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Degenerate higher order scalar-tensor theories beyond Horndeski up to cubic order

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ABSTRACT: We present all scalar-tensor Lagrangians that are cubic in second derivatives of a scalar field, and that are degenerate, hence avoiding Ostrogradsky instabilities. Thanks to the existence of constraints, they propagate no more than three degrees of freedom, despite having higher order equations of motion. We also determine the viable combinations of previously identified quadratic degenerate Lagrangians and the newly established cubic ones. Finally, we study whether the new theories are connected to known scalar-tensor theories such as Horndeski and beyond Horndeski, through conformal or disformal transformations.

KEYWORDS: Classical Theories of Gravity, Cosmology of Theories beyond the SM

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1 Introduction

General Relativity (GR) is the unique consistent classical theory for a massless, self-interacting spin two field in four dimensional spacetime [1]. It describes accurately gravitational phenomena spanning a large interval of scales, from short distances probed by table top experiments, to large distances probed by astronomy and astrophysics [2]. By including

a positive cosmological constant term to the Einstein-Hilbert action, GR can also describe the current acceleration of the universe, but only if one is willing to accept the enormous fine tuning that observations require on the value of the cosmological constant [3]. Attempts to avoid such fine tuning motivate the study of gravitational theories more general than GR, the simplest option being scalar-tensor theories of gravity (see e.g. [4] for a review). Theories that involve derivative scalar interactions, in the family of Galileons [5], are characterised by interesting screening effects, as for example the Vainshtein mechanism [6], which are able to reduce the strength of the scalar fifth force to a value compatible with present constraints on deviations from GR.

Intriguingly, although the subject has been studied for many decades by now, we still do not know the structure of the *most general* consistent scalar-tensor theory, i.e. a theory describing a scalar interacting with a spin-2 tensor field in four dimensions. Horndeski [7] analysed the most general actions for scalar-tensor theories which lead to second order equations of motion (EOMs), and avoid Ostrogradsky instabilities [8, 9]. In four dimensional spacetime, this condition allows one to consider actions which contain at most three powers of second derivatives of the scalar field. However, as realised only recently, there also exist viable theories “beyond Horndeski” [10–12], which do not suffer from the Ostrogradsky instability even though the corresponding Euler-Lagrange equations are higher order. Such theories have interesting consequences for cosmology and astrophysics. In particular, they lead to a breaking of the Vainshtein mechanism *inside* matter, which can modify the structure of nonrelativistic stars [13–18], as well as that of relativistic ones [19].

The aim of the present paper is to determine the maximal generalization of Horndeski theories in four dimensions, by which we mean all scalar-tensor theories that contain at most three powers of second derivatives of the scalar field, and that propagate at most three degrees of freedom.

As demonstrated in [20], a systematic way to identify scalar-tensor theories that contain at most three degrees of freedom, i.e. without Ostrogradsky ghost, is to consider Lagrangians that are degenerate, i.e. whose Hessian matrix — obtained by taking the second derivatives of the Lagrangian with respect to velocities — is degenerate. For scalar-tensor theories, such a degeneracy can depend on the specific coupling between the metric and the scalar field. From the Hamiltonian point of view, the degeneracy of the Lagrangian translates into the existence of constraints on phase space, in addition to the usual Hamiltonian and momentum constraints due to diffeomorphism invariance, and explains why one degree of freedom is eliminated, even if the equations of motion are higher order. A detailed Hamiltonian analysis confirms the direct link between this degeneracy and the elimination of the Ostrogradsky ghost [21]. For Lagrangians depending on the accelerations of *several* variables, the degeneracy of the Lagrangian is not sufficient to eliminate the multiple Ostrogradsky ghosts and extra conditions must be imposed, as shown in [22] for classical mechanics systems (see also [23] for a slightly different approach, reaching the same conclusion). The singularity of the Hessian matrix (this time obtained by taking the second derivatives of the Lagrangian with respect to the lapse and shift) finds application also in other contexts like massive gravity: indeed, it is this condition that provides the tertiary constraint necessary to remove the Boulware-Deser ghost mode [24].

The degeneracy criterium, which provides a powerful and simple method to identify viable theories, was used in [20] to identify all scalar tensor theories whose Lagrangian depends quadratically on second order derivatives of a scalar field. Degenerate higher derivative Lagrangians, later dubbed EST (Extended Scalar Tensor) in [26], or DHOST (Degenerate Higher Order Scalar Tensor) in [27], include Horndeski theories as well as their extensions “beyond Horndeski”. As stressed in [20] and [25], only specific combinations of Horndeski theories and of their extensions beyond Horndeski are (Ostrogradsky) ghost-free. Quadratic degenerate theories are further studied in [26–28], in particular how they change under disformal transformations of the metric.

In the present work, we extend the systematic classification of degenerate theories to include Lagrangians that possess a cubic dependence on second order derivatives, so to find the most general extension of Horndeski scalar-tensor theory of gravity. We also allow for non-minimal couplings with gravity and show that the *only* viable Lagrangian, among all possible ones involving the Riemann tensor contracted with the second derivative of the scalar field, is of the form $G^{\mu\nu}\nabla_\mu\nabla_\nu\phi$. The class of theories we consider thus encompasses Horndeski Lagrangians and our analysis confirms that all Horndeski theories are degenerate, as expected. We also find new classes of cubic Lagrangians that are degenerate. In total, we identify seven classes of minimally coupled cubic theories, and two classes of non-minimally coupled cubic theories. We study in which cases it is possible to combine any of these cubic theories with the previously identified quadratic theories to obtain more general Lagrangians. We investigate which cubic theories admit a well-defined Minkowski limit, i.e. when the metric is frozen to its Minkowski value. We also study whether the new cubic theories are related to known Lagrangians through conformal or disformal transformations. Technical appendixes contain details of the calculations leading to the results we present in the main text.

2 Degenerate Scalar-Tensor theories

Scalar-Tensor theories involving second order derivatives of the scalar field in the action are generally plagued by an Ostrogradsky instability, unless the Lagrangian is degenerate, i.e. there is a primary constraint that leads to the removal of the additional undesired mode.¹ In order to study these theories, it is useful to recast the action into ordinary first order form via the introduction of a suitable auxiliary variable. This can be done by replacing all first order derivatives $\nabla_\mu\phi$ by the components of a vector field A_μ , as first explained in [20], and by imposing the relation

$$A_\mu = \nabla_\mu\phi, \tag{2.1}$$

¹In this paper we do not perform a full Hamiltonian analysis (see refs. [21] and [29] for Hamiltonian formulations of beyond Horndeski theories); however, we expect that, in general, the primary constraint is second-class and leads to a secondary constraint that is also second-class, so that both constraints remove one degree of freedom, as shown explicitly in [21] for the quadratic case. Note that the primary constraint can also be first-class in some very particular cases.

using a Lagrangian multiplier. Therefore, after introducing the general action we investigate, we will focus on its kinetic structure by identifying the time derivatives of the fields contained in $\nabla_\mu A_\nu$.

2.1 Action

In this paper we consider the most general action involving quadratic and cubic powers of the second derivative of the scalar field:

$$S[g, \phi] = \int d^4x \sqrt{-g} \left(f_2 R + C_{(2)}^{\mu\nu\rho\sigma} \phi_{\mu\nu} \phi_{\rho\sigma} + f_3 G_{\mu\nu} \phi^{\mu\nu} + C_{(3)}^{\mu\nu\rho\sigma\alpha\beta} \phi_{\mu\nu} \phi_{\rho\sigma} \phi_{\alpha\beta} \right) \quad (2.2)$$

where the functions f_2 and f_3 depend only on ϕ and $X \equiv \nabla_\mu \phi \nabla^\mu \phi$ (we use a mostly plus convention for the spacetime metric). The tensors $C_{(2)}$ and $C_{(3)}$ are the most general tensors constructed with the metric $g_{\mu\nu}$ and the first derivative of the scalar field $\phi_\mu \equiv \nabla_\mu \phi$.

As we will see in detail in the next subsection, when written in terms of the auxiliary variable A_μ , each second derivative of ϕ yields terms linear in velocities. By contrast, the curvature depends quadratically on the velocities of the metric and one can introduce terms non-minimally coupled to gravity, such as $f_2 R$ and $f_3 G_{\mu\nu} \phi^{\mu\nu}$, leading to second or third powers in velocities respectively. A priori, one could also envisage many more terms of this kind involving the Riemann tensor contracted in various ways. However, as shown in appendix A, the *only* viable Lagrangians among all the possible ones with appropriate powers in velocities, turn out to be these two (up to integrations by parts).

Note that one could also include in our general action (2.2) terms of the form $P(X, \phi)$ or terms depending linearly on $\phi^{\mu\nu}$. We have not included such terms explicitly because they do not modify the degeneracy conditions, but one should keep in mind that they can always be added to the Lagrangians that will be identified in our analysis.

Due to the way the tensors $C_{(2)}$ and $C_{(3)}$ are contracted in the action, one can always impose, without loss of generality, the symmetry relations:

$$C_{(2)}^{\mu\nu\rho\sigma} = C_{(2)}^{\rho\sigma\mu\nu} = C_{(2)}^{\nu\mu\rho\sigma} \quad \text{and} \quad C_{(3)}^{\mu\nu\rho\sigma\alpha\beta} = C_{(3)}^{\rho\sigma\mu\nu\alpha\beta} = C_{(3)}^{\mu\nu\alpha\beta\rho\sigma} = C_{(3)}^{\nu\mu\alpha\beta\rho\sigma}. \quad (2.3)$$

