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Fock representations of Q-deformed commutation relations

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Abstract

We consider Fock representations of the Q-deformed commutation relations

$$\partial_s \partial_t^\dagger = Q(s,t)\partial_t^\dagger \partial_s + \delta(s,t), \quad s,t \in T.$$

Here $T := \mathbb{R}^d$ (or more generally T is a locally compact Polish space), the function $Q : T^2 \to \mathbb{C}$ satisfies $|Q(s,t)| \leq 1$ and $Q(s,t) = \overline{Q(t,s)}$, and

$$\int_{T^2} h(s)g(t)\delta(s,t)\,\sigma(ds)\sigma(dt) := \int_T h(t)g(t)\,\sigma(dt),$$

σ being a fixed reference measure on T. In the case where $|Q(s,t)| \equiv 1$, the Q-deformed commutation relations describe a generalized statistics studied by Liguori and Mintchev (1995). These generalized statistics contain anyon statistics as a special case (with $T = \mathbb{R}^2$ and a special choice of the function Q). The related Q-deformed Fock space $\mathcal{F}(\mathcal{H})$ over $\mathcal{H} := L^2(T \to \mathbb{C}, \sigma)$ is constructed. An explicit form of the orthogonal projection of $\mathcal{H}^{\otimes n}$ onto the n-particle space $\mathcal{F}_n(\mathcal{H})$ is derived. A scalar product in $\mathcal{F}_n(\mathcal{H})$ is given by an operator $\mathcal{P}_n \geq 0$ in $\mathcal{H}^{\otimes n}$ which is strictly positive on $\mathcal{F}_n(\mathcal{H})$. We realize the smeared operators $\partial_t^\dagger$ and ∂_t as creation and annihilation operators in $\mathcal{F}(\mathcal{H})$, respectively. Additional Q-commutation relations are obtained between the creation operators and between the annihilation operators. They are of the form $\partial_s^\dagger \partial_t^\dagger = Q(t,s)\partial_t^\dagger \partial_s^\dagger$, $\partial_s \partial_t = Q(t,s)\partial_t \partial_s$, valid for those $s,t \in T$ for which $|Q(s,t)| = 1$.

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1 Introduction

The aim of the paper is to construct Fock representations of the Q-commutation relations

$$\partial_s \partial_t^\dagger = Q(s,t)\partial_t^\dagger \partial_s + \delta(s,t), \quad s,t \in T. \tag{1}$$

Here $T = \mathbb{R}^d$, or more generally, T is a locally compact Polish space, the function $Q : T^2 \to \mathbb{C}$ is Hermitian, i.e., $Q(s,t) = \overline{Q(t,s)}$, and satisfies $|Q(s,t)| \le 1$, ∂_t and $\partial_t^\dagger$ are operator-valued distributions, adjoint of each other, and

$$\int_{T^2} \delta(s,t) f(s,t)\, \sigma(ds)\, \sigma(dt) := \int_T f(t,t)\, \sigma(dt),$$

where σ is a fixed Radon meaure on X (typically $\sigma(dt) = dt$ being the Lebesgue measure if $T = \mathbb{R}^d$). We will call (1) the *Q-deformed commutation relations*, or just Q-CR.

For a function Q satisfying $|Q(s,t)| \equiv 1$, a Fock representation of the Q-CR was constructed by Liguori and Mintchev [24]. In that case, creation operators $\partial_t^\dagger$ and annihilation operators ∂_t satisfy the additional commutation relations:

$$\partial_s^\dagger \partial_t^\dagger = Q(t,s)\partial_t^\dagger \partial_s^\dagger, \quad \partial_s \partial_t = Q(t,s)\partial_t \partial_s. \tag{2}$$

The term *Fock representation* means that, for each annihilation operator, one has $\partial_t \Omega = 0$, where Ω is the vacuum vector.

In the present study, relations (2) will hold for those $s,t \in T$ which satisfy $|Q(s,t)| = 1$. Note that, under the assumption that the function Q is Hermitian, the commutation relations (2) are consistent if and only if $|Q(s,t)| = 1$.

For the first time, an interpolation between the canonical (bosonic) commutation relations (CCR) and the canonical (fermionic) anticommutation relations (CAR) was rigorously constructed in [7]. Let $\mathcal{H}$ be a separable Hilbert space and let $q \in (-1,1)$. On a q-deformed Fock space over $\mathcal{H}$, Bożejko and Speicher [7] constructed q-creation operators $a^+(f)$ (in fact $a^+(f)$ were *free* creation operators), and q-annihilation operators $a^-(f) := (a^+(f))^*$, for $f \in \mathcal{H}$, which satisfy the q-commutation relations:

$$a^-(f)a^+(g) = qa^+(g)a^-(f) + (f,g)_{\mathcal{H}}, \quad f,g \in \mathcal{H}. \tag{3}$$

The limits $q = 1$ and $q = -1$ correspond to the boson and fermion statistics, respectively, thus giving the CCR and CAR. The case $q = 0$ corresponds to the creation and annihilation operators acting in the full Fock space; these operators are particularly important for models of free probability, see e.g. [30, 5]. Aspects of noncommutative probability related to the general q-commutation relations (3) were discussed e.g. in [7, 4, 1].

By using probabilistic methods, Speicher [36] proved existence of a representation of the (discrete) q_{ij}-commutation relations of the form

$$\partial_i \partial_j^\dagger = q_{ij} \partial_j^\dagger \partial_i + \delta_{ij} \tag{4}$$

with $-1 \le q_{ij} = q_{ji} \le 1$, $i, j \in \mathbb{N}$, and $(\partial_i^\dagger)^* = \partial_i$. Bożejko and Speicher [8] constructed a Fock representation of the following commutation relations between creation operators $\partial_j^\dagger$ and annihilation operators ∂_i, with $i, j \in \mathbb{N}$:

$$\partial_i \partial_j^\dagger = \sum_{k,l} q_{jl}^{ik}\, \partial_k^\dagger \partial_l + \delta_{i,j}. \tag{5}$$

They showed that, if the operator Ψ given by the matrix $(q_{jl}^{ik})_{i,j,k,l}$ is self-adjoint, satisfies the braid relations, and has norm < 1, then there exists a Fock representation of the commutation relations (5). As a consequence, they obtained a Fock representation of the q_{ij}-commutation relations (4) even for complex q_{ij} with $\overline{q_{ij}} = q_{ji}$ and $\sup_{i,j} |q_{ij}| = \|\Psi\| < 1$. By taking the weak limit of corresponding operator algebras, Bożejko and Speicher [8] also derived existence of a representation of the q_{ij}-commutation relations (4) with $\sup_{i,j} |q_{ij}| = \|\Psi\| = 1$. Also Jørgensen, Schmitt and Werner [18, 19] considered representations of the commutation relations (5).

In the case where $\|\Psi\| = 1$, Jørgensen, Proskurin, and Samoĭlenko [20] found, for $n \ge 2$, the kernel of the nonnegative operator which determines the scalar product in the n-particle space of the Fock space corresponding to the commutation relations (5). The papers [8] and [20] taken together give then a Fock representation of the commutation relations (4) with $\sup_{i,j} |q_{ij}| = 1$.

Properties of the algebras generated by such operators were studied by many authors. In the context of C^*-algebras, let us mention the works by Dykema and Nica [11] and Kennedy and Nica [21] (who studied relations of the C^*-algebras generated by the q-commutation relations with the Cuntz algebra), Jørgensen, Schmitt and Werner [18, 19] (who studied the Wick order generated C^*-algebras), Proskurin and Samoilenko [32] (who studied general Wick *-algebras). There are also a number of studies of the q-commutation relations in the context of von Neumann algebras, in particular, by Lust-Piquard [25] (who studied properties of the Riesz transform), Królak [22], Nou [31], Śniady [35], Ricard [33] (who studied factoriality problems), Shlyakhtenko [34] (who studied Voiculescu's free entropy for families of q-Gaussian operators), and Bożejko [2] (who studied positivity of the symmetrization operators constructed through a self-adjoint Yang–Baxter operator $\Psi \geq -1$). Also Dabrowski [10], Guionnet and Shlyakhtenko [17], and Nelson and Zeng [28, 29] proved that q-factors or, more generally, q_{ij}-factors are isomorphic to the free group factors ($q = 0$) for small values of q or q_{ij}, respectively. Another possible generalization of the commutation relations (3) related to the group of signed permutations can be found in [3]

All the above mentioned investigations are of *discrete* type, so that the set T is at most countable. As we have already mentioned above, in the *continuous* setting, a Fock representation of the Q-CR (1), (2), called a *generalized statistics*, was constructed by Liguori and Mintchev [24], see also [14, 15, 16, 13, 6]. A rigorous meaning of these commutation relations is given by smearing them with functions from $\mathcal{H} := L^2(T \to \mathbb{C}, \sigma)$. More precisely, defining for $f \in \mathcal{H}$ operators $a^+(f) := \int_T \sigma(dt)\, f(t)\partial_t^\dagger$ and $a^-(f) := \int_T \sigma(dt)\, \overline{f(t)}\partial_t$, we get the commutation relations:

$$\begin{aligned}
a^-(f)a^+(g) &= \int_{T^2} \sigma(ds)\, \sigma(dt)\, \overline{f(s)}g(t)Q(s,t)\partial_t^\dagger\partial_s + \int_T \overline{f(t)}g(t)\, \sigma(dt),\\
a^+(f)a^+(g) &= \int_{T^2} \sigma(ds)\, \sigma(dt)\, f(s)g(t)Q(t,s)\partial_t^\dagger\partial_s^\dagger,\\
a^-(f)a^-(g) &= \int_{T^2} \sigma(ds)\, \sigma(dt)\, \overline{f(s)g(t)}Q(t,s)\partial_t\partial_s,
\end{aligned}$$

where $f, g \in \mathcal{H}$. (Of course, the operator-valued integrals in these relations should be given a rigorous meaning.)

From the physical point of view, the most important case of a generalized statistics is the anyon statistics, where $T = \mathbb{R}^2$ and the function $Q(s,t)$ is determined by a complex parameter q with $|q| = 1$, namely,

$$Q(s,t) = \begin{cases} q, & \text{if } s^1 < t^1, \\ \bar{q}, & \text{if } s^1 > t^1. \end{cases} \tag{6}$$

Here, $s = (s^1, s^2), t = (t^1, t^2) \in \mathbb{R}^2$. Note that the value of the function Q on the set $\{(s,t) \in T^2 \mid s^1 = t^1\}$ does not matter for the Fock representation of the Q-CR. For an explanation as to why such commutation relations describe an anyon statistic, we refer the reader to Liguori and Mintchev's paper [24] and to Goldin and Sharp's paper [16].

Goldin and Majid [13] proved the following anyonic exclusion principle, which generalizes Pauli's exclusion principle for fermions: If $q^m = 1$ and $q \neq 1$, then the creation operators $a^+(f)$ in the Fock representation of the anyon commutation relations satisfy $a^+(f)^m = 0$, or equivalently, the Q-symmetrization of the function $f^{\otimes m}$ is equal to zero.

In [26], non-Fock representations of the anyon commutation relations have been constructed, whose vacuum states are gauge-invariant quasi-free. Note that, for those representations, the (real) value of the function $Q(s,t)$ for $s = t$ must be specified.

Let us mention that anyon systems have also been considered in the discrete setting, i.e., when $T \subset \mathbb{N}$, see e.g. [12, 23, 13]. It should be, however, mentioned that, when discussing the anyons in the discete setting, Goldin and Majid [13] dropped the assumption that the annihilation operator is adjoint of the creation operator, and proved an anyonic exclusion principle for their model.

In this paper, we study the continuous case with a function Q satisfying $|Q(s,t)| \le 1$. This natural choice of Q contains generalized statistics and the relations (3) as special cases. We would also like to draw the reader's attention to the study by Merberg [27], where the case $Q : T^2 \to (-1,1)$ was considered and factoriality of the related von Neumann algebras generated by the Q-Gaussian operators was discussed.

