X, the discrete topology is given by $\mathcal{T}_{\mathrm{d}} = 2^X$ and the indiscrete topology is given by $\mathcal{T}_{\mathrm{i}} = \{\emptyset, X\}$.

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A *topological space* is a set equipped with a topology. Given a set-indexed family $(X_i, \mathcal{T}_i)_{i\in I}$, the coproduct $\coprod_{i\in I}(X_i, \mathcal{T}_i)$ is the set $\bigcup_{i\in I}(\{i\} \times X_i)$ equipped with the smallest topology $\mathcal{T}$ satisfying $\{(\{i\} \times U) \mid U \in \mathcal{T}_i\}$ is a subset of $\mathcal{T}$ for all $i \in I$. The product $\prod_{i\in I}(X_i, \mathcal{T}_i)$ is the set $\prod_{i\in I} X_i$ equipped with the smallest topology containing $\{\prod_{i\in I} U_i \mid (\forall i \in I)\, U_i \in \mathcal{T}_i, |\{i \in I \mid U_i \neq X_i\}| < |\mathbb{N}|\}$. For a topological space $(X, \mathcal{T})$ and a subset $Y \subseteq X$, the subspace topology on Y is defined as $\mathcal{T}_Y = \{U \cap Y \mid U \in \mathcal{T}\}$.

A function f between topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ is a function $f : X \longrightarrow Y$. It is continuous, if it satisfies $f^{-1}(U) \in \mathcal{T}$ for all $U \in \mathcal{S}$. The injection $\iota_j : (X_j, \mathcal{T}_j) \longrightarrow \coprod_{i\in I}(X_i, \mathcal{T}_i)$ defined through $\iota_j(x) = (j, x)$ is continuous, as well as the projection $\pi_j : \prod_{i\in I}(X_i, \mathcal{T}_i) \longrightarrow (X_j, \mathcal{T}_j)$ to the jth entry. The inclusion $\hookrightarrow$ of $(Y, \mathcal{T}_Y)$ in $(X, \mathcal{T})$ for $Y \subseteq X$ is also continuous. If $f : (X, \mathcal{T}) \longrightarrow (Y, \mathcal{S})$ is continuous, and $Z \subseteq X$, so is $f_{|Z} : (Z, \mathcal{T}_Z) \longrightarrow (Y, \mathcal{S})$.

For a family of continuous functions $(f_i : (X_i, \mathcal{T}_i) \longrightarrow (Y_i, \mathcal{S}_i))_{i\in I}$ we define by $\coprod_{i\in I} f_i(i, x) = (i, f_i(x))$ their *coproduct* $\coprod_{i\in I} f_i : \coprod_{i\in I}(X_i, \mathcal{T}_i) \longrightarrow \coprod_{i\in I}(Y_i, \mathcal{S}_i)$. The function $\coprod_{i\in I} f_i$ is continuous. Analogously, $\prod_{i\in I} f_i : \prod_{i\in I}(X_i, \mathcal{T}_i) \longrightarrow \prod_{i\in I}(Y_i, \mathcal{S}_i)$ is defined through $\prod_{i\in I} f_i(\prod_{i\in I} x_i) = \prod_{i\in I} f_i(x_i)$. As abbreviation, $f_1 \times f_2$ stands for $\prod_{i\in\{1,2\}} f_i$. For continuous functions $f : (X, \mathcal{T}) \longrightarrow (Y, \mathcal{S})$ and $g : (Y, \mathcal{S}) \longrightarrow (Z, \mathcal{R})$, the composition $g \circ f : (X, \mathcal{T}) \longrightarrow (Z, \mathcal{R})$ is continuous.

As the specific topologies are not relevant for the rest of the paper, we will use the notation $\underline{X}$ to indicate that a set X is equipped with a certain topology. Subsets are equipped with the restriction of the topology of the superset and (co)products of sets with the (co)product topology. A standard reference on topology is the book [6].

2.2 Order and lattice theory

A *preorder* on a class is a binary relation $\preceq$ that is reflexive and transitive. Each preorder defines an equivalence relation $\cong$ via $a \cong b \leftrightarrow a \preceq b \wedge b \preceq a$. On the equivalence classes regarding $\cong$, $\preceq$ becomes a partial order, as it is antisymmetric. In the following, we will not distinguish between a preorder and the partial order on its equivalence classes, the interpretation will be clear from the context.

A partially ordered class $(\mathcal{P}, \preceq)$ is said to be an *α-complete join-semilattice*, for a cardinal α, if for each $P \subseteq \mathcal{P}$ with $|P| < \alpha$ there is an element $\sup P \in \mathcal{P}$ so that $x \preceq \sup P$ holds for all $x \in P$, and where for each $z \in \mathcal{P}$ satisfying $(\forall y \in P)\ y \preceq z$ also $\sup P \preceq z$ is true. The dual notion is an *α-complete meet-semilattice*, where the existence of $\inf P$ with $\inf P \preceq x$ for $x \in P$ is required, so that $(\forall y \in P)\ z \preceq y$ implies $z \preceq \inf P$.

If a partially ordered class is an α-complete join-semilattice for all cardinals α, it is called a *complete join-semilattice*. A partially ordered class that is both an α-complete join-semilattice and an α-complete meet-semilattice is called an *α-complete lattice*. The definition of a complete lattice is straightforward.

If $\mathcal{Q}$ is a subclass of $\mathcal{P}$, then $(\mathcal{Q}, \preceq)$ is called a *sub-join-semilattice* of $(\mathcal{P}, \preceq)$, if $\sup P \in \mathcal{Q}$ holds for all $P \subseteq \mathcal{Q}$, the definition of sub-meet-semilattices and sub-lattices is straight-forward.

By choosing $P = \emptyset$, each α-complete join-semilattice has a least element $\sup \emptyset$, and every α-complete meet-semilattice has a maximal element $\inf \emptyset$. Note that not all results on partially ordered sets are valid for proper classes. An important distinction is that a complete join-semilattice that is defined over a set is also a complete lattice, while this is generally not true for an underlying proper class[1)].

Among the realm of further interesting properties of (semi)lattices is distributivity. While distributivity in the common sense is only defined for lattices, there are several possible ways of extending distributivity to semilattices. We will call a complete join-semilattice *distributive*, if $x \leq \sup_{i\in I} y_i$ implies the existence of a family $(x_i)_{i\in I}$ satisfying $x_i \leq y_i$ for all $i \in I$ and $x = \sup_{i\in I} x_i$. For complete lattices, distributivity as defined above is equivalent to the more familiar equation $\inf\{x, \sup_{i\in I} y_i\} = \sup_{i\in I}(\inf\{x, y_i\})$.

A treatment on lattices over sets can be found in [5] or [9].

2.3 Partial functions and problems

While we use the word *function* only to denote total functions, it will be convenient to use partial functions, as well. A *partial function* $f :\subseteq X \longrightarrow Y$ is a function $f : Z \longrightarrow Y$ with $Z \subseteq X$. A partial function $f :\subseteq \underline{X} \longrightarrow \underline{Y}$ will be called *continuous*, if $f : \underline{Z} \longrightarrow \underline{Y}$ is continuous. A statement such as $f(x) = g(x)$ for

1) For a set $P \subseteq \mathcal{P}$, the infimum can be constructed as $\inf P = \sup\{x \in \mathcal{P} \mid (\forall y \in P)\ x \preceq y\}$, provided the right side exists. If the join-semilattice $(\mathcal{P}, \preceq)$ is complete, the right side will exist, as long as $\{x \in \mathcal{P} \mid (\forall y \in P)\ x \preceq y\}$ is a set. This is guaranteed only if $\mathcal{P}$ is a set.

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partial functions means that either both sides are undefined (that is x is not a member of the respective subspace), or equal.

For some applications, functions are not necessarily an adequate formalisation for the notion of a problem to be solved. In some cases, a problem can be represented by a binary relation linking instances with solutions. We will employ an even more general notion, defining a problem[2)] $P : \underline{X} \longrightarrow \underline{Y}$ to be a set of partial functions from $\underline{X}$ to $\underline{Y}$. It is straight-forward to identify a function with the singleton set containing it, which allows us to consider functions as a special case of problems. For relations, we will choose another way, and call problems satisfying a certain locality property relations.

Definition 2.1 A problem $P : \underline{X} \longrightarrow \underline{Y}$ is called a *relation*, if $f \in P$ follows for all partial functions $f :\subseteq \underline{X} \longrightarrow \underline{Y}$ that fulfill $(\forall x \in \underline{X})\ (\exists g_x \in P)\ f(x) = g_x(x)$.

Especially, both the problem $\emptyset$ and the problem $\{\emptyset\}$ (with all possible domains and codomains) are considered as relations due to our definition, the latter being the set containing only the nowhere defined function. The problem $\emptyset$ is the problem without solutions, the problem $\{\emptyset\}$ is the problem without instances. The notion of problems was taken from [22].

