DIFFERENTIAL SMOOTHNESS OF SKEW POLYNOMIAL RINGS

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Abstract. It is shown that, under some natural assumptions, the tensor product of differentially smooth algebras and the skew-polynomial rings over differentially smooth algebras are differentially smooth.

1. Introduction

The study of smoothness of algebras goes back at least to Grothendieck's EGA. The concept of a formally smooth commutative (topological) algebra introduced in there [5, Définition 19.3.1] was later extended to the non-commutative case by Schelter in [12]. An algebra is formally smooth if and only if the kernel of the multiplication map is projective as a bimodule. As argued by Schelter himself, this notion arose as a replacement of a far too general definition based on the finiteness of the global dimension. Although it plays an important role in non-commutative geometry (see e.g. [4], where such algebras are termed *quasi-free*), the notion of formal smoothness seems to be too restrictive. The too crude notion of smoothness based on the finiteness of the global dimensions was refined in [13], where a Noetherian algebra was said to be smooth provided that it had a finite global dimension equal to the homological dimension of all its simple modules. From the homological perspective probably most satisfying is the notion of *homological smoothness* introduced in [14]: an algebra is homologically smooth provided it admits a finite resolution by finitely generated projective bimodules. Algebras of this kind display a Poincaré type duality between Hochschild homology and cohomology, and retain many properties characteristic of co-ordinate algebras of smooth varieties (see e.g. [8], where this last point is strongly argued for).

A different and more constructive approach to smoothness of algebras was taken in [3]. In this approach the smoothness of an algebra A is related to the existence of a specific differential graded algebra (with A as the degree-zero part) whose size is aligned with the rate of growth of A measured by the Gelfand-Kirillov dimension, and which satisfies a strict version of the Poincaré duality in terms of an isomorphism with the corresponding complex of integral forms [2] (see Section 2 for precise definition). In view of this direct use of differential graded algebras this kind of smoothness is referred to as *differential smoothness*. The main advantage of this approach is its concreteness: a differentially smooth algebra comes equipped with a well-behaved differential structure and with the precisely defined concept of integration. Examples of differentially smooth algebras include the coordinate algebras of the quantum group $SU_q(2)$, the quantum 2-sphere (see [2]), the non-commutative pillow algebra, the quantum cone algebras (see

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[3]), the quantum polynomial algebras (see [6]), and Hopf algebra domains of Gelfand-Kirillov dimension 2 that are not PI (see [1]). Although many of these examples are known to be also homologically smooth, the relationship between the differential and other types of smoothness is not clear yet.

At the root of difficulties with comparing differential and other types of smoothness is the constructive nature of the former, which prevents one from using functorial or just existential arguments. In this paper we make a few steps toward resolving such difficulties and present two general constructions which lead from differentially smooth to differentially smooth algebras. First, we show that – under some natural assumptions on differential structures and algebras – the tensor product of differentially smooth algebras is differentially smooth. This allows one to deduce quickly smoothness of polynomial and Laurent polynomial rings without necessity of constructing a specific differential structure (it suffices to have such a structure for polynomials in one variable). Second, again under some natural assumptions, we prove that the skew-polynomial rings over a smooth algebra are smooth.

2. Preliminaries

Let $\mathbb{F}$ be a field. By a *differential calculus* over an $\mathbb{F}$-algebra R we mean a differential graded algebra $(\Omega R, d)$ (i.e. a graded algebra with the degree-one square-zero linear map $d : \Omega R \to \Omega R$ satisfying the graded Leibniz rule) such that:

(a) $\Omega R = \bigoplus_{n\in\mathbb{N}} \Omega^n R$, i.e. it is non-negatively graded, and $\Omega^0 R = R$,
(b) For all $n \in \mathbb{N}$,

$$\Omega^n R = R \underbrace{d(R)d(R)\cdots d(R)}_{n\text{-times}}.$$

The requirement (b) is called the *density condition*. A differential calculus $(\Omega R, d)$ over R is said to be *connected*, provided $\ker(d\,|_R) = \mathbb{F}$. It is said to have *dimension* N or to be N*-dimensional* provided

$$\Omega^N R \neq 0 \qquad \text{and} \qquad \Omega^n R = 0, \quad \text{for all } n > N.$$

An N-dimensional differential calculus $(\Omega R, d)$ over R is said to *admit a volume form*, provided $\Omega^N R$ is isomorphic to R as both a left and right R-module (but not necessarily as an R-bimodule). Any free generator v of $\Omega^N R$ as a right and left R-module is referred to as a *volume form*. Associated to a volume form v are two maps:

(a) the right R-module *co-ordinate isomorphism*:

$$\pi_{\mathsf{v}} : \Omega^N R \to R, \qquad \pi_{\mathsf{v}}(\mathsf{v}r) = r; \tag{2.1}$$

(b) the *twisting* algebra automorphism:

$$\theta_{\mathsf{v}} : R \to R, \qquad r \mapsto \pi_{\mathsf{v}}(r\mathsf{v}). \tag{2.2}$$

To any right R-linear homomorphism $\varphi : \Omega^n R \to R$ we associate a family of right R-module maps

$$\ell^k_\varphi : \Omega^k R \to \mathrm{Hom}_R(\Omega^{n-k}R, R), \quad \gamma \mapsto [\gamma' \mapsto \varphi(\gamma\gamma')], \qquad k \in \{0, 1, \ldots, n\}. \tag{2.3}$$

The right R-multiplication on the space of right R-linear maps $\mathrm{Hom}_R(\Omega^k R, R)$ is defined by $(\psi r)(\gamma) = \psi(r\gamma)$, for all $\psi \in \mathrm{Hom}_R(\Omega^k R, R)$, $r \in R$ and $\gamma \in \Omega^k R$. The

maps $\ell^k_{\pi_{\mathsf{v}}}$ associated to a volume form co-ordinate isomorphism (2.1) are R-bimodule homomorphisms, provided the left R-multiplication on $\mathrm{Hom}_R(\Omega^k R, R)$ is defined via the twisting automorphism θ_{v},

$$(r\psi)(\gamma) = \theta_{\mathsf{v}}(r)\psi(\gamma).$$

Following [3] a differential calculus with a volume N-form v is said to be *integrable* provided all bimodule homomorphisms $\ell^k_{\pi_{\mathsf{v}}}$, $k \in \{0, 1, \ldots, N\}$ are invertible. This is equivalent to the existence of a complex of integrable forms [2] isomorphic to $(\Omega R, d)$ (see [3, Theorem 2.2]). To relieve the notation we will write ℓ^k_{v} or simply ℓ^k for $\ell^k_{\pi_{\mathsf{v}}}$. For the future use we thus record that if $\mathsf{v} \in \Omega^N R$ is a volume form, then

$$\ell^k_{\mathsf{v}} : \Omega^k R \to \mathrm{Hom}_R(\Omega^{N-k} R, R), \quad \gamma \mapsto [\gamma' \mapsto \pi_{\mathsf{v}}(\gamma\gamma')], \qquad k \in \{0, 1, \ldots, N\}. \tag{2.4}$$

Given an affine $\mathbb{F}$-algebra R with generating subspace $\mathcal{V}$, let us write $\mathcal{V}(n)$ for the subspace of R spanned by 1 and all words in generators of R of length at most n. The *Gelfand-Kirillov dimension* of R is a real number defined as

$$\mathrm{GKdim}(A) := \inf\{t \mid \dim \mathcal{V}(n) \leq n^t,\ n \gg 0\}, \tag{2.5}$$

if it exists, and is defined as infinity otherwise. The Gelfand-Kirillov dimension of an arbitrary $\mathbb{F}$-algebra R is by definition the supremum of the Gelfand-Kirillov dimensions of affine $\mathbb{F}$-subalgebras of R (see [11, 8.1.16]). Although it is not generally true that the Gelfand-Kirillov dimension of the tensor product of two algebras R, S is equal to the sum of their (finite) Gelfand-Kirillov dimensions, it is, however, the case that if $\mathrm{GKdim}(R) \leq 2$ or $\mathrm{GKdim}(S) \leq 2$, then

$$\mathrm{GKdim}(R \otimes S) = \mathrm{GKdim}(R) + \mathrm{GKdim}(S);$$

see [9, Proposition 3.12]. We refer the reader to [9] or [11, Chapter 8] for a detailed discussion of the Gelfand-Kirillov dimension, which, in the case of a commutative Noetherian algebra is a very good measure of geometric dimension of the underlying affine space.