In Section 2 we present a construction of the Fock representation of the Q-CR (1). To this end, we construct a certain Q-deformed Fock space over $\mathcal{H} = L^2(T \to \mathbb{C}, \sigma)$, denoted by $\mathcal{F}(\mathcal{H})$. We describe the n-particle subspaces, $\mathcal{F}_n(\mathcal{H})$, of $\mathcal{F}(\mathcal{H})$. As a set, each $\mathcal{F}_n(\mathcal{H})$ is a subset of $\mathcal{H}^{\otimes n} = L^2(T^n \to \mathbb{C}, \sigma^{\otimes n})$ and consists of all functions $f^{(n)} \in \mathcal{H}^{\otimes n}$ that are Q-*quasisymmetric*, meaning that, a.e. for each $k \in \{1, \dots, n-1\}$,

$$f^{(n)}(t_1, \dots, t_n) = Q(t_k, t_{k+1}) f(t_1, \dots, t_{k+1}, t_k, \dots, t_n) \tag{7}$$

provided $|Q(t_k, t_{k+1})| = 1$. We derive an explicit formula for the orthogonal projection of $\mathcal{H}^{\otimes n}$ onto $\mathcal{F}_n(\mathcal{H})$. A scalar product in $\mathcal{F}_n(\mathcal{H})$ is given by an operator $\mathcal{P}_n \ge 0$ in $\mathcal{H}^{\otimes n}$ which is strictly positive on $\mathcal{F}_n(\mathcal{H})$. We then realize $a^+(f)$, $a^-(f)$ ($f \in \mathcal{H}$) as creation and annihilation operators acting in the Q-deformed Fock space $\mathcal{F}(\mathcal{H})$. These operators satisfy the Q-CR (1). Additionally, due to the Q-symmetry (7) in each $\mathcal{F}_n(\mathcal{H})$, we get the following commutation relations between the creation operators and between the annihilation operators:

$$\begin{aligned} \partial_s^\dagger \partial_t^\dagger &= Q(t,s) \partial_t^\dagger \partial_s^\dagger, \quad \text{if } |Q(s,t)| = 1, \\ \partial_s \partial_t &= Q(t,s) \partial_t \partial_s, \quad \text{if } |Q(s,t)| = 1. \end{aligned} \tag{8}$$

We note that, by choosing T to be a discrete set and σ to be the counting measure on T, one can apply our results in a discrete setting. In fact, the explicit description of the n-particle space $\mathcal{F}_n(\mathcal{H})$, explicit formula for the orthogonal projection of $\mathcal{H}^{\otimes n}$ onto $\mathcal{F}_n(\mathcal{H})$, and the additional commutation relations (8) appear to be new results even in the discrete setting.

We finish Section 2 with a proposition that shows that discrete anyons of fermion type satisfy the anyonic exclusion principle, compare with [13].

In Section 3, we prove the results formulated in Section 2.

2 Construction of the Fock representation of Q-CR

In this section, we will construct a Fock representation of the commutation relation (1), and we will note that the additional commutation relations (8) then also hold.

2.1 Operator $\mathcal{P}_n$

Let T be a locally compact Polish space, let $\mathcal{B}(T)$ denote the Borel σ-algebra on T, and let σ be a Radon measure on $(T, \mathcal{B}(T))$. Let $E \in \mathcal{B}(T^2)$ be a symmetric subset of T^2: if $(s,t) \in E$ then $(t,s) \in E$. We assume that $\sigma^{\otimes 2}(E) = 0$. Denote $T^{(2)} := T^2 \setminus E$, which is also a symmetric set. We fix a complex-valued measurable function

$$Q : T^{(2)} \to \{z \in \mathbb{C} : |z| \le 1\}$$

which is Hermitian: for all $(s,t) \in T^{(2)}$, we have $Q(s,t) = \overline{Q(t,s)}$. This function is defined $\sigma^{\otimes 2}$-almost everywhere on T^2.

Remark 1. The case where $|Q(s,t)| = 1$ for all $(s,t) \in T^{(2)}$ corresponds to a generalized statistics studied by Liguori and Mintchev [24]. The special case where $T = \mathbb{R}^2$, $\sigma(dt) = dt$ is the Lebesgue measure on T, $E = \{(s,t) \in T^2 \mid s^1 = t^1\}$, and the function Q is defined by formula (6) with $q \in \mathbb{C}$, $|q| = 1$, corresponds to anyon statistics, see [24, 16, 13]. The choice $Q(s,t) = q$ for all $(s,t) \in T^{(2)} = T^2$ with $q \in (-1,1)$ corresponds to the q-commutations (3), see [7].

Let us consider an operator Ψ which transforms a measurable function $f^{(2)} : T^{(2)} \to \mathbb{C}$ into the function

$$(\Psi f^{(2)})(s,t) := Q(s,t) f^{(2)}(t,s), \quad (s,t) \in T^{(2)}. \tag{9}$$

Analogously to $T^{(2)}$, we define, for $n \ge 3$,

$$T^{(n)} := \big\{(t_1, \dots, t_n) \in T^n : (t_i, t_j) \notin E \text{ for all } 1 \le i < j \le n\big\}.$$

It is clear that $\sigma^{\otimes n}(T^n \setminus T^{(n)}) = 0$. The operator Ψ can be extended to a transformation of functions $f^{(n)} : T^{(n)} \to \mathbb{C}$ by setting, for $k \in \{1, \dots, n-1\}$,

$$(\Psi_k f^{(n)})(t_1, \dots, t_n) := Q(t_k, t_{k+1}) f^{(n)}(t_1, \dots, t_{k-1}, t_{k+1}, t_k, t_{k+2}, \dots, t_n). \tag{10}$$

Let $\mathcal{H} := L^2(T \to \mathbb{C}, \sigma)$ be the complex L^2-space over T. We agree that the scalar product $(\cdot, \cdot)_{\mathcal{H}}$ is antilinear in the first dot and linear in the second. For $n \ge 2$, the nth tensor power of $\mathcal{H}$, denoted by $\mathcal{H}^{\otimes n}$, can be identified with the complex L^2-space $L^2(T^{(n)} \to \mathbb{C}, \sigma^{\otimes n})$. Each Ψ_k is a contraction in $\mathcal{H}^{\otimes n}$. The following trivial lemma shows that the operators Ψ_k are self-adjoint and satisfy the braid relations.

Lemma 2. *The operators Ψ_k satisfy:*

$$\begin{aligned}\Psi_k^* &= \Psi_k,\\ \Psi_k\Psi_l &= \Psi_l\Psi_k \quad \text{if } |k-l|\geq 2,\\ \Psi_k\Psi_{k+1}\Psi_k &= \Psi_{k+1}\Psi_k\Psi_{k+1}.\end{aligned} \tag{11}$$

Let S_n denote the symmetric group on $\{1,\dots,n\}$. Represent a permutation $\pi\in S_n$ as an arbitrary product of adjacent transpositions,

$$\pi=\pi_{j_1}\cdots\pi_{j_m}, \tag{12}$$

where $\pi_j:=(j,j+1)\in S_n$ for $1\le j\le n-1$. A permutation $\pi\in S_n$ can be represented (not in a unique way, in general) as a reduced product of a minimal number of adjacent transpositions, i.e., in the form (12) with a minimal m. This number m is then called the length of π, denoted by $|\pi|$. It is well known that $|\pi|$ is equal to the number of inversions of π, i.e., the number of $1\le i<j\le n$ such that $\pi(i)>\pi(j)$.

The mapping $\pi_k\mapsto\Psi_{\pi_k}:=\Psi_k$ can be multiplicatively extended to S_n by setting

$$S_n\ni\pi\mapsto\Psi_\pi:=\Psi_{j_1}\cdots\Psi_{j_m}. \tag{13}$$

Although representation (12) of $\pi\in S_n$ in a reduced form is not unique, the formulas (11) yield that the extension (13) is well defined, i.e., it does not depend on the representation. (This fact also follows from the proof of Proposition 3 below.)

We will use the notations $\mathbf{t}^{(n)}:=(t_1,\dots,t_n)\in T^{(n)}$, $\mathbf{t}_\pi^{(n)}:=(t_{\pi(1)},\dots,t_{\pi(n)})$ for $\pi\in S_n$.

Proposition 3. *For each $\pi\in S_n$ and $f^{(n)}\in\mathcal{H}^{\otimes n}$, we have*

$$(\Psi_\pi f^{(n)})(\mathbf{t}^{(n)})=Q_{\pi^{-1}}(\mathbf{t}^{(n)})f^{(n)}(\mathbf{t}_\pi^{(n)}), \tag{14}$$

where

$$Q_\pi(\mathbf{t}^{(n)}):=\prod_{\substack{1\le i<j\le n\\ \pi(i)>\pi(j)}}Q(t_i,t_j),\quad \mathbf{t}^{(n)}\in T^{(n)}. \tag{15}$$

For $n\geq 2$, we define an operator $\mathcal{P}_n$ on $\mathcal{H}^{\otimes n}$ by

$$\mathcal{P}_n:=\frac{1}{n!}\sum_{\pi\in S_n}\Psi_\pi. \tag{16}$$

The operator $\mathcal{P}_n$ is a self-adjoint contraction in $\mathcal{H}^{\otimes n}$, since so are the operators Ψ_k.

The following result is a special case of Theorem 1.1 in [8].

Theorem 4 ([8])**.** *For each $n\geq 2$, we have $\mathcal{P}_n\geq 0$.*

For any $f^{(n)}, g^{(n)} \in \mathcal{H}^{\otimes n}$, we define

$$(f^{(n)}, g^{(n)})_{\mathcal{F}_n(\mathcal{H})} := (\mathcal{P}_n f^{(n)}, g^{(n)})_{\mathcal{H}^{\otimes n}} . \tag{17}$$

We consider the factor space

$$\mathcal{F}_n(\mathcal{H}) := \mathcal{H}^{\otimes n} / \big\{ f^{(n)} \in \mathcal{H}^{\otimes n} : (f^{(n)}, f^{(n)})_{\mathcal{F}_n(\mathcal{H})} = 0 \big\},$$

and define a scalar product on $\mathcal{F}_n(\mathcal{H})$ by (17).

Below, for a bounded linear operator L in a Hilbert space $\mathfrak{H}$, we denote by $\mathrm{Ker}(L)$ and $\mathrm{Ran}(L)$ the kernel of L and the range of L, respectively. Recall that $\mathrm{Ker}(L)$ is a closed linear subspace of $\mathfrak{H}$ and, if L is self-adjoint,

$$\mathfrak{H} = \mathrm{Ker}(L) \oplus \overline{\mathrm{Ran}(L)},$$

where $\overline{\mathrm{Ran}(L)}$ denotes the closure of the linear subspace $\mathrm{Ran}(L)$. The following lemma only uses the fact that $\mathcal{P}_n \geq 0$.

Lemma 5. *(i) We have*

$$\big\{ f^{(n)} \in \mathcal{H}^{\otimes n} : (f^{(n)}, f^{(n)})_{\mathcal{F}_n(\mathcal{H})} = 0 \big\} = \mathrm{Ker}(\mathcal{P}_n).$$

(ii) For each $f^{(n)} \in \overline{\mathrm{Ran}(\mathcal{P}_n)}$, $f^{(n)} \neq 0$,

$$(f^{(n)}, f^{(n)})_{\mathcal{F}_n(\mathcal{H})} > 0.$$

By Lemma 5, we can identify $\mathcal{F}_n(\mathcal{H})$ with the set $\overline{\mathrm{Ran}(\mathcal{P}_n)}$ equipped with scalar product (17).

The result below follows from Theorem 2 and Remark 4 in [20].

Theorem 6 ([20])**.** *We have*

$$\mathrm{Ker}(\mathcal{P}_n) = \overline{\sum_{k=1}^{n-1} \mathrm{Ker}(\mathbf{1} + \Psi_k)}, \tag{18}$$

i.e., the kernel of $\mathcal{P}_n$ is equal to the closure of the linear span of the subspaces $\mathrm{Ker}(\mathbf{1} + \Psi_k)$, $k = 1, \dots, n-1$.

