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A geometrical approach to degenerate scalar-tensor theories

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Abstract: Degenerate scalar-tensor theories are recently proposed covariant theories of gravity coupled with a scalar field. Despite being characterised by higher order equations of motion, they do not propagate more than three degrees of freedom, thanks to the existence of constraints. We discuss a geometrical approach to degenerate scalar-tensor systems, and analyse its consequences. We show that some of these theories emerge as a certain limit of DBI Galileons. In absence of dynamical gravity, these systems correspond to scalar theories enjoying a symmetry which is different from Galileon invariance. The scalar theories have however problems concerning the propagation of fluctuations around a time dependent background. These issues can be tamed by breaking the symmetry by hand, or by minimally coupling the scalar with dynamical gravity in a way that leads to degenerate scalar-tensor systems. We show that distinct theories can be connected by a relation which generalizes Galileon duality, in certain cases also when gravity is dynamical. We discuss some implications of our results in concrete examples. Our findings can be helpful for assessing stability properties and understanding the non-perturbative structure of systems based on degenerate scalar-tensor systems.

Keywords: Classical Theories of Gravity, Gauge Symmetry, Space-Time Symmetries

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Contents

1 Introduction

Galileons are a class of scalar theories with derivative self interactions, characterised by equations of motion (EOMs) of second order, and satisfying a symmetry transformation $\phi \to \phi + a + b_\mu x^\mu$ [1]. They have several features motivating their study: their structure is stable under loop corrections thanks to non-renormalization theorems [2, 3] and is closed under a duality [4]. Their S-matrix has special, distinctive properties [5–7]. They exhibit superluminal behaviour around certain sources, possibly providing consistent theoretical set-ups to study such phenomenon. When coupled with gravity, Galileon symmetry is normally broken; on the other hand, it is possible to covariantize Galileons in such a way that they maintain second order EOMs [8, 9]. Covariant Galileons are especially interesting for their cosmological applications: they can lead to self accelerating cosmologies, and at the same time 'hide' the presence of a scalar fifth force against local measurements of gravitational interactions, through an efficient Vainshtein screening mechanism. See [10, 11] for reviews. Galileons and their covariantized versions have a geometrical interpretation, arising as certain limits of the so called DBI Galileons, which describe the features of probe branes embedded in an extra dimensional set-up [12–15].

The fact that Galilean symmetry is normally broken when coupling Galileons with gravity does not necessarily mean that the structure of the resulting actions is not protected: in certain situations, gravitational interactions can break Galilean symmetries in a soft way, yet preserving some of the desired features of Galileons. See e.g. [16–18] for examples on this respect. When coupled with gravity demanding second order EOMs, Galileons admit as maximal extension the theories of Horndeski [19, 20], which are the most general scalar-tensor theories characterised by second order EOMs.

Interestingly, it has been recently realized that Horndeski theories are *not* the most general covariant scalar-tensor theories with at most three degrees of freedom (that is, theories that do not propagate additional ghostly modes). Other possibilities arise, when considering degenerate scalar-tensor set-ups. The existence of primary constraints prevents the propagation of additional degrees of freedom, even for theories characterised by equations of motion of order higher than two. Examples are the theories of beyond Horndeski (bH) [21–23]. Such systems have been recently further generalised to the so called Extended Scalar Tensor/(Degenerate Higher Order Scalar Tensor) theories, EST/(DHOST) in [24–26], using methods and tools developed in seminal papers by Langlois and Noui [24, 27]. Some of bH (or more generally EST) theories are known to be related with standard Horndeski Lagrangians through conformal or disformal transformations; others, instead, seem to lie at isolated points in the space of scalar-tensor theories [25, 28, 29]. The study of these theories is still in its infancy, but by now we know that they can have consequences for cosmology and screening mechanisms: they lead to a breaking of the Vainshtein mechanism inside matter, modifying the internal structure of non relativistic stars [30–34].

The aim of this work is to address the following questions:

- Is there some relation between bH or EST actions and other known scalar-tensor theories with a well understood extra dimensional origin? In particular, are there any connections with extra dimensional models as Dirac-Born-Infeld (DBI) and Dvali-Gabadadze-Porrati (DGP) set-ups? Addressing this question would allow one to set these theories in a broader context, and to apply to them results and geometrical techniques developed when studying other systems.

- Is the structure of bH or EST theories protected by some symmetry, at least in certain cases? And is this structure closed under a duality, as it happens for Galileon Lagrangians? This information can be important to examine the stability of these theories under loop corrections, and to understand whether their distinctive non linear properties can be connected through dualities to features of simpler systems.

We do some preliminary steps towards answering the previous points. It has been noticed already in [22] that a particular choice of the free functions characterising bH Lagrangians give theories which, in absence of dynamical gravity, reduce to quartic and quintic Galileons. This implies that a naive covariantization of standard quartic and quintic Galileons (when expressed in the specific form $L_4^{\rm gal,1}$ and $L_5^{\rm gal,1}$, using the nomenclature of [35]) is ghost-free. In absence of gravity, these bH theories acquire standard Galileon invariance. See also [36] for a geometrical derivation of degenerate scalar-tensor theories in the context of mimetic gravity theories [37].

Motivated by a construction in terms of a probe brane in an extra dimensional set-up, we show here that *other* examples of theories belonging to the bH class provide set-ups which, with no dynamical gravity, respect a symmetry different from Galileon symmetry. Namely

$$\delta\pi = \pi\,\omega^\mu\,\partial_\mu\pi\,, \tag{1.1}$$

with ω^μ a constant vector. Moreover, special cases of bH Lagrangians at different orders are connected by a duality transformation, which generalizes the standard Galileon duality.

We organise our work in successive steps, to build up the tools necessary to discuss our results:

- We start in section 2 with a review of a determinantal approach to standard Galileon Lagrangians, pointing out that it is particularly convenient to make manifest how Galileon dualities connect different Galileon actions.

- We then discuss in section 3 a novel perspective to DBI Galileons based on a determinantal approach to these systems. This is different with respect to the usual approach which obtains DBI Galileons starting from specific curvature invariants that form the action for a brane probing extra dimensional space. Our point of view is convenient for discussing dualities among DBI Galileon Lagrangians, and for making contact with degenerate scalar-tensor theories when coupling with gravity.

- Section 4 studies a novel, particular limit of DBI Galileons which we dub extreme relativistic. In order for the limit to be well defined, we have to consider extra dimensional space times with different signatures, depending on whether the vector normal to the probe brane is time like or space like. The extreme relativistic limit is opposite to the non relativistic limit of DBI Galileon actions, which provides standard Galileons. The resulting scalar theories have peculiar features which we point out, and are characterised by the field dependent symmetry of eq. (1.1). Moreover, scalar theories of different orders are again connected by dualities. The scalar theories have problems since fluctuations around interesting backgrounds do not have proper kinetic terms: this issue can be tamed by weakly breaking the symmetry.

- Section 5 shows that a minimal covariantization of the DBI Galileons in the extreme relativistic limit gives consistent scalar-tensor theories of gravity, despite being characterized by equations of motion of order higher than two. Indeed, the resulting system corresponds to a particular case of beyond Horndeski Lagrangians. We also discuss how to further generalise our results to include extended scalar-tensor theories. This construction provides a geometrical perspective to degenerate scalar-tensor theories. The strong coupling problems we met in section 4 are automatically solved when the scalar theories are coupled with gravity. On the other hand, the scalar symmetry gets usually broken, although for certain cases some of its properties can be preserved. Moreover, we show that in certain cases different classes of such theories can be connected by dualities, also when dynamical gravity is turned on. Our results can be helpful for assessing stability properties or understanding the non-perturbative structure of degenerate scalar-tensor systems.

We conclude in section 6 discussing possible follow ups for our work, while two appendixes contain further technical details.

2 Symmetries and dualities for Galileon Lagrangians

To set the stage, in this section we succinctly review basic properties of Galileon scalar theories in four dimensional flat space, in absence of gravity, adopting a method that will be useful for what comes next. The use of an approach based on the Levi Civita symbol, discussed for example in [38], is particularly suitable for studying dualities among the actions we consider, as well as for investigating their symmetries.

2.1 The Galileon system

Galileon theories are described by scalar actions which lead to equations of motion of second order, and satisfy a Galileon symmetry. There are several ways to express Galileon actions (see for example the reviews [35, 38]). Here, we adopt a determinantal form whose basic building block is the scalar action

$$\mathcal{S} = \mathcal{N} \int d^4x \, (\partial\pi)^2 \, \det\left[\mathbf{1} + c\,\mathbf{\Pi}\right], \tag{2.1}$$

with $\mathcal{N}$ an overall normalization, c a dimensionful constant, $\mathbf{1}$ the unit matrix in four dimensions, and $\mathbf{\Pi}$ the 4×4 symmetric matrix whose components are

$$\mathbf{\Pi} = \Pi_\mu^{\nu} = \partial_\mu \partial^\nu \, \pi \,.$$

A determinantal form for action (2.1) is convenient once we recall the definition of determinant of a matrix $\mathbf{M}$ in terms of the antisymmetric Levi Civita symbol:

$$\det \mathbf{M} = \frac{1}{4!} \, M_{\mu_1}^{\nu_1} \, M_{\mu_2}^{\nu_2} \, M_{\mu_3}^{\nu_3} \, M_{\mu_4}^{\nu_4} \, \epsilon_{\nu_1\nu_2\nu_3\nu_4} \, \epsilon^{\mu_1\mu_2\mu_3\mu_4} \,. \tag{2.2}$$

This implies that

$$\begin{aligned} \det\left[\mathbf{1} + \mathbf{M}\right] = {} & 1 + [M] + \frac{1}{2}\left([M]^2 - \left[M^2\right]\right) + \frac{1}{6}\left([M]^3 - 3\,[M]\left[M^2\right] + 2\left[M^3\right]\right) \\ & + \frac{1}{24}\left([M]^4 - 6\,[M]^2\left[M^2\right] + 3\left[M^2\right]^2 + 8\,[M]\left[M^3\right] - 6\left[M^4\right]\right) \end{aligned} \tag{2.3}$$

where $[M] = \operatorname{tr}\mathbf{M}$. Using this fact, our action reads

$$\mathcal{S} = \mathcal{N}\int d^4x\, (\partial\pi)^2 \Bigg[1 + c\,[\Pi] + \frac{c^2}{2}\left([\Pi]^2 - \left[\Pi^2\right]\right) + \frac{c^3}{6}\left([\Pi]^3 - 3\,[\Pi]\left[\Pi^2\right] + 2\left[\Pi^3\right]\right)\Bigg], \tag{2.4}$$

plus a total derivative. See also [39] for other uses of a determinantal approach for studying conformal Galileons. Action (2.4) contains a combination of each Galileon Lagrangian (quadratic, cubic, etc), but with fixed coefficients depending on powers of the parameter c. We do not include the tadpole contribution. It is straightforward to prove two key properties of Galileon actions: the corresponding equations of motion (EOMs) are second order, and the theory enjoys a coordinate-dependent Galileon symmetry

$$\delta\pi = w_\mu\, x^\mu + s \tag{2.5}$$

for constant quantities w_μ, s, which is a symmetry of the action up to boundary terms.

In order to get a Galileonic system with the preferred coefficients — say d_i — in front of each Galileon Lagrangian, we can sum three copies of action (2.4) — each one depending on a parameter c_i, with $i = 1, 2, 3$.

Explicitly, we write

$$\mathcal{S}_{\text{tot}} = \sum_{i=1}^{3} S_i = \mathcal{N}\sum_{i=1}^{3}\int d^4x\, (\partial\pi)^2 \det\left[\mathbf{1} + c_i\mathbf{\Pi}\right], \tag{2.6}$$

where each Galileon Lagrangian S_i is characterized by an a priori different constant c_i. Expanding the determinant in this expression, we find

$$\mathcal{S}_{\text{tot}} = \mathcal{N}\int d^4x\, (\partial\pi)^2 \Bigg[d_2 + d_3\,[\Pi] + \frac{d_4}{2}\left([\Pi]^2 - \left[\Pi^2\right]\right) + \frac{d_5}{6}\left([\Pi]^3 - 3\,[\Pi]\left[\Pi^2\right] + 2\left[\Pi^3\right]\right)\Bigg], \tag{2.7}$$

with

$$d_2 = 3\mathcal{N}, \tag{2.8}$$

$$d_3 = \mathcal{N}\left(c_1 + c_2 + c_3\right), \tag{2.9}$$

$$d_4 = \mathcal{N}\left(c_1^2 + c_2^2 + c_3^2\right), \tag{2.10}$$

$$d_5 = \mathcal{N}\left(c_1^3 + c_2^3 + c_3^3\right). \tag{2.11}$$

The previous system of algebraic equations can be solved for c_i in terms of d_i by means of Newton identities.

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2.2 The duality

The structure of Galileon Lagrangians is invariant under a *duality*, a field transformation which connects Galileon theories of different orders (each order defined in terms of the number of powers of second derivatives on the scalar field). The properties and consequences of Galileon duality have been introduced in [4] and studied at length in various works: see e.g. [5, 40–46]. The duality among different Galileon Lagrangians can be an important tool to shed light on the non-perturbative structure of Galileons, for example to understand physical consequences of superluminality, and its connections with screening mechanisms; see e.g. [10] for a review on these topics. In this subsection, we briefly review this subject at a formal level, to demonstrate that a determinantal approach makes more manifest the duality of Galileon actions.

