SKEW DERIVATIONS ON GENERALIZED WEYL ALGEBRAS

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Abstract. A wide class of skew derivations on degree-one generalized Weyl algebras $R(a, \varphi)$ over a ring R is constructed. All these derivations are twisted by a degree-counting extensions of automorphisms of R. It is determined which of the constructed derivations are Q-skew derivations. The compatibility of these skew derivations with the natural $\mathbb{Z}$-grading of $R(a, \varphi)$ is studied. Additional classes of skew derivations are constructed for generalized Weyl algebras given by an automorphism φ of a finite order. Conditions that the central element a that forms part of the structure of $R(a, \varphi)$ need to satisfy for the orthogonality of pairs of aforementioned skew derivations are derived. In addition local nilpotency of constructed derivations is studied. General constructions are illustrated by description of all skew derivations (twisted by a degree-counting extension of the identity automorphism) of generalized Weyl algebras over the polynomial ring in one variable and with a linear polynomial as the central element.

1. Introduction

This paper is devoted to the construction of a class of skew derivations of *degree-one generalized Weyl algebras*. In ring theory generalized Weyl algebras arose in the analysis of classification of simple $sl(2)$-modules in [2] and were introduced and initially studied by Bavula in a series of papers [3], [4], [5], [6], [7], [8], and also appeared in [15]. From a different perspective, degree-one generalized Weyl algebras appeared in non-commutative algebraic geometry [20], [17] (there they were called *rank-one hyperbolic algebras*). Since their introduction these algebras have become a subject of intensive study motivated in particular by the fact that many examples of algebras arising from quantum group theory or non-commutative geometry fall into this class. Degree-one generalized Weyl algebras $R(a, \varphi)$ are obtained as polynomial extensions of a ring R by adjoining two additional generators that satisfy relations determined by an automorphism φ of R and an element a in the centre of R (see Section 2 for the precise definition), and they can be understood as generalizations of skew Laurent polynomial rings.

The motivation for this study, results of which are being presented to the reader herewith, comes from non-commutative differential geometry, where skew derivations often play the role of vector fields (cf. [18, Section 4.4]) and may be used to equip non-commutative spaces with (exterior) differential structures. Recall that a vector field on a smooth manifold X can be defined as a linear endomorphism of the algebra of smooth functions on X that satisfies the Leibniz rule. The classic formula

$$df(\chi) = \chi(f), \tag{1.1}$$

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where f is a smooth function on X and χ is a vector field connects vector fields with the definition of the exterior derivative d. In non-commutative differential geometry, the philosophy of which is based on intepretation of non-commutative algebras A as algebras of functions on *non-commutative* manifolds or varieties, one starts with an exterior derivation d as a part of a differential graded algebra with the zero-degree coinciding with A; vector fields are then secondary objects to differential forms (in opposition to the standard textbook approach to classical differential geometry). If one tries to preserve formulae such as (1.1) (or to define vector fields through such correspondence with exterior derivative), one quickly realises that usually the non-commutativity of A forces one to abandon hope for a non-commutative vector field χ to satisfy the Leibniz rule. It happens quite often however that, at least locally, the usual Leibniz rule for χ can be replaced by the twisted Leibniz rule thus making χ a skew derivation. Conversely, skew derivations can be used to define an exterior derivation that satisfies the usual Leibniz rule and takes values in a suitably defined module of one-forms (see Section 2 for more details).

As the Leibniz rule for skew derivations studied here is twisted by an automorphism, one first should make a choice of a suitable automorphism. Automorphism groups of generalized Weyl algebras have been studied in special cases, for example in the case of quantum generalized Weyl algebras [9], [19], [21], [16] or generalized down-up algebras [13], to mention but a few. Our aim, however, is to work in a general degree-one situation, and hence we construct skew derivations twisted by automorphisms that can be defined for any generalized Weyl algebra over R. Such automorphisms are determined by an automorphism σ of R compatible with the data defining the generalized Weyl algebra, and a central unit μ in R (see Lemma 2.3 for details). We term them *degree-counting extensions of* σ of *coarseness* μ.

In the main Section 3 of the present paper we construct a wide class of skew derivations (twisted by degree-counting extensions of $\sigma \in \mathrm{Aut}(R)$) on degree-one generalized Weyl algebras $R(a,\varphi)$. Each element in this class is determined by the datum comprising a system of skew derivations of R and elements of R, all of which are required to satisfy a set of natural conditions (see Theorem 3.1). We term the skew derivation on $R(a,\varphi)$ associated to precisely one of the above data, an *elementary derivation* (these are of three types depending on the type of the initial datum and also carry an integer weight reflecting the stadard $\mathbb{Z}$-grading of $R(a,\varphi)$). Individually, each assignment of a skew derivation on $R(a,\varphi)$ to a skew derivation on R defines an injective map of twisted degree-one Hochschild cohomology groups. We show also that our construction affords one a full classification of skew derivations which send R to a positive (respectively, negative) part of $R(a,\varphi)$ (the positivity or negativity is defined with respect to a natural $\mathbb{Z}$-grading) and vanish on one of the extending generators of $R(a,\varphi)$ as well as all those skew derivations which vanish on R. Next we determine which of the constructed skew derivations are Q-skew derivations and we also derive sufficient and necessary conditions for compatibility of skew derivations with the natural $\mathbb{Z}$-grading of $R(\varphi;a)$ as maps of a fixed degree. Departing from the general case, we focus on algebras associated to automorphisms φ of finite order, and construct additional classes of skew derivations on them.

Keeping in mind that skew derivations can be used to construct first-order differential calculi provided they satisfy particular *orthogonality conditions* (see Section 2 for explanation), in Section 4 we derive sufficient conditions for the orthogonality of pairs of skew derivations constructed in Theorem 3.1. The bulk of these conditions involves the pairwise co-primeness of a with $\varphi^i(a)$, which incidentally is crucial for the statement of the Kashiwara theorem for generalized Weyl algebras [17, 2.2 Theorem]. In this way some of the results of [11], where orthogonal systems of skew derivations were studied for generalized Weyl algebras over a polynomial ring in one variable, can be reproduced as special cases of a far more general situation.

In Section 5 we study local nilpotency of elementary skew derivations. In particular we show that a locally nilpotent derivation on R induces a locally nilpotent elementary skew-derivation on $R(a,\varphi)$ in the non-zero weight case. We give an example which illustrates that the same cannot be generally said in the zero-weight case. We also derive sufficient conditions which ensure that constructed locally nilpotent derivations satisfy the assumptions of the Bergen-Grzeszczuk theorem [10, Theorem 1] which allows one to describe $R(a,\varphi)$ as an Ore extension of the subring of invariants.

In the final Section 6 we focus on generalized Weyl algebras over the polynomial ring in one indeterminate h with coefficients from a field $\mathbb{K}$, and with a linear polynomial as the central element. The automorphism φ is chosen to be the map rescaling h by a non-zero scalar $q \in \mathbb{K}$. We classify all skew derivations twisted by a degree-counting extension of the identity automorphism of any coarseness and show in this way that Theorem 3.1 gives the full classification in this case. We finish with a number of examples in which we construct orthogonal pairs of skew derivations on the quantum disc algebra, i.e. the generalized Weyl algebra over $\mathbb{K}[h]$ given by the central element $a = 1 - h$ and the automorphism $h \mapsto qh$.

2. Preliminaries

Let R be an associative ring with unit, and let M be an R-bimodule. Recall that the tensor algebra generated by M, $T_R(M)$, is an $\mathbb{N}$-graded algebra,

$$T_R(M) = R \oplus \bigoplus_{k>0} M^{\otimes_R k},$$

with the product given by the concatenation and the natural isomorphisms $R \otimes_R M \cong M \cong M \otimes_R R$. The algebra $T_R(M)$ has the following universal property: for any ring homomorphism $\phi_0 : R \to B$ and any R-bimodule map $\phi_1 : M \to B$, where the R-bimodule structure of B is given through ϕ_0, there exists a unique ring homomorphism $\phi : T_R(M) \to B$, which restricts to ϕ_0 on R and ϕ_1 on M; see e.g. [1, Chapter 1]. By the free polynomial ring in an indeterminate x with coefficients in R, $R\langle x\rangle$, we mean the tensor algebra generated by the free rank-one R-bimodule RxR (x is a free generator). Any ring automorphism σ_0 of R gives rise to a ring homomorphism $\sigma_0 : R \to R\langle x\rangle$, and any element $axb \in RxR$ induces a bimodule homomorphism

$$\sigma_1 : RxR \to R\langle x\rangle, \qquad rxs \mapsto \sigma_0(r)axb\sigma_0(s). \tag{2.1}$$

By the universal property, there exists unique ring endomorphism $\sigma : R\langle x\rangle \to R\langle x\rangle$, extending σ_0 and σ_1. One easily checks that if σ_0 is an automorphism and a, b are units in R, then σ is an automorphism.

The free polynomial ring in more than one indeterminate is defined iteratively, in particular:

$$R\langle x, y\rangle = R\langle x\rangle\langle y\rangle = T_R(RxR \oplus RyR).$$

Given an associative, unital ring R, a ring automorphism $\varphi : R \to R$ and an element a of the centre of R, the associated *degree-one generalized Weyl algebra* $R(a, \varphi)$ is defined as the quotient of the free polynomial ring $R\langle x, y\rangle$ by the relations:

$$xy = \varphi(a), \quad yx = a, \quad xr = \varphi(r)x, \quad yr = \varphi^{-1}(r)y, \tag{2.2}$$

for all $r \in R$. Every element of $R(a, \varphi)$ can be uniquely written as $r + \sum_{k>0} r_k x^k + \sum_{l>0} s_l y^l$, where $r, r_k, s_l \in R$. In the sequel, by a generalized Weyl algebra we always mean a degree-one generalized Weyl algebra.

If R is a $\mathbb{Z}$-graded algebra, then $R(a, \varphi)$ can also be made into a $\mathbb{Z}$-graded algebra provided that φ is a degree-preserving automorphism and a is a homogenous element. Specifically, if the degree of a is d, then x can be set to have, say, a positive degree m and y to have degree $d - m$. We refer to this grading of $R(a, \varphi)$ as the *(d, m)-type grading*. In particular if R is concentrated in the degree zero (or, simply, not treated as graded), then we set

$$R(a, \varphi)_0 = R, \quad R(a, \varphi)_+ = \{\sum_{m>0} r_m x^m \mid r_m \in R\}, \quad R(a, \varphi)_- = \{\sum_{m>0} r_m y^m \mid r_m \in R\},$$

so that

$$R(a, \varphi) = R(a, \varphi)_- \oplus R(a, \varphi)_0 \oplus R(a, \varphi)_+.$$

We refer to $R(a, \varphi)_+$ (respectively, $R(a, \varphi)_-$) as to the *positive* (respectively, *negative*) part of $R(a, \varphi)$. When R is treated as concentrated in degree 0, we refer to the $(0, 1)$-type grading of $R(a, \varphi)$ as to the *standard grading*. It is clear that every generalized Weyl algebra can be given the standard grading.

Although the definition of $R(a, \varphi)$ is not invariant under the exchange of generators x and y, one easily checks that the following map

$$\Psi : R(a, \varphi) \to R(\varphi(a), \varphi^{-1}), \quad x \mapsto y, \quad y \mapsto x, \quad \Psi|_R = \mathrm{id}_R, \tag{2.3}$$

(where we denote the generators of two generalized Weyl algebras by the same letters) is an isomorphism of algebras; see [9, 2.7 Lemma (i)]. We refer to this isomorphism as to the *x-y symmetry* (it is called a *Fourier transform* in [17]). This symmetry allows one to deduce counterparts of various statements, without any additional effort.

For any ring A, a *(right) skew derivation* is a pair (∂, σ) consisting of a ring endomorphism $\sigma : A \to A$ and an additive map $\partial : A \to A$ that satisfies the σ-twisted Leibniz rule, for all $a, b \in A$,

$$\partial(ab) = \partial(a)\sigma(b) + a\partial(b). \tag{2.4}$$

Clearly, if (∂_1, σ) and (∂_2, σ) are skew derivations, then so are $(\partial_1 + \partial_2, \sigma)$ and $(-\partial_i, \sigma)$. Obviously, $(0, \sigma)$ is a skew derivation. Hence the set $\mathrm{Der}_\sigma(A)$ of all skew derivations (∂, σ) of A with a fixed σ is an abelian group.

With exception of Lemma 2.1, we will always assume that σ is an automorphism.

To any element $b \in A$ one can associate the corresponding *inner* skew derivation (∂_b, σ) given by the σ-twisted commutator with b, i.e., for all $a \in A$,

$$\partial_b(a) = b\sigma(a) - ab.$$

The assignment $b \mapsto (\partial_b, \sigma)$ defines an additive map

$$\Delta : A \to \mathrm{Der}_\sigma(A). \tag{2.5}$$

Given a skew derivation (∂,σ) and a central unit $Q\in A$, invariant under σ, both $\sigma\circ\partial\circ\sigma^{-1}$ and $Q\,\partial$ are σ-twisted skew-derivations; (∂,σ) is called a *skew Q-derivation* provided

$$\sigma\circ\partial\circ\sigma^{-1}=Q\,\partial. \tag{2.6}$$

Any inner skew derivation (∂_b,σ) is a skew 1-derivation.

Given a ring automorphism σ of A and an A-bimodule M, we write M_σ for the A-bimodule with right A-action twisted by σ, i.e. defined by

$$m\cdot a:=m\sigma(a),\qquad \text{for all } a\in A,\, m\in M.$$

With this notation a pair (∂,σ) is a skew derivation on A if and only if ∂ is an A_σ-valued derivation of A. Again, for an A-bimodule M we denote by M^A the abelian group

$$M^A:=\{m\in M\mid \forall a\in A,\, am=ma\}.$$

Obviously M^A is a module over the centre $Z(A)=A^A$ of A. We will most frequently use this notation in the case of an A-bimodule A_σ, where σ is an automorphism of A. In this case

$$A^A_\sigma:=\{b\in A\mid \forall a\in A,\, ab=b\sigma(a)\}$$

is referred to as the *σ-twisted centre of A*. Note that if φ is a ring automorphism commuting with σ, then if $b\in A^A_\sigma$, then $\varphi^n(b)\in A^A_\sigma$, for all $n\in\mathbb{Z}$.

Recall that, for a ring A and an A-bimodule M, the M-valued Hochschild cohomology of A, $HH(A,M)$, is the cohomology of the complex $\mathfrak{b}: HC(A,M)^n\to HC(A,M)^{n+1}$, where $HC(A,M)^n=\mathrm{Hom}(A^{\otimes n},M)$, the group of additive homomorphisms from $A^{\otimes n}$ to M, and

$$(\mathfrak{b}f)(a_0\otimes\cdots\otimes a_n)=a_0f(a_1\otimes\cdots\otimes a_n)+\sum_{k=1}^{n}(-1)^n f(a_0\otimes\cdots\otimes a_{k-1}a_k\otimes\cdots\otimes a_n)$$
$$+(-1)^{n+1}f(a_0\otimes\cdots\otimes a_{n-1})a_n. \tag{2.7}$$

In particular, $HH^0(A,M)=M^A$ and $HC^1(A,M)$ is the group of M-valued derivations on A, while the image of $\mathfrak{b}: HC^0(A,M)\to HC^1(A,M)$ consists of all inner derivations. Thus, the kernel of the map Δ (2.5) is simply equal to $HH^1(A,A_\sigma)$.