We will now give an explicit description of the space $\mathcal{F}_n(\mathcal{H}) = \overline{\mathrm{Ran}(\mathcal{P}_n)}$. We denote

$$\Theta := \big\{ (s,t) \in T^{(2)} : |Q(s,t)| = 1 \big\}, \quad \Theta' := T^{(2)} \setminus \Theta = \big\{ (s,t) \in T^{(2)} : |Q(s,t)| < 1 \big\}. \tag{19}$$

Theorem 7. *The space $\mathcal{F}_n(\mathcal{H}) = \overline{\mathrm{Ran}(\mathcal{P}_n)}$ is equal (as a set) to the subspace of $\mathcal{H}^{\otimes n}$ consisting of all $f^{(n)} \in \mathcal{H}^{\otimes n}$ that are Q-quasisymmetric, i.e., formula* (7) *holds for each $k \in \{1, \dots, n-1\}$ and for $\sigma^{\otimes n}$-a.a. $(t_1, \dots, t_n) \in T^{(n)}$ such that $|Q(t_k, t_{k+1})| = 1$, i.e., for $\sigma^{\otimes n}$-a.a. $(t_1, \dots, t_n) \in T_k^{(n)}$, where*

$$T_k^{(n)} := \big\{ (t_1, \dots, t_n) \in T^{(n)} : (t_k, t_{k+1}) \in \Theta \big\} . \tag{20}$$

2.2 Orthogonal projection onto $\overline{\mathrm{Ran}(\mathcal{P}_n)}$.

We will now describe the orthogonal projection $\mathbb{P}_n$ of $\mathcal{H}^{\otimes n}$ onto $\overline{\mathrm{Ran}(\mathcal{P}_n)} = \mathcal{F}_n(\mathcal{H})$. For this purpose, we define a function

$$R(s,t) := \begin{cases} Q(s,t), & \text{if } (s,t) \in \Theta, \\ 0, & \text{if } (s,t) \in \Theta'. \end{cases}$$

Observe that

$$|R(s,t)| = \begin{cases} 1, & \text{if } (s,t) \in \Theta, \\ 0, & \text{if } (s,t) \in \Theta', \end{cases}$$

and that the function R is Hermitian. Hence, for each $\pi \in S_n$, similarly to the operator $\Psi_\pi : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n}$ defined in subsec. 2.1 for the function $Q(s,t)$, we may define an operator $\Phi_\pi : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n}$ for the function $R(s,t)$. By Proposition 3, we get

$$(\Phi_\pi f^{(n)})(\mathbf{t}^{(n)}) = R_{\pi^{-1}}(\mathbf{t}^{(n)}) f^{(n)}(\mathbf{t}^{(n)}_\pi), \tag{21}$$

where

$$R_\pi(\mathbf{t}^{(n)}) := \prod_{\substack{1 \le i < j \le n \\ \pi(i) > \pi(j)}} R(t_i, t_j), \quad \mathbf{t}^{(n)} \in T^{(n)}. \tag{22}$$

Let $\pi \in S_n$ and let $\mathbf{t}^{(n)} \in T^{(n)}$ be such that, for some $1 \le i < j \le n$, we have $\pi(i) > \pi(j)$ and $(t_i, t_j) \in \Theta'$. Then, it follows from (22) that $R_\pi(\mathbf{t}^{(n)}) = 0$. Otherwise, i.e., if such i and j do not exist, we get $|R_\pi(\mathbf{t}^{(n)})| = 1$.

Given $\mathbf{t}^{(n)} \in T^{(n)}$, we define a splitting

$$S_n = S_n^1(\mathbf{t}^{(n)}) \sqcup S_n^0(\mathbf{t}^{(n)})$$

of the set S_n into two disjoint subsets:

$$\begin{aligned} S_n^1(\mathbf{t}^{(n)}) &: = \{\pi \in S_n : |R_{\pi^{-1}}(\mathbf{t}^{(n)})| = 1\}, \\ S_n^0(\mathbf{t}^{(n)}) &: = \{\pi \in S_n : |R_{\pi^{-1}}(\mathbf{t}^{(n)})| = 0\}. \end{aligned} \tag{23}$$

Let $c_n(\mathbf{t}^{(n)}) := |S_n^1(\mathbf{t}^{(n)})|$ denote the cardinality. We define an operator $\mathbb{P}_n : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n}$ by setting, for each $f^{(n)} \in \mathcal{H}^{\otimes n}$,

$$\begin{aligned} (\mathbb{P}_n f^{(n)})(\mathbf{t}^{(n)}) &: = \frac{1}{c_n(\mathbf{t}^{(n)})} \sum_{\pi \in S_n^1(\mathbf{t}^{(n)})} (\Phi_\pi f^{(n)})(\mathbf{t}^{(n)}) \\ &= \frac{1}{c_n(\mathbf{t}^{(n)})} \sum_{\pi \in S_n^1(\mathbf{t}^{(n)})} R_{\pi^{-1}}(\mathbf{t}^{(n)}) f^{(n)}(\mathbf{t}^{(n)}_\pi). \end{aligned} \tag{24}$$

Theorem 8. *For each $n \geq 2$, the operator $\mathbb{P}_n$ is the orthogonal projection of $\mathcal{H}^{\otimes n}$ onto $\overline{\mathrm{Ran}(\mathcal{P}_n)} = \mathcal{F}_n(\mathcal{H})$.*

The corollary below is a straightforward consequence of Theorem 8.

Corollary 9. *For each $n \geq 2$,*

$$\mathbb{P}_n \mathcal{P}_n = \mathcal{P}_n \mathbb{P}_n = \mathcal{P}_n.$$

We will also need the following result about the operators $\mathbb{P}_n$, which follows from Theorem 8 and its proof.

Corollary 10. *For each $n \geq 2$ and $k \in \{1, \dots, n-1\}$, we have*

$$\mathbb{P}_n = \mathbb{P}_n(\mathbb{P}_k \otimes \mathbb{P}_{n-k}). \tag{25}$$

Here we denote by $\mathbb{P}_1 := \mathbf{1}$ the identity operator in $\mathcal{H}$.

Remark 11. For $f^{(n)} \in \mathcal{F}_n(\mathcal{H})$ and $g^{(m)} \in \mathcal{F}_m(\mathcal{H})$, we may define a *$Q$-quasisymmetric tensor product* of $f^{(n)}$ and $g^{(m)}$ by

$$f^{(n)} \circledast g^{(m)} := \mathbb{P}_{n+m}(f^{(n)} \otimes g^{(m)}).$$

Then Corollary 10 implies that the Q-quasisymmetric tensor product $\circledast$ is associative.

2.3 Creation and annihilation operators and their Q-commutation relations

Recall that we have defined complex Hilbert spaces $\mathcal{F}_n(\mathcal{H})$ for $n \geq 2$. Let also $\mathcal{F}_1(\mathcal{H}) := \mathcal{H}$ and $\mathcal{F}_0(\mathcal{H}) := \mathbb{C}$. We define a Q-deformed Fock space to be the Hilbert space

$$\mathcal{F}(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \mathcal{F}_n(\mathcal{H})\, n!\,.$$

Thus, every $f \in \mathcal{F}(\mathcal{H})$ is represented as $f = (f^{(n)})_{n=0}^{\infty}$, where $f^{(n)} \in \mathcal{F}_n(\mathcal{H})$, and the norm of f is given by

$$\|f\|^2_{\mathcal{F}(\mathcal{H})} := \sum_{n=0}^{\infty} \|f^{(n)}\|^2_{\mathcal{F}_n(\mathcal{H})}\, n!\,.$$

The vector $\Omega := (1, 0, 0, \dots)$ is called the vacuum.

Let $\mathcal{F}_{\mathrm{fin}}(\mathcal{H}) \subset \mathcal{F}(\mathcal{H})$ be the subspace consisting of all finite sequences of the form $f = (f^{(0)}, f^{(1)}, \dots, f^{(k)}, 0, 0, \dots)$ for some $k \in \mathbb{N}$. The subspace $\mathcal{F}_{\mathrm{fin}}(\mathcal{H})$ is evidently dense in $\mathcal{F}(\mathcal{H})$.

For each $h \in \mathcal{H}$, we define a creation operator $a^+(h) : \mathcal{F}_{\mathrm{fin}}(\mathcal{H}) \to \mathcal{F}_{\mathrm{fin}}(\mathcal{H})$ by setting

$$a^+(h)\Omega := h, \qquad a^+(h)f^{(n)} := \mathbb{P}_{n+1}(h \otimes f^{(n)}), \quad f^{(n)} \in \mathcal{F}_n(\mathcal{H}),\ n \in \mathbb{N}. \tag{26}$$

The domain of the adjoint operator of $a^+(h)$ in $\mathcal{F}(\mathcal{H})$ contains $\mathcal{F}_{\mathrm{fin}}(\mathcal{H})$, and furthermore the annihilation operator $a^-(h) := (a^+(h))^* \restriction \mathcal{F}_{\mathrm{fin}}(\mathcal{H})$ also maps $\mathcal{F}_{\mathrm{fin}}(\mathcal{H})$ into itself.

The following proposition gives an explicit form of the action of the annihilation operator.

Proposition 12. *For each $h \in \mathcal{H}$, we have $a^-(h)\Omega = 0$, $a^-(h)g = (h, g)_{\mathcal{H}}$ for $g \in \mathcal{H}$, and*

$$\begin{aligned} &(a^-(h)f^{(n)})(t_1, \dots, t_{n-1}) \\ &\quad = \sum_{k=1}^{n} \mathbb{P}_{n-1}\left[\int_T \overline{h(s)} \left(\prod_{i=1}^{k-1} Q(s, t_i)\right) f^{(n)}(t_1, \dots, t_{k-1}, s, t_k, \dots, t_{n-1})\, \sigma(ds)\right] \end{aligned} \tag{27}$$

for any $f^{(n)} \in \mathcal{F}_n(\mathcal{H})$, $n \geq 2$. In formula (27), the operator $\mathbb{P}_{n-1}$ acts on the function of $t_1, \dots, t_{n-1}$ variables. Furthermore, for any $g^{(n)} \in \mathcal{H}^{\otimes n}$, $n \geq 2$,

$$\begin{aligned} &\big(a^-(h)\mathbb{P}_n g^{(n)}\big)(t_1, \dots, t_{n-1}) \\ &\quad = \sum_{k=1}^{n} \mathbb{P}_{n-1}\left[\int_T \overline{h(s)} \left(\prod_{i=1}^{k-1} Q(s, t_i)\right) g^{(n)}(t_1, \dots, t_{k-1}, s, t_k, \dots, t_{n-1})\, \sigma(ds)\right]. \end{aligned} \tag{28}$$

For $t \in T$, we now informally define creation and annihilation operators at point t, denoted by $\partial_t^\dagger$ and ∂_t, respectively. A rigorous meaning to these operators is given through smearing them with functions $h \in \mathcal{H}$:

$$a^+(h) = \int_T \sigma(dt)\, h(t)\partial_t^\dagger, \quad a^-(h) = \int_T \sigma(dt)\, \overline{h(t)}\, \partial_t. \tag{29}$$

So we have the following informal equalities:

$$\begin{aligned} \partial_t^\dagger f^{(n)} &= \mathbb{P}_{n+1}(\delta_t \otimes f^{(n)}), \\ \partial_t f^{(n)}(t_1, \dots, t_{n-1}) &= \sum_{k=1}^{n} \mathbb{P}_{n-1}\left[\left(\prod_{i=1}^{k-1} Q(t, t_i)\right) f^{(n)}(t_1, \dots, t_{k-1}, t, t_k, \dots, t_{n-1})\right], \end{aligned}$$

where δ_t denotes the delta function at t.