Galileon duality in flat space is based on a field dependent map among two sets of coordinates, x^μ and $\tilde{x}^\mu$,[1]

$$x^\mu \Rightarrow \tilde{x}^\mu = x^\mu + \frac{1}{\Lambda_S^3}\partial^\mu\pi\,, \tag{2.12}$$

where Λ_S is a parameter with dimension of an energy scale, introduced for dimensional reasons. For simplicity, we choose units for which

$$\Lambda_S = 1 \tag{2.13}$$

since here (and in what follows) we are more interested to exhibit symmetries and dualities, rather than discussing their physical consequences (strong coupling scales, etc).

By taking the differential of eq. (2.12)

$$dx^\mu \Rightarrow d\tilde{x}^\mu = \left(\delta^\mu_\nu + \Pi^\mu_\nu\right)\, dx^\nu\,, \tag{2.14}$$

we get the Jacobian J^μ_ν for this transformation

$$J^\mu_\nu = \frac{d\tilde{x}^\mu}{dx^\nu} = \left(\delta^\mu_\nu + \Pi^\mu_\nu\right)\,. \tag{2.15}$$

We assume that such coordinate change is invertible, in the sense that a second scalar field $\tilde{\pi}$ exists, which sends $\tilde{x}^\mu$ back to x^μ (see [46] for details)

$$\tilde{x}^\mu \Rightarrow x^\mu = \tilde{x}^\mu - \tilde{\partial}^\mu\tilde{\pi}\,. \tag{2.16}$$

We call $\tilde{\pi}$ the *dual field* of π.

The requirement of invertibility of this transformation means that applying the transformations (2.12) and (2.16) in succession we go back to the original coordinates: the duality is defined as a map which sends the derivative of the scalar π to the derivative of $\tilde{\pi}$

$$\partial^\mu\pi \Rightarrow \tilde{\partial}^\mu\tilde{\pi} = \partial^\mu\pi\,. \tag{2.17}$$

[1] Throughout this work, we use the symbol $\Rightarrow$ for denoting the duality transformation.

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So the *derivative of* π (and not π) is a scalar under the duality transformation. On the other hand, π and its dual transform as

$$\pi(x) \Rightarrow \tilde{\pi}(\tilde{x}) = \pi(x) + \frac{1}{2}\left(\partial\pi(x)\right)^2 , \tag{2.18}$$

$$\tilde{\pi}(\tilde{x}) \Rightarrow \pi(x) = \tilde{\pi}(\tilde{x}) - \frac{1}{2}\left(\tilde{\partial}\tilde{\pi}(\tilde{x})\right)^2 . \tag{2.19}$$

The second derivative of π transforms non-trivially (as a 'covariant vector') under duality: using a matrix notation, $\mathbf{\Pi} \equiv \Pi_\mu^\nu$ we can write

$$\mathbf{\Pi} \Rightarrow \tilde{\mathbf{\Pi}} = \left[\mathbf{1} + \mathbf{\Pi}\right]^{-1} \mathbf{\Pi} . \tag{2.20}$$

where the $[\dots]^{-1}$ denotes the inverse of a matrix. Collecting these results, it is straightforward to determine how the Galileon system (2.1) changes under the action of duality:

$$\begin{aligned}
\mathcal{S} = \mathcal{N} \int d^4x \, (\partial\pi)^2 \det\left[\mathbf{1} + c\,\mathbf{\Pi}\right] \Rightarrow \mathcal{N} \int d^4\tilde{x} \, \left(\tilde{\partial}\tilde{\pi}\right)^2 & \det\left[\mathbf{1} + c\,\tilde{\mathbf{\Pi}}\right] , \\
= \mathcal{N} \int d^4x \, \det\left[\mathbf{1} + \mathbf{\Pi}\right] (\partial\pi)^2 & \det\left[\mathbf{1} + c\,\left[\mathbf{1} + \mathbf{\Pi}\right]^{-1} \mathbf{\Pi}\right] , \\
= \mathcal{N} \int d^4x \, (\partial\pi)^2 & \det\left[(\mathbf{1} + \mathbf{\Pi})\left(\mathbf{1} + c\,\left[\mathbf{1} + \mathbf{\Pi}\right]^{-1} \mathbf{\Pi}\right)\right] , \\
= \mathcal{N} \int d^4x \, (\partial\pi)^2 & \det\left[\mathbf{1} + (c+1)\,\mathbf{\Pi}\right] = \tilde{\mathcal{S}} .
\end{aligned} \tag{2.21}$$

The structure of this determinantal action remains the same, **but** the coefficient in front of the matrix $\mathbf{\Pi}$ within the determinant argument passes from the value c to $(c+1)$. This is the only change induced by applying the duality.

We can then combine different dual actions $\tilde{\mathcal{S}}$, to form an arbitrary combination of Galileon Lagrangians with arbitrary coefficients (as done around eq. (2.7)). Comparing the results before and after applying the duality transformation, we find that the overall coefficients in front of each Galileon Lagrangian are mapped to their 'dual' values

$$\tilde{d}_2 = 3\mathcal{N} , \tag{2.22}$$

$$\tilde{d}_3 = \mathcal{N}\left(c_1 + c_2 + c_3 + 3\right) , \tag{2.23}$$

$$\tilde{d}_4 = \mathcal{N}\left((c_1+1)^2 + (c_2+1)^2 + (c_3+1)^2\right) , \tag{2.24}$$

$$\tilde{d}_5 = \mathcal{N}\left((c_1+1)^3 + (c_2+1)^3 + (c_3+1)^3\right) , \tag{2.25}$$

where the d_i are the quantities introduced in eq. (2.7) and following. Hence we have the relations

$$\tilde{d}_2 = d_2 , \tag{2.26}$$

$$\tilde{d}_3 = d_2 + d_3 , \tag{2.27}$$

$$\tilde{d}_4 = d_2 + 2d_3 + d_4 , \tag{2.28}$$

$$\tilde{d}_5 = d_2 + 3d_3 + 3d_4 + d_5 . \tag{2.29}$$

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We checked that these results are in agreement with [4].[2] Note in passing that (2.21) provides a simple expression for the dual of free fields, once we select $c_i = 0$.

3 Symmetries and dualities for DBI Galileons

3.1 Some motivations

One motivation for introducing Galileons is to find a 'ghost-free' version of a system describing the physics of the DGP brane-world [47]. It is then natural to ask whether Galileon actions have a geometrical description in terms of a brane embedded in higher dimensional space. This was achieved in [12], and generalised in [13, 14], introducing a class of theories called DBI Galileons. They enjoy symmetries generalising Galileon transformations (in absence of dynamical gravity). In this work we show that the same approach can be used to find a relation between DBI Galileons and a subclass of beyond Horndeski and EST theories. First of all, however, to pave the way we need to reconsider DBI Galileons from a perspective which is slightly different from the one of [12].

The approach of [12] starts from the fact that brane actions can be built by means of gravitational Lovelock and Gibbons-Hawking terms, which describe derivative self-interactions for a scalar controlling the position of the probe brane in the higher dimensional bulk. The resulting scalar actions are automatically ghost free, since they are built in terms of specific combinations (DBI, Lovelock, Gibbons-Hawking) of the brane induced metric, ensuring that the scalar equations of motion are at most second order. In addition, this scalar action enjoys symmetries inherited from isometries of the higher dimensional space. These symmetries, associated with the properties of the extra dimensional geometry, can reduce to Galileon symmetries in appropriate, 'small first derivative' limits.

On the other hand, recently it has been realised that degenerate scalar-tensor theories exist, which although characterized by higher order EOMs, are nevertheless consistent thanks to the existence of primary constraints. These are the theories of beyond Horndeski, or more in general EST/DHOST [24–26]. It is natural to ask whether these theories admit some sort of geometrical interpretation. This is one of the purposes of our work, and we start in this section to examine scalar theories which will be related to degenerate scalar-tensor theories, once coupled to dynamical gravity. In particular, in this section we build on the results of [12–14], but we discuss convenient, determinantal expressions for DBI Galileons. This allows us to express such actions in a more compact form, and to exhibit symmetries and dualities among them. Our method for expressing the DBI Galileon system does *not directly rely* on using Lovelock or Gibbons-Hawking combinations: on the other hand, it provides consistent theories in absence of dynamical gravity. Our approach will then be used in section 5 to make manifest the connection between DBI Galileons and beyond Horndeski/EST theories, once the system is minimally coupled with dynamical gravity.

[2] The dictionary between the notation of [4] and ours is as follows: $d_2 = -6b_2, d_3 = 3b_3, d_4 = -4b_4, d_5 = 15b_5, \tilde{d}_2 = -6p_2, \tilde{d}_3 = -3p_3, \tilde{d}_4 = -4p_4, \tilde{d}_5 = -15p_5$, where we renamed their c_i's to b_i's. These relations take care of differences in the definitions of the Galileon Lagrangians — total derivatives and global coefficients — as well as of a sign difference in the definition of the dual field.

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We start discussing Poincaré DBI Galileons — so called since they correspond to scalar actions associated with a brane embedded in a 5d bulk with Poincaré symmetry — to then continue analysing AdS DBI Galileons.

3.2 Poincaré DBI Galileons

Poincaré DBI Galileons [12, 48, 49] are scalar theories with second order EOMs, enjoying a symmetry that generalises Galileon invariance. We call such symmetry Poincaré induced symmetry, being inherited from a global Poincaré symmetry in five dimensions. Namely, their action and the symmetry they satisfy are inherited from a five dimensional description in terms of a probe brane in a 5d flat space. The system can also be described using a determinantal approach, which generalises the one applied in the previous section to standard Galileons.

The action we are interested in reads

$$S = \mathcal{N} \int d^4x \, \frac{1}{\gamma} \det\left[\, \delta_\mu^\nu + c\,\gamma \left(\Pi_\mu^\nu - \gamma^2\, \partial_\mu \pi\, \partial^\lambda \pi\, \Pi_\lambda^\nu \right)\right], \tag{3.1}$$

with

$$\gamma = \frac{1}{\sqrt{\kappa_0^2 + X}}\,. \tag{3.2}$$

$\mathcal{N}$, c, κ_0 are constants, and

$$X = (\partial\pi)^2\,. \tag{3.3}$$

The motivation for considering this action will be clearer in what follows. When expanding the determinant, one finds a sum of four actions, weighted by different powers of the constant parameter c, from zero to three. A direct calculation shows that each of them reproduces the structure of the DBI Galileon Lagrangians presented in [12], once selecting $\kappa_0 = 1$. Hence, action (3.1) succinctly contain all DBI Galileons in flat space. More explicitly, using eq. (2.3), we expand the determinant and get

$$\begin{aligned} S = \mathcal{N} \int d^4x \, \frac{1}{\gamma}\Big(1 + c\gamma\left([\Pi] - \gamma^2[\Phi]\right) + \frac{c^2\gamma^2}{2}\left([\Pi]^2 - [\Pi^2] + 2\gamma^2[\Phi^2] - 2\gamma^2[\Phi][\Pi]\right) \\ + \frac{c^3\gamma^3}{6}\Big[[\Pi]^3 + 2[\Pi^3] - 3[\Pi^2][\Pi] + 3\gamma^2\Big(2[\Pi][\Phi^2] - 2[\Phi^3] - [\Phi][\Pi]^2 + [\Phi][\Pi^2]\Big)\Big]\Big)\,, \end{aligned} \tag{3.4}$$

where we use the notation $[\Phi^n] = \operatorname{tr}(\partial\pi\, \Pi^n\, \partial\pi)$. The coefficient of each power of c corresponds to one of the Lagrangians for DBI Galileons, when choosing $\kappa_0 = 1$. A linear combination of these actions is able to provide DBI Galileons with arbitrary coefficients (as described in the previous section for the standard Galileon case). Working within a determinantal approach allows us to make more transparent dualities and symmetries for DBI Galileons.

Notice that we are discussing a slight generalisation of the actions of [12], which includes a free constant parameter κ_0 in the definition of the γ factor.[3] We do so because once

[3] The generalisation we consider is a particular case of the actions discussed in [13, 14]. It is simple to see that this additional free parameter can be obtained from the standard DBI Galileon case, by a rescaling $\pi \to \pi/\kappa_0$.

coupled with gravity, appropriate choices of this parameter κ_0 make manifest the connection between these actions and degenerate scalar-tensor theories. This will be discussed in section 5.

Action (3.1) leads to second order equations of motion for the scalar field, thanks to the properties of the determinant. Additionally, this action is invariant (up to boundary terms) under a scalar symmetry (here ω^μ is an arbitrary constant vector)

$$\delta\pi = \kappa_0^2\,\omega_\mu\,x^\mu + \pi\,\omega^\mu\,\partial_\mu\,\pi\,, \tag{3.5}$$

and under a duality, as we discuss in the next two subsections.

3.2.1 Geometrical interpretation, and underlying symmetry

As we mentioned, the scalar Lagrangian (3.1) is invariant under the scalar transformation (3.5), up to boundary terms (an additional, shift symmetry $\pi \to \pi$+const is also satisfied, but we do not consider it here). This can be proved by a direct computation. In the limit of small field derivatives, this transformation reduces to Galileon symmetry, at least when $\kappa_0 \neq 0$ (more on this later).

Alternatively, this symmetry can be understood 'geometrically' in terms of an action for a probe brane embedded in a higher dimensional bulk, using arguments similar to [12] — further developed in [13, 14] — that we briefly review here, and accommodate to our discussion.

The transformation (3.5) is associated with a symmetry for a probe brane in 5d flat space, inherited from a global isometry in five dimensions. In particular, eq. (3.5) is associated with boosts in five dimensions. To see this fact more explicitly, we consider a 5d bulk with flat metric

$$g^{(5)}_{MN}\,dX^M dX^N = \kappa_0^2\,\eta_{\mu\nu}\,dX^\mu dX^\nu + dy^2\,, \tag{3.6}$$

where $X^5 = y$. We introduce a constant parameter κ_0^2 in front of the 4d slices in the 5d metric. Still, the 5d metric is flat, and have the same number of isometries of Minkowski space. As we will discuss in section 4, the parameter κ_0 is associated with the 'maximal speed' allowed by causality for motion along the extra dimension.