The version of smoothness studied in the present text is recalled in the following

Definition 2.1 ([3])**.** An affine algebra R of integer Gelfand-Kirillov dimension N is said to be *differentially smooth*, if there exists a connected, N-dimensional, integrable differential calculus on R.

Let R be an algebra and σ an algebra automorphism of R. By a *skew-polynomial ring over* R we mean the algebra $R[z;\sigma]$ generated additionally by z and the relations $zr = \sigma(r)z$, for all $r \in R$. Similarly the Laurent skew-polynomial ring $R[z^{\pm 1};\sigma]$ is defined. As was the case for tensor product algebras, it is not generally true that $\mathrm{GKdim}(R[z;\sigma]) = \mathrm{GKdim}(R) + 1$ (see [11, Example 8.2.16]). The equality holds, whenever σ is *locally algebraic*, i.e. if for all $r \in R$, the set $\{\sigma^n(r) \mid n \in \mathbb{N}\}$ is contained in a finite dimensional subspace of R (see [10, Proposition 1]).

As we will often make statements that apply equally well to the skew-polynomial and the Laurent skew-polynomial rings, we reserve the symbol $R[z^\bullet;\sigma]$ to denote either $R[z;\sigma]$ or $R[z^{\pm 1};\sigma]$.

3. Differential smoothness of the tensor product of algebras

The aim of this section is to prove that, under some mild and geometrically natural assumptions, tensor product of integrable differential calculi on two algebras gives an integrable calculus on the tensor product algebra.

Suppose that $(\Omega R, d_R)$, where $\Omega R = \bigoplus_{k=0}^{N} \Omega^k R$, is an N-dimensional differential calculus on an $\mathbb{F}$-algebra R, and that $(\Omega S, d_S)$, where $\Omega S = \bigoplus_{k=0}^{M} \Omega^k S$, is an M-dimensional differential calculus on an $\mathbb{F}$-algebra S. Consider $T := R \otimes S$ and

$$\Omega T := \Omega R \otimes \Omega S = \bigoplus_{n=0}^{N+M} \left(\underbrace{\sum_{i=0}^{k} \Omega^i R \otimes \Omega^{k-i} S}_{=:\Omega^k T} \right).$$

Components $\Omega^i R$, resp. $\Omega^j S$, are considered to be zero if i or j are not within their limits. ΩT becomes a differential graded algebra, with graded multiplication defined as

$$(\omega \otimes \nu)(\omega' \otimes \nu) = (-1)^{|\nu||\omega'|} \omega\omega' \otimes \nu\nu', \tag{3.1}$$

for homogeneous elements $\omega, \omega', \nu, \nu'$, and extended differential d_T of ΩT defined by

$$d_T(\omega \otimes \nu) := \omega \otimes d_S(\nu) \ + (-1)^i d_R(\omega) \otimes \nu, \tag{3.2}$$

for all $\omega \in \Omega^i R$ and $\nu \in \Omega^j S$. By the density condition

$$\omega = \sum_t r_0^t d_R(r_1^t) \cdots d_R(r_i^t), \qquad \nu = \sum_u s_0^u d_S(s_1^u) \cdots d_S(s_j^u),$$

hence, in view of (3.1) and (3.2),

$$\omega \otimes \nu = \sum_{t,u} (r_0^t \otimes s_0^u) d_T(r_1^t \otimes 1) \cdots d_T(r_i^t \otimes 1) d_T(1 \otimes s_1^u) \cdots d_T(1 \otimes s_j^u).$$

Therefore, the differential graded algebra $(\Omega T, d_T)$ is a differential calculus on T.

Proposition 3.1. *Let R and S be algebras with integrable differential calculi $(\Omega R, d_R)$ and $(\Omega S, d_S)$. Suppose that ΩR is a finitely generated projective right R-module and that ΩS is a finitely generated projective right S-module. Then $(\Omega R \otimes \Omega S, d)$ is an integrable differential calculus for $R \otimes S$.*

Proof. We write $T := R \otimes S$ and assume that (Ω^R, d_R) has dimension N and (Ω^S, d_S) has dimension M. Note that for homogeneous $\omega \otimes \nu \in \Omega^i R \otimes \Omega^{k-i} S$ and $\omega' \otimes \nu' \in \Omega^j R \otimes \Omega^{k'-j} S$ we have

$$(\omega \otimes \nu)(\omega' \otimes \nu') = (-1)^{(k-i)j} \underbrace{\omega\,\omega'}_{\in \Omega^{i+j} R} \otimes \underbrace{\nu\,\nu'}_{\in \Omega^{k+k'-i-j} S} \in \Omega^{k+k'} T.$$

Since $\Omega^{i+j} R = 0$ for $i + j > N$ and $\Omega^{N+M-i-j} S = 0$ for $i + j < N$, we have for all k:

$$\left(\Omega^i R \otimes \Omega^{k-i} S\right) \left(\Omega^j R \otimes \Omega^{N+M-k-j} S\right) = 0, \qquad \forall j \neq N - i.$$

This means in particular

$$\left(\Omega^i R \otimes \Omega^{k-i} S\right) \Omega^{N+M-k} T = \left(\Omega^i R \otimes \Omega^{k-i} S\right) \left(\Omega^{N-i} R \otimes \Omega^{M-(k-i)} S\right), \tag{3.3}$$

for all $i \leq k$.

Since $\Omega^{N+M}T = \Omega^N R \otimes \Omega^M S$, for all $\varphi_1 \in \mathrm{Hom}_R(\Omega^N R, R)$ and $\varphi_2 \in \mathrm{Hom}_S(\Omega^M S, S)$, $\varphi := \varphi_1 \otimes \varphi_2 \in \mathrm{Hom}_T(\Omega^{N+M}T, T)$, and so we can consider the maps (2.3), $\ell^k_\varphi : \Omega^k T \to \mathrm{Hom}_T(\Omega^{N+M-k}T, T)$. By equation (3.3),

$$\ell^k_\varphi \left(\Omega^i R \otimes \Omega^{k-i} S\right) \subseteq \mathrm{Hom}_T(\Omega^{N-i} R \otimes \Omega^{M-(k-i)} S, T).$$

In particular

$$\ell^k_\varphi(\omega \otimes \nu)(\omega' \otimes \nu') = \varphi_1(\omega\omega') \otimes \varphi_2(\nu \otimes \nu') = \ell^i_{\varphi_1}(\omega)(\omega') \otimes \ell^{k-i}_{\varphi_2}(\nu)(\nu'),$$

for all $\omega \in \Omega^i R, \omega' \in \Omega^{N-i} R$, $\nu \in \Omega^{k-i} S, \nu' \in \Omega^{M-k+i} S$, where $\ell^i_{\varphi_1}$ and $\ell^{k-i}_{\varphi_2}$ are defined by (2.3). Identifying $\ell^i_{\varphi_1} \otimes \ell^{k-i}_{\varphi_2} \in \mathrm{Hom}_R(\Omega^i R, R) \otimes \mathrm{Hom}_S(\Omega^{k-i} S, S)$ with an element of $\mathrm{Hom}_T(\Omega^i R \otimes \Omega^{k-i} S, T)$, we obtain $\ell^k_\varphi = \sum_{i=0}^k \ell^i_{\varphi_1} \otimes \ell^{k-i}_{\varphi_2}$, since $\Omega^k T = \bigoplus_{i=0}^k \Omega^i R \otimes \Omega^{k-i} S$.