As a consequence, they can be expressed as

$$C_{(2)}^{\mu\nu\rho\sigma} = \langle \langle a_1 g^{\mu\rho} g^{\nu\sigma} + a_2 g^{\mu\nu} g^{\rho\sigma} + a_3 \phi^\mu \phi^\nu g^{\rho\sigma} + a_4 \phi^\mu \phi^\rho g^{\nu\sigma} + a_5 \phi^\mu \phi^\nu \phi^\rho \phi^\sigma \rangle \rangle, \quad (2.4)$$

$$\begin{aligned} C_{(3)}^{\mu\nu\rho\sigma\alpha\beta} = \langle \langle & b_1 g^{\mu\nu} g^{\rho\sigma} g^{\alpha\beta} + b_2 g^{\mu\nu} g^{\rho\alpha} g^{\sigma\beta} + b_3 g^{\mu\rho} g^{\nu\alpha} g^{\sigma\beta} + b_4 g^{\mu\nu} g^{\rho\sigma} \phi^\alpha \phi^\beta \\ & + b_5 g^{\mu\nu} g^{\rho\alpha} \phi^\sigma \phi^\beta + b_6 g^{\mu\rho} g^{\nu\sigma} \phi^\alpha \phi^\beta + b_7 g^{\nu\rho} g^{\sigma\alpha} \phi^\mu \phi^\beta + b_8 g^{\mu\rho} \phi^\nu \phi^\sigma \phi^\alpha \phi^\beta \\ & + b_9 g^{\mu\nu} \phi^\rho \phi^\sigma \phi^\alpha \phi^\beta + b_{10} \phi^\mu \phi^\nu \phi^\rho \phi^\sigma \phi^\alpha \phi^\beta \rangle \rangle, \end{aligned} \quad (2.5)$$

where the functions a 's and b 's depend only on ϕ and X . The notation $\langle \langle \dots \rangle \rangle$ means that these expressions are symmetrised so to satisfy eq (2.3). Explicitly, we have

$$C_{(2)}^{\mu\nu\rho\sigma} \phi_{\mu\nu} \phi_{\rho\sigma} + C_{(3)}^{\mu\nu\rho\sigma\alpha\beta} \phi_{\mu\nu} \phi_{\rho\sigma} \phi_{\alpha\beta} = \sum_{i=1}^5 a_i L_i^{(2)} + \sum_{i=1}^{10} b_i L_i^{(3)}, \quad (2.6)$$

where

$$\begin{aligned} L_1^{(2)} &= \phi_{\mu\nu}\phi^{\mu\nu}, & L_2^{(2)} &= (\Box\phi)^2, & L_3^{(2)} &= (\Box\phi)\phi^\mu\phi_{\mu\nu}\phi^\nu, \\ L_4^{(2)} &= \phi^\mu\phi_{\mu\rho}\phi^{\rho\nu}\phi_\nu, & L_5^{(2)} &= (\phi^\mu\phi_{\mu\nu}\phi^\nu)^2; \end{aligned} \quad (2.7)$$

and

$$\begin{aligned} L_1^{(3)} &= (\Box\phi)^3, & L_2^{(3)} &= (\Box\phi)\phi_{\mu\nu}\phi^{\mu\nu}, & L_3^{(3)} &= \phi_{\mu\nu}\phi^{\nu\rho}\phi_\rho^\mu, \\ L_4^{(3)} &= (\Box\phi)^2\phi_\mu\phi^{\mu\nu}\phi_\nu, & L_5^{(3)} &= \Box\phi\phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^\rho, & L_6^{(3)} &= \phi_{\mu\nu}\phi^{\mu\nu}\phi_\rho\phi^{\rho\sigma}\phi_\sigma, \\ L_7^{(3)} &= \phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^{\rho\sigma}\phi_\sigma, & L_8^{(3)} &= \phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^\rho\phi_\sigma\phi^{\sigma\lambda}\phi_\lambda, \\ L_9^{(3)} &= \Box\phi(\phi_\mu\phi^{\mu\nu}\phi_\nu)^2, & L_{10}^{(3)} &= (\phi_\mu\phi^{\mu\nu}\phi_\nu)^3. \end{aligned} \quad (2.8)$$

Introducing the auxiliary variable A_μ as in (2.1), the general action (2.2) becomes

$$\begin{aligned} S[g, \phi; A_\mu, \lambda^\mu] &= \int d^4x \sqrt{-g} \left(f_2 R + C_{(2)}^{\mu\nu\rho\sigma} \nabla_\mu A_\nu \nabla_\rho A_\sigma \right. \\ &\quad \left. + f_3 G_{\mu\nu} \nabla^\mu A^\nu + C_{(3)}^{\mu\nu\rho\sigma\alpha\beta} \nabla_\mu A_\nu \nabla_\rho A_\sigma \nabla_\alpha A_\beta + \lambda^\mu (\phi_\mu - A_\mu) \right), \end{aligned} \quad (2.9)$$

where the tensors $C_{(2)}^{\mu\nu\rho\sigma}$ and $C_{(3)}^{\mu\nu\rho\sigma\alpha\beta}$ are now expressed in terms of A_μ and ϕ . Clearly, the two Lagrangians (2.2) and (2.9) are equivalent.

Although we do not perform explicitly a Hamiltonian analysis here,² let us briefly comment about the role of the Lagrangian multipliers λ^μ and the relations they enforce. Since the action (2.9) does not involve the velocities of λ^μ , the corresponding conjugate momenta p_μ appear in the total Hamiltonian H_T as primary constraints that weakly vanish. The evolution of p_i gives the secondary constraints $\phi_i - A_i \approx 0$. By contrast, the evolution of p_0 allows one to solve for the multiplier used in H_T to impose the other primary constraint $\pi - \lambda^0 \approx 0$, where π is the momentum of ϕ . The evolution of $\pi - \lambda^0$, on the other hand, fixes the multiplier associated with p_0 . All these constraints are *second class* and therefore can be consistently imposed in the Hamiltonian analysis; in particular the constraints $\phi_i - A_i \approx 0$ enables us to eliminate the velocity of A_i in favour of the spatial derivative of A_0 , as explained in detail in the next section. It is thus clear that the constraints that follow from the λ^μ in (2.9) do not get mixed up with the (potential) extra primary constraint necessary to eliminate the Ostrogradsky mode, which characterises degenerate theories.

2.2 Covariant ADM decomposition

In order to study the kinetic structure of the action (2.9), we must perform a 3 + 1 decomposition of its building blocks. We now assume the existence of an arbitrary slicing of spacetime with 3-dimensional spacelike hypersurfaces. We introduce the unit vector n^μ normal to the spacelike hypersurfaces, which is time-like and satisfies the normalization condition $n_\mu n^\mu = -1$. This induces a three-dimensional metric, corresponding to the projection tensor on the spatial hypersurfaces, defined by

$$h_{\mu\nu} \equiv g_{\mu\nu} + n_\mu n_\nu. \quad (2.10)$$

²All the details about the complete Hamiltonian analysis of quadratic theories can be found in [21].

Following the construction of [20], we define the spatial and normal projection of A_μ , respectively

$$\hat{A}_\mu \equiv h_\mu^\nu A_\nu, \quad A_* \equiv A_\mu n^\mu. \quad (2.11)$$

Let us now introduce the time direction vector $t^\mu = \partial/\partial t$ associated with a time coordinate t that labels the slicing of spacelike hypersurfaces. One can always decompose t^μ as

$$t^\mu = N n^\mu + N^\mu, \quad (2.12)$$

thus defining the lapse function N and the shift vector N^μ orthogonal to n^μ . We also define the “time derivative” of any spatial tensor as the spatial projection of its Lie derivative with respect to t^μ . In particular, we have

$$\dot{A}_* \equiv t^\mu \nabla_\mu A_*, \quad \dot{\hat{A}}_\mu \equiv h_\mu^\nu \mathcal{L}_t \hat{A}_\nu, \quad \dot{h}_{\mu\nu} \equiv h_\mu^\alpha h_\nu^\beta \mathcal{L}_t h_{\alpha\beta}. \quad (2.13)$$

Due to the symmetric property of $\nabla_\mu A_\nu = \nabla_\nu A_\mu$, it is possible to express $\dot{\hat{A}}_\mu$ in terms of $D_\mu A_*$ and $\dot{h}_{\mu\nu}$, therefore the only velocities (time derivative of the fields) involved in $\nabla_\mu A_\nu$ are

$$\dot{A}_* = N V_* + N^\mu D_\mu A_*, \quad \dot{h}_{\mu\nu} = 2(N K_{\mu\nu} + D_{(\mu} N_{\nu)}) , \quad (2.14)$$

where $V_* \equiv n^\mu \nabla_\mu A_*$, $K_{\mu\nu}$ is the extrinsic curvature tensor and D_μ denotes the 3-dimensional covariant derivative associated with the spatial metric $h_{\mu\nu}$.

Instead of using the velocities $\dot{h}_{\mu\nu}$ and \dot{A}_* , it is convenient to work with the covariant objects $K_{\mu\nu}$ and V_* and interpret them as “covariant velocities” associated with the fields $h_{\mu\nu}$ and A_* . Working with these covariant quantities allows us to avoid dealing with the lapse and the shift vector.

Using these definitions, as well as the property $\nabla_\mu A_\nu = \nabla_\nu A_\mu$, the 3+1 covariant decomposition of $\nabla_\mu A_\nu$ is given by

$$\nabla_\mu A_\nu = D_\mu \hat{A}_\nu - A_* K_{\mu\nu} + 2(n_{(\mu} K_{\nu)\rho} \hat{A}^\rho - n_{(\mu} D_{\nu)} A_*) + n_\mu n_\nu (V_* - \hat{A}_\rho a^\rho), \quad (2.15)$$

where $a^\mu \equiv n^\nu \nabla_\nu n^\mu$ is the acceleration vector. One can rewrite (2.15) as

$$\nabla_\mu A_\nu = \lambda_{\mu\nu} V_* + \Lambda_{\mu\nu}{}^{\rho\sigma} K_{\rho\sigma} + D_\mu \hat{A}_\nu - 2 n_{(\mu} D_{\nu)} A_* - \lambda_{\mu\nu} \hat{A}_\rho a^\rho, \quad (2.16)$$

with

$$\lambda_{\mu\nu} \equiv n_\mu n_\nu, \quad \Lambda_{\mu\nu}{}^{\rho\sigma} \equiv -A_* h_{(\mu}^\rho h_{\nu)}^\sigma + 2 n_{(\mu} h_{\nu)}^{(\rho} \hat{A}^{\sigma)}. \quad (2.17)$$

These two tensors fully characterise the velocity structure of the building block $\nabla_\mu A_\nu$ that appears in the action (2.9) and will play an essential role in deriving the degeneracy conditions.

2.3 Horndeski Lagrangians and kinetic structure of the action

As an example of theories of the type (2.2), and as a useful step for the general case, let us first consider the particular case of the so-called quartic and quintic Horndeski Lagrangians³

$$L_4^H = f_2 R - 2f_{2X}(\Box\phi^2 - \phi^{\mu\nu}\phi_{\mu\nu}), \quad (2.18)$$

$$L_5^H = f_3 G_{\mu\nu}\phi^{\mu\nu} + \frac{1}{3}f_{3X}(\Box\phi^3 - 3\Box\phi\phi_{\mu\nu}\phi^{\mu\nu} + 2\phi_{\mu\nu}\phi^{\mu\sigma}\phi^\nu{}_\sigma), \quad (2.19)$$

which correspond, respectively, to a quadratic and a cubic Lagrangian in our terminology. Indeed, they are of the form (2.2), with

$$a_1 = -a_2 = 2f_{2X}, \quad a_3 = a_4 = a_5 = 0, \quad (2.20)$$

and

$$3b_1 = -b_2 = \frac{3}{2}b_3 = f_{3X}, \quad b_i = 0 \quad (i = 4, \dots, 10). \quad (2.21)$$

It is instructive to extract the kinetic part of these two Lagrangians as the result will be useful for the general case. The kinetic structure of the original Lagrangians (2.18) and (2.19) is the same as the following ones

$$L_{4\text{kin}}^H = C_{(2)H}^{\mu\nu\rho\sigma}\phi_{\mu\nu}\phi_{\rho\sigma} \quad \text{and} \quad L_{5\text{kin}}^H = C_{(3)H}^{\mu\nu\rho\sigma\alpha\beta}\phi_{\mu\nu}\phi_{\rho\sigma}\phi_{\alpha\beta}, \quad (2.22)$$

with⁴

$$C_{(2)H}^{\mu\nu\rho\sigma} = \frac{2}{A_*^2} \left[-f_2 h^{\nu[\mu} h^{\rho]\sigma} + 2f_{2X} \left(A^2 h^{\nu[\mu} h^{\rho]\sigma} - \hat{A}^2 \hat{P}^{\nu[\mu} \hat{P}^{\rho]\sigma} \right) \right], \quad (2.23)$$

$$C_{(3)H}^{\mu\nu\rho\sigma\alpha\beta} = -\frac{2f_{3X}}{A_*^2} \left(A^2 h^{\mu[\nu} h^{|\rho|\sigma} h^{|\alpha|\beta]} - \hat{A}^2 \hat{P}^{\mu[\nu} \hat{P}^{|\rho|\sigma} \hat{P}^{|\alpha|\beta]} \right), \quad (2.24)$$

where $A^2 \equiv A_\mu A^\mu$, $\hat{A}^2 \equiv \hat{A}_\mu \hat{A}^\mu$ and we have introduced the projection tensor (orthogonal to the directions n^μ and \hat{A}^μ)

$$\hat{P}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{\hat{A}^2} \hat{A}_\mu \hat{A}_\nu. \quad (2.25)$$

Notice that the tensors (2.23) and (2.24) are orthogonal to the vector n^μ , therefore the kinetic terms do not contain the velocity V_* . This is the peculiarity of Horndeski Lagrangians which reflects in second order equations of motions.