A 4d brane embedded in the 5d bulk is characterised by a brane embedding, $X^M(x^\mu)$, which maps the four brane dimensions into the five bulk dimensions. We foliate the bulk in terms of slices y =const; the brane embedding is chosen as

$$\begin{aligned} X^\mu &= x^\mu\,, \\ y &= \pi(x)\,, \end{aligned} \tag{3.7}$$

and fixes the gauge associated with the freedom to reparameterise the foliation. The scalar field π is a modulus which geometrically corresponds to the position of the brane along the fifth bulk coordinate. See figure 1.

The brane induced geometry can be deduced from the information we provided. The induced brane metric is

$$g_{\mu\nu} = \frac{\partial X^M}{\partial x^\mu}\frac{\partial X^N}{\partial x^\nu}g^{(5)}_{MN} \;=\; \kappa_0^2\eta_{\mu\nu} + \partial_\mu\pi\partial_\nu\pi\,. \tag{3.8}$$

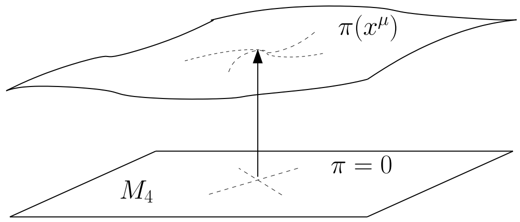

Figure 1. Brane geometry with respect to a bulk foliation $y = \text{const}$.

The matrix inverse of the induced brane metric is

$$g^{\mu\nu} = \frac{1}{\kappa_0^2}\left(\eta^{\mu\nu} - \gamma^2 \partial^\mu \pi \partial^\nu \pi\right), \tag{3.9}$$

where recall that $\gamma = (\kappa_0^2 + X)^{-1/2}$. It satisfies the relation

$$g^{\mu\,\alpha}\, g_{\alpha\nu} = \delta^\mu_\nu\,. \tag{3.10}$$

The square root of the metric determinant is

$$\sqrt{-g} = \frac{\kappa_0^3}{\gamma}\,. \tag{3.11}$$

Another tensorial quantity of interest is proportional to the 'brane extrinsic curvature', a tensor defining intrinsic properties of the brane geometry which can be computed using standard definitions [50]. In this case, it results

$$K_{\mu\nu} = -\gamma\,\kappa_0\,\Pi_{\mu\nu}\,. \tag{3.12}$$

An interesting property of the quantities $g_{\mu\nu}$, $K_{\mu\nu}$ is that they transform as tensors with respect to the transformation (3.5), which we rewrite here

$$\delta\pi = \kappa_0^2\,\omega_\mu\, x^\mu + \pi\,\omega^\mu\,\partial_\mu\,\pi\,. \tag{3.13}$$

Namely, when applying the scalar transformation (3.13), these quantities transform as

$$g_{\mu\nu} \;\to\; \xi^\alpha\,\partial_\alpha g_{\mu\nu} + \partial_\mu \xi^\alpha\, g_{\alpha\nu} + \partial_\nu \xi^\alpha\, g_{\alpha\mu}\,, \tag{3.14}$$

(and analogously for $K_{\mu\nu}$) with

$$\xi^\alpha = \pi\,\omega^\alpha\,, \tag{3.15}$$

and ω^α being the constant vector of eq. (3.13).

The scalar symmetry (3.13) can be interpreted as geometrically associated with isometries of the embedding 5d geometry: this viewpoint has been developed in a comprehensive way in [13, 14]. Suppose we have a bulk isometry associated with a Killing vector $\mathcal{V}^A$: the probe brane action should enjoy this symmetry as well, in the form of a symmetry transformation for the scalar field π. On the other hand, we have to take into account that by choosing the embedding (3.7) we are selecting a specific gauge, associated with our freedom of choosing the brane coordinates. As explained in [13, 14], we need to include a ‘compensating’ gauge transformation to the field π, for ensuring that the brane action is invariant under bulk isometry. In total, the scalar transformation inherited from the bulk isometry, which is a symmetry of the brane action, reads

$$\delta\pi = \mathcal{V}^5(x,\pi) - \kappa_0\, \mathcal{V}^\mu \partial_\mu \pi\,, \tag{3.16}$$

where the second term in the right hand side is associated with the compensating gauge transformation. Applying these arguments to our case, we find the scalar symmetry (3.13).

A consequence of the symmetry is that any action which is a scalar built in terms of the tensors $g_{\mu\nu}$, $K_{\mu\nu}$ is invariant under the transformation (3.5). Among the symmetry preserving actions, we find the determinantal action (3.1) we considered in the previous subsection, which can be expressed as

$$S = \frac{\mathcal{N}}{\kappa_0^3} \int d^4x\, \sqrt{-g}\; \det\left[\delta^\mu_\nu - \kappa_0\, c\, g^{\mu\,\alpha} K_{\alpha\,\nu}\right], \tag{3.17}$$

$$= \mathcal{N} \int d^4x\, \frac{1}{\gamma} \det\left[\,\delta^\mu_\nu + c\,\gamma \left(\Pi^\mu_\nu - \gamma^2\, \partial^\mu\pi\, \partial_\lambda\pi\, \Pi^\lambda_\nu\right)\right]. \tag{3.18}$$

Action (3.18) coincides with eq. (3.1). Hence it is a scalar action in flat 4d space, built with the tensors $g_{\mu\nu}$, $K_{\mu\nu}$; and it is invariant under symmetry (3.5) up to boundary terms: calling $\mathbf{K} = g^{\mu\,\alpha} K_{\alpha\,\nu}$, expanding the determinant we find combinations of traces $\mathrm{tr}\mathbf{K}$, $\mathrm{tr}\mathbf{K^2}$, $\dots$, and their powers, Remarkably, as stated previously, we get all the four combinations that correspond to DBI Galileons. The determinantal action, additionally, is useful for exhibiting a duality for DBI galileons as we discuss in the next subsection.

It is important to emphasize again that our derivation of DBI Galileons is different with respect to the approach of [12]. In that case, DBI Galileons are obtained starting from combinations of curvature invariants that automatically ensure that the EOMs for the fields involved are second order (Lovelock and Gibbons-Hawking terms); this remains true when gravity is made dynamical (i.e. the flat metric $\eta_{\mu\nu}$ on 4d slices $y = const$ is promoted to a dynamical metric $q_{\mu\nu}$), and allows one to obtain covariantized versions of Galileons [12].

In our approach, working with the determinantal action (3.18), we are not ensured that the EOMs remain of second order, once the theory is minimally coupled with gravity by making gravity dynamical. Indeed, when coupled with gravity, the system is characterised by higher order EOMs. On the other hand, as we will see in section 5, when $\kappa_0 \to 0$ a primary constraint arises, which prevents the propagation of an additional degree of freedom associated with an Ostrogradsky instability: the resulting theory belongs to the class of beyond Horndeski theories of gravity, or more generally to EST.

3.2.2 The duality

Action (3.18) satisfies a duality which generalises the Galileon duality we reviewed in section 2.2. Our determinantal approach to DBI Galileon makes this duality manifest, and as far as we are aware we are the first to discuss in this particular way a duality for Poincaré DBI galileons. (See also [40, 45].)

Our arguments here are very similar in spirit to the discussion of duality for standard Galileons, as developed in section 2.2. The determinantal action (3.1) can be expressed in several equivalent ways

$$S = \mathcal{N} \int d^4x \, \frac{1}{\gamma} \det \left[\delta_\mu^\nu + c\,\gamma \left(\Pi_\mu^\nu - \gamma^2 \, \partial_\mu\pi \, \partial^\lambda\pi \, \Pi_\lambda^\nu \right) \right] , \tag{3.19}$$

$$= \mathcal{N} \int d^4x \, \frac{1}{\gamma} \det \left[\delta_\mu^\nu + c \, \partial_\mu \left(\gamma \partial^\nu \pi \right) \right] , \tag{3.20}$$

$$= \frac{\mathcal{N}}{\kappa_0^3} \int d^4x \sqrt{-g} \, \det \left[\delta_\nu^\mu - \kappa_0 \, c \, g^{\mu\sigma} \, K_{\sigma\,\nu} \right] , \tag{3.21}$$

where in the last line we write the action in a 'geometric form', exploiting the concept of induced metric and extrinsic curvature on a probe brane, as discussed in the previous subsection. Recall that $\gamma^{-1} = \sqrt{\kappa_0^2 + X}$.

We introduce a field dependent coordinate map which is at the basis of the duality

$$x^\mu \;\Rightarrow\; \tilde{x}^\mu = x^\mu + \gamma \, \partial^\mu \pi \,, \tag{3.22}$$

(where, as for the case of Galileons, section 2.2, we choose units which set to one the parameter Λ_S which is needed for dimensional reasons). We demand that there exists a 'dual' scalar $\tilde{\pi}$, which maps through duality the tilde coordinates $\tilde{x}^\mu$ back to the original coordinates x^μ:

$$\tilde{x}^\mu \;\Rightarrow\; x^\mu = \tilde{x}^\mu - \tilde{\gamma} \, \tilde{\partial}^\mu \tilde{\pi} \,. \tag{3.23}$$

Comparing eqs (3.22) and (3.23), we find

$$\tilde{\gamma} \, \tilde{\partial}^\mu \tilde{\pi} = \gamma \, \partial^\mu \pi \,. \tag{3.24}$$

The simplest way to satisfy the previous condition (3.24) is to impose

$$\tilde{\partial}^\mu \tilde{\pi} = \partial^\mu \pi \,, \tag{3.25}$$

that is, the duality maps the derivative to the scalar π to the derivative of the dual $\tilde{\pi}$.

The Jacobian associated with map (3.22) is

$$J_\nu^\mu \equiv \frac{d\tilde{x}^\mu}{dx^\nu} = \delta_\nu^\mu + \gamma \left(\delta_\rho^\mu - \gamma^2 \partial^\mu \pi \partial_\rho \pi \right) \Pi_\nu^\rho \,, \tag{3.26}$$

$$= \delta_\nu^\mu - \kappa_0 \, g^{\mu\,\alpha} K_{\alpha\,\nu} \,. \tag{3.27}$$

Eq (3.25) implies (indexes are raised/lowered with flat Minkowski metric)

$$\tilde{\partial}_\mu \tilde{\pi} \, d\tilde{x}^\mu = \partial_\mu \pi \, J_\nu^\mu \, dx^\nu = \partial_\mu \pi \, dx^\mu + \partial_\mu \pi \, \partial_\nu \left(\gamma \partial^\mu \pi \right) dx^\nu \,. \tag{3.28}$$

Integrating, we find a non local relation among the field π and its dual

$$\pi(x) \;\Rightarrow\; \tilde{\pi}(\tilde{x}) = \pi(x) + \int^x d\bar{x}^\lambda \, \partial_\lambda \left(\gamma \partial^\rho \pi \right) \, \partial_\rho \pi \,. \tag{3.29}$$

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The second derivative of the scalar, on the other hand, transforms under duality by means of the inverse Jacobian:

$$\partial_\mu\left(\gamma\partial^\nu\pi\right) \qquad \Rightarrow \qquad \tilde{\partial}_\mu\left(\tilde{\gamma}\tilde{\partial}^\nu\tilde{\pi}\right)=\tilde{\partial}_\mu\left(\gamma\partial^\nu\pi\right)=\left(J^{-1}\right)_\mu^\lambda\,\partial_\lambda\left(\gamma\partial^\nu\pi\right). \tag{3.30}$$

Notice that, using the definitions of induced brane geometrical quantities

$$g_{\mu\nu} = \kappa_0^2\eta_{\mu\nu}+\partial_\mu\pi\partial_\nu\pi \tag{3.31}$$

$$\kappa_0\, g^{\mu\alpha}\,K_{\alpha\nu} = -\partial^\mu\left(\gamma\partial_\nu\pi\right). \tag{3.32}$$

The results so far imply that

$$d^4x \Rightarrow d^4\tilde{x}=(\det J)\;d^4x\,, \tag{3.33}$$

$$\sqrt{-g} \Rightarrow \sqrt{-\tilde{g}}=\sqrt{-g}\,, \tag{3.34}$$

$$g^{\mu\alpha}K_{\alpha\nu} \Rightarrow \tilde{g}^{\mu\alpha}\tilde{K}_{\alpha\nu}=\left(J^{-1}\right)^\mu_{\;\rho}\,g^{\rho\alpha}\,K_{\alpha\nu}\,. \tag{3.35}$$

These ingredients are sufficient for finding how our original action transforms under duality (here $K^\mu_\nu=g^{\mu\alpha}K_{\alpha\nu}$)

$$\begin{aligned}\frac{\mathcal{N}}{\kappa_0^3}\int d^4x\,\sqrt{-g}\,\det\left[\delta^\mu_\nu-\kappa_0\,c\,K^\mu_\nu\right] \Rightarrow& \frac{\mathcal{N}}{\kappa_0^3}\int d^4\tilde{x}\,\sqrt{-\tilde{g}}\,\det\left[\delta^\mu_\nu-\kappa_0\,c\,\tilde{K}^\mu_{\;\nu}\right]\\ =& \frac{\mathcal{N}}{\kappa_0^3}\int d^4x\,\det J\,\sqrt{-g}\,\det\left[\delta^\mu_\nu-\kappa_0\,c\,\left(J^{-1}\right)^\mu_{\;\beta}\,K^\beta_\nu\right]\\ =& \frac{\mathcal{N}}{\kappa_0^3}\int d^4x\,\sqrt{-g}\,\det\left[\delta^\mu_\nu-\kappa_0\,(c+1)\,K^\mu_\nu\right].\end{aligned} \tag{3.36}$$

Hence the structure of the action remains the same, and the only change is a shift in the constant $c\Rightarrow c+1$ which appears within the determinant. This result generalises the standard Galileon duality that we reviewed in section 2.2.