Suppose that $\mathsf{v}_R \in \Omega^N R$ and $\mathsf{v}_S \in \Omega^M S$ are volume forms with corresponding coordinate isomorphisms $\pi_{\mathsf{v}_R} : \Omega^N R \to R$ and $\pi_{\mathsf{v}_S} : \Omega^N S \to S$. Then $\mathsf{v} = \mathsf{v}_R \otimes \mathsf{v}_S$ is a volume form for $\Omega^{N+M}T$ with isomorphism $\pi_{\mathsf{v}} = \pi_{\mathsf{v}_R} \otimes \pi_{\mathsf{v}_S}$. We have already seen that $\ell^k_{\mathsf{v}} = \sum_{i=0}^k \ell^i_{\mathsf{v}_R} \otimes \ell^{k-i}_{\mathsf{v}_S}$, for all $0 < k < N + M$.

By assumption the maps $\ell^i_{\mathsf{v}_R}$ and $\ell^j_{\mathsf{v}_S}$ are bijective for all $0 \le i, j \le k$. Hence also

$$\ell^i_{\mathsf{v}_R} \otimes \ell^{k-i}_{\mathsf{v}_S} : \Omega^i R \otimes \Omega^{k-i} S \longrightarrow \mathrm{Hom}_R(\Omega^{n-i} R, R) \otimes \mathrm{Hom}_S(\Omega^{m-(k-i)} S, S),$$

is bijective for all $0 \le i \le k$.

If $\Omega^{N-i} R$ and $\Omega^{M-(k-i)} S$ are finitely generated projective as right R-modules, respectively as right S-modules, then by [15, 15.9],

$$\mathrm{Hom}_R(\Omega^{N-i} R, R) \otimes \mathrm{Hom}_S(\Omega^{M-(k-i)} S, S) = \mathrm{Hom}_T(\Omega^{N-i} R \otimes \Omega^{M-(k-i)} S, T),$$

and the maps $\ell^i_{\mathsf{v}_R} \otimes \ell^{k-i}_{\mathsf{v}_S}$ between $\Omega^i R \otimes \Omega^{k-i} S$ and $\mathrm{Hom}_T(\Omega^{N-i} R \otimes \Omega^{M-(k-i)} S, T)$ are bijections. Thus

$$\ell^k_{\mathsf{v}} : \underbrace{\bigoplus_{i=0}^k \Omega^i R \otimes \Omega^{k-i} S}_{\Omega^k T} \xrightarrow{\sum \ell^i_{\mathsf{v}_R} \otimes \ell^{k-i}_{\mathsf{v}_S}} \underbrace{\bigoplus_{i=0}^k \mathrm{Hom}_R\left(\Omega^{N-i} R, R\right) \otimes \mathrm{Hom}_S\left(\Omega^{M-(k-i)} S, S\right)}_{\mathrm{Hom}_T\left(\bigoplus_{i=0}^k \Omega^{N-i} R \otimes \Omega^{M-(k-i)} S, T\right)}$$

is a bijection. Since

$$\bigoplus_{i=0}^k \Omega^{N-i} R \otimes \Omega^{M-(k-i)} S = \bigoplus_{j=0}^{N-k} \Omega^j R \otimes \Omega^{N+M-k-j} S = \Omega^{N+M-k} T,$$

where components of ΩR or ΩS are zero if their degrees are not within the limits, we eventually conclude that ℓ^k_π is a bijection between $\Omega^k T$ and $\mathrm{Hom}_T(\Omega^{N+M-k} T, T)$. □

Proposition 3.1 yields

Corollary 3.2. *If R and S are differentially smooth algebras with respect to calculi which are finitely generated projective as right modules and*

$$\mathrm{GKdim}(R \otimes S) = \mathrm{GKdim}(R) \ + \mathrm{GKdim}(S),$$

then the tensor product algebra $R \otimes S$ is differentially smooth.

Proof. We only need to check whether the connectedness of ΩR and ΩS implies the connectedness of ΩT. Let $x = \sum_{i,j} \alpha_{ij} r_i \otimes s_j \in \ker d_T \subseteq R \otimes S$, where the sets $\{r_i\}$ and $\{s_j\}$ are linearly independent and $\alpha_{ij} \in \mathbb{F}$. If ΩR is connected, then in view of the definition (3.2), $\sum_i \alpha_{ij} r_i$ is a scalar multiple of 1, for all j, i.e. $x = \sum_j 1 \otimes \beta_j s_j$, for some scalars β_j. If, furthermore ΩS is connected, then the definition (3.2) implies that $\sum_j \beta_j s_j$ is a scalar multiple of 1, hence x is a scalar multiple of $1 \otimes 1$. Therefore, ΩT is connected. The assertion then follows by Proposition 3.1. □

Corollary 3.3. *Let R be a differentially smooth algebra with respect to a differential calculus ΩR that is finitely generated and projective over R. Then extensions of the form $R[x_1, \dots, x_n, y_1^{\pm 1}, \dots, y_m^{\pm 1}]$ are also differentially smooth.*

Proof. Both the polynomial algebra $\mathbb{F}[x]$ and the Laurent polynomial algebra $\mathbb{F}[y^{\pm 1}]$ have Gelfand-Kirillov dimension one and are smooth. In the case of $\mathbb{F}[x]$ a connected one-dimensional integrable differential calculus is (freely as a module) generated by the volume one-form $\mathsf{v} = d(x)$ and the associated twisting automorphism $\theta_{\mathsf{v}}(f(x)) = f(qx)$, where q is any non-zero element of $\mathbb{F}$ (this determines fully the structure of $\Omega\mathbb{F}[x] = \mathbb{F}[x] \oplus \Omega^1\mathbb{F}[x]$). In the case of $\mathbb{F}[y^{\pm 1}]$, the volume form can be chosen as $\mathsf{v} = y^{-1}d(y)$ and then θ_v is the identity map. Since

$$R[x_1, \dots, x_n, y_1^{\pm 1}, \dots, y_m^{\pm 1}] = R \otimes \mathbb{F}[x_1] \otimes \dots \otimes \mathbb{F}[x_n] \otimes \mathbb{F}[y_1^{\pm 1}] \otimes \dots \otimes \mathbb{F}[y_m^{\pm 1}],$$

and $\mathrm{GKdim}\,(\mathbb{F}[x_i]) = \mathrm{GKdim}\,\big(\mathbb{F}[y_j^{\pm 1}]\big) = 1 \leq 2$, the assertion follows by a repeated use of Corollary 3.2 and [9, Proposition 3.12]. □

