Paper:
Wu , B. & Wu, J. (2018). Characterizing the path-indpendent property of the Girsanov density for degenerated stochastic differential equations. *Statistics & Probability Letters, 133*(2), 71-79.
http://dx.doi.org/https://doi.org/10.1016/j.spl.2017.10.005

CHARACTERIZING THE PATH-INDEPENDENT PROPERTY OF THE GIRSANOV DENSITY FOR DEGENERATED STOCHASTIC DIFFERENTIAL EQUATIONS

Abstract. In this paper, we derive a characterisation theorem for the path-independent property of the density of the Girsanov transformation for *degenerated* stochastic differential equations (SDEs), extending the characterisation theorem of [13] for the non-degenerated SDEs. We further extends our consideration to non-Lipschitz SDEs with jumps and with degenerated diffusion coefficients, which generalises the corresponding characterisation theorem established in [10].

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geqslant 0})$ be a filtered probability space. Let $d, m \in \mathbb{N}$ be fixed. We are concerned with the following SDE

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t, \quad t \geq 0 \tag{1}$$

where

$$b : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d, \quad \sigma : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^{d\otimes m}$$

$(W_t)_{t\geqslant 0}$ is an m-dimensional $\{\mathcal{F}_t\}_{t\geqslant 0}$-Brownian motion. Under standard usual conditions, e.g. the two coefficients b and σ satisfy linear growth and local Lipschitz conditions (for the second variable), there is a unique solution to the above SDE (1) for a given initial data X_0, see, e.g., [3].

The celebrated Girsanov theorem provides a very powerful tool to solve SDEs under the name of the *Girsanov transformation* or the *transformation of the drift*. We use $|\cdot|$ and $\langle\cdot,\cdot\rangle$ to denote the Euclidean norm and scalar product of vectors in $\mathbb{R}^m$ or $\mathbb{R}^d$, respectively. Let $\gamma : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^m$ be a measurable function such that the following exponential integrability along the paths of the solution $(X_t)_{t\geqslant 0}$ holds (also known as Novikov condition)

$$\mathbb{E}\left(\exp\left\{-\frac{1}{2}\int_0^t |\gamma(s, X_s)|^2\mathrm{d}s + \int_0^t \langle\gamma(s, X_s), \mathrm{d}W_s\rangle\right\}\right) < \infty, \quad t \geqslant 0. \tag{2}$$

Then, Girsanov theorem ([2, 3, 4]) says that for any arbitrarily fixed $T > 0$

$$\tilde{W}_t := W_t - \int_0^t \gamma(s, X_s)\mathrm{d}s, \quad t \in [0, T] \tag{3}$$

is an m-dimensional $\{\mathcal{F}_t\}_{t\in[0,T]}$-Brownian motion under the probability measure

$$Q_T := \exp\left\{-\frac{1}{2}\int_0^T |\gamma(s, X_s)|^2\mathrm{d}s + \int_0^T \langle\gamma(s, X_s), \mathrm{d}W_s\rangle\right\} \cdot \mathbb{P}. \tag{4}$$

AMS Subject Classification(2010): 60H10; 35Q53.

Keywords: degenerated stochastic differential equations (SDEs), Girsanov transformation, non-Lipschitz SDEs with jumps, semi-linear partial integro-differential equation of parabolic type.

*This work was partly supported by NSF of China (No. 11371099).

Moreover, the solution $(X_t)_{t\in[0,T]}$ fulfils the following SDE

$$\mathrm{d}X_t = [b(t,X_t) + \sigma(t,X_t)\gamma(t,X_t)]\mathrm{d}t + \sigma(t,X_t)\mathrm{d}\tilde{W}_t, \quad t \in [0,T]. \tag{5}$$

Now let us assume that (along the paths of the solution $(X_t)_{t\geqslant 0}$)

$$b(t,X_t) + \sigma(t,X_t)\gamma(t,X_t) = 0, \quad a.s.\ \forall\ t \geqslant 0. \tag{6}$$

Equivalently, $b \in Im(\sigma)$, where $Im(\sigma)$ is the imagine space of σ. Then

$$\mathrm{d}X_t = \sigma(t,X_t)\mathrm{d}\tilde{W}_t. \tag{7}$$

We are interested in the path-independent property for the exponent of the Girsanov density of Q_T for any fixed $T > 0$. That is, whether there exists a scalar function $v : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ such that

$$Z_t := \frac{1}{2}\int_0^t |\gamma(s,X_s)|^2\mathrm{d}s + \int_0^t \langle -\gamma(s,X_s), \mathrm{d}W_s\rangle = v(t,X_t) - v(0,X_0), \quad t \geq 0. \tag{8}$$

This problem arises from a number of studies in economics, finance as well as from stochastic mechanics, just mention a few, see [13, 15] (and references therein).

If $\sigma : [0,\infty)\times\mathbb{R}^d \to \mathbb{R}^{d\otimes d}$ (i.e., taking $m = d$) is non-degenerate, that is, the $d\times d$-matrix $\sigma(t,x)$ is invertible for any $(t,x) \in [0,\infty)\times\mathbb{R}^d$, a characterisation of the path-independent property has been obtained in [13].

Throughout the article, we assume that σ, b satisfy the Hörmander's condition, i.e. if the associated Lie algebra spans, in any point, the whole $\mathbb{R}^d$. In particular, we allow σ to be degenerate.