The complex $(HC(A,A_\sigma),\mathfrak{b})$, where σ is a ring automorphism of A, contains a subcomplex, which will play a special role in the discussion of skew derivations on generalized Weyl algebras. Let φ be an automorphism of A commuting with σ, and let μ be an element of the centre of A. Set

$$HC^n_{\sigma;\mu,\varphi}(A):=\{f\in\mathrm{Hom}(A^n,A)\mid \varphi^{-1}\circ f\circ\varphi^{\otimes n}=\mu\, f\}.$$

Then the Hochschild coboundary $\mathfrak{b}$ (2.7) for $M=A_\sigma$ restricts to $HC_{\sigma;\mu,\varphi}(A)$. The cohomology of the resulting complex is denoted by $HH_{\sigma;\,\mu,\varphi}(A)$, and we refer to it as a *doubly twisted Hochschild cohomology of A*. In case $\mu=1$, $\varphi=\mathrm{id}$ this is the standard twisted Hochschild cohomology of A, denoted by $HH_\sigma(A)$

Let $(\partial_i,\sigma_i)_{i=1}^n$ be a (finite) family of skew derivations of a ring A. We say that it forms an *orthogonal system of skew derivations* provided there exist two finite sets $\{a_{i\,t}\},\{b_{i\,t}\}\subset A$ such that,

$$\sum_t a_{i\,t}\partial_k(b_{i\,t})=\delta_{ik},\quad \text{for all}\quad i,k=1,\dots,n. \tag{2.8}$$

Note that this is equivalent to to the existence of three finite sets $\{a_{i\,t}\}, \{b_{i\,t}\}, \{c_{i\,t}\}$ of elements of A such that

$$\sum_t a_{i\,t}\partial_k(b_{i\,t})\sigma_k\left(\sigma_i^{-1}\left(c_{i\,t}\right)\right) = \delta_{ik}, \quad \text{for all} \quad i,k = 1,\dots,n. \tag{2.9}$$

Indeed, obviously (2.8) implies (2.9). On the other hand, if (2.9) holds, then the twisted Leibniz rules yield

$$\sum_t a_{i\,t}\partial_k\left(b_{i\,t}\sigma_i^{-1}\left(c_{i\,t}\right)\right) - \sum_t a_{i\,t}b_{i\,t}\partial_k\left(\sigma_i^{-1}\left(c_{i\,t}\right)\right) = \sum_t a_{i\,t}\partial_k(b_{i\,t})\sigma_k\left(\sigma_i^{-1}\left(c_{i\,t}\right)\right) = \delta_{ik},$$

hence $\{a_{i\,t}, -a_{i\,t}b_{i\,t}\}, \{b_{i\,t}\sigma_i^{-1}\left(c_{i\,t}\right), \sigma_i^{-1}\left(c_{i\,t}\right)\}$ are the required two sets.

As explained, for example in [12, Section 3], orthogonal systems of skew derivations on A can be used to form first order differential calculi on A. By the latter we mean a pair consisting of an A-bimodule Ω and an Ω-valued derivation $d : A \to \Omega$, such that $\Omega = Ad(A)$. Given an orthogonal system of skew derivations $(\partial_i, \sigma_i)_{i=1}^n$, Ω and d are defined by,

$$\Omega = \bigoplus_{i=1}^n A_{\sigma_i}, \qquad d : a \mapsto \left(\partial_i(a)\right)_{i=1}^n. \tag{2.10}$$

While the σ-twisted Leibniz rule ensures that the map d in (2.10) is a derivation, the orthogonality conditions (2.8) are equivalent to the density of Ω: $\Omega = Ad(A)$.

The following lemma is well-known (see e.g. similar [1, Lemma 1.8]), we include its proof for completeness.

Lemma 2.1. *Let $A = T_R(M)$, be the tensor algebra generated by an R-bimodule M, and let $\sigma : T_R(M) \to T_R(M)$ be the unique endomorphism extending a pair $\sigma_0 : R \to R \subset T_R(M)$ (a ring map), $\sigma_1 : R \to T_R(M)$ (an R-bimodule homomorphism). Let (δ_0, σ_0) be a (right) skew-derivation on R, and let*

$$\delta_1 : M \to T_R(M)$$

be an additive map such that, for all $r, s \in R$, $m \in M$

$$\delta_1(rms) = \delta_0(r)\sigma_1(ms) + r\delta_1(m)\sigma_0(s) + rm\delta_0(s). \tag{2.11}$$

Then there exists (unique) skew-derivation (δ, σ) on $T_R(M)$, extending δ_0 and δ_1.

Proof. Define

$$\hat{\delta} : R \oplus \bigoplus_{k>0} M^{\otimes_{\mathbb{Z}} k} \longrightarrow T_R(M),$$

as δ_0 on R, and

$$\hat{\delta}(m_1 \otimes \cdots \otimes m_n) = \sum_{k=1}^n m_1 \otimes_R \cdots \otimes_R \delta_1(m_k) \otimes_R \sigma(m_{k+1} \otimes_R \cdots \otimes_R m_n).$$

Thanks to the σ_0-twisted Leibniz rule and (2.11), $\hat{\delta}$ is coequalised by all the maps defining tensor product over R. Consequently there is a unique map $\delta : T_R(M) \to T_R(M)$. Again, thanks to the σ_0-twisted Leibniz rule and (2.11), the resulting δ satisfies the σ-twisted Leibniz rule, hence the pair (δ, σ) is a skew-derivation as claimed. □

Remark 2.2. In the case of the free polynomial ring $R\langle x\rangle$, given a skew-derivation (δ_0, σ_0) on R and σ_1 as in (2.1), a suitable δ_1 is determined by any $c \in R\langle x\rangle$ simply through the use of (2.11) (thanks to the fact that $M = RxR$ is a free left and right A-module):

$$\delta_1(rxs) = \delta_0(r)\sigma_1(xs) + rc\sigma_0(s) + rx\delta_0(s).$$

Hence, given a skew derivation (δ_0, σ_0) on R, any choice of elements $a, b \in R$ (for the definition of σ_1), and any choice of $c \in R\langle x\rangle$ determines a skew-derivation on $R\langle x\rangle$ (if we want the resulting σ to be an automorphism, we need σ_0 to be an automorphism and a, b to be units). The procedure can be iterated for polynomial rings in more than one indeterminates.

We will use this freedom of extending skew derivations from R to $R\langle x, y\rangle$ in the construction of skew derivations on generalized Weyl algebras $R(a, \varphi)$. By checking that a skew derivation δ on $R\langle x, y\rangle$ (possibly obtained by extending a skew derivation on R) respects the defining relations of $R(a, \varphi)$, which is equivalent to checking the invariance of a suitable ideal in $R\langle x, y\rangle$ under δ, we ensure that it descends to the skew derivation on $R(a, \varphi)$.

In this paper we investigate skew derivations of generalized Weyl algebras $R(a, \varphi)$ related to a particular class of automorphisms of $R(a, \varphi)$ (compare [9, 2.7 Lemma (iii)]).

Lemma 2.3. *Given $R(a, \varphi)$, let σ be a ring automorphism of R such that*

$$\sigma \circ \varphi = \varphi \circ \sigma, \qquad \sigma(a) = a. \tag{2.12}$$

Then, for any central unit μ in R, the map σ extends to the automorphism σ_μ of $R(a, \varphi)$ by

$$\sigma_\mu(x) = \mu^{-1}x, \qquad \sigma_\mu(y) = y\,\mu = \varphi^{-1}(\mu)y. \tag{2.13}$$

Proof. Equations (2.13) specify a bimodule map $RxR \oplus RyR \to R\langle x, y\rangle$, and hence (by the universal property) there is ring automorphism of $R\langle x, y\rangle$ which restricts to σ and the bimodule map determined by (2.13). One easily verifies that resulting automorphism vanishes on the ideal that defines $R(a, \varphi)$ through relations (2.2), hence σ_μ is an automorphism of $R(a, \varphi)$ as required. □

Thinking about $R(a, \varphi)$ as a $\mathbb{Z}$-graded algebra we feel justified in making the following

Definition 2.4. An automorphism σ_μ described in Lemma 2.3 is called a *degree-counting extension* of the automorphism σ of R (of *coarseness* μ).

3. Skew derivations on generalized Weyl algebras

In this section first we describe a wide class of skew derivations on a generalized Weyl-algebra $R(a, \varphi)$, twisted by the degree-counting extension of an automorphism σ of R of a general coarseness μ. Next we determine which of the constructed derivations are Q-skew derivations. Finally, we construct additional skew derivations, when the automorphism φ has a finite order.

Theorem 3.1. *Let $R(a, \varphi)$ be a generalized Weyl algebra and let σ be an automorphism of R commuting with φ and fixing a. Let σ_μ be the degree-counting extension of σ of coarseness μ, and consider the following data:*

(a) *skew derivations on* R $(\alpha_i,\ \varphi^i\circ\sigma)_{i\in\mathbb{Z}}$, *such that, for all* $i\in\mathbb{Z}$,

$$\alpha_i\circ\varphi=\varphi^i(\mu)\varphi\circ\alpha_i, \tag{3.1}$$

and there exists $c\in R^R_\sigma$ *such that*

$$\alpha_0(a)=a\varphi^{-1}(c); \tag{3.2}$$

(b) *elements* $c_i\in R^R_{\varphi^i\circ\sigma}$ *and* $b_i\in\left(R\setminus R^R_{\varphi^i\circ\sigma}\right)\cup\{0\}$, $i\in\mathbb{Z}$;

(c) *a set* I *of positive integers such that, for all* $r\in R$, *the sets* $\{i\in I\mid \alpha_{\pm i}(r)\neq 0\}$ *are finite and the sequences* $(c_i)_{\pm i\in I}$, $(b_i)_{\pm i\in I}$ *are finitely supported.*

Given above data, define,

$$\begin{aligned}\partial(r) &= \sum_{m\in I\cup\{0\}}\left(\alpha_m(r)+b_m\varphi^m\circ\sigma(r)-rb_m\right)x^m\\ &\quad+\sum_{n\in I}\left(\alpha_{-n}(r)+b_{-n}\varphi^{-n}\circ\sigma(r)-rb_{-n}\right)y^n, \qquad \text{for all } r\in R,\end{aligned} \tag{3.3a}$$

$$\begin{aligned}\partial(x)&=\sum_{m\in I\cup\{0\}}\left(c_m-\varphi(b_m)+\varphi^m(\mu^{-1})b_m\right)x^{m+1}\\ &\quad+\sum_{n\in I}\varphi\left(\alpha_{-n}(a)+\varphi^{-n-1}(\mu^{-1})\left(\varphi^{-1}(b_{-n})\varphi^{-n}(a)-ac_{-n}\right)-b_{-n}a\right)y^{n-1},\end{aligned} \tag{3.3b}$$

$$\begin{aligned}\partial(y)&=\sum_{n\in I}\left(c_{-n}+\varphi^{-n-1}(\mu)b_{-n}-\varphi^{-1}(b_{-n})\right)y^{n+1}+\left(\varphi^{-1}(\mu c-b_0)+\varphi^{-1}(\mu)b_0+\tilde{c}_0\right)y\\ &\quad+\sum_{m\in I}\varphi^{m-1}(\mu)\left(\alpha_m(a)-\varphi^{-1}\left(c_m+\varphi^m(\mu^{-1})b_m\right)a+b_m\varphi^m(a)\right)x^{m-1},\end{aligned} \tag{3.3c}$$

where $\tilde{c}_0\in R^R_\sigma$ *is a solution to the equation* $(\tilde{c}_0+\varphi^{-1}(\mu c_0))a=0$. *Then* ∂ *extends to a skew derivation* (∂,σ_μ) *on* $R(a,\varphi)$.

Proof. Since any $R(a,\varphi)$ can be viewed as a $\mathbb{Z}$-graded algebra with the standard grading, the summands in (3.3) can be separated according to their degrees, thus yielding

(i) The zero-degree case:

$$\partial_0(r)=\alpha_0(r)+b_0\sigma(r)-rb_0,\qquad \partial_0(x)=\left(c_0+\mu^{-1}b_0-\varphi(b_0)\right)x, \tag{3.4a}$$

$$\partial_0(y)=\left(\varphi^{-1}(\mu c-b_0)+\varphi^{-1}(\mu)b_0+\tilde{c}_0\right)y\,. \tag{3.4b}$$

(ii) The positive degree case ($m>0$):

$$\partial_m(r)=\left(\alpha_m(r)+b_m\varphi^m\circ\sigma(r)-rb_m\right)x^m, \tag{3.5a}$$

$$\partial_m(x)=\left(c_m+\varphi^m(\mu^{-1})b_m-\varphi(b_m)\right)x^{m+1}, \tag{3.5b}$$

$$\partial_m(y)=\varphi^{m-1}(\mu)\left(\alpha_m(a)-\varphi^{-1}\left(c_m+\varphi^m(\mu^{-1})b_m\right)a+b_m\varphi^m(a)\right)x^{m-1}. \tag{3.5c}$$

(iii) The negative degree case ($n>0$):

$$\partial_{-n}(r)=\left(\alpha_{-n}(r)+b_{-n}\varphi^{-n}\circ\sigma(r)-rb_{-n}\right)y^n, \tag{3.6a}$$

$$\partial_{-n}(x)=\varphi\left(\alpha_{-n}(a)+\varphi^{-n-1}(\mu^{-1})\left(\varphi^{-1}(b_{-n})\varphi^{-n}(a)-ac_{-n}\right)-b_{-n}a\right)y^{n-1}, \tag{3.6b}$$

$$\partial_{-n}(y)=\left(c_{-n}+\varphi^{-n-1}(\mu)b_{-n}-\varphi^{-1}(b_{-n})\right)y^{n+1}. \tag{3.6c}$$

We will prove that the homogeneous maps defined by (3.4), (3.5) and (3.6) extend to the elements of $\mathrm{Der}_{\sigma_\mu}(R(a,\varphi))$.

Each of the maps ∂_i can be itself split into three parts,

$$\partial_i = \partial_i^\alpha + \partial_i^c + \partial_i^b, \tag{3.7}$$

where ∂_i^α is obtained from ∂_i by setting $b_i = c_i = 0$, ∂_i^c is obtained from ∂_i by setting $\alpha_i = b_i = 0$, and ∂_i^b is obtained from ∂_i by setting $\alpha_i = c_i = 0$. The extension of the last of these is simply an inner derivation corresponding to $b_m x^m$, for the non-negative degree and $b_{-n}y^n$, for the negative one, thus only former two require more attention. We treat these cases in two separate lemmas.

Lemma 3.2. *An additive map $\partial_m^\alpha : R(a,\varphi) \to R(a,\varphi)$ is a σ_μ-twisted skew derivation of positive standard degree m and such that $\partial_m^\alpha(x) = 0$ if and only if there exists a skew derivation $(\alpha_m, \varphi^m \circ \sigma)$ of R such that $\alpha_m \circ \varphi = \varphi^m(\mu)\varphi \circ \alpha_m$, and, for all $r \in R$,*

$$\partial_m^\alpha(r) = \alpha_m(r)\, x^m, \qquad \partial_m^\alpha(y) = \varphi^{m-1}(\mu)\alpha_m(a)\, x^{m-1} = \alpha_m(a)\, x^{m-1}\mu. \tag{3.8}$$

An additive map $\partial_0^\alpha : R(a,\varphi) \to R(a,\varphi)$ is a σ_μ-twisted skew derivation of standard degree 0 and such that $\partial_0^\alpha(x) = 0$ if and only if there exists a skew derivation (α_0, σ) of R and $c \in R_\sigma^R$, such that $\alpha_0 \circ \varphi = \mu\varphi \circ \alpha_0$ and $\alpha_0(a) = \varphi^{-1}(c)a$, and, for all $r \in R$,

$$\partial_0^\alpha(r) = \alpha_0(r), \qquad \partial_0^\alpha(y) = \varphi^{-1}(\mu\, c)\, y. \tag{3.9}$$

An additive map $\partial_{-n}^\alpha : R(a,\varphi) \to R(a,\varphi)$ is a σ_μ-twisted skew derivation of negative standard degree $-n$ and such that $\partial_{-n}^\alpha(y) = 0$ if and only if there exists a skew derivation $(\alpha_{-n}, \varphi^{-n} \circ \sigma)$ of R such that $\alpha_{-n} \circ \varphi = \varphi^{-n}(\mu)\varphi \circ \alpha_{-n}$, and, for all $r \in R$,

$$\partial_{-n}^\alpha(r) = \alpha_{-n}(r)\, y^n, \qquad \partial_{-n}^\alpha(x) = \varphi\left(\alpha_{-n}(a)\right)\, y^{n-1}.$$

Proof. The positive degree m skew derivation $(\partial_m^\alpha, \sigma_\mu)$ that vanishes on x is necessarily of the form

$$\partial_m^\alpha(r) = \alpha_m(r)x^m, \qquad \partial_m^\alpha(x) = 0, \qquad \partial_m^\alpha(y) = c_m x^{m-1},$$

where α_m is an additive endomorphism of R and $c_m \in R$. By the twisted Leibniz rule, for all $r, s \in R$,

$$\partial_m^\alpha(rs) = \partial_m^\alpha(r)\sigma(s) + r\partial_m^\alpha(s),$$

hence, by (2.2),

$$\alpha_m(rs)x^m = \alpha_m(r)x^m\sigma(s) + r\alpha_m(s)x^m = \left(\alpha_m(r)\varphi^m(\sigma(s)) + r\alpha_m(s)\right)x^m,$$

which is equivalent to the fact that $(\alpha_m, \varphi^m \circ \sigma)$ is a skew derivation of R. Next, for all $r \in R$,

$$\begin{aligned} 0 &= \partial_m^\alpha(xr - \varphi(r)x) = x\alpha_m(r)x^m - \alpha_m(\varphi(r))x^m\mu^{-1}\, x \\ &= \left(\varphi(\alpha_m(r)) - \varphi^m(\mu^{-1})\alpha_m(\varphi(r))\right)x^{m+1}, \end{aligned}$$

by (2.2) and the centrality of μ. This yields necessarily that $\alpha_m \circ \varphi = \varphi^m(\mu)\varphi \circ \alpha_m$. Furthermore,

$$0 = \partial_m(yx - a) = c_m x^{m-1}\mu^{-1}\, x - \alpha_m(a)x^m = \left(\varphi^{m-1}\left(\mu^{-1}\right)c_m - \alpha_m(a)\right)x^m,$$

by (2.2) and the centrality of μ . This fixes $c_m = \varphi^{m-1}(\mu)\,\alpha_m(a)$, and thus proves the necessity of the stated form of ∂_m^α.