Let us now turn to the general action (2.2). In order to extract its kinetic part, it is convenient to re-express the curvature terms in the action as Horndeski Lagrangians so that one can use the results above. The action (2.2) is thus rewritten as

$$S[g, \phi] = \int d^4x \sqrt{-g} \left(L_4^H + \tilde{C}_{(2)}^{\mu\nu\rho\sigma} \phi_{\mu\nu} \phi_{\rho\sigma} + L_5^H + \tilde{C}_{(3)}^{\mu\nu\rho\sigma\alpha\beta} \phi_{\mu\nu} \phi_{\rho\sigma} \phi_{\alpha\beta} \right), \quad (2.26)$$

³We use the subscripts 4 and 5, referring to ‘quartic’ and ‘quintic’, only for the Horndeski Lagrangians themselves. According to our terminology, for all other associated variables we use instead the subscripts (2) and (3) referring to ‘quadratic’ or ‘cubic’ types of theory.

⁴Note that $h^{\mu[\nu} h^{\rho]\sigma}$ denotes the anti-symmetrisation on (ν, ρ) and $h^{\mu[\nu} h^{|\rho|\sigma} h^{|\alpha|\beta]}$ denotes the anti-symmetrisation on the second index of each tensor h . The same notation holds for the terms defined with the projector \hat{P} .

where the tensors $\tilde{C}_{(2)}^{\mu\nu\rho\sigma}$ and $\tilde{C}_{(3)}^{\mu\nu\rho\sigma\alpha\beta}$ are of the form (2.4)–(2.5) with the new functions

$$\tilde{a}_1 = a_1 - 2f_{2X}, \quad \tilde{a}_2 = a_2 + 2f_{2X}, \quad (2.27)$$

$$\tilde{b}_1 = b_1 - \frac{1}{3}f_{3X}, \quad \tilde{b}_2 = b_2 + f_{3X}, \quad \tilde{b}_3 = b_3 - \frac{2}{3}f_{3X}, \quad (2.28)$$

while all the other functions remain unchanged.

Replacing the Lagrangians L_4^H and L_5^H in (2.26) with the kinetically equivalent ones (2.22), one finds that the kinetic structure of the total action is described by the tensors

$$C_{(2)}^{\mu\nu\rho\sigma} = C_{(2)H}^{\mu\nu\rho\sigma} + \tilde{C}_{(2)}^{\mu\nu\rho\sigma} \quad \text{and} \quad C_{(3)}^{\mu\nu\rho\sigma\alpha\beta} = C_{(3)H}^{\mu\nu\rho\sigma\alpha\beta} + \tilde{C}_{(3)}^{\mu\nu\rho\sigma\alpha\beta}. \quad (2.29)$$

Only these tensors are relevant for the degeneracy conditions, which we derive below.

2.4 Degeneracy conditions and primary constraints

We now introduce the Hessian matrix of the Lagrangian with respect to the velocities V_* and K_{ij} . This matrix can be written in the form (introducing a factor 1/2 for convenience)

$$\mathbb{H} = \begin{pmatrix} \mathcal{A} & \mathcal{B}^{ij} \\ \mathcal{B}^{kl} & \mathcal{K}^{ij,kl} \end{pmatrix}, \quad (2.30)$$

with

$$\mathcal{A} \equiv \frac{1}{2} \frac{\partial^2 L}{\partial V_*^2}, \quad \mathcal{B}^{ij} \equiv \frac{1}{2} \frac{\partial^2 L}{\partial V_* \partial K_{ij}}, \quad \mathcal{K}^{ij,kl} \equiv \frac{1}{2} \frac{\partial^2 L}{\partial K_{ij} \partial K_{kl}}. \quad (2.31)$$

The degeneracy of the theory is associated with the degeneracy of its Hessian matrix, i.e. $\det \mathbb{H} = 0$. Equivalently, one can find a non trivial null eigenvector (v_0, \mathcal{V}_{kl}) such that

$$v_0 \mathcal{A} + \mathcal{B}^{kl} \mathcal{V}_{kl} = 0, \quad v_0 \mathcal{B}^{ij} + \mathcal{K}^{ij,kl} \mathcal{V}_{kl} = 0. \quad (2.32)$$

These conditions translate into the existence of a primary constraint, which takes the form

$$v_0 \pi_* + \mathcal{V}_{ij} \pi^{ij} + \dots \approx 0, \quad (2.33)$$

where we have introduced the ‘‘covariant momenta’’ conjugated respectively to A_* and $h_{\mu\nu}$,

$$\pi_* \equiv \frac{\delta L}{\delta V_*}, \quad \pi^{ij} \equiv \frac{\delta L}{\delta K_{ij}}, \quad (2.34)$$

and the dots indicate momentum-independent terms, involving only the fields and their spatial derivatives. Note that we will always assume $v_0 \neq 0$ since we are interested in removing the Ostrogradsky mode: therefore in the following we will fix $v_0 = 1$ without loss of generality.

It is important to keep in mind that the primary constraint (2.33) is a *scalar* constraint involving only the scalar components of π^{ij} , i.e. $\mathcal{V}_{ij} \pi^{ij}$. It is indeed responsible for removing the scalar Ostrogradsky mode. However, there could still be extra primary constraints in the vector sector of π^{ij} , which can further reduce the number of degrees of freedom (dof) (as

pointed out in [27] and further stressed in [28]). Indeed, as we will show in what follows, some classes of theories that possess the constraint (2.33), also enjoy the two following primary constraints:

$$\hat{A}_i \hat{P}_{jk} \pi^{ij} + \dots \approx 0, \quad (2.35)$$

where we have used the projector (2.25). These constraints remove the two helicity-2 dof present in the metric sector, leaving the theory with only one dof.

In order to compute the Hessian matrix of (2.26), one needs to keep all terms quadratic and cubic in the velocities. The Hessian matrix decomposes into its quadratic and cubic contributions denoted

$$\mathbb{H}_{(2)} = \begin{pmatrix} \mathcal{A}_{(2)} & \mathcal{B}_{(2)}^{ij} \\ \mathcal{B}_{(2)}^{kl} & \mathcal{K}_{(2)}^{ij,kl} \end{pmatrix}, \quad \mathbb{H}_{(3)} = \begin{pmatrix} \mathcal{A}_{(3)} & \mathcal{B}_{(3)}^{ij} \\ \mathcal{B}_{(3)}^{kl} & \mathcal{K}_{(3)}^{ij,kl} \end{pmatrix}, \quad (2.36)$$

with

$$\mathcal{A}_{(2)} \equiv C_{(2)}^{\mu\nu\rho\sigma} \lambda_{\mu\nu} \lambda_{\rho\sigma}, \quad \mathcal{B}_{(2)}^{ij} \equiv C_{(2)}^{\mu\nu\rho\sigma} \lambda_{\mu\nu} \Lambda_{\rho\sigma}^{ij}, \quad \mathcal{K}_{(2)}^{ij,kl} \equiv C_{(2)}^{\mu\nu\rho\sigma} \Lambda_{\mu\nu}^{ij} \Lambda_{\rho\sigma}^{kl}, \quad (2.37)$$

$$\mathcal{A}_{(3)} \equiv 3C_{(3)}^{\mu\nu\rho\sigma\alpha\beta} \lambda_{\mu\nu} \lambda_{\rho\sigma} \phi_{\alpha\beta}, \quad \mathcal{B}_{(3)}^{ij} \equiv 3C_{(3)}^{\mu\nu\rho\sigma\alpha\beta} \lambda_{\mu\nu} \Lambda_{\rho\sigma}^{ij} \phi_{\alpha\beta}, \quad \mathcal{K}_{(3)}^{ij,kl} \equiv 3C_{(3)}^{\mu\nu\rho\sigma\alpha\beta} \Lambda_{\mu\nu}^{ij} \Lambda_{\rho\sigma}^{kl} \phi_{\alpha\beta}. \quad (2.38)$$

Introducing the tensor

$$L_{\mu\nu} \equiv \lambda_{\mu\nu} + \Lambda_{\mu\nu}^{ij} \mathcal{V}_{ij}, \quad (2.39)$$

the conditions (2.32) (with $v_0 = 1$) for purely quadratic theories read

$$C_{(2)}^{\mu\nu\rho\sigma} \lambda_{\mu\nu} L_{\rho\sigma} = 0, \quad C_{(2)}^{\mu\nu\rho\sigma} \Lambda_{\mu\nu}^{ij} L_{\rho\sigma} = 0. \quad (2.40)$$

On the other hand, the cubic Hessian matrix contains velocities. Therefore, the degeneracy conditions must be satisfied for arbitrary values of $\phi_{\alpha\beta}$. This implies that $\phi_{\alpha\beta}$ can be “factorised” and the conditions (2.32) in the cubic case are analogous to the quadratic ones, namely

$$C_{(3)}^{\mu\nu\rho\sigma\alpha\beta} \lambda_{\mu\nu} L_{\rho\sigma} = 0, \quad C_{(3)}^{\mu\nu\rho\sigma\alpha\beta} \Lambda_{\mu\nu}^{ij} L_{\rho\sigma} = 0. \quad (2.41)$$

The above equations mean that, in order to get a degenerate Lagrangian, the projections of the tensors $C_{(2)}^{\mu\nu\rho\sigma} L_{\rho\sigma}$ or $C_{(3)}^{\mu\nu\rho\sigma\alpha\beta} L_{\rho\sigma}$, respectively via $\lambda_{\mu\nu}$ and $\Lambda_{\mu\nu}^{ij}$, must vanish. As shown in appendix B, this implies that these tensors are necessarily of the form

$$C_{(2)}^{\mu\nu\rho\sigma} L_{\rho\sigma} = 2\sigma A^{(\mu} \hat{A}^{\nu)}, \quad (2.42)$$

and

$$C_{(3)}^{\mu\nu\rho\sigma\alpha\beta} L_{\alpha\beta} = 4\sigma_1 A^{(\mu} h^{\nu)(\rho} A^{\sigma)} + 4\sigma_2 A^{(\mu} \hat{A}^{\nu)} A^{(\rho} \hat{A}^{\sigma)}, \quad (2.43)$$

where σ , σ_1 and σ_2 are arbitrary scalar quantities.