Let $\Lambda := \{(t,X_t) \in [0,\infty) \times \mathbb{R}^d : t \geqslant 0, \omega \in \Omega\} \subset [0,\infty) \times \mathbb{R}^d$, the support of the solution. In particular, we have $\Lambda = [0,\infty) \times \mathbb{R}^d$ if b and σ satisfy the Hörmander's conditions. Then by using Itô's formula to $v(t,X_t)$ viewing as the composition of v : $[0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ with the semimartingale $(X_t)_{t\geqslant 0}$, the utilising the uniqueness of Doob-Meyer decomposition for continuous semimartingales, we can derive for any $t \geqslant 0$ the following

$$\gamma(t,X_t) = -\sigma^*(t,X_t)\nabla v(t,X_t) \tag{9}$$

and

$$\frac{1}{2}|\gamma(t,X_t)|^2 = \frac{\partial v}{\partial t}(t,X_t) + \langle\nabla v(t,X_t), b(t,X_t)\rangle + \frac{1}{2}Tr[(\sigma\sigma^*(t,X_t)\nabla^2 v(t,X_t))] \tag{10}$$

where $\sigma^*(t,x)$ stands for the transposed matrix of $\sigma(t,x)$, ∇ and ∇^2 stand for the gradient and Hessian operators with respect to the second variable, respectively. Moreover, we get

$$\begin{cases} \gamma(t,x) = -\sigma^*(t,x)\nabla v(t,x) \quad (I) \\ \frac{1}{2}|\gamma(t,x)|^2 = \frac{\partial v}{\partial t}(t,x) + \langle\nabla v(t,x), b(t,x)\rangle + \frac{1}{2}Tr[(\sigma\sigma^*(t,x)\nabla^2 v(t,x))] \quad (II) \end{cases} \tag{11}$$

for any $(t,x) \in [0,\infty) \times \Lambda$. Putting (I) into (II) and (6) yield the following nonlinear parabolic PDE of the (reversible) HJB type

$$\begin{cases} \frac{\partial v}{\partial t}(t,x) = -\frac{1}{2}\{Tr[(\sigma\sigma^*(t,x)\nabla^2 v(t,x))] + |\sigma^*(t,x)\nabla v(t,x)|^2\}, \\ \sigma(t,x)\sigma^*(t,x)\nabla v(t,x) = b(t,x), \quad (t,x) \in [0,\infty) \times \Lambda. \end{cases} \tag{12}$$

Remark 1.1. *All above derivations are reciprocal, namely, that gives a characterisation of path-independence property.*

Theorem 1.2. *Assume that $\gamma : [0,\infty)\times\mathbb{R}^d \to \mathbb{R}^d$ is a function satisfying (6). Then there exists a scalar function $v \in C^{1,2}((0,\infty)\times\mathbb{R}^d \to \mathbb{R}^d)$ such that*

$$\frac{1}{2}\int_0^t |\gamma(s,X_s)|^2 \mathrm{d}s + \int_0^t \langle -\gamma(s,X_s), \mathrm{d}W_s\rangle = v(t,X_t) - v(0,X_0) \tag{13}$$

if and only if (12) holds.

Proof. By the previous argument, we only show the sufficiency. Since

$$v \in C^{1,2}((0,\infty)\times\mathbb{R}^d \to \mathbb{R}^d),$$

we know that $v(t,X_t)$ is a continuous semimartinagle of X_t. Thus we ahve

$$\begin{aligned}\mathrm{d}v(t,X_t) = &\left\{\frac{\partial v}{\partial t}(t,X_t) + \langle \nabla v(t,X_t), b(t,X_t)\rangle + \frac{1}{2}Tr[(\sigma\sigma^*(t,X_t)\nabla^2 v(t,X_t))]\right\}\mathrm{d}t \\ &+ \langle \sigma^*(t,X_t)\nabla v(t,X_t), \mathrm{d}W_t\rangle.\end{aligned} \tag{14}$$

Combining this with (12), we get

$$\begin{aligned}&\mathrm{d}v(t,X_t) \\ &= \left\{-\frac{1}{2}|\sigma^*(t,X_t)\nabla v(t,X_t)|^2 + \langle \nabla v(t,X_t), b(t,X_t)\rangle\right\} + \langle \sigma^*(t,X_t)\nabla v(t,X_t), \mathrm{d}W_t\rangle \\ &= \frac{1}{2}|\gamma(t,X_t)|^2\mathrm{d}t + \langle -\gamma(t,X_t), \mathrm{d}W_t\rangle.\end{aligned} \tag{15}$$

This implies (13). □

Corollary 1.3. *Assume that $\gamma : [0,\infty)\times\mathbb{R}^d \to \mathbb{R}^d$ is a function satisfying (6). Then there exist a function $f \in C^2(\mathbb{R}\to\mathbb{R})$ and a scalar function $v \in C^{1,2}((0,\infty)\times\mathbb{R}^d \to \mathbb{R}^d)$ such that*

$$\frac{1}{2}\int_0^t |\gamma(s,X_s)|^2 \mathrm{d}s + \int_0^t \langle -\gamma(s,X_s), \mathrm{d}W_s\rangle = f(v(t,X_t)) - f(v(0,X_0)) \tag{16}$$

if and only if

$$\left\{\begin{aligned} &f'(v)(t,x)\tfrac{\partial v}{\partial t}(t,x) = -\tfrac{1}{2}\big\{f'(v)(t,x)Tr[(\sigma\sigma^*(t,x)\nabla^2 v(t,x))] \\ &\qquad\qquad + f''(v)(t,x)|\sigma^*(t,x)\nabla v(t,x)|^2 + |f'(v)(t,x)|^2|\sigma^*(t,x)\nabla v(t,x)|^2\big\}, \\ &b(t,x) = f'(v)(t,x)\sigma(t,x)\sigma^*(t,x)\nabla v(t,x), \quad (t,x)\in[0,\infty)\times\Lambda.\end{aligned}\right. \tag{17}$$