Using (26) and Corollary 10, we see that, for any $g, h \in \mathcal{H}$ and $f^{(n)} \in \mathcal{F}_n(\mathcal{H})$,

$$a^+(g)a^+(h)f^{(n)} := \mathbb{P}_{n+2}(g \otimes h \otimes f^{(n)}). \tag{30}$$

In view of (29) and (30), for each $\varphi^{(2)} \in \mathcal{H}^{\otimes 2}$, we can naturally define an operator

$$\int_{T^2} \sigma(ds)\, \sigma(dt)\, \varphi^{(2)}(s, t)\, \partial_s^\dagger \partial_t^\dagger : \mathcal{F}_{\mathrm{fin}}(\mathcal{H}) \to \mathcal{F}_{\mathrm{fin}}(\mathcal{H})$$

by setting

$$\int_{T^2}\sigma(ds)\,\sigma(dt)\,\varphi^{(2)}(s,t)\,\partial_s^\dagger\partial_t^\dagger f^{(n)} := \mathbb{P}_{n+2}(\varphi^{(2)}\otimes f^{(n)}) \tag{31}$$

for $f^{(n)}\in\mathcal{F}_n(\mathcal{H})$. In particular, choosing $\varphi^{(2)}=g\otimes h$ with $g,h\in\mathcal{H}$, we get

$$\int_{T^2}\sigma(ds)\,\sigma(dt)\,g(s)h(t)\,\partial_s^\dagger\partial_t^\dagger = a^+(g)a^+(h).$$

Remark 13. Note that we also accept the natural formula

$$\int_{T^2}\sigma(ds)\,\sigma(dt)\,\varphi^{(2)}(s,t)\,\partial_t^\dagger\partial_s^\dagger = \int_{T^2}\sigma(ds)\,\sigma(dt)\,\varphi^{(2)}(t,s)\,\partial_s^\dagger\partial_t^\dagger. \tag{32}$$

Similarly, using also Proposition 12, we may define, for each $\varphi^{(2)}\in\mathcal{H}^{\otimes 2}$, linear operators

$$\int_{T^2}\sigma(ds)\,\sigma(dt)\,\varphi^{(2)}(s,t)\,\partial_s\partial_t : \mathcal{F}_{\mathrm{fin}}(\mathcal{H})\to\mathcal{F}_{\mathrm{fin}}(\mathcal{H}),$$
$$\int_{T^2}\sigma(ds)\,\sigma(dt)\,\varphi^{(2)}(s,t)\,\partial_s^\dagger\partial_t : \mathcal{F}_{\mathrm{fin}}(\mathcal{H})\to\mathcal{F}_{\mathrm{fin}}(\mathcal{H}).$$

Note that

$$\begin{aligned}\left(\int_{T^2}\sigma(ds)\,\sigma(dt)\,\varphi^{(2)}(s,t)\,\partial_s^\dagger\partial_t^\dagger\right)^* &= \int_{T^2}\sigma(ds)\,\sigma(dt)\,\overline{\varphi^{(2)}(s,t)}\,\partial_t\partial_s\\ &= \int_{T^2}\sigma(ds)\,\sigma(dt)\,\overline{\varphi^{(2)}(t,s)}\,\partial_s\partial_t.\end{aligned} \tag{33}$$

Also, for any $g,h\in\mathcal{H}$, we denote

$$\int_{T^2}\sigma(ds)\,\sigma(dt)\,g(s)h(t)\partial_s\partial_t^\dagger := a^-(\overline{g})a^+(h).$$

We will now present the commutation relations for the creation and annihilation operators.

Theorem 14 (Q-CR)**.** *The operators $\partial_t^\dagger$, ∂_t ($t\in T$) satisfy the (informal) commutations relations* (1) *and* (8)*. Rigorously, this means the following: for any $g,h\in\mathcal{H}$,*

$$\int_{T^2}\sigma(ds)\,\sigma(dt)\,g(s)h(t)\partial_s\partial_t^\dagger = \int_T g(t)h(t)\,\sigma(dt) + \int_{T^2}\sigma(ds)\,\sigma(dt)\,g(s)h(t)Q(s,t)\partial_t^\dagger\partial_s, \tag{34}$$

and for any function $\varphi^{(2)}\in\mathcal{H}^{\otimes 2}$ that vanishes a.e. in Θ' (see (19)*),*

$$\int_{T^2}\sigma(ds)\,\sigma(dt)\,\varphi^{(2)}(s,t)\,\partial_s^\dagger\partial_t^\dagger = \int_{T^2}\sigma(ds)\,\sigma(dt)\,\varphi^{(2)}(s,t)Q(t,s)\,\partial_t^\dagger\partial_s^\dagger, \tag{35}$$

$$\int_{T^2}\sigma(ds)\,\sigma(dt)\,\varphi^{(2)}(s,t)\,\partial_s\partial_t = \int_{T^2}\sigma(ds)\,\sigma(dt)\,\varphi^{(2)}(s,t)Q(t,s)\,\partial_t\partial_s. \tag{36}$$

We finish this section with several remarks.

Remark 15. We can naturally identify the diagonal $\Delta := \{(s,t) \in T^2 \mid s = t\}$ with T. Denote by $\tilde{\sigma}$ the measure σ on Δ. We may consider $\tilde{\sigma}$ as a measure on T^2 which is equal to zero outside of Δ. Denote

$$\mathfrak{G} := L^2(T^2 \to \mathbb{C}, \sigma^{\otimes 2}) \cap L^1(T^2 \to \mathbb{C}, \tilde{\sigma}).$$

In view of (34), for each $\varphi^{(2)} \in \mathfrak{G}$, we may define an operator

$$\int_{T^2} \sigma(ds)\, \sigma(dt)\, \varphi^{(2)}(s,t)\, \partial_s \partial_t^\dagger : \mathcal{F}_{\mathrm{fin}}(\mathcal{H}) \to \mathcal{F}_{\mathrm{fin}}(\mathcal{H}),$$

which satisfies

$$\int_{T^2} \sigma(ds)\, \sigma(dt)\, \varphi^{(2)}(s,t) \partial_s \partial_t^\dagger = \int_T \varphi^{(2)}(t,t)\, \sigma(dt) + \int_{T^2} \sigma(ds)\, \sigma(dt)\, \varphi^{(2)}(s,t) Q(s,t) \partial_t^\dagger \partial_s.$$

Remark 16. Denote $B(\varphi) := a^+(\varphi) + a^-(\varphi)$. The family of operators $(B(\varphi))_{\varphi \in \mathcal{H}}$ can be thought of as a *noncommutative Brownian motion* (or a *noncommutative Gaussian white noise*). Let $\mathcal{P}$ denote the complex unital $*$-algebra generated by $(B(\varphi))_{\varphi \in \mathcal{H}}$, i.e., the algebra of noncommutative polynomials in the variables $B(\varphi)$. We define a vacuum state on $\mathcal{P}$ by $\tau(p) := (p\Omega, \Omega)_{\mathcal{F}(\mathcal{H})}$, $p \in \mathcal{P}$. By analogy with the proofs of Theorem 4.4 in [8] and Corollary 4.9 in [6], one can prove the following result: the state τ is tracial (i.e., it satisfies $\tau(p_1 p_2) = \tau(p_2 p_1)$ for all $p_1, p_2 \in \mathcal{P}$) if and only if the function Q is real-valued, i.e., $Q : T^{(2)} \to [-1, 1]$.

Remark 17. The results of this section hold, in particular, in the case where $\sigma^{\otimes 2}(\Theta') = 0$, i.e., when $|Q(s,t)| < 1$ for $\sigma^{\otimes 2}$-a.a. $(s,t) \in T^2$. Then, for each $n \geq 2$, the equality $\mathcal{F}_n(\mathcal{H}) = \mathcal{H}^{\otimes n}$ holds (in the sense of sets). Evidently, there are no commutation relations (35), (36) in this case. Note also that, if $|Q(s,t)| \leq r < 1$ for some number $0 < r < 1$, then the creation operators $a^+(h)$ and the annihilation operators $a^-(h)$ ($h \in \mathcal{H}$) are bounded in $\mathcal{F}(\mathcal{H})$, see Theorem 3.1, (ii) in [8].

2.4 Discrete setting: the anyonic exclusion principle

We will now make several observations about the discrete setting. We may choose T to be a finite or countable set and σ to be the counting measure on T, i.e., $\sigma(\{t\}) = 1$ for each $t \in T$. Hence, the space $\mathcal{H}$ becomes the complex ℓ^2-space over T, i.e., $\mathcal{H} = \ell^2(T \to \mathbb{C})$. We obviously have $T^{(2)} = T^2$, so that the function $Q(s,t)$ is defined for all $(s,t) \in T^2$. Thus, we have, in particular, constructed Fock representations of the discrete commutation relations (4) with additional commutation relations between $\partial_s^\dagger$, $\partial_t^\dagger$ and between ∂_s, ∂_t for those pairs $(s,t) \in T^2$ for which $|Q(s,t)| = 1$. (Note that, in this case, the operators $\partial_t^\dagger$, ∂_t have a rigorous meaning.)

Since the function Q is Hermitian, we have $Q(t,t) \in \mathbb{R}$ for each $t \in T$. Hence, $|Q(t,t)| = 1$ if and only if either $Q(t,t) = 1$ or $Q(t,t) = -1$. In the first case, we just get the tautological commutation relation $(\partial_t^\dagger)^2 = (\partial_t^\dagger)^2$. In the second case, we get $(\partial_t^\dagger)^2 = -(\partial_t^\dagger)^2$, so that $(\partial_t^\dagger)^2 = \partial_t^2 = 0$. If the latter formulas hold for all $t \in T$, then we may call the corresponding commutation relations the *discrete Q-CR of fermion type*.

For the discrete Q-CR of fermion type, the operators $\partial_t^\dagger$, ∂_t become bounded in $\mathcal{F}(\mathcal{H})$ and have norm equal to 1, see [8], Corollary 3.2 and Remark after it. Hence, for each $h \in \ell^1(T \to \mathbb{C})$,

$$\|a^+(h)\| = \|a^-(h)\| \le \|h\|_{\ell^1(T\to\mathbb{C})}.$$

Let us now assume that $T \subset \mathbb{N}$ and fix $q \in \mathbb{C}$, $|q| = 1$. We consider the function

$$Q(s,t) := \begin{cases} q, & \text{if } s > t \\ \bar{q}, & \text{if } s < t \\ -1, & \text{if } s = t, \end{cases}$$

The corresponding Q-CR describe a *discrete anyon system of fermion type.* Note that $|Q(s,t)| = 1$ for all $(s,t) \in T^2$, hence $\mathcal{P}_n = \mathbb{P}_n$ is the projection of $\mathcal{H}^{\otimes n}$ onto $\mathcal{F}_n(\mathcal{H})$.

Theorem 18 (Anyonic exclusion principle)**.** *Consider a discrete anyon system of fermion type. Let $m \in \mathbb{N}$, $m \ge 2$. Assume that the parameter $q \in \mathbb{C}$, $q \ne 1$, is an mth root of unity, i.e., $q^m = 1$. Then, for any $h \in \mathcal{H}$, we have*

$$a^+(h)^m = a^-(h)^m = 0. \tag{37}$$

3 Proofs

In this section we collect the proofs of the results from Section 2.

Proof of Proposition 3. We start with the following crucial lemma.

Lemma 19. *Let $\rho = \pi_l \eta$ be a reduced representation of a permutation $\rho \in S_n$. Then*

$$Q_\rho(t_1,\dots,t_n) = Q(t_{\eta^{-1}(l)}, t_{\eta^{-1}(l+1)}) Q_\eta(t_1,\dots,t_n), \quad (t_1,\dots,t_n) \in T^{(n)}. \tag{38}$$

Proof. Let

$$L_\rho := Q_\rho(t_1,\dots,t_n) = \prod_{\substack{1\le i<j\le n\\ \rho(i)>\rho(j)}} Q(t_i,t_j), \quad L_\eta := Q_\eta(t_1,\dots,t_n) = \prod_{\substack{1\le i<j\le n\\ \eta(i)>\eta(j)}} Q(t_i,t_j).$$

Let $1 \le u < v \le n$. We consider the following cases.