By solving the conditions (2.42), one recovers the quadratic theories identified in [20]; they are summarised in appendix C. Conditions (2.43) are solved in detail in appendix D and in the next section we report the various classes of purely cubic theories. Then, we will consider the possibility to merge quadratic and cubic theories. In this case the additional condition to impose is that $L_{\mu\nu}$ is the same in (2.42) and (2.43), i.e. we have to use the same \mathcal{V}_{ij} .

3 Classification of cubic theories

The degeneracy conditions for quadratic theories (i.e. with $f_3 = b_i = 0$) have already been solved and the corresponding theories identified in [20]. These quadratic theories were then examined in more details in [26–28]. In this section we thus focus our attention on the purely cubic theories, i.e. characterized by $f_2 = a_i = 0$. Solving the degeneracy conditions here is much more involved than in the quadratic case and rewriting them in the tensorial form (2.41) is instrumental to obtain the full classification. Below, we simply present the full classification, indicating for each class the free functions among the b_i and the constraints satisfied by the other functions. All the cubic theories we identify are summarised at the end of the section in table 1. The details of how we have identified these classes are given in appendix D, where the reader can also find the explicit expression of the null eigenvectors associated with the degeneracy. The latter are indispensable to identify the healthy combinations of quadratic and cubic Lagrangians, which will be given in the next section.

3.1 Minimally coupled theories

We start with the minimally coupled case, corresponding to $f_3 = 0$. There are seven different classes of theories.

- **³M-I**: Four free functions b_1, b_2, b_3 and b_4 (with $9b_1 + 2b_2 \neq 0$). All the other functions are determined as follows:

$$\begin{aligned}
 b_5 &= -\frac{2}{X}b_2, & b_6 &= \frac{9b_1b_3 + 3b_4X(b_2 + b_3) - 2b_2^2}{X(9b_1 + 2b_2)}, \\
 b_7 &= -\frac{3}{X}b_3, & b_8 &= \frac{9b_1b_3 - 6b_4X(b_2 + b_3) + 6b_2b_3 + 4b_2^2}{X^2(9b_1 + 2b_2)}, \\
 b_9 &= \frac{1}{X^2(9b_1 + 2b_2)^2} \left[3b_4^2X^2(9b_1 + 3b_2 + b_3) - 2b_4X(9b_1(b_2 - b_3) + 4b_2^2) \right. \\
 &\quad \left. + 24b_1b_2^2 + 54b_1^2b_2 + 27b_1^2b_3 + 4b_2^3 \right], \\
 b_{10} &= \frac{1}{X^3(9b_1 + 2b_2)^3} \left[3b_4^3X^3(9b_1 + 3b_2 + b_3) - 6b_2b_4^2X^2(9b_1 + 3b_2 + b_3) \right. \\
 &\quad \left. + 2b_4X(81b_1^2(b_2 + b_3) + 18b_1b_2(3b_2 + 2b_3) + 2b_2^2(5b_2 + 3b_3)) \right. \\
 &\quad \left. - 2(54b_1^2b_2(b_2 + 2b_3) + 4b_1b_2^2(7b_2 + 9b_3) + 81b_1^3b_3 + 4b_2^3(b_2 + b_3)) \right]. \quad (3.1)
 \end{aligned}$$

This class includes the pure quintic beyond Horndeski Lagrangian:

$$\begin{aligned}
 L_5^{\text{bH}} &= f(\phi, X) \left[X((\square\phi)^3 - 3\square\phi\phi_{\mu\nu}\phi^{\mu\nu} + 2\phi_{\mu\nu}\phi^{\nu\rho}\phi_\rho^\mu) \right. \\
 &\quad \left. - 3\left((\square\phi)^2\phi_\mu\phi^{\mu\nu}\phi_\nu - 2\square\phi\phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^\rho - \phi_{\mu\nu}\phi^{\mu\nu}\phi_\rho\phi^{\rho\sigma}\phi_\sigma + 2\phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^{\rho\sigma}\phi_\sigma\right) \right], \quad (3.2)
 \end{aligned}$$

which corresponds to the choice of functions

$$\frac{b_1}{X} = -\frac{b_2}{3X} = \frac{b_3}{2X} = -\frac{b_4}{3} = \frac{b_5}{6} = \frac{b_6}{3} = -\frac{b_7}{6} = f. \quad (3.3)$$

The above combination is special as it leaves the Lagrangian linear in V_* , therefore in (2.38) $\mathcal{A}_3 = 0$.

Notice that in this class $9b_1 + 2b_2 \neq 0$. The condition $9b_1 + 2b_2 = 0$ leads to the next three classes.

- **³M-II.** Three free functions b_1, b_3, b_6 (with $9b_1 - 2b_3 \neq 0$). All the other functions are given by

$$\begin{aligned} b_2 &= -\frac{9}{2}b_1, & b_4 &= -\frac{3}{X}b_1, & b_5 &= \frac{9}{X}b_1, \\ b_7 &= -\frac{3}{X}b_3, & b_8 &= \frac{3b_3 - 2b_6X}{X^2}, \\ b_9 &= \frac{9b_1(b_3 + 2b_6X) - 81b_1^2 - 2b_6^2X^2}{3X^2(9b_1 - 2b_3)}, \\ b_{10} &= \left[18b_6X(-12b_1b_3 + 27b_1^2 + 2b_3^2) - 36b_3(-8b_1b_3 + 18b_1^2 + b_3^2) \right. \\ &\quad \left. - 12b_3b_6^2X^2 + 4b_6^3X^3 \right] \left[9X^3(9b_1 - 2b_3)^2 \right]^{-1}, \end{aligned}$$

In this class $9b_1 - 2b_3 \neq 0$. The case $9b_1 - 2b_3 = 0$ (and $9b_1 + 2b_2 = 0$) is described by the next two classes.

- **³M-III.** A single free function b_1 . All the other functions are determined in terms of b_1 as follows:

$$\begin{aligned} b_2 &= -\frac{9}{2}b_1, & b_3 &= \frac{9}{2}b_1, & b_4 &= -\frac{3b_1}{X}, & b_5 &= \frac{9}{X}b_1, & b_6 &= \frac{9b_1}{2X}, \\ b_7 &= -\frac{27}{2X}b_1, & b_8 &= \frac{9b_1}{2X^2}, & b_9 &= -\frac{3b_1}{2X^2}, & b_{10} &= -\frac{b_1}{X^3}. \end{aligned}$$

- **³M-IV.** Five free functions $b_1, b_4, b_5, b_8, b_{10}$. The other functions are given by

$$\begin{aligned} b_2 &= -\frac{9}{2}b_1, & b_3 &= \frac{9}{2}b_1, & b_6 &= -3b_4 - \frac{9}{2X}b_1, \\ b_7 &= -3b_5 + \frac{27}{2X}b_1, & b_9 &= \frac{3b_1 - 2X(2b_4 + b_5)}{2X^2}. \end{aligned}$$

- **³M-V.** Two free functions, b_1 and b_4 , while the other functions are given by

$$b_2 = b_3 = b_5 = b_6 = b_7 = b_8 = 0, \quad b_9 = \frac{b_4^2}{3b_1}, \quad b_{10} = \frac{b_4^3}{27b_1^2}. \quad (3.4)$$

There is only one (scalar) dof that propagates due to the primary constraints (2.35), and their associated secondary constraints.

- **³M-VI.** Six free functions b_1, b_4, b_5, b_8, b_9 and b_{10} . All the other functions vanish:

$$b_2 = b_3 = b_6 = b_7 = 0. \quad (3.5)$$

Again, there is only one (scalar) dof that propagates.

- **³M-VII.** Four free functions b_5, b_7, b_8 and b_{10} . The remaining functions vanish, except b_9 :

$$b_1 = b_2 = b_3 = b_4 = b_6 = 0, \quad b_9 = -\frac{b_5}{X}. \quad (3.6)$$

3.2 Non-minimally coupled theories

We now consider the purely cubic Lagrangians with $f_3 \neq 0$. There are two classes of theories.

- ³N-I. In addition to f_3 , the functions b_1 and b_4 are free (with the only restriction $b_1 \neq 0$). The other functions are determined as follows:

$$\begin{aligned}
 b_2 &= -3b_1, & b_3 &= 2b_1, & b_6 &= -b_4, \\
 b_5 &= \frac{2(f_{3X} - 3b_1)^2 - 2b_4 f_{3X} X}{3b_1 X}, & b_7 &= \frac{2b_4 f_{3X} X - 2(f_{3X} - 3b_1)^2}{3b_1 X}, \\
 b_8 &= \frac{2(3b_1 + b_4 X - f_{3X})((f_{3X} - 3b_1)^2 - b_4 f_{3X} X)}{9b_1^2 X^2}, \\
 b_9 &= \frac{2b_4(3b_1 + b_4 X - f_{3X})}{3b_1 X}, & b_{10} &= \frac{2b_4(3b_1 + b_4 X - f_{3X})^2}{9b_1^2 X^2}.
 \end{aligned} \tag{3.7}$$

Quintic Horndeski (2.19), as well as the combination of quintic Horndeski plus quintic beyond Horndeski (3.2), is included in this class of models.

- ³N-II. Free functions b_5 , b_8 and b_{10} , in addition to f_3 . The other functions are given by

$$\begin{aligned}
 b_1 &= b_2 = b_3 = 0, & b_7 &= -b_5, \\
 b_4 &= -b_6 = \frac{f_{3X}}{X}, & b_9 &= -\frac{2f_{3X} + Xb_5}{X^2}.
 \end{aligned}$$

3.3 Minkowski limit

Here we discuss which ones among the classes of theories described above admit a healthy Minkowski limit, i.e. the limit where the metric is given by $g_{\mu\nu} = \eta_{\mu\nu}$ and the metric fluctuations are ignored. In this limit, only the scalar sector is dynamical and the Hessian matrix reduces to its purely scalar component, i.e. \mathcal{A} . For cubic theories, the degeneracy is thus expressed by the condition $\mathcal{A}_{(3)} = 0$, which imposes the relations

$$b_1 = -\frac{b_2}{3} = \frac{b_3}{2}, \quad b_4 = -\frac{b_5}{2} = -b_6 = \frac{b_7}{2}, \quad b_8 = b_9 = b_{10} = 0. \tag{3.8}$$

The only classes that satisfy these conditions are

- ³M-I: Beyond Horndeski theory,
- ³N-I: Beyond Horndeski and Horndeski theory,
- ³N-II: Imposing also $b_5 = -2f_{3X}/X$ and $b_8 = b_{10} = 0$.