Proof. According to Theorem 1.2, we know that (16) is equivalent to

$$\left\{\begin{aligned} &\tfrac{\partial f(v)}{\partial t}(t,x) = -\tfrac{1}{2}\{Tr[(\sigma\sigma^*(t,x)\nabla^2 f(v)(t,x))] + |\sigma^*(t,x)\nabla f(v)(t,x)|^2\}, \\ &\sigma(t,x)\sigma^*(t,x)\nabla f(v)(t,x) = b(t,x), \quad (t,x)\in[0,\infty)\times\Lambda.\end{aligned}\right.$$

Since

$$\begin{aligned}&Tr[(\sigma\sigma^*(t,x)\nabla^2 f(v)(t,x))] = Tr[(\sigma\sigma^*(t,x)\nabla(f'(v)(t,x)\nabla v(t,x))] \\ &= f'(v)(t,x)Tr[(\sigma\sigma^*(t,x)\nabla^2 v(t,x))] + f''(v)(t,x)|\sigma^*(t,x)\nabla v(t,x)|^2\end{aligned}$$

and

$$\sigma^*(t,x)\nabla f(v)(t,x) = f'(v)\sigma^*(t,x)\nabla v(t,x).$$

Combining all the above equalities, we conclude that (16) is equivalent to (17).

□

Example 1.4. *Under the conditions of Corollary 1.3, we have the following examples of the function f*

(a) If $f(x) = x$, then

$$\frac{1}{2}\int_0^t |\gamma(s,X_s)|^2 \mathrm{d}s + \int_0^t \langle -\gamma(s,X_s), \mathrm{d}W_s\rangle = v(t,X_t) - v(0,X_0) \tag{18}$$

if and only if

$$\sigma(t,x)\sigma^*(t,x)\nabla v(t,x) = b(t,x), \quad (t,x) \in [0,\infty)\times\Lambda \tag{19}$$

and v satisfies the following time-reversed KPZ type equation,

$$\frac{\partial v}{\partial t}(t,x) = -\frac{1}{2}\{Tr[(\sigma\sigma^*(t,x)\nabla^2 v(t,x))] + |\sigma^*(t,x)\nabla v(t,x)|^2\}, \quad (t,x)\in[0,\infty)\times\Lambda. \tag{20}$$

In particular, if σ is invertible, this covers the result obtained in [13].

(b) If $f(x) = \log|x|$, for $x \neq 0$, then

$$\frac{1}{2}\int_0^t |\gamma(s,X_s)|^2 \mathrm{d}s + \int_0^t \langle -\gamma(s,X_s), \mathrm{d}W_s\rangle = \log\frac{v(t,X_t)}{v(0,X_0)} \tag{21}$$

if and only if

$$\sigma(t,x)\sigma^*(t,x)\nabla v(t,x) = v(t,x)b(t,x), \quad (t,x)\in[0,\infty)\times\Lambda, \tag{22}$$

and v satisfies the following time-reversed heat kernel type equation,

$$\frac{\partial v}{\partial t}(t,x) = -\frac{1}{2}Tr[(\sigma\sigma^*(t,x)\nabla^2 v(t,x))], \quad (t,x)\in[0,\infty)\times\Lambda. \tag{23}$$

In particular, if $\sigma = Id$, then we have

$$\frac{1}{2}\int_0^t |b(s,X_s)|^2 \mathrm{d}s + \int_0^t \langle b(s,X_s), \mathrm{d}W_s\rangle = \log\left|\frac{v(t,X_t)}{v(0,X_0)}\right| \tag{24}$$

if and only if

$$\nabla v(t,x) = v(t,x)b(t,x), \quad (t,x)\in[0,\infty)\times\mathbb{R}^d, \tag{25}$$

and v satisfies the standard heat kernel equation,

$$\frac{\partial v}{\partial t}(t,x) = -\frac{1}{2}\Delta v(t,x), \quad (t,x)\in[0,\infty)\times\mathbb{R}^d. \tag{26}$$

(c) If $f(x) = x^{2k+1}, k\in\mathbb{N}\cup\{0\}$, or $f(x) = x^{2k+1}, k\in\mathbb{Z}$ for $x\neq 0$, then

$$\frac{1}{2}\int_0^t |\gamma(s,X_s)|^2 \mathrm{d}s + \int_0^t \langle -\gamma(s,X_s), \mathrm{d}W_s\rangle = v^{2k+1}(t,X_t) - v^{2k+1}(0,X_0) \tag{27}$$

if and only if

$$(2k+1)v^{2k}(t,x)\sigma(t,x)\sigma^*(t,x)\nabla v(t,x) = b(t,x), \quad (t,x)\in[0,\infty)\times\Lambda, \tag{28}$$

and v satisfies the following time-reversed HJB equation,

$$\begin{aligned}\frac{\partial v}{\partial t}(t,x) = -\frac{1}{2}\Big\{&Tr[(\sigma\sigma^*(t,x)\nabla^2 v(t,x))] \\ &+ \frac{(2k+1)v^{2k+1}(t,x)+2k}{v(t,x)}|\sigma^*(t,x)\nabla v(t,x)|^2\Big\}.\end{aligned} \tag{29}$$