- If $\eta(u), \eta(v) \notin \{l, l+1\}$, then both $\eta(u), \eta(v)$ are fixed points for π_l. Consequently, $\rho(u) = \eta(u)$ and $\rho(v) = \eta(v)$, so that $\rho(u) > \rho(v)$ if and only if $\eta(u) > \eta(v)$. Hence, the term $Q(t_u, t_v)$ appears in L_ρ if and only if it appears in L_η.

- If $\eta(u) \in \{l, l+1\}$ and $\eta(v) \notin \{l, l+1\}$, then $\rho(v) = v \notin \{l, l+1\}$ and, since $\rho(u) = (\pi_l \eta)(u) \in \{l, l+1\}$, the order between $\eta(u)$ and $\eta(v)$ is the same as between $\rho(u)$ and $\rho(v)$. Thus, the term $Q(t_u, t_v)$ appears in L_ρ if and only if it appears in L_η.

- The case $\eta(u) \notin \{l, l+1\}$ and $\eta(v) \in \{l, l+1\}$ is analogous to the previous one.

- Consider the case $\eta(u) = l$ and $\eta(v) = l+1$. Then the term $Q(t_u, t_v)$ does not appear in L_η. Further, $\rho(u) = (\pi_l \eta)(u) = \pi_l(l) = l+1$ and $\rho(v) = (\pi_l \eta)(v) = \pi_l(l+1) = l$, so that $\rho(u) > \rho(v)$. Hence, the term $Q(t_u, t_v)$ appears in L_ρ. But we also have $Q(t_{\eta^{-1}(l)}, t_{\eta^{-1}(l+1)}) = Q(t_u, t_v)$ on the right hand side of equality (38).

- Finally, consider the case $\eta(u) = l+1$ and $\eta(v) = l$. But then $\rho(u) = (\pi_l \eta)(u) = l$ and $\rho(v) = (\pi_l \eta)(v) = l+1$. Thus, η changes the order of the pair (u, v), while ρ does not. Therefore, η has more inversions than ρ: $|\eta| > |\rho|$. But this contradicts the assumption that ρ is in the reduced form. Thus, this case is impossible.

□

We will now prove the proposition by induction on the length of a permutation π. If $|\pi| = 1$, then $\pi = \pi_k$ for some $k \in \{1, \dots, n-1\}$. In this case, the statement trivially follows from the definition of Ψ_k, see (10). Assume that the statement holds for each permutation of length m. Let π be a a permutation of length $m+1$, and let $\pi = \varphi \pi_l$ be a reduced representation of π. Hence, the length of the permutation φ is m. Denote $\eta := \varphi^{-1}$ and $\rho := \pi^{-1}$, so that $\rho = \pi_l \eta$. Then, for each $f^{(n)} \in \mathcal{H}^{\otimes n}$, by using the induction's assumption and Lemma 19, we get

$$\begin{aligned}
&(\Psi_\pi f^{(n)})(t_1, \dots, t_n) = (\Psi_\varphi \Psi_l f^{(n)})(t_1, \dots, t_n) \\
&\quad = Q_\eta(t_1, \dots, t_n)(\Psi_l f^{(n)})(t_{\varphi(1)}, \dots, t_{\varphi(n)}) \\
&\quad = Q_\eta(t_1, \dots, t_n) Q(t_{\varphi(l)}, t_{\varphi(l+1)}) f^{(n)}(t_{\varphi(1)}, \dots, t_{\varphi(l+1)}, t_{\varphi(l)}, \dots, t_{\varphi(n)}) \\
&\quad = Q_\rho(t_1, \dots, t_n) f^{(n)}(t_{\pi(1)}, \dots, t_{\pi(n)}).
\end{aligned}$$

□

Proof of Lemma 5. (i) Since $\mathcal{P}_n$ is self-adjoint and $\mathcal{P}_n \geq 0$, we can write $\mathcal{P}_n = (\sqrt{\mathcal{P}_n})^2$. Let $f^{(n)} \in \mathcal{H}^{\otimes n}$ be such that

$$0 = (f^{(n)}, f^{(n)})_{\mathcal{F}_n(\mathcal{H})} = \|\sqrt{\mathcal{P}_n} f^{(n)}\|^2_{\mathcal{H}^{\otimes n}}.$$

Hence, $f^{(n)} \in \operatorname{Ker}(\sqrt{\mathcal{P}_n})$. But $\operatorname{Ker}\sqrt{\mathcal{P}_n} \subset \operatorname{Ker}\mathcal{P}_n$, which implies

$$\left\{f^{(n)} \in \mathcal{H}^{\otimes n} \mid (f^{(n)}, f^{(n)})_{\mathcal{F}_n(\mathcal{H})} = 0\right\} \subset \operatorname{Ker}(\mathcal{P}_n).$$

The inverse inclusion trivially follows from (17).

(ii) Let $f^{(n)} \in \overline{\operatorname{Ran}(\mathcal{P}_n)}$ be such that $(f^{(n)}, f^{(n)})_{\mathcal{F}_n(\mathcal{H})} = 0$. By part (i), $f^{(n)} \in \operatorname{Ker}(\mathcal{P}_n)$. But $\overline{\operatorname{Ran}(\mathcal{P}_n)} \perp \operatorname{Ker}(\mathcal{P}_n)$. Hence, $\overline{\operatorname{Ran}(\mathcal{P}_n)} \cap \operatorname{Ker}(\mathcal{P}_n) = \{0\}$, and so $f^{(n)} = 0$. $\square$

Proof of Theorem 7. Using (18), we have

$$\overline{\operatorname{Ran}(\mathcal{P}_n)} = \left(\sum_{k=1}^{n-1} \operatorname{Ker}(\mathbf{1} + \Psi_k)\right)^{\perp} = \bigcap_{k=1}^{n-1} \operatorname{Ker}(\mathbf{1} + \Psi_k)^{\perp} = \bigcap_{k=1}^{n-1} \overline{\operatorname{Ran}(\mathbf{1} + \Psi_k)}. \tag{39}$$

For $l \in \mathbb{N}$ and $k \in \{1, \dots, n-1\}$, we denote

$$T_{k,l}^{(n)} := \left\{(t_1, \dots, t_n) \in T^{(n)} : \frac{l-1}{l} \le |Q(t_k, t_{k+1})| < \frac{l}{l+1}\right\}$$

and recall the definition of $T_k^{(n)}$, see (20). Then, for each $k \in \{1, \dots, n-1\}$, we have the orthogonal decomposition

$$\mathcal{H}^{\otimes n} = \left(\bigoplus_{l=1}^{\infty} L^2(T_{k,l}^{(n)} \to \mathbb{C}, \sigma^{\otimes n})\right) \oplus L^2(T_k^{(n)} \to \mathbb{C}, \sigma^{\otimes n}). \tag{40}$$

Each of the spaces on the right-hand side of (40) is invariant for the operator $\mathbf{1} + \Psi_k$. On each space $L^2(T_{k,l}^{(n)} \to \mathbb{C}, \sigma^{\otimes n})$, the norm of the operator Ψ_k is bounded by $\frac{l}{l+1} < 1$. Hence, the operator $\mathbf{1} + \Psi_k$ is invertible in this space. Therefore the kernel of the operator $\mathbf{1} + \Psi_k$ restricted to $L^2(T_{k,l}^{(n)} \to \mathbb{C}, \sigma^{\otimes n})$ is trivial:

$$\operatorname{Ker}(\mathbf{1} + \Psi_k) \cap L^2(T_{k,l}^{(n)} \to \mathbb{C}, \sigma^{\otimes n}) = \{0\} \quad \text{for each } l \in \mathbb{N}.$$

Let $f^{(n)} \in L^2(T_k^{(n)} \to \mathbb{C}, \sigma^{\otimes n})$. Consider the decomposition $f^{(n)} = f_{k,+}^{(n)} + f_{k,-}^{(n)}$ with

$$f_{k,\pm}^{(n)}(t_1, \dots, t_n) := \frac{1}{2}\big[f^{(n)}(t_1, \dots, t_n) \pm Q(t_k, t_{k+1}) f^{(n)}(t_1, \dots, t_{k+1}, t_k, \dots, t_n)\big].$$

One can easily see that $f_{k,+}^{(n)}$ and $f_{k,-}^{(n)}$ are orthogonal and $f_{k,+}^{(n)} \in \operatorname{Ran}(\mathbf{1} + \Psi_k)$. Hence $f_{k,-}^{(n)} \in \operatorname{Ker}(\mathbf{1} + \Psi_k)$. Therefore, the orthogonal projection of $L^2(T_k^{(n)} \to \mathbb{C}, \sigma^{\otimes n})$ onto $\operatorname{Ker}(\mathbf{1} + \Psi_k)$, denoted by $D_k^{(n)}$, is given by

$$(D_k^{(n)} f^{(n)})(t_1, \dots, t_n) = \frac{1}{2}\big[f^{(n)}(t_1, \dots, t_n) - Q(t_k, t_{k+1}) f^{(n)}(t_1, \dots, t_{k+1}, t_k, \dots, t_n)\big].$$

Hence, the orthogonal projection of $\mathcal{H}^{\otimes n}$ onto $\operatorname{Ker}(\mathbf{1}+\Psi_k)$, denoted by $E_k^{(n)}$, is given by

$$\begin{aligned}&(E_k^{(n)} f^{(n)})(t_1,\dots,t_n)\\&\quad=\frac{1}{2}\chi_{T_k^{(n)}}(t_1,\dots,t_n)\big[f^{(n)}(t_1,\dots,t_n)-Q(t_k,t_{k+1})f^{(n)}(t_1,\dots,t_{k+1},t_k,\dots,t_n)\big],\end{aligned}$$

where χ_A denotes the indicator function of a set A. Therefore, the orthogonal projection of $\mathcal{H}^{\otimes n}$ onto $\operatorname{Ker}(\mathbf{1}+\Psi_k)^\perp=\overline{\operatorname{Ran}(\mathbf{1}+\Psi_k)}$, denoted by $F_k^{(n)}$, is given by

$$\begin{aligned}&(F_k^{(n)} f^{(n)})(t_1,\dots,t_n)=\chi_{T^{(n)}\setminus T_k^{(n)}}(t_1,\dots,t_n)f^{(n)}(t_1,\dots,t_n)\\&\quad+\frac{1}{2}\chi_{T_k^{(n)}}(t_1,\dots,t_n)\big[f^{(n)}(t_1,\dots,t_n)+Q(t_k,t_{k+1})f^{(n)}(t_1,\dots,t_{k+1},t_k,\dots,t_n)\big].\end{aligned}$$

Thus, the set $\overline{\operatorname{Ran}(\mathbf{1}+\Psi_k)}$ consists of all functions from $\mathcal{H}^{\otimes n}$ that are Q-quasi-symmetric in the t_k,t_{k+1}-variables on the set $T_k^{(n)}$, i.e., for $\sigma^{\otimes n}$-a.a $(t_1,\dots,t_n)\in T_k^{(n)}$, equality (7) holds. From here and formula (39), the theorem follows.

□

Proof of Theorem 8. We start with the following lemma.

Lemma 20. *(i) Let $\mathbf{t}^{(n)}\in T^{(n)}$. Then $\pi\in S_n^1(\mathbf{t}^{(n)})$ if and only if $\pi^{-1}\in S_n^1(\mathbf{t}_\pi^{(n)})$.*

(ii) Let $\mathbf{t}^{(n)}\in T^{(n)}$, let $\pi\in S_n^1(\mathbf{t}^{(n)})$, and let $\nu\in S_n^1(\mathbf{t}_\pi^{(n)})$. Then $\varphi:=\pi\nu\in S_n^1(\mathbf{t}^{(n)})$.

(iii) For each $\mathbf{t}^{(n)}\in T^{(n)}$ and $\pi\in S_n^1(\mathbf{t}^{(n)})$, we have $c_n(\mathbf{t}^{(n)})=c_n(\mathbf{t}_\pi^{(n)})$.

Proof. (i) By (22),

$$R_{\pi^{-1}}(\mathbf{t}^{(n)})=\overline{R_\pi(\mathbf{t}_\pi^{(n)})}. \tag{41}$$

From here the statement follows.