2.4 Strongly zero-dimensional metrisable spaces

For applying the results of the present paper to computable analysis, the topological spaces of particular importance are the strongly zero-dimensional metrisable spaces. The most important examples for this class are the spaces $\underline{\alpha^{\mathbb{N}}}$ for a cardinal number α. The set $\alpha^{\mathbb{N}}$ is defined as $\alpha^{\mathbb{N}} = \{f \mid f : \mathbb{N} \longrightarrow \alpha\}$, with the topology derived from the metric $d(f, g) = 2^{-\min\{n \in \mathbb{N} \mid f(n) \neq g(n)\}}$. Of particular relevance is $\underline{\mathbb{N}^{\mathbb{N}}}$ as it serves as foundation for the theory of representations. A *representation* of a set X is defined as a surjective partial function $\delta :\subseteq \mathbb{N}^{\mathbb{N}} \longrightarrow X$.

We will now define a *strongly zero-dimensional metrisable space* as a topological space that admits a metric d, so that the range of d is $\{0\} \cup \{2^{-n} \mid n \in \mathbb{N}\}$. Clearly, each space $\underline{\alpha^{\mathbb{N}}}$ is a strongly zero-dimensional metrisable space. On the other hand, each strongly zero-dimensional metrisable space with weight α is homeomorphic to a subspace of $\underline{\alpha^{\mathbb{N}}}$.

Subspaces, coproducts and countable products of strongly zero-dimensional metrisable spaces are strongly zero-dimensional metrisable spaces. For each $\alpha > 1$, all coproducts of not more than α subspaces of $\underline{\alpha^{\mathbb{N}}}$ are homeomorphic to a subspace of $\underline{\alpha^{\mathbb{N}}}$, the same holds for countable products. The results in this subsection are due to [10] and [7].

3 Definitions

A function f is *many-one reducible* to a function g, if there is a computable function G with $f = g \circ G$. Analogously, $\leq_0$ reducibility is defined using continuous functions. Clearly, the codomain of all functions to be compared with $\leq_0$ has to be fixed. Additionally, a topology on the codomain is not needed.

Definition 3.1 Let $f : \underline{X} \longrightarrow \underline{Z}$ and $g : \underline{Y} \longrightarrow \underline{Z}$ be functions. Then $f \leq_0 g$ holds, if there is a continuous function $G : \underline{X} \longrightarrow \underline{Y}$ with $f = g \circ G$.

The primary application of $\leq_0$ is given by its interpretation as translatability of representations ([23], [22], and [17]). In this case, the domain is usually restricted to be subspaces of $\underline{\mathbb{N}^{\mathbb{N}}}$. Some results for comparing the discontinuity of functions in general with $\leq_0$ can be found in [10].

The version of bounded Turing reducibility that is analogous to $\leq_2$-reducibility states that f is reducible to g, if f can be computed using a single oracle call to g. To replace oracle calls with compositions of functions, the continuous function $\Delta_{\underline{X}} : \underline{X} \longrightarrow (\underline{X} \times \underline{X})$ defined through $\Delta_{\underline{X}}(x) = (x, x)$ has to be introduced for topological spaces $\underline{X}$. Furthermore, the identity on a topological space $\underline{X}$ is denoted by $\mathrm{id}_{\underline{X}}$.

Definition 3.2 [3)] Let $f : \underline{X_1} \longrightarrow \underline{Y_1}$ and $g : \underline{X_2} \longrightarrow \underline{Y_2}$ be functions. Then $f \leq_2 g$ holds, if there are continuous partial functions $F :\subseteq \underline{X_1} \times \underline{Y_2} \longrightarrow \underline{Y_1}$, $G :\subseteq \underline{X_1} \longrightarrow \underline{X_2}$ with $f = F \circ (\mathrm{id}_{\underline{X_1}} \times (g \circ G)) \circ \Delta_{\underline{X_1}}$.

---

2) There is a close analogy to the notion of mass problems used by Medvedev [11].

3) In some of the older literature, $\leq_2$ is called Wadge-reducibility ($\leq_{\mathrm{w}}$), recently the name Weihrauch-reducibility ($\leq_{\mathrm{W}}$) was suggested in [3].

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Note that in Definition 3.2, G could also be required to be a function, while requiring F to be a function leads to a different reducibility, as pointed out in [14, Subsection 1.6.3], using an example from [10, Theorem 2.5.5].

The Definitions 3.1, 3.2 are often restricted through placing certain conditions on the occurring topological spaces. For $\leq_2$, [22], [19], [12], [13] only consider subspaces[4)] of certain products of $\underline{\mathbb{N}}$ and $\underline{\mathbb{N}^{\mathbb{N}}}$ or equivalent spaces, [14] restricts considerations to metric spaces, while [2] studies computable metric spaces. [10] presents some results for $\leq_2$ restricted to functions with strongly zero-dimensional metrisable spaces as domain and discrete codomain.

While any restrictions on the kind of topological spaces to be considered can be employed for $\leq_0$, as the definition of $\leq_2$ contains some products of the involved spaces, as well as partial functions, it seems reasonable to restrict $\leq_2$ only to classes of topological spaces that are closed under formation of binary products and subspaces.

An extension of 3.2 to problems is presented in [22], the same approach can also be used for extending $\leq_0$ to problems. The uniform approach employed here, as the functions F, G in the following definitions do not depend on g, is justified by the interpretation of problems as sets of possible solutions.

Definition 3.3 Let $P : \underline{X} \longrightarrow \underline{Z}$ and $Q : \underline{Y} \longrightarrow \underline{Z}$ be problems. Define $P \leq_0 Q$, if there is a continuous partial function $G :\subseteq \underline{X} \longrightarrow \underline{Y}$ satisfying $g \circ G \in P$ for all $g \in Q$.

Definition 3.4 Let $P : \underline{X_1} \longrightarrow \underline{Y_1}$ and $Q : \underline{X_2} \longrightarrow \underline{Y_2}$ be problems. Define $P \leq_2 Q$, if there are continuous partial functions F, G with $F \circ (\mathrm{id}_{\underline{X_1}} \times (g \circ G)) \circ \Delta_{\underline{X_1}} \in P$ for all $g \in G$.

It is easy to see that Definitions 3.3 and 3.4 extend the Definitions 3.1 and 3.2 when functions are identified with the singleton set containing them. Note especially, that while G was required to be a continuous function in Definition 3.1, but a continuous partial function in Definition 3.3, in the case of singleton sets of functions, G turns to be a (total) function even in Definition 3.3.

There are further variants of $\leq_2$ that are not restrictions of Definition 3.4, such as the realizer reducibility introduced in [2] or the reducibility for multi-valued functions generalizing realizer reducibility on represented metric spaces from [8]. However, the corresponding partial order of Brattka's realizer reducibility is isomorphic to the restriction of $\leq_2$ as defined here for relations to subspaces of $\underline{\mathbb{N}^{\mathbb{N}}}$.

In the following, we will study equivalence classes for both $\leq_0$ and $\leq_2$. The class of equivalence classes of functions regarding $\leq_i$ is denoted by $\mathbb{F}_i$, the class of equivalence classes of relations by $\mathbb{R}_i$ and the class of equivalence classes for problems by $\mathbb{P}_i$ for $i \in \{0, 2\}$. Note that despite not having been defined explicitly, the reducibilities for relations are obtained as restrictions of the reducibilities for problems.

4 The induced partially ordered classes

4.1 Suprema for $\leq_2$

Since every preorder induces a partial order on its equivalence classes, in particular $(\mathbb{F}_2, \leq_2)$ is a partially ordered class. As will be proven below, it is even a complete join-semilattice. We start with recalling a definition from Subsection 2.1.

Definition 4.1 Let $(f_i : \underline{X_i}, \longrightarrow \underline{Y_i})_{i \in I}$ be a set-indexed family of functions between topological spaces. Define $\coprod_{i \in I} f_i : \coprod_{i \in I} \underline{X_i} \longrightarrow \coprod_{i \in I} \underline{Y_i}$ through $\coprod_{i \in I} f_i(i, x) = (i, f_i(x))$.

Theorem 4.2 $(\mathbb{F}_2, \leq_2)$ *is a complete join-semilattice. The suprema are given by* $\sup S = \coprod_{f \in S} f$.

Proof. We have to show the following claims:

1. For all $j \in I$, $f_j \leq_2 \coprod_{i \in I} f_i$.

Define $G(x) = (j, x)$, $F(x, i, y) = y$. Both functions are continuous with respect to the relevant topologies.

2. $f_i \leq_2 g$ for all $i \in I$ implies $\coprod_{i \in I} f_i \leq_2 g$.

$f_i \leq_2 g$ implies the existence of suitably defined continuous functions F_i, G_i with $f_i(x) = F_i(x, g(G_i(x)))$. Define F through $F(i, x, y) = (i, F_i(x, y))$ and G through $G(i, x) = G_i(x)$. The properties of the coproduct of topological spaces ensure that F and G are continuous with respect to the relevant topologies. □

Theorem 4.2 can be transferred to restrictions of $\leq_2$ to suitable classes of topological spaces, as long as these are closed under formation of coproducts. While not all natural examples are closed under arbitrary coproducts, the following theorem provides results for almost all studied restrictions.

4) The consideration of subspaces is hidden in the use of partial functions.

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Corollary 4.3 *The partial order induced by the restriction of $\leq_2$ to a class of topological spaces that is closed under formation of α-coproducts, is an α-complete join-semilattice.*

Starting from the definition of $\coprod$ for functions, a definition of $\coprod$ for problems can obtained. For that, we need to define the *coproduct of a family of partial functions*, which can be done by reading partial functions instead of functions in Definition 4.1. A separate definition for relations will not be given, but can be obtained as a special case of the following.