4. Differential smoothness of skew-polynomial rings

The aim of this section is to prove the following

Theorem 4.1. *Let R be a an algebra with an integrable differential calculus $(\Omega R, d)$ such that ΩR is a finitely generated right R-module. For any automorphism σ of R that extends to a degree-preserving automorphism of ΩR, which commutes with d, there exists an integrable differential calculus $(\Omega A, d)$ on the skew-polynomial ring $A = R[z; \sigma]$ and the Laurent skew-polynomial ring $A = R[z^{\pm 1}; \sigma]$. If R is differentially smooth with respect to $(\Omega R, d)$ and $\mathrm{GKdim}(A) = \mathrm{GKdim}(R)+1$, then A is also differentially smooth.*

Recall that the trivial extension of an algebra A by an A-bimodule M is the algebra B isomorphic to $A \oplus M$ as a vector space and with the multiplication

$$(a, m)(a', m') = (aa', am' + ma'), \qquad \text{for all } a, a' \in A,\ m, m' \in M.$$

If ν is an automorphism of an algebra A, then we will denote by A^ν, the A-bimodule with the multiplication

$$a \cdot b \cdot a' := ab\nu(a'), \qquad \text{for all } a, a', b \in A.$$

Furthermore, we write $M[z]$ (respectively $M[z^{\pm 1}]$) for the direct sum of identical copies of a bimodule M labelled by all natural numbers (resp. integers), with the elements of the summand corresponding to n written as mz^n, $m \in M$. As was the case of skew-polynomial rings $M[z^\bullet]$ denotes either $M[z]$ or $M[z^{\pm 1}]$.

Lemma 4.2. *Let $(\Omega R, d)$ be an N-dimensional differential calculus on an algebra R and let σ be a degree-preserving automorphism of ΩR that commutes with d. Denote also by σ the restriction of σ to R, and let $S = \Omega R[z^\bullet; \sigma]$ and $A = R[z^\bullet; \sigma]$ be the corresponding skew-polynomial rings. Define the automorphism $\overline{\sigma}$ of S by*

$$\overline{\sigma}(\omega z^n) = (-1)^{|\omega|}\sigma(\omega)z^n, \tag{4.1}$$

for all homogeneous $\omega \in \Omega R$ and integers n. Then the trivial extension $\Omega A = S \oplus S^{\overline{\sigma}}$ is an $N+1$-dimensional differential calculus on $R[z^\bullet; \sigma]$ with differential

$$d(\omega z^n, \nu z^m) = \left(d(\omega)z^n, (-1)^{|\omega|}\omega\partial_z(z^n) + d(\nu)z^m\right), \tag{4.2}$$

for all homogeneous $\omega, \nu \in \Omega R$, where ∂_z denotes the (formal) derivative of polynomials. The grading on ΩA is given by

$$|(\omega z^n, 0)| = |\omega|, \qquad |(0, \nu z^m)| = |\nu| + 1, \tag{4.3}$$

for all homogeneous $\omega, \nu \in \Omega R$ with $\nu \neq 0$.

Proof. Using the fact that the differential map d in ΩR raises degree of a form by one, one easily checks that the map defined in (4.2) is square-zero. Note that, by equation (4.2),

$$d(z, 0) = (0, 1),$$

hence the generator $(0, 1)$ of the S-bimodule $S^{\overline{\sigma}} \subset \Omega A$ can be denoted by dz, and $(\omega z^n, \nu z^m)$ can be interpreted as the differential form $\omega z^n + \nu z^m dz$. Using this interpretation the equation (4.2) comes out as

$$d(\omega z^n + \nu z^m dz) = d(\omega)z^n + \left((-1)^{|\omega|}\omega\partial_z(z^n) + d(\nu)z^m\right)dz. \tag{4.4}$$

Furthermore, the multiplication in $S \oplus S^{\overline{\sigma}}$ says concretely that, for all $\nu \in \Omega^k R$,

$$(0, 1)(\omega z^n, 0) = (0, \overline{\sigma}(\omega z^n)) = (0, (-1)^{|\omega|}\sigma(\omega)z^n),$$

meaning

$$dz\omega z^n = (-1)^{|\omega|}\sigma(\omega)z^n dz = \overline{\sigma}(\omega z^n)dz. \tag{4.5}$$

The structure of a trivial extension pays tribute to the fact that $\Omega A dz$ is a square-zero ideal of the algebra of differential forms ΩA, hence

$$dzdz = 0. \tag{4.6}$$

The equations (4.5) and (4.6) determine fully the algebra structure of ΩA.

We need to check that the map d defined by (4.4) satisfies the graded Leibniz rule. Let us take any homogeneous $\omega, \nu \in \Omega R$ and compute

$$\begin{aligned}
d(\omega z^n \nu z^m) &= d(\omega\sigma^n(\nu)z^{n+m}) \\
&= d(\omega\sigma^n(\nu))z^{n+m} + (-1)^{|\omega|+|\nu|}\omega\sigma^n(\nu)\partial_z(z^{n+m})dz \\
&= d(\omega)\sigma^n(\nu)z^{n+m} + (-1)^{|\omega|}\omega\sigma^n(d(\nu))z^{n+m} \\
&\quad +(-1)^{|\omega|+|\nu|}\omega\sigma^n(\nu)(\partial_z(z^n)z^m + z^n\partial_z(z^m))dz \\
&= d(\omega)z^n\nu z^m + (-1)^{|\omega|}\omega z^n d(\nu)z^m \\
&\quad +(-1)^{|\omega|+|\nu|}(\omega\partial_z(z^n)\sigma(\nu)z^m + \omega z^n\nu\partial_z(z^m))dz \\
&= \left(d(\omega)z^n + (-1)^{|\omega|}\omega\partial_z(z^n)dz\right)\nu z^m \\
&\quad +(-1)^{|\omega|}\omega z^n\left(d(\nu)z^m + (-1)^{|\nu|}\nu\partial_z(z^m)dz\right) \\
&= d(\omega z^n)\nu z^m + (-1)^{|\omega|}\omega z^n d(\nu z^m),
\end{aligned}$$

where we used the definition of multiplication of the skew-polynomial algebra, (4.4) and the fact that both d on ΩR and ∂_z satisfy the (graded) Leibniz rule. Next

$$\begin{aligned}
d(\omega z^n dz\ \nu z^m) &= (-1)^{|\nu|}d(\omega\sigma^{n+1}(\nu)z^{n+m}dz) \\
&= (-1)^{|\nu|}d(\omega\sigma^{n+1}(\nu))z^{n+m}dz \\
&= (-1)^{|\nu|}[d(\omega)\sigma^{n+1}(\nu) + (-1)^{|\omega|}\omega d(\sigma^{n+1}(\nu))]z^{n+m}dz \\
&= d(\omega z^n)\overline{\sigma}(\nu z^m)dz + (-1)^{|\omega|}\omega z^n\overline{\sigma}(d(\nu z^m))dz \\
&= d(\omega z^n dz)\ \nu z^m + (-1)^{|\omega|}\omega z^n dz\ d(\nu z^m),
\end{aligned}$$

by (4.5) and (4.4). Finally we can compute:

$$\begin{aligned}
d(\omega z^n\ \nu z^m dz) &= d(\omega\sigma^n(\nu)z^{n+m}dz) \\
&= d(\omega\sigma^n(\nu))z^{n+m}dz \\
&= [d(\omega)\sigma^n(\nu) + (-1)^{|\omega|}\omega\sigma^n(d(\nu))]z^{n+m}dz \\
&= d(\omega z^n)\nu z^m dz + (-1)^{|\omega|}\omega z^n d(\nu z^m dz).
\end{aligned}$$