(ii) Assume that $\varphi\notin S_n^1(\mathbf{t}^{(n)})$. Then there exist $i<j$ such that $\varphi^{-1}(i)>\varphi^{-1}(j)$ and $R(t_i,t_j)=0$. Let us consider two cases.

Case 1: $\pi^{-1}(i)>\pi^{-1}(j)$. But then (22) implies that $R_{\pi^{-1}}(\mathbf{t}^{(n)})=0$, hence $\pi\notin S_n^1(\mathbf{t}^{(n)})$, which is a contradiction.

Case 2: $\pi^{-1}(i)<\pi^{-1}(j)$. We then have

$$\nu^{-1}(\pi^{-1}(i))=\varphi^{-1}(i)>\varphi^{-1}(j)=\nu^{-1}(\pi^{-1}(j)).$$

By (22),

$$R_{\nu^{-1}}(\mathbf{t}_\pi^{(n)}):=\prod_{\substack{1\le a<b\le n\\ \nu^{-1}(a)>\nu^{-1}(b)}} R(t_{\pi(a)},t_{\pi(b)}).$$

Choose $a=\pi^{-1}(i)$ and $b=\pi^{-1}(j)$. Then $a<b$, $\nu^{-1}(a)>\nu^{-1}(b)$, and

$$R(t_{\pi(a)},t_{\pi(b)})=R(t_i,t_j)=0.$$

Therefore, $R_{\nu^{-1}}(\mathbf{t}_\pi^{(n)})=0$, which implies $\nu\notin S_n^1(\mathbf{t}_\pi^{(n)})$. This is again a contradiction. Thus, we must have $\varphi\in S_n^1(\mathbf{t}^{(n)})$.

(iii) By part (ii), if $\nu\in S_n^1(\mathbf{t}_\pi^{(n)})$, then $\pi\nu\in S_n^1(\mathbf{t}^{(n)})$. Hence, $c_n(\mathbf{t}_\pi^{(n)})\le c_n(\mathbf{t}^{(n)})$. On the other hand, by part (i), $\pi^{-1}\in S_n^1(\mathbf{t}_\pi^{(n)})$. Hence, by part (i), if $\mu\in S_n^1(\mathbf{t}^{(n)})$ then $\pi^{-1}\mu\in S_n^1(\mathbf{t}_\pi^{(n)})$. Hence, $c_n(\mathbf{t}^{(n)})\le c_n(\mathbf{t}_\pi^{(n)})$. □

We first show that the operator $\mathbb{P}_n$ is self-adjoint. By (21)–(24), we can write the operator $\mathbb{P}_n$ in the form

$$(\mathbb{P}_n f^{(n)})(\mathbf{t}^{(n)})=\frac{1}{c_n(\mathbf{t}^{(n)})}\sum_{\pi\in S_n}R_{\pi^{-1}}(\mathbf{t}^{(n)})f^{(n)}(\mathbf{t}_\pi^{(n)}).$$

Hence using Lemma 20, (iii) and (41), we get, for any $f^{(n)},g^{(n)}\in\mathcal{H}^{\otimes n}$,

$$\begin{aligned}
(\mathbb{P}_n f^{(n)},g^{(n)})_{\mathcal{H}^{\otimes n}}&=\sum_{\pi\in S_n}\int_{T^{(n)}}\frac{1}{c_n(\mathbf{t}^{(n)})}\overline{R_{\pi^{-1}}(\mathbf{t}^{(n)})f^{(n)}(\mathbf{t}_\pi^{(n)})}\,g^{(n)}(\mathbf{t}^{(n)})\,\sigma^{\otimes n}(d\mathbf{t}^{(n)})\\
&=\sum_{\pi\in S_n}\int_{T^{(n)}}\frac{1}{c_n(\mathbf{t}_{\pi^{-1}}^{(n)})}\overline{R_{\pi^{-1}}(\mathbf{t}_{\pi^{-1}}^{(n)})f^{(n)}(\mathbf{t}^{(n)})}\,g^{(n)}(\mathbf{t}_{\pi^{-1}}^{(n)})\,\sigma^{\otimes n}(d\mathbf{t}^{(n)})\\
&=\sum_{\pi\in S_n}\int_{T^{(n)}}\frac{1}{c_n(\mathbf{t}_{\pi}^{(n)})}\overline{R_{\pi}(\mathbf{t}_{\pi}^{(n)})f^{(n)}(\mathbf{t}^{(n)})}\,g^{(n)}(\mathbf{t}_{\pi}^{(n)})\,\sigma^{\otimes n}(d\mathbf{t}^{(n)})\\
&=\sum_{\pi\in S_n}\int_{T^{(n)}}\frac{1}{c_n(\mathbf{t}_{\pi}^{(n)})}R_{\pi^{-1}}(\mathbf{t}^{(n)})\,\overline{f^{(n)}(\mathbf{t}^{(n)})}\,g^{(n)}(\mathbf{t}_{\pi}^{(n)})\,\sigma^{\otimes n}(d\mathbf{t}^{(n)})\\
&=\int_{T^{(n)}}\overline{f^{(n)}(\mathbf{t}^{(n)})}\sum_{\pi\in S_n^1(\mathbf{t}^{(n)})}\frac{1}{c_n(\mathbf{t}_{\pi}^{(n)})}R_{\pi^{-1}}(\mathbf{t}^{(n)})\,g^{(n)}(\mathbf{t}_{\pi}^{(n)})\,\sigma^{\otimes n}(d\mathbf{t}^{(n)})\\
&=\int_{T^{(n)}}\overline{f^{(n)}(\mathbf{t}^{(n)})}\,\frac{1}{c_n(\mathbf{t}^{(n)})}\sum_{\pi\in S_n^1(\mathbf{t}^{(n)})}R_{\pi^{-1}}(\mathbf{t}^{(n)})\,g^{(n)}(\mathbf{t}_{\pi}^{(n)})\,\sigma^{\otimes n}(d\mathbf{t}^{(n)})\\
&=(f^{(n)},\mathbb{P}_n g^{(n)})_{\mathcal{H}^{\otimes n}}.
\end{aligned}\tag{42}$$

Thus, $\mathbb{P}_n^*=\mathbb{P}_n$.

Our next aim is to prove that $\mathbb{P}_n^2=\mathbb{P}$, which will imply that $\mathbb{P}_n$ is an orthogonal projection in $\mathcal{H}^{\otimes n}$. For $f^{(n)}\in\mathcal{H}^{\otimes n}$, we have, by Lemma 20, (ii) and (iii),

$$\begin{aligned}
(\mathbb{P}_n^2 f^{(n)})(\mathbf{t}^{(n)})&=\frac{1}{c_n(\mathbf{t}^{(n)})}\sum_{\pi\in S_n^1(\mathbf{t}^{(n)})}\frac{1}{c_n(\mathbf{t}_\pi^{(n)})}\sum_{\nu\in S_n^1(\mathbf{t}_\pi^{(n)})}(\Phi_\pi\Phi_\nu f^{(n)})(\mathbf{t}^{(n)})\\
&=\frac{1}{c_n(\mathbf{t}^{(n)})^2}\sum_{\pi\in S_n^1(\mathbf{t}^{(n)})}\sum_{\nu\in S_n^1(\mathbf{t}_\pi^{(n)})}(\Phi_\pi\Phi_\nu f^{(n)})(\mathbf{t}^{(n)})\\
&=\frac{1}{c_n(\mathbf{t}^{(n)})^2}\sum_{\varphi\in S_n^1(\mathbf{t}^{(n)})}\sum_{\substack{\pi\in S_n^1(\mathbf{t}^{(n)}),\,\nu\in S_n^1(\mathbf{t}_\pi^{(n)})\\ \pi\nu=\varphi}}(\Phi_\pi\Phi_\nu f^{(n)})(\mathbf{t}^{(n)}).
\end{aligned}\tag{43}$$

Let $\varphi \in S_n^1(\mathbf{t}^{(n)})$ and $\pi \in S_n^1(\mathbf{t}^{(n)})$. By Lemma 20, (i), we have $\pi^{-1} \in S_n^1(\mathbf{t}_\pi^{(n)})$. Hence, by Lemma 20, (ii), we get $\nu := \pi^{-1}\varphi \in S_n^1(\mathbf{t}_\pi^{(n)})$. From here and (43) we get:

$$(\mathbb{P}_n^2 f^{(n)})(\mathbf{t}^{(n)}) = \frac{1}{c_n(\mathbf{t}^{(n)})^2} \sum_{\varphi \in S_n^1(\mathbf{t}^{(n)})} \sum_{\pi \in S_n^1(\mathbf{t}^{(n)})} (\Phi_\pi \Phi_{\pi^{-1}\varphi} f^{(n)})(\mathbf{t}^{(n)}). \tag{44}$$

Lemma 21. *Let $\mathbf{t}^{(n)} \in T^{(n)}$, and $\pi \in S_n^1(\mathbf{t}^{(n)})$, and $\nu \in S_n^1(\mathbf{t}_\pi^{(n)})$. Then, for each $f^{(n)} \in \mathcal{H}^{\otimes n}$,*

$$(\Phi_\pi \Phi_\nu f^{(n)})(\mathbf{t}^{(n)}) = (\Phi_{\pi\nu} f^{(n)})(\mathbf{t}^{(n)}) \tag{45}$$

Proof. We first note that equality (45) explicitly means that

$$R_{\pi^{-1}}(\mathbf{t}^{(n)}) R_{\nu^{-1}}(\mathbf{t}_\pi^{(n)}) f^{(n)}(\mathbf{t}_{\pi\nu}^{(n)}) = R_{\nu^{-1}\pi^{-1}}(\mathbf{t}^{(n)}) f^{(n)}(\mathbf{t}_{\pi\nu}^{(n)}),$$

which is equivalent to the equality

$$R_{\pi^{-1}}(\mathbf{t}^{(n)}) R_{\nu^{-1}}(\mathbf{t}_\pi^{(n)}) = R_{\nu^{-1}\pi^{-1}}(\mathbf{t}^{(n)}). \tag{46}$$

Since $\pi \in S_n^1(\mathbf{t}^{(n)})$, $\nu \in S_n^1(\mathbf{t}_\pi^{(n)})$, and $\pi\nu \in S_n^1(\mathbf{t}^{(n)})$, we have

$$|R_{\pi^{-1}}(\mathbf{t}^{(n)})| = 1, \quad |R_{\nu^{-1}}(\mathbf{t}_\pi^{(n)})| = 1, \quad |R_{\nu^{-1}\pi^{-1}}(\mathbf{t}^{(n)})| = 1. \tag{47}$$

We define a Hermitian function $G : T^{(2)} \to \mathbb{C}$ by

$$G(s,t) := \begin{cases} R(s,t), & \text{if } |R(s,t)| = 1, \\ 1, & \text{if } R(s,t) = 0. \end{cases} \tag{48}$$

For each $\pi \in S_n$, similarly to the operator Ψ_π defined for the function Q and to the operator Φ_π defined for the function R, we define an operator $\Gamma_\pi : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n}$ for the function G. Thus,

$$(\Gamma_\pi f^{(n)})(\mathbf{t}^{(n)}) = G_{\pi^{-1}}(\mathbf{t}^{(n)}) f^{(n)}(\mathbf{t}_\pi^{(n)}),$$

where

$$G_\pi(\mathbf{t}^{(n)}) := \prod_{\substack{1 \le i < j \le n \\ \pi(i) > \pi(j)}} G(t_i, t_j), \quad \mathbf{t}^{(n)} \in T^{(n)}. \tag{49}$$

For an adjacent transposition $\pi_j = (j, j+1)$, we denote $\Gamma_j := \Gamma_{\pi_j}$. By Lemma 2, the operators Γ_j satisfy the braid relations. Furthermore, since $|G(s,t)| = 1$ for all $(s,t) \in T^{(2)}$, we get $\Gamma_j^2 = \mathbf{1}$. Using e.g. [9], we therefore conclude that the operators Γ_π with $\pi \in S_n$ form a unitary representation of S_n, i.e., for any $\pi, \nu \in S_n$, it holds that $\Gamma_\pi \Gamma_\nu = \Gamma_{\pi\nu}$, and in fact, for all $\mathbf{t}^{(n)} \in T^{(n)}$,

$$G_{\pi^{-1}}(\mathbf{t}^{(n)}) G_{\nu^{-1}}(\mathbf{t}_\pi^{(n)}) = G_{\nu^{-1}\pi^{-1}}(\mathbf{t}^{(n)}). \tag{50}$$

But if $\mathbf{t}^{(n)} \in T^{(n)}$, $\pi \in S_n^1(\mathbf{t}^{(n)})$, and $\nu \in S_n^1(\mathbf{t}_\pi^{(n)})$, then formulas (47)–(50) imply (46). □

Now formula (44) and Lemma 21 yield the equality

$$(\mathbb{P}_n^2 f^{(n)})(\mathbf{t}^{(n)}) = \frac{1}{c_n(\mathbf{t}^{(n)})^2} \sum_{\varphi\in S_n^1(\mathbf{t}^{(n)})} \sum_{\pi\in S_n^1(\mathbf{t}^{(n)})} (\Phi_\varphi f^{(n)})(\mathbf{t}^{(n)})$$
$$= \frac{1}{c_n(\mathbf{t}^{(n)})} \sum_{\varphi\in S_n^1(\mathbf{t}^{(n)})} (\Phi_\varphi f^{(n)})(\mathbf{t}^{(n)}) = (\mathbb{P}_n f^{(n)})(\mathbf{t}^{(n)}). \tag{51}$$

Thus, $\mathbb{P}_n$ is an orthogonal projection in $\mathcal{H}^{\otimes n}$.