The above calculations confirm that the map defined by (3.8) is compatible with the two of relations (2.2). To check the compatibility with the remaining two, we first compute, for all $r \in R$,

$$\begin{aligned}
\partial_m^\alpha(yr) &= \alpha_m(a)\,x^{m-1}\mu\sigma(r) + y\alpha_m(r)x^m \\
&= \alpha_m(a)\varphi^{m-1}(\sigma(r))x^{m-1}\mu + \varphi^{-1}(\alpha_m(r))ax^{m-1} \\
&= \alpha_m(a)\varphi^{m-1}(\sigma(r))x^{m-1}\mu + \varphi^{m-1}(\mu)\alpha_m(\varphi^{-1}(r))ax^{m-1} \\
&= \left(\alpha_m(a)\varphi^m\left(\sigma(\varphi^{-1}(r))\right) + a\alpha_m(\varphi^{-1}(r))\right)x^{m-1}\mu = \alpha_m\left(a\varphi^{-1}(r)\right)x^{m-1}\mu,
\end{aligned}$$

where the first equality follows by the definition of ∂_m^α through equation (3.8). The second and the third equalities follow by (2.2), the centrality of μ and (3.1), while the third one holds since α_m is a $\varphi^m \circ \sigma$-skew derivation. On the other hand, using (3.8), (2.2), that σ fixes central a, and that α_m is a $\varphi^m \circ \sigma$-skew derivation we compute

$$\begin{aligned}
\partial_m^\alpha\left(\varphi^{-1}(r)y\right) &= \alpha_m(\varphi^{-1}(r))x^m y\,\mu + \varphi^{-1}(r)\alpha_m(a)\,x^{m-1}\mu \\
&= \left(\alpha_m(\varphi^{-1}(r))\varphi^m(a)x^{m-1} + \varphi^{-1}(r)\alpha_m(a)x^{m-1}\right)\mu \\
&= \alpha_m\left(a\varphi^{-1}(r)\right)x^{m-1}\mu.
\end{aligned}$$

Therefore $\partial_m^\alpha(yr - \varphi^{-1}(r)y) = 0$, as required. Finally,

$$\begin{aligned}
\partial_m^\alpha(xy - \varphi(a)) &= x\alpha_m(a)\,x^{m-1}\mu - \alpha_m(\varphi(a))x^m \\
&= \left(\varphi^m(\mu)\varphi(\alpha_m(a)) - \alpha_m(\varphi(a))\right)x^m = 0,
\end{aligned}$$

by (3.8), (2.2) and (3.1). Thus, the σ_μ-skew derivation property of ∂_m^α is compatible with relations (2.2), and hence ∂_m^α is a σ_μ-skew derivation on $R(a,\varphi)$ as explained in Remark 2.2.

In the degree-zero case, ∂_0^α is necessarily of the form

$$\partial_0^\alpha(r) = \alpha_0(r), \qquad \partial_0^\alpha(x) = 0, \qquad \partial_0^\alpha(y) = c_0 y,$$

where α_0 is an additive endomorphism of R and $c_0 \in R$. The twisted Leibniz rule of ∂_0^α restricted to R is equivalent with the twisted Leibniz rule of α_0. The fact that ∂_0^α vanishes on $xr - \varphi(r)x$ implies that $\alpha_0 \circ \varphi = \mu\varphi \circ \alpha_0$. Applying ∂_0^α to $yx - a = 0$, one obtains that

$$\alpha_0(a) = c_0\varphi^{-1}(\mu^{-1})a,$$

hence $\alpha_0(a) = \varphi^{-1}(c)a$, where $c = \varphi(c_0)\mu^{-1}$ and provides one with the stated form of $\partial_0^\alpha(y)$. Finally, the condition $\partial_0^\alpha(yr - \varphi^{-1}(r)y) = 0$, implies that $c \in R_\sigma^R$.

Conversely, assume that ∂_0^α vanishes on x and is defined as in (3.9). Since (α_0, σ) is a skew derivation on R, and σ_μ restricted to R is equal to σ, ∂_0^α restricted to R satisfies the σ_μ-twisted Leibniz rule. We need to check that ∂_0^α extended to the whole of $R(a,\varphi)$ by the σ_μ-twisted Leibniz rule preserves all the relations (2.2). First, let us compute

$$\partial_0^\alpha(xy - \varphi(a)) = x\varphi^{-1}(\mu c)y - \alpha_0\left(\varphi(a)\right) = \mu c\varphi(a) - \mu\varphi(\alpha_0(a)) = 0,$$

where the first equality follows by the definition of ∂_0^α and the σ_μ-twisted Leibniz rule, while the second one follows by relations (2.2), the fact that α_0 μ-commutes with φ, and the centrality of μ. Next:

$$\partial_0^\alpha(yx - a) = \varphi^{-1}\left(\mu c\right)y\mu^{-1}x - \alpha_0(a) = \varphi^{-1}\left(c\right)a - \alpha_0(a) = 0,$$

where the first equality follows by the definition of ∂_0^α via the twisted Leibniz rule, and the second one by (2.2) and the centrality of μ. Next, for all $r \in R$,

$$\partial_0^\alpha(xr - \varphi(r)x) = x\alpha_0(r) - \alpha_0(\varphi(r))\mu^{-1}x = (\varphi(\alpha_0(r)) - \varphi(\alpha_0(r)))\,x = 0.$$

Here, as before, the first equality follows by the definition of ∂_0^α and the σ_μ-twisted Leibniz rule, while the second one follows by (3.1) and centrality of μ. Finally, for all $r \in R$,

$$\begin{aligned}\partial_0^\alpha(yr - \varphi^{-1}(r)y) &= \varphi^{-1}(\mu c)y\sigma(r) + y\alpha_0(r) - \alpha_0(\varphi^{-1}(r))\varphi^{-1}(\mu)y - \varphi^{-1}(r)\varphi^{-1}(\mu c)y \\ &= \left(\varphi^{-1}(\mu c)\varphi^{-1}(\sigma(r)) + \varphi^{-1}(\alpha_0(r)) - \varphi^{-1}(\alpha_0(r)) - \varphi^{-1}(\mu rc)\right) y \\ &= \varphi^{-1}(\mu)\left(\varphi^{-1}(rc) - \varphi^{-1}(rc)\right) y = 0.\end{aligned}$$

Again, the first equality follows by the definition of ∂_0^α and the σ_μ-twisted Leibniz rule. The second equality is a consequence of (2.2), (3.1) and the centrality of μ, while the third one follows by (2.12). Since $c \in R_\sigma^R$, the final equality is obtained. Thus, ∂_0^α vanishes on all generators of the ideal in $R\langle x, y\rangle$ that defines $R(a,\varphi)$, hence ∂_0^α extends as a σ_μ-twisted derivation to the whole of $R(a,\varphi)$; cf. Remark 2.2.

The negative degree case follows by the x-y symmetry. More precisely, by the x-y symmetry, a negative degree skew derivation $(\partial_{-n}^\alpha, \sigma_\mu)$ in $R(a,\varphi)$ corresponds to the positive degree skew derivation $(\partial_n^\alpha, \sigma_{\varphi^{-1}(\mu^{-1})})$ in $R(\varphi(a), \varphi^{-1})$. The negative degree conditions and formulae are simply translations of the positive degree case. □

Lemma 3.3. *An additive map $\partial_i^c : R(a,\varphi) \to R(a,\varphi)$ is a σ_μ-twisted skew derivation of standard degree i and such that $\partial_i^c(R) = 0$ if and only if there exists $c_i \in R^R_{\varphi^i\circ\sigma}$ such that*

$$\partial_i^c(x) = c_i\, x^{i+1}, \qquad \partial_i^c(y) = -\varphi^{m-1}(\mu)\varphi^{-1}(c_i)a\, x^{i-1},$$

if i is positive, or

$$\partial_i^c(x) = -\varphi^i(\mu^{-1})\varphi(ac_i)\, y^{-i-1}, \qquad \partial_i^c(y) = c_i\, y^{-i+1},$$

if i is negative, or

$$\partial_0^c(x) = c_0\, x, \qquad \partial_0^c(y) = \tilde{c}_0\, y,$$

where $\tilde{c}_0 \in R_\sigma^R$ is a solution to the equation $(\tilde{c}_0 + \varphi^{-1}(\mu c_0))a = 0$.

Proof. Let $i = m$ be a positive integer. Since ∂_m^c is a degree-m map, necessarily,

$$\partial_m^c(x) = c_m\, x^{m+1}, \qquad \partial_m^c(y) = \tilde{c}_m\, x^{m-1},$$

for some $c_m, \tilde{c}_m \in R$. In view of the defining relations (2.2), the twisted Leibniz rule and the fact that ∂_m^c vanishes on R imply, for all $r \in R$,

$$0 = \partial_m^c(xr - \varphi(r)x) = c_m x^{m+1}\sigma(r) - \varphi(r)c_m x^{m+1} = \left(c_m\varphi^{m+1}(\sigma(r)) - \varphi(r)c_m\right)\, x^{m+1},$$

therefore $c_m \in R^R_{\varphi^m\circ\sigma}$. By the same token,

$$0 = \partial_m^c(yx - a) = \tilde{c}_m\, x^{m-1}\mu^{-1}x + yc_m\, x^{m+1} = \left(\varphi^{m-1}(\mu^{-1})\tilde{c}_m + \varphi^{-1}(c_m)a\right) x^m,$$

which implies that $\tilde{c}_m = -\varphi^{m-1}(\mu)\varphi^{-1}(c_m)a$, as required. This proves the necessity of the stated form of a homogeneous skew derivation of positive degree. Assuming the stated form of ∂^c_m, the above calculations confirm that ∂^c_m is compatible with two of the defining relations (2.2). The remaining two can be checked as follows,

$$\begin{aligned}\partial^c_m(yr-\varphi^{-1}(r)y) &= -\varphi^{m-1}(\mu)\varphi^{-1}(c_m)ax^{m-1}\sigma(r)+\varphi^{-1}(r)\varphi^{m-1}(\mu)\varphi^{-1}(c_m)ax^{m-1}\\ &= -\varphi^{-1}\left(\varphi^m(\mu)\left(c_m\varphi^m\circ\sigma(r)-rc_m\right)\right)a\,x^{m-1}=0,\end{aligned}$$

since $c_m\in R^R_{\varphi^m\circ\sigma}$. Finally,

$$\begin{aligned}\partial^c_m(xy-\varphi(a)) &= c_m x^{m+1}y\mu - x\varphi^{m-1}(\mu)\varphi^{-1}(c_m)ax^{m-1}\\ &= \varphi^m(\mu)\left(c_m\varphi^{m+1}(a)-c_m\varphi(a)\right)x^m=0,\end{aligned}$$

by the centrality of a, and the facts that $c_m\in R^R_{\varphi^m\circ\sigma}$ and $\sigma(a)=a$.

The negative degree case follows by the x-y symmetry. For the degree zero case, necessarily

$$\partial^c_0(x)=c_0\,x,\qquad \partial^c_0(y)=\tilde{c}_0\,y.$$

As in the positive degree case, the condition $\partial^c_0(xr-\varphi(r)x)=0$ implies that $c_0\in R^R_\sigma$; by the x-y symmetry, or directly from $\partial^c_0(yr-\varphi^{-1}(r)y)=0$, one obtains that also $\tilde{c}_0\in R^R_\sigma$. The relation $\partial^c_0(yx-a)=0$ is equivalent to $(\tilde{c}_0+\varphi^{-1}(\mu c_0))a=0$. The sufficiency is checked in the similar way as for the positive degree case. □

In view of Lemma 3.2 and Lemma 3.3, and the discussion preceding the former, the map ∂ is a locally finite sum of σ_μ-twisted skew derivations (homogeneous with respect to the standard grading), hence it is a skew derivation of $R(a,\varphi)$. This completes the proof of the theorem. □

Remark 3.4. Clearly, if a is a regular element of R, then c is uniquely determined (if it exists) by equation (3.2), and $\tilde{c}_0=-\varphi^{-1}(\mu c_0)$.

Remark 3.5. Note that the existence of a regular element in $R^R_{\varphi^i\circ\sigma}$ implies that, for all z in the centre of R, $\sigma(z)=z$.

Remark 3.6. Since the automorphism σ commutes with φ and $\sigma(a)=a$, the generalized Weyl algebra $R(a,\varphi)$ can be restricted to $S(a,\varphi)$, where

$$S:=\{s\in R\mid \sigma(s)=s\}\subseteq R,$$

is the fixed point subalgebra of R. If also $\sigma(\mu)=\mu$ (which e.g. is necessarily the case if there is a regular element in R^R_σ, see Remark 3.5), then σ_μ, restricted to $S(a,\varphi)$, is the degree-counting extension of the identity automorphism of S of coarseness μ. In this case the skew derivations listed in Theorem 3.1 restrict to skew derivations on $S(a,\varphi)$.

Definition 3.7. The skew derivations listed in equations (3.4)–(3.6) will be referred to as *elementary*. The integer index m of ∂_m is called a *weight*. When needed, the components ∂^α_m, ∂^c_m, ∂^b_m defined through (3.7) will be further qualified as the *α-type*, *c-type* and the *inner-type*, respectively.

Remark 3.8. Obviously, there is no need to demand that a non-zero b_i be not in $R^R_{\varphi^i\circ\sigma}$ in the definition of an inner-type elementary derivation. However, if $b_i\in R^R_{\varphi^i\circ\sigma}$, then the restriction of ∂^b_i to R vanishes and hence the contribution to ∂ coming from an inner-type elementary derivation can be absorbed into that coming from a c-type elementary derivation.

The degree-i c-derivation in turn is inner if and only if there exists $r_i \in R$ such that

$$c_i = \begin{cases} \varphi^i\left(\mu^{-1}\right) r_i - \varphi\left(r_i\right), & \text{if } i \geq 0, \\ \varphi^i\left(\mu\right) r_i - \varphi^{-1}\left(r_i\right), & \text{if } i < 0, \end{cases}$$

and also $\tilde{r}_0$ such that $\tilde{c}_0 = \mu \tilde{r}_i - \varphi^{-1}\left(\tilde{r}_i\right)$ in the zero-degree case.