Definition 4.4 Let $(P_i : \underline{X_i} \longrightarrow \underline{Y_i})_{i\in I}$ be a set-indexed family of problems. Then we define $\coprod_{i\in I} P_i$ as $\{\coprod_{i\in I} f_i \mid (\forall i \in I)\ f_i \in P_i\}$.

Theorem 4.5 $(\mathbb{P}_2, \leq_2)$ *is a complete join-semilattice. The suprema are given by* $\sup S = \coprod_{P\in S} P$.

P r o o f. We have to show the following claims:

1. For all $j \in I$, $P_j \leq_2 \coprod_{i\in I} P_i$.

Define $G(x) = (j, x)$ and $F(x, i, y) = y$. Both functions are continuous with respect to the relevant topologies. For each $\coprod_{i\in I} f_i \in \coprod_{i\in I} P_i$, $F(x, \coprod_{i\in I} f_i(G(x))) = f_j(x)$ and $f_j \in P_j$ hold, proving the statement.

2. $P_i \leq_2 Q$ for all $i \in I$ implies $\coprod_{i\in I} P_i \leq_2 Q$.

If $P_i \leq_2 Q$, then there exists suitably defined continuous functions F_i, G_i with $x \longmapsto F_i(x, g(G_i(x))) \in P_i$ for all $g \in Q$. Define F through $F(i, x, y) = (i, F_i(x, y))$ and G through $G(i, x) = G_i(x)$. The properties of the coproduct ensure that F and G are continuous with respect to the relevant topologies. $x \longmapsto F(i, x, g(G(i, x)))$ for any $g \in Q$ and fixed $i \in I$ is in P_i, so $(i, x) \longmapsto F(i, x, g(G(i, x)))$ is in $\coprod_{i\in I} P_i$. □

To cover relations, a new definition of suprema is not needed. Straight-forward observation of the relevant definitions is sufficient to obtain the next proposition and the following corollary:

Proposition 4.6 *If the problem P_i is a relation for all $i \in I$, then $\coprod_{i\in I} P_i$ is also a relation.*

Corollary 4.7 $(\mathbb{R}_2, \leq_2)$ *is a sub-join-semilattice of* $(\mathbb{P}_2, \leq_2)$*. In particular,* $(\mathbb{R}_2, \leq_2)$ *is a complete join-semilattice with suprema given by* $\sup S = \coprod_{R\in S} R$.

Through identifying a function f with the problem $\{f\}$, the partially ordered class $(\mathbb{F}_2, \leq_2)$ is a substructure of the partially ordered class $(\mathbb{R}_2, \leq_2)$. As suprema are formed in a compatible fashion, the complete join-semilattice $(\mathbb{F}_2, \leq_2)$ is even a sub-join-semilattice of $(\mathbb{R}_2, \leq_2)$, and hence of $(\mathbb{P}_2, \leq_2)$.

Similar statements to Corollary 4.3 can be phrased and proved for $(\mathbb{R}_2, \leq_2)$ and $(\mathbb{P}_2, \leq_2)$, which is not exercised here.

As the coproduct of an empty family of topological spaces is the space $(\emptyset, \{\emptyset\})$, the minimal element in $(\mathbb{F}_2, \leq_2)$ is the equivalence class containing exactly the functions with domain $\emptyset$. The minimal element in $(\mathbb{P}_2, \leq_2)$ is the equivalence class containing all problems that contain a function with domain $\emptyset$. The continuous functions with non-empty domain form the second-least element of $(\mathbb{F}_2, \leq_2)$, the problems containing a continuous function with non-empty domain are the second-least element of $(\mathbb{P}_2, \leq_2)$.

Definition 4.8 A function $f : \underline{X} \longrightarrow \underline{Y}$ is *sequentially continuous*, if $\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n)$ holds for each sequence $(x_n)_{n\in\mathbb{N}}$ in X. A topological space $\underline{X}$ is called *sequential*, if every sequentially continuous function on $\underline{X}$ is continuous.

The restriction to sequential topological spaces even yields a third-least equivalence class of functions containing the function $\mathrm{cf} : \underline{\mathbb{N}^\mathbb{N}} \longrightarrow \{0, 1\}$ with $\mathrm{cf}^{-1}(\{1\}) = \{0^\mathbb{N}\}$. This can be rephrased to yield a characterization of sequential topological spaces:

Theorem 4.9 *A topological space $\underline{X}$ is sequential, if and only if* $\mathrm{cf} \leq_2 f$ *holds for all discontinuous functions $f : \underline{X} \longrightarrow \underline{Y}$ with some topological space $\underline{Y}$.*

P r o o f. As all continuous functions are sequentially continuous, our claim is equivalent to: $\mathrm{cf} \leq_2 f$ holds, if and only if f is not sequentially continuous.

As sequential continuity is preserved under composition and products, sequential continuity of g and $f \leq_2 g$ implies sequential continuity of f. It is trivial to see that cf is not sequentially continuous, hence, one direction of our equivalence.

For the other direction, assume that f is not sequentially continuous. Then there is a converging sequence $(x_n)_{n\in\mathbb{N}}$ in $\underline{X}$ with

$$f(\lim_{n\to\infty} x_n) \neq \lim_{n\to\infty} f(x_n).$$

The limit on the right side might not exist, and both limits are not necessarily unique. In any case, this means that there is an open neighbourhood U of $f(\lim_{n\to\infty} x_n)$, and a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ with $f(x_{n_k}) \notin U$ for all $k \in \mathbb{N}$.

Define a continuous function $G : \underline{\mathbb{N}^\mathbb{N}} \longrightarrow \underline{X}$ by $G(x) = x_{n_k}$, if $d(0^\mathbb{N}, x) = 2^{-k}$, and $G(0^\mathbb{N}) = \lim_{n\to\infty} x_n$. Due to the properties of a metric, G is continuous. A partial function $F : \underline{X} \times \underline{Y} \longrightarrow \{0, 1\}$ is defined via $F(0^\mathbb{N}, y) = 1$ and $F(x, y) = 0$ for $x \neq 0^\mathbb{N}$. The domain of F is

$$(\{0^\mathbb{N}\} \times \{\lim_{n\to\infty} x_n\}) \cup ((\mathbb{N}^\mathbb{N} \setminus \{0^\mathbb{N}\}) \times (X \setminus U)).$$

To show that F is continuous on its domain, we have to show that both $(\{0^\mathbb{N}\} \times \{\lim_{n\to\infty} x_n\})$ and $((\mathbb{N}^\mathbb{N}\setminus\{0^\mathbb{N}\})\times(X\setminus U))$ are closed subsets of their union. By $(\{0^\mathbb{N}\} \times \{\lim_{n\to\infty} x_n\}) = (\{0^\mathbb{N}\} \times X) \cap \mathrm{dom}(F)$ and $((\mathbb{N}^\mathbb{N} \setminus \{0^\mathbb{N}\}) \times (X \setminus U)) = (\mathbb{N}^\mathbb{N} \times (X \setminus U)) \cap \mathrm{dom}(F)$ this is indeed the case. Now F and G witness $\mathrm{cf} \leq_2 f$. □

However, even if one regards only problems with domain $\underline{\mathbb{N}^\mathbb{N}}$, there exists a decreasing chain between the continuous problems and $\{\mathrm{cf}\}$, as shown in [21, Section 4].

For $(\mathbb{P}_2, \leq_2)$, there exists a maximal element, this contains all empty problems. For functions, however, no maximal element exists, proving that $(\mathbb{F}_2, \leq_2)$ is not an α-complete meet-semilattice and therefore not an α-complete lattice for any $\alpha > 0$. This claim follows from the examples given at the end of Subsection 5.1 utilizing the concept of Basesize. Note that all specific problems mentioned here are relations, so the statements hold for $(\mathbb{R}_2, \leq_2)$, as well.

4.2 Infima for $\leq_2$

Recently, Brattka and Gherardi presented a construction of binary infima of relations for computable Weihrauch reducibility in [3], which can be transferred to the topological setting in a straight-forward manner.

Definition 4.10 Define $\bigwedge_{i\in I} R_i : \prod_{i\in I} \underline{X_i} \longrightarrow \coprod_{i\in I} \underline{Y_i}$, where $(R_i : \underline{X_i} \longrightarrow \underline{Y_i})_{i\in I}$ is a set-indexed family of relations, by $f \in \bigwedge_{i\in I} R_i$, if for all $\overline{x} \in \prod_{i\in I} \underline{X_i}$ we have $f(\overline{x}) = (i, y)$ and there is a $g \in R_i$ with $g(\pi_i(\overline{x})) = y$.

Theorem 4.11 $(\mathbb{R}_2, \leq_2)$ *is a complete meet-semilattice. The infima are given by* $\inf A = \bigwedge_{R\in A} R$.

Proof. We have to show the following claims:

1. For all $j \in I$, $\bigwedge_{i\in I} R_i \leq_2 R_j$ holds.

For $g \in R_j$, we have $\overline{x} \longmapsto \iota_j(g(\pi_j(\overline{x}))) \in \bigwedge_{i\in I} R_i$. As both the projection π_j and the injection ι_j are continuous, this concludes the proof.