3.3 DBI Galileons in a maximally symmetric extra dimensional space

Some of the results we discussed in the previous subsections can be generalised to a set of scalar actions associated with branes probing curved 5d space times, as for example AdS or dS spaces, which maintain a four dimensional flat slicing. The new feature introduced by such versions of DBI Galileons is an explicit dependence of the action on the field π (and not only on its derivatives) and a generalisation of the symmetries reviewed earlier. These actions fall in the class of conformal Galileons, in the nomenclature of [13, 14].

In order to discuss these theories, we use the convenient geometrical approach introduced in [13, 14]. We consider for definiteness a curved 5d space time with warped metric

$$ds^2_{(5)}=\kappa_0^2\,f^2(y)\,\eta_{\mu\nu}\,dX^\mu dX^\nu+dy^2. \tag{3.37}$$

We examine the same brane configuration as in the previous subsection

$$X^\mu = x^\mu, \tag{3.38}$$

$$y = \pi(x^\mu). \tag{3.39}$$

The associated brane induced metric results

$$g_{\mu\nu} = f^2(\pi)\, \kappa_0^2\, \eta_{\mu\nu} + \partial_\mu\pi\partial_\nu\pi\,, \tag{3.40}$$

with inverse

$$g^{\mu\nu} = \frac{1}{f^2\,\kappa_0^2}\left(\eta^{\mu\nu} - \frac{\partial^\mu\pi\,\partial^\nu\pi}{\kappa_0^2\,f^2 + X}\right). \tag{3.41}$$

The brane extrinsic curvature is

$$K_{\mu\nu} = -\frac{\kappa_0\, f}{\sqrt{\kappa_0^2 f^2 + X}}\left(\partial_\mu\partial_\nu\pi - \frac{2f'}{f}\,\partial_\mu\pi\partial_\nu\pi - \kappa_0^2\, f\, f'\,\eta_{\mu\nu}\right). \tag{3.42}$$

We construct an action with the very same geometric structure as the one of the previous subsection (see eq. (3.18)),

$$\mathcal{S} = \frac{\mathcal{N}}{\kappa_0^3}\int d^4x\,\sqrt{-g}\,\det\left[\delta_\mu^\nu - c\,\kappa_0\, g^{\mu\alpha}\, K_{\alpha\nu}\right], \tag{3.43}$$

but with the new induced metric and brane extrinsic curvature given in eqs (3.40), (3.42). As explained in section 3.2, and in detail in [13, 14], being a scalar built in terms of the tensors $g_{\mu\nu}$, $K_{\mu\nu}$, this action is invariant under any scalar symmetry associated with the isometries of the 5d space under consideration.

As a concrete example, that turns to be relevant for what comes next, we consider a probe Minkowski brane embedded in a 5d AdS bulk. This embedding is described by the warp factor

$$f(\pi) = e^{-\frac{\pi}{\ell}}\,, \tag{3.44}$$

in eq. (3.37), with ℓ the AdS radius. In the limit $\ell \to \infty$ we recover $f = 1$ and the Poincaré DBI Galileons of subsection 3.2. Plugging (3.41), (3.42) and (3.44) in (3.43) we obtain

$$\begin{aligned}\mathcal{S} = \mathcal{N}\int d^4x\,\frac{e^{-3\pi/\ell}}{\gamma}\,\det\Big[&\delta_\nu^\mu + c\,\gamma\, e^{\pi/\ell}\left(\eta^{\mu\alpha} - \gamma^2\partial^\mu\pi\,\partial^\alpha\pi\right)\\ &\times\left(\partial_\alpha\partial_\nu\pi + \frac{2}{\ell}\,\partial_\alpha\pi\partial_\nu\pi + \frac{\kappa_0^2}{\ell}\,e^{-2\pi/\ell}\,\eta_{\alpha\nu}\right)\Big],\end{aligned} \tag{3.45}$$

with

$$\gamma = \frac{1}{\sqrt{(\kappa_0^2\, e^{-2\pi/\ell} + X)}}. \tag{3.46}$$

This action is invariant under the following transformation of the scalar field, which is a symmetry inherited from the isometries of the AdS bulk

$$\delta\pi = \kappa_0^2\, w_\mu x^\mu + \partial_\mu\pi\left(\frac{\ell}{2}\left(e^{2\pi/\ell} - 1\right)\, w^\mu + \frac{\kappa_0^2\, x^2}{2\ell}\, w^\mu - \frac{\kappa_0^2}{\ell}\,(wx)\, x^\mu\right). \tag{3.47}$$

When expanding the determinant, one finds a set of actions which are related to the AdS DBI Galileons discussed in [12].[4] The advantage of such geometrical approach, developed in [12–14], is that it makes more manifest the symmetries associated with the action (3.43). For the case of AdS DBI Galileons, however, it is not clear whether a duality exists which connects actions of different order. The case of a brane embedded in de Sitter space is discussed in appendix A.

4 An extreme relativistic limit

We examine in this section a certain limit of the DBI Galileons in flat space, which we dub extreme relativistic, which satisfies a symmetry different from Galileon symmetry. As we will learn in the next section, the resulting theories are particularly interesting when coupled with dynamical gravity, since they are related with beyond Horndeski and other degenerate scalar-tensor theories.

When discussing Poincaré DBI Galileons, we considered a five dimensional flat metric, characterised by a parameter κ_0 as

$$ds^2 = \kappa_0^2\,\eta_{\mu\nu}dx^\mu dx^\nu + dy^2\,. \tag{4.1}$$

Physically, the parameter κ_0 is a 'warp factor' controlling the maximal velocity along the fifth dimension. The speed of light $\mathbf{v}^y_{\text{light}}$ along the extra dimension y is

$$\mathbf{v}^y_{\text{light}} = \kappa_0\,, \tag{4.2}$$

in units where on the four dimensional slices $y = const$ the speed of light $\mathbf{v}^{4d}_{\text{light}}$ is $\mathbf{v}^{4d}_{\text{light}} = 1$.

The dynamics of the brane in the extra dimension depends on the value of κ_0. Recall that, for Poincaré DBI Galileons, we are dealing with the brane action

$$S = \mathcal{N}\int d^4x\,\frac{1}{\gamma}\,\text{det}\left[\,\delta_\mu^\nu + c\,\gamma\left(\Pi_\mu^\nu - \gamma^2\,\partial_\mu\pi\,\partial^\lambda\pi\,\Pi_\lambda^\nu\right)\right], \tag{4.3}$$

where $\gamma^{-1} = \sqrt{\kappa_0^2 + X}$. The action is invariant under the scalar transformation

$$\delta\pi = \kappa_0^2\,\omega_\mu\,x^\mu + \pi\,\omega^\mu\,\partial_\mu\,\pi\,. \tag{4.4}$$

We can distinguish three physically distinct cases:

- Standard DBI Galileons: $\kappa_0 = 1$. The speed of light along the extra dimension is the same as in the four dimensional slices. The scalar action can be interpreted geometrically in terms of a brane probing an extra dimensional space time, as reviewed in section 3.2. It enjoys the symmetry (4.4) for $\kappa_0 = 1$ which generalises the Galilean symmetry adding a relativistic correction to it.

[4] One finds the Lagrangians for AdS DBI Galileons expanding the determinant of a slightly different action, given by

$$\mathcal{S} = \int d^4x\,\sqrt{-g}\,\left\{\text{det}\,[\delta^\mu_\nu - c\,\kappa_0\,g^{\mu\alpha}\,K_{\alpha\nu}] - \frac{6\,c^2\,\kappa_0^2}{\ell^2} + \frac{3\,c^3\,\kappa_0^3}{2\,\ell^2}\,K\right\}, \tag{3.48}$$

when setting $\kappa_0 = 1$. On the other hand, both actions (3.43) and (3.48) are invariant under the scalar symmetry (3.47) and have second order equations of motion.

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- 5d non relativistic limit: $\kappa_0 \to \infty$. In this limit, the speed of light along the extra dimension is infinite, see eq. (4.2). We expect then that there is no causal bound on the speed along the extra dimensions. Indeed the theory that we get corresponds to standard Galileons: the first derivatives of the brane modulus π are small, and relativistic corrections are negligible. In this limit, the factor $\gamma \simeq \kappa_0^{-1} \to 0$. To find a meaningful action, at the same time we then select large values for the constants $\mathcal{N}$ and c such that $\mathcal{N}/\kappa_0 = 2\bar{\mathcal{N}} = const$ and $c/\kappa_0 = \bar{c} = const$. In this limit, we then find the following action for the system

$$S_{\kappa_0 \to \infty} = \bar{\mathcal{N}} \int d^4x \, (\partial\pi)^2 \, \det\left[\delta_\mu^\nu + \bar{c}\,\Pi_\mu^\nu\right] \tag{4.5}$$

plus a total derivative, which corresponds to the standard Galileon action of eq. (2.1).

- 5d extreme relativistic limit: $\kappa_0 \to 0$. In this limit, the speed of light along the extra dimension y vanishes[5]: this is a peculiar limit, where causality forbids a motion along the extra dimension y. The brane action results

$$S_{\kappa_0 \to 0} = \mathcal{N} \int d^4x \sqrt{X} \, \det\left[\delta_\nu^\mu + c\frac{1}{\sqrt{X}}\left(\delta_\rho^\mu - \frac{\partial^\mu\pi\partial_\rho\pi}{X}\right)\partial^\rho\partial_\nu\pi\right]. \tag{4.6}$$

In order to have a well defined square root, $X > 0$, and this implies that we need to focus brane actions with space like scalar first derivatives $\partial\pi$. In this extreme relativistic limit, the action has still a symmetry

$$\delta\pi = \pi\,\omega^\mu\,\partial_\mu\,\pi\,, \tag{4.7}$$

which corresponds to the relativistic, field dependent contributions of eq. (4.4) with $\kappa_0 = 0$. We will focus on this system in what follows.

4.1 The scalar theory in the extreme relativistic limit

Action (4.6) geometrically describes a brane configuration embedded in a five dimensional space time where the speed of light along the extra dimension, $v^y_{\rm light} = \kappa_0$, vanishes since $\kappa_0 \to 0$: causality would seem to require us to select $X > 0$ in order to have a well defined square root. Quantities $X > 0$ and $X < 0$ are respectively space like and time like with respect to the four vector $\partial\pi$ relative to the four flat dimensions at $y = const$.

On the other hand, at the formal level, the system allows us to also consider the case where $X < 0$, that is a time like scalar derivative $\partial\pi$. If $X < 0$, we can define $\mathcal{N}$ and c to be purely imaginary (say, $\mathcal{N} = i\,\tilde{\mathcal{N}}$, $c = -i\,\tilde{c}$ with $\tilde{\mathcal{N}}$, $\tilde{c}$ real constants) so to compensate for the imaginary 'i factor' associated with the square root, and get a real action. The action for a time like scalar, $X < 0$, has the same structure as before:

$$S = \tilde{\mathcal{N}} \int d^4x \sqrt{-X} \, \det\left[\delta_\nu^\mu + \frac{\tilde{c}}{\sqrt{-X}}\left(\delta_\rho^\mu - \frac{\partial^\mu\pi\partial_\rho\pi}{X}\right)\partial^\rho\partial_\nu\pi\right], \tag{4.8}$$

[5]This is reminiscent to what happens in a black hole geometry, where the 'speed of light' vanishes at the black hole ergosurface where the coefficient g_{tt} of the time coordinate vanishes. It would be interesting to pursue this analogy further and reformulate the $\kappa_0 \to 0$ limit as approaching a special point on some examples of 5d geometries.

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and enjoys the same symmetry as eq. (4.6). A possible geometrical interpretation for this set-up can be found considering a probe brane embedded in a five dimensional space time with *two* time directions. One is the usual T, the other is a time like (Wick rotated) version of the extra dimensional coordinate y, that we dub $\tilde{y}$. The five dimensional metric to consider is

$$ds^2 = -\kappa_0^2\, dT^2 + \kappa_0^2\, d\vec{X}^2 - d\tilde{y}^2 \tag{4.9}$$

We can define — analogously as explained in section 3.2.1 — an embedding $X^\mu = x^\mu$, $\tilde{y} = \pi$. Calculations can be carried on straightforwardly following the very same steps as section 3.2.1, finding that the action (3.17) leads to action (4.8), once substituting the new expressions for induced metric and extrinsic curvature. While such geometrical derivation of action (4.8) can be useful for determining symmetries and dualities for our system, its physical relevance deserves further study, since the physical meaning of bulk space-times with multiple time directions is not clear to us. On the other hand, let us point out that theories equipped with two time directions in extra dimensions have been considered in string/M-theory contexts, see e.g. [51] and references therein.