This shows that there is a new theory, ³N-II, which propagates three degrees of freedom on curved spacetime and has a healthy Minkowski limit. On the other hand, theories that do not satisfy (3.8) could still have a healthy decoupling limit around a non-trivial background.

Minimally coupled theories				
Classification	# dof	Free functions	Minkowski limit	Examples
³ M-I	3	$i=1,2,3,4$	✓ (bH)	bH, $\Omega \otimes \text{bH}^{(1)}$
³ M-II	3	$i=1,3, 6$	X	
³ M-III	3	$i=1$	X	
³ M-IV	3	$i=1,4,5,8,10$	X	
³ M-V	1	$i=1,4$	X	
³ M-VI	1	$i=1,4,5,8,9,10$	X	
³ M-VII	3	$i=5,7,8,10$	X	

Non-minimally coupled theories				
Classification	# dof	Free functions	Minkowski limit	Examples
³ N-I	3	$f_3, i=1,4$	✓ (H, bH)	H, H+bH, $\Gamma \otimes \text{H}^{(2)}$, $(\Omega, \Gamma) \otimes \text{H}^{(3)}$
³ N-II	3	$f_3, i=5,8,10$	✓	

Table 1. Summary of all cubic degenerate classes. The subscript i in free functions indicates which functions among b_i are free. Examples: (1): theories obtained by the generalised conformal transformation (Ω) from beyond Horndeski (bH). (2): theories obtained by the generalised disformal transformation (Γ) from Horndeski (H). This is equivalent to a combination of Horndeski and beyond Horndeski [25]. (3): theories obtained by the generalised conformal and disformal transformation from Horndeski. See section 5 for discussions about the generalised conformal and disformal transformation.

4 Merging quadratic with cubic theories

In this section we wish to determine all the theories of the form (2.2), i.e. quadratic plus cubic Lagrangians, that are degenerate. Adding two degenerate Lagrangians does not always yield a degenerate one. This is the case only if the null eigenvectors associated with the two Lagrangians coincide. Therefore, in order to see whether the combination of two Lagrangians is viable, one needs to compare their eigenvectors, which are all listed in appendix C for quadratic theories and in appendix D for cubic ones, and check when they are equal.

We present four tables describing all the different possibilities for merging quadratic and cubic theories. We indicate with ✓ theories that can be freely combined, with X theories that cannot be combined, and with (n) theories that can be combined imposing the additional condition(s) (n) listed below each table.

Minimally coupled quadratic plus minimally coupled cubic theories.

	³ M-I	³ M-II	³ M-III	³ M-IV	³ M-V	³ M-VI	³ M-VII
² M-I	(1)	(2)	✓	X	(3)	X	X
² M-II	X	X	✓	✓	X	X	(4)
² M-III	X	X	X	X	✓	✓	(5)

Conditions:

- (1). $b_4 = \frac{-6a_1b_1+4a_2b_2+a_3X(9b_1+2b_2)}{2X(a_1+3a_2)}$
- (2). $b_6 = \frac{3(6a_1b_1+4a_2b_3+a_3X(2b_3-9b_1))}{4X(a_1+3a_2)}$
- (3). $b_4 = -\frac{3b_1(2a_1-3a_3X)}{2X(a_1+3a_2)}$
- (4). $b_7 = -3b_5$
- (5). $b_7 = 0$ (1 dof)

Notice that condition (5) eliminates also the 2 tensor dof, leaving the joined classes ${}^2\text{M-III} + {}^3\text{M-VII}$ with only one scalar dof.

The quartic beyond Horndeski theory L_4^{bH} is included in ${}^2\text{M-I}$, while the quintic beyond Horndeski theory L_5^{bH} (3.2) is included ${}^3\text{M-I}$. They satisfy the condition (1) thus the combination $L_4^{\text{bH}} + L_5^{\text{bH}}$ is still viable [20, 25].

Non-minimally coupled quadratic plus minimally coupled cubic theories.

	${}^3\text{M-I}$	${}^3\text{M-II}$	${}^3\text{M-III}$	${}^3\text{M-IV}$	${}^3\text{M-V}$	${}^3\text{M-VI}$	${}^3\text{M-VII}$
${}^2\text{N-I}$	(1) & (3)	(1) & (6)	(1)	X	(1) & (4)	X	
${}^2\text{N-II}$	X	X	X	X	X	X	(7)
${}^2\text{N-III}$	(3)	(6)	✓	X	(4)	X	X
${}^2\text{N-IV}$	(2) & (3)	(2) & (6)	(2)	X	(5)	X	X

Conditions:

- (1). $a_3 = -\frac{8a_1f_2X}{f_2} + \frac{6a_1+4f_2X}{X} - \frac{4f_2}{X^2}$
- (2). $a_3 = \frac{12a_2f_2X}{f_2} - \frac{8(a_2-f_2X)}{X} - \frac{6f_2}{X^2}$
- (3). $b_4 = \frac{2f_2X(9b_1+2b_2)}{f_2} - \frac{2(6b_1+b_2)}{X}$
- (4). $b_4 = 6b_1 \left(\frac{3f_2X}{f_2} - \frac{2}{X} \right)$
- (5). $b_4 = \frac{3b_1(X(a_3X+4f_2X)-2f_2)}{2X(a_2X+f_2)}$
- (6). $b_6 = \frac{3(6b_1f_2-9b_1f_2X-X-b_3f_2+2b_3f_2X)}{f_2X}$
- (7). $b_7 = -b_5$

The quartic Horndeski theory L_4^{H} (2.18) is included in ${}^2\text{N-I}$. The combination $L_4^{\text{H}} + L_5^{\text{bH}}$ does *not* satisfy the conditions (1) and (3), thus this combination is not degenerate [20, 25].

Minimally coupled quadratic plus non-minimally coupled cubic theories.

	² M-I	² M-II	² M-III
³ N-I	X	X	X
³ N-II	X	X	X

The quintic Horndeski theory L_5^H (2.19) is included in ³N-I. As can be seen from the table, it is not possible to combine ³N-I and ²M-I thus the combination $L_5^H + L_4^{bH}$ is not viable [25].

Non-minimally coupled quadratic plus non-minimally coupled cubic theories.

	² N-I	² N-II	² N-III	² N-IV
³ N-I	(1)	X	X	X
³ N-II	X	✓	X	X

Conditions:

$$(1). \quad b_4 = \frac{-a_1 f_{3X} X - 6b_1 f_2 + 6b_1 f_{2X} X + 2f_2 f_{3X}}{f_2 X}$$

$$a_3 = \frac{2(b_1(9a_1 f_2 X - 12a_1 f_{2X} X^2 + 6f_2 f_{2X} X - 6f_2^2) + 2f_{3X}(f_2 - a_1 X)^2)}{3b_1 f_2 X^2}$$

The classes ²N-I and ³N-I contain three free functions each, thus the combination ²N-I + ³N-I contains four free functions due to the conditions (1). In the next section, we show that this theory can be obtained by the generalised conformal and disformal transformation from $L_4^H + L_5^H$.

5 Conformal and disformal transformation

We now investigate which ones among the cubic theories can be obtained from known Lagrangians through conformal and disformal transformations. The same analysis for quadratic theories can be found in [26, 27]. First we identify the class of theories minimally coupled with gravity (i.e. $f_3 = 0$) that can be obtained from beyond Horndeski (3.2) by a conformal transformation. Then, we study the class of theories that can be obtained from Horndeski theory (2.19) by a conformal together with a disformal transformation.

5.1 Conformal transformation on Beyond Horndeski

It was shown in [25] that under the generalised disformal transformation

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + \Gamma(X)\phi_\mu\phi_\nu, \tag{5.1}$$

beyond Horndeski theory is transformed into itself:

$$\bar{L}_5^{bH}[\bar{f}] = L_5^{bH}[f], \tag{5.2}$$

where $f = \bar{f}/(1 + X\Gamma)^{7/2}$. On the other hand, under the generalised conformal transformation

$$\bar{g}_{\mu\nu} = \Omega(X)g_{\mu\nu}, \quad (5.3)$$

it transforms as

$$\bar{L}_5^{\text{bH}}[\bar{f}] = L_5^{\text{bH}}[f] + \sum_i \hat{b}_i L_i^{(3)}, \quad (5.4)$$

where

$$\begin{aligned} f &= \frac{\bar{f}}{\Omega^2}, & \hat{b}_4 &= -\hat{b}_6 = \frac{3\bar{f}X\Omega_X}{\Omega^3}, & \hat{b}_8 &= \frac{6\bar{f}\Omega_X}{\Omega^3}, \\ \hat{b}_9 &= \frac{6\bar{f}\Omega_X(X\Omega_X - \Omega)}{\Omega^4}, & \hat{b}_{10} &= \frac{6\bar{f}\Omega_X^2(X\Omega_X - \Omega)}{\Omega^5}, \end{aligned} \quad (5.5)$$

and the other \hat{b}_i vanish. In terms of the total b_i , this gives

$$\begin{aligned} b_1 &= Xf, & b_2 &= -3Xf, & b_3 &= 2Xf, & b_4 &= -b_6 = -3f + \frac{3fX\Omega_X}{\Omega}, & b_5 &= -b_7 = 6f, \\ b_8 &= \frac{6f\Omega_X}{\Omega}, & b_9 &= \frac{6f\Omega_X(X\Omega_X - \Omega)}{\Omega^2}, & b_{10} &= \frac{6f\Omega_X^2(X\Omega_X - \Omega)}{\Omega^3}. \end{aligned} \quad (5.6)$$

These b 's satisfy conditions (3.1), thus this theory is included in class ³M-I.

