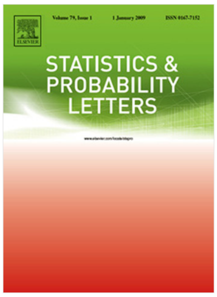
Paper:

Wu , B. & Wu, J. (2018). Characterizing the path-indpendent property of the Girsanov density for degenerated stochastic differential equations. *Statistics & Probability Letters, 133*(2), 71-79.
http://dx.doi.org/https://doi.org/10.1016/j.spl.2017.10.005



Characterizing the path-independent property of the Girsanov density for degenerated stochastic differential equations

Bo Wu, Jiang-Lun Wu

PII: S0167-7152(17)30319-X
DOI: https://doi.org/10.1016/j.spl.2017.10.005
Reference: STAPRO 8046

To appear in: *Statistics and Probability Letters*

Received date : 13 January 2017
Revised date : 5 July 2017
Accepted date : 10 October 2017

Please cite this article as: Wu B., Wu J.-L., Characterizing the path-independent property of the Girsanov density for degenerated stochastic differential equations. *Statistics and Probability Letters* (2017), https://doi.org/10.1016/j.spl.2017.10.005

CHARACTERIZING THE PATH-INDEPENDENT PROPERTY OF THE GIRSANOV DENSITY FOR DEGENERATED STOCHASTIC DIFFERENTIAL EQUATIONS

BO WU[1] AND JIANG-LUN WU[2]

1. School of Mathematical Sciences, Fudan University
Shanghai 200433, China
wubo@fudan.edu.cn
2. Department of Mathematics, College of Science, Swansea University
Singleton Park, Swansea SA2 8PP, UK
j.l.wu@swansea.ac.uk

Abstract. In this paper, we derive a characterisation theorem for the path-independent property of the density of the Girsanov transformation for *degenerated* stochastic differential equations (SDEs), extending the characterisation theorem of [13] for the non-degenerated SDEs. We further extends our consideration to non-Lipschitz SDEs with jumps and with degenerated diffusion coefficients, which generalises the corresponding characterisation theorem established in [10].

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geqslant 0})$ be a filtered probability space. Let $d, m \in \mathbb{N}$ be fixed. We are concerned with the following SDE

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t, \quad t \geq 0 \tag{1}$$

where

$$b : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d, \quad \sigma : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^{d\otimes m}$$

$(W_t)_{t\geqslant 0}$ is an m-dimensional $\{\mathcal{F}_t\}_{t\geqslant 0}$-Brownian motion. Under standard usual conditions, e.g. the two coefficients b and σ satisfy linear growth and local Lipschitz conditions (for the second variable), there is a unique solution to the above SDE (1) for a given initial data X_0, see, e.g., [3].

The celebrated Girsanov theorem provides a very powerful tool to solve SDEs under the name of the *Girsanov transformation* or the *transformation of the drift*. We use $|\cdot|$ and $\langle\cdot,\cdot\rangle$ to denote the Euclidean norm and scalar product of vectors in $\mathbb{R}^m$ or $\mathbb{R}^d$, respectively. Let $\gamma : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^m$ be a measurable function such that the following exponential integrability along the paths of the solution $(X_t)_{t\geqslant 0}$ holds (also known as Novikov condition)

$$\mathbb{E}\left(\exp\left\{-\int_0^t |\gamma(s, X_s)|^2 \mathrm{d}s + \int_0^t \langle\gamma(s, X_s), \mathrm{d}W_s\rangle\right\}\right) < \infty, \quad t \geqslant 0. \tag{2}$$

AMS Subject Classification(2010): 60H10; 35Q53.

Keywords: degenerated stochastic differential equations (SDEs), Girsanov transformation, non-Lipschitz SDEs with jumps, semi-linear partial integro-differential equation of parabolic type.

*This work was partly supported by NSF of China (No. 11371099).

Then, Girsanov theorem ([2, 3, 4]) says that for any arbitrarily fixed $T > 0$

$$\tilde{W}_t := W_t - \int_0^t \gamma(s, X_s)\mathrm{d}s, \quad t \in [0, T] \tag{3}$$

is an m-dimensional $\{\mathcal{F}_t\}_{t\in[0,T]}$-Brownian motion under the probability measure

$$Q_T := \exp\left\{ -\int_0^T |\gamma(s, X_s)|^2 \mathrm{d}s + \int_0^T \langle \gamma(s, X_s), \mathrm{d}W_s \rangle \right\} \cdot \mathbb{P}. \tag{4}$$

Moreover, the solution $(X_t)_{t\in[0,T]}$ fulfils the following SDE

$$\mathrm{d}X_t = [b(t, X_t) + \sigma(t, X_t)\gamma(t, X_t)]\mathrm{d}t + \sigma(t, X_t)\mathrm{d}\tilde{W}_t, \quad t \in [0, T]. \tag{5}$$

Now let us assume that (along the paths of the solution $(X_t)_{t\geqslant 0}$)

$$b(t, X_t) - \sigma(t, X_t)\gamma(t, X_t) = 0, \quad a.s.\ \forall\ t \geqslant 0. \tag{6}$$

Equivalently, $b \in Im(\sigma)$, where $Im(\sigma)$ is the imagine space of σ. Then

$$\mathrm{d}X_t = \sigma(t, X_t)\mathrm{d}\tilde{W}_t. \tag{7}$$

We are interested in the path-independent property for the exponent of the Girsanov density of Q_T for any fixed $T > 0$. That is, whether there exists a scalar function $v : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ such that

$$Z_t := \frac{1}{2}\int_0^t |\gamma(s, X_s)|^2 \mathrm{d}s + \int_0^t \langle \gamma(s, X_s), \mathrm{d}W_s \rangle = v(t, X_t) - v(0, X_0), \quad t \geq 0. \tag{8}$$

This problem arises from a number of studies in economics, finance as well as from stochastic mechanics, just mention a few, see [13, 15] (and references therein).