This proves that $(\Omega A, d)$ is a differential graded algebra. It is clear that $\Omega^{N+1}A = \Omega^N R[z^\bullet]dz \neq 0$ and there are no components $\Omega^n A$ if $n > N+1$, hence $(\Omega A, d)$ has dimension $N+1$. Since every element of $(\Omega A, d)$ can be written as a linear combination of $\omega z^n + \nu z^m dz$, with $\omega, \nu \in \Omega R$ and ΩR satisfies the density condition (over R), also ΩA satisfies this condition (over A). Therefore, $(\Omega A, d)$ is an $N+1$-dimensional calculus as claimed. □

Lemma 4.3. *In the set-up of Lemma 4.2 assume that $(\Omega R, d)$ has a volume N-form v with the twisting automorphism θ_v and the co-ordinate isomorphism π_v. Let $u = \pi_\mathsf{v}(\sigma(\mathsf{v}))$ and define the map*

$$\bar{\theta} : R[z^\bullet;\sigma] \to R[z^\bullet;\sigma], \qquad \sum_i a_i z^i \mapsto \sum_i \theta_\mathsf{v}(a_i)(uz)^i. \tag{4.7}$$

Then:

(1) The map $\bar{\theta}$ is an algebra automorphism of $R[z^\bullet;\sigma]$.

(2) $\Omega\left(R[z^\bullet;\sigma]\right)$ has a volume form $\mathsf{v}dz$ with the twisting automorphism $\overline{\sigma}^{-1}\circ\bar{\theta}$.

Proof. First note that there exists $v \in R$ such that $\sigma^{-1}(\mathsf{v}) = \mathsf{v}v$. Hence $u\sigma(v) = 1 = v\sigma^{-1}(u)$, i.e. u is invertible and the map $\bar{\theta}$ is well defined also in the Laurent case. Furthermore $\bar{\theta}$ is invertible, with the inverse

$$\bar{\theta}^{-1} : R[z;\sigma] \to R[z;\sigma], \qquad \sum_i a_i z^i \mapsto \sum_i \theta_{\mathsf{v}}^{-1}(a_i)(u^{-1}z)^i.$$

Since, for all $a \in R$, $a\mathsf{v} = \mathsf{v}\theta_{\mathsf{v}}(a)$ and $\sigma(\mathsf{v}) = \mathsf{v}\pi_{\mathsf{v}}(\sigma(\mathsf{v})) = \mathsf{v}u$,

$$\mathsf{v}\theta_{\mathsf{v}}(\sigma(a))u = \sigma(a)\mathsf{v}u = \sigma(a)\sigma(\mathsf{v}) = \sigma(a\mathsf{v}) = \sigma(\mathsf{v}\theta_{\mathsf{v}}(a)) = \sigma(\mathsf{v})\sigma(\theta_{\mathsf{v}}(a)) = \mathsf{v}u\sigma(\theta_{\mathsf{v}}(a)).$$

Hence

$$\theta_{\mathsf{v}}(\sigma(a))u = u\sigma(\theta_{\mathsf{v}}(a))$$

holds for all a in R, and therefore

$$\bar{\theta}(z)\bar{\theta}(a) = uz\theta_{\mathsf{v}}(a) = u\sigma(\theta_{\mathsf{v}}(a))z = \theta_{\mathsf{v}}(\sigma(a))uz = \bar{\theta}(\sigma(a))\bar{\theta}(z),$$

and

$$\begin{aligned}\bar{\theta}(z^{-1})\bar{\theta}(a) &= z^{-1}u^{-1}\theta_{\mathsf{v}}(a) = z^{-1}\sigma(\theta_{\mathsf{v}}(\sigma^{-1}(a)))u^{-1} \\ &= \theta_{\mathsf{v}}(\sigma^{-1}(a))z^{-1}u^{-1} = \bar{\theta}(\sigma^{-1}(a))\bar{\theta}(z^{-1}),\end{aligned}$$

This implies that $\bar{\theta}$ is an algebra map and completes the proof of the first assertion.

To prove the second assertion, first let us write A for $R[z^\bullet;\sigma]$. As $\Omega^N R = \mathsf{v}R$, we have $\left(\Omega^N R\right)[z^\bullet] = \mathsf{v}A$, and thus $\Omega^{N+1}A = \left(\Omega^N R\right)[z^\bullet]dz = \mathsf{v}Adz$, i.e. any element of $\Omega^{N+1}A$ is of the form $\mathsf{v}fdz$, for some $f \in A$. By (4.5), $\mathsf{v}fdz = \mathsf{v}dz\overline{\sigma}^{-1}(f)$, hence $\Omega^{N+1}A = \mathsf{v}dzA$. Let $f = \sum a_i z^i \in A$ be such that $\mathsf{v}dzf = 0$, then $\mathsf{v}\overline{\sigma}(f) = 0$, which implies $f = 0$. Thus $\Omega^{N+1}A = \mathsf{v}dzA$ is a free rank one right A-module. Moreover, for any element $f \in A$,

$$f\mathsf{v}dz = \mathsf{v}\bar{\theta}(f)dz = \mathsf{v}dz\overline{\sigma}^{-1}(\bar{\theta}(f)),$$

which also shows that $A\mathsf{v}dz = \mathsf{v}dzA$, and hence $\mathsf{v}dz$ is a free generator on both sides, and that the twisting automorphism has the stated form. □

Lemma 4.4. *Let $R \subseteq S$ be a ring extension and $\sigma \in \mathrm{Aut}(S)$ such that the restriction of σ to R is an automorphism of R. Consider the skew-polynomial ring $S[z^\bullet;\sigma]$ and its subring $R[z^\bullet;\sigma]$. Let M be a σ-stable, right R-submodule of S and consider the right $R[z^\bullet;\sigma]$-submodule $M[z^\bullet]$ of $S[z^\bullet;\sigma]$.*

(1) The additive map $\psi : \mathrm{Hom}_R(M,R)[z^\bullet] \to \mathrm{Hom}_{R[z;\sigma]}\left(M[z^\bullet], R[z^\bullet;\sigma]\right)$ given by

$$fz^k \mapsto \psi_{fz^k} : [mz^i \mapsto \sigma^k(f(m))z^{k+i}], \qquad \forall f \in \mathrm{Hom}_R(M,R),$$

is well-defined and injective.

(2) If M is a finitely generated right R-module and $\sigma(M) = M$, then ψ is bijective.

Proof. (1) We will first show that ψ_{fz^k} is a right $R[z^\bullet;\sigma]$-linear map. For all $m \in M$, $r \in R$ and $i, j \in \mathbb{Z}$,

$$\begin{aligned}\psi_{fz^k}(mz^i)rz^j &= \sigma^k(f(m))z^{k+i}rz^j \\ &= \sigma^k(f(m\sigma^i(r)))z^{k+i+j} = \psi_{fz^k}\left(m\sigma^i(r)z^{i+j}\right) = \psi_{fz^k}\left(mz^irz^j\right)\end{aligned}$$

If $\psi_{\sum_k f_k z^k} = 0$ for some $\sum_k f_k z^k \in \mathrm{Hom}_R(M,R)[z^\bullet]$, then, for all $m \in M$,

$$\sum_k \sigma^k\left(f_k\left(m\right)\right) z^k = \psi_{\sum_k f_k z^k}(m) = 0,$$

i.e. $\sigma^k(f_k(m)) = 0$, for all k. Hence $f_k = 0$ for all k, showing that ψ is injective.