Expanding the determinants in eqs (4.6) and (4.8) we find four scalar Lagrangians in flat space

$$\mathcal{L}_1 = \Lambda^2 \sqrt{|X|} \tag{4.10}$$

$$\mathcal{L}_2 = \Lambda \left([\Pi] - \frac{1}{X}[\Phi] \right) \tag{4.11}$$

$$\mathcal{L}_3 = \frac{1}{\sqrt{|X|}} \left([\Pi]^2 - [\Pi^2] + \frac{2}{X} \left([\Phi^2] - [\Phi][\Pi] \right) \right) \tag{4.12}$$

$$\mathcal{L}_4 = \frac{1}{\Lambda X} \left([\Pi]^3 + 2[\Pi^3] - 3[\Pi^2][\Pi] + \frac{3}{X} \left(2[\Pi][\Phi^2] - 2[\Phi^3] - [\Phi][\Pi]^2 + [\Phi][\Pi^2] \right) \right), \tag{4.13}$$

which describe both the cases of X positive or negative. As before, we use the notation $[\Pi^n] = \operatorname{tr}(\Pi^n)$ and $[\Phi^n] = \operatorname{tr}(\partial\pi\, \Pi^n\, \partial\pi)$. We include an energy scale Λ to make explicit the dimension of each operator. Each of these four Lagrangians enjoys the scalar symmetry

$$\delta\pi = \pi\, w^\mu \partial_\mu \pi = \frac{1}{2} w^\mu \partial_\mu \pi^2 \tag{4.14}$$

with w^μ an arbitrary constant four vector, which leaves the action invariant up to boundary terms. This transformation lacks the linear 'coordinate dependent' part which characterises Galileon symmetries (the '$\delta\pi = w_\mu x^\mu$') hence the system is qualitatively different from Galileons, and we do not reduce to Galileon actions in any 'small derivative' limit. Additionally, the four actions are also connected by a duality, as discussed in section 3.2 (whose results remain valid in the $\kappa_0 \to 0$ limit).

Taken by themselves, these scalar actions are quite peculiar: there is no limit in which the scalar has standard kinetic terms, since standard kinetic terms are not compatible with symmetry (4.14). Some of these Lagrangians are non analytic, since they contain the square root of X, and all of them contain powers of $1/X$. On the other hand, such scalar theories might make sense when expanded around some background which solves the equations of motion, or by coupling to other fields like gravity. We now discuss a simple,

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concrete example to develop these points further, and to assess the physical relevance of these systems.

4.1.1 An explicit example: part I

In the time-like case $X < 0$ these theories seem to have problematic causal properties, which can be fixed by slightly breaking the scalar symmetry (as we are going to discuss now), or by enlarging the system by coupling it to other fields, as dynamical gravity (as we discuss in the next section, see in particular section 5.3.1).

We analyse a concrete example that is simple, but illustrative. We consider a linear combination $\mathcal{L}_C$ of the Lagrangians (4.10)–(4.13) with constant dimensionless coefficients

$$-\mathcal{L}_C = \alpha_1 \mathcal{L}_1 + \alpha_2 \mathcal{L}_2 + \alpha_3 \mathcal{L}_3 + \alpha_4 \mathcal{L}_4 \,. \tag{4.15}$$

We start determining some homogeneous background around which we expand our theory. Any scalar configuration which is linear in the coordinates solves the equations of motion. This can be checked by direct computation, or by using symmetry arguments. Denote with $\bar{\pi} = c_\mu x^\mu$ one scalar configuration, with c_μ arbitrary vector. Applying the scalar transformation (4.14), we obtain $\bar{\pi} + \delta\bar{\pi} = (1 + \omega^\rho c_\rho)\, c_\mu x^\mu$: so a symmetry transformation sends this configuration to an arbitrary other one with a linear profile, but with a different vector c_μ. Then, since $\bar{\pi} = 0$ is a solution, also any $\bar{\pi} = c_\mu x^\mu$ must be solution. In order to preserve three dimensional spatial isotropy, we select a background configuration that is linear in time:

$$\bar{\pi}(t) = P_0^2\, t \tag{4.16}$$

with P_0 an arbitrary constant with dimension of a mass. (We put a P_0^2 in the previous formula to assign the correct dimension to the scalar field.) We examine the dynamics of fluctuations around $\bar{\pi}(t)$:

$$\pi(t, \vec{x}) = \bar{\pi}(t) + \hat{\pi}(t, \vec{x}) \,. \tag{4.17}$$

An homogeneous background $\bar{\pi}(t)$ spontaneously breaks the symmetry (4.14) down to a residual symmetry

$$\delta\pi = \pi\, w^i \partial_i \pi \,, \tag{4.18}$$

with w^i an arbitrary *three* spatial vector. Indeed, transformation (4.18) leaves invariant any function $\bar{\pi}(t)$, and only acts on the fluctuations $\hat{\pi}(t, \vec{x})$ introduced in eq. (4.17). In the limit of small fluctuations, the residual symmetry (4.18) acts at linear order on $\hat{\pi}(t, \vec{x})$ as

$$\delta\hat{\pi}(t,\, \vec{x}) = \bar{\pi}(t)\, w^i \partial_i \hat{\pi}(t,\, \vec{x}) \,. \tag{4.19}$$

Expanding the combination $\mathcal{L}_C$ at quadratic order in small fluctuation $\hat{\pi}$ around the background $\bar{\pi}(t)$, we do not find a standard kinetic term for the scalar fluctuation, but instead the quantity

$$\mathcal{L}^{\text{quad}} = q(t)\, (\vec{\nabla}\hat{\pi})^2 \tag{4.20}$$

where $q(t)$ depends on $\bar{\pi}(t)$, and on the coefficients α_i characterising the combination $\mathcal{L}_C$ we selected. Such quadratic Lagrangian for fluctuations only contain spatial derivatives of

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$\hat{\pi}$, and lacks the time derivative piece $\dot{\hat{\pi}}^2$. This fact is easier to understand in terms of the residual symmetry (4.19): while the quadratic Lagrangian (4.20) is invariant under this transformation (up to boundary terms), a term like $\dot{\hat{\pi}}^2$ is not. The system described by the quadratic Lagrangian (4.20) is degenerated, and does not satisfy the conditions of Leray's theorem for having a well defined Cauchy's problem [52] (healthier degenerate systems will be discussed in the next section, when coupling with gravity).

A way out is to break *explicitly* symmetry (4.14), for example by including a standard kinetic term with small overall coefficient. We add such a term to our Lagrangian

$$-\tilde{\mathcal{L}}_C = \alpha_0 X + \alpha_1 \mathcal{L}_1 + \alpha_2 \mathcal{L}_2 + \alpha_3 \mathcal{L}_3 + \alpha_4 \mathcal{L}_4 \,. \tag{4.21}$$

The first term proportional to α_0 breaks symmetry (4.14); in this case, again our homogeneous solution $\bar{\pi}(t)$ of eq. (4.16) solves background equations of motion, since the term proportional to α_0 has a Galilean symmetry. Studying the dynamics of quadratic fluctuations, associated with this Lagrangian, we find a healthy kinetic term for the fluctuation $\hat{\pi}$ at quadratic level, if $\alpha_{0,1}$ are positive:

$$\tilde{\mathcal{L}}^{\rm quad} = \alpha_0 \left(\dot{\hat{\pi}}^2 - c_\pi^2 \, \partial_j \hat{\pi} \, \partial^j \hat{\pi} \right) \,, \tag{4.22}$$

$$c_\pi^2 = 1 - \frac{\alpha_1 \, \Lambda^2}{2 \, P_0^2 \, \alpha_0} \,. \tag{4.23}$$

By an appropriate choice of the quantities, $\alpha_{0,\,1}$ and P_0, we can ensure that

$$0 \, < \, c_\pi \, \le \, 1 \,, \tag{4.24}$$

so that fluctuations get healthy kinetic terms. We can also then canonically normalize the field

$$\hat{\pi} \, \to \, \frac{1}{\sqrt{2 \, \alpha_0}} \hat{\pi} \,, \tag{4.25}$$

so to have a canonical kinetic terms, with a speed of sound different from unity. In order to have a consistent system when α_0 is small — with $0 < c_\pi \le 1$ — we have to require that the energy scale of the background solution P_0 is larger than the scale entering Λ in the Lagrangian.

We can also proceed and examine higher order self interactions for the fluctuations. We limit to turn on the coefficients α_0, α_1 in eq. (4.21), while setting to zero the remaining α_i. After canonically normalize the field, as in eq. (4.25), we find that the Lagrangian expanded up to fourth order results

$$\tilde{\mathcal{L}} = \frac{1}{2} \left[\dot{\hat{\pi}}^2 - c_\pi^2 \, (\nabla \hat{\pi})^2 \right] - \frac{\sqrt{\alpha_0} \, \left(1 - c_\pi^2\right)^2}{\sqrt{2} \, \alpha_1 \, \Lambda^2} \, \dot{\hat{\pi}} \, (\nabla \hat{\pi})^2 + \frac{\left(1 - c_\pi^2\right)^3 \, \alpha_0 \, (\nabla \hat{\pi})^2 \, \left(4 \dot{\hat{\pi}}^2 + (\nabla \hat{\pi})^2 \right)}{4 \, \alpha_1^2 \, \Lambda^4} + \dots \tag{4.26}$$

with c_π given in eq. (4.23). As long as the sound speed lies in the interval (4.24), the system is defined also in a small α_0 regime, since interactions are suppressed.

On the other hand, symmetry (4.18) is explicitly broken by the contribution of the kinetic term. If we would like to recover the symmetry, in a regime where α_0 is very small, we

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need to go outside the safe interval (4.24) for the sound speed. Consider Lagrangian (4.26) in a regime where $\alpha_0 \to 0$, $c_\pi \to \infty$, such that the combination $\alpha_0\, c_\pi^2 = const \equiv 2\,\beta\,\alpha_1^2$ for some arbitrary constant β. Moreover, let us rescale $\hat{\pi} \to \hat{\pi}/c_\pi$. In such limit, the Lagrangian (4.26) becomes

$$\tilde{\mathcal{L}}_{\rm sym} = -\frac{1}{2}\,(\nabla\hat{\pi})^2 - \frac{\sqrt{\beta}}{\Lambda^2}\,\dot{\hat{\pi}}\,(\nabla\hat{\pi})^2 - \frac{\beta}{2\,\Lambda^4}\,(\nabla\hat{\pi})^2\,\left(4\dot{\hat{\pi}}^2 + (\nabla\hat{\pi})^2\right) + \dots \tag{4.27}$$

and each term respect symmetry (4.18) at the corresponding order in perturbations.

It would be interesting to analyse whether the symmetries and the properties of our scalar actions can protect their structure under corrections, for example against scalar self loops, leading to non-renormalization theorems as it happens for Galileons. In fact, this kind of questions have been recently reconsidered for a wide class of derivatively coupled theories [53], using simple yet powerful methods based on power counting techniques (see e.g. [54] for a review). The scalar transformation that we consider — eq. (4.14) — is a symmetry of the action only up to boundary terms. Usually set-ups with this property are protected under quantum corrections, as discussed in [53]. Breaking it spontaneously by selecting a non-trivial homogeneous scalar background should not spoil these features. An explicit symmetry breaking (as done by adding a kinetic term to the Lagrangian in eq. (4.21)) might still lead to a system where corrections can be kept under control: in the limit in which the explicit symmetry breaking parameters are small (for our previous example, $\alpha_0 \ll \alpha_1$), one expects quantum corrections to be small, at most proportional to α_0 and its powers [55]. It would be interesting to concretely develop these arguments in our specific example, where we know that, in the limit of α_0 small, the background profile P_0/Λ must be large. We will return to these issues from a different perspective in the next section 5, where we will study the related topic of what happens to our scalar system when coupling with gravity.

4.2 A generalization

If we consider a field redefinition

$$\pi \to f(\pi) \tag{4.28}$$

and apply it to action (4.8), we obtain a new action which explicit depends on π (and not only on its derivatives) thanks to an overall factor in front of the determinant. It is given by

$$S = \mathcal{N}\int d^4x\, f'(\pi)\sqrt{X}\,\det\left[\delta^\mu_\nu + c\frac{1}{\sqrt{X}}\left(\delta^\mu_\rho - \frac{\partial^\mu\pi\partial_\rho\pi}{X}\right)\partial^\rho\partial_\nu\pi\right]. \tag{4.29}$$

Such action satisfies a symmetry which generalizes (4.14), and is given by

$$\delta\pi = f(\pi)\,w^\mu\partial_\mu\pi \tag{4.30}$$

for constant vector w^μ. As a byproduct, this fact implies that the equations of motion associated with action (4.29) are invariant under constant rescaling of the field π: $\pi \to \lambda\pi$. This since in this case $f = \lambda\pi$, $f' = \lambda$, and the constant λ goes in front of the integral in eq. (4.29), without affecting the equations of motion.

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4.3 Extreme relativistic limit of DBI Galileons in AdS space

All what we said so far can be straightforwardly extended to the case of DBI Galileons embedded in AdS space, which is another system with an interesting geometrical interpretation (see section 3.3). Taking the $\kappa_0 \to 0$ limit of action (3.45), we get

$$S = \mathcal{N} \int d^4x \, e^{-3\pi/\ell} \sqrt{X} \, \det \left[\delta^\mu_\nu + c \frac{e^{-\pi/\ell}}{\sqrt{X}} \left(\delta^\mu_\rho - \frac{\partial^\mu \pi \partial_\rho \pi}{X} \right) \partial^\rho \partial_\nu \pi \right] . \tag{4.31}$$

This action is symmetric under the field-dependent transformation (the $\kappa_0 \to 0$ limit of eq. (3.47))

$$\delta\pi = \frac{\ell}{2} \left(e^{2\pi/\ell} - 1 \right) w^\mu \partial_\mu \pi \, , \tag{4.32}$$

for arbitrary constant vector ω^μ. Again, this transformation lacks the linear 'coordinate dependent' part which characterises Galileon symmetries (the '$\delta\pi = w_\mu x^\mu$'); hence the system is qualitatively different from Galileons. Also for the AdS case, the set-up admits a simple generalisation: the structure of action (4.31) is the same by doing a field redefinition

$$\pi \to -\ell \, \ln h(\pi), \tag{4.33}$$

for arbitrary function h. The action becomes

$$S = \int d^4x \, h^3 \, h' \sqrt{X} \, \det \left[\delta^\mu_\nu + c \frac{h}{\sqrt{X}} \left(\delta^\mu_\rho - \frac{\partial^\mu \pi \partial_\rho \pi}{X} \right) \partial^\rho \partial_\nu \pi \right] . \tag{4.34}$$

The associated symmetry becomes

$$\delta\pi = \frac{\ell}{2} \left(\frac{1}{h^2} - 1 \right) w^\mu \partial_\mu \pi \, . \tag{4.35}$$

The resulting action and symmetry, eqs (4.34) and (4.35), are similar, although not identical, to the system discussed in the previous subsection 4.2.