If $\sigma : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^{d\otimes d}$ (i.e., taking $m = d$) is non-degenerate, that is, the $d \times d$-matrix $\sigma(t, x)$ is invertible for any $(t, x) \in [0, \infty) \times \mathbb{R}^d$, a characterisation of the path-independent property has obtained in [13].

Throughout the article, we assume that σ, b satisfy the Hörmander's condition, i.e. if the associated Lie algebra spans, in any point, the whole $\mathbb{R}^d$. In particular, we allow σ be degenerate.

Let $\Lambda_t := \{(t, X_t) \in [0, \infty) \times \mathbb{R}^d : t \geqslant 0, \omega \in \Omega\} \subset [0, \infty) \times \mathbb{R}^d$, the support of the solution. In particular, for each $t > 0$, we have $\Lambda_t = [0, \infty) \times \mathbb{R}^d$ if $b(t, x)$ and $\sigma(t, x)$ satisfy the Hörmander's conditions. Then by using Itô's formula to $v(t, X_t)$ viewing as the composition of $v : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ with the semimartingale $(X_t)_{t\geqslant 0}$, the utilising the uniqueness of Doob-Meyer decomposition for continuous semimartingales, we can derive for any $t \geqslant 0$ the following

$$\gamma(t, X_t) = \sigma^*(t, X_t)\nabla v(t, X_t) \tag{9}$$

and

$$\frac{1}{2}|\gamma(t, X_t)|^2 = \frac{\partial v}{\partial t}(t, X_t) + \langle \nabla v(t, X_t), b(t, X_t) \rangle + \frac{1}{2}Tr[(\sigma\sigma^*(t, X_t)\nabla^2 v(t, X_t))] \tag{10}$$

where $\sigma^*(t, x)$ stands for the transposed matrix of $\sigma(t, x)$, ∇ and ∇^2 stand for the gradient and Hessian operators with respect to the second variable, respectively. Moreover, we get

$$\left\{ \begin{array}{l} \gamma(t, x) = \sigma^*(t, x)\nabla v(t, x) \quad (I) \\ \frac{1}{2}|\gamma(t, x)|^2 = \frac{\partial v}{\partial t}(t, x) + \langle \nabla v(t, x), b(t, x) \rangle + \frac{1}{2}Tr[(\sigma\sigma^*(t, x)\nabla^2 v(t, x))] \quad (II) \end{array} \right. \tag{11}$$

for any $(t,x) \in [0,\infty) \times \Lambda$. Putting (I) into (II) and (6) yield the following nonlinear parabolic PDE of the (reversible) HJB type

$$\left\{ \begin{array}{l} \frac{\partial v}{\partial t}(t,x) = -\frac{1}{2}\{Tr[(\sigma\sigma^*(t,x)\nabla^2 v(t,x))] + |\sigma^*(t,x)\nabla v(t,x)|^2\}, \\ \sigma(t,x)\sigma^*(t,x)\nabla v(t,x) = b(t,x), \quad (t,x) \in [0,\infty) \times \Lambda. \end{array} \right. \tag{12}$$

Remark 1.1. *All above derivations are reciprocal, namely, that gives a characterisation of path-independence property.*

Theorem 1.2. *Assume that $\gamma : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d$ is a function satisfying (6). Then there exists a scalar function $v \in C^{1,2}((0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d)$ such that*

$$\frac{1}{2}\int_0^t |\gamma(s,X_s)|^2 \mathrm{d}s + \int_0^t \langle \gamma(s,X_s), \mathrm{d}W_s\rangle = v(t,X_t) - v(0,X_0) \tag{13}$$

if and only if (12) holds.

Proof. By the previous argument, we only show that the sufficiency. Since

$$v \in C^{1,2}((0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d),$$

we know that $v(t,X_t)$ is a continuous semimartinagle of X_t. Thus we ahve

$$\begin{aligned} \mathrm{d}v(t,X_t) = &\left\{\frac{\partial v}{\partial t}(t,X_t) + \langle \nabla v(t,X_t), b(t,X_t)\rangle + \frac{1}{2}Tr[(\sigma\sigma^*(t,X_t)\nabla^2 v(t,X_t))]\right\}\mathrm{d}t \\ &+ \langle \sigma^*(t,X_t)\nabla v(t,X_t), \mathrm{d}W_t\rangle. \end{aligned} \tag{14}$$

Combining this with (12), we get

$$\begin{aligned} &\mathrm{d}v(t,X_t) \\ &= \left\{-\frac{1}{2}|\sigma^*(t,X_t)\nabla v(t,X_t)|^2 + \langle \nabla v(t,X_t), b(t,X_t)\rangle\right\} + \langle \sigma^*(t,X_t)\nabla v(t,X_t), \mathrm{d}W_t\rangle \\ &= \frac{1}{2}|\gamma(t,X_t)|^2\mathrm{d}t + \langle \gamma(t,X_t), \mathrm{d}W_t\rangle. \end{aligned} \tag{15}$$