(2) For any k, define an additive map

$$p_k : R[z;\sigma] \to R, \qquad p_k\left(\sum r_i z^i\right) = \sigma^{-k}(r_k).$$

The maps p_k are right R-linear, because

$$p_k\left(\sum r_i z^i r'\right) = p_k\left(\sum r_i \sigma^i(r') z^i\right) = \sigma^{-k}(r_k) r' = p_k\left(\sum r_i z^i\right) r'.$$

Suppose $\{b_1, \ldots, b_n\}$ is a generating set for M_R. Let $f \in \mathrm{Hom}_{R[z^\bullet;\sigma]}\left(M[z^\bullet], R[z^\bullet;\sigma]\right)$. There exist a finite indexing set $I \subset \mathbb{Z}$ and elements $\beta_{ik} \in R$, for $1 \leq i \leq n$ and $k \in I$, such that $f(b_i) = \sum_{k\in I} \beta_{ik} z^k$. For all $k \in I$, define right R-linear maps $f_k \in \mathrm{Hom}_R(M,R)$ by the composition $f_k = p_k \circ f$, i.e. $f_k(b_i) = \sigma^{-k}(\beta_{ik})$, for all i. Let $m \in M$. There are $\lambda_i \in R$ such that $m = \sum_{i=1}^n b_i \lambda_i$. Then, for any j,

$$\begin{aligned} f(mz^j) &= \sum_{i=1}^n f(b_i)\lambda_i z^j = \sum_{k\in I}\sum_{i=1}^n \beta_{ik} z^k \lambda_i z^j \\ &= \sum_{k\in I}\sum_{i=1}^n \sigma^k(f_k(b_i))\sigma^k(\lambda_i) z^{k+j} = \sum_{k\in I} \sigma^k(f_k(m)) z^{k+j} = \psi_{\sum_{k\in I} f_k z^k}\left(mz^j\right). \end{aligned}$$

Hence $f = \psi\left(\sum_{k\in I} f_k z^k\right)$. □

Corollary 4.5. *Let σ be an automorphism of degree 0 of a graded algebra $\Omega = \bigoplus_{k=0}^\infty \Omega^k$. Set $R = \Omega^0$. If Ω^k is finitely generated as right R-module, then*

$$\mathrm{Hom}_R(\Omega^k, R)[z^\bullet] \simeq \mathrm{Hom}_{R[z^\bullet;\sigma]}(\Omega^k[z^\bullet], R[z^\bullet;\sigma]).$$

With these assertions at hand we can now prove Theorem 4.1.

Proof of Theorem 4.1. Let us denote by v a volume form for ΩR, with the corresponding co-ordinate isomorphism $\pi_\mathsf{v} : \Omega^N R \to R$. By Lemma 4.3, $\mathsf{v}dz$ is a volume form for the differential calculus ΩA on $A = R[z^\bullet, \sigma]$ constructed in Lemma 4.2, and let $\pi_{\mathsf{v}dz} : \Omega^{N+1} A \to A$ be the corresponding co-ordinate isomorphism. For all $\omega \in \Omega^N R$,

$$\pi_{\mathsf{v}dz}(\omega z^i dz) = \pi_{\mathsf{v}dz}\left(\mathsf{v}\pi_\mathsf{v}(\omega) dz\right) z^i = \sigma^{-1}(\pi_\mathsf{v}(\omega)) z^i. \tag{4.8}$$

Consider the maps $\ell^k_\mathsf{v} : \Omega^k R \to \mathrm{Hom}_R(\Omega^{N-k}R, R)$, $\ell_{\mathsf{v}dz} : \Omega^k A \to \mathrm{Hom}_A(\Omega^{N+1-k}A, A)$ associated to respective volume forms by (2.4). We canonically extend the ℓ^k_v to maps from $\Omega^k R[z^\bullet]$ to $\mathrm{Hom}_R(\Omega^{N-k}R, R)[z^\bullet]$, by acting on the coefficients. To show that A is differentially smooth, we need to show that the maps $\ell^k_{\mathsf{v}dz} : \Omega^k A \to \mathrm{Hom}_A\left(\Omega^{N-k}A, A\right)$ are bijective. The bijectivity of $\ell^k_{\mathsf{v}dz}$ is clear for $k = 0$ or $k = N+1$, since $\mathsf{v}dz$ is a volume form for ΩA. Let $1 \leq k \leq N$ and recall that $\Omega^k A = (\Omega^k R)[z^\bullet] \oplus (\Omega^{k-1}R)[z^\bullet]dz$. It is not difficult to see that the image of $(\Omega^k R)[z^\bullet]$ under $\ell^k_{\mathsf{v}dz}$ lies in $\mathrm{Hom}_A(\left(\Omega^{N-k}R\right)[z^\bullet]dz, A)$, while the image of $(\Omega^{k-1}R)[z^\bullet]dz$ under $\ell^k_{\mathsf{v}dz}$ lies in $\mathrm{Hom}_A(\left(\Omega^{N+1-k}R\right)[z^\bullet], A)$.

For later use we define, for any $i \in \mathbb{Z}$, invertible elements $v_i \in R$, such that $\sigma^i(\mathsf{v}) = v_i \mathsf{v}$. Then $v_{i+j}\mathsf{v} = \sigma^i(\sigma^j(\mathsf{v})) = \sigma^i(v_j) v_i \mathsf{v}$ for any $i, j \in \mathbb{Z}$, i.e. $v_{i+j} = \sigma^i(v_j) v_i$ from

which it follows that $\sigma^i(v_{-i})$ is the inverse of v_i. The following equation shows the commutation of σ and π_{v}. Let $\omega \in \Omega^N R$. Then

$$\sigma^{-i}(\pi_{\mathsf{v}}(\omega)) = \pi_{\mathsf{v}}(\mathsf{v}\sigma^{-i}(\pi_{\mathsf{v}}(\omega))) = \pi_{\mathsf{v}}(\sigma^{-i}(v_i \mathsf{v} \pi_{\mathsf{v}}(\omega))) = \pi_{\mathsf{v}}(\sigma^{-i}(v_i\omega)). \tag{4.9}$$

Furthermore, the invertible elements v_i give rise to a linear isomorphism

$$\Phi : (\Omega^{k-1}R)[z^\bullet]dz \to (\Omega^{k-1}R)[z^\bullet], \qquad \omega z^i dz \mapsto (-1)^{N-k+1}\sigma^{-(i+1)}(v_{i+1}\omega)z^i.$$