Example 3.9. It is well-known (see, for example [14, 4.6.8 Lemma]) that any derivation of the Weyl algebra A_1 (over a field $\mathbb{K}$ of characteristic zero) is inner. The algebra A_1 is an example of a generalized Weyl algebra with $R = \mathbb{K}[h]$, the polynomial ring in one indeterminate, $\varphi(f(h)) = f(h+1)$ and $a = h$. One might therefore ask whether there exist any non-inner σ_μ-skew derivations on A_1. First observe that since $\sigma(a) = a$, we are immediately forced to set σ to be the identity automorphism of $\mathbb{K}[h]$. Obviously, since σ is the identity it commutes with φ, hence it has a degree-counting extension σ_μ, where μ is any non-zero element of $\mathbb{K}$ (non-zero scalars are the only units in $\mathbb{K}[h]$). Since we are interested in the twisted case, we assume that $\mu \neq 1$.

To obtain an α-type derivation we need first to determine which skew-derivations of $\mathbb{K}[h]$ satisfy condition (3.1). Assume that $\alpha(h) = b(h) = \sum_{k=0}^{n} b_k h^k$. Then (3.1) evaluated at h gives

$$\sum_{k=0}^{n} b_k h^k = \mu \sum_{k=0}^{n} b_k (h+1)^k. \tag{3.10}$$

Comparing the coefficients at corresponding powers of h and using the fact that $\mu \neq 1$ one quickly finds that (3.10) has only the trivial solution $b_0 = \ldots = b_n = 0$. Hence there are no non-trivial α-type derivations on A_1.

Consider any σ_μ-derivation ∂_m of non-negative standard degree m. Necessarily, $\partial_m(x) = c(h)x^{m+1}$, for some $c(h) = \sum_{k=0}^{n} c_h h^n \in \mathbb{K}[h]$. Let us consider further the following system of equations for the $j_k \in \mathbb{K}$,

$$\left(\mu^{-1} - 1\right) j_k - \sum_{i=1}^{n-k} \binom{k+i}{i} j_{k+i} = c_k, \qquad k = 0, \ldots, n. \tag{3.11}$$

Arranging the unknown j_k in the descending indices order, one immediately finds that the matrix of coefficients of (3.11) is lower-triangular with non-zero diagonal entries $\mu^{-1} - 1$, hence it has non-zero determinant $\left(\mu^{-1} - 1\right)^{n+1}$. The fact that the system of equations (3.11) can always be solved means that there exists a polynomial $j(h) = \sum_{k=0}^{n} j_k h^k$ such that

$$c(h) = \mu^{-1} j(h) - j(h+1), \tag{3.12}$$

irrespective of the choice of $c(h)$. Let ∂_m^b be an inner σ_μ-derivation associated to $b = j(h)x^m$. Then

$$\partial_m^b(x) = \mu^{-1} j(h) x^{m+1} - x j(h) x^m = \left(\mu^{-1} j(h) - j(h+1)\right) x^{m+1} = c(h) x^{m+1},$$

by (2.2) and (3.12), i.e.

$$(\partial_m^b - \partial_m)(x) = 0. \tag{3.13}$$

Since skew derivations form an Abelian group, $\partial_m^b - \partial_m$ is a σ_μ-skew derivation which vanishes on x by (3.13). By Lemma 3.2, $\partial_m^b - \partial_m$ is necessarily an α-type derivation, hence it is zero by the preceding discussion, and thus $\partial_m = \partial_m^b$ is an inner σ_μ-skew derivation. The negative degree case is dealt with in a similar way (or follows by the x-y symmetry).

Therefore, we conclude that all σ_μ-skew derivations of A_1 are inner.

The construction of Theorem 3.1 can be given cohomological interpretation.

Corollary 3.10. *In the set-up of Theorem 3.1, for all non-zero m, the assignment $\alpha_m \mapsto \partial_m^\alpha$ induces an injective map*

$$HH^1_{\varphi^m\circ\sigma;\,\varphi^{m-1}(\mu),\varphi}(R) \longrightarrow HH^1_{\sigma_\mu}\left(R(a,\varphi)\right),$$

of (doubly in the domain) twisted Hochschild cohomology groups.

Proof. Since $(\alpha_m, \varphi^m\circ\sigma)$ is a skew derivation, it is an element of $HC^1(R, R_{\varphi^m\circ\sigma})$, the first of conditions (3.1) implies that $\alpha_m \in HC^1_{\varphi^m\circ\sigma;\,\varphi^{m-1}(\mu),\varphi}(R)$. If α_m is inner with respect to $s \in HC^0_{\varphi^m\circ\sigma;\,\varphi^{m-1}(\mu),\varphi}(R)$, i.e. an element of R such that $s = \varphi^m(\mu)\varphi(s)$, then one easily checks that ∂_m^α is inner with respect to sx^m. This proves the existence of the map between cohomology groups.

If ∂_m^α is inner, then for all $r\in R$,

$$\begin{aligned}
\partial_m^\alpha(r) &= \sum_k s_k x^k\sigma(r) + \sum_l r_l y^l\sigma(r) - \sum_k r s_k x^k - \sum_l r r_l y^l \\
&= \sum_k \left(s_k\varphi^k(\sigma(r)) - rs_k\right)x^k + \sum_l \left(r_l\varphi^{-l}(\sigma(r)) - rr_l\right)y^l = \alpha_m(r)x^m,
\end{aligned}$$

which implies that $r_l = 0$ for all l, and $s_k = 0$ for all $k \neq m$. Hence

$$\alpha_m(r) = s_m\varphi^m(\sigma(r)) - rs_m,$$

i.e. α_m is inner. Therefore, the constructed map is an additive monomorphism, as stated. □

The proof of Theorem 3.1, specifically Lemma 3.2 and Lemma 3.3, provides one with almost full classification of skew derivations of a generalized Weyl algebra.

Corollary 3.11. *Let $R(a,\varphi)$ be a generalized Weyl algebra with $a\in R$ neither zero nor a zero divisor, and let σ be an automorphism of R commuting with φ and fixing a. Let σ_μ be the degree-counting extension of σ of coarseness μ. If (∂,σ_μ) is a skew derivation of $R(a,\varphi)$ such that either*

(i) $\partial(R)\subset R(a,\varphi)_+$ and $\partial(x) = 0$, or

(ii) $\partial(R)\subset R(a,\varphi)_-$ and $\partial(y) = 0$, or

(iii) $\partial(R) = 0$,

then it is a sum of elementary derivations as in Theorem 3.1.

Proof. In the first case necessarily,

$$\partial(r) = \sum_{m>0}\alpha_m(r)\,x^m, \qquad \text{for all } r\in R.$$

Setting $\partial(x) = 0$, assuming the general form

$$\partial(y) = \sum_{i\geq 0} c_i x^i + \sum_{j>0} d_j y^j,$$

and demanding $\partial(yx-a)=0$, one obtains:

$$\begin{aligned}\sum_{m>0}\alpha_m(a)\,x^m &= \sum_{i\geq 0}\varphi^i\left(\mu^{-1}\right)c_i x^{i+1}+\sum_{j>0}\varphi^{-j}\left(\mu^{-1}\right)d_j y^j x\\ &= \sum_{i\geq 0}\varphi^i\left(\mu^{-1}\right)c_i x^{i+1}+\sum_{j>0}\varphi^{-j}\left(\mu^{-1}\right)d_j\varphi^{-j+1}(a)y^{j-1}.\end{aligned}$$

This implies that $d_j=0$, for all j, while $c_{m-1}=\varphi^{m-1}(\mu)\,\alpha_m(a)$, and Lemma 3.2 affirms the necessity of conditions listed in Theorem 3.1. The second case is deduced by the x-y symmetry. Since every skew derivation on $R(a,\varphi)$ is a locally finite sum of homogeneous (with respect to the standard grading) skew derivations, the third case follows directly from Lemma 3.3. □

So far we have made no restrictions on the central unit $\mu\in R$, which determined the coarseness of the degree-counting automorphism. In all examples we have in mind, however, where typically R is an algebra over a field and μ is a scalar parameter, μ is a central element in the whole of the generalized Weyl algebra $R(a,\varphi)$, or equivalently, $\varphi(\mu)=\mu$. Furthermore, if μ is scalar, also $\sigma(\mu)=\mu$. Having these typical applications in mind and to avoid undue complications in the formulae we make this assumption in the following proposition.

Proposition 3.12. *Let $R(a,\varphi)$ be a generalized Weyl algebra and let σ be an automorphism of R commuting with φ and fixing a. Let σ_μ be the degree-counting extension of σ of coarseness μ such that $\varphi(\mu)=\mu=\sigma(\mu)$, and let Q be a central unit in R such that $\varphi(Q)=\sigma(Q)=Q$. If all the $b_i=0$, $\pm i\in I$, then the skew derivation (∂,σ_μ) (3.3) is a skew Q-derivation if and only if*

(a) For all $\pm i\in I\cup\{0\}$, $(\alpha_i,\varphi^i\circ\sigma)$ are skew Q-derivations;

(b) $\sigma(c_i)=\mu^iQ\,c_i$, for all $\pm i\in I\cup\{0\}$, $\sigma(\tilde{c}_0)=Q\,\tilde{c}_0$ and $\sigma(c)=Q\,c$.

If at least one of the b_i, $\pm i\in I\cup\{0\}$, is not zero, then the skew derivation (∂,σ_μ) (3.3) is a skew Q-derivation if and only if, in addition to (a) and (b), $Q=1$.

Proof. First note that $(\alpha_i,\varphi^i\circ\sigma)$ is a skew Q-derivation if and only if

$$\sigma\circ\alpha_i\circ\sigma^{-1}=\mu^iQ\alpha_i. \tag{3.14}$$

Indeed, in view of the repeated use of (3.1),

$$\varphi^i\circ\sigma\circ\alpha_i\circ\sigma^{-1}\circ\varphi^{-i}=\mu^{-i}\,\sigma\circ\alpha_i\circ\sigma^{-1},$$

hence $\varphi^i\circ\sigma\circ\alpha_i\circ\sigma^{-1}\circ\varphi^{-i}=Q\alpha_i$ if and only if the condition (3.14) is fulfilled. Observe that, since σ fixes a and commutes with φ, conditions (3.14) imply,

$$\sigma\left(\alpha_i\left(\varphi^k(a)\right)\right)=\mu^iQ\,\alpha_i\left(\varphi^k(a)\right), \tag{3.15}$$

for all $i,k\in\mathbb{Z}$.

In deriving the sufficient and necessary conditions for (∂,σ_μ) to be skew Q-derivations we freely use the possibility of interpreting $R(a,\varphi)$ as a $\mathbb{Z}$-graded algebra with the standard grading, so that all the equalities must hold degree-wise. Assume that all the

b_i in the definition of the skew derivation (∂, σ_μ) (3.3) are equal to zero. Then,

$$\begin{aligned}\sigma_\mu \circ \partial \circ \sigma_\mu^{-1}(r) &= \sum_{m\in I\cup\{0\}} \sigma\left(\alpha_m(\sigma^{-1}(r))\right) \sigma_\mu(x^m) + \sum_{n\in I} \sigma\left(\alpha_{-n}(\sigma^{-1}(r))\right) \sigma_\mu(y^n) \\ &= \sum_{m\in I\cup\{0\}} \mu^{-m}\sigma\left(\alpha_m(\sigma^{-1}(r))\right) x^m + \sum_{n\in I} \sigma\mu^n\left(\alpha_{-n}(\sigma^{-1}(r))\right) y^n.\end{aligned}$$

This is equal to $Q\partial(r)$ if and only if (3.14) holds or, equivalently, all the $(\alpha_i, \varphi^{-1}\circ\sigma)$ are skew Q-derivations. In this way we obtain both the sufficiency and necessity of hypothesis (a). In particular the equality (3.15) must be true, and in view of this and using (3.1), we can compute

$$\begin{aligned}&\sigma_\mu \circ \partial \circ \sigma_\mu^{-1}(x) = \mu\,\sigma_\mu\left(\partial(x)\right) \\ &= \mu\left(\sum_{m\in I\cup\{0\}} \sigma(c_m)\sigma_\mu(x^{m+1}) + \sum_{n\in I}\sigma\left(\varphi(\alpha_{-n}(a) - \varphi^{-n-1}(\mu^{-1})ac_{-n})\right)\sigma_\mu(y^{n-1})\right) \\ &= \sum_{m\in I\cup\{0\}} \mu^{m+1}Qc_m\mu^{-m-1}x^{m+1} + \mu\sum_{n\in I}\varphi\left(\sigma(\alpha_{-n}(a) - \varphi^{-n-1}(\mu^{-1})ac_{-n})\right)\mu^{n-1}y^{n-1} \\ &= \sum_{m\in I\cup\{0\}} Qc_m x^{m+1} + \mu\sum_{n\in I}\varphi\left(\mu^{-n}Q\alpha_{-n}(a) - \varphi^{-n-1}(\mu^{-1})a\mu^{-n}Qc_{-n}\right)\mu^{n-1}y^{n-1} \\ &= \sum_{m\in I\cup\{0\}} Qc_m x^{m+1} + \sum_{n\in I} Q\varphi\left(\alpha_{-n}(a) - \varphi^{-n-1}(\mu^{-1})ac_{-n}\right)y^{n-1}.\end{aligned}$$

This is equal to $Q\partial(x)$ if and only in the conditions (b) for non-negative i are satisfied. In a similar way, comparing $\sigma_\mu\circ\partial\circ\sigma_\mu^{-1}(y)$ with $Q\partial(y)$ one derives the sufficiency and the necessity of the remaining conditions in (b).

Inclusion of an elementary skew derivation of inner type forces Q to be 1. This completes the proof of the proposition. □

Remark 3.13. If a is a regular element of R, then the last condition in hypothesis (b) of Proposition 3.12, i.e. that $\sigma(c) = Qc$, follows from the hypothesis (a).

Proposition 3.14. *Let R be a $\mathbb{Z}$-graded ring and consider $R(a,\varphi)$ as a $\mathbb{Z}$-graded ring with the (d,k)-type grading. Let σ be an automorphism of the graded ring R commuting with φ and fixing a. Let σ_μ be the degree-counting extension of σ of coarseness μ of degree 0. Let (∂,σ_μ) be the skew derivation associated as in Theorem 3.1 to the data α_i, b_i, c_i, I. Assume that all non-zero b_i have degree l. Then ∂ is a map of degree l if and only if,*

$$\deg(\alpha_i) = l - ik + \frac{i-|i|}{2}d, \quad \deg(c_i) = l, \qquad \forall \pm i \in I\cup\{0\}. \tag{3.16}$$

Proof. Indeed, we need here to check $\deg(\alpha_i)$ in three cases where i is zero, positive and negative respectively. Notice that the last term of the first equality in (3.16) will disappear in the first two cases. Suppose that ∂ is a map of degree l, then, remembering that $\deg(b_i) = l$ and that both σ and φ are degree-zero maps, (3.4)–(3.6) give:

$$\deg(\alpha_0) = \deg(\partial_0) = l, \quad \deg(c_0) = l,$$
$$\deg(\alpha_m) = \deg(c_m) = \deg(\partial_m) - \deg(x^m) = l - mk,$$

and

$$\deg(\alpha_{-n}) = \deg(c_{-n}) = \deg(\partial_{-n}) - \deg(y^n) = l - n(d-k) = l + nk - nd.$$

Put together this gives (3.16).

On the other hand, if (3.16) holds, then (3.4)–(3.6) clearly imply that ∂ is a map of degree l. □

Additional classes of skew derivations can be constructed for generalized Weyl algebras associated to automorphisms of finite orders.