This implies (13). ☐

Corollary 1.3. *Assume that $\gamma : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d$ is a function satisfying (6). Then there exist a function $f \in C^2(\mathbb{R} \to \mathbb{R})$ and a scalar function $v \in C^{1,2}((0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d)$ such that*

$$\frac{1}{2}\int_0^t |\gamma(s,X_s)|^2 \mathrm{d}s + \int_0^t \langle \gamma(s,X_s), \mathrm{d}W_s\rangle = f(v(t,X_t)) - f(v(0,X_0)) \tag{16}$$

if and only if

$$\left\{ \begin{array}{l} f'(v)(t,x)\frac{\partial v}{\partial t}(t,x) = -\frac{1}{2}\big\{f'(v)(t,x)Tr[(\sigma\sigma^*(t,x)\nabla^2 v(t,x))] \\ \qquad\qquad +f''(v)(t,x)|\sigma^*(t,x)\nabla v(t,x)|^2 + |f'(v)(t,x)|^2|\sigma^*(t,x)\nabla v(t,x)|^2\big\}, \\ b(t,x) = f'(v)(t,x)\sigma(t,x)\sigma^*(t,x)\nabla v(t,x), \quad (t,x) \in [0,\infty) \times \Lambda. \end{array} \right. \tag{17}$$

Proof. According to Theorem 1.2, we know that (16) is equivalent to

$$\left\{ \begin{array}{l} \frac{\partial f(v)}{\partial t}(t,x) = -\frac{1}{2}\{Tr[(\sigma\sigma^*(t,x)\nabla^2 f(v)(t,x))] + |\sigma^*(t,x)\nabla f(v)(t,x)|^2\}, \\ \sigma(t,x)\sigma^*(t,x)\nabla f(v)(t,x) = b(t,x), \quad (t,x) \in [0,\infty) \times \Lambda. \end{array} \right.$$

Since

$$\begin{aligned}&Tr[(\sigma\sigma^*(t,x)\nabla^2 f(v)(t,x))] = Tr[(\sigma\sigma^*(t,x)\nabla(f'(v)(t,x)\nabla v(t,x))]\\ &= f'(v)(t,x)Tr[(\sigma\sigma^*(t,x)\nabla^2 v(t,x))] + f''(v)(t,x)|\sigma^*(t,x)\nabla v(t,x)|^2\end{aligned}$$

and

$$\sigma^*(t,x)\nabla f(v)(t,x) = f'(v)\sigma^*(t,x)\nabla v(t,x).$$

Combining all the above equalities, we conclude that (16) is equivalent to (17).

□

Example 1.4. *Under the conditions of Corollary 1.3, we have the following examples of the function f*

(a) If $f(x) = x$, then

$$\frac{1}{2}\int_0^t |\gamma(s,X_s)|^2 \mathrm{d}s + \int_0^t \langle \gamma(s,X_s), \mathrm{d}W_s\rangle = v(t,X_t) - v(0,X_0) \tag{18}$$

if and only if

$$\sigma(t,x)\sigma^*(t,x)\nabla v(t,x) = b(t,x), \quad (t,x) \in [0,\infty)\times\Lambda \tag{19}$$

and v satisfies the following time-reversed KPZ type equation,

$$\frac{\partial v}{\partial t}(t,x) = -\frac{1}{2}\{Tr[(\sigma\sigma^*(t,x)\nabla^2 v(t,x))] + |\sigma^*(t,x)\nabla v(t,x)|^2\}, \quad (t,x)\in[0,\infty)\times\Lambda. \tag{20}$$

In particular, if σ is invertible, this covers the result obtained in [13].

(b) If $f(x) = \log|x|$, for $x \neq 0$, then

$$\frac{1}{2}\int_0^t |\gamma(s,X_s)|^2 \mathrm{d}s + \int_0^t \langle \gamma(s,X_s), \mathrm{d}W_s\rangle = \log\frac{v(t,X_t)}{v(0,X_0)} \tag{21}$$

if and only if

$$\sigma(t,x)\sigma^*(t,x)\nabla v(t,x) = v(t,x)b(t,x), \quad (t,x)\in[0,\infty)\times\Lambda, \tag{22}$$

and v satisfies the following time-reversed heat kernel type equation,

$$\frac{\partial v}{\partial t}(t,x) = -\frac{1}{2}Tr[(\sigma\sigma^*(t,x)\nabla^2 v(t,x))], \quad (t,x)\in[0,\infty)\times\Lambda. \tag{23}$$

In particular, if $\sigma = Id$, then we have

$$\frac{1}{2}\int_0^t |b(s,X_s)|^2 \mathrm{d}s + \int_0^t \langle b(s,X_s), \mathrm{d}W_s\rangle = \log\left|\frac{v(t,X_t)}{v(0,X_0)}\right| \tag{24}$$

if and only if

$$\nabla v(t,x) = v(t,x)b(t,x), \quad (t,x)\in[0,\infty)\times\mathbb{R}^d, \tag{25}$$

and v satisfies the standard heat kernel equation,

$$\frac{\partial v}{\partial t}(t,x) = -\frac{1}{2}\Delta v(t,x), \quad (t,x)\in[0,\infty)\times\mathbb{R}^d. \tag{26}$$

(c) If $f(x) = x^{2k+1}, k\in\mathbb{N}\cup\{0\}$, or $f(x) = x^{2k+1}, k\in\mathbb{Z}$ for $x\neq 0$, then

$$\frac{1}{2}\int_0^t |\gamma(s,X_s)|^2 \mathrm{d}s + \int_0^t \langle \gamma(s,X_s), \mathrm{d}W_s\rangle = v^{2k+1}(t,X_t) - v^{2k+1}(0,X_0) \tag{27}$$

if and only if

$$(2k+1)v^{2k}(t,x)\sigma(t,x)\sigma^*(t,x)\nabla v(t,x) = b(t,x), \quad (t,x)\in[0,\infty)\times\Lambda, \tag{28}$$

and v satisfies the following time-reversed HJB equation,

$$\begin{aligned}\frac{\partial v}{\partial t}(t,x) = -\frac{1}{2}\Big\{&Tr[(\sigma\sigma^*(t,x)\nabla^2 v(t,x))]\\&+\frac{(2k+1)v^{2k+1}(t,x)+2k}{v(t,x)}|\sigma^*(t,x)\nabla v(t,x)|^2\Big\}.\end{aligned} \tag{29}$$