2. If $S \leq_2 R_i$ holds for all $i \in I$, then $S \leq_2 \bigwedge_{i\in I} R_i$ is implied.

Suppose that the witnesses for $S \leq_2 R_i$ are G_i and F_i. Define G by $G(x) = \prod_{i\in I} G_i(x)$, and define F by $F(x, (j, y)) = F_j(x, y)$. Then F and G witness $S \leq_2 \bigwedge_{i\in I} R_i$. □

An extension of the concept to problems is possible, also, by utilizing injections and projections even more:

Definition 4.12 Define $\sqcap_{i\in I} P_i : \prod_{i\in I} \underline{X_i} \longrightarrow \coprod_{i\in I} \underline{Y_i}$, where $(P_i : \underline{X_i} \longrightarrow \underline{Y_i})_{i\in I}$ is a set-indexed family of problems, by $f \in \sqcap_{i\in I} P_i$, if there is an $j \in I$ and a $g \in P_i$ with $f = \iota_j \circ g \circ \pi_j$.

Theorem 4.13 $(\mathbb{P}_2, \leq_2)$ *is a complete meet-semilattice. The infima are given by* $\inf A = \sqcap_{P\in A} P$.

Proof. We have to show the following claims:

1. For all $j \in I$, $\sqcap_{i\in I} P_i \leq_2 P_j$ holds.

Both injections and projections are continuous.

2. If $Q \leq_2 P_i$ holds for all $i \in I$, then $Q \leq_2 \sqcap_{i\in I} P_i$ is implied.

Suppose that the witnesses for $Q \leq_2 P_i$ are G_i and F_i. Define G by $G(x) = \prod_{i\in I} G_i(x)$, and define F by $F(x, (j, y)) = F_j(x, y)$. Then F and G witness $Q \leq_2 \sqcap_{i\in I} P_i$. □

Note that despite the similarity of Definition 4.10 and Definition 4.12 the former is not a special case of the latter: Even if all problems P_i are relations, for $|I| > 1$ the problem $\sqcap_{i\in I} P_i$ is not a relation. In particular, the conjecture that $(\mathbb{R}_2, \leq_2)$ is not a sub-lattice of $(\mathbb{P}_2, \leq_2)$, although both share suprema, seems reasonable to us.

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4.3 Suprema and infima for $\leq_0$

Using a very similar construction to Definition 4.1, suprema can be introduced for all variations of $\leq_0$ studied here. Again, we will start with considering functions only.

Definition 4.14 Let $(f_i : \underline{X_i} \longrightarrow \underline{Z})_{i \in I}$ be a set-indexed family. Define $\uparrow f_i \uparrow_{i \in I} : \coprod_{i \in I} \underline{X_i} \longrightarrow \underline{Z}$ through $\uparrow f_i \uparrow_{i \in I}(i, x) = f_i(x)$.

Theorem 4.15 $(\mathbb{F}_0, \leq_0)$ *is a complete join-semilattice. The suprema are given by* $\sup A = \uparrow f \uparrow_{f \in A}$.

Proof. We have to show the following claims:

1. For all $j \in I$, $f_j \leq_0 \uparrow f_i \uparrow_{i \in I}$.
Choose $G : \underline{X_j} \longrightarrow \coprod_{i \in I} \underline{X_i}$ defined through $G(x) = (j, x)$.

2. If $f_j \leq_0 g$ holds for all $j \in I$, then $\uparrow f_i \uparrow_{i \in I} \leq_0 g$ holds.
There are continuous functions G_j, so that $f_j = g \circ G_j$ holds for each $j \in J$. Define G by $G(i, x) = G_i(x)$. G is continuous, and satisfies $\uparrow f_i \uparrow_{i \in I} = g \circ G$. □

For representations, binary suprema[5)] for $\leq_0$ have already been introduced in [22]. Taking into consideration that $\mathbb{N}^\omega$ and $\mathbb{N}^\omega \coprod \mathbb{N}^\omega$ are homeomorphic, Definition 4.14 extends [22, Definition 3.3.11], while the Theorem 4.15 extends [22, Theorem 3.3.12 1.]. As the restriction of $\leq_0$ to functions with domain in a class of topological spaces closed under formation of α-coproducts yields an α-complete join-semilattice, also countable suprema exist for representations.

By extending [22, Definition 3.3.7], a definition of binary infima for representations, $(\mathbb{F}_0, \leq_0)$ is shown to lead to a complete lattice.

Definition 4.16 Let $(f_i : \underline{X} \longrightarrow \underline{Z})_{i \in I}$ be a set-indexed family of functions. Define $\downarrow f_i \downarrow_{i \in I} : \mathfrak{P} \longrightarrow \underline{Z}$, where $\mathfrak{P} = \{\prod_{i \in I} x_i \in \prod_{i \in I} \underline{X_i} \mid (\forall i \in I)(\forall j \in I)\ f_i(x_i) = f_j(x_j)\}$ is equipped with the restriction of the usual product topology, through $\downarrow f_i \downarrow_{i \in I}(\prod_{i \in I} x_i) = f_{i_0}(x_{i_0})$ for an arbitrary fixed $i_0 \in I$.

Theorem 4.17 $(\mathbb{F}_0, \leq_0)$ *is a complete meet-semilattice. The infima are given by* $\inf A = \downarrow f \downarrow_{f \in A}$.

Proof. We have to show the following claims:

1. For all $j \in I$, $\downarrow f_i \downarrow_{i \in I} \leq_0 f_j$.
Choose $G : \mathfrak{P} \longrightarrow \underline{X_j}$ as the restriction of the projection π_j to the jth entry.

2. Let $g : \underline{Y} \longrightarrow \underline{Z}$ be a function. If $g \leq_0 f_i$ holds for all $i \in I$, then $g \leq_0 \downarrow f_i \downarrow_{i \in I}$ follows.
Assume the existence of continuous functions G_i, so that $g = f_i \circ G_i$ holds. Then $f_i(G_i(y)) = f_j(G_j(y))$ for all $i, j \in I$, $y \in Y$. Thus a continuous function $G : \underline{Y} \longrightarrow \mathfrak{P}$ can be defined via $G(y) = \prod_{i \in I} G_i(y)$. G satisfies $g = \downarrow f_i \downarrow_{i \in I} \circ G$. □

The definition of suprema can be extended to relations and problems in the usual manner, as exercised below.

Definition 4.18 Define $\uparrow P_i \uparrow_{i \in I} = \{\uparrow f_i \uparrow_{i \in I} \mid (\forall i \in I)\ f_i \in P_i\}$, where $(P_i : \underline{X_i} \longrightarrow \underline{Z})_{i \in I}$ is a set-indexed family of problems.

Theorem 4.19 $(\mathbb{P}_0, \leq_0)$ *is a complete join-semilattice. The suprema are given by* $\sup A = \uparrow P \uparrow_{P \in A}$.

Proof. We have to show the following claims:

1. For all $j \in I$, $P_j \leq_0 \uparrow P_i \uparrow_{i \in I}$.
Choose $G : \underline{X_j} \longrightarrow \coprod_{i \in I} \underline{X_i}$ defined through $G(x) = (j, x)$. Then $\uparrow f_i \uparrow_{i \in I} \circ G = f_j$ holds, so from $\uparrow f_i \uparrow_{i \in I} \in \uparrow P_i \uparrow_{i \in I}$ follows $\uparrow f_i \uparrow_{i \in I} \circ G \in P_j$.

2. $P_i \leq_0 Q$ for all $i \in I$ implies $\uparrow P_i \uparrow_{i \in I} \leq_0 Q$.
There are continuous functions G_j, so that $g \circ G_j \in P_j$ holds for each $j \in J$ and each $g \in Q$. Define G through $G(i, x) = G_i(x)$. G is continuous, and satisfies $\uparrow g \circ G_j \uparrow_{i \in I} = g \circ G$, and thus $g \circ G \in \uparrow P_i \uparrow_{i \in I}$ for each $g \in Q$. □

Theorem 4.20 *If the problem P_i is a relation for all $i \in I$, then $\uparrow P_i \uparrow_{i \in I}$ is also a relation.*

Corollary 4.21 $(\mathbb{R}_0, \leq_0)$ *is a sub-join-semilattice of* $(\mathbb{P}_0, \leq_0)$. *In particular,* $(\mathbb{R}_0, \leq_0)$ *is a complete join-semilattice with suprema given by* $\sup A = \uparrow R \uparrow_{R \in A}$.

5) In addition, a definition of countable suprema and infima is given by Weihrauch, but, as the focus in [22] is on computable reducibilities, the fact that the constructions given are (co)limits for the continuous reducibilities was not pointed out explicitly.

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As $(\mathbb{F}_0, \leq_0)$ is a complete lattice, there is a smallest and greatest element. The smallest element is the inclusion of the empty set in $\underline{Z}$, the greatest element is the identity $\mathrm{id} : (Z, \{\emptyset, Z\}) \longrightarrow \underline{Z}$. Constant functions are equivalent, if and only if they have the same image, and incomparable otherwise. Each constant function is a second-smallest element.

Considering problems does not change the results from the last paragraph much, the empty problem is even greater than $\{\mathrm{id}\}$, but the equivalence class including $\{\mathrm{id}\}$ is the unique second-greatest element.