Proposition 3.15. *Let $R(a,\varphi)$ be a generalized Weyl algebra and let σ be an automorphism of R commuting with φ and fixing a. Let σ_μ be the degree-counting extension of σ of coarseness μ. Assume that φ has finite order D, i.e.*

$$\varphi^D = \mathrm{id}, \tag{3.17}$$

and let

$$(\alpha_i, \sigma)_{i\in\mathbb{Z}},$$

be skew derivations on R such that, for all i,

$$\alpha_i \circ \varphi = \mu\, \varphi \circ \alpha_i, \qquad \alpha_i(a) \in R^R_\sigma. \tag{3.18}$$

Let $I \subseteq \mathbb{N}\setminus\{0\}$ be such that, for all $r \in R$, the sets $\{i \in I \mid \alpha_{\pm i}(r) \neq 0\}$ are finite. For all $b_m, c_n \in R^R_\sigma$, define

$$\partial(r) = \sum_{m\in I} \alpha_m(r)\, x^{mD} + \sum_{n\in I} \alpha_{-n}(r)\, y^{nD}, \qquad \text{for all } r \in R, \tag{3.19a}$$

$$\partial(x) = \sum_{m\in I} b_m x^{mD+1} + \mu^{-1} \sum_{n\in I} \left(\alpha_{-n}(\varphi(a)) - \varphi\left(c_n\right) a\right) y^{nD-1}, \tag{3.19b}$$

$$\partial(y) = \varphi^{-1}(\mu) \sum_{m\in I} \left(\alpha_m(a) - \varphi^{-1}\left(b_m\right) a\right) x^{mD-1} + \sum_{n\in I} c_n\, y^{nD+1}. \tag{3.19c}$$

Then ∂ extends to a skew derivation (∂, σ_μ) on $R(a,\varphi)$.

Proof. As was the case for Theorem 3.1, we prove that, for all m and n, the following maps extend to the derivations of $R(a,\varphi)$,

$$\partial_m(r) = \alpha_m(r)\, x^{mD},\ \ \partial_m(x) = b_m x^{mD+1},\ \ \partial_m(y) = \varphi^{-1}(\mu)\left(\alpha_m(a) - \varphi^{-1}\left(b_m\right) a\right) x^{mD-1},$$

$$\partial_{-n}(r) = \alpha_{-n}(r)\, y^{nD},\ \ \partial_{-n}(x) = \mu^{-1}\left(\alpha_{-n}(\varphi(a)) - \varphi\left(c_n\right) a\right) y^{nD-1},\ \ \partial_{-n}(y) = c_n\, y^{nD+1}.$$

First, since (α_m, σ) is a skew derivation on R, and σ_μ restricted to R is equal to σ, ∂_m satisfies the σ_μ-twisted Leibniz rule. We need to check that ∂_m extended to the whole of $R(a,\varphi)$ by the σ_μ-twisted Leibniz rule preserves all relations (2.2). In view of (3.17) we make constant use of the fact that all powers of φ can be calculated modulo D and start by computing

$$\begin{aligned}\partial_m(xy - \varphi(a)) &= b_m x^{mD+1} y\mu + x\varphi^{-1}(\mu)(\alpha_m(a) - \varphi^{-1}(b_m)a)x^{mD-1} - \alpha_m(\varphi(a))x^{mD}\\ &= \mu b_m \varphi(a) x^{mD} + \mu\varphi(\alpha_m(a))x^{mD} - \mu b_m\varphi(a)x^{mD} - \mu\varphi(\alpha_m(a))x^{mD} = 0,\end{aligned}$$

where (2.2), (3.18) and the centrality of $\mu \in R$ were used. In a similar way one easily finds that,

$$\begin{aligned}\partial_m(yx - a) &= \varphi^{-1}(\mu)\left(\alpha_m\left(a\right) - \varphi^{-1}(b_m)a\right)x^{mD-1}\mu^{-1}x + y b_m x^{mD+1} - \alpha_m(a)x^{mD}\\ &= \alpha_m(a)x^{mD} - \varphi^{-1}(b_m)ax^{mD} + \varphi^{-1}(b_m)ax^{mD} - \alpha_m(a)x^{mD} = 0.\end{aligned}$$

Furthermore, for all $r \in R$,

$$\begin{aligned}\partial_m(xr-\varphi(r)x) &= b_m x^{mD+1}\sigma(r) + x\alpha_m(r)x^{mD} - \alpha_m(\varphi(r))x^{mD}\mu^{-1}x - \varphi(r)b_m x^{mD+1}\\ &= b_m\varphi(\sigma(r))x^{mD+1} + \varphi(\alpha_m(r))x^{mD+1} - \mu^{-1}\alpha_m(\varphi(r))x^{mD+1}\\ &\quad -\varphi(r)b_m x^{mD+1} = b_m\varphi(\sigma(r))x^{mD+1} - \varphi(r)b_m x^{mD+1} = 0,\end{aligned}$$

where the first equality follows by the definition of ∂_m via the twisted Leibniz rule. The second one follows by (2.2) and the centrality of μ, the third one by (3.18), while the last one follows by the fact that $b_m \in R^R_\sigma$. In a similar way,

$$\begin{aligned}\partial_m(yr-\varphi^{-1}(r)y) &= \varphi^{-1}(\mu)\left(\alpha_m(a)-\varphi^{-1}\left(b_m\right)a\right)x^{mD-1}\sigma(r) + y\alpha_m(r)x^{mD}\\ &\quad -\alpha_m(\varphi^{-1}(r))x^{mD}y\mu - \varphi^{-1}(r)\varphi^{-1}(\mu)\left(\alpha_m(a)-\varphi^{-1}\left(b_m\right)a\right)x^{mD-1}\\ &= \varphi^{-1}(\mu)\alpha_m(a)\varphi^{-1}(\sigma(r))x^{mD-1} - \varphi^{-1}(\mu)\varphi^{-1}(b_m)a\varphi^{-1}(\sigma(r))x^{mD-1}\\ &\quad +\varphi^{-1}(\alpha_m(r))ax^{mD-1} - \varphi^{-1}(\mu)\alpha_m(\varphi^{-1}(r))\varphi^{mD}(a)x^{mD-1}\\ &\quad -\varphi^{-1}(\mu)\varphi^{-1}(r)\alpha_m(a)x^{mD-1} + \varphi^{-1}(\mu)\varphi^{-1}(r)\varphi^{-1}(b_m)ax^{mD-1} = 0.\end{aligned}$$

Thus, ∂_m vanishes on all generators of the ideal in $R\langle x, y\rangle$ that defines $R(a,\varphi)$, hence ∂_m extends to a σ_μ-twisted derivation to the whole of $R(a,\varphi)$. The fact that ∂_{-n} extends to the whole of $R(a,\varphi)$ as a σ_μ-derivations follows by the x-y symmetry. □

4. Orthogonal pairs of α-type elementary skew derivations

The orthogonality of a system of skew derivations on a given ring A relies heavily on the structure of A, and – in general – very little can be said even in the case of rather explicitly defined generalized Weyl algebras over R if the ring R is not specified. The cases of R being a polynomial ring in one and two variables are discussed in some detail in [11]. Here, rather than specifying R, we would like to concentrate on a general case, and in such a case at least some examples of pairs of orthogonal skew derivations (included in the families described in Theorem 3.1) can be given. We start with the following simple observation.

Lemma 4.1. *Let $(\partial_i,\sigma_i)_{i=1}^n$ be a family of skew derivations on a ring A. If there exist $\{b_1, b_2, \ldots, b_n\} \subset A$ such that*

$$A\partial_i(b_i)A = A, \qquad \partial_k(b_i) = 0, \quad \text{for all } i \neq k, \tag{4.1}$$

then $(\partial_i,\sigma_i)_{i=1}^n$ is an orthogonal system of skew derivations.

Proof. The fact that the ideal generated by $\partial_i(b_i)$ is equal to A is equivalent to the existence of finite subsets $\{a_{i\,t}\}, \{c_{i\,t}\}$ of elements of A such that,

$$1 = \sum_t a_{i\,t}\partial_i(b_i)c_{i\,t} = \sum_t a_{i\,t}\partial_i\left(b_i\sigma_i^{-1}\left(c_{i\,t}\right)\right) - \sum_t a_{i\,t}b_i\partial_i\left(\sigma_i^{-1}\left(c_{i\,t}\right)\right),$$

where the second equality follows by the σ_i-twisted Leibniz rule. This gives condition (2.8) with $i = k$. If $i \neq k$, then,

$$\begin{aligned}\sum_t a_{i\,t}\partial_k\left(b_i\sigma_i^{-1}\left(c_{i\,t}\right)\right) &- \sum_t a_{i\,t}b_i\partial_k\left(\sigma_i^{-1}\left(c_{i\,t}\right)\right)\\ &= \sum_t a_{i\,t}b_i\partial_k\left(\sigma_i^{-1}\left(c_{i\,t}\right)\right) - \sum_t a_{i\,t}b_i\partial_k\left(\sigma_i^{-1}\left(c_{i\,t}\right)\right) = 0,\end{aligned}$$

by the σ_k-twisted Leibniz rule and since $\partial_k(b_i) = 0$. This confirms that (2.8) holds also for $i \neq k$. □

In the following, by saying that two elements $r, s \in R$ are *coprime* we will mean that the ideals generated by them are coprime, i.e. that

$$RrR + RsR = R.$$

Proposition 4.2. *Let $R(a, \varphi)$ be a generalized Weyl algebra and let σ, $\bar{\sigma}$ be automorphisms of R commuting with φ and fixing a. Let σ_μ, $\bar{\sigma}_{\bar{\mu}}$ be their degree-counting extensions with respective coarseness μ and $\bar{\mu}$. Choose a positive integer N such that a is coprime with $\varphi^i(a)$, for all $i \in \{1, 2, \ldots, 2N-1\}$, fix $m, n \in \{0, 1, \ldots, N\}$ and consider the following data:*

(a) A skew derivation $(\alpha, \sigma \circ \varphi^{m+1})$ of R such that
   - *(i) $\alpha(a)$ is in the centre of R,*
   - *(ii) $\alpha(a)$ is coprime with $\varphi^j(a)$, $j \in \{-m-1, -m, \ldots, 0, m+1, m+2, \ldots, 2m\}$ and with $\varphi^{-m}(\alpha(a))$,*
   - *(iii) $\alpha \circ \varphi = \varphi^{m+1}(\mu)\, \varphi \circ \alpha$.*

(b) A skew derivation $(\bar{\alpha}, \bar{\sigma} \circ \varphi^{-n-1})$ of R such that
   - *(i) $\bar{\alpha}(a)$ is in the centre of R,*
   - *(ii) $\varphi^{n+1}(\bar{\alpha}(a))$ is coprime with $\varphi^j(a)$, $j \in \{-n-1, -n, \ldots, 0, n+1, n+2, \ldots, 2n\}$ and with $\varphi(\bar{\alpha}(a))$,*
   - *(iii) $\bar{\alpha} \circ \varphi = \varphi^{-n-1}(\bar{\mu})\, \varphi \circ \bar{\alpha}$.*

Then the elementary α-type skew derivations (∂, σ_μ) and $(\bar{\partial}, \bar{\sigma}_{\bar{\mu}})$ of $R(a, \varphi)$ associated to α, $\bar{\alpha}$ form an orthogonal pair.

Proof. Explicitly, the α-type elementary weight $m+1$ and $-n-1$ respectively skew derivations ∂, $\bar{\partial}$ are given by

$$\partial(r) = \alpha(r) x^{m+1}, \qquad \partial(x) = 0, \qquad \partial(y) = \alpha(a)\, x^m \mu\,, \tag{4.2a}$$

$$\bar{\partial}(r) = \bar{\alpha}(r) y^{n+1}, \qquad \bar{\partial}(x) = \varphi\left(\bar{\alpha}\left(a\right)\right) y^n, \qquad \bar{\partial}(y) = 0, \tag{4.2b}$$

for all $r \in R$, and then extended to the whole of $R(a, \varphi)$ by the twisted Leibniz rules. We will show that $\partial(y)$ and $\bar{\partial}(x)$ generate ideals both equal to $R(a, \varphi)$ and then use Lemma 4.1 to conclude that (∂, σ_μ) and $(\bar{\partial}, \bar{\sigma}_{\bar{\mu}})$ form an orthogonal pair. Observe that, in view of (2.2) and the centrality of $\alpha(a)$,

$$y^m \partial(y) = \mu \varphi^{-m}\left(\alpha\left(a\right)\right) \varphi^{-m+1}\left(a\right) \cdots \varphi^{-1}(a) a,$$

and

$$\partial(y) y^m = \varphi^m(\mu) \alpha(a) \varphi(a) \varphi^2(a) \cdots \varphi^m(a).$$

Hence the ideal generated by $\partial(y)$ is equal to the whole of $R(a, \varphi)$, provided

$$R\varphi^{-m}\left(\alpha\left(a\right)\right) \varphi^{-m+1}\left(a\right) \cdots \varphi^{-1}(a) a + R\alpha(a) \varphi(a) \varphi^2(a) \cdots \varphi^m(a) = R. \tag{4.3}$$

In a similar way,

$$x^n \bar{\partial}(x) = \varphi^{n+1}\left(\bar{\alpha}\left(a\right)\right) \varphi^n\left(a\right) \cdots \varphi^2(a) \varphi(a),$$

and

$$\bar{\partial}(x) x^n = \varphi\left(\bar{\alpha}(a)\right) a \varphi^{-1}(a) \cdots \varphi^{-n+1}(a).$$

Hence the ideal generated by $\bar{\partial}(x)$ is equal to the whole of $R(a,\varphi)$, provided

$$R\varphi\left(\bar{\alpha}(a)\right)a\varphi^{-1}(a)\cdots\varphi^{-n+1}(a)+R\varphi^{n+1}\left(\bar{\alpha}\left(a\right)\right)\varphi^{n}\left(a\right)\cdots\varphi^{2}(a)\varphi(a)=R. \tag{4.4}$$

Since a is coprime with all $\varphi^i(a)$, $i\in\{1,2,\ldots,2N-1\}$, $\varphi^{-i}(a)$ is coprime with $\varphi^j(a)$, for all $i\in\{0,1,\ldots,N-1\}$ and $j\in\{1,2,\ldots,N\}$, and hence,

$$R=Ra+R\varphi^j(a)=\big(R\varphi^{-1}(a)+R\varphi^j(a)\big)\,a+R\varphi^j(a)=R\varphi^{-1}(a)a+R\varphi^j(a),$$

where the last equality is a consequence of $R\varphi^j(a)a\subseteq R\varphi^j(a)$. Next,

$$\begin{aligned}R &= R\varphi^{-1}(a)a+R\varphi^j(a)=\big(R\varphi^{-2}(a)+R\varphi^j(a)\big)\,\varphi^{-1}(a)a+R\varphi^j(a)\\ &= R\varphi^{-2}(a)\varphi^{-1}(a)a+R\varphi^j(a).\end{aligned}$$

Repeating this sufficiently many times, one concludes that

$$R=R\varphi^{-i}(a)\cdots\varphi^{-2}(a)\varphi^{-1}(a)a+R\varphi^j(a), \tag{4.5}$$

for all $i\in\{0,1,\ldots,N-1\}$ and $j\in\{1,2,\ldots,N\}$. Similarly, starting with $R=Ra+R\alpha(a)$, and using that $\alpha(a)$ is coprime with $\varphi^{-m}(\alpha(a))$ and all $\varphi^{-i}(a)$, where $i\in\{0,1,\ldots,m-1\}$, by the same arguments one obtains that

$$R=R\varphi^{-m}(\alpha(a))\varphi^{-m+1}(a)\cdots\varphi^{-2}(a)\varphi^{-1}(a)a+R\alpha(a). \tag{4.6}$$