(d) If $f(x)=\tan(x), x\in(-\frac{\pi}{2},\frac{\pi}{2})$, then

$$\frac{1}{2}\int_0^t |\gamma(s,X_s)|^2 \mathrm{d}s + \int_0^t \langle \gamma(s,X_s), \mathrm{d}W_s\rangle = \tan(v(t,X_t)) - tan(v(0,X_0)) \tag{30}$$

if and only if

$$\sigma(t,x)\sigma^*(t,x)\nabla v(t,x) = \cos^2(v(t,x))b(t,x), \quad (t,x)\in[0,\infty)\times\Lambda, \tag{31}$$

and v satisfies the following time-reversed HJB equation,

$$\begin{aligned}\frac{\partial v}{\partial t}(t,x) = -\frac{1}{2}\Big\{&Tr[(\sigma\sigma^*(t,x)\nabla^2 v(t,x))]\\&+\frac{[\cos(v(t,x))+\sin(v(t,x))]^2}{\cos^2(v(t,x))}|\sigma^*(t,x)\nabla v(t,x)|^2\Big\}.\end{aligned} \tag{32}$$

Proof. It is obvious for (a). We only prove (b)((c) may be similarly handed). By Corollary 1.3, we know that (21) is equivalent to

$$\begin{cases} \frac{1}{v(t,x)}\frac{\partial v}{\partial t}(t,x) = -\frac{1}{2}\{\frac{1}{v(t,x)}Tr[(\sigma\sigma^*(t,x)\nabla^2 v(t,x))] \\ \qquad\qquad -\frac{1}{v^2(t,x)}|\sigma^*(t,x)\nabla v(t,x)|^2 + \frac{1}{v^v(t,x)}|\sigma^*(t,x)\nabla v(t,x)|^2\} \\ \sigma(t,x)\sigma^*(t,x)\nabla v(t,x) = v(t,x)b(t,x), \quad (t,x)\in[0,\infty)\times\Lambda \end{cases} \tag{33}$$

which are just (22) and (23), respectively. ☐

Example 1.5. [Gruschin operator] *Let $b(t,z)=(-xt,-x^k yt)^T, z=(x,y)\in\mathbb{R}^2, t\geqslant 0$ and $\sigma(t,z)$ be given by*

$$\sigma(t,z)=\begin{pmatrix}1 & 0\\ 0 & x^k\end{pmatrix}, \quad k\in\mathbb{N}, z=(x,y)\in\mathbb{R}^2, t\geqslant 0. \tag{34}$$

Then $b\in Im(\sigma)$ and the Hörmander's condition holds for $\mathcal{H}=\{\frac{\partial}{\partial x}, x^k\frac{\partial}{\partial y}\}$ with commutators up to order k. Define the subelliptic diffusion operator

$$L = X^2 + Y^2 + b(t,\cdot).$$

Let $\gamma(t,z)=(-xt,-yt)^*$ and X_s be the associated L-diffusion process, then $b(t,z)=\sigma(t,z)\gamma(t,z)$. Assume that $v\in C^{1,2}((0,\infty)\times\mathbb{R}^2\to\mathbb{R}^2)$ fulfills the following

$$\frac{1}{2}\int_0^t |\gamma(s,X_s)|^2 \mathrm{d}s + \int_0^t \langle \gamma(s,X_s), \mathrm{d}W_s\rangle = v(t,X_t) - v(0,X_0). \tag{35}$$

Then, by Theorem 1.2, we know that v satisfies the equation (12).

The following two examples come from [14].

Example 1.6. [Kohn operator] *Consider the three-dimensional Heisenberg group realized as $\mathbb{R}^3$ equipped with the group multiplication*

$$(x,y,z)(x',y',z') := (x+x', y+y', z+z'+(xy'-x'y)/2),$$

which is a Lie group with left-invariant orthonormal frame $\{X, Y, Z\}$, *where*

$$X = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z},\ Y = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z},\ Z = [X, Y] = \frac{\partial}{\partial z}$$

Then the Kohn-Laplacian is $\Delta_H := X^2 + Y^2$. *Let*

$$b(t, u) = (xt, yt, \frac{z(x-y)}{2}t), \quad u = (x, y, z) \in \mathbb{R}^3, t \geqslant 0$$

and $\sigma(t, z)$ *be given by*

$$\sigma(t, u) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{y}{2} & \frac{x}{2} & 0 \end{pmatrix}, \quad u = (x, y, z) \in \mathbb{R}^3, t \geqslant 0. \tag{36}$$

Define the subelliptic diffusion operator

$$L = X^2 + Y^2 + b(t, \cdot).$$

Let $\gamma(t, z) = (xt, yt, zt)^*$ *and* X_s *be the associated* L*-diffusion process, then* $b \in Im(\sigma)$ *and* $b(t, z) = \sigma(t, z)\gamma(t, z)$. *Then, the Hörmander's condition holds for* $\mathcal{H} = \{\frac{\partial}{\partial x}, x^k\frac{\partial}{\partial y}\}$ *with commutators up to order* k. *Assume that* $v \in C^{1,2}((0, \infty) \times \mathbb{R}^2 \to \mathbb{R}^2)$ *fulfills the following*

$$\frac{1}{2}\int_0^t |\gamma(s, X_s)|^2 \mathrm{d}s + \int_0^t \langle \gamma(s, X_s), \mathrm{d}W_s \rangle = v(t, X_t) - v(0, X_0) \tag{37}$$