It remains to prove that $\mathrm{Ran}(\mathbb{P}_n) = \mathcal{F}_n(\mathcal{H})$. Let $f^{(n)} \in \mathcal{F}_n(\mathcal{H})$. Theorem 7 and the construction of the Φ_π operators imply that, for $\sigma^{\otimes n}$-a.a. $\mathbf{t}^{(n)} \in T^{(n)}$ and for each $\pi \in S_n^1(\mathbf{t}^{(n)})$, we have $(\Phi_\pi f^{(n)})(\mathbf{t}^{(n)}) = f^{(n)}(\mathbf{t}^{(n)})$. Hence, by (24), $\mathbb{P}_n f^{(n)} = f^{(n)}$, i.e., $f^{(n)} \in \mathrm{Ran}(\mathbb{P}_n)$.

Finally, we have to prove the inclusion $\mathrm{Ran}(\mathbb{P}_n) \subset \mathcal{F}_n(\mathcal{H})$. This means that, for any $f^{(n)} \in \mathcal{H}^{\otimes n}$ and $k \in \{1, \dots, n-1\}$,

$$(\Phi_k \mathbb{P}_n f^{(n)})(\mathbf{t}^{(n)}) = (\mathbb{P}_n f^{(n)})(\mathbf{t}^{(n)}) \quad \text{for } \sigma^{\otimes n}\text{-a.a. } \mathbf{t}^{(n)} \in T_k^{(n)}. \tag{52}$$

The proof of (52) is similar to the proof of the equality $\mathbb{P}_n^2 = \mathbb{P}$ (formulas (43), (44), and (51)), so we omit it. □

Proof of Corllary 10. We start with the following lemma

Lemma 22. *For each $n \in \mathbb{N}$, we have*

$$\mathbb{P}_{n+1}(\mathbb{P}_n \otimes \mathbf{1}) = \mathbb{P}_{n+1}, \tag{53}$$
$$\mathbb{P}_{n+1}(\mathbf{1} \otimes \mathbb{P}_n) = \mathbb{P}_{n+1}. \tag{54}$$

Proof. We will only prove equality (53), since the proof of (54) is similar. For a permutation $\nu \in S_n$, we denote by $\nu \otimes \mathrm{id}$ the permutation from S_{n+1} defined by $(\nu \otimes \mathrm{id})(i) := \nu(i)$ for $i \in \{1, \dots, n\}$ and $(\nu \otimes \mathrm{id})(n+1) := n+1$. Analogously to the proof of Theorem 8, we get, for any $f^{(n+1)} \in \mathcal{H}^{\otimes(n+1)}$,

$$(\mathbb{P}_{n+1}(\mathbb{P}_n \otimes \mathbf{1}) f^{(n+1)})(\mathbf{t}^{(n+1)})$$
$$= \frac{1}{c_{n+1}(\mathbf{t}^{(n+1)})} \sum_{\pi\in S_{n+1}^1(\mathbf{t}^{(n+1)})} \frac{1}{c_n(t_{\pi(1)}, \dots, t_{\pi(n)})} \sum_{\nu\in S_n^1(t_{\pi(1)},\dots,t_{\pi(n)})} (\Phi_{\pi(\nu\otimes \mathrm{id})} f^{(n+1)})(\mathbf{t}^{(n+1)})$$
$$= \frac{1}{c_{n+1}(\mathbf{t}^{(n+1)})} \sum_{i=1}^{n+1} \sum_{\substack{\varphi\in S_{n+1}^1(\mathbf{t}^{(n+1)})\\ \varphi(n+1)=i}} \sum_{\substack{\pi\in S_{n+1}^1(\mathbf{t}^{(n+1)})\\ \pi(n+1)=i\\ \nu\in S_n^1(t_{\pi(1)},\dots,t_{\pi(n)})\\ \pi(\nu\otimes\mathrm{id})=\varphi}} \frac{1}{c_n(t_{\pi(1)}, \dots, t_{\pi(n)})} (\Phi_\varphi f^{(n+1)})(\mathbf{t}^{(n+1)}). \tag{55}$$

Let $\varphi,\pi\in S^1_{n+1}(\mathbf{t}^{(n+1)})$ be such that $\pi(n+1)=\nu(n+1)=i$. Then $\nu':=\pi^{-1}\varphi\in S^1_{n+1}(\mathbf{t}^{(n+1)}_\pi)$ and $\nu'(n+1)=n+1$. Therefore, $\nu'=\nu\otimes\mathrm{id}$, where $\nu\in S^1_n(t_{\pi(1)},\dots,t_{\pi(n)})$. Hence, by (55),

$$(\mathbb{P}_{n+1}(\mathbb{P}_n\otimes\mathbf{1})f^{(n+1)})(\mathbf{t}^{(n+1)})$$
$$=\frac{1}{c_{n+1}(\mathbf{t}^{(n+1)})}\sum_{i=1}^{n+1}\sum_{\substack{\varphi\in S^1_{n+1}(\mathbf{t}^{(n+1)})\\ \varphi(n+1)=i}}(\Phi_\varphi f^{(n+1)})(\mathbf{t}^{(n+1)})\sum_{\substack{\pi\in S^1_{n+1}(\mathbf{t}^{(n+1)})\\ \pi(n+1)=i}}\frac{1}{c_n(t_{\pi(1)},\dots,t_{\pi(n)})}.$$

Therefore, it is sufficient to prove that, for any $\mathbf{t}^{(n+1)}\in T^{(n+1)}$ and $i\in\{1,\dots,n+1\}$,

$$\sum_{\substack{\pi\in S^1_{n+1}(\mathbf{t}^{(n+1)})\\ \pi(n+1)=i}}\frac{1}{c_n(t_{\pi(1)},\dots,t_{\pi(n)})}=1. \tag{56}$$

To this end, we denote

$$S^1_{n,i}(\mathbf{t}^{(n+1)}):=\{\pi\in S^1_n(\mathbf{t}^{(n+1)}):\pi(n+1)=i\},$$

and let $c_{n+1,i}(\mathbf{t}^{(n+1)}):=|S^1_{n,i}(\mathbf{t}^{(n+1)})|$. We state that, for any $\mathbf{t}^{(n+1)}\in T^{(n+1)}$ and $\pi\in S^1_{n,i}(\mathbf{t}^{(n+1)})$,

$$c_{n+1,i}(\mathbf{t}^{(n+1)})=c_n(t_{\pi(1)},\dots,t_{\pi(n)}). \tag{57}$$

Indeed, if $\nu\in S^1_n(t_{\pi(1)},\dots,t_{\pi(n)})$, then $\nu\otimes\mathrm{id}\in S^1_{n+1}(\mathbf{t}^{(n+1)}_\pi)$. Therefore, $\pi(\nu\otimes\mathrm{id})\in S^1_{n+1}(\mathbf{t}^{(n+1)})$ and

$$\big(\pi(\nu\otimes\mathrm{id})\big)(n+1)=\pi(n+1)=i.$$

Hence, $\pi(\nu\otimes\mathrm{id})\in S^1_{n+1,i}(\mathbf{t}^{(n+1)})$. So $c_n(t_{\pi(1)},\dots,t_{\pi(n)})\le c_{n+1,i}(\mathbf{t}^{(n+1)})$. On the other hand, take any $\varphi\in S^1_{n+1,i}(\mathbf{t}^{n+1})$. Let $\nu':=\pi^{-1}\varphi$. As shown above, $\nu'=\nu\otimes\mathrm{id}$, where $\nu\in S^1_n(t_{\pi(1)},\dots,t_{\pi(n)})$. Hence, $c_{n+1,i}(\mathbf{t}^{(n+1)})\le c_n(t_{\pi(1)},\dots,t_{\pi(n)})$, and formula (57) is proven. Finally, formula (57) implies (56). $\square$

Lemma 23. *For $k\in\mathbb{N}$, we denote by $\mathbf{1}_k$ the identity operator in $\mathcal{H}^{\otimes k}$. Then, for each $n\ge 2$,*

$$\mathbb{P}_{n+k}(\mathbb{P}_n\otimes\mathbf{1}_k)=\mathbb{P}_{n+k}, \tag{58}$$
$$\mathbb{P}_{n+k}(\mathbf{1}_k\otimes\mathbb{P}_n)=\mathbb{P}_{n+k}. \tag{59}$$

Proof. We will again only prove the first formula, (58), the proof of (59) being similar. We prove (58) by induction on k. For $k=1$, formula (58) becomes (53). Let $k\ge 2$ and assume that formula (53) holds for $k-1$. We then have

$$\mathbb{P}_{n+k}(\mathbb{P}_n\otimes\mathbf{1}_k)=\mathbb{P}_{n+k}(\mathbb{P}_{n+k-1}\otimes\mathbf{1})(\mathbb{P}_n\otimes\mathbf{1}_{k-1}\otimes\mathbf{1})$$

$$
\begin{aligned}
&= \mathbb{P}_{n+k}\big[\big(\mathbb{P}_{n+k-1}(\mathbb{P}_n \otimes \mathbf{1}_{k-1})\big) \otimes \mathbf{1}\big] \\
&= \mathbb{P}_{n+k}(\mathbb{P}_{n+k-1} \otimes \mathbf{1}) = \mathbb{P}_{n+k}.
\end{aligned}
$$

□

Using Lemma 23, we get, for $n \geq 2$ and $k \in \{1, \dots, n-1\}$

$$
\mathbb{P}_n(\mathbb{P}_k \otimes \mathbb{P}_{n-k}) = \mathbb{P}_n(\mathbb{P}_k \otimes \mathbf{1}_{n-k})(\mathbf{1}_k \otimes \mathbb{P}_{n-k}) = \mathbb{P}_n(\mathbf{1}_k \otimes \mathbb{P}_{n-k}) = \mathbb{P}_n.
$$

□

Proof of Proposition 12. The result below was shown in the proof of Theorem 3.1 in [8].