4.4 Distributivity

Theorem 4.22 $(\mathbb{F}_2, \leq_2)$ *is distributive.*

Proof. We assume functions $f : \underline{X} \longrightarrow \underline{Y}$ and $g_i : \underline{X_i} \longrightarrow \underline{Y_i}$ for $i \in I$ satisfying $f \leq_2 \coprod_{i\in I} g_i$. There are continuous partial functions $F :\subseteq \underline{X} \times (\coprod_{i\in I} \underline{Y_i}) \longrightarrow \underline{Y}$ and $G :\subseteq \underline{X} \longrightarrow \coprod_{i\in I} \underline{X_i}$ with $f(x) = F(x, [\coprod_{i\in I} g_i](G(x)))$ for all $x \in \underline{X}$. We can assume that G is a continuous function. If $\underline{I}$ is the set I with the discrete topology, then the function $\rho : \coprod_{i\in I} \underline{X_i} \longrightarrow \underline{I}$ defined via $\rho(i, x) = i$ is continuous, and so is $\rho \circ G$. The set $O_i = (\rho \circ G)^{-1}(\{i\})$ for $i \in I$ thus is a open and closed subset of $\underline{X}$.

We use f_i to denote the restriction of f to the set O_i. As set-inclusions are continuous, each f_i fulfills $f_i \leq_2 f$, implying $\coprod_{i\in I} f_i \leq_2 f$. Suitable restrictions of F and G also yield $f_i \leq_2 g_i$ for all $i \in I$. It remains to prove $f \leq_2 \coprod_{i\in I} f_i$. If we use the continuous function $\varrho : \underline{X} \times (\coprod_{i\in I} \underline{O_i}) \longrightarrow \underline{X}$ defined by $\varrho(x, i, y) = y$, the identity

$$f(x) = \varrho(x, [\textstyle\coprod_{i\in I} f_i]((\rho \circ G)(x), x))$$

shows the remaining claim. □

Theorem 4.23 $(\mathbb{R}_2, \leq_2)$ *is distributive.*

Proof. To extend the proof of Theorem 4.22 to relations, we need exactly the locality condition specified in Definition 2.1 to show that the constructed function is a member of the relevant relation. □

For exactly this reason, the theorem is not extended to general problems.

Theorem 4.24 $(\mathbb{F}_0, \leq_0)$ *is distributive.*

Proof. The proof is exactly parallel to the proof of Theorem 4.22. □

Theorem 4.25 $(\mathbb{R}_0, \leq_0)$ *is distributive.*

Proof. The proof is exactly parallel to the proof of Theorem 4.23. □

5 Suprema and characteristic numbers

5.1 Level and Basesize

An important tool in the study of the discontinuity of functions are certain characteristic numbers that are compatible with $\leq_2$ (and hence with $\leq_0$). Here, two variants of the Level as introduced in [10], as well as Basesize introduced in [14] will be considered. Called *cardinality of discontinuity*, Basesize was studied extensively in [24].

Definition 5.1 Let $f : \underline{X} \longrightarrow \underline{Y}$ be a function. For an ordinal number α, inductively define the sets $\mathcal{L}^1_\alpha(f) \subseteq X$ via

$$\mathcal{L}^1_0(f) = X, \qquad \mathcal{L}^1_{\alpha+1}(f) = \{x \in \mathcal{L}^1_\alpha(f) \mid f_{|\mathcal{L}^1_\alpha(f)} \text{ is discontinuous at } x\},$$
$$\mathcal{L}^1_\gamma(f) = \textstyle\bigcap_{\alpha<\gamma} \mathcal{L}^1_\alpha(f) \quad \text{for a limit ordinal } \gamma.$$

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Definition 5.2 Let $f : \underline{X} \longrightarrow \underline{Y}$ be a function. For an ordinal number α, define inductively the sets $\mathcal{L}^2_\alpha(f) \subseteq X$ via

$$\mathcal{L}^2_0(f) = X, \qquad \mathcal{L}^2_{\alpha+1}(f) = \text{cl}\left(\{x \in \mathcal{L}^2_\alpha(f) \mid f_{|\mathcal{L}^2_\alpha(f)} \text{ is discontinuous at } x\}\right),$$
$$\mathcal{L}^2_\gamma(f) = \text{cl} \bigcap_{\alpha<\gamma} \mathcal{L}^2_\alpha(f) \quad \text{for a limit ordinal } \gamma.$$

Definition 5.3 Let $f : \underline{X} \longrightarrow \underline{Y}$ be a function and $x \in X$. Define $\text{lev}^i(f,x) = \min\{\alpha \mid x \notin \mathcal{L}^i_\alpha(f)\}$ and $\text{Lev}^i(f) = \min\{\alpha \mid \mathcal{L}^i_\alpha = \emptyset\}$ for $i \in \{1,2\}$.

The formulation of statements involving the Level of a function usually is simplified by assuming that a non-existing Level is comparable with the normal $\leq$ relation for ordinal numbers, and is greater than all ordinal numbers. This agreement extends to suprema and minima of suitable classes of ordinal numbers.

Proposition 5.4 $\text{Lev}^i(f) = \sup\{\text{lev}^i(f,x) \mid x \in X\}$.

Theorem 5.5 *If $f \leq_2 g$ holds, then* $\text{Lev}^i(f) \leq \text{Lev}^i(g)$.

P r o o f. This is the statement of [10, Korollar 2.4.3]. □

When trying to define the Level of a problem, two main criteria should be employed. First, the Level of a singleton problem should be identical to the Level of the function it contains. Second, the result of Theorem 5.5 should remain valid when functions are replaced by problems. An elegant way[6)] of reaching both criteria is presented in the following definition, which already was given by Hertling in [10, Section 1.3].

Definition 5.6 Let P be a problem. Define $\text{Lev}^i(P) = \min\{\text{Lev}^i(f) \mid f \in P\}$ for $i \in \{1,2\}$.

Theorem 5.7 *If $P \leq_2 Q$ holds, then* $\text{Lev}^i(P) \leq \text{Lev}^i(Q)$ *follows for* $i \in \{1,2\}$.

P r o o f. If $P \leq_2 Q$ holds, there are continuous functions F, G with $x \longmapsto F(x, g(G(x))) \in P$ for all $g \in Q$. Choose a special $g \in Q$, so that $\text{Lev}^i(g) = \text{Lev}^i(Q)$ is fulfilled. Clearly, $x \longmapsto F(x, g(G(x))) \leq_2 g$ is true, so from Theorem 5.5 results $\text{Lev}^i(x \longmapsto F(x, g(G(x)))) \leq \text{Lev}^i(g) = \text{Lev}^i(Q)$. The claim now follows from Definition 5.6. □

The third characteristic number to be considered is Basesize. Basesize extends the notion of k-continuity explored in [21]. Its definition for functions was first presented in [14]. In contrast to the Level, the Basesize of a function is a cardinal number.

Definition 5.8 Let $f : \underline{X} \longrightarrow \underline{Y}$ be a function. A partition for f is a partition p of X, so that $f_{|U}$ is continuous for all $U \in p$. The Basesize of f is defined as the least cardinality of a partition for f and denoted by $\text{bas}(f)$.

Theorem 5.9 *For functions $f : \underline{X} \longrightarrow \underline{Y}$, $g : \underline{U} \longrightarrow \underline{V}$, $f \leq_2 g$ implies* $\text{bas}(f) \leq \text{bas}(g)$.

P r o o f. Let $\{A_i \mid i \in I\}$ be a partition for g with minimal cardinality. Let F, G be continuous partial functions with $f(x) = F(x, g(G(x)))$ for all $x \in X$. Then $\{G^{-1}(A_i) \mid i \in I\}$ is a partition of X, and as $g \circ G$ is continuous when restricted to $G^{-1}(A_i)$, so is f. So $\{G^{-1}(A_i) \mid i \in I\}$ is a partition for f. □

The two variants of the Level and Basesize are linked with an inequality. All combinations of equality and strict inequality are possible.

Theorem 5.10 $\text{bas}(f) \leq \text{Lev}^1(f) \leq \text{Lev}^2(f)$.

P r o o f. By Definition 5.1, for $D_\alpha = \mathcal{L}^1_\alpha(f) \setminus \mathcal{L}^1_{\alpha+1}(f)$, the restriction $f_{|D_\alpha}$ is continuous. As we have $\text{dom}(f) = \bigcup_{\alpha=0}^{\text{Lev}^1(f)} D_\alpha$, the sets D_α form a partition for f. Thus, the first inequality is proven. The second inequality is the statement of [10, Satz 1.1.7 (1)]. □

When trying to define the Basesize of a problem, both the goals and the method to achieve them are completely analogous to the same task for the Level.

Definition 5.11 For a problem P, define $\text{bas}(P) = \min\{\text{bas}(f) \mid f \in P\}$.

---

6) The validity of Theorem 5.5 gives $\min\{\text{Lev}^i(f) \mid f \in P\}$ as an upper bound for $\text{Lev}^i(P)$, but the two criteria are not sufficient to uniquely determine Definition 5.6.

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Theorem 5.12 *For problems P, Q, if $P \leq_2 Q$, then* $\mathrm{bas}(P) \leq \mathrm{bas}(Q)$.

P r o o f. Choose $g \in Q$ with $\mathrm{bas}(g) = \mathrm{bas}(Q)$. There exists an $f \in P$ with $f \leq_2 g$. This implies that $\mathrm{bas}(P) \leq \mathrm{bas}(f) \leq \mathrm{bas}(g) = \mathrm{bas}(Q)$ holds. □

Clearly, the inequalities in Theorem 5.10 hold for problems, too.