(d) If $f(x)=\tan(x), x\in(-\frac{\pi}{2},\frac{\pi}{2})$, then

$$\frac{1}{2}\int_0^t|\gamma(s,X_s)|^2\mathrm{d}s+\int_0^t\langle-\gamma(s,X_s),\mathrm{d}W_s\rangle=\tan(v(t,X_t))-tan(v(0,X_0)) \tag{30}$$

if and only if

$$\sigma(t,x)\sigma^*(t,x)\nabla v(t,x)=\cos^2(v(t,x))b(t,x),\quad (t,x)\in[0,\infty)\times\Lambda, \tag{31}$$

and v satisfies the following time-reversed HJB equation,

$$\begin{aligned}\frac{\partial v}{\partial t}(t,x)=-\frac{1}{2}\Big\{&Tr[(\sigma\sigma^*(t,x)\nabla^2v(t,x))]\\&+\frac{[\cos(v(t,x))+\sin(v(t,x))]^2}{\cos^2(v(t,x))}|\sigma^*(t,x)\nabla v(t,x)|^2\Big\}.\end{aligned} \tag{32}$$

Proof. It is obvious for (a). We only prove (b)((c) may be similarly handed). By Corollary 1.3, we know that (21) is equivalent to

$$\left\{\begin{array}{l}\frac{1}{v(t,x)}\frac{\partial v}{\partial t}(t,x)=-\frac{1}{2}\{\frac{1}{v(t,x)}Tr[(\sigma\sigma^*(t,x)\nabla^2v(t,x))]\\ \qquad\qquad -\frac{1}{v^2(t,x)}|\sigma^*(t,x)\nabla v(t,x)|^2+\frac{1}{v^v(t,x)}|\sigma^*(t,x)\nabla v(t,x)|^2\}\\ \sigma(t,x)\sigma^*(t,x)\nabla v(t,x)=v(t,x)b(t,x),\quad (t,x)\in[0,\infty)\times\Lambda\end{array}\right. \tag{33}$$

which are just (22) and (23), respectively. □

Example 1.5. [HJE] *Let $v(t,x)$ be the solution to the following Burgers equation on $[0,\infty)\times\mathbb{R}$:*

$$\frac{\partial}{\partial t}v(t,x)=-\frac{1}{2}\triangle_x v(t,x)-\frac{1}{2}\left|\nabla_x v(t,x)\right|^2.$$

Consider the stochastic differential equation

$$\mathrm{d}X_t=\mathrm{d}W_t+\nabla_x v(t,X_t)\mathrm{d}t.$$

By Theorem 1.2, it is easy to check that $v(t,x)$ satisfies the equation (13), were $\gamma(t,x):=-\nabla_x v(t,x)$.

The following two examples come from [14].

Example 1.6. [Gruschin operator] *Let $b(t,z)=(-xt,-x^kyt)^T, z=(x,y)\in\mathbb{R}^2, t\geqslant 0$ and $\sigma(t,z)$ be given by*

$$\sigma(t,z)=\begin{pmatrix}1&0\\0&x^k\end{pmatrix},\quad k\in\mathbb{N}, z=(x,y)\in\mathbb{R}^2, t\geqslant 0. \tag{34}$$

Then $b\in Im(\sigma)$ and the Hörmander's condition holds for $\mathcal{H}=\{\frac{\partial}{\partial x},x^k\frac{\partial}{\partial y}\}$ with commutators up to order k. Define the subelliptic diffusion operator

$$L=X^2+Y^2+b(t,\cdot).$$

Let $\gamma(t,z)=(xt,yt)^T$ and X_s be the associated L-diffusion process, then $b(t,z)=-\sigma(t,z)\gamma(t,z)$. Assume that $v\in C^{1,2}((0,\infty)\times\mathbb{R}^2\to\mathbb{R}^2)$ fulfills the following

$$\frac{1}{2}\int_0^t|\gamma(s,X_s)|^2\mathrm{d}s+\int_0^t\langle-\gamma(s,X_s),\mathrm{d}W_s\rangle=v(t,X_t)-v(0,X_0). \tag{35}$$

Then, by Theorem 1.2, we know that v satisfies the equation (12).

Example 1.7. [Kohn operator] *Consider the three-dimensional Heisenberg group realized as $\mathbb{R}^3$ equipped with the group multiplication*

$$(x,y,z)(x',y',z') := (x+x', y+y', z+z'+(xy'-x'y)/2),$$

which is a Lie group with left-invariant orthonormal frame $\{X, Y, Z\}$, where

$$X = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z},\ Y = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z},\ Z = [X, Y] = \frac{\partial}{\partial z}$$

Then the Kohn-Laplacian is $\Delta_H := X^2 + Y^2$. Let

$$b(t,u) = (xt, yt, \frac{z(x-y)}{2}t), \quad u = (x,y,z) \in \mathbb{R}^3, t \geqslant 0$$

and $\sigma(t,z)$ be given by

$$\sigma(t,u) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{y}{2} & \frac{x}{2} & 0 \end{pmatrix}, \quad u = (x,y,z) \in \mathbb{R}^3, t \geqslant 0. \tag{36}$$

Define the subelliptic diffusion operator

$$L = X^2 + Y^2 + b(t,\cdot).$$

Let $\gamma(t,z) = (-xt, -yt, -zt)^*$ and X_s be the associated L-diffusion process, then $b \in Im(\sigma)$ and $b(t,z) = -\sigma(t,z)\gamma(t,z)$. Then, the Hörmander's condition holds for $\mathcal{H} = \{\frac{\partial}{\partial x}, x^k\frac{\partial}{\partial y}\}$ with commutators up to order k. Assume that $v \in C^{1,2}((0,\infty)\times\mathbb{R}^2 \to \mathbb{R}^2)$ fulfills the following

$$\frac{1}{2}\int_0^t |\gamma(s,X_s)|^2 \mathrm{d}s + \int_0^t \langle -\gamma(s,X_s), \mathrm{d}W_s\rangle = v(t,X_t) - v(0,X_0) \tag{37}$$

Then, by theorem 1.2, we know that v satisfies the equation (12).