Then, by theorem 1.2, we know that v *satisfies the equation (12).*

2. Non-Lipschitz SDEs with jumps

2.1. The characterisation theorem for SDEs with continuous diffusions on $\mathbb{R}^d$. Let $(\mathbb{U}, \|\cdot\|_{\mathbb{U}})$ be a finite dimensional normed space endowed with its Borel σ-algebra $\mathscr{U}$. Let ν be a σ-finite measure defined on $(\mathbb{U}, \mathscr{U})$. Let us fix $\mathbb{U}_0 \in \mathscr{U}$ with $\nu(\mathbb{U} \setminus \mathbb{U}_0) < \infty$ and $\int_{\mathbb{U}_0} \|u\|_{\mathbb{U}}^2 \nu(\mathrm{d}u) < \infty$. Furthermore, let $\lambda : [0, \infty) \times \mathbb{U} \to (0, 1]$ be a given measurable function. Then, following e.g. [3, 4], there exists a non-negative integer valued $(\mathcal{F}_t)_{t\geqslant 0}$-Poisson random measure $N_\lambda(\mathrm{d}t, \mathrm{d}u)$ on the given filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F}_t)_{t\geqslant 0})$ with intensity $\mathbb{E}(N_\lambda(\mathrm{d}t, \mathrm{d}u)) = \lambda(t, u)\mathrm{d}t\nu(\mathrm{d}u)$. Set

$$\tilde{N}_\lambda(\mathrm{d}t, \mathrm{d}u) := N_\lambda(\mathrm{d}t, \mathrm{d}u) - \lambda(t, u)\mathrm{d}t\nu(\mathrm{d}u)$$

that is, $\tilde{N}_\lambda(\mathrm{d}t, \mathrm{d}u)$ stands for the compensated $(\mathcal{F}_t)_{t\geqslant 0}$-predictable martingale measure of $N_\lambda(\mathrm{d}t, \mathrm{d}u)$.

We are concerned with the following SDE on $\mathbb{R}^d$

$$\begin{cases} \mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}B_t + \int_{\mathbb{U}_0} f(t, X_{t-}, u)\tilde{N}_\lambda(\mathrm{d}t, \mathrm{d}u), & t \in (0, T], \\ X_0 = x_0 \in \mathbb{R}^d, \end{cases} \tag{38}$$

for any given $T > 0$, where b, σ are Borel measurable as given in the previous section, $(B_t)_{t\geqslant 0}$ is an m-dimensional $\{\mathcal{F}_t\}_{t\geqslant 0}$-Brownian motion, $f : [0, T] \times \mathbb{R}^d \times \mathbb{U}_0 \mapsto \mathbb{R}^d$ is Borel measurable, and $\tilde{N}_\lambda$ is the compensated $(\mathcal{F}_t)_{t\geqslant 0}$-predictable martingale measure of an induced $\{\mathcal{F}_t\}_{t\geqslant 0}$-Poisson random measure given above which is independent of $(B_t)_{t\geqslant 0}$. This equation arises in nonlinear filtering and has been considered recently in [11, 8, 9] (see also the monograph [12]).

The characterisation theorem for path-independent property of Girsanov density for the above equation with non-degenerated σ was established in [10]. More precisely, under the following conditions

($\mathbf{H}_1$) There exists $\lambda_0 \in \mathbb{R}$ such that for all $x, y \in \mathbb{R}^d$ and $t \in [0, T]$

$$2\langle x-y, b(t,x)-b(t,y)\rangle + \|\sigma(t,x)-\sigma(t,y)\|^2 \leqslant \lambda_0 |x-y|^2 \kappa(|x-y|),$$

where κ is a positive continuous function, bounded on $[1,\infty)$ and satisfying

$$\lim_{x\downarrow 0} \frac{\kappa(x)}{\log x^{-1}} = \delta < \infty.$$

($\mathbf{H}_2$) There exists $\lambda_1 > 0$ such that for all $x \in \mathbb{R}^d$ and $t \in [0, T]$

$$|b(t,x)|^2 + \|\sigma(t,x)\|^2 \leqslant \lambda_1 (1+|x|)^2.$$

($\mathbf{H}_3$) $b(t,x)$ is continuous in x and there exists $\lambda_2 > 0$ such that

$$\langle \sigma(t,x)h, h\rangle \geqslant \sqrt{\lambda_2}|h|^2, \qquad t \in [0,T], \quad x, h \in \mathbb{R}^d. \tag{39}$$

($\mathbf{H}_f$) For all $x, y \in \mathbb{R}^d$ and $t \in [0, T]$,

$$\int_{\mathbb{U}_0} \big|f(t,x,u) - f(t,y,u)\big|^2 \nu(\mathrm{d}u) \leqslant 2|\lambda_0||x-y|^2\kappa(|x-y|)$$

and for $q = 2$ and 4

$$\int_{\mathbb{U}_0} |f(t,x,u)|^q \, \nu(\mathrm{d}u) \leqslant \lambda_1 (1+|x|)^q.$$

Qiao and Wu in [10] proved a characterisation theorem, where a partial integer-differential equation (PIDE) as the main characterizing equation was derived. We notice that the assumption (H3) on the diffusion coefficient σ is too strong. Here we aim to relax this condition. First of all, we let σ to be $d \times m$-matrix-valued for $d, m \in \mathbb{N}$, i.e., σ is in general not square matrix-valued. And σ, b satisfy the Hörmander's condition.