In the following, examples will be constructed showing that all combinations of Basesize and Level not ruled out by Theorem 5.10 can occur.

Let $\mathcal{N} = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Given an ordinal number λ, let $\mathcal{M}_\lambda$ be the set of order-preserving functions from $(\lambda, \leq)$ to $(\mathcal{N}, \geq)$. For each $x \in \mathcal{M}_\lambda$, let $U_x \subseteq \mathcal{M}_\lambda$ be defined via $U_x = \{y \in \mathcal{M}_\lambda \mid (\forall \nu \in \lambda)\, y(\nu) \geq x(\nu)\}$, and equip $\mathcal{M}_\lambda$ with the topology induced by the base[7)] $\{U_x \mid x \in \mathcal{M}_\lambda\}$.

For $c \in \mathcal{M}$, let $F(c) \in \lambda + 1$ denote the least element with $c(F(c)) = 0$ or λ if no such element exists. Given further a cardinal number β with $\beta \leq \lambda$, we define a function $R_{\lambda\beta} : (\lambda + 1) \longrightarrow \beta$ using ordinal left division with remainder. $R_{\lambda\beta}(\alpha)$ shall be the uniquely determined ordinal number less than β, so that there is an ordinal ζ with $\alpha = \beta\zeta + R_{\lambda\beta}(\alpha)$. The restriction $\beta \leq \lambda$ ensures the surjectivity of $R_{\lambda\beta}$.

Now a function $f_{\lambda\beta} : \underline{\mathcal{M}}_\lambda \longrightarrow \underline{\beta}$ is defined as $f_{\lambda\beta} = R_{\lambda\beta} \circ F$, where $\underline{\beta}$ is the set β equipped with the discrete topology.

Theorem 5.13 $\mathrm{Lev}^1(f_{\lambda\beta}) = \lambda$ *and* $\mathrm{bas}(f_{\lambda\beta}) = \beta$.

P r o o f. The first statement follows from the observation that $\mathcal{L}^1_\alpha(f_{\lambda\beta}) = \{c \in \mathcal{M}_\lambda \mid F(c) \geq \alpha\}$, which we will continue to prove. For $\alpha = 0$, the claim is trivially true.

If $F(c) = \nu$ holds for some $c \in \mathcal{M}_\lambda$, then we have $F(x) \leq \nu$ for all $x \in U_c$. In particular, the restriction $(f_{\lambda\beta})_{|A}$ to some set A is always continuous in the points $x \in A$ with $(\forall y \in A)\, F(x) \leq F(y)$. On the other hand, define x^+ for some x with $F(x) = \alpha + 1$ by $x^+(\nu) = x(\nu)$ for $\nu \neq \alpha$, and $x^+(\alpha) = 0$. Then $x^+ \in \mathcal{M}_\lambda$ holds, additionally x^+ is in every neighborhood of x in some subset of $\mathcal{M}_\lambda$ that contains both x and x^+. Thus, on such a subset, $f_{\lambda\beta}$ is discontinuous in x. This shows the claim for all successor ordinals $\alpha + 1$.

For limit ordinals δ, the claim follows from the definition of $\mathcal{L}^1_\delta(f_{\lambda\beta}) = \bigcap_{\alpha<\delta} \mathcal{L}^1_\alpha(f_{\lambda\beta})$. This concludes the proof of $\mathrm{Lev}^1(f_{\lambda\beta}) = \lambda$.

$\mathrm{bas}(f_{\lambda\beta}) \leq \beta$ is clear. It remains to show $\beta \leq \mathrm{bas}(f_{\lambda\beta})$. For that, note that we can consider $\mathcal{M}_{\lambda_1}$ as a subset of $\mathcal{M}_{\lambda_2}$ for $\lambda_1 \leq \lambda_2$ by extending $g \in \mathcal{M}_{\lambda_1}$ to g' with $g'(x) = g(x)$ for $x < \lambda_1$ and $g'(x) = 0$ otherwise. Especially, this makes $f_{\beta\beta}$ a restriction of $f_{\lambda\beta}$, so proving $\beta \leq \mathrm{bas}(f_{\beta\beta})$ is sufficient.

For that, consider the elements $x_\nu \in \mathcal{M}_\beta$ with $x_\nu(\alpha) = 1$ for $\alpha < \nu$ and $x_\nu(\alpha) = 0$ for $\nu \leq \alpha \leq \beta$. Then x_ν is in every neighborhood of x_μ for $\nu < \mu$, and the presence of x_ν makes $f_{\beta\beta}$ discontinuous in x_μ. Thus, if $\mathcal{P}$ is a partition of $f_{\beta\beta}$, no $P \in \mathcal{P}$ may contain x_ν, x_μ with $\nu < \mu$. This contradicts the assumption $|\mathcal{P}| < \beta$ and concludes the proof. □

Corollary 5.14 $(\mathbb{F}_2, \leq_2)$ *has no greatest element.*

P r o o f. Let $g : \underline{X} \longrightarrow \underline{Y}$ be a representative of the greatest element of $(\mathbb{F}_2, \leq_2)$. Obviously, we have $\mathrm{bas}(g) \leq |X|$. Abbreviate $\beta = |X|^+$. Consider the function $f_{\beta\beta}$. Due to assumption, we have $f_{\beta\beta} \leq_2 g$, so Theorem 5.9 yields $\mathrm{bas}\, f_{\beta\beta} \leq_2 |X|$. Together with Theorem 5.13, we have $|X|^+ \leq |X|$, an obvious contradiction. Thus, there is no greatest element in $(\mathbb{F}_2, \leq_2)$. □

5.2 Permutability of characteristic numbers and suprema

In this subsection we show that the characteristic numbers defined above commute with suprema, that is, the supremum of the characteristic numbers of some family of functions, relations or problems is the characteristic number of the supremum of these. While the proofs are done only for $\leq_2$, the corresponding results for $\leq_0$ are direct consequences (as it is true for the previous subsection).

---

7) The intersection $\bigcap_{x\in A} U_x$ for some $A \subseteq \mathcal{M}_\lambda$ is identical to U_y, where $y \in \mathcal{M}_\lambda$ is defined by requiring $y(\nu) = \max_{x\in A} x(\nu)$ for all $\nu \in \lambda$.

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Theorem 5.15 $\mathrm{Lev}^1(\coprod_{i\in I} f_i) = \sup\{\mathrm{Lev}^1(f_i) \mid i \in I\}$.

P r o o f. Assume that for each $i \in I$ the domain of f_i is $\underline{X_i}$, so the domain of $\coprod_{i\in I} f_i$ is $\coprod_{i\in I} \underline{X_i}$. As for each $j \in I$, the set X_j is open and closed in $\coprod_{i\in I} \underline{X_i}$, $\coprod_{i\in I} f_i$ is continuous in (j, x) if and only if f_j is continuous in x. The same is true for all restrictions. Thus, $\mathcal{L}^1_\alpha(\coprod_{i\in I} f_i) = \bigcup_{i\in I}\{i\} \times \mathcal{L}^1_\alpha(f_i)$ follows. So $\mathcal{L}^1_\alpha(\coprod_{i\in I} f_i) = \emptyset$ is true if and only if $\mathcal{L}^1_\alpha(f_i) = \emptyset$ holds for all $i \in I$. □

Theorem 5.16 $\mathrm{Lev}^2(\coprod_{i\in I} f_i) = \sup\{\mathrm{Lev}^2(f_i) \mid i \in I\}$.

P r o o f. To prove the claim, the proof of Theorem 5.15 needs to be slightly modified. To this end, note $\mathrm{cl} \coprod_{i\in I} U_i = \coprod_{i\in I} \mathrm{cl}\, U_i$. □

Theorem 5.17 $\mathrm{bas}(\coprod_{i\in I} f_i) = \sup\{\mathrm{bas}(f_i) \mid i \in I\}$.

P r o o f. The first fact in the proof of Theorem 4.2 together with Theorem 5.9 yields

$$\mathrm{bas}(\coprod_{i\in I} f_i) \geq \sup\{\mathrm{bas}(f_i) \mid i \in I\}.$$

Now assume an index set J with $|J| = \sup\{\mathrm{bas}(f_i) \mid i \in I\}$. For each $i \in I$, there is a subset J_i of J, so that there is a partition $\{U_{ij} \mid j \in J_i\}$ for f_i. Define $U_{ij} = \emptyset$ for $j \in J \setminus J_i$. A partition for $\coprod_{i\in I} f_i$ can be obtained as $\{\bigcup_{i\in I}\{i\} \times U_{ij} \mid j \in J\}$, proving the other direction of the equality. □

Again, by building on the result for functions presented in the theorems above, the results can also be obtained for problems. Interestingly, the proof is uniform and not dependent on the specific characteristic number used. This can be regarded as further strengthening the definition of Level and Basesize for problems.