Using equations (4.8) and (4.9) we compute, for all $\omega z^i \in \Omega^{k-1}R[z^\bullet]$ and $\omega' z^j \in \Omega^{N-k+1}R[z^\bullet]$,

$$\begin{aligned}
\ell^k_{\mathsf{v}dz}(\omega z^i dz)(\omega' z^j) &= \pi_{\mathsf{v}dz}(\omega z^i dz \omega' z^j) = (-1)^{|\omega'|}\pi_{\mathsf{v}dz}(\omega\sigma^{i+1}(\omega')z^{i+j}dz) \\
&= (-1)^{|\omega'|}\sigma^{-1}(\pi_{\mathsf{v}}(\omega\sigma^{i+1}(\omega')))z^{i+j} \\
&= (-1)^{|\omega'|}\sigma^{i}(\sigma^{-(i+1)}(\pi_{\mathsf{v}}(\omega\sigma^{i+1}(\omega'))))z^{i+j} \\
&= (-1)^{|\omega'|}\sigma^{i}(\pi_{\mathsf{v}}(\sigma^{-(i+1)}(v_{i+1}\omega\sigma^{i+1}(\omega'))))z^{i+j} \\
&= (-1)^{|\omega'|}\sigma^{i}(\pi_{\mathsf{v}}(\sigma^{-(i+1)}(v_{i+1}\omega)\omega')z^{i+j} \\
&= \psi(\ell^{k-1}_{\mathsf{v}}((-1)^{|\omega'|}\sigma^{-(i+1)}(v_{i+1}\omega z^i)))(\omega' z^j) \\
&= \psi(\ell^{k-1}_{\mathsf{v}}(\Phi(\omega z^i dz)))(\omega' z^j),
\end{aligned}$$

where $\psi : \mathrm{Hom}_R(\Omega^{N-k+1}R, R)[z^\bullet] \to \mathrm{Hom}_A\left(\Omega^{N-k+1}R[z^\bullet], A\right)$ is the homomorphism from Lemma 4.4. Hence we have shown that the following diagram commutes:

$$\begin{array}{ccc}
\left(\Omega^{k-1}R\right)[z^\bullet]dz & \xrightarrow{\ell^k_{\mathsf{v}dz}} & \mathrm{Hom}_A(\Omega^{N+1-k}R[z^\bullet], A) \\
{\scriptstyle\Phi}\downarrow & & \uparrow{\scriptstyle\psi} \\
\left(\Omega^{k-1}R\right)[z^\bullet] & \xrightarrow[\ell^{k-1}_{\mathsf{v}}]{} & \mathrm{Hom}_R(\Omega^{N+1-k}R, R)[z^\bullet].
\end{array} \tag{4.10}$$

In a way similar to the definition of Φ we define the linear isomorphism $\Phi' : \left(\Omega^k R\right)[z^\bullet] \to \left(\Omega^k R\right)[z^\bullet]$ by

$$\Phi'(\omega z^i) = \sigma^{-(i+1)}(v_{i+1}\omega)z^i \qquad \text{for all } \omega \in \Omega^k R.$$

Moreover, the map $\varphi : \Omega^{N-k}R[z^\bullet]dz \to \Omega^{N-k}R[z^\bullet]$, given by $\varphi(\omega z^i dz) = \sigma^{-1}(\omega)z^i$, is an isomorphism of right A-modules. The adjoint map of φ is the isomorphism

$$\varphi^* : \mathrm{Hom}_A\left(\Omega^{N-k}R\right)[z^\bullet], A) \to \mathrm{Hom}_A(\Omega^{N-k}R[z^\bullet]dz, A),$$

$$f \mapsto f \circ \varphi : [\omega z^i dz \mapsto f(\sigma^{-1}(\omega)z^i)].$$

Using again equation (4.9), Φ' and φ^* we compute, for all $\omega z^i \in \Omega^k R[z^\bullet], \omega' z^j \in \Omega^{N-k}R[z^\bullet]$,

$$\begin{aligned}
\ell^k_{\mathsf{v}dz}(\omega z^i)(\omega' z^j dz) &= \pi_{\mathsf{v}dz}(\omega z^i \omega' z^j dz) = \pi_{\mathsf{v}dz}(\omega\sigma^i(\omega')z^{i+j}dz) \\
&= \sigma^{-1}(\pi_{\mathsf{v}}(\omega\sigma^i(\omega')))z^{i+j} = \sigma^i(\sigma^{-(i+1)}(\pi_{\mathsf{v}}(\omega\sigma^i(\omega'))))z^{i+j} \\
&= \sigma^i(\pi_{\mathsf{v}}(\sigma^{-(i+1)}(v_{i+1}\omega\sigma^i(\omega')))z^{i+j} \\
&= \sigma^i(\pi_{\mathsf{v}}(\sigma^{-(i+1)}(v_{i+1}\omega)\sigma^{-1}(\omega')))z^{i+j} \\
&= \psi(\ell^k_{\mathsf{v}}(\sigma^{-(i+1)}(v_{i+1}\omega)z^i))(\sigma^{-1}(\omega')z^j) \\
&= \psi(\ell^k_{\mathsf{v}}(\Phi'(\omega z^i)))(\varphi(\omega' z^j dz)) = \varphi^*(\psi(\ell^k_{\mathsf{v}}(\Phi'(\omega z^i))))(\omega' z^j dz).
\end{aligned}$$

Therefore, the following diagram commutes:

$$\begin{array}{ccc} \big(\Omega^k R\big)\,[z^\bullet] & \xrightarrow{\ell^k_{\mathsf{v}dz}} & \mathrm{Hom}_A(\Omega^{N-k}R[z^\bullet]dz, A) \\ {\scriptstyle\Phi'}\big\downarrow & & \big\uparrow{\scriptstyle\varphi^*\circ\psi} \\ \big(\Omega^k R\big)\,[z^\bullet] & \xrightarrow[\ell^k_{\mathsf{v}}]{} & \mathrm{Hom}_R(\Omega^{N-k}R, R)[z^\bullet]. \end{array} \tag{4.11}$$

Assume for all k that the maps ℓ^k_{v} are bijective and that the right R-modules $\Omega^k R$ are finitely generated. Then ψ is bijective by Corollary 4.5 and hence $\psi\circ\ell^{k-1}_{\mathsf{v}}\circ\Phi$ and $\varphi^*\circ\psi\circ\ell^k_{\mathsf{v}}\circ\Phi'$ are bijective maps. Since the diagrams (4.10) and (4.11) commute, also $\ell^k_{\mathsf{v}dz}:\Omega^k A\to\mathrm{Hom}_A(\Omega^{N+1-k}A, A)$ is bijective.

Finally, if ΩR is a connected calculus, then vanishing of the first component in (4.2), implies that if $d(f)=0$, for $f=\sum_i a_i z^i\in R[z^\bullet;\sigma]$, then $f\in\mathbb{F}[z^\bullet]$ (i.e. f has scalar coefficients only). The second component in (4.2) is simply $\partial_z(f)$, hence it vanishes if and only if f is a scalar multiple of the identity in A. Therefore, the calculus ΩA is also connected. This completes the proof of the theorem. □

Example 4.6. For any non-zero $q\in\mathbb{F}$, let us define A_q as an algebra generated by x, y, z and relations

$$xy=yx,\qquad xz=qzy,\qquad yz=zx. \tag{4.12}$$

The algebra A_q is differentially smooth. Similarly, the algebra B_q, generated by x, y and invertible z subject to relations (4.12), is differentially smooth.

Proof. The algebras A_q and B_q are both skew-polynomial rings, $A_q=\mathbb{F}[x,y][z;\sigma]$, $B_q=\mathbb{F}[x,y][z^{\pm1};\sigma]$, where the automorphism σ of $\mathbb{F}[x,y]$ is given by

$$\sigma(x)=y,\qquad \sigma(y)=qx.$$

The polynomial algebra $\mathbb{F}[x,y]$ is differentially smooth with the usual commutative differential calculus $\Omega(\mathbb{F}[x,y])$,

$$xdx=dxx,\qquad xdy=dyx,\qquad ydx=dxy,\qquad ydy=dyy,$$

$$dxdy=-dydx,\qquad (dx)^2=(dy)^2=0.$$

The automorphism σ extends to an automorphism of $\Omega(\mathbb{F}[x,y])$ by requesting it commute with d, i.e.

$$\sigma(dx)=dy,\qquad \sigma(dy)=qdx.$$

Since $\Omega(\mathbb{F}[x,y])$ is finitely generated as a right $\mathbb{F}[x,y]$-module and

$$\mathrm{GKdim}(A_q)=\mathrm{GKdim}(B_q)=3=\mathrm{GKdim}(\mathbb{F}[x,y])+1,$$

Theorem 4.1 yields the differential smoothness of A_q and B_q. □

Remark 4.7. We notice in passing that B_1 in Example 4.6 contains the down-up algebra $A(0,1,0)$ [7] as a proper subalgebra and hence the assertion of Example 4.6 can be a starting point in determining whether $A(0,1,0)$ is differentially smooth.