2. Non-Lipschitz SDEs with jumps

2.1. **The characterisation theorem for SDEs with continuous diffusions on $\mathbb{R}^d$.**

Let $(\mathbb{U}, \|\cdot\|_{\mathbb{U}})$ be a finite dimensional normed space endowed with its Borel σ-algebra $\mathscr{U}$. Let ν be a σ-finite measure defined on $(\mathbb{U}, \mathscr{U})$. Let us fix $\mathbb{U}_0 \in \mathscr{U}$ with $\nu(\mathbb{U}\setminus\mathbb{U}_0) < \infty$ and $\int_{\mathbb{U}_0} \|u\|_{\mathbb{U}}^2\,\nu(\mathrm{d}u) < \infty$. Furthermore, let $\lambda : [0,\infty)\times\mathbb{U} \to (0,1]$ be a given measurable function. Then, following e.g. [3, 4], there exists a non-negative integer valued $(\mathcal{F}_t)_{t\geqslant 0}$-Poisson random measure $N_\lambda(\mathrm{d}t, \mathrm{d}u)$ on the given filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F}_t)_{t\geqslant 0})$ with intensity $\mathbb{E}(N_\lambda(\mathrm{d}t,\mathrm{d}u)) = \lambda(t,u)\mathrm{d}t\nu(\mathrm{d}u)$. Set

$$\tilde{N}_\lambda(\mathrm{d}t,\mathrm{d}u) := N_\lambda(\mathrm{d}t,\mathrm{d}u) - \lambda(t,u)\mathrm{d}t\nu(\mathrm{d}u)$$

that is, $\tilde{N}_\lambda(\mathrm{d}t,\mathrm{d}u)$ stands for the compensated $(\mathcal{F}_t)_{t\geqslant 0}$-predictable martingale measure of $N_\lambda(\mathrm{d}t,\mathrm{d}u)$.

We are concerned with the following SDE on $\mathbb{R}^d$

$$\begin{cases} \mathrm{d}X_t = b(t,X_t)\mathrm{d}t + \sigma(t,X_t)\mathrm{d}B_t + \int_{\mathbb{U}_0} f(t,X_{t-},u)\tilde{N}_\lambda(\mathrm{d}t,\mathrm{d}u), & t \in (0,T], \\ X_0 = x_0 \in \mathbb{R}^d, \end{cases} \tag{38}$$

for any given $T > 0$, where b, σ are Borel measurable as given in the previous section, $(B_t)_{t\geqslant 0}$ is an m-dimensional $\{\mathcal{F}_t\}_{t\geqslant 0}$-Brownian motion, $f : [0,T]\times\mathbb{R}^d\times\mathbb{U}_0 \mapsto \mathbb{R}^d$ is Borel measurable, and $\tilde{N}_\lambda$ is the compensated $(\mathcal{F}_t)_{t\geqslant 0}$-predictable martingale measure of an induced $\{\mathcal{F}_t\}_{t\geqslant 0}$-Poisson random measure given above which is independent of $(B_t)_{t\geqslant 0}$. This equation arises in nonlinear filtering and has been considered recently in [11, 8, 9] (see also the monograph [12]).

The characterisation theorem for path-independent property of Girsanov density for the above equation with non-degenerated σ was established in [10]. More precisely, under the following conditions

($\mathbf{H}_1$) There exists $\lambda_0 \in \mathbb{R}$ such that for all $x, y \in \mathbb{R}^d$ and $t \in [0, T]$

$$2\langle x - y, b(t, x) - b(t, y)\rangle + \|\sigma(t, x) - \sigma(t, y)\|^2 \leqslant \lambda_0 |x - y|^2 \kappa(|x - y|),$$

where κ is a positive continuous function, bounded on $[1, \infty)$ and satisfying

$$\lim_{x\downarrow 0} \frac{\kappa(x)}{\log x^{-1}} = \delta < \infty.$$

($\mathbf{H}_2$) There exists $\lambda_1 > 0$ such that for all $x \in \mathbb{R}^d$ and $t \in [0, T]$

$$|b(t, x)|^2 + \|\sigma(t, x)\|^2 \leqslant \lambda_1 (1 + |x|)^2.$$

($\mathbf{H}_3$) $b(t, x)$ is continuous in x and there exists $\lambda_2 > 0$ such that

$$\langle \sigma(t, x)h, h\rangle \geqslant \sqrt{\lambda_2}|h|^2, \qquad t \in [0, T], \quad x, h \in \mathbb{R}^d. \tag{39}$$

($\mathbf{H}_f$) For all $x, y \in \mathbb{R}^d$ and $t \in [0, T]$,

$$\int_{\mathbb{U}_0} \left| f(t, x, u) - f(t, y, u) \right|^2 \nu(\mathrm{d}u) \leqslant 2|\lambda_0||x - y|^2 \kappa(|x - y|)$$

and for $q = 2$ and 4

$$\int_{\mathbb{U}_0} |f(t, x, u)|^q \, \nu(\mathrm{d}u) \leqslant \lambda_1 (1 + |x|)^q.$$

Qiao and Wu in [10] proved a characterisation theorem, where a partial integer-differential equation (PIDE) as the main characterizing equation was derived. We notice that the assumption (H3) on the diffusion coefficient σ is too strong. Here we aim to relax this condition. First of all, we let σ to be $d \times m$-matrix-valued for $d, m \in \mathbb{N}$, i.e., σ is in general not square matrix-valued. And σ, b satisfy the Hörmander's condition.