5.2 Conformal and disformal transformation on Horndeski

The generalised conformal and disformal transformation

$$\bar{g}_{\mu\nu} = \Omega(X)g_{\mu\nu} + \Gamma(X)\phi_\mu\phi_\nu, \quad (5.7)$$

transforms the Horndeski action as

$$\bar{L}_5^{\text{H}}[\bar{f}_3] = L_5^{\text{H}}[f_3] + L_5^{\text{bH}}[f] + \sum_i b_i L_i^{(3)}, \quad (5.8)$$

where

$$f_3 = \frac{\bar{f}_3\sqrt{\Omega}}{\sqrt{\Omega + X\Gamma}} + \int \bar{f}_3 \frac{(\Omega - X\Omega_X)\Gamma + X\Omega\Gamma_X}{2\sqrt{\Omega}(\Omega + X\Gamma)^{3/2}} dX, \quad (5.9)$$

$$f = \frac{\bar{f}_3\bar{X}\sqrt{\Omega}(\Omega_X + X\Gamma_X)}{3(\Omega + X\Gamma)^{5/2}}, \quad (5.10)$$

$$b_4 = -b_6 = \frac{\bar{f}_3\bar{X}\Omega_X\sqrt{\Omega}}{(\Omega + X\Gamma)^{5/2}}, \quad (5.11)$$

$$b_5 = -b_7 = \frac{2\bar{f}_3\bar{X}\Omega_X[X(\Omega_X + X\Gamma_X) - \Omega]}{\sqrt{\Omega}(\Omega + X\Gamma)^{5/2}}, \quad (5.12)$$

$$b_8 = \frac{2X\bar{f}_3\bar{X}\Omega_X[\Omega_X(\Omega_X + X\Gamma_X) + \Omega\Gamma_X]}{\Omega^{3/2}(\Omega + X\Gamma)^{5/2}}, \quad (5.13)$$

$$b_9 = -\frac{2X\bar{f}_3\bar{X}\Omega_X\Gamma_X}{\sqrt{\Omega}(\Omega + X\Gamma)^{5/2}}, \quad (5.14)$$

$$b_{10} = -\frac{2X\bar{f}_3\bar{X}\Omega_X^2\Gamma_X}{\Omega^{3/2}(\Omega + X\Gamma)^{5/2}}. \quad (5.15)$$

and the other b_i vanish. One can check that this theory satisfies the conditions (3.7), thus it is included in class ${}^3\text{N-I}$. Theories in class ${}^3\text{N-I}$ have three free functions. On the other hand, the action (5.8) contains \bar{f}_3 , Ω and Γ . Thus there is the same number of free functions. Indeed we can relate $\bar{f}_{3\bar{X}}$, Ω_X and Γ_X to f_{3X} , f and b_4 as

$$\bar{f}_{3\bar{X}} = \frac{f_{3X} (\Omega + X\Gamma)^{5/2}}{\sqrt{\Omega} [\Omega - X (\Omega_X + X\Gamma_X)]}, \quad (5.16)$$

$$\Omega_X = \frac{b_4 \Omega}{3Xf + f_{3X}}, \quad (5.17)$$

$$\Gamma_X = \frac{\Omega (3f - b_4)}{X (3Xf + f_{3X})}. \quad (5.18)$$

Thus, theories in class ${}^3\text{N-I}$ can be mapped to Horndeski if the transformation (5.7) is invertible.

Finally we consider the generalised conformal and disformal transformation from $L_4^{\text{H}} + L_5^{\text{H}}$. Using the result for the transformation of L_4^{H} obtained in [26, 27], we can show that this theory corresponds to the combination of ${}^2\text{N-I}$ and ${}^3\text{N-I}$ and satisfies the condition (1). This theory has four free functions, which correspond to \bar{f}_2 , \bar{f}_3 , Ω and Γ . Thus this theory can be regarded as the ‘‘Jordan frame’’ version of the Horndeski theory where the gravitational part of the Lagrangian is described by Horndeski with the metric $\bar{g}_{\mu\nu}$, $L_4^{\text{H}}[\bar{g}] + L_5^{\text{H}}[\bar{g}]$, while the matter is non-minimally coupled through $g_{\mu\nu}$. By performing the generalised conformal and disformal transformation, the gravitational action is described by the combination of ${}^2\text{N-I}$ and ${}^3\text{N-I}$ and the metric is minimally coupled to matter.

6 Conclusions

In this paper, we presented all Ostrogradsky ghost-free theories that are at most cubic in the second derivative of the scalar field, and that propagate at most three degrees of freedom. Extending Horndeski’s results, we have found new Lagrangians, which lead to higher order equations of motion but avoid Ostrogradsky instabilities by means of constraints that prevent the propagation of dangerous extra degrees of freedom.

In order to achieve our results, we used the degeneracy criterium introduced in [20], and classified the Lagrangians that are degenerate, i.e. whose Hessian matrix, obtained by taking the second derivatives of the Lagrangian with respect to velocities, is degenerate. In total, we identified seven classes of minimally coupled cubic theories and two classes of non-minimally coupled cubic theories, which contain as subclasses all known scalar-tensor theories which are cubic in second derivatives of the scalar field. We also investigated which cubic theories admit a well-defined Minkowski limit, i.e. when the metric is frozen to its Minkowski value. Our results are summarised in the table 1.

We then studied in which cases it is possible to combine any of these cubic theories with the previously identified quadratic ones. Note that one can also add arbitrary terms of the form $P(X, \phi)$ and $Q(X, \phi)\square\phi$ without changing the degeneracy of the total Lagrangian. We confirmed the previous finding that the combination of quartic or quintic

beyond Horndeski with a different Horndeski is not viable. Finally, we studied whether our cubic theories are related to known Lagrangians through generalised conformal or disformal transformations. We identified the theory, with four free functions, that is obtained by the generalised conformal and disformal transformation from the combination of quartic and quintic Horndeski Lagrangians.

Various interesting developments are left for the future. First, phenomenological aspects of these new theories should be investigated, in particular studying the existence of stable cosmological FLRW solutions — possibly self-accelerating — and their properties, by using for instance the effective description of dark energy (see e.g. [30] for a review and [31] for a recent generalization that includes non-minimal couplings to matter). It would also be worth analysing possible distinctive features of screening mechanisms in these set-ups. Secondly, on the theory side, it would be interesting to analyse further generalizations of scalar-tensor theories containing higher powers of second derivatives of the scalar field. Such theories do not admit a well-defined Minkowski limit, and some explicit examples have been discussed in [26] and in [32]. A more complete classification using the techniques we presented should be feasible, and left for future investigations.

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A Curvature dependent Lagrangians

Curvature tensors depend quadratically on the extrinsic curvature so, according to the kinetic structure presented in section 2.2, their combination with the second derivative of the scalar field yields cubic powers in velocities. All the possible quadratic and cubic terms in velocities involving the curvature are

$$L_R = \sum_{i=1}^2 L_2[f_i] + \sum_{i=3}^9 L_3[f_i], \tag{A.1}$$

where

$$L_2[f_1] = f_1 R_{\mu\nu} \phi^\mu \phi^\nu, \quad L_2[f_2] = f_2 R; \tag{A.2}$$

and

$$L_3[f_3] = f_3 R_{\mu\nu} \phi^{\mu\nu}, \tag{A.3}$$

$$L_3[f_4] = f_4 R \square \phi, \quad L_3[f_5] = f_5 R \phi_\mu \phi^{\mu\nu} \phi_\nu, \tag{A.4}$$

$$L_3[f_6] = f_6 R_{\mu\nu} \phi^\mu \phi^\nu \square \phi, \quad L_3[f_7] = f_7 R_{\mu\nu} \phi^\mu \phi^\nu \phi_\rho \phi^{\rho\sigma} \phi_\sigma, \tag{A.5}$$

$$L_3[f_8] = f_8 R_{\mu\nu} \phi^{\mu\rho} \phi_\rho \phi_\nu, \quad L_3[f_9] = f_9 R_{\mu\nu\rho\sigma} \phi^\mu \phi^{\nu\rho} \phi^\sigma, \tag{A.6}$$

where f_i are arbitrary functions of ϕ and X . Only one of the two quadratic Lagrangians in (A.2) is independent, since it is possible to express one in terms of the other through integrations by parts: we worked with $L_2[f_2]$. Also the cubic Lagrangians (A.3)–(A.6) are not all independent: we can obtain $L_3[f_9]$ from $L_3[f_6]$ and $L_3[f_8]$ using integrations by parts, and $L_3[f_8]$ from $L_3[f_4]$, $L_3[f_5]$ and $L_3[f_3]$ using also the Bianchi identity. Therefore, we are left with five cubic independent Lagrangians (A.3)–(A.5). To keep contact with Horndeski theory, without loss of generality it is useful to replace (A.3) with the following expression

$$L_3[f_3] = f_3 G_{\mu\nu} \phi^{\mu\nu}, \tag{A.7}$$

that we studied in the main text.

In this appendix we concentrate separately on the four remaining cubic non-minimally coupled Lagrangians (A.4)–(A.5). What characterises these Lagrangians in comparison with (A.7) is that they all feature time (and space) derivatives of the extrinsic curvature. This indicates the possible presence of additional Ostrogradsky modes, this time coming from the metric sector of the theory, unless there are suited extra primary constraints that remove them.

The covariant 3+1 decomposition of (A.4)–(A.5) shows that the only components of the extrinsic curvature that acquire time derivatives are the scalar ones:

$$E \equiv \frac{\hat{A}^i \hat{A}^j}{\hat{A}^2} K_{ij}, \quad F \equiv \hat{P}^{ij} K_{ij}. \tag{A.8}$$

Their covariant velocities appear in the form

$$V_E \equiv n^\mu \nabla_\mu E, \quad V_F \equiv n^\mu \nabla_\mu F, \tag{A.9}$$

in analogy to what we encountered in section 2.2. Therefore, applying the same kind of field redefinition used for the scalar field (2.1), Lagrangians (A.4)–(A.5) generally propagate two more Ostrogradsky modes, E and F , in addition to A_* . To avoid their propagation, we need two more primary constraints.

Defining the conjugate momenta associated to the new fields

$$\pi_E \equiv \frac{\delta L}{\delta V_E}, \quad \pi_F \equiv \frac{\delta L}{\delta V_F}, \tag{A.10}$$

for the set of Lagrangians (A.4)–(A.5) we obtain

$$\pi_* = \alpha V_E + \beta V_F + \dots, \quad \pi_E = \alpha V_*, \quad \pi_F = \beta V_*, \tag{A.11}$$

where

$$\alpha = -2f_4 - X f_6 + A_*^2 (2f_5 + X f_7), \quad \beta = -2f_4 + A_*^2 (2f_5 + f_6) - A_*^4 f_7, \tag{A.12}$$

and the dots in π_* represent non relevant terms. From the form of the momenta (A.11), it is clear that a total of three primary constraints can only be obtained in the trivial way, i.e.

$$\pi_* \approx 0, \quad \pi_E \approx 0, \quad \pi_F \approx 0. \tag{A.13}$$

Hence $\alpha = \beta = 0$ and, due to the Lorentz invariance of f_i , relations (A.12) give

$$f_4 = f_5 = f_6 = f_7 = 0. \tag{A.14}$$

B Tensorial structure implied by the degeneracy conditions

First, we show the equivalence between the relations (2.40) and (2.42) for quadratic theories. It is simple to show that (2.40) is equivalent to

$$M_{\rho\sigma}^{\alpha\beta} C_{(2)}^{\rho\sigma}(L) = 0 \quad \text{with} \quad C_{(2)}^{\rho\sigma}(L) \equiv C_{(2)}^{\mu\nu\rho\sigma} L_{\mu\nu} \quad (\text{B.1})$$

and

$$M_{\rho\sigma}^{\alpha\beta} \equiv -A_* g_{(\rho}^{\alpha} g_{\sigma)}^{\beta} + 2n_{(\rho} g_{\sigma)}^{(\alpha} A^{\beta)}. \quad (\text{B.2})$$

Indeed, decomposing (B.1) in the directions $n_\alpha n_\beta$, $h_\alpha^i h_\beta^j$ and $n_\alpha h_\beta^i$ leads to the equations (2.40). As a consequence, $C_{(2)}^{\rho\sigma}(L)$ is necessarily in the kernel of M viewed as an operator acting on symmetric 4 dimensional matrices. A matrix $V^{\mu\nu}$ is in the kernel of M when

$$M_{\mu\nu}^{\alpha\beta} V^{\mu\nu} = 0 \iff A_* V^{\alpha\beta} = 2n_\mu V^{\mu(\alpha} A^{\beta)} \quad (\text{B.3})$$

$$\iff \exists V^\mu \text{ s.t. } V^{\alpha\beta} = V^{(\alpha} A^{\beta)} \text{ with } V_\mu n^\mu = 0. \quad (\text{B.4})$$

Furthermore, the only available vector V^μ in the theory which is orthogonal to n_μ is in the direction \hat{A}^μ . Hence, there exists a scalar σ such that

$$C_{(2)}^{\mu\nu\rho\sigma} L_{\rho\sigma} = 2\sigma A^{(\mu} \hat{A}^{\nu)}, \quad (\text{B.5})$$

which is the relation (2.42).