Let $\gamma : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^m$ be a measurable function such that the following condition **($\mathbf{H}_{\gamma,\lambda}$)** holds

$$\mathbb{E}\Big[\exp\Big\{\frac{1}{2}\int_0^T |\gamma(s,X_s)|^2 \,\mathrm{d}s + \int_0^T \int_{\mathbb{U}_0} \Big(\frac{1-\lambda(s,u)}{\lambda(s,u)}\Big)^2 \lambda(s,u)\nu(\mathrm{d}u)\mathrm{d}s\Big\}\Big] < \infty.$$

Set

$$\begin{aligned}
\Gamma_t : &= \exp\Big\{-\int_0^t \langle \gamma(s,X_s), \mathrm{d}B_s\rangle - \frac{1}{2}\int_0^t |\gamma(s,X_s)|^2 \,\mathrm{d}s \\
&\qquad -\int_0^t \int_{\mathbb{U}_0} \log\lambda(s,u) N_\lambda(\mathrm{d}s,\mathrm{d}u) - \int_0^t \int_{\mathbb{U}_0} (1-\lambda(s,u))\nu(\mathrm{d}u)\mathrm{d}s\Big\}, \\
M_t : &= -\int_0^t \langle \gamma(s,X_s), \mathrm{d}B_s\rangle + \int_0^t \int_{\mathbb{U}_0} \frac{1-\lambda(s,u)}{\lambda(s,u)} \tilde{N}_\lambda(\mathrm{d}s,\mathrm{d}u),
\end{aligned}$$

and then (Γ_t) is the Doléans-Dade exponential of (M_t), see e.g., [2].

Under ($\mathbf{H}_1$), ($\mathbf{H}_2$) and **($\mathbf{H}_f$)**, it is well known that there exists a unique strong solution to Eq.(38) (cf. [12, Theorem 170, p.140]). This solution will be denoted by X_t. In the following, we define the support of a random vector ([6]) and then present a result about the support of X_t under the above assumptions.

Definition 2.1. *The support of a random vector Y is defined as*

$$\mathrm{supp}(Y) := \{x \in \mathbb{R}^d | (\mathbb{P} \circ Y^{-1})(B(x,r)) > 0, \ \textit{for all}\ r > 0\}$$

where $B(x,r) := \{y \in \mathbb{R}^d | |y-x| < r\}$, the open ball centered at x with radius r.

Under $(\mathbf{H}_{\gamma,\lambda})$, (M_t) is a locally square integrable martingale. Moreover, $M_t - M_{t-} > -1$ a.s. and

$$\begin{aligned}
&\mathbb{E}\Big[\exp\Big\{\frac{1}{2} < M^c, M^c >_T + < M^d, M^d >_T\Big\}\Big]\\
= \ &\mathbb{E}\Big[\exp\Big\{\frac{1}{2}\int_0^T |\gamma(s,X_s)|^2\,\mathrm{d}s + \int_0^T \int_{\mathbb{U}_0} \Big(\frac{1-\lambda(s,u)}{\lambda(s,u)}\Big)^2 \lambda(s,u)\nu(\mathrm{d}u)\mathrm{d}s\Big\}\Big] < \infty,
\end{aligned}$$

where M^c and M^d are continuous and purely discontinuous martingale parts of (M_t), respectively. Thus, it follows from [7, Theorem 6] that (Λ_t) is an exponential martingale. Define a measure $\tilde{\mathbb{P}}$ via

$$\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = \Lambda_T.$$

By the Girsanov theorem for Brownian motions and random measures, one can obtain that under the measure $\tilde{\mathbb{P}}$ the system (38) is transformed into the following

$$\mathrm{d}X_t = [b(t,X_t) + \sigma(t,X_t)\gamma(t,X_t)]\mathrm{d}t + \sigma(t,X_t)\mathrm{d}\tilde{B}_t + \int_{\mathbb{U}_0} f(t,X_{t-},u)\tilde{N}(\mathrm{d}t,\mathrm{d}u),$$

Now let us assume that (along th paths of $(X_t)_{t\geqslant 0}$)

$$b(t,X_t) + \sigma(t,X_t)\gamma(t,X_t) = 0.$$

Then we get

$$\mathrm{d}X_t = \sigma(t,X_t)\mathrm{d}\tilde{B}_t + \int_{\mathbb{U}_0} f(t,X_{t-},u)\tilde{N}(\mathrm{d}t,\mathrm{d}u),$$

where

$$\tilde{B}_t := B_t + \int_0^t \gamma(s,X_s)\mathrm{d}s, \quad \tilde{N}(\mathrm{d}t,\mathrm{d}u) := N_\lambda(\mathrm{d}t,\mathrm{d}u) - \mathrm{d}t\nu(\mathrm{d}u).$$

Next, we set

$$\begin{aligned}
Y_t \ &:= \ -\log\Gamma_t = \int_0^t \langle\gamma(s,X_s), \mathrm{d}B_s\rangle + \frac{1}{2}\int_0^t |\gamma(s,X_s)|^2\,\mathrm{d}s\\
&\quad + \int_0^t \int_{\mathbb{U}_0} \log\lambda(s,u)N_\lambda(\mathrm{d}s,\mathrm{d}u) + \int_0^t \int_{\mathbb{U}_0} (1-\lambda(s,u))\nu(\mathrm{d}u)\mathrm{d}s.
\end{aligned}$$

Clearly, (Y_t) is a one-dimensional stochastic process with the following stochastic differential form

$$\begin{aligned}
\mathrm{d}Y_t \ &= \ \langle\gamma(t,X_t), \mathrm{d}B_t\rangle + \frac{1}{2}|\gamma(t,X_t)|^2\,\mathrm{d}t\\
&\quad + \int_{\mathbb{U}_0} \log\lambda(t,u)N_\lambda(\mathrm{d}t,\mathrm{d}u) + \int_{\mathbb{U}_0} (1-\lambda(t,u))\nu(\mathrm{d}u)\mathrm{d}t.
\end{aligned}$$

Let $\Lambda := \mathrm{supp}((t,X_t), t \geqslant 0)$. Then we have the following.