Let $\gamma : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^m$ be a measurable function such that the following condition **($\mathbf{H}_{\gamma,\lambda}$)** holds

$$\mathbb{E}\Big[\exp\Big\{\frac{1}{2}\int_0^T |\gamma(s, X_s)|^2 \, \mathrm{d}s + \int_0^T \int_{\mathbb{U}_0} \left(\frac{1 - \lambda(s, u)}{\lambda(s, u)}\right)^2 \lambda(s, u)\nu(\mathrm{d}u)\mathrm{d}s\Big\}\Big] < \infty.$$

Set

$$\begin{aligned}
\Gamma_t : &= \exp\Big\{ - \int_0^t \langle \gamma(s, X_s), \mathrm{d}B_s\rangle - \frac{1}{2}\int_0^t |\gamma(s, X_s)|^2 \, \mathrm{d}s \\
&\qquad - \int_0^t \int_{\mathbb{U}_0} \log \lambda(s, u) N_\lambda(\mathrm{d}s, \mathrm{d}u) - \int_0^t \int_{\mathbb{U}_0} (1 - \lambda(s, u))\nu(\mathrm{d}u)\mathrm{d}s \Big\}, \\
M_t : &= - \int_0^t \langle \gamma(s, X_s), \mathrm{d}B_s\rangle + \int_0^t \int_{\mathbb{U}_0} \frac{1 - \lambda(s, u)}{\lambda(s, u)} \tilde{N}_\lambda(\mathrm{d}s, \mathrm{d}u),
\end{aligned}$$

and then (Γ_t) is the Doléans-Dade exponential of (M_t), see e.g., [2].

Under ($\mathbf{H}_1$), ($\mathbf{H}_2$) and **($\mathbf{H}_f$)**, it is well known that there exists a unique strong solution to Eq.(38) (cf. [12, Theorem 170, p.140]). This solution will be denoted by X_t. In the following, we define the support of a random vector ([6]) and then present a result about the support of X_t under the above assumptions.

Definition 2.1. *The support of a random vector Y is defined as*

$$\text{supp}(Y) := \{x \in \mathbb{R}^d | (\mathbb{P} \circ Y^{-1})(B(x,r)) > 0, \text{ for all } r > 0\}$$

where $B(x,r) := \{y \in \mathbb{R}^d | |y - x| < r\}$, the open ball centered at x with radius r.

Under $(\mathbf{H}_{\gamma,\lambda})$, (M_t) is a locally square integrable martingale. Moreover, $M_t - M_{t-} > -1$ a.s. and

$$\begin{aligned}
& \mathbb{E}\Big[\exp\Big\{\frac{1}{2} < M^c, M^c >_T + < M^d, M^d >_T\Big\}\Big] \\
= & \mathbb{E}\Big[\exp\Big\{\frac{1}{2}\int_0^T |\gamma(s,X_s)|^2 \, \mathrm{d}s + \int_0^T \int_{\mathbb{U}_0} \Big(\frac{1-\lambda(s,u)}{\lambda(s,u)}\Big)^2 \lambda(s,u)\nu(\mathrm{d}u)\mathrm{d}s\Big\}\Big] < \infty,
\end{aligned}$$

where M^c and M^d are continuous and purely discontinuous martingale parts of (M_t), respectively. Thus, it follows from [7, Theorem 6] that (Λ_t) is an exponential martingale. Define a measure $\tilde{\mathbb{P}}$ via

$$\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = \Lambda_T.$$

By the Girsanov theorem for Brownian motions and random measures, one can obtain that under the measure $\tilde{\mathbb{P}}$ the system (38) is transformed into the following

$$\mathrm{d}X_t = [b(t,X_t) + \sigma(t,X_t)\gamma(t,X_t)]\mathrm{d}t + \sigma(t,X_t)\mathrm{d}\tilde{B}_t + \int_{\mathbb{U}_0} f(t,X_{t-},u)\tilde{N}(\mathrm{d}t,\mathrm{d}u),$$

Now let us assume that (along th paths of $(X_t)_{t\geqslant 0}$)

$$b(t,X_t) + \sigma(t,X_t)\gamma(t,X_t) = 0.$$

Then we get

$$\mathrm{d}X_t = \sigma(t,X_t)\mathrm{d}\tilde{B}_t + \int_{\mathbb{U}_0} f(t,X_{t-},u)\tilde{N}(\mathrm{d}t,\mathrm{d}u),$$

where

$$\tilde{B}_t := B_t + \int_0^t \gamma(s,X_s)\mathrm{d}s, \quad \tilde{N}(\mathrm{d}t,\mathrm{d}u) := N_\lambda(\mathrm{d}t,\mathrm{d}u) - \mathrm{d}t\nu(\mathrm{d}u).$$

Next, we set

$$\begin{aligned}
Y_t \;:=\; & -\log\Gamma_t = \int_0^t \langle\gamma(s,X_s), \mathrm{d}B_s\rangle + \frac{1}{2}\int_0^t |\gamma(s,X_s)|^2 \, \mathrm{d}s \\
& + \int_0^t \int_{\mathbb{U}_0} \log\lambda(s,u) N_\lambda(\mathrm{d}s,\mathrm{d}u) + \int_0^t \int_{\mathbb{U}_0} (1-\lambda(s,u))\nu(\mathrm{d}u)\mathrm{d}s.
\end{aligned}$$

Clearly, (Y_t) is a one-dimensional stochastic process with the following stochastic differential form

$$\begin{aligned}
\mathrm{d}Y_t \;=\; & \langle\gamma(t,X_t), \mathrm{d}B_t\rangle + \frac{1}{2}|\gamma(t,X_t)|^2 \, \mathrm{d}t \\
& + \int_{\mathbb{U}_0} \log\lambda(t,u) N_\lambda(\mathrm{d}t,\mathrm{d}u) + \int_{\mathbb{U}_0} (1-\lambda(t,u))\nu(\mathrm{d}u)\mathrm{d}t.
\end{aligned}$$

Let $\Lambda := \text{supp}((t,X_t), t \geqslant 0)$. Then we have the following.