Since $\alpha(a)$ is coprime with $\varphi^j(a)$, for all $j\in\{m+1,\ldots,2m\}$, $\varphi^{-m}(\alpha(a))$ is coprime with $\varphi^j(a)$, for all $j\in\{1,\ldots,m\}$. Bearing in mind that $m\le N$, (4.5) implies that

$$\begin{aligned}R &= \big(R\varphi^{-m}(\alpha(a))+R\varphi^j(a)\big)\,\varphi^{-i}(a)\cdots\varphi^{-2}(a)\varphi^{-1}(a)a+R\varphi^j(a)\\ &= R\varphi^{-m}(\alpha(a))\varphi^{-m+1}(a)\cdots\varphi^{-2}(a)\varphi^{-1}(a)a+R\varphi^j(a),\end{aligned} \tag{4.7}$$

for all $j\in\{1,\ldots,m\}$. Starting with (4.6) and then repeatedly using (4.7) we thus obtain

$$\begin{aligned}R &= R\varphi^{-m}(\alpha(a))\varphi^{-m+1}(a)\cdots\varphi^{-2}(a)\varphi^{-1}(a)a+R\alpha(a)\\ &= R\varphi^{-m}(\alpha(a))\varphi^{-m+1}(a)\cdots\varphi^{-2}(a)\varphi^{-1}(a)a\\ &\quad+\big(R\varphi^{-m}(\alpha(a))\varphi^{-m+1}(a)\cdots\varphi^{-2}(a)\varphi^{-1}(a)a+R\varphi(a)\big)\,\alpha(a)\\ &= R\varphi^{-m}(\alpha(a))\varphi^{-m+1}(a)\cdots\varphi^{-2}(a)\varphi^{-1}(a)a+R\varphi(a)\alpha(a)\\ &= R\varphi^{-m}(\alpha(a))\varphi^{-m+1}(a)\cdots\varphi^{-2}(a)\varphi^{-1}(a)a\\ &\quad+\big(R\varphi^{-m}(\alpha(a))\varphi^{-m+1}(a)\cdots\varphi^{-2}(a)\varphi^{-1}(a)a+R\varphi^2(a)\big)\,\varphi(a)\alpha(a)\\ &= R\varphi^{-m}(\alpha(a))\varphi^{-m+1}(a)\cdots\varphi^{-2}(a)\varphi^{-1}(a)a+R\varphi^2(a)\varphi(a)\alpha(a)\\ &= \ldots\\ &= R\varphi^{-m}(\alpha(a))\varphi^{-m+1}(a)\cdots\varphi^{-2}(a)\varphi^{-1}(a)a+R\varphi^m(a)\cdots\varphi^2(a)\varphi(a)\alpha(a),\end{aligned}$$

i.e. the required equation (4.3). Replacing m by n and $\alpha(a)$ by $\varphi^{n+1}(\bar{\alpha}(a))$ in the above arguments, one finds that also (4.4) holds. Now Lemma 4.1 implies that (∂,σ_μ) and $(\bar{\partial},\sigma_{\bar{\mu}})$ form an orthogonal pair. □

Remark 4.3. Note that since, for all central elements r,s of R, $Rrs\subseteq Rr$, Rs, equality (4.3) implies hypothesis (a)(ii), while (4.4) implies hypothesis (b)(ii) in Proposition 4.2.

Remark 4.4. Following [6], $R(a,\varphi)$ is said to *satisfy condition (C)* if, for all maximal ideals $\mathfrak{p}$ in R and all $n \in \mathbb{Z} \setminus \{0\}$, $\varphi^n(\mathfrak{p}) \neq \mathfrak{p}$. As observed in [6, p. 226], if R is a Dedekind ring, then condition (C) implies the existence of N such that a is coprime with $\varphi^i(a)$, for all $i \in \{1, 2, \ldots, 2N-1\}$.

5. Local nilpotency of elementary α-type derivations

A skew-derivation (∂, σ) on a ring R is said to be *locally nilpotent* if for all $r \in R$ there are $k \in \mathbb{N}$ such that $\partial^k(r) = 0$. The existence of locally nilpotent derivations may have significant bearing on the structure of R. If R is an algebra over a field $\mathbb{K}$, and (∂, σ) is a Q-skew derivation with $Q \in \mathbb{K}^*$ not a root of unity, and there exists an element $T \in R$ such that $\partial(T) = 1$, then the Bergen-Grzeszczuk theorem [10, Theorem 1] identifies R as a specific Ore extension of the ring

$$R^{\partial} := \{r \in R \mid \partial(r) = 0\}$$

of invariants of (∂, σ).

In this section we prove that, under mild condition, local nilpotency of $(\alpha_i, \varphi^i \circ \sigma)$, $i \neq 0$, implies local nilpotency of the associated elementary α-type derivation. We draw conclusions about the structure of $R(a,\varphi)$ in case the hypotheses of the Bergen-Grzeszczuk theorem are met, and also identify severe obstructions which prevent the weight-zero α-type elementary derivation to be locally nilpotent. Throughout this section we assume that $\varphi(\mu) = \mu$.

Proposition 5.1. *Let $(\partial_i^\alpha, \sigma_\mu)$ be a the α-type elementary skew derivation of $R(a,\varphi)$ of non-zero weight associated to a skew derivation $(\alpha_i, \varphi^i \circ \sigma)$ of R as in Theorem 3.1. If $\alpha_i(\mu) = 0$ and α_i is a locally nilpotent skew derivation, then also $(\partial_i^\alpha, \sigma_\mu)$ is a locally nilpotent skew derivation.*

Proof. We concentrate on the non-negative weight case, the other case follows by the x-y symmetry. Using repeatedly the definition of ∂_m^α, and the fact that $\alpha_i(\mu) = 0$ (and our standing assumption that $\varphi(\mu) = \mu$) one easily finds that, for all $r \in R$ and $k, l \in \mathbb{N}$,

$$\partial_m^{\alpha\, k}(rx^l) = \mu^{-k\left(l+m\frac{k-1}{2}\right)} \alpha_m^k(r) x^{km+l}. \tag{5.1}$$

Since α_m is locally nilpotent, for any $\sum_l r_i x^l \in R(a,\varphi)$ there exists $k \in \mathbb{N}$ such that $\partial_m^{\alpha\, k}\left(\sum_l r_l x^l\right) = 0$.

Observe that, for all $r \in R$ and $l \in \mathbb{N}$,

$$\partial_m^\alpha(ry^l) = \begin{cases} c_m^l(r) x^{m-l}, & \text{if } l \leq m, \\ c_m^l(r) y^{l-m}, & \text{if } l > m, \end{cases} \tag{5.2}$$

for some $c_m^l(r) \in R$. If $l \leq m$, then the nilpotency of ∂_m^α on ry^l follows by the nilpotency of α_m and by (5.1). In the other case, let k and s be such that

$$l = km + s, \qquad s \leq m.$$

The repeated use of the formula (5.2) yields

$$\partial_m^{\alpha\, k}(ry^{km+s}) = c_m^s\left(c_m^{m+s}\left(\cdots\left(c_m^{km+s}(r)\right)\cdots\right)\right) x^{m-s},$$

and the nilpotency of ∂_m^α on ry^l follows by the nilpotency of α_m and by (5.1). Therefore, ∂_m^α is a locally nilpotent skew derivation on $R(a,\varphi)$. □

Corollary 5.2. *Let R be an algebra over a field $\mathbb{K}$ and let (α,σ) be a skew q-derivation, where $q\in\mathbb{K}^*$ is not a root of unity. Let φ be an automorphism of R commuting with σ and let a be a central element of R. If there exist a positive integer m, $\mu\in\mathbb{K}^*$ and $T\in R$, such that $\alpha\circ\varphi=\mu\,\varphi\circ\alpha$, $\sigma(a)=\varphi^m(a)$, and*

$$\mu^m\alpha(T)\varphi^m(a)\cdot\ldots\cdot\varphi(a)+T\sum_{k=0}^{m-1}\mu^{m-k}\varphi^{-k}\left(\alpha(a)\varphi^{m-1}(a)\cdot\ldots\cdot\varphi(a)\right)=1, \tag{5.3}$$

then $R(a,\varphi)$ is (isomorphic to) a skew-polynomial ring (i.e. the Ore extension of B) $B[z;\delta,\hat\sigma]$, where B is the subalgebra of invariants of the weight-m α-type elementary skew-derivation $(\partial_m^\alpha,(\varphi^{-m}\circ\sigma)_\mu)$ associated to (α,σ), $\hat\sigma$ is the restriction of the inverse of $(\varphi^{-m}\circ\sigma)_\mu$ to B, and

$$\delta(b)=Ty^m\hat\sigma(b)-bTy^m.$$

Proof. By Proposition 3.12, $(\partial_m^\alpha,(\varphi^{-m}\circ\sigma)_\mu)$ is a skew q-derivation. The condition (5.3) is equivalent to $c_m^m(T)=1$, where $c_m^m(r)$ is defined in (5.2), hence $\partial_m^\alpha(Ty^m)=1$. Hence all the hypotheses of (the right-twisted skew derivation version of) the Bergen-Grzeszczuk theorem [10, Theorem 1] are satisfied and the corollary follows. We note that indeterminate z can be identified as $Ty^m\in R(a,\varphi)$. □

While the transmission of the local nilpotency of $(\alpha_i,\varphi^i\circ\sigma)$ to the local nilpotency of an elementary α-type derivation of non-zero weight depends neither on R nor on the additional properties of α_i (bar $\alpha_i(\mu)=0$), the possibility of the zero weight elementary α-type derivation to be locally nilpotent puts severe restrictions on R or α_0. We illustrate this in the case of an algebra over a field and a skew q-derivation, where q is not a root of unity.

Lemma 5.3. *Let R be an algebra over the field $\mathbb{K}$ and let (α,σ) be a locally nilpotent skew q-derivation, where $q\in\mathbb{K}^*$ is not a root of unity. Let $\mu\in\mathbb{K}^*$ and let φ be an algebra automorphism of R commuting with σ and such that $\alpha\circ\varphi=\mu\,\varphi\circ\alpha$. Let a be central element of R, for which there exists a regular element $c\in R_\sigma^R$ such that $\alpha(a)=ca$. Then the weight 0 elementary α-type skew derivation of $R(a,\varphi)$ determined by*

$$\partial(r)=\alpha(r),\qquad\partial(x)=0,\qquad\partial(y)=\mu c\,y,$$

is not locally nilpotent.

Proof. Since $c\in R_\sigma^R$, $0=c\sigma(c)-cc=c(\sigma(c)-c)$, and the regularity of c implies that $\sigma(c)=c$. In view of the σ-invariance of c and the fact that α is a skew q-derivation one easily finds that, for all $k\in\mathbb{N}$,

$$\alpha^k(c)c=q^kc\alpha^k(c). \tag{5.4}$$

Applying ∂ k-times to y, and using (5.4), we find that

$$\partial^k(y)=\sum_{i=0}^{k-1}r_iy,$$

where $r_i\in R$ are such that

$$r_ic=q^icr_i,\qquad i=1,\ldots,k-1. \tag{5.5}$$

In particular $r_0 = \mu^k c^k$. If ∂ is locally nilpotent, then there exists k such that $\partial^k(y) = 0$, which yields the equation

$$\sum_{i=0}^{k-1} r_i = 0 \tag{5.6}$$

Multiplying equation (5.6) by c^i, $l = 1, \ldots, k$ from the right, commuting all the c to the left with the help of (5.5), and using the regularity of c we end up with the following system of k equations for the r_i,

$$\sum_{i=0}^{k-1} q^{il} r_i = 0, \qquad l = 0, 1, \ldots, k-1.$$

The matrix of the coefficients for this system has the Vandermonde form

$$\begin{pmatrix} 1 & q & q^2 & \cdots & q^{k-1} \\ 1 & q^2 & q^4 & \cdots & q^{2(k-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & q^k & q^{2k} & \cdots & q^{k(k-1)} \end{pmatrix},$$

with the determinant $\prod_{1<i<j\leq k}(q^j - q^i)$ which is non-zero, since q is not a root of unity. This implies that all the $r_i = 0$, in particular $r_0 = \mu^k c^k = 0$, which contradicts the regularity of c. □

6. Skew derivations of the quantum disc and quantum plane algebras

In this section we apply the results of Section 3 to generalized Weyl algebras over a polynomial ring in one variable associated to linear polynomials and known as the *quantum disc algebra* and the *quantum plane* or the *quantum polynomial ring in two variables*. In particular, we show that all skew σ_μ-derivations on these algebras are classified by Theorem 3.1 as long as the deformation parameter is not a root of unity.

Let $\mathbb{K}$ be a field and q a non-zero element of $\mathbb{K}$. The *coordinate algebra of the quantum disc* $D_q(x, y)$ or the *quantum disc algebra* is a $\mathbb{K}$-algebra generated by x, y and the relation

$$xy - q\, yx = 1 - q. \tag{6.1}$$

The *quantum polynomial ring in two variables* or the *quantum plane algebra* is a $\mathbb{K}$-algebra $\mathbb{K}_q[x, y]$ generated by x, y and the relation

$$xy = q\, yx. \tag{6.2}$$

Both algebras have $\mathbb{K}$-linear bases given by all monomials $y^m x^n$. $D_q(x, y)$ and $\mathbb{K}_q[x, y]$ are examples of generalized Weyl algebras. Consider the following automorphism of the polynomial algebra in one variable,

$$\varphi : \mathbb{K}[h] \to \mathbb{K}[h], \qquad f(h) \mapsto f(qh). \tag{6.3}$$

Then

$$D_q(x, y) = \mathbb{K}[h](1 - h, \varphi) \quad \text{and} \quad \mathbb{K}_q[x, y] = \mathbb{K}[h](h, \varphi). \tag{6.4}$$

Non-zero scalar multiples of the identity are the only units of the polynomial algebra $\mathbb{K}[h]$. We choose such a multiple μ. Since an automorphism σ considered in Lemma 2.3

should satisfy $\sigma(1-h) = 1-h$, in the case of $D_q(x,y)$ or $\sigma(h) = h$ in the $\mathbb{K}_q[x,y]$-case, σ must be the identity automorphism. Thus σ_μ is fully determined by (2.13).

Remark 6.1. Note that, up to isomorphism, $D_q(x,y)$ and $\mathbb{K}_q[x,y]$ are the only two generalized Weyl algebras over $\mathbb{K}[h]$ corresponding to the automorphism (6.3) and a linear polynomial. Indeed, the relations

$$yx = \alpha + \beta\, h, \qquad xy = \alpha + q\beta\, h, \qquad \beta \neq 0,$$

yield

$$xy - qyx = (1-q)\alpha.$$

If $\alpha = 0$ we obtain $\mathbb{K}_q[x,y]$, while if $\alpha \neq 0$, by rescaling the generators we obtain $D_q(x,y)$.

Theorem 6.2. *Assume that a non-zero $q \in \mathbb{K}$ is not a root of unity, and let A be either the disc algebra $D_q(x,y)$ or the quantum polynomial ring $\mathbb{K}_q[x,y]$. Set $h = 1 - yx$ if $A = D_q(x,y)$ or $h = yx$ if $A = \mathbb{K}_q[x,y]$, and let μ be a non-zero element of $\mathbb{K}$.*

(1) For all $f(h) \in \mathbb{K}[h]$, the map ∂ on generators of A given by

$$\partial(x) = f(h)x, \qquad \partial(y) = -\mu f(q^{-1}h)y, \tag{6.5}$$

extends to a skew derivation (∂, σ_μ) of A. These are the only σ_μ-derivations such that $\partial(h) = 0$. They are inner if and only if there is no $d \in \{0, \ldots, \deg(f)\}$ such that $\mu = q^{-d}$, and the coefficient f_d in $f(h) = \sum_k f_k h^k$ is not zero.

(2) If there exists $d \in \mathbb{N}$ such that

$$\mu = q^{-d+1}, \tag{6.6}$$

then:

(a) for all $a(x) \in \mathbb{K}[x]$ and $b(y) \in \mathbb{K}[y]$, the map given by

$$\partial(x) = h^d\, b(y), \qquad \partial(y) = h^d\, a(x), \tag{6.7}$$

extends to a skew derivation (∂, σ_μ) of A. All these derivations are inner if $d \neq 0$, and they are not inner if $d = 0$.

(b) If $d \geq 1$, then for all $\lambda \in \mathbb{K}^*$, the map given by

$$\partial(x) = 0, \qquad \partial(y) = \lambda h^{d-1}\, y, \tag{6.8}$$

extends to a non-inner skew derivation on $\mathbb{K}_q[x,y]$.