Theorem 2.2. *Let* $v : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ *be a scalar function which is* C^1 *with respect to the first variable and* C^2 *with respect to the second variable. Then*

$$\begin{aligned} v(t,X_t) &= v(0,x_0) + \int_0^t \langle \gamma(s,X_s), \mathrm{d}B_s\rangle + \frac{1}{2}\int_0^t |\gamma(s,X_s)|^2\,\mathrm{d}s \\ &\quad + \int_0^t\int_{\mathbb{U}_0} \log\lambda(s,u) N_\lambda(\mathrm{d}s,\mathrm{d}u) + \int_0^t\int_{\mathbb{U}_0} (1-\lambda(s,u))\nu(\mathrm{d}u)\mathrm{d}s, \end{aligned} \quad (40)$$

equivalently,

$$Y_t = v(t,X_t) - v(0,x_0), \quad t \in [0,T]$$

holds if and only if

$$b(t,x) = (\sigma\sigma^*\nabla v)(t,x), \qquad (t,x) \in \Lambda, \quad (41)$$

$$\lambda(t,u) = \exp\{v(t,x+f(t,x,u)) - v(t,x)\}, \quad (t,x,u) \in \Lambda \times \mathbb{U}_0, \quad (42)$$

and v *satisfies the following time-reversed partial integro-differential equation (PIDE),*

$$\begin{aligned} \frac{\partial}{\partial t} v(t,x) &= -\frac{1}{2}[Tr(\sigma\sigma^*)\nabla^2 v](t,x) - \frac{1}{2}|\sigma^*\nabla v|^2(t,x) - \int_{\mathbb{U}_0}\Big[e^{v(t,x+f(t,x,u))-v(t,x)} - 1 \\ &\quad - \langle f(t,x,u), \nabla v(t,x)\rangle e^{v(t,x+f(t,x,u))-v(t,x)}\Big]\nu(\mathrm{d}u). \end{aligned} \quad (43)$$

Proof. Following the line of [8]. To the reader's convenience, we give the detailed proof here. The proof of necessity. By (40),

$$\begin{aligned} \mathrm{d}v(t,X_t) &= \left[\frac{1}{2}|\gamma(t,X_t)|^2 + \int_{\mathbb{U}_0}\Big(\lambda(t,u)\log\lambda(t,u) + \big(1-\lambda(t,u)\big)\Big)\nu(\mathrm{d}u)\right]\mathrm{d}t \\ &\quad + \int_{\mathbb{U}_0} \log\lambda(t,u)\tilde{N}_\lambda(\mathrm{d}t,\mathrm{d}u) + \langle\gamma(t,X_t), \mathrm{d}B_t\rangle. \end{aligned} \quad (44)$$

It is clear from (44) that $v(t,X_t)$ is a càdlàg semimartingale with a predictable finite variation part. On the other hand, note that X_t satisfies Equation (38) and $v(t,x)$ is a $C^{1,2}$-function, by applying the Itô formula to the composition process $v(t,X_t)$, one could obtain the following

$$\begin{aligned} \mathrm{d}v(t,X_t) &= \frac{\partial}{\partial t}v(t,X_t)\mathrm{d}t + \langle b, \nabla v\rangle(t,X_t)\mathrm{d}t + \frac{1}{2}[Tr(\sigma\sigma^*)\nabla^2 v](t,X_t)\mathrm{d}t \\ &\quad + \int_{\mathbb{U}_0}\Big[v(t,X_{t-}+f(t,X_{t-},u)) - v(t,X_{t-}) \\ &\qquad - \langle f(t,X_{t-},u), \nabla v(t,X_{t-})\rangle\Big]\lambda(t,u)\nu(\mathrm{d}u)\mathrm{d}t \\ &\quad + \int_{\mathbb{U}_0}[v(t,X_{t-}+f(t,X_{t-},u)) - v(t,X_{t-})]\,\tilde{N}_\lambda(\mathrm{d}t,\mathrm{d}u) \\ &\quad + \langle(\sigma^*\nabla v)(t,X_t), \mathrm{d}B_t\rangle. \end{aligned} \quad (45)$$

Thus, (45) is another decomposition of the semimartingale $v(t,X_t)$. By uniqueness for decomposition of the semimartingale, it holds that for $t \in [0,T]$,

$$\begin{aligned} \gamma(t,X_t) &= (\sigma^*\nabla v)(t,X_t), \\ \log\lambda(t,u) &= v(t,X_{t-}+f(t,X_{t-},u)) - v(t,X_{t-}), \quad u \in \mathbb{U}_0, \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{2}\left|\gamma(t,X_t)\right|^2+\int_{\mathbb{U}_0}\Big(\lambda(t,u)\log\lambda(t,u)+\big(1-\lambda(t,u)\big)\Big)\nu(\mathrm{d}u)\\ =\ &\frac{\partial}{\partial t}v(t,X_t)+\langle b,\nabla v\rangle(t,X_t)+\frac{1}{2}[Tr(\sigma\sigma^*)\nabla^2 v](t,X_t)\\ &+\int_{\mathbb{U}_0}\Big[v(t,X_{t-}+f(t,X_{t-},u))-v(t,X_{t-})\\ &\qquad -\langle f(t,X_{t-},u),\nabla v(t,X_{t-})\rangle\Big]\lambda(t,u)\nu(\mathrm{d}u),\quad a.s.. \end{aligned}$$