5 Minimal coupling with gravity: degenerate scalar-tensor theories

Our flat space 'extreme relativistic' Lagrangians with $\kappa_0 \to 0$ can be minimally coupled to gravity in a consistent way, by promoting the flat four dimensional slices to arbitrarily curved slices with dynamical four dimensional metric. This relates our systems to beyond Horndeski [21, 22] and extended scalar-tensor theories [24–26], providing a geometrical perspective to the latter systems.

5.1 From DBI Galileons to beyond Horndeski theories

We can minimally couple with gravity the extreme relativistic actions (4.6) and (4.31), promoting the flat four dimensional metric tensor $\eta_{\mu\nu}$ to a dynamical tensor $q_{\mu\nu}$, and writing respectively

$$S = \mathcal{N} \int d^4x \, \sqrt{-q} \, \sqrt{X} \, \det \left[\delta^\mu_\nu + c \frac{1}{\sqrt{X}} \left(\delta^\mu_\rho - \frac{\partial^\mu \pi \partial_\rho \pi}{X} \right) \nabla^\rho \partial_\nu \pi \right] , \tag{5.1}$$

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and

$$S=\mathcal{N}\int d^4x\, e^{-3\pi/\ell}\sqrt{-q}\,\sqrt{X}\,\det\left[\delta^\mu_\nu+c\frac{e^{-\pi/\ell}}{\sqrt{X}}\left(\delta^\mu_\rho-\frac{\partial^\mu\pi\partial_\rho\pi}{X}\right)\nabla^\rho\partial_\nu\pi\right]. \tag{5.2}$$

Eq. (5.2) reduces to (5.1) in the limit of infinite AdS radius $\ell\to\infty$. For simplicity, in what follows we focus on the Poincaré limit $\ell\to\infty$, although similar considerations can be done for the AdS case as well.

Interestingly, the scalar-tensor theories one obtains by expanding the determinants in the previous expressions are consistent, although the associated EOMs are generally of higher order. The set of Lagrangians one finds corresponds to Lagrangian densities (4.10)–(4.13), with standard derivatives replaced by covariant derivatives. Such scalar-tensor theories belong to the class of beyond Horndeski theories [21, 22]. It is easier to check this fact using the idempotent 'projection tensor'

$$P^\mu_\nu=\delta^\mu_\nu-\frac{\nabla^\mu\pi\nabla_\nu\pi}{X}\,, \tag{5.3}$$

which satisfies the relation $P^\nu_\mu\,\nabla_\nu\pi=0$. Using this quantity, it is possible to prove (see section IIB of [29]) that the theories of beyond Horndeski can also be expressed in terms of a determinantal expression. They can be written as

$$S=\int d^4x\sqrt{-q}\,A(\pi,X)\,\det\left[\delta^\mu_\nu+B(\pi,X)P^\rho_\nu\,\nabla_\rho\partial^\mu\,\pi\right], \tag{5.4}$$

where A, B are arbitrary function of π, X, and ∇ denotes covariant derivative with respect to a 4d metric $q_{\mu\nu}$. Although these theories are characterised by EOMs of order higher than two, they propagate at most three dofs. Action (5.2) belongs to this class of theories: hence it does not propagate more than three degrees of freedom. We emphasize that actions as (5.4) do not need supplementary gravitational counterterms for being consistent. This fact relates a limit of DBI Galileons with beyond Horndenski.

We can also investigate geometrically the covariantization procedure for the extreme relativistic limit of DBI Galileons, in terms of a probe brane in an five dimensional set-up. Recall that for studying Poincaré DBI Galileons we consider a five dimensional metric as

$$ds^2_{(5)}=\kappa_0^2\,\eta_{\mu\nu}\,dX^\mu dX^\nu+dy^2 \tag{5.5}$$

i.e. the four dimensional slices y =constant have flat metric. We discuss here the possibility of promoting the metric of 4d slices to a dynamical field, writing

$$ds^2_{(5)}\;=\kappa_0^2\,q_{\mu\nu}\,dX^\mu dX^\nu+dy^2 \tag{5.6}$$

with $q_{\mu\nu}$ a dynamical tensor. We choose again the usual foliation associated with a brane embedded on this geometry, $X^\mu=\kappa_0\,x^\mu$, $y=\pi$. The induced metric on the brane is

$$g_{\mu\nu}=\kappa_0^2\,q_{\mu\nu}+\partial_\mu\pi\partial_\nu\pi\,, \tag{5.7}$$

with determinant

$$\sqrt{-g}=\kappa_0^3\,\sqrt{-q}\,\sqrt{\kappa_0^2+X}\,, \tag{5.8}$$

where

$$X = q^{\mu\nu}\nabla_\mu\pi\nabla_\nu\pi\,. \tag{5.9}$$

Its inverse is (4d indexes are raised with $q^{\mu\nu}$)

$$g^{\mu\nu} = \frac{1}{\kappa_0^2}\left(q^{\mu\nu} - \frac{\partial^\mu\pi\,\partial^\nu\pi}{\kappa_0^2 + X}\right)\,. \tag{5.10}$$

The extrinsic curvature tensor is

$$K_{\mu\nu} = -\frac{\kappa_0}{\sqrt{\kappa_0^2 + X}}\left(\nabla_\mu\nabla_\nu\pi\right)\,. \tag{5.11}$$

We now consider the extreme relativistic limit

$$\kappa_0 \to 0\,. \tag{5.12}$$

Most of the geometrical quantities written above become singular in this limit: or they vanish, or they become infinite. On the other hand, the determinantal action (3.45)

$$\mathcal{S} = \frac{\mathcal{N}}{\kappa_0^3}\int d^4x\,\sqrt{-g}\,\det\left[\delta_\mu^\nu - c\,\kappa_0\,g^{\mu\alpha}\,K_{\alpha\nu}\right]\,, \tag{5.13}$$

as a whole has a smooth limit, since taking $\kappa_0 \to 0$ we obtain the regular expression

$$S = \mathcal{N}\int d^4x\,\sqrt{-q}\,\sqrt{X}\,\det\left[\delta_\nu^\mu + c\frac{1}{\sqrt{X}}\left(\delta_\rho^\mu - \frac{\partial^\mu\pi\partial_\rho\pi}{X}\right)\nabla^\rho\partial_\nu\pi\right]\,, \tag{5.14}$$

which indeed coincides with action (5.1). So we learn that when taking the limit $\kappa_0 \to 0$, a brane action built with appropriate combinations of geometrical quantities leads to sensible scalar-tensor theories. We then find a connection between certain DBI Galileons and a special case of beyond Horndeski theories.

5.2 The duality

We now discuss a way to extend the duality transformation presented in section 3.2.2 to the case of curved space time. Our aim is to proceed as much as possible along the same steps we followed in discussing duality with non-dynamical gravity. When coupling the scalar with dynamical gravity, however, defining the action of a duality is a very delicate matter, as already pointed out for the case of standard Galileons in [46] (see also [44]). The main issue is how to define the transformation of the dynamical metric $q_{\mu\nu}(x)$ under duality. We study here a particular case of duality, which is nevertheless sufficient for finding a novel relation among the different Lagrangians of eqs (4.10)–(4.13), once they are minimally coupled with gravity.

Consider the field dependent map among two sets of coordinates

$$x^\mu \;\Rightarrow\; \tilde{x}^\mu = x^\mu + \frac{1}{\sqrt{X}}\,\partial^\mu\pi\,, \tag{5.15}$$

(where, as for the case of Galileons, we choose units which set to one the parameter Λ_S which is needed for dimensional reasons.) Recall that $X = q^{\mu\nu}\,\nabla_\mu\pi\,\nabla_\nu\pi$.

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The duality is defined as the transformation which maps the line element dx^μ of the first set of coordinates, with the line element $d\tilde{x}^\mu$ of the second set of coordinates as follows

$$dx^\mu \quad \Rightarrow \quad d\,\tilde{x}^\mu = d\,x^\mu + d\left(\frac{1}{\sqrt{X}}\,\partial^\mu\pi\right) = d\,x^\mu + \nabla_\nu\left(\frac{1}{\sqrt{X}}\,\partial^\mu\pi\right)\,dx^\nu\,. \tag{5.16}$$

The transformation matrix between these line elements is

$$J^\mu_\nu \equiv \frac{d\tilde{x}^\mu}{dx^\nu} = \delta^\mu_\nu + \nabla_\nu\left(\frac{1}{\sqrt{X}}\partial^\mu\pi\right)\,. \tag{5.17}$$

We demand that the dual metric $\tilde{q}_{\mu\nu}(\tilde{x})$ is a *scalar* under duality:

$$\tilde{q}_{\mu\nu}(\tilde{x}) = q_{\mu\nu}(x)\,. \tag{5.18}$$

We also demand that there exists a dual scalar field $\tilde{\pi}$, which we can use for mapping back the coordinates $\tilde{x}^\mu$ to x^μ:

$$\tilde{x}^\mu \;\Rightarrow\; x^\mu = \tilde{x}^\mu - \frac{1}{\sqrt{\tilde{X}}}\,\tilde{\partial}^\mu\tilde{\pi}\,. \tag{5.19}$$

Comparing eqs (5.19) and (5.15), and using (5.18), we find that the simplest way to satisfy our conditions is to impose that the derivative of π is a scalar under duality

$$\tilde{\partial}^\mu\tilde{\pi}(\tilde{x}) = \partial^\mu\pi(x)\,. \tag{5.20}$$

This is analogous to what we have seen in the case of flat metric; also, these results imply that the *induced* four dimensional metric $g_{\mu\nu}$ is scalar under duality (indexes are raised/lowered with curved metric $q_{\mu\nu}$)

$$g_{\mu\nu}(x) = \kappa_0^2 q_{\mu\nu} + \partial_\mu\pi\partial_\nu\pi \;\;\Rightarrow\;\; \tilde{g}_{\mu\nu}(\tilde{x}) = g_{\mu\nu}(x)\,, \tag{5.21}$$

as happens in flat space.

Eq. (5.20) implies

$$\tilde{\partial}_\mu\tilde{\pi}\,d\tilde{x}^\mu = \;\partial_\mu\pi\,J^\mu_\nu\,dx^\nu = \partial_\mu\pi\,dx^\mu + \partial_\mu\pi\,\nabla_\nu\left(\gamma\partial^\mu\pi\right)dx^\nu\,. \tag{5.22}$$

Integrating both sides of the previous relation, we find a non local relation among the field π and its dual

$$\pi(x) \;\Rightarrow\; \tilde{\pi}(\tilde{x}) = \pi(x) + \int^x d\bar{x}^\lambda\,\nabla_\lambda\left(\gamma\partial^\rho\pi\right)\,\partial_\rho\pi\,. \tag{5.23}$$

The results so far imply that, under the action of the duality,

$$d^4x \Rightarrow d^4\tilde{x} = (\det J)\;d^4x = \det\left[\delta^\mu_\nu + \nabla_\nu\left(\frac{1}{\sqrt{X}}\partial^\mu\pi\right)\right]d^4x\,, \tag{5.24}$$

$$X \Rightarrow \tilde{X} = X\,. \tag{5.25}$$

We meet a serious problem however, with respect to how to define in a consistent way the dual of the second derivative of the scalar field. This since the expression $\nabla_\mu\,\partial^\nu\,\pi$ contains a covariant derivative, which does not transform properly as a vector, because the metric does not transform as a tensor under duality. Hence, following this route, we can not define a dual version of the entire set of scalar-tensor actions we examined.

Less ambitiously, we can nevertheless define a dual version of the action

$$S = \int d^4x \sqrt{-g} \tag{5.26}$$

$$= \int d^4x \sqrt{-q}\,\sqrt{X} \tag{5.27}$$

associated with Lagrangian (4.10) minimally coupled with gravity. It is

$$\tilde{S} = \int d^4\tilde{x} \sqrt{-\tilde{q}}\,\sqrt{\tilde{X}} \tag{5.28}$$

$$= \int d^4x \sqrt{-q}\,\sqrt{X}\,\det\left[\delta^\mu_\nu + \nabla_\nu\left(\frac{1}{\sqrt{X}}\partial^\mu\pi\right)\right] , \tag{5.29}$$

which is a particular case of eq. (5.1) for $c = 1$. Expanding the determinant, we find a minimal coupling with gravity of all the actions (4.10)–(4.13), with fixed coefficients: our duality maps scalar and metric fields in such a way to generate all our actions starting from the simplest among them. It would be interesting to extend these findings to determine an action of the duality which applies to all the scalar-tensor actions we have studied.