Theorem 2.2. *Let $v : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ be a scalar function which is C^1 with respect to the first variable and C^2 with respect to the second variable. Then*

$$\begin{aligned} v(t,X_t) &= v(0,x_0) + \int_0^t \langle -\gamma(s,X_s), \mathrm{d}B_s\rangle + \frac{1}{2}\int_0^t |\gamma(s,X_s)|^2\,\mathrm{d}s \\ &\quad + \int_0^t\int_{\mathbb{U}_0} \log\lambda(s,u) N_\lambda(\mathrm{d}s,\mathrm{d}u) + \int_0^t\int_{\mathbb{U}_0}(1-\lambda(s,u))\nu(\mathrm{d}u)\mathrm{d}s, \quad (40) \end{aligned}$$

equivalently,

$$Y_t = v(t,X_t) - v(0,x_0), \quad t\in[0,T]$$

holds if and only if

$$\begin{aligned} b(t,x) &= (\sigma\sigma^*\nabla v)(t,x), \qquad (t,x)\in\Lambda, \quad (41) \\ \lambda(t,u) &= \exp\{v(t,x+f(t,x,u)) - v(t,x)\}, \quad (t,x,u)\in\Lambda\times\mathbb{U}_0, \quad (42) \end{aligned}$$

and v satisfies the following time-reversed partial integro-differential equation (PIDE),

$$\begin{aligned} \frac{\partial}{\partial t}v(t,x) &= -\frac{1}{2}[Tr(\sigma\sigma^*)\nabla^2 v](t,x) - \frac{1}{2}|\sigma^*\nabla v|^2(t,x) - \int_{\mathbb{U}_0}\Big[e^{v(t,x+f(t,x,u))-v(t,x)} - 1 \\ &\quad -\langle f(t,x,u),\nabla v(t,x)\rangle e^{v(t,x+f(t,x,u))-v(t,x)}\Big]\nu(\mathrm{d}u). \quad (43) \end{aligned}$$

Proof. Following the line of [8]. To the reader's convenience, we give the detailed proof here.

(a) **Necessity:** By (40),

$$\begin{aligned} \mathrm{d}v(t,X_t) &= \left[\frac{1}{2}|\gamma(t,X_t)|^2 + \int_{\mathbb{U}_0}\Big(\lambda(t,u)\log\lambda(t,u) + \big(1-\lambda(t,u)\big)\Big)\nu(\mathrm{d}u)\right]\mathrm{d}t \\ &\quad + \int_{\mathbb{U}_0}\log\lambda(t,u)\tilde{N}_\lambda(\mathrm{d}t,\mathrm{d}u) + \langle\gamma(t,X_t),\mathrm{d}B_t\rangle. \quad (44) \end{aligned}$$

It is clear from (44) that $v(t,X_t)$ is a càdlàg semimartingale with a predictable finite variation part. On the other hand, note that X_t satisfies Equation (38) and $v(t,x)$ is a $C^{1,2}$-function, by applying the Itô formula to the composition process $v(t,X_t)$, one could obtain the following

$$\begin{aligned} \mathrm{d}v(t,X_t) &= \frac{\partial}{\partial t}v(t,X_t)\mathrm{d}t + \langle b,\nabla v\rangle(t,X_t)\mathrm{d}t + \frac{1}{2}[Tr(\sigma\sigma^*)\nabla^2 v](t,X_t)\mathrm{d}t \\ &\quad + \int_{\mathbb{U}_0}\Big[v(t,X_{t-}+f(t,X_{t-},u)) - v(t,X_{t-}) \\ &\qquad -\langle f(t,X_{t-},u),\nabla v(t,X_{t-})\rangle\Big]\lambda(t,u)\nu(\mathrm{d}u)\mathrm{d}t \\ &\quad + \int_{\mathbb{U}_0}[v(t,X_{t-}+f(t,X_{t-},u)) - v(t,X_{t-})]\,\tilde{N}_\lambda(\mathrm{d}t,\mathrm{d}u) \\ &\quad + \langle(\sigma^*\nabla v)(t,X_t),\mathrm{d}B_t\rangle. \quad (45) \end{aligned}$$

Thus, (45) is another decomposition of the semimartingale $v(t,X_t)$. By uniqueness for decomposition of the semimartingale, it holds that for $t\in[0,T]$,

$$\begin{aligned} \gamma(t,X_t) &= (\sigma^*\nabla v)(t,X_t), \\ \log\lambda(t,u) &= v(t,X_{t-}+f(t,X_{t-},u)) - v(t,X_{t-}), \quad u\in\mathbb{U}_0, \end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2}|\gamma(t,X_t)|^2+\int_{\mathbb{U}_0}\Big(\lambda(t,u)\log\lambda(t,u)+\big(1-\lambda(t,u)\big)\Big)\nu(\mathrm{d}u)\\
=\ &\frac{\partial}{\partial t}v(t,X_t)+\langle b,\nabla v\rangle(t,X_t)+\frac{1}{2}[Tr(\sigma\sigma^*)\nabla^2 v](t,X_t)\\
&+\int_{\mathbb{U}_0}\Big[v(t,X_{t-}+f(t,X_{t-},u))-v(t,X_{t-})\\
&\quad-\langle f(t,X_{t-},u),\nabla v(t,X_{t-})\rangle\Big]\lambda(t,u)\nu(\mathrm{d}u),\quad a.s..
\end{aligned}$$