Note that (t,X_t) runs through Λ, thus, we have that

$$\begin{aligned} \gamma(t,x) &= (\sigma^*\nabla v)(t,x), && (t,x)\in\Lambda, && (46)\\ \log\lambda(t,u) &= v(t,x+f(t,x,u))-v(t,x), && (t,x,u)\in\Lambda\times\mathbb{U}_0, && (47) \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{2}\left|\gamma(t,x)\right|^2+\int_{\mathbb{U}_0}\Big(\lambda(t,u)\log\lambda(t,u)+\big(1-\lambda(t,u)\big)\Big)\nu(\mathrm{d}u)\\ =\ &\frac{\partial}{\partial t}v(t,x)+\langle b,\nabla v\rangle(t,x)+\frac{1}{2}[Tr(\sigma\sigma^*)\nabla^2 v](t,x)\\ &+\int_{\mathbb{U}_0}\Big[v(t,x+f(t,x,u))-v(t,x)\\ &\qquad -\langle f(t,x,u),\nabla v(t,x)\rangle\Big]\lambda(t,u)\nu(\mathrm{d}u). \qquad (48) \end{aligned}$$

It is easy to see that (46) and (47) correspond to (41) and (42), respectively, which together with (48) further yields the PIDE (43).

Next, let us show sufficiency. Assume that there exists a $C^{1,2}$-function $v(t,x)$ satisfying (41), (42) and (43). For the composition process $v(t,X_t)$, the Itô formula admits us to get (45). Combining (41), (42) and (43) with (45), we have

$$\begin{aligned} \mathrm{d}v(t,X_t) &= \left[\frac{1}{2}\left|\gamma(t,X_t)\right|^2+\int_{\mathbb{U}_0}\Big(\big(\lambda(t,u)\log\lambda(t,u)\big)\lambda(t,u)+\big(1-\lambda(t,u)\big)\Big)\nu(\mathrm{d}u)\right]\mathrm{d}t\\ &\quad+\int_{\mathbb{U}_0}\log\lambda(t,u)\tilde{N}_\lambda(\mathrm{d}t,\mathrm{d}u)+\langle\sigma^{-1}(t,X_t)b(t,X_t),\mathrm{d}B_t\rangle\\ &= \langle\gamma,\mathrm{d}B_t\rangle+\frac{1}{2}\left|\sigma^{-1}(t,X_t)b(t,X_t)\right|^2\mathrm{d}t\\ &\quad+\int_{\mathbb{U}_0}\log\lambda(t,u)N_\lambda(\mathrm{d}t,\mathrm{d}u)+\int_{\mathbb{U}_0}(1-\lambda(t,u))\nu(\mathrm{d}u)\mathrm{d}t. \end{aligned}$$

The proof is completed. ☐

Acknowledgments The authors are grateful to Prof. Feng-Yu Wang and Dr Huijie Qiao for stimulating discussions.

References

[1] B. Hambly, M. Mariapragassam and Ch. Reisinger: A forward equation for barrier options under the Brunick&Shreve Markovian projection, arXiv:1411.3618 [q-fin.MF].

[2] C. Dellacherie and P. A. Meyer: *Probabilities and Potential B: Theory of Martingales*. North-Holland, Amsterdam/New York/Oxford, 1982.

[3] N. Ikeda and S. Watanabe: *Stochastic Differential Equations and Diffusion Processes*, 2nd ed., North-Holland/Kodanska, Amsterdam/Tokyo, 1989.

[4] J. Jacod and A.N. Shiryaev: *Limit Theorems for Stochastic Processes*. Springer-Verlag, Berlin, 1987.

[5] A. Matoussi, W. Sabbagh and C. Zhou: The obstacle problem for semilinear parabolic partial integro-differential equations, arXiv:1307.0875v3 [math.PR]

[6] K. R. Parthasarathy: *Probability Measures on Metric Spaces*. AMS Chelsea Publishing, 2005.

[7] P. E. Protter and K. Shimbo: No arbitrage and general semimartingales. (http://legacy.orie.cornell.edu/protter/WebPapers/nagirsanov8.pdf)

[8] H. Qiao: Exponential ergodicity for SDEs with jumps and non-Lipschitz coefficients, *J. Theor. Probab.*, 27 (2014) 137-152.

[9] H. Qiao and J. Duan: Nonlinear filtering of stochastic dynamical systems with Lévy noises, *Advances in Applied Probability* 47 (2015) 902-918.

[10] H. Qiao and J.-L. Wu: Characterising the path-independence of the Girsanov transformation for non-Lipschnitz SDEs with jumps, submitted.

[11] H. Qiao and X. Zhang: Homeomorphism flows for non-Lipschitz stochastic differential equations with jumps, *Stochastic Processes and their Applications*118 (2008) 2254-2268.

[12] R. Situ: *Theory of Stochastic Differential Equations with Jumps and Applications*. Series: Mathematical and Analytical Techniques with Applications to Engineering. Springer-Verlag, Berlin, Heidelberg, NewYork, 2005.

[13] A. Truman, F.-Y. Wang, J.-L. Wu, W. Yang: A link of stochastic differential equations to nonlinear parabolic equations, *SCIENCE CHINA Mathematics*, 55 (2012) 1971-1976.

[14] F.- Y. Wang, *Analysis for diffusion processes on Riemannian manifolds,* World Scientific, 2014.

[15] J.-L. Wu and W. Yang: On stochastic differential equations and a generalised Burgers equation, pp 425-435 in *Stochastic Analysis and Its Applications to Finance: Essays in Honor of Prof. Jia-An Yan (eds. T S Zhang, X Y Zhou)*, Interdisciplinary Mathematical Sciences, Vol. 13, World Scientific, Singapore, 2012.