5.3 The symmetry

We can now ask about the fate of the flat space symmetries we discussed in the previous sections. In the presence of dynamical gravity, as in action (5.1), we normally break the scalar symmetry

$$\delta\pi = \pi\, w^\mu \partial_\mu \pi \tag{5.30}$$

since the equations of motion for the dynamical metric field are not necessarily invariant under such transformation. On the other hand, two classes of general arguments can be made. The first set of considerations concerns systems in which gravity is still not dynamical, but with a non trivial fixed metric $\bar{q}_{\mu\nu}$ on the 4d slices. If such space 5d space time has still some isometries, it is possible to use the techniques of [13, 14] for constructing a scalar transformation which generalises (5.30) and is a symmetry of the action. If, on the other hand, the 5d space time does not admit any isometry, it might still be possible to describe it as a 'small perturbation' of some symmetric space time. A second kind of considerations can be made when gravity becomes dynamical, and the metric $q_{\mu\nu}(x)$ on the 4d slices is a dynamical field with its own equations of motion. In this set-up, the equations of motion for $q_{\mu\nu}$ normally break the scalar symmetry (5.30). In some situations, the symmetry could be broken by gravity in a soft way, and some of its properties might be maintained. Such arguments have been used in phenomenological approaches of Galileons to cosmology, see e.g. [16–18]. Moreover, as advocated for example in [12], one could try to promote the global 5d Poincaré and AdS symmetries considered so far to local symmetries, and analyse their physical consequences for the brane induced action. It would be interesting to develop these considerations by studying concrete systems, to explicitly understand the fate of symmetries when gravity is turned on. In the present context, we limit ourselves to reconsider the simple, explicit example of section 4.1.1, for understanding the behavior of fluctuations when the set-up is coupled with dynamical gravity, and at what extent the symmetry is broken.

5.3.1 An explicit example: part II

We focus on time like systems with $X<0$, and reconsider the example of section 4.1.1 in the present context. We analyse the dynamics of fluctuations around a time dependent homogeneous background $\bar{\pi}(t)$ which solves the equations of motion for scalar and metric fields. We have seen in section 4.1.1 that, in absence of gravity, a residual symmetry prevents us from having canonical kinetic terms for scalar fluctuations around our background profile. When gravity is turned on, the situation can be improved. The symmetry gets dynamically broken by gravitational effects, and fluctuations acquire kinetic terms, thanks to a kinetic mixing among the scalar and the metric sectors. Let us see these facts more explicitly.

We focus for simplicity on a scalar-tensor action based of Lagrangian density $\mathcal{L}_3$ of eq. (4.12), to which we add an Einstein-Hilbert (EH) term

$$\mathcal{S}_3=\int d^4x\,\sqrt{-q}\left[\mu^2\,R-\frac{\alpha_3}{\sqrt{|X|}}\left([\Pi]^2-[\Pi^2]+\frac{2}{X}\left([\Phi^2]-[\Phi][\Pi]\right)\right)\right]\,, \tag{5.31}$$

where all derivatives are covariant derivatives, μ is a constant with dimension mass, and α_3 a dimensionless constant. An EH term is included since it does not break the symmetry further than what is done by the covariant derivatives in (5.31): it will be nevertheless important when discussing fluctuations. An EH term does not introduce ghosts in this case: this since such term belongs to quartic Horndeski theories, it can be merged with no harm with the combination proportional to α_3, which belong to quartic beyond Horndeski (see e.g. [22, 24, 29] for details and related discussions). The resulting theory is disformally related to Horndeski, but only in absence of matter; we do not discuss here such disformal transformation.

We are interested to study configurations which admit Minkowski space as metric background. Einstein equations, when evaluated on a Minkowski background, impose the following condition on the scalar field:

$$\begin{aligned}0=&\frac{1}{2}\eta_{\mu\nu}\left([\Pi]^2-[\Pi^2]+2\pi^{,\alpha}[\Pi]_{,\alpha}\right)-\pi_{,\alpha\mu\nu}\pi^{,\alpha}-[\Pi]\Pi_{\mu\nu}\\&+\frac{1}{X}\Big[-\eta_{\mu\nu}\left([\Pi][\Phi]+\pi^{,\alpha}\pi^{,\beta}\pi^{,\gamma}\pi_{,\alpha\beta\gamma}+3[\Phi^2]\right)+[\Phi]\Pi_{\mu\nu}+2\pi^{,\alpha}\pi^{,\beta}\Pi_{\alpha\mu}\Pi_{\beta\nu}\\&+2\pi^{,\alpha}\pi^{,\beta}\pi_{,\alpha\beta(\nu}\pi_{\mu)}+2\pi^{,\alpha}\Pi_{\alpha\beta}\Pi^{\beta}{}_{(\mu}\pi_{,\nu)}-\pi^{,\alpha}\pi_{,\mu}\pi_{,\nu}[\Pi]_{,\alpha}+\frac{1}{2}([\Pi]^2-3[\Pi^2])\pi_{,\mu}\pi_{,\nu}\Big]\\&+\frac{1}{X^2}\left[3g_{\mu\nu}[\Phi]^2-6[\Phi]\pi^{,\alpha}\Pi_{\alpha(\mu}\pi_{,\nu)}+3[\Phi^2]\pi_{,\mu}\pi_{,\nu}\right]\,.\end{aligned} \tag{5.32}$$

Such condition breaks the scalar symmetry as in (5.30). On the other hand such system of equations still admits a scalar solution which is linear in the coordinates in Minkowski space:

$$\pi=c_\mu x^\mu \tag{5.33}$$

for arbitrary constant vector c_μ. This because such configuration satisfies the scalar equation (as explained section 4.1.1) and at the same time satisfies condition (5.32) (because each of its terms contain second derivatives, hence it vanishes when evaluated on a linear scalar configuration). It would be interesting to investigate whether this fact can be associated with some remnant of a symmetry. Hence we are allowed to select a time dependent homogeneous profile

$$\pi_0(t) = P_0^2 t \tag{5.34}$$

(see eq. (4.16)) as background scalar solution, with P_0 is some arbitrary parameter with dimension of a mass.

We study the dynamics of fluctuations (scalar, tensor) around this scalar profile and Minkowski space. Scalar and tensor fluctuations are defined at linear level around our background as:

$$q_{\mu\nu}\, dx^\mu dx^\nu \,=\, -\,(1+2N)\; dt^2 + 2\, B_{,i}\, dx^i\, dt + [(1+2\zeta)\,\delta_{ij} + h_{ij}]\; dx^i\, dx^j \,, \tag{5.35}$$

$$\pi \,=\, P_0^2\, t + \hat{\pi}\,. \tag{5.36}$$

Here N, B are the standard ADM constraints, $\hat{\pi}$, ζ scalar fluctuations, and h_{ij} transverse traceless tensor fluctuations. Constraint equations impose the following relations

$$N \,=\, 0\,, \tag{5.37}$$

$$B \,=\, \frac{2\,\alpha_3\,\hat{\pi}}{\mu^2+2P_0^2\alpha_3} + \psi \qquad \text{with} \qquad \nabla^2\psi = \frac{6\,\alpha_3\,\ddot{\hat{\pi}}}{(\mu^2+2P_0^2\,\alpha_3)}\,, \tag{5.38}$$

$$\dot{\hat{\pi}} \,=\, -\frac{\mu^2+2P_0^2\alpha_3}{2\alpha_3}\zeta\,. \tag{5.39}$$

After imposing the constraints, we find that the quadratic Lagrangian for scalar fluctuations contain only one propagating mode, ζ, whose quadratic Lagrangian is

$$\mathcal{L}^{(2)}_\zeta = 6\alpha_3\, P_0^2 \left[\dot{\zeta}^2 - \left(\frac{\mu^2+2\alpha_3 P_0^2}{3\,\alpha_3 P_0^2}\right)(\partial_i\zeta)^2\right]\,. \tag{5.40}$$

The mode still propagates in the limit $\mu\to 0$: the kinetic mixing among scalar fluctuations and the constraints, induced by the covariant derivatives in action (5.31) is sufficient for fully breaking the symmetry and give dynamics to scalar fluctuations. In order to avoid ghosts, one imposes $\alpha_3>0$.

The quadratic action for tensors, on the other hand, results

$$\mathcal{L}^{(2)}_h = \frac{\mu^2+\alpha_3\,P_0^2}{4}\left(\dot{h}_{ij}^2 - \frac{\mu^2}{\mu^2+\alpha_3 P_0^2}\,(\partial_l h_{ij})^2\right)\,. \tag{5.41}$$

In order for tensors to propagate with no strong coupling issues (as pointed out in [28, 56]), we need $\mu\neq 0$. The resulting system propagates three healthy degrees of freedom. Notice that both the sound speeds c_ζ, c_h are less than one, if α_3, μ^2 are positive quantities. It would be interesting to understand whether gravity breaks the scalar symmetry (5.30) in some spontaneous way, and whether (around Minkowski space) there are some remnants of the scalar symmetry that are also a symmetry of the gravitational equations of motion (5.32). We plan to investigate this subject in a separate publication.

5.4 The relation with a broader class of EST theories

Beyond Horndeski are not the only scalar-tensor theories, with higher order equations of motion, which are made consistent by the existence of primary constraints preventing the propagation of additional degrees of freedom. Generalisations of this case have been studied recently, and have been dubbed EST [26] or DHOST [25], using an approach developed by Langlois and Noui [24]. In particular, in [26] it has been pointed out that a class of consistent extensions of beyond Horndeski theories can be built in terms of the projection operators P_μ^ν introduced in (5.3).

We introduce the two index tensor

$$Q_\mu^\nu \equiv P_\mu^\alpha \nabla_\alpha \nabla^\nu \pi \,, \tag{5.42}$$

and consider a scalar-tensor theory described by an action which is a combination of scalar quantities formed with $Q_{\mu\nu}$, like

$$S = \int d^4x \sqrt{-q} \left[A_1(\pi, X)\, Q_\mu^\mu + A_2(\pi, X) \left(Q_\mu^\mu \right)^2 + A_3(\pi, X) \left(Q_\mu^\nu \, Q_\nu^\mu \right) + \dots \right] \tag{5.43}$$

for arbitrary functions A_i. Thanks to the existence of a primary constraint, these actions propagate at most three degrees of freedom. See appendix B — based on [26] — for full details.

At the light of these facts, we can use the results we obtained in the previous sections to determine a geometrical perspective for these particular cases of EST theories. We consider a probe brane in AdS bulk, as described in section 3.3. We have seen in the previous subsection that the $\kappa_0 \to 0$ limit of the combination $\kappa_0\, g^{\mu\alpha} K_{\alpha\nu}$ reads

$$\lim_{\kappa_0 \to 0} \kappa_0\, g^{\mu\alpha} K_{\alpha\nu} = \frac{e^{-\pi/\ell}}{\sqrt{X}} P_\alpha^\mu \, \nabla^\alpha \partial_\nu \pi = \frac{e^{-\pi/\ell}}{\sqrt{X}} \, Q_\nu^\mu \tag{5.44}$$

So it is proportional to Q_ν^μ. (The limit $\ell \to \infty$ corresponds to a brane embedded in a Poincaré bulk.) On the other hand,

$$\frac{1}{\kappa_0^3} \sqrt{-g} = \sqrt{-q}\, e^{-3\pi/\ell} \sqrt{\kappa_0^2\, e^{-2\pi/\ell} + X} \tag{5.45}$$

This implies that any probe brane action built as a scalar related to $\kappa_0\, g^{\mu\alpha} K_{\alpha\nu}$ automatically propagates at most three degrees of freedom in the limit $\kappa_0 \to 0$, being a special case of an EST action (5.43). For example, using the results of the previous sections, any action of the form (a_i are constant parameters)

$$\begin{aligned} S = \lim_{\kappa_0 \to 0} \frac{\mathcal{N}}{\kappa_0^3} \int d^4x \sqrt{-g} \Big[& 1 + a_1 \, \mathrm{tr} \left(\kappa_0\, g^{\mu\alpha} K_{\alpha\nu} \right) \\ & + a_2 \, \mathrm{tr} \left(\kappa_0\, g^{\mu\alpha} K_{\alpha\nu} \right)^2 + a_3 \, \mathrm{tr}^2 \left(\kappa_0\, g^{\mu\alpha} K_{\alpha\nu} \right) + \dots \Big] \end{aligned} \tag{5.46}$$

belongs to EST theories, and consequently propagates no more than three degrees of freedom.

Do these actions satisfy some symmetries in certain limits, which can protect their structure against loop corrections, and lead for example to non renormalization theorems?

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It depends, and a geometrical approach in terms of a brane probing an extra dimensional space can be useful to investigate this question. First, let us discuss the case of gravity *not* dynamical. If these theories correspond to super relativistic ($\kappa_0 \to 0$) limits of probe brane actions, the existence of symmetries depend on the presence of isometries in the bulk space time, as discussed in [13, 14] and reviewed in the previous sections. However, in general, actions as eq. (5.46) do not admit a 'decoupling limit' around Minkowski space where gravitational dynamics can be set to zero and only scalar dynamics can be considered (unless they reduce to beyond Horndeski), since Minkowski space is not a solution of the equations. On the other hand, scalar theories can be well defined around some non trivial backgrounds with isometries, and exhibit new symmetries which are different from the ones considered here. Second, for the case with dynamical gravity, the same considerations of the previous sections apply. Gravity tends to break all scalar symmetries, but there might be cases where such symmetries are broken only in a soft way, or generalisations which promote the scalar symmetries to full scalar-tensor symmetries. We intend to develop these interesting issues in a future analysis.

6 Discussion

In this work, we investigated a geometrical approach to degenerate scalar-tensor theories, with the main aim to investigate symmetries and dualities that they satisfy. Using such view point, we found a connection between beyond Horndeski (and more general degenerate scalar-tensor theories of gravity) and a certain limit of DBI Galileons.