Note that (t,X_t) runs through Λ, thus, we have that

$$\gamma(t,x)=(\sigma^*\nabla v)(t,x),\qquad (t,x)\in\Lambda, \tag{46}$$

$$\log\lambda(t,u)=v(t,x+f(t,x,u))-v(t,x),\quad (t,x,u)\in\Lambda\times\mathbb{U}_0, \tag{47}$$

and

$$\begin{aligned}
&\frac{1}{2}|\gamma(t,x)|^2+\int_{\mathbb{U}_0}\Big(\lambda(t,u)\log\lambda(t,u)+\big(1-\lambda(t,u)\big)\Big)\nu(\mathrm{d}u)\\
=\ &\frac{\partial}{\partial t}v(t,x)+\langle b,\nabla v\rangle(t,x)+\frac{1}{2}[Tr(\sigma\sigma^*)\nabla^2 v](t,x)\\
&+\int_{\mathbb{U}_0}\Big[v(t,x+f(t,x,u))-v(t,x)\\
&\quad-\langle f(t,x,u),\nabla v(t,x)\rangle\Big]\lambda(t,u)\nu(\mathrm{d}u).
\end{aligned}\tag{48}$$

It is easy to see that (46) and (47) correspond to (41) and (42), respectively, which together with (48) further yields the PIDE (43).

(b) **Sufficiency:** Assume that there exists a $C^{1,2}$-function $v(t,x)$ satisfying (41), (42) and (43). For the composition process $v(t,X_t)$, the Itô formula admits us to get (45). Combining (41), (42) and (43) with (45), we have

$$\begin{aligned}
\mathrm{d}v(t,X_t)\ =\ &\Big[\frac{1}{2}|\gamma(t,X_t)|^2+\int_{\mathbb{U}_0}\Big(\big(\lambda(t,u)\log\lambda(t,u)\big)\lambda(t,u)+\big(1-\lambda(t,u)\big)\Big)\nu(\mathrm{d}u)\Big]\mathrm{d}t\\
&+\int_{\mathbb{U}_0}\log\lambda(t,u)\tilde{N}_\lambda(\mathrm{d}t,\mathrm{d}u)+\langle\sigma^{-1}(t,X_t)b(t,X_t),\mathrm{d}B_t\rangle\\
=\ &\langle\gamma,\mathrm{d}B_t\rangle+\frac{1}{2}\left|\sigma^{-1}(t,X_t)b(t,X_t)\right|^2\mathrm{d}t\\
&+\int_{\mathbb{U}_0}\log\lambda(t,u)N_\lambda(\mathrm{d}t,\mathrm{d}u)+\int_{\mathbb{U}_0}(1-\lambda(t,u))\nu(\mathrm{d}u)\mathrm{d}t.
\end{aligned}$$

The proof is completed. □

Acknowledgments The authors are grateful to Prof. Feng-Yu Wang and Dr Huijie Qiao for stimulating discussions.

References

[1] B. Hambly, M. Mariapragassam and Ch. Reisinger: A forward equation for barrier options under the Brunick&Shreve Markovian projection, arXiv:1411.3618 [q-fin.MF].

[2] C. Dellacherie and P. A. Meyer: *Probabilities and Potential B: Theory of Martingales.* North-Holland, Amsterdam/New York/Oxford, 1982.

[3] N. Ikeda and S. Watanabe: *Stochastic Differential Equations and Diffusion Processes*, 2nd ed., North-Holland/Kodanska, Amsterdam/Tokyo, 1989.

[4] J. Jacod and A.N. Shiryaev: *Limit Theorems for Stochastic Processes.* Springer-Verlag, Berlin, 1987.

[5] A. Matoussi, W. Sabbagh and C. Zhou: The obstacle problem for semilinear parabolic partial integro-differential equations, arXiv:1307.0875v3 [math.PR]

[6] K. R. Parthasarathy: *Probability Measures on Metric Spaces.* AMS Chelsea Publishing, 2005.

[7] P. E. Protter and K. Shimbo: No arbitrage and general semimartingales. (http://legacy.orie.cornell.edu/protter/WebPapers/nagirsanov8.pdf)

[8] H. Qiao: Exponential ergodicity for SDEs with jumps and non-Lipschitz coefficients, *J. Theor. Probab.*, 27 (2014) 137-152.

[9] H. Qiao and J. Duan: Nonlinear filtering of stochastic dynamical systems with Lévy noises, *Advances in Applied Probability* 47 (2015) 902-918.

[10] H. Qiao and J.-L. Wu: Characterising the path-independence of the Girsanov transformation for non-Lipschnitz SDEs with jumps, submitted.

[11] H. Qiao and X. Zhang: Homeomorphism flows for non-Lipschitz stochastic differential equations with jumps, *Stochastic Processes and their Applications*118 (2008) 2254-2268.

[12] R. Situ: *Theory of Stochastic Differential Equations with Jumps and Applications.* Series: Mathematical and Analytical Techniques with Applications to Engineering. Springer-Verlag, Berlin, Heidelberg, NewYork, 2005.

[13] A. Truman, F.-Y. Wang, J.-L. Wu, W. Yang: A link of stochastic differential equations to nonlinear parabolic equations, *SCIENCE CHINA Mathematics*, 55 (2012) 1971-1976.

[14] F.- Y. Wang, *Analysis for diffusion processes on Riemannian manifolds,* World Scientific, 2014.

[15] J.-L. Wu and W. Yang: On stochastic differential equations and a generalised Burgers equation, pp 425-435 in *Stochastic Analysis and Its Applications to Finance: Essays in Honor of Prof. Jia-An Yan (eds. T S Zhang, X Y Zhou)*, Interdisciplinary Mathematical Sciences, Vol. 13, World Scientific, Singapore, 2012.