Theorem 5.18 *Let* $\mathrm{num} \in \{\mathrm{Lev}^1, \mathrm{Lev}^2, \mathrm{bas}\}$. *Then it follows that*

$$\mathrm{num}(\coprod_{i\in I} P_i) = \sup\{\mathrm{num}(P_i) \mid i \in I\}.$$

P r o o f. According to Definition 4.4, $\mathrm{num}(\coprod_{i\in I} P_i) = \mathrm{num}(\{\coprod_{i\in I} P_i \mid (\forall i \in I)\ f_i \in P_i\})$. By Definition 5.6 or 5.11 it follows that

$$\mathrm{num}(\{\coprod_{i\in I} f_i \mid (\forall i \in I)\ f_i \in P_i\}) = \min\{\mathrm{num}(\coprod_{i\in I} f_i) \mid (\forall i \in I)\ f_i \in P_i\}.$$

Applying Theorem 5.15, 5.16 or 5.17, we obtain

$$\min\{\mathrm{num}(\coprod_{i\in I} f_i) \mid (\forall i \in I)\ f_i \in P_i\} = \min\{\sup\{\mathrm{num}(f_i) \mid i \in I\} \mid f_i \in P_i\}.$$

$\min$ and $\sup$ commute, so in the next step we have

$$\min\{\sup\{\mathrm{num}(f_i) \mid i \in I\} \mid f_i \in P_i\} = \sup\{\min\{\mathrm{num}(f_i) \mid f_i \in P_i\} \mid i \in I\}.$$

Another application of Definition 5.6 or 5.11 results in

$$\sup\{\min\{\mathrm{num}(f_i) \mid f_i \in P_i\} \mid i \in I\} = \sup\{\mathrm{num}(P_i) \mid i \in I\}.$$ □

6 Additional observations

6.1 A continuous version of truth-table reducibility

For some applications the limitation of having only one call to the oracle will be too strict, so a continuous version of truth-table reducibility, meaning the possibility of making any finite number of parallel oracle calls, is desirable. The notion of n parallel calls to an oracle f can be replaced by the notion of one call to the oracle $f^n := \prod_{i=1}^n f$. The extension to any finite number of calls is accomplished by taking the supremum over all n. As abbreviation, we define $\overline{f} := \coprod_{n\in\mathbb{N}} f^n$. Then the *corresponding reducibility* can be defined:

Definition 6.1 For two functions f, g, let $f \leq_{\mathrm{ct}} g$ hold, if $f \leq_2 \overline{g}$ holds.

The properties of $\leq_{\mathrm{ct}}$ are derived from the properties of $\leq_2$, as $\overline{\ }$ is a closure operator for $\leq_2$, as the following theorem shows.

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Theorem 6.2 *The operator* $\overline{}$ *satisfies the following properties:*

1. $f \leq_2 \overline{f}$*;*
2. $f \leq_2 g$ *implies* $\overline{f} \leq_2 \overline{g}$*;*
3. $\overline{f} \equiv_2 \overline{\overline{f}}$.

Proof.

1. Trivial.

2. By taking the n-fold products of the continuous partial functions witnessing $f \leq_2 g$, we have $f^n \leq_2 g^n$. The properties of suprema then yield the claim.

3. We only have to show $\overline{\overline{f}} \leq_2 \overline{f}$. For the proof we utilize the following distributivity law, which generalises [14, Theorem 2.2.5.5]:

$$f \times \coprod_{i\in I} g_i \equiv_2 \coprod_{i\in I}(f \times g_i).$$

Iterated application yields $\coprod_{n\in\mathbb{N}} f^n \equiv_2 (\coprod_{n\in\mathbb{N}} f^n)^m$ for all $m \in \mathbb{N}$. The claim now follows again from the general properties of suprema. □

Obviously, $\coprod$ can also be considered as supremum in the partially ordered set induced by $\leq_{\mathrm{ct}}$, yielding yet another complete join-semilattice. Binary suprema, however, are also given by the product in this case, as they are equivalent regarding $\leq_{\mathrm{ct}}$. Again it is possible to define $\leq_{\mathrm{ct}}$ for relations and problems as well. The join-semilattices corresponding to $\leq_{\mathrm{ct}}$ are quotients of the respective join-semilattices for $\leq_2$.

Other examples of closure operators for $\leq_2$ implying other reducibilities have been studied in [3] and [16]. Some applications for the $\overline{}$-operator can be found in [15].

6.2 From problems to relations

The relationship between the different infima operators $\sqcap$ and $\bigwedge$ for problems and relations can be understood best by considering the following operator, which assigns a relation to each problem:

Definition 6.3 Given a problem $P : \underline{X} \longrightarrow \underline{Y}$, define the relation $\mathcal{R}(P) : \underline{X} \longrightarrow \underline{Y}$ by

$$\mathcal{R}(P) = \{f :\subseteq \underline{X} \longrightarrow \underline{Y} \mid (\forall x \in X)\ (\exists g_x \in P)\ f(x) = g_x(x)\}.$$

Theorem 6.4 $\mathcal{R}$ *is a co-closure operator* (*interior operator*)*, i.e. it satisfies:*

1. $\mathcal{R}(P) \leq_2 P$*;*
2. $P \leq_2 Q$ *implies* $\mathcal{R}(P) \leq_2 \mathcal{R}(Q)$*;*
3. $\mathcal{R}(P) = P$*, if* P *is a relation.*

Proof.

1. We have $P \subseteq \mathcal{R}(P)$ by choosing $g_x = f$ for all $x \in X$. This implies $\mathcal{R}(P) \leq_2 P$.
2. This follows from the locality in the definition of $\leq_2$.
3. According to Definition 2.1, $\mathcal{R}(P) = P$ is equivalent to P being a relation. □

By using $\mathcal{R}$, we can now simply state the relationship as

$$\bigwedge = \mathcal{R} \circ \sqcap.$$

6.3 A note on computable reducibilities

Due to issues of cardinality, our constructions of uncountable infima and uncountable suprema are inapplicable for the computable versions of $\leq_0$ and $\leq_2$, as the resulting spaces are no longer representable. However, despite the equivalence of $\underline{\mathbb{N}^\mathbb{N}}$ and $\coprod_{n\in\mathbb{N}} \underline{\mathbb{N}^\mathbb{N}}$, already the existence of countable infima and suprema breaks down, as countable products and coproducts of computable functions are not necessarily computable.

Existence of suprema and infima in the computable case is ensured for finite sets. In addition, considering countable coproducts still makes sense. For example, the operator $\overline{}$ forms a closure-operator also in the computable setting. While the properties of suprema cannot be invoked anymore, the functions actually realizing the reductions do happen to be computable. We use $\leq_{\mathrm{W}}$ to refer to the computable version of $\leq_2$:

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Theorem 6.5 *The operator* $\overline{}$ *satisfies the following properties:*

1. $f \leq_W \overline{f}$;
2. $f \leq_W g$ *implies* $\overline{f} \leq_W \overline{g}$;
3. $\overline{f} \equiv_W \overline{\overline{f}}$.

Proof.

1. The coproduct injections are computable.

2. If the computable partial functions F and G witness $f \leq_W g$, then F' and $\overline{G}$ witness $\overline{f} \leq_W \overline{g}$, where F' is defined as $F'(i, x, i, y) = F^i(x, y)$.

3. We only have to show $\overline{\overline{f}} \leq_2 \overline{f}$. Consider the computable function G defined as

$$G(n, ((i_1, x_1), (i_2, x_2), \dots, (i_n, x_n))) = (\textstyle\sum_{k=1}^n i_k, \langle x_1, x_2, \dots, x_n \rangle).$$

Then G witnesses the claim. □

6.4 Decomposing functions

When a function f is expressed as a supremum of some functions f_i, apparently all questions regarding the discontinuity of f can be answered by examining the functions f_i. An example for this is the notion of C_∞-continuous functions introduced in [12], which corresponds to the supremum of the Ω_n-continuous functions.

For functions defined on a strongly zero-dimensional metrisable space whose Level exists and is a countable limit-ordinal, a general procedure to find an expression as a supremum of less discontinuous functions will be given below. We consider the function $f : \underline{X} \longrightarrow \underline{Y}$, where $\underline{X}$ is assumed to be metrisable and strongly zero-dimensional. We set $\gamma = \mathrm{Lev}^2(f)$, and let $(\gamma_n)_{n\in\mathbb{N}}$ be an arbitrary sequence satisfying $\gamma_n \leq \gamma$ for all $n \in \mathbb{N}$, as well as $\lim_{i\to\infty} \gamma_i = \gamma$. Further, L_n shall denote the set $\mathcal{L}^2_{\gamma_n}(f)$, and f_n the restriction of f to $X \setminus L_n$.

Theorem 6.6 $f \cong_2 \coprod_{n\in\mathbb{N}} f_n$.

Proof. As each f_n is a restriction of f, for all $n \in \mathbb{N}$, directly $f_n \leq_2 f$ can be obtained. The second fact in the proof of Theorem 4.2 yields $\coprod_{n\in\mathbb{N}} f_n \leq_2 f$.

For the other direction, let d be a metric on X that induces its topology. As $\underline{X}$ is strongly zero-dimensional, the range of d can assumed to be $\mathcal{N} = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$, equipped with the restriction of the usual Euclidean topology on the real field. For a subset $A \subseteq X$, the function $x \longmapsto d(x, A)$ is a continuous function from $\underline{X}$ to $\underline{\mathcal{N}}$. The function $L : \underline{X} \longrightarrow \prod_{n\in\mathbb{N}} \underline{\mathcal{N}}$, defined by $L(x)(n) = d(x, L_n)$ is also continuous. $\prod_{n\in\mathbb{N}} \underline{\mathcal{N}}$ is homeomorphic to $\underline{\{0,1\}^{\mathbb{N}}}$ using ι as homeomorphism, which is defined via $\iota(w)(\langle n, m \rangle) = 1$ if $w(n) = \frac{1}{m}$.