The generalization to cubic theories is rather immediate. Let us show that (2.43) and (2.41) are equivalent. Following the same strategy as previously, we first show that (2.41) is equivalent to

$$M_{\rho\sigma}^{\alpha\beta} C_{(3)}^{\mu\nu\rho\sigma}(L) = 0 \quad \text{with} \quad C_{(3)}^{\mu\nu\rho\sigma}(L) \equiv C_{(3)}^{\mu\nu\rho\sigma\gamma\delta} L_{\gamma\delta}, \quad (\text{B.6})$$

with M defined as in the quadratic case by (B.2).

Now, both M and $C_{(3)}(L)$ can be viewed as operators acting on symmetric 4 dimensional matrices. Thus, (B.6) means that $C_{(3)}(L)$ and M are orthogonal, or equivalently that the image of $C_{(3)}(L)$ lies in the kernel of M . To go further, we recall that the kernel of M is defined by (B.3). The vector space orthogonal to n^μ is three dimensional and a basis is given by h_μ^γ where γ labels the elements of the basis (only 3 out of the 4 components of h_μ^γ are independent). Thus, if we use the notation $V_\gamma^{\mu\nu}$ for a basis of $\text{Ker}(M)$ where γ labels the elements of the basis, then $V_\gamma^{\mu\nu} = h_\gamma^{(\mu} A^{\nu)}$ which is clearly of the form (B.3). Hence, due to symmetries, $C_{(3)}(L)$ can be written as

$$C_{(3)}^{\mu\nu\rho\sigma}(L) = m^{\gamma\delta} V_\gamma^{\mu\nu} V_\delta^{\rho\sigma} \quad (\text{B.7})$$

where $m^{\gamma\delta}$ is a symmetric matrix. Due to the covariance, the symmetric matrix m is necessarily of the form $m^{\gamma\delta} = 4\sigma_1 g^{\gamma\delta} + 4\sigma_2 A^\gamma A^\delta$ where σ_1 and σ_2 are scalars. Notice that there is no components of the form $A^{(\gamma} n^{\delta)}$ nor of the form $n^\gamma n^\delta$ in $m^{\gamma\delta}$ because $V_\gamma^{\mu\nu} n^\gamma = 0$. As a conclusion, (B.6) is true if and only if there exist scalars σ_1 and σ_2 such that:

$$C_{(3)}^{\mu\nu\rho\sigma}(L) = 4\sigma_1 A^{(\mu} h^{\nu)(\rho} A^{\sigma)} + 4\sigma_2 A^{(\mu} \hat{A}^{\nu)} A^{(\rho} \hat{A}^{\sigma)}. \quad (\text{B.8})$$

C Quadratic theories

We review the quadratic theories proposed in [20] and further classified in [26] and [27].

C.1 Minimally coupled theories

- **²M-I.** Three free functions a_1 , a_2 , and a_3 , together with

$$a_4 = -2 \frac{a_1}{X}, \quad a_5 = \frac{4a_1(a_1 + 2a_2) - 4a_1a_3X + 3a_3^2X^2}{4(a_1 + 3a_2)X^2}. \quad (\text{C.1})$$

We assume $a_2 \neq -a_1/3$. This case includes beyond Horndeski theory. The corresponding null eigenvector is given by

$$v_1 = -\frac{X(2a_2 + a_3X)}{A_*(2a_1(A_*^2 + 2X) + 2a_2(2A_*^2 + 5X) - a_3X(A_*^2 + X))}, \quad (\text{C.2})$$

$$v_2 = \frac{-2a_1 - 4a_2 + a_3X}{A_*(2a_1(A_*^2 + 2X) + 2a_2(2A_*^2 + 5X) - a_3X(A_*^2 + X))}. \quad (\text{C.3})$$

This class was called M-I in [26] and IIIa in [27].

- **²M-II.** Three free functions a_1 , a_4 , a_5 and

$$a_2 = -\frac{a_1}{3}, \quad a_3 = \frac{2a_1}{3X}. \quad (\text{C.4})$$

The corresponding null eigenvector is given by

$$v_1 = \frac{X}{A_*(A_*^2 + X)}, \quad v_2 = -\frac{1}{A_*(A_*^2 + X)}. \quad (\text{C.5})$$

This class was called M-II in [26] and IIIb in [27].

- **²M-III.** Four free functions a_2 , a_3 , a_4 , a_5 and the unique condition

$$a_1 = 0. \quad (\text{C.6})$$

The eigenvector is given by

$$v_1 = -\frac{X}{2A_*(A_*^2 + X)}, \quad v_2 = \frac{-2A_*^2 + X}{2A_*(A_*^2 + X)^2}. \quad (\text{C.7})$$

This class was called M-III in [26] and IIIc in [27].

For minimally coupled quadratic theories, the vector components of π_{ij} (i.e. $\hat{A}_i \hat{P}_{jk} \pi^{ij}$) are proportional to a_1 , therefore, as noticed in [28], this class propagates only one scalar dof.

C.2 Non-minimally coupled theories

► ²**N-I.** Three free functions f_2, a_1 and a_3 . The conditions are

$$a_2 = -a_1 \neq -\frac{f_2}{X}, \quad (\text{C.8})$$

$$a_4 = \frac{1}{8(f_2 - a_1 X)^2} [4f_2 (3(a_1 - 2f_{2X})^2 - 2a_3 f_2) - a_3 X^2 (16a_1 f_{2X} + a_3 f_2) + 4X (3a_1 a_3 f_2 + 16a_1^2 f_{2X} - 16a_1 f_{2X}^2 - 4a_1^3 + 2a_3 f_2 f_{2X})], \quad (\text{C.9})$$

$$a_5 = \frac{1}{8(f_2 - a_1 X)^2} (2a_1 - a_3 X - 4f_{2X}) [a_1 (2a_1 + 3a_3 X - 4f_{2X}) - 4a_3 f_2]. \quad (\text{C.10})$$

The combination of Horndeski and beyond Horndeski theories is included in this class. The corresponding null eigenvector is given by

$$v_1 = D A_* (a_1 X - f_2) (2a_1 - a_3 X - 4f_{2X}), \quad (\text{C.11})$$

$$v_2 = D A_* (a_1 (2a_1 + a_3 X - 4f_{2X}) - 2a_3 f_2), \quad (\text{C.12})$$

with

$$D^{-1} \equiv a_1 (A_*^2 (-a_3 X^2 + 2f_2 + 12f_{2X} X) + A_*^4 (4f_{2X} - a_3 X) - 6f_2 X + 8f_{2X} X^2) - 2a_1^2 (3A_*^2 X + A_*^4) + f_2 ((A_*^2 + X) (a_3 (2A_*^2 + X) - 4f_{2X}) + 4f_2).$$

This class was called N-I in [26] and Ia in [27].

► ²**N-II.** Three free functions f_2, a_4, a_5 and

$$a_2 = -a_1 = -\frac{f_2}{X}, \quad a_3 = \frac{2(f_2 - 2X f_{2X})}{X^2} \quad (\text{C.13})$$

The corresponding null eigenvector is given by

$$v_1 = 0, \quad v_2 = -\frac{A_*}{(A_*^2 + X)^2}. \quad (\text{C.14})$$

This class was called N-II in [26] and Ib in [27].

For non-minimally coupled quadratic theories, the vector components of π_{ij} are instead proportional to $f_2 - X a_1$, hence also here there are not tensor modes [28].

► ²**N-III.** Three free functions f_2, a_1 and a_2 . The conditions are

$$a_1 + a_2 \neq 0, \quad \text{and} \quad a_1 \neq \frac{f_2}{X}, \quad (\text{C.15})$$

$$a_3 = \frac{4f_{2X}(a_1 + 3a_2)}{f_2} - \frac{2(a_1 + 4a_2 - 2f_{2X})}{X} - \frac{4f_2}{X^2}, \quad (\text{C.16})$$

$$a_4 = \frac{2f_2}{X^2} + \frac{8f_{2X}^2}{f_2} - \frac{2(a_1 + 2f_{2X})}{X}, \quad (\text{C.17})$$

$$a_5 = \frac{2}{f_2^2 X^3} [4f_2^3 + f_2^2 X (3a_1 + 8a_2 - 12f_{2X}) + 8f_2 f_{2X} X^2 (f_{2X} - a_1 - 3a_2) + 6f_{2X}^2 X^3 (a_1 + 3a_2)]. \quad (\text{C.18})$$

The corresponding null eigenvector is given by

$$v_1 = \frac{X(f_2 - 2f_{2X}X)}{A_*(2A_*^2(f_2 - f_{2X}X) + X(3f_2 - 2f_{2X}X))}, \quad (\text{C.19})$$

$$v_2 = \frac{2f_{2X}X - 2f_2}{A_*(2A_*^2(f_2 - f_{2X}X) + X(3f_2 - 2f_{2X}X))}. \quad (\text{C.20})$$

This class was called N-III (i) in [26] and IIa in [27].

► ²N-IV. Three free functions f_2 , a_2 and a_3 . The conditions are

$$a_1 + a_2 \neq 0 \quad (\text{C.21})$$

$$a_1 = \frac{f_2}{X}, \quad (\text{C.22})$$

$$a_4 = \frac{8f_{2X}^2}{f_2} - \frac{4f_{2X}}{X}, \quad (\text{C.23})$$

$$a_5 = \frac{1}{4f_2X^3(f_2 + a_2X)} [f_2a_3^2X^4 - 4f_2^3 - 8f_2^2X(a_2 - 2f_{2X}) - 4f_2X^2(4f_{2X}(f_{2X} - 2a_2) + a_3f_2) + 8f_{2X}X^3(a_3f_2 - 4a_2f_{2X})]. \quad (\text{C.24})$$

The corresponding null eigenvector is given by

$$v_1 = -2EX(a_2X + f_2)(f_2 - 2f_{2X}X), \quad (\text{C.25})$$

$$v_2 = E \frac{4X(a_2X + f_2)(f_2 - 2f_{2X}X) - A_*^2(f_2X(4a_2 + a_3X - 4f_{2X}) - 8a_2f_{2X}X^2 + 2f_2^2)}{A_*^2 + X}, \quad (\text{C.26})$$

with

$$E^{-1} \equiv A_*^3(f_2X(4a_2 + a_3X - 4f_{2X}) - 8a_2f_{2X}X^2 + 2f_2^2) + A_*X^2(2a_2(f_2 - 4f_{2X}X) + a_3f_2X - 4f_2f_{2X}). \quad (\text{C.27})$$

This class was called N-III (ii) in [26] and IIb in [27].