The statement of Theorem 4.1 can be iterated in the following way.

Proposition 4.8. *Let R be an algebra with an integrable differential calculus $(\Omega R, d)$ such that ΩR is a finitely generated right R-module. Let σ be an automorphism of R that extends to a degree-preserving automorphism of ΩR, which commutes with d. Let $(\Omega A, d)$ be the integrable differential calculus on $A = R[z^\bullet; \sigma]$ constructed via Theorem 4.1.*

(1) For any $q \in \mathbb{F}^*$, the map σ extends to an automorphism of the differential graded algebra $(\Omega A, d)$, by

$$\sigma_q : \Omega A \to \Omega A, \qquad \omega f(z) \mapsto \sigma(\omega) f(qz), \quad \omega f(z) dz \mapsto q\sigma(\omega) f(qz) dz.$$

(2) If R is differentially smooth with respect to $(\Omega R, d)$ and $\mathrm{GKdim}(A) = \mathrm{GKdim}(R) + 1$, then $A[t^\bullet; \sigma_q] = R[z^\bullet; \sigma][t^\bullet; \sigma_q]$ is also differentially smooth.

Proof. That σ_q is an algebra automorphism is established by a routine calculation. To check that σ_q commutes with d, first observe that

$$\partial_z \circ \sigma_q = q\, \sigma_q \circ \partial_z. \tag{4.13}$$

Hence, for all $f, g \in \mathbb{F}[z^\bullet]$ and homogeneous $\omega, \nu \in \Omega R$,

$$\begin{aligned} \sigma_q \left(d\left(\omega f + \nu g dz\right)\right) &= \sigma(d\omega)\sigma_q(f) + q\left((-1)^{|\omega|}\sigma(\omega)\sigma_q(\partial_z(f)) + \sigma(d\nu)\sigma_q(g)\right) dz \\ &= d\left(\sigma(\omega)\right)\sigma_q(f) + (-1)^{|\omega|}\sigma(\omega)\partial_z(\sigma_q(f))dz + d\left(\sigma(\nu)\right)\sigma_q(gdz) \\ &= d\left(\sigma_q\left(\omega f + \nu g dz\right)\right), \end{aligned}$$

where the second equality follows by (4.13) and the fact that σ commutes with d. This completes the proof of the first statement.

Since ΩR is finitely generated as a right R module and

$$\Omega^k A = \Omega^k R[z^\bullet; \sigma] \oplus \Omega^{k-1} R[z^\bullet; \sigma]^{\bar{\sigma}},$$

also ΩA is finitely generated as a right A-module. ΩA is integrable of dimension $\mathrm{GKdim}(R) + 1$, hence, by Theorem 4.1 $A[t^\bullet; \sigma_q]$ admits an integrable calculus of dimension $\mathrm{GKdim}(R) + 2$. Furthermore, the automorphism σ_q is locally algebraic, hence $\mathrm{GKdim}(A[t^\bullet; \sigma_q]) = \mathrm{GKdim}(A) + 1$ by [10, Proposition 1], and since $\mathrm{GKdim}(A) = \mathrm{GKdim}(R) + 1$ the second assertion follows. □

Proposition 4.8 leads to a quick proof of the differential smoothness of special cases of algebras whose differential smoothness was established in [6].

Corollary 4.9. *The coordinate algebra of the non-commutative n-dimensional affine space, i.e. the algebra $\mathbb{F}_{\mathbf{q}}[x_1, \ldots, x_n]$ generated by $x_1, \ldots, x_n$ subject to the relations*

$$x_i x_j = q_i x_j x_i, \qquad \text{for all } i < j,$$

where $\mathbf{q} = (q_1, \ldots, q_{n-1}) \in (\mathbb{F}^*)^{n-1}$, is differentially smooth.

Proof. $\mathbb{F}_{\mathbf{q}}[x_1, \ldots, x_n]$ is an iterated skew polynomial ring. Starting with the polynomial ring $\mathbb{F}[x_1]$, which is differentially smooth by the usual commutative differential structure, and applying first Theorem 4.1 and then its iteration Proposition 4.8 sufficiently many times (with a different q at each step), we conclude that $\mathbb{F}_{\mathbf{q}}[x_1, \ldots, x_n]$ is differentially smooth, as claimed. □

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References

[1] T. Brzeziński, *Differential smoothness of affine Hopf algebras of Gelfand-Kirillov dimension two*, Colloq. Math. **139** (2015), 111–119.
[2] T. Brzeziński, L. El Kaoutit & C. Lomp, *Non-commutative integral forms and twisted multi-derivations*, J. Noncommut. Geom. **4** (2010), 281–312.
[3] T. Brzeziński & A. Sitarz, *Smooth geometry of the noncommutative pillow, cones and lens spaces*, arXiv:1410.6587 to appear in J. Noncomm. Geom.
[4] J. Cuntz and D. Quillen, *Algebra extensions and nonsingularity*, J. Amer. Math. Soc. **8** (1995), 251–289.
[5] A. Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. I.* Inst. Hautes Études Sci. Publ. Math. **20** (1964).
[6] S. Karaçuha & C. Lomp, *Integral calculus on quantum exterior algebras*, Int. J. Geom. Methods Mod. Phys. **11** (2014), 1450026
[7] E. Kirkman, I.M. Musson & D.S. Passman, *Noetherian down-up algebras*, Proc. Amer. Math. Soc. **127** (1999), no. 11, 3161–3167.
[8] U. Krähmer, *On the Hochschild (co)homology of quantum homogeneous spaces*, Israel J. Math. **189** (2012), 237–266.
[9] G.R. Krause & T.H. Lenagan, *Growth of Algebras and Gelfand-Kirillov Dimension*, Revised edition, American Mathematical Society, Providence, RI (2000).
[10] A. Leroy, J. Matczuk & J. Okniński, *On the Gelfand-Kirillov dimension of normal localizations and twisted polynomial rings*, [in:] *Perspectives in Ring Theory* (F. van Oystaeyen and L. Le Bruyn, eds.), Kluwer Academic Publishers, 1988, pp. 205–214.
[11] J.C. McConnell & J. C. Robson, *Noncommutative Noetherian Rings*. Revised edition, American Mathematical Society, Providence, RI (2001).
[12] W.F. Schelter, *Smooth algebras,* J. Algebra **103** (1986), 677–685.
[13] J.T. Stafford & J.J. Zhang, *Homological properties of (graded) Noetherian PI rings,* J. Algebra **168** (1994), 988–1026.
[14] M. Van den Bergh, *A relation between Hochschild homology and cohomology for Gorenstein rings*, Proc. Amer. Math. Soc. **126** (1998), 1345–1348; Erratum: Proc. Amer. Math. Soc. **130** (2002), 2809–2810.
[15] R. Wisbauer, *Modules and algebras: bimodule structure and group actions on algebras*, Pitman Monographs and Surveys in Pure and Applied Mathematics 81, Longman, Harlow, 1996.

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