By definition, each set L_n is closed, so as $\underline{X}$ is metrisable, $d(x, L_n) = 0$ is equivalent to $x \in L_n$. Since $\bigcap_{n\to\infty} L_n = \emptyset$, for each $x \in X$ there is an n with $x \notin L_n$, so there is an m with $d(x, L_n) = \frac{1}{m}$. So for each x, the sequence $\iota(L(x))$ contains a 1. The function which takes a sequence $w \in \{0,1\}^\omega \setminus \{0^\omega\}$ and returns the least number n, so that w_n is 1, is computable and thus continuous. The function $\langle n, m \rangle \longmapsto n$ is computable and thus continuous. Composition of all these functions yields a continuous function $\mathfrak{L} : \underline{X} \longrightarrow \underline{\mathbb{N}}$ which satisfies $x \notin L_{\mathfrak{L}(x)}$.

Each $x \in X$ thus satisfies $x \in \mathrm{dom}(f_{\mathfrak{L}(x)})$. Therefore, $x \longmapsto [\coprod_{n\in\mathbb{N}} f_n](\mathfrak{L}(x), x)$ is well-defined. Composition with a projection yields $f(x) = \mathrm{pr}([\coprod_{n\in\mathbb{N}} f_n](\mathfrak{L}(x), x))$, and as both $\mathfrak{L}$ and pr are continuous, this shows $f \leq_2 \coprod_{n\in\mathbb{N}} f_n$. □

In other cases, the decomposition is already present in the definitions. Typical examples here are dimensions, in many instances, some problem parameterized with some natural number in the role of a dimension will be the supremum over all problems with the parameter fixed to some value.

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6.5 Defining admissibility via suprema of $\leq_0$

Admissibility is a desirable property of representations which can be considered central to computable analysis. In [17], Schröder extends the definition of admissibility that e.g. can be found in [22] to a more general case, yielding the following definition:

Definition 6.7 A surjective partial function $\delta :\subseteq \underline{\mathbb{N}^{\mathbb{N}}} \longrightarrow \underline{X}$ is called *admissible*, if it is continuous and $\rho \leq_0 \delta$ for all continuous surjective partial functions $\rho :\subseteq \underline{\mathbb{N}^{\mathbb{N}}} \longrightarrow \underline{X}$.

Note the following two observations. If $f \leq_0 g$ holds, and f is surjective, so is g. If g_i is continuous for $i \in I$, so is $\uparrow g_i \uparrow_{i\in I}$. Then admissibility can be rephrased as a maximality statement regarding the partial order[8] $\leq_0$. We use $\mathcal{C}_{\rm p}(\underline{X}, \underline{Y})$ to denote the set of continuous partial functions from $\underline{X}$ to $\underline{Y}$.

Proposition 6.8 *A partial function $\delta :\subseteq \underline{\mathbb{N}^{\mathbb{N}}} \longrightarrow \underline{X}$ is admissible, if $\delta \cong_0 \uparrow\rho\uparrow_{\rho\in\mathcal{C}_{\rm p}(\underline{\mathbb{N}^{\mathbb{N}}},\underline{X})}$ holds.*

While Definition 6.8 does not seem to be more useful than Definition 6.7 for practical purposes, it does clearly show the order-theoretic nature of admissibility. Also, Definition 6.8 invites the following extension:

Definition 6.9 A partial function $f :\subseteq \underline{Y} \longrightarrow \underline{X}$ is called *admissible*, if $f \cong_0 \uparrow g\uparrow_{g\in\mathcal{C}(\underline{Y},\underline{X})}$ holds.

In [17] the topological spaces $\underline{X}$ admitting an admissible representation following Definitions 6.7 or 6.8 were characterized as those T_0-spaces with a countable pseudobase. A generalization of the question lies at hand: Given a topological space $\underline{Y}$, for which topological spaces $\underline{X}$ is there an admissible partial function $f : \underline{Y} \longrightarrow \underline{X}$? We conclude with giving a trivial answer for a certain subcase: For a discrete space $\underline{D}$, there is an admissible (partial) function $f : \underline{D} \longrightarrow \underline{X}$ if and only if $|X| \leq |D|$ holds, as admissibility then coincides with surjectivity. As the class of topological spaces where the underlying sets do not exceed a certain cardinality is not cartesian closed, this example can be considered as a demonstration that $\underline{\mathbb{N}^{\mathbb{N}}}$ is especially suitable as domain for representations.

6.6 Generalizing $\leq_0$ in category theory

The simple Definition 3.1 can easily be formulated in the framework of category theory. Given a category $\mathcal{L}$, a subcategory $\mathcal{K}$ of $\mathcal{L}$ and an object $Z \in \mathcal{L}$, a partial order $\leq_0$ can be defined on the class of morphisms in $\mathcal{L}$ with codomain Z:

Definition 6.10 For morphisms $u : X \longrightarrow Z$, $v : Y \longrightarrow Z$, $u, v \in \mathcal{L}$, let $u \leq_0 v$ hold, if there is a morphism $G \in \mathcal{K}$ with $v = u \circ G$.

While it is not necessary that $\mathcal{K}$ includes all objects from $\mathcal{L}$ for Definition 6.10 to be valid, this requirement certainly makes $\leq_0$ more useful, so it will be adopted in the following. Note that the trivial case $\mathcal{K} = \mathcal{L}$ is a worthwhile object of study on its own, just as $\leq_0$ can be fruitfully used to compare continuous functions only.

For studying suprema for $\leq_0$, we require that $\mathcal{L}$ has arbitrary coproducts and that $\mathcal{K}$ is closed in $\mathcal{L}$ under formation of coproducts. We recall the definition of coproducts in category theory:

Definition 6.11 Given a family $(A_i)_{i\in I}$ of objects in a category $\mathcal{L}$, an object A together with morphisms $\mu_i : A_i \longrightarrow A$ is called the *coproduct* of the $(A_i)_{i\in I}$, if for every family of morphisms $(f_i : A_i \longrightarrow Z)$ there is a unique morphism $f : A \longrightarrow Z$ satisfying $f_i = f \circ \mu_i$ for all $i \in I$.

We claim that this uniquely determined morphism f is the supremum of the morphisms f_i. As $\mathcal{K}$ was required to include all objects and to be closed under formation of coproducts, $\mathcal{K}$ includes all morphisms μ_i, proving $f_i \leq_0 f$ for all $i \in I$. If there is a morphism $g \in \mathcal{L}$ with morphisms $G_i \in \mathcal{K}$ for $i \in I$ satisfying $g = f_i \circ G_i$, then $g = f \circ (\mu_i \circ G_i)$ follows. Thus $f_i \leq_0 g$ for all $i \in I$ implies $f \leq_0 g$, proving f to be the supremum of the f_i.

Studying infima will require the existence of arbitrary pullbacks in $\mathcal{L}$, and the closure of $\mathcal{K}$ in $\mathcal{L}$ under formation of pullbacks, albeit in a very strong sense. Again, we start with recalling the definition of pullbacks:

Definition 6.12 Given a family $(f_i : A_i \longrightarrow Z)_{i\in I}$ of morphisms in $\mathcal{L}$. The *pullback* of the f_i is a family of morphisms $(p_i : P \longrightarrow A_i)_{i\in I}$ satisfying $f_i \circ p_i = f_j \circ p_j$ for all $i, j \in I$, so that if $(q_i : Q \longrightarrow A_I)_{i\in I}$ is another family of morphisms with $f_i \circ q_i = f_j \circ q_j$, there is a unique morphism $\lambda : Q \longrightarrow P$ with $q_i = p_i \circ \lambda$ for all $i \in I$.

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[8] This was already noted in [17].

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The infimum of the family f_i in the definition above is given by the morphism $f = f_i \circ p_i$ (which does not depend on i), as long as $p_i \in \mathcal{K}$ for all $i \in I$ and $\lambda \in \mathcal{K}$ are fulfilled. $f \leq_0 f_i$ is clear. Suppose $g \leq_0 f_i$ for all $i \in I$, so there are morphisms G_i with $g = f_i \circ G_i$. This implies $f_i \circ G_i = f_j \circ G_j$, so there is a λ with $G_i = p_i \circ \lambda$, thus $g = (f_i \circ p_i) \circ \lambda$ holds, establishing $g \leq_0 f$.

Partially ordered classes can easily be expressed as categories. If $(K, \preceq)$ is a partially ordered class, the associated partial-order-category has the elements of K as objects, and contains a unique morphism $u : A \longrightarrow B$ if and only if $A \preceq B$ holds. Composition of morphisms is defined in a straightforward fashion. Infima in the partially ordered class are pullbacks in the partial-order-category, and suprema in the partially ordered class are coproducts in the partial-order category.

The previous definition of $\leq_0$ is obtained from the version given here by choosing $\mathcal{K}$ to be the category Top of topological spaces and continuous functions, and $\mathcal{L}$ to be the category of topological spaces and arbitrary functions, which is equivalent to Set.

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