(3) The (combinations of the) above maps together with the inner-type derivations exhaust all σ_μ-skew derivations on A contained in Theorem 3.1. Every σ_μ-skew derivation on A is of this type.

Proof. First we study all possible σ_μ-skew derivations on A that satisfy the assumptions of Theorem 3.1. Since in our case σ is the identity map, we first determine φ^n-skew derivations of the polynomial algebra $\mathbb{K}[h]$. The action of φ^n on any element of $\mathbb{K}[h]$ results in rescaling the h by q^n. Thus any φ^n-skew derivation ∂_n of $\mathbb{K}[h]$ takes the form of a multiple of an appropriate Jackson's derivative (understood as the ordinary derivative in case $n = 0$),

$$\partial_n(f(h)) = a_n(h) f'_{q^n}(h) := a_n(h) \frac{f(q^n h) - f(h)}{(q^n - 1)\, h}. \tag{6.9}$$

Requesting that $\partial_n \circ \varphi = \mu\, \varphi \circ \partial_n$, and evaluating it at h, yields the constraint

$$q a_n(h) = \mu\, a_n(qh), \tag{6.10}$$

which has the following solutions: either

(i) $a_n(h) = 0$ and there are no restrictions on μ, or else
(ii) there exists $d \in \mathbb{N}$ such that $\mu = q^{-d+1}$ (see (6.6)) and then $a_n(h)$ is a scalar multiple of h^d.

These skew derivations provide us with only choices of maps α_i in Theorem 3.1. In the case (i), all elementary α-type derivations ∂_i^α, $i \in \mathbb{Z} \setminus \{0\}$ are trivial, and we are thus left with ∂_0,

$$\partial_0(x) = f(h)x, \qquad \partial_0(y) = -\mu\, \varphi^{-1}(f(h))y = -\mu\, f(q^{-1}h)y.$$

This proves the first part of statement (1). In the case (ii) we obtain

$$\partial_m^\alpha(x) = 0, \quad \partial_m^\alpha(y) \sim h^d x^{m-1}, \qquad \partial_{-n}^\alpha(x) \sim h^d y^{n-1}, \quad \partial_{-n}^\alpha(y) = 0.$$

Combining these solutions we obtain the first part of statement (2)(a). If $d \geq 1$, then the monomial $a_0 h^d$ contains factor h, and hence gives rise to the α-type weight zero elementary derivation on $\mathbb{K}_q[x, y]$,

$$\partial_0^\alpha(x) = 0, \quad \partial_0^\alpha(y) \sim h^{d-1}\, y,$$

which establishes the first part of the statement (2)(b).

Since $\mathbb{K}[h]$ is a commutative ring, its twisted centre $\mathbb{K}[h]_{\varphi^i}^{\mathbb{K}[h]}$ is non-trivial only when $i = 0$, in which case it is the centre of $\mathbb{K}[h]$ i.e. the whole of $\mathbb{K}[h]$. Thus there are no non-trivial elementary c-derivations of non-zero weight; in the weight zero case, they are precisely as in (6.5). By Lemma 3.3 these are the only derivations which vanish on the whole of $\mathbb{K}[h]$ (or, equivalently in this case on h), which proves the second part of statement (1).

Skew derivations (6.5) are inner, provided there exists $j(h) \in \mathbb{K}[h]$ such that

$$f(h) = \mu^{-1} j(h) - j(qh). \tag{6.11}$$

This equation can be solved if and only if $\mu \neq q^{-k}$, $k = 0, \ldots, \deg(f)$, for all those k with non-zero coefficients at h^k in $f(h)$, and the solution is

$$j(h) = \sum_{k=0}^{\deg(f)} \frac{f_k}{\mu^{-1} - q^k} h^k, \qquad \text{where} \quad f(h) = \sum_{k=0}^{\deg(f)} f_k h^k,$$

(if $\mu = q^{-k}$ and $f_k = 0$, then there is no h^k-term in $j(h)$). This proves the third part of statement (1).

In the case of derivations (6.7), since the polynomials b and a are mutually independent, we can treat the cases $a = 0$ and $b = 0$ separately. In the first case, define

$$j(y) = q^{-d} \sum_{n=0}^{\deg(b)} \frac{b_n}{q^{-n-1} - 1} y^{n+1},$$

where $b(y) = \sum_n b_n y^n$. One can easily check that both for the quantum disc and the quantum polynomial ring, if $d > 0$, then

$$h^d b(y) = q^{d-1} h^{d-1} j(y) x - x h^{d-1} j(y), \qquad q^{d-1} h^{d-1} j(y) y - y h^{d-1} j(y) = 0,$$

i.e. the derivation $\partial(x) = h^d b(y)$, $\partial(y) = 0$ is inner. If $d = 0$, then all combinations of x with polynomials in y will produce a polynomial in h of degree at least one, hence they cannot be equal to $b(y)$. The other case, $b(y) = 0$, is dealt with in a similar way. This proves the second statement in (2)(a).

Finally we look at derivations (6.8). Since $\partial(x) = 0$, in view of equation (6.11) the derivation (6.8) can be inner if and only if $j(h) = j_{d-1}h^{d-1}$. In this case, however, we would have

$$\partial(y) = q^{1-d} j_{d-1}h^{d-1}y - j_{d-1}yh^{d-1} = 0,$$

which contradicts that $\partial(y) \neq 0$. This completes the proof of statement (2)(b).

The first part of assertion (3) follows from the necessity of solutions to constraints arising from the assumptions of Theorem 3.1. It remains to prove that Theorem 3.1 provides one with the full classification of σ_μ-derivations on A.

A general skew-derivation ∂ on A is determined from its values on generators x and y. Note that the twisted Leibniz rule implies that $\partial(q - 1) = 0$, hence in either the quantum plane or the quantum disc case the only constraints on the definition of ∂ come from

$$\partial(xy) = q\partial(yx), \tag{6.12}$$

i.e.

$$\partial(x)\sigma_\mu(y) + x\partial(y) = q\left(\partial(y)\sigma_\mu(x) + y\partial(x)\right) \tag{6.13}$$

Since A is a $\mathbb{Z}$-graded algebra (with respect to the standard grading) and (6.12) is a homogenous relation (in degree zero), to classify all σ_μ-skew derivations on A suffices it to classify all the homogeneous ones (a general skew derivation is necessarily a locally finite sum of homogeneous components).

A degree $n \in \mathbb{Z}$ skew derivation is of the form

$$\partial_n(x) = X_{n+1} \qquad \partial_n(y) = Y_{n-1},$$

where $X_i, Y_i \in A$, $i \in \mathbb{Z}$ are homogeneous, degree i elements of A. The homogenous degree n components of (6.13) come out as

$$\mu X_{n+1}y + xY_{n-1} - qyX_{n+1} - q\mu^{-1}Y_{n-1}x = 0. \tag{6.14}$$

The classes of solutions to (6.14) depend on the degree (more specifically, whether $n = 0$ or not), on the specific algebra (whether the quantum plane or the quantum disc), and on the value of μ (specifically, whether μ is a power of q lesser than or equal to 1). Thus we presently proceed to discuss various cases.

If n is positive, then $X_{n+1} = f(h)x^{n+1}$ and $Y_{n-1} = g(h)x^{n-1}$, for some $f(h), g(h) \in \mathbb{K}[h]$, (which, of course are different for different n). Using these explicit forms of X_{n+1} and Y_{n-1}, the equation (6.14) becomes

$$g(qh) - q\mu^{-1}g(h) = \left(qf\left(q^{-1}h\right) - \mu q^{n+1}f(h)\right)h, \tag{6.15}$$

in the quantum plane $\mathbb{K}_q[x, y]$-case, and

$$g(qh) - q\mu^{-1}g(h) = qf\left(q^{-1}h\right)(1 - h) - \mu f(h)\left(1 - q^{n+1}h\right), \tag{6.16}$$

in the quantum disc $D_q(x, y)$-case. Writing equations (6.15) and (6.16) out in components of polynomials f at powers of h we thus obtain

$$\left(1 - q\mu^{-1}\right)g_0 = 0, \qquad \left(q^{k-1} - \mu^{-1}\right)g_k = \left(q^{-k+1} - q^n\mu\right)f_{k-1}, \tag{6.17}$$

in the quantum plane $\mathbb{K}_q[x,y]$-case, and

$$\left(q\mu^{-1}-1\right)g_0=(\mu-q)\,f_0,\quad \left(q\mu^{-1}-q^k\right)g_k=\left(\mu-q^{-k+1}\right)f_k+q\left(q^{-k+1}-q^n\mu\right)f_{k-1},\tag{6.18}$$

in the quantum disc case. If $\mu \neq q^{1-d}$ for all $d \in \mathbb{N}$, then in both cases the choice of $f(h)$ fully determines $g(h)$. Explicitly,

$$g(h) = -\mu f\left(q^{-1}h\right) - \sum_{k=1}^{\deg(f)+1} \frac{q^{-k+1}-q^n\mu}{q^{k-1}-\mu^{-1}} f_{k-1}h^k, \qquad \text{for } D_q(x,y), \tag{6.19a}$$

$$g(h) = \sum_{k=1}^{\deg(f)+1} \frac{q^{-k+1}-q^n\mu}{q^{k-1}-\mu^{-1}} f_{k-1}h^k, \qquad \text{for } \mathbb{K}_q[x,y]. \tag{6.19b}$$

We claim that all these skew derivations are elementary ones of inner type. Since $g(h)$ is determined from $f(h)$, to decide whether ∂_n is an inner-type elementary derivation suffices it to determine whether there exists $j(h) = \sum_k j_k h^k \in \mathbb{K}[h]$ such that

$$f(h)x^{n+1} = \partial_n(x) = \mu^{-1}j(h)x^{n+1} - xj(h)x^n, \quad \text{i.e.} \quad f(h) = \mu^{-1}j(h) - j(qh), \tag{6.20}$$

(see (6.11)). The comparison of coefficients at powers of h in (6.20) gives the following system of equations

$$\left(\mu^{-1} - q^k\right) j_k = f_k. \tag{6.21}$$

Therefore, the j_k are determined provided $\mu \neq q^{-k}$ (at least for all k for which the coefficients at h^k in $f(h)$ are non-zero), which is always the case if $\mu \neq q^{1-d}$ for all $d \in \mathbb{N}$. Thus we conclude that if $\mu \neq q^{1-d}$ for all $d \in \mathbb{N}$, all the positive degree skew-derivations are inner (and inner-type, since the twisted centres of A are trivial for $n \neq 0$). The assertion for negative degrees follows by the x-y symmetry (note that while applying the x-y symmetry we will need to change not only q to q^{-1} but also μ to μ^{-1} which will maintain the relations between q and μ intact).

In the zero-degree case, $X_1 = f(h)x$ and $Y_{-1} = g(h)y$, for some $f(h), g(h) \in \mathbb{K}[x]$ and the equation (6.14) takes the form

$$\mu f(h)xy + g(qh)xy - qf\left(q^{-1}h\right)yx - q\mu^{-1}g(h)yx = 0. \tag{6.22}$$

In the quantum plane case equation (6.22) is equivalent to

$$g(qh) - \mu^{-1}g(h) = f\left(q^{-1}h\right) - \mu f(h), \tag{6.23}$$

or, comparing the coefficients at respective powers of h,

$$\left(q^k - \mu^{-1}\right) g_k = \left(q^{-k} - \mu\right) f_k. \tag{6.24}$$

If $\mu \neq q^{-d}$, for all $d \in \mathbb{N}$, then equation (6.24) for $g(h)$ has a unique solution

$$g(h) = -\mu f\left(q^{-1}h\right), \tag{6.25}$$

which means that ∂_0 is a c-type elementary derivation as in (6.5).

In the quantum disc case, equation (6.22) is equivalent to

$$\left(\mu f(h) + g(qh)\right)(1 - qh) = q\left(f\left(q^{-1}h\right) + \mu^{-1}g(h)\right)(1 - h). \tag{6.26}$$

Since $q \neq 1$, equation (6.26) has a solution if and only if there exists $b(h) \in \mathbb{K}[h]$ such that

$$\mu f(h) + g(qh) = b(h)(1 - h), \qquad q\left(f\left(q^{-1}h\right) + \mu^{-1}g(h)\right) = b(h)(1 - qh). \tag{6.27}$$

Replacing h by $q^{-1}h$ in the first of equations (6.27), and comparing it with the second one we conclude that (6.27) imply the following equation for $b(h)$

$$b(h)\,(1-qh) = q\mu^{-1}b\left(q^{-1}h\right)\left(1-q^{-1}h\right). \tag{6.28}$$

Comparing coefficients, one easily finds that if $\mu \neq q^{1-d}$, for all $d \in \mathbb{N}$, then $b(h) = 0$, so, in view of the second of equations (6.27), $g(h) = -\mu f\left(q^{-1}h\right)$. Therefore, also in this case the derivations of degree zero are necessarily of c-type as in (6.5).

Next, assume that there exists $d \in \mathbb{N}$ such that $\mu = q^{-d}$. We start our discussion with the quantum disc case. For a positve degree n derivation, the equation (6.18) for $k = d+1$ gives

$$0 = q\left(q^{-d} - q^{n-d}\right)f_d,$$

and hence $f_d = 0$, since $n > 0$. The polynomial $g(h)$ is determined by the choice of $f(h)$ except for the term $g_{d+1}h^{d+1}$ (this is a contribution to ∂ from a derivation of the type (6.7) with $b(y) = 0$ and $a(x) \sim x^{n-1}$). Despite this indeterminacy we can still decide whether ∂_n is an inner skew-derivation. In this case, the possibility of finding $j(h)$ satisfying (6.20) boils down to solving equations (6.21) in the form $\left(q^d - q^k\right)j_k = f_k$. Since $f_d = 0$, these can always be solved for all k and with no restrictions on j_d. In particular,

$$j_{d+1} = \frac{f_{d+1}}{q^d - q^{d+1}}. \tag{6.29}$$

The corresponding equation for $\partial_n(y)$ is

$$g(h)x^{n-1} = \partial_n(y) = q^{-d}j(h)x^n y - yj(h)x^n = \left(q^{-d}j(h)(1-q^n h) - j(q^{-1}h)(1-h)\right)x^{n-1},$$

or for coefficients

$$g_k = \left(q^{-d} - q^{-k}\right)j_k - \left(q^{n-d} - q^{-k+1}\right)j_{k-1}.$$

In particular, using (6.29),

$$g_{d+1} = \left(q^{-d} - q^{-d-1}\right)j_{d+1} + \left(q^{-d-1} - q^{n-d}\right)j_d = -q^{-2d-1}f_{d+1} - \left(q^{-d} - q^{n-d}\right)j_d.$$

Since $n > 0$, and we have freedom in choosing j_d, there is a choice (determined by the above equation), which ensures that ∂_n is an inner type elementary derivation irrespective of the value of g_{d+1}.

For the quantum plane algebra, if $\mu = q^{-d}$, equations (6.17) lead to $f_d = 0$ and leave g_{d+1} undetermined. Equations (6.21) can always be solved with no constraints on j_d. Answering whether $\partial_n(y)$ can be expressed as a twisted commutator with $j(h)x^n$ reduces to solving the system of equations

$$g_k = q^{-k+1}\left(q^n - 1\right)j_{k-1}.$$

In particular, when $k = d+1$ this determines so far undetermined value of j_d, and thus allows one to express both $\partial_n(x)$ and $\partial_n(y)$ as inner skew derivations induced by the same $j(h)$. The negative degree case follows by the x-y symmetry.