Lemma 24 ([8])**.** *Let a bounded linear operator $\mathcal{R}_n : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n}$ be defined by*

$$
\mathcal{R}_n := \mathbf{1}_n + \Psi_1 + \Psi_1\Psi_2 + \dots + \Psi_1\Psi_2 \cdots \Psi_{n-1}. \tag{60}
$$

Then, for $n \in \mathbb{N}$,

$$
(n+1)\mathcal{P}_{n+1} = (\mathbf{1} \otimes \mathcal{P}_n)\mathcal{R}_{n+1}. \tag{61}
$$

Analogously to $\mathcal{F}_{\mathrm{fin}}(\mathcal{H})$ we define a linear space $\mathbf{F}_{\mathrm{fin}}(\mathcal{H})$ that consists of finite sequence $(f^{(0)}, f^{(1)}, \dots, f^{(n)}, \dots)$ with $f^{(i)} \in \mathcal{H}^{\otimes i}$. For $h \in \mathcal{H}$, we define a linear operator $A^-(h) : \mathbf{F}_{\mathrm{fin}}(\mathcal{H}) \to \mathbf{F}_{\mathrm{fin}}(\mathcal{H})$ by setting

$$
(A^-(h)f^{(n)})(t_1, \dots, t_{n-1}) := \int_T \overline{h(s)} f^{(n)}(s, t_1, \dots, t_{n-1})\, \sigma(ds)
$$

for $f^{(n)} \in \mathcal{H}^{\otimes n}$, $n \in \mathbb{N}$, and $A^-(h)(1, 0, 0, \dots) := 0$.

Let $f^{(n)} \in \mathcal{F}_n(\mathcal{H})$, $g^{(n+1)} \in \mathcal{H}^{\otimes(n+1)}$, and $h \in \mathcal{H}$. Then, by (26) and Lemmas 9 and 24, we get

$$
\begin{aligned}
&\big(a^+(h)f^{(n)}, \mathbb{P}_{n+1}g^{(n+1)}\big)_{\mathcal{F}_{n+1}(\mathcal{H})} (n+1)! \\
&\quad = \big(\mathcal{P}_{n+1}\mathbb{P}_{n+1}(h \otimes f^{(n)}), \mathbb{P}_{n+1}g^{(n+1)}\big)_{\mathcal{H}^{\otimes(n+1)}} (n+1)! \\
&\quad = \big(\mathcal{P}_{n+1}(h \otimes f^{(n)}), g^{(n+1)}\big)_{\mathcal{H}^{\otimes(n+1)}} (n+1)! \\
&\quad = \big(\mathcal{R}^*_{n+1}(\mathbf{1} \otimes \mathcal{P}_n)(h \otimes f^{(n)}), g^{(n+1)}\big)_{\mathcal{H}^{\otimes(n+1)}} n! \\
&\quad = \big(h \otimes (\mathcal{P}_n f^{(n)}), \mathcal{R}_{n+1}g^{(n+1)}\big)_{\mathcal{H}^{\otimes(n+1)}} n! \\
&\quad = \big(\mathcal{P}_n f^{(n)}, A^-(h)\mathcal{R}_{n+1}g^{(n+1)}\big)_{\mathcal{H}^{\otimes n}} n! \\
&\quad = \big(f^{(n)}, \mathbb{P}_n A^-(h)\mathcal{R}_{n+1}g^{(n+1)}\big)_{\mathcal{F}_n(\mathcal{H})} n!\,.
\end{aligned}
$$

From here both formulas (27) and (28) follow. □

Remark 25. Note that formula (28) can now be written in the form

$$a^-(h)\mathbb{P}_n g^{(n)} = \mathbb{P}_{n-1}A^-(h)\mathcal{R}_n g^{(n)} \tag{62}$$

for $h \in \mathcal{H}$ and $g^{(n)} \in \mathcal{H}^{\otimes n}$.

Proof of Theorem 14. By choosing an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of $\mathcal{H}$ and writing the infinite matrix of the operator Ψ (see (9)) in terms of the orthonormal basis $(e_n \otimes e_m)_{n,m\in\mathbb{N}}$ of $\mathcal{H}^{\otimes 2}$, one can derive the commutation relation (34) from Section 3 of [8]. For the reader's convenience, we will now present a complete proof of this commutation relation without use of an orthonormal basis.

Let $g, h \in \mathcal{H}$ and $f^{(n)} \in \mathcal{F}_n(\mathcal{H})$. By formulas (26) and (62), we get

$$a^-(g)a^+(h)f^{(n)} = \mathbb{P}_n A^-(g)\mathcal{R}_{n+1}(h \otimes f^{(n)}). \tag{63}$$

By (60),

$$\mathcal{R}_{n+1} = \mathbf{1}_{n+1} + \Psi_1(\mathbf{1} \otimes \mathcal{R}_n). \tag{64}$$

Formulas (63) and (64) yield

$$a^-(g)a^+(h)f^{(n)} = (g,h)_{\mathcal{H}} f^{(n)} + \mathbb{P}_n u^{(n)}, \tag{65}$$

where

$$u^{(n)} := A^-(g)\Psi_1\big(h \otimes (\mathcal{R}_n f^{(n)})\big).$$

A direct calculation shows that

$$u^{(n)}(t_1,\dots,t_n) = \int_T \sigma(ds)\,\overline{g(s)}\,h(t_1)Q(s,t_1)\big(\mathcal{R}_n f^{(n)}\big)(s,t_2,\dots,t_n). \tag{66}$$

On the other hand, using additionally (54), we get

$$a^+(h)a^-(g)f^{(n)} = \mathbb{P}_n\left(h \otimes (\mathbb{P}_{n-1}A^-(g)\mathcal{R}_n f^{(n)})\right) = \mathbb{P}_n v^{(n)}, \tag{67}$$

where

$$v^{(n)} := h \otimes (A^-(g)\mathcal{R}_n f^{(n)}).$$

Note that

$$v^{(n)}(t_1,\dots,t_n) = \int_T \sigma(ds)\,\overline{g(s)}\,h(t_1)\big(\mathcal{R}_n f^{(n)}\big)(s,t_2,\dots,t_n). \tag{68}$$

Formulas (65)–(68) prove (34).

Corollary 10 and formula (31) show that, for each $\varphi^{(2)} \in \mathcal{H}^{\otimes 2}$ and $f^{(n)} \in \mathcal{F}_n(\mathcal{H})$,

$$\int_{T^2} \sigma(ds)\,\sigma(dt)\,\varphi^{(2)}(s,t)\,\partial_s^\dagger\partial_t^\dagger f^{(n)} = \mathbb{P}_{n+2}\big((\mathbb{P}_2\varphi^{(2)}) \otimes f^{(n)}\big). \tag{69}$$

By Theorem 7, since $\varphi^{(2)}$ has support in Θ, we get $\mathbb{P}_2\Psi\varphi^{(2)} = \mathbb{P}_2\varphi^{(2)}$. Hence, formulas (32) and (69) imply

$$\begin{aligned}\int_{T^2} \sigma(ds)\,\sigma(dt)\,\varphi^{(2)}(s,t)\,\partial_s^\dagger\partial_t^\dagger f^{(n)} &= \int_{T^2} \sigma(ds)\,\sigma(dt)\,Q(s,t)\varphi^{(2)}(t,s)\,\partial_s^\dagger\partial_t^\dagger f^{(n)}\\ &= \int_{T^2} \sigma(ds)\,\sigma(dt)\,\varphi^{(2)}(s,t)Q(t,s)\,\partial_t^\dagger\partial_s^\dagger f^{(n)},\end{aligned}$$

which gives (35).

Finally, formula (36) is obtained by taking the adjoint operators on the left and right hand sides of formula (35), see (33). □

Proof of Theorem 18. Using Corollary 10, we get, for each $f^{(n)} \in \mathcal{F}_n(\mathcal{H})$ and $h \in \mathcal{H}$,

$$a^+(h)^m f^{(n)} = \mathbb{P}_{m+n}(h^{\otimes m} \otimes f^{(n)}) = \mathbb{P}_{m+n}((\mathbb{P}_m h^{\otimes m}) \otimes f^{(n)}).$$

Hence, it suffices to prove that $\mathbb{P}_m(h^{\otimes m}) = 0$.

Denote by $(e_t)_{t\in T}$ the canonical orthonormal basis in $\mathcal{H} = \ell^2(T \to \mathbb{C})$, i.e., $e_t(s) = 1$ if $s = t$ and $e_t(s) = 0$ if $s \neq t$. In view of (9), we get

$$\Psi e_s \otimes e_t = Q(t,s) e_t \otimes e_s, \quad (s,t) \in T^2.$$

Note that the operators $(\Psi_\pi)_{\pi\in S_m}$ form a unitary representation of the group S_m, see the proof of Lemma 21. Therefore, for each $k \in \{1,\dots,m-1\}$, we have $\mathbb{P}_m\Psi_k = \mathbb{P}_m$. Hence, for any $t_1,\dots,t_m \in T$,

$$\mathbb{P}_m(e_{t_1} \otimes \cdots \otimes e_{t_{k-1}} \otimes e_{t_{k+1}} \otimes e_{t_k} \otimes e_{t_{k+2}} \otimes \cdots \otimes e_{t_m}) = Q(t_k,t_{k+1})\mathbb{P}_n(e_{t_1} \otimes \cdots \otimes e_{t_m}).$$

This implies that

$$\mathbb{P}_m(e_{t_1} \otimes \cdots \otimes e_{t_m}) = 0 \quad \text{if } |\{t_1,\dots,t_m\}| < m \tag{70}$$

(i.e., if some index t_i appears twice or more times). Analogously, for any $(t_1,\dots,t_m) \in T^m$ and $\pi \in S_m$,

$$\mathbb{P}_m(e_{t_{\pi(1)}} \otimes \cdots \otimes e_{t_{\pi(m)}}) = Q_\pi(t_1,\dots,t_m)\mathbb{P}_m(e_{t_1} \otimes \cdots \otimes e_{t_m}). \tag{71}$$

Let $h = \sum_{t\in T} h_t e_t \in \mathcal{H}$. We get, by (70) and (71),

$$\begin{aligned}\mathbb{P}_m h^{\otimes m} &= \sum_{t_1,\dots,t_m\in T} h_{t_1}\cdots h_{t_m}\mathbb{P}_m(e_{t_1} \otimes \cdots \otimes e_{t_m})\\ &= \sum_{\substack{t_1,\dots,t_m\in T\\ t_i\neq t_j \text{ if } i\neq j}} h_{t_1}\cdots h_{t_m}\mathbb{P}_m(e_{t_1} \otimes \cdots \otimes e_{t_m})\end{aligned}$$

$$= \sum_{\substack{t_1,\dots,t_m \in T \\ t_1 < t_2 \cdots < t_m}} \sum_{\pi \in S_m} h_{t_{\pi(1)}} \cdots h_{t_{\pi(m)}} \mathbb{P}_m(e_{t_{\pi(1)}} \otimes \cdots \otimes e_{t_{\pi(m)}})$$
$$= \sum_{\substack{t_1,\dots,t_m \in T \\ t_1 < t_2 \cdots < t_m}} h_{t_1} \cdots h_{t_m} \sum_{\pi \in S_m} \mathbb{P}_m(e_{t_{\pi(1)}} \otimes \cdots \otimes e_{t_{\pi(m)}})$$
$$= \sum_{\substack{t_1,\dots,t_m \in T \\ t_1 < t_2 \cdots < t_m}} h_{t_1} \cdots h_{t_m} \left(\sum_{\pi \in S_m} Q_\pi(t_1, \dots, t_m) \right) \mathbb{P}_m(e_{t_1} \otimes \cdots \otimes e_{t_m}), \tag{72}$$

where we used that $h_{t_{\pi(1)}} \cdots h_{t_{\pi(m)}} = h_{t_1} \cdots h_{t_m}$ for any permutation $\pi \in S_m$. It can be easily proven by induction on m that, for any $t_1, \dots, t_m \in T$ with $t_1 < t_2 \cdots < t_m$, we have

$$\sum_{\pi \in S_m} Q_\pi(t_1, \dots, t_m) = [m]_q! \,. \tag{73}$$

Here we used the notation, for $m \in \mathbb{N}$ and $q \neq 1$,

$$[m]_q! := \prod_{i=1}^{m} [i]_q, \quad \text{where } [i]_q := 1 + q + q^2 + \cdots + q^{i-1} = \frac{1 - q^i}{1 - q}.$$

Since $q^m = 1$, we get $[m]_q! = 0$. Hence, the theorem follows from (72) and (73). $\square$

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