We started presenting a perspective on DBI Galileons based on a determinantal approach. In absence of dynamical gravity, a particular limit of DBI Galileons — which we called extreme relativistic — leads to classes of scalar theories with a field dependent symmetry, that are connected by dualities. These theories reveal problematic properties when one computes the kinetic terms of fluctuations around a given background. Such problems can be tamed by weakly breaking the symmetry, by hand or by coupling the scalar theory to gravity. In the latter case, we showed that a minimal covariantization of DBI Galileons in the extreme relativistic limit leads to beyond Horndeski systems, or more in general to degenerate scalar-tensor theories which are consistent despite having equations motion of order higher than two. Our results indicate that degenerate scalar-tensor theories can admit a geometrical interpretation in terms of particular limits of DBI Galileon set-ups, and that (in absence of dynamical gravity) they enjoy symmetries which are different from Galileons. Moreover, different special cases of beyond Horndeski theories are connected by a duality, in some cases also with dynamical gravity. Our results can be helpful for assessing the stability properties or understanding the non-perturbative structure of systems based on degenerate scalar-tensor theories.

Our results can be extended in several directions. Our geometrical construction of degenerate scalar-tensor theories in terms of branes probing extra dimensions indicates that special theories can be obtained when the brane probes specific points in the extra dimensions, where the coefficient in front of the time coordinate vanishes. It would be interesting to examine this observation further, investigating in more general terms scalar-tensor theories obtained from branes placed in special locations of the embedding space time. Other possible developments concern symmetries and dualities. We have shown that the scalar symmetry is normally broken when gravity is made dynamical: it would be interesting to find concrete systems or situations — specific subclasses of our theories, for example expanded around specific configurations — where the symmetry breaking can be soft and controllable, and some of the features of the symmetry can be maintained. Also, we have studied some special case of duality when gravity is dynamical: it would be interesting to extend the discussion to study dualities connecting other examples of degenerate scalar-tensor theories. Finally, it would be interesting to study whether additional fields can be included in these systems, still preserving the properties that we determined. We plan to investigate these questions in future studies.

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A de Sitter case

In this appendix we discuss an alternative, model dependent way to incorporate a case of beyond Horndeski in a brane world scenario described by an action of the same determinantal form that we introduced in the main body of the text, e.g. eq. (3.21). The physical effect of the limit that we discuss here is the same as in the case $\kappa \to 0$: to get a situation where the derivative terms of π dominate the induced metric. However, it is important to emphasize that this alternative limit is formally different from $\kappa \to 0$ and only works for specific brane/bulk configurations.

By construction, action (3.21) inherits a global symmetry from the Killing symmetries of the bulk [13, 14], and propagates the right number of degrees of freedom in the limit to beyond Horndeski. The explicit form of the scalar symmetry depends on the isometries of the bulk metric and the geometry of the brane, as explained in [14], where maximally symmetric embeddings are worked out in detail. Among all the possible configurations, there are two that admit a limit $f(\pi) \to 0$ controlled only by the (A)dS radius, irrespectively of the value of κ_0. These two cases are the following:

1. A Minkowski brane embedded in an AdS_5 bulk. This is the same geometrical configuration that we discussed in section 3.3. Here $f(\pi) = e^{-\pi/\ell}$, and the limit $f \to 0$ is achieved when $\ell \to 0$ and $\pi > 0$. Under these considerations, the results presented in section 3.3 are recovered by redefining $\pi = \pi/\kappa_0$ and $\ell = \kappa_0 \ell$.

2. A dS_4 brane embedded in a dS_5 bulk. Here $f = \ell \sin(\pi/\ell)$ and the limit $f \to 0$ corresponds to $\ell \to 0$. In contrast to the previous case, this limit cannot be related

to a limit taken with κ_0 by a redefinition of ℓ and π. We describe this model in detail below.

The set-up of point 2 has the induced metric

$$g_{\mu\nu} = f(\pi)^2 q_{\mu\nu} + \nabla_\mu\pi\nabla_\nu\pi\,, \tag{A.1}$$

with $f = \ell\sin\left(\frac{\pi}{\ell}\right)$ and $q_{\mu\nu}$ the metric on the dS_4 slices that foliate the dS_5 bulk. Indices are raised and lowered with $q_{\mu\nu}$. The extrinsic curvature $K_{\mu\nu}$ is constructed according to (3.42), but replacing $\eta_{\mu\nu}$ with $q_{\mu\nu}$. The contraction of the matrix inverse of (A.1) with the extrinsic curvature takes the form

$$g^{\mu\alpha}K_{\alpha\nu} = \frac{\gamma}{f^2}\left(\delta^\mu_\alpha - \frac{\gamma^2}{f^2}\nabla^\mu\pi\nabla_\alpha\pi\right)\left(-\nabla^\alpha\nabla_\nu\pi + ff'\delta^\alpha_\nu + 2\frac{f}{f'}\nabla^\alpha\pi\nabla_\nu\pi\right), \tag{A.2}$$

with $\gamma = f/\sqrt{f^2+X}$. For this embedding, the action can be put in beyond Horndeski form by taking the limit $\ell\to 0$. To see this, first note that

$$\lim_{\ell\to 0}\left(\delta^\mu_\alpha - \frac{\gamma^2}{f^2}\nabla^\mu\pi\nabla_\alpha\pi\right)\nabla^\alpha\pi = 0\,. \tag{A.3}$$

Now, since $ff' \sim \mathcal{O}(\ell)$, in the limit $\ell\to 0$ the action is dominated by

$$S_{dSG} = \int d^4x\sqrt{-q}\,\ell^3\sin^3(\pi/\ell)\sqrt{X}\det\left[\delta^\mu_\nu + \frac{c_1}{\ell\sin(\pi/\ell)\sqrt{X}}\left(\delta^\mu_\alpha - \frac{\nabla^\mu\pi\nabla_\alpha\pi}{X}\right)\nabla^\alpha\nabla_\nu\pi\right], \tag{A.4}$$

which is well-defined if $|c_1/\ell|$ is kept finite when taking the limit $\ell\to 0$. This action belongs to beyond Horndeski, cf. (5.4). The symmetries of this action — the limit $\ell\to 0$ of the transformations generated by the Killing vectors of a dS_4 brane embedded in a dS_5 bulk derived in [14] — are

$$\delta_+\pi = -\cot\left(\frac{\pi}{\ell}\right)\partial_u\pi, \tag{A.5}$$

$$\delta_-\pi = -(u^2+y^2)\cot\left(\frac{\pi}{\ell}\right)\partial_u\pi - 2u\cot\left(\frac{\pi}{\ell}\right)y^i\partial_i\pi, \tag{A.6}$$

$$\delta_i\pi = -y_i\cot\left(\frac{\pi}{\ell}\right)\partial_u\pi - u\cot\left(\frac{\pi}{\ell}\right)\partial_i\pi, \tag{A.7}$$

where u and y^i ($y^2 = \delta_{ij}y^iy^j$, $i = 1,2,3$) are coordinates defined in terms of the bulk coordinates, such that the induced metric can be written as

$$ds^2 = d\pi^2 + \ell^2\sin^2\left(\frac{\pi}{\ell}\right)\left[\frac{1}{u^2}(-du^2+dy^2)\right]. \tag{A.8}$$

These symmetries differ from the symmetries for finite ℓ only in that the Killing vectors have lost their ∂_π components, these components are associated to the part of the symmetry transformation that does not depend on π.

B Existence of a primary constraint

In this appendix, we review the main arguments that prove that actions of the form (5.43) (which generalise and include Beyond Horndeski systems as discussed in eq. (5.1)) are free of Ostrogradsky instabilities, and propagate at most three degrees of freedom. We consider the quantities $X = \nabla_\mu \pi \nabla^\mu \pi$ and

$$Q_\mu^\nu = \left(\delta_\mu^\rho - \frac{\nabla_\mu \pi \nabla^\rho \pi}{X} \right) \nabla_\rho \nabla^\nu \pi \,, \tag{B.1}$$

and examine arbitrary scalar-tensor actions (calling $q_{\mu\nu}$ the metric tensor) of the form

$$S = \int d^4 x \sqrt{-q} \left[B_1 + B_2 \, Q_\mu^\mu + B_3 \left(Q_\mu^\mu \right)^2 + B_4 \left(Q_\mu^\nu \, Q_\nu^\mu \right) + \dots \right] , \tag{B.2}$$

where the B_i are arbitrary functions of π, X. An immediate issue arises: action (B.2) contains second derivatives of the scalar field. Hence, besides the metric and the scalar π, actions as (B.2) would seem to propagate an additional, fourth mode mode — related with the scalar velocity $\dot{\pi}$ — which is associated with an Ostrogradsky instability. In this appendix, we show that this issue does not actually apply for actions (B.2): there exists a primary constraint which relates the dynamics of the scalar velocity with the dynamics of the metric, so to have a system which propagates at most three — and not four — degrees of freedom.

We do so using the geometrical approach introduced by Langlois and Noui, and further developed in [24]; in particular, we review the arguments as presented in [24, 26]. We decompose the four dimensional space time in $3+1$ dimensional slices: we assume there exists a foliation of space time on t = const hypersurfaces. We can then define on each hypersurface a 'time vector' t^μ as

$$t^\mu = N \, n^\mu + N^\mu \,, \tag{B.3}$$

with n^μ the normal, and N and N^μ respectively the lapse and shift vector. Such time vector determines the time evolution of the fields involved. The $3+1$ decomposition allows us to consider two quantities which further characterise the hypersurface geometry:

$$h_\mu^\nu = \delta_\mu^\nu + n^\nu n_\mu \,, \tag{B.4}$$

is the induce metric on the hypersurface; while

$$K_{\mu\nu} = \frac{1}{2N} \left(\dot{h}_{\mu\nu} - \nabla_{(\mu} N_{\nu)} \right) , \tag{B.5}$$

is the hypersurface extrinsic curvature. Here dot indicates the Lie derivative

$$\dot{h}_{\mu\nu} = t^\rho \, \nabla_\rho \, h_{\mu\nu} \,. \tag{B.6}$$

Instead of using $\nabla_\mu \pi$ in the scalar-tensor action, it is convenient to express it in terms of a vector A_μ defined as

$$A_\mu = \nabla_\mu \pi \,, \tag{B.7}$$

so that $X = A_\mu A^\mu$ and

$$Q^\nu_\mu = \left(\delta^\rho_\mu - \frac{A_\mu A^\rho}{X}\right) \nabla_\rho A^\nu \,. \tag{B.8}$$

After expressing the action in terms of A_μ, it is easy to 'go back' to an expression in terms of π only, if one wishes to do so, by imposing relation (B.7) by means of a Lagrange multiplier. The $3+1$ decomposition of space time can be implemented on the vector A_μ and its covariant derivative as

$$A_\mu = -A_\star \, n_\mu + \hat{A}_\mu \,, \tag{B.9}$$

$$\nabla_\mu A_\nu = D_\mu \, \hat{A}_\nu - A_\star \, K_{\mu\nu} + n_{(\mu} \left(K_{\nu)\rho} \, \hat{A}^\rho - D_{\nu)} A_\star\right) n_\mu n_\nu \left(V_\star - \hat{A}_\rho a^\rho\right) \,, \tag{B.10}$$

where $(\dots)$ on the index denotes symmetrization (with no numerical factors in front) and $a^\rho = n^\sigma \nabla_\sigma n^\rho$ is the acceleration vector. We have to consider three quantities which characterise the time flow of the fields described by the action. The first is time derivative of the metric, conveniently described by the extrinsic curvature $K_{\mu\nu}$: there are generically two degrees of freedom associated with this quantity (as expected for a spin 2 massless tensor). The second is the time derivative of the scalar, described by $A_\star$ (one dof). The third one is the time derivative of the scalar time derivative (one dof), controlled by the quantity

$$V_\star = n^\mu A_\mu = \frac{1}{N}\left(\dot{A}_\star - N^\mu \, \nabla_\mu A_\star\right) \,. \tag{B.11}$$

Hence action (B.2) propagates 4 dofs, unless there are constraint conditions. In what follow, in order to identify the kinetic terms in the action and express everything in terms of covariant quantities, it is easier to work directly with the extrinsic curvature $K_{\mu\nu}$ and with $V_\star$, rather than the velocities $\dot{h}_{\mu\nu}$ and $\dot{A}_\star$. We now show that constraint conditions exist in the form of primary constraints, by proving that a linear combination of the conjugate momenta

$$\pi_\star = \frac{1}{\sqrt{-q}} \frac{\delta S}{\delta V_\star} \,, \tag{B.12}$$

$$\pi^\alpha_\mu = \frac{1}{\sqrt{-q}} \frac{\delta S}{\delta K^\mu_\alpha K_\star} \,, \tag{B.13}$$

vanishes. This fact forbids the propagation of a fourth mode.

Using the definition of projection tensor, one has the important relation

$$A_\star \, P^\nu_\mu \, n_\nu = P^\nu_\mu \, \hat{A}_\nu \,. \tag{B.14}$$

Moreover,

$$\frac{\delta Q^\nu_\mu}{\delta V_\star} = P^\rho_\mu \, n_\rho n^\nu \,, \tag{B.15}$$

$$\frac{\delta Q^\nu_\mu}{\delta K^\beta_\alpha} \, \hat{A}_\beta \, \hat{A}^\alpha = P^\rho_\mu \left(-A_\star \hat{A}_\rho \hat{A}^\nu + \hat{A}^2 \, n_\rho \, \hat{A}^\nu + \hat{A}^2 \, \hat{A}_\rho \, n^\nu\right) \,. \tag{B.16}$$

Using the fact that the action S in eq. (B.2) is a sum of powers of traces of Q^ν_μ and its powers, as well as relation (B.14), one finds the following linear relation among conjugate momenta (with the notation $\approx$ we mean weak inequality, that is inequality in the phase space of constraints)

$$A_\star \left(2\hat{A}^2 - A_\star^2\right)\, \pi_\star - \hat{A}_\rho\, \hat{A}^\sigma\, \pi^\rho_\sigma \,\approx\, 0\,. \tag{B.17}$$

Hence, there exists a primary constraint which forbids the propagation of a fourth mode for theories described by an action (B.2). The theories that we are investigate propagate at most three degrees of freedom.

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