Still assume that $\mu = q^{-d}$, for some $d \in \mathbb{N}$. In the quantum plane case, the equation (6.24) for $k = d$ is automatically satisfied; for all other values of k, the choice of $f(h)$ determines $g(h)$ through (6.25). Therefore, any degree zero skew derivation on $\mathbb{K}_q[x,y]$ is determined from

$$\partial_0(x) = f(h)x + \nu h^d x, \qquad \partial_0(y) = -q^{-d}f\left(q^{-1}h\right)y + \lambda h^d y, \tag{6.30}$$

where $f(h) = \sum_k f_k h^k$, $f_d = 0$, and $\lambda, \nu \in \mathbb{K}$. The $\nu h^d x$-term can be absorbed into the definition of $f(h)$ leading to the modification of the $\lambda h^d y$-term, but still leaving the choice of scalar unconstrained. In this way any degree-zero derivations in this case is a combination of a c-type derivation as in (6.5) and an α-type derivation (6.8) both of weight zero.

In the quantum disc case, if $\mu = q^{-d+1}$, then comparing coefficients in (6.28) one obtains the recurrence relation

$$b_0 = \ldots = b_{d-1} = 0, \qquad b_{d+k+1} = \frac{q^{k+2}-1}{q^{k+1}-1} b_{d+k}, \quad k = 0, 1, \ldots, \tag{6.31}$$

with seemingly no restrictions on b_d. Should $b_d \neq 0$, the recurrence (6.31) would result in infinitely many non-zero terms b_i. Thus, in order to ensure that $b(h)$ is a polynomial we are forced to set $b_d = 0$ and, consequently $b(h) = 0$. The second of equations (6.27) then yields $g(h) = -q^{-d+1} f(q^{-1}h)$, and hence ∂_0 is a c-type elementary derivation as in (6.5).

It remains only to study the non-zero degree derivations for $\mu = q$. Both equations (6.17) and (6.18) can be solved for all g_k, $k > 0$ and there is no restriction on g_0. Thus $g(h) - g_0$ is fully determined by $f(h)$ (and given by (6.19)) and since a suitable $j(h)$ satisfying equations (6.21) can be found, the derivation determined by $\partial(x) = f(h)x^{n+1}$, $\partial(y) = (g(h) - g_0)\, x^{n-1}$ is of inner type. For he remaining part, $x \mapsto 0$, $y \mapsto g_0 x^{n-1}$, so it is of α-type as in (6.7).

This completes the proof of the theorem. □

We end the paper with examples of pairs of orthogonal skew-derivations on the quantum plane and the quantum disc.

Example 6.3. In this example we apply Proposition 4.2 to discuss orthogonal pairs of skew derivations on the quantum disc and polynomial algebras. Note that Proposition 4.2 is applicable only to elementary skew derivations leading to derivations of the type (6.7). Since in this case the skew derivations α_i on $\mathbb{K}[h]$ evaluated at h (in the case of the quantum plane) or $1-h$ (in the case of the disc) are proportional to h^d, the co-primeness requirements of Proposition 4.2 immediately imply that $d = 0$, and hence $\mu = q$. Hence the elementary skew derivations ∂, $\bar{\partial}$ can be given by

$$\partial(x) = 0, \qquad \partial(y) = c\, x^m, \tag{6.32a}$$

$$\bar{\partial}(x) = \bar{c} y^n, \qquad \bar{\partial}(y) = 0, \tag{6.32b}$$

for all $m, n \in \mathbb{N}$, and non-zero elements $c, \bar{c}$ of $\mathbb{K}$. Henceforth we need to consider the quantum plane and quantum polynomial ring cases separately.

(i) In the $\mathbb{K}_q[x, y]$-case, $a = h$, hence it is never coprime with $\varphi^i(a) = q^i h$, and thus only $m = n = 0$ in (6.32) gives an orthogonal pair.

(ii) In the $D_q(x, y)$-case, $a = 1 - h$, hence $\varphi^i(a) = 1 - q^i h$ is always coprime with a as long as q is not a root of unity (which is assumed in Theorem 6.2 and hence in this example). Thus (∂, σ_q), $(\bar{\partial}, \sigma_q)$ form an orthogonal pair for any choice of m and n.

As a next example, we construct orthogonal pairs of skew derivations on the quantum disc algebra in the case $\mu = q$ (q not a root of unity) that are not already included in Example 6.3. In general, a homogeneous σ_q-skew derivation is the sum of an inner type elementary derivation and an α-type derivation (6.7). Explicitly, let $c(h), \bar{c}(h) \in \mathbb{K}[h]$ be polynomials that have root zero and $e, \bar{e} \in \mathbb{K}$. For any $n, m > 1$,

$$\partial(x) = c_q'(h)x^n, \qquad \partial(y) = q^2\left((h-1)\,c_q'\left(q^{-1}h\right) + [n-1]_q c\left(q^{-1}h\right) + e\right)x^{n-2}, \quad (6.33)$$

yields the most general degree $n-1>0$ σ_q-skew derivation on $D_q(x,y)$, while

$$\bar{\partial}(x) = q^{-2}\left((qh-1)\,\bar{c}_q'(qh) + [m-1]_{q^{-1}}\bar{c}(qh) + \bar{e}\right)y^{m-2}, \quad \bar{\partial}(y) = \bar{c}_q'(h)\,y^m, \quad (6.34)$$

induces the $-m+1<0$-degree one. Here, for any non-zero scalar γ and a positive integer k, $[k]_\gamma$ denotes the γ-integer,

$$[k]_\gamma = 1 + \gamma + \gamma^2 + \ldots + \gamma^{k-1},$$

and, $f_q'(\gamma h)$ is the Jackson's q-derivative of the polynomial $g(h) := f(\gamma h)$ as in (6.9), i.e.

$$f_q'(\gamma h) = \frac{f(q\gamma h) - f(\gamma h)}{(q-1)\,h}.$$

To determine whether ∂ and $\bar{\partial}$ form an orthogonal pair would require a detailed analysis of the roots of $c(h)$ and $\bar{c}(h)$. In the following example we choose these polynomials in such a way that the orthogonality is fully determined by the properties of q.

Example 6.4. Assume that a non-zero $q \in \mathbb{K}$ is not a root of unity. For any $n, m \in \mathbb{N}$ such that $m, n > 1$ and for any non-zero $c, \bar{c} \in \mathbb{K}$, consider the following pair of σ_q-skew derivations on $D_q(x,y)$, defined on the generators by

$$\partial(x) = cx^n, \qquad \partial(y) = -q[n]_q c\,(1-h)x^{n-2}. \quad (6.35a)$$

$$\bar{\partial}(x) = -q^{-1}[m]_q\bar{c}(1-q^{-m+2}h)y^{m-2}, \qquad \bar{\partial}(y) = \bar{c}y^m. \quad (6.35b)$$

For all $k, l \in \mathbb{N}$ define

$$q_{kl} = \frac{1-[k]_q[l]_q q^{-k+1}}{1-[k]_q[l]_q}.$$

If

$$q_{kl} \neq q^i, \qquad i \in \{-2k+3, -2k+4, \ldots, -k+1, l, l+1, \ldots, 2l-3, 2l-1\} \quad (6.36a)$$

and

$$q_{kl} \neq q^{2l-2}q_{lk}, \quad (6.36b)$$

for $(k,l) = (m,n)$ and $(k,l) = (n,m)$, then ∂ and $\bar{\partial}$ form an orthogonal pair.

These derivations are obtained from (6.33), (6.34) by setting

$$c(h) = ch, \quad e = c - [n]_q, \qquad \bar{c}(h) = \bar{c}h, \quad \bar{e} = q\,(1-[m]_q)\,\bar{c}.$$

Proof. We will construct sets of elements of $D_q(x,y)$ which will satisfy orthogonality conditions (2.9) for skew derivations (6.35). Observe that

$$\frac{q}{[m]_q}y^n\bar{\partial}(x) + (1-q^{-m-n+2}h)y^{n-2}\bar{\partial}(y)$$
$$= -\bar{c}y^n(1-q^{-m+2}h)y^{m-2} + \bar{c}(1-q^{-m-n+2}h)y^{m+n-2} = 0,$$

and

$$\frac{q}{[m]_q}\bar{\partial}(x)y^n + \bar{\partial}(y)(1-q^2h)y^{n-2}$$
$$= -\bar{c}(1-q^{-m+2}h)y^{m+n-2} + \bar{c}y^m(1-q^{-2}h)y^{n-2} = 0.$$

On the other hand

$$\begin{aligned}
&\frac{q}{[m]_q}\; y^n\partial(x)+(1-q^{-m-n+2}h)y^{n-2}\partial(y)\\
&\quad= \frac{q}{[m]_q}cy^nx^n - q[n]_q c\,(1-q^{-m-n+2}h)y^{n-2}(1-h)x^{n-2}\\
&\quad= \frac{q}{[m]_q}c\prod_{k=0}^{n-2}\left(1-q^{-k}h\right)\left(1-q^{-n+1}h-[m]_q[n]_q\left(1-q^{-m-n+2}h\right)\right)\\
&\quad= qc\left(\frac{1}{[m]_q}-[n]_q\right)\prod_{k=0}^{n-2}\left(1-q^{-k}h\right)\left(1-q^{-n+1}\frac{1-[m]_q[n]_q q^{-m+1}}{1-[m]_q[n]_q}h\right)\\
&\quad= qc\left(\frac{1}{[m]_q}-[n]_q\right)\prod_{k=0}^{n-2}\left(1-q^{-k}h\right)\left(1-q^{-n+1}q_{mn}h\right), \qquad (6.37)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{q}{[m]_q}\partial(x)y^n+\partial(y)(1-q^2h)y^{n-2}\\
&\quad= \frac{q}{[m]_q}cx^ny^n - q[n]_q c\,(1-h)x^{n-2}(1-q^2h)y^{n-2}\\
&\quad= \frac{q}{[m]_q}c(1-q^nh)\prod_{k=1}^{n-2}\left(1-q^{k}h\right)\left(1-q^{n-1}h-[m]_q[n]_q\,(1-h)\right)\\
&\quad= qc\left(\frac{1}{[m]_q}-[n]_q\right)(1-q^nh)\prod_{k=1}^{n-2}\left(1-q^{k}h\right)\left(1-q^{n-1}\frac{1-[m]_q[n]_q q^{-n+1}}{1-[m]_q[n]_q}h\right)\\
&\quad= qc\left(\frac{1}{[m]_q}-[n]_q\right)(1-q^nh)\prod_{k=1}^{n-2}\left(1-q^{k}h\right)\left(1-q^{n-1}q_{nm}h\right). \qquad (6.38)
\end{aligned}$$

By (6.36) $q_{mn}\neq q^i$, for all $i\in\{n,\dots,2n-3,2n-1\}$, $q_{nm}\neq q^i$, for all $i\in\{-2n+3,-2n+4,\dots,-n+1\}$ and $q_{mn}\neq q^{2n-2}q_{nm}$. Combining this with the fact that q is not a root of unity, we conclude that polynomials in h that appear in (6.37) and (6.38) have no roots in common. Therefore a polynomial combination of them can be found giving 1, and thus the first set of elements that satisfy (2.9) can be constructed.

In a similar way,

$$\begin{aligned}
(1-q^mh)x^{m-2}\partial(x) &+ \frac{q^{-1}}{[n]_q}x^m\partial(y)\\
&= c(1-q^mh)x^{m+n-2}-cx^m(1-h)x^{n-2}=0,
\end{aligned}$$

and

$$\begin{aligned}
\partial(x)(1-q^{-n}h)x^{m-2} &+ \frac{q^{-1}}{[n]_q}\partial(y)x^m\\
&= cx^n(1-q^{-n}h)x^{m-2}-c(1-h)x^{n+m-2}=0.
\end{aligned}$$

On the other hand

$$
\begin{aligned}
&(1-q^m h)x^{m-2}\bar{\partial}(x) + \frac{q^{-1}}{[n]_q}x^m\bar{\partial}(y) \\
&\qquad = -q^{-1}[m]_q\bar{c}\,(1-q^m h)x^{m-2}(1-q^{-m+2}h)y^{m-2} + \frac{q^{-1}}{[n]_q}\bar{c}x^m y^m \\
&\qquad = \frac{q^{-1}}{[n]_q}\bar{c}(1-q^m h)\prod_{k=1}^{m-2}\left(1-q^k h\right)\left(-[m]_q[n]_q\,(1-h)+1-q^{m-1}h\right) \\
&\qquad = q^{-1}\bar{c}\left(\frac{1}{[n]_q}-[m]_q\right)(1-q^m h)\prod_{k=1}^{m-2}\left(1-q^k h\right) \\
&\qquad\qquad\qquad \times\left(1-q^{m-1}\frac{1-[m]_q[n]_q q^{-m+1}}{1-[m]_q[n]_q}h\right) \\
&\qquad = q^{-1}\bar{c}\left(\frac{1}{[n]_q}-[m]_q\right)(1-q^m h)\prod_{k=1}^{m-2}\left(1-q^k h\right)\left(1-q^{m-1}q_{mn}h\right), \qquad (6.39)
\end{aligned}
$$

and

$$
\begin{aligned}
&\bar{\partial}(x)(1-q^{-n}h)x^{m-2} + \frac{q^{-1}}{[n]_q}\bar{\partial}(y)x^m \\
&\qquad = -q^{-1}[m]_q\bar{c}\,(1-q^{-m+2}h)y^{m-2}(1-q^{-n}h)x^{m-2} + \frac{q^{-1}}{[n]_q}\bar{c}y^m x^m \\
&\qquad = \frac{q^{-1}}{[n]_q}\bar{c}\prod_{k=0}^{m-2}\left(1-q^{-k}h\right)\left(-[m]_q[n]_q\left(1-q^{-n-m+2}h\right)+1-q^{-m+1}h\right) \\
&\qquad = q^{-1}\bar{c}\left(\frac{1}{[n]_q}-[m]_q\right)\prod_{k=0}^{m-2}\left(1-q^{-k}h\right)\left(1-q^{-m+1}\frac{1-[m]_q[n]_q q^{-n+1}}{1-[m]_q[n]_q}h\right) \\
&\qquad = q^{-1}\bar{c}\left(\frac{1}{[n]_q}-[m]_q\right)\prod_{k=0}^{m-2}\left(1-q^{-k}h\right)\left(1-q^{-m+1}q_{nm}h\right). \qquad (6.40)
\end{aligned}
$$

As before, since q is not a root of unity and by (6.36), polynomials in h that appear in (6.39) and (6.40) have no roots in common. Thus a polynomial combination of them can be found giving 1 and then the second set of elements that satisfy (2.9) can be constructed. This proves the statement. □

The final example illustrates that in some situations the conditions in Example 6.4 that q needs to satisfy are satisfied (almost) automatically.

Example 6.5. In the setup of Example 6.4, let us take $\mathbb{K}=\mathbb{C}$ and $q \neq 1$ a positive real number. Furthermore, let $m=n$. In this case, the conditions (6.36) reduce to

$$
q_{nn} \neq q^i, \qquad i \in \{-2n+3, -2n+4, \dots, -n+1, n, n+1, \dots, 2n-3, 2n-1\}.
$$

By using the definitions of $[n]_q$ and q_{nn}, one easily finds that

$$
q_{nn} = q^{-n}\frac{[n+1]_q}{[n]_q+1}.
$$

Assuming that $q \neq 1$, for $i \leq -n$, the critical equation

$$q^{-n}\frac{[n+1]_q}{[n]_q+1} = q^i, \tag{6.41}$$

is equivalent to

$$[-i-n+1]_q[n]_q + [-i-n]_q = 0.$$

If $i \leq -n$, and q is positive, all the q-numbers on the left hand side are positive, and thus there are no solutions. For $i = -n+1$, (6.41) is equivalent to $(q-1)^2 = 0$, and thus has no solutions for q other than 1. Finally, for positive i and $q \neq 1$ the equation (6.41) is equivalent to

$$q[n]_q[n+i-1]_q + [n+i]_q = 0,$$

which again has no positive solutions for q.

Therefore, the following pair of σ_q-skew derivations on $D_q(x,y)$, defined on the generators by

$$\partial(x) = cx^n, \qquad \partial(y) = -q[n]_q c\,(1-h)x^{n-2}.$$
$$\bar\partial(x) = -q^{-1}[n]_q\bar c(1-q^{-n+2}h)y^{n-2}, \qquad \bar\partial(y) = \bar c y^n.$$

is orthogonal.

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