Due to the condition $f_2 - Xa_1 = 0$, only one scalar dof is present in this class [28].

D Identifying cubic theories

In this appendix we solve in details the conditions (2.43) for purely cubic theories. Let us first note that, since \mathcal{V}_{ij} lies in the hyper-surface orthogonal to n^μ , it can be decomposed as follows

$$\mathcal{V}_{ij} = v_1 h_{ij} + v_2 \hat{A}_i \hat{A}_j, \quad (\text{D.1})$$

where v_1 and v_2 are scalar quantities. The tensor $L_{\mu\nu}$, introduced in (2.39), can thus be written as

$$\begin{aligned} L_{\mu\nu} &\equiv \lambda_{\mu\nu} + \Lambda_{\mu\nu}^{ij} \mathcal{V}_{ij} = \lambda_{\mu\nu} + v_1 \Lambda_{\mu\nu}^{ij} h_{ij} + v_2 \Lambda_{\mu\nu}^{ij} \hat{A}_i \hat{A}_j, \\ &= \ell_1 n_\mu n_\nu + 2\ell_2 n_{(\mu} A_{\nu)} - A_*(v_1 g_{\mu\nu} + v_2 A_\mu A_\nu), \end{aligned} \quad (\text{D.2})$$

with

$$\ell_1 \equiv 1 + A_* v_1 + A_*(2X + A_*^2)v_2 \quad \text{and} \quad \ell_2 \equiv v_1 + Xv_2. \quad (\text{D.3})$$

D.1 Minimally coupled theories

A long but straightforward calculation shows that the tensorial relation (2.43) leads to the following 12 equations:

$$\ell_1 b_2 = 0, \quad \ell_1 b_3 = 0, \quad \ell_1 b_6 = 0, \quad (\text{D.4})$$

$$(\ell_1 A_* + X\ell_2)b_5 + 2\ell_2 b_2 = 0, \quad (\ell_1 A_* + X\ell_2)b_7 + 3\ell_2 b_3 = 0, \quad (\text{D.5})$$

$$\ell_1 b_7 = 12(m_1 + A_*^2 m_2), \quad (\ell_1 A_* + X\ell_2)b_8 + \ell_2(2b_6 + b_7) = 12A_* m_2, \quad (\text{D.6})$$

$$\begin{aligned} & \ell_1(-b_9 + 3A_*^2 b_{10}) + 2\ell_2 A_*(3Xb_{10} + b_8 + b_9) \\ -A_*[(v_1 + Xv_2)(2b_8 + 3Xb_{10}) + (4v_1 + Xv_2)b_9 + 2v_2 b_6 + v_2 b_7] &= 12m_2, \end{aligned} \quad (\text{D.7})$$

$$\begin{aligned} & \ell_1(-b_4 + A_*^2 b_9) + \ell_2 A_*(2b_9 X + 2b_4 + b_5) \\ -A_*[b_4(4v_1 + Xv_2) + (b_5 + Xb_9)(v_1 + Xv_2) + (b_6 v_1 + b_2 v_2)] &= 0, \end{aligned} \quad (\text{D.8})$$

$$\begin{aligned} & \ell_1(-b_5 + b_8 A_*^2) + 2\ell_2 A_*(b_8 X + b_5 + b_7) \\ -A_*[b_5(4v_1 + Xv_2) + 3b_3 v_2 + b_7(3v_1 + 2Xv_2) + b_8 X(v_1 + Xv_2)] &= 12m_1, \end{aligned} \quad (\text{D.9})$$

$$\begin{aligned} & \ell_1(-3b_1 + b_4 A_*^2) + 2\ell_2 A_*(b_4 X + 3b_1) \\ -A_*[3b_1(4v_1 + v_2 X) + 2b_2 v_1 + b_4 X(v_1 + Xv_2)] &= 0, \end{aligned} \quad (\text{D.10})$$

$$\begin{aligned} & \ell_1(-b_2 + b_6 A_*^2) + 2\ell_2 A_*(b_6 X + b_2) \\ -A_*[b_2(4v_1 + Xv_2) + 3b_3 v_1 + b_6 X(v_1 + Xv_2)] &= 0. \end{aligned} \quad (\text{D.11})$$

The two equations in (D.6) enable us to solve for σ_1 and σ_2 , yielding

$$\sigma_1 = \frac{1}{12} [(\ell_1 - A_* \ell_2)b_7 - A_*(\ell_1 A_* + X\ell_2)b_8 - 2A_* \ell_2 b_6], \quad (\text{D.12})$$

$$\sigma_2 = \frac{1}{12A_*} [(\ell_1 A_* + X\ell_2)b_8 + \ell_2(2b_6 + b_7)]. \quad (\text{D.13})$$

Hence, if we replace these expressions in the previous system, we end up with 10 equations for the 10 unknown b_i . These 10 equations can be written in a matrix form as follows:

$$\left(\begin{array}{c|c} & \\ \hline \mathbf{A} & \mathbf{0} \\ \hline \mathbf{C} & \mathbf{B} \end{array} \right) \begin{pmatrix} b_2 \\ b_3 \\ b_6 \\ b_5 \\ b_7 \\ b_1 \\ b_4 \\ b_8 \\ b_9 \\ b_{10} \end{pmatrix} = 0, \quad (\text{D.14})$$

where

$$\mathbf{A} \equiv \begin{pmatrix} \ell_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \ell_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ell_1 & 0 & 0 & 0 \\ 2\ell_2 & 0 & 0 & T & 0 & 0 \\ 0 & 3\ell_2 & 0 & 0 & T & 0 \\ S & -3A_*v_1 & A_*T & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{B} \equiv \begin{pmatrix} A_*T & 0 & 0 & 0 \\ 0 & 2A_*T & 0 & 0 \\ S & 0 & A_*T & 0 \\ 0 & -T & SA_* & 3A_*^2T \end{pmatrix}, \quad (\text{D.15})$$

$$\mathbf{C} \equiv \begin{pmatrix} -2A_*v_1 & 0 & 0 & 0 & 0 & 3S \\ 0 & -3A_*v_2 & 2A_*\ell_2 & S & -\ell_1 + XA_*v_2 & 0 \\ -A_*v_2 & 0 & -A_*v_1 & 0 & 0 & 0 \\ 0 & 0 & -2(A_*^2v_2 + \ell_2) & 0 & -(A_*^2v_2 + \ell_2) & 0 \end{pmatrix},$$

and $\mathbf{0}$ denotes a 6×4 matrix of zeros. We have also introduced the notation $T \equiv \ell_1 A_* + X\ell_2$ and $S \equiv -\ell_1 + \ell_2 A_* - 3v_1 A_*$.

The resolution of the system depends on the rank of the matrices \mathbf{A} and \mathbf{B} . To solve the system, it is useful to separate the vector in (D.14) into two pieces

$$\mathbf{b}_+ = (b_2, b_3, b_6, b_5, b_7, b_1) \quad \text{and} \quad \mathbf{b}_- = (b_4, b_8, b_9, b_{10}). \quad (\text{D.16})$$

Hence, we solve successively the following two matrix equations

$$\mathbf{A}\mathbf{b}_+ = 0 \quad \text{and} \quad \mathbf{C}\mathbf{b}_+ + \mathbf{B}\mathbf{b}_- = 0. \quad (\text{D.17})$$

We can distinguish several cases, depending on whether ℓ_1 or T vanish.

► $\ell_1 = 0$ and $T \neq 0$

In that case v_2 is related to v_1 by

$$v_2 = -\frac{1 + A_*v_1}{A_*(A_*^2 + 2X)}. \quad (\text{D.18})$$

The matrix \mathbf{A} is highly degenerate with rank=3 and $\mathbf{A}\mathbf{b}_+ = 0$ produces 3 conditions. Two of them give

$$b_5 = -\frac{2}{X}b_2, \quad b_7 = -\frac{3}{X}b_3, \quad (\text{D.19})$$

and the third one is

$$A_*(A_*^2(2b_2 + 3b_3 - Xb_6) + X(5b_2 + 6b_3 - Xb_6))v_1 = -X(b_2 + Xb_6). \quad (\text{D.20})$$

Equation (D.20) plus the four remaining equations $\mathbf{C}\mathbf{b}_+ + \mathbf{B}\mathbf{b}_- = 0$ are solved into three sectors.

³M-I). $9b_1 + 2b_2 \neq 0$:

$$b_6 = \frac{9b_1b_3 + 3b_4X(b_2 + b_3) - 2b_2^2}{X(9b_1 + 2b_2)},$$

$$\begin{aligned}
 b_8 &= \frac{9b_1b_3 - 6b_4X(b_2 + b_3) + 6b_2b_3 + 4b_2^2}{X^2(9b_1 + 2b_2)}, \\
 b_9 &= \frac{1}{X^2(9b_1 + 2b_2)^2} \left[3b_4^2X^2(9b_1 + 3b_2 + b_3) - 2b_4X(9b_1(b_2 - b_3) + 4b_2^2) \right. \\
 &\quad \left. + 24b_1b_2^2 + 54b_1^2b_2 + 27b_1^2b_3 + 4b_2^3 \right], \\
 b_{10} &= \frac{1}{X^3(9b_1 + 2b_2)^3} \left[3b_4^3X^3(9b_1 + 3b_2 + b_3) - 6b_2b_4^2X^2(9b_1 + 3b_2 + b_3) \right. \\
 &\quad \left. + 2b_4X(81b_1^2(b_2 + b_3) + 18b_1b_2(3b_2 + 2b_3) + 2b_2^2(5b_2 + 3b_3)) \right. \\
 &\quad \left. - 2(54b_1^2b_2(b_2 + 2b_3) + 4b_1b_2^2(7b_2 + 9b_3) + 81b_1^3b_3 + 4b_2^3(b_2 + b_3)) \right],
 \end{aligned}$$

and therefore

$$v_1 = -\frac{X(3b_1 + b_4X)}{A_*(A_*^2(6b_1 + 2b_2 - b_4X) + X(15b_1 + 4b_2 - b_4X))} \quad (\text{D.21})$$

As a conclusion, we end up with four free parameters b_1, b_2, b_3 and b_4 .

³M-II). $9b_1 + 2b_2 = 0$ and $9b_1 - 2b_3 \neq 0$:

$$\begin{aligned}
 b_2 &= -\frac{9}{2}b_1, & b_4 &= -\frac{3}{X}b_1, & b_8 &= \frac{3b_3 - 2b_6X}{X^2}, \\
 b_9 &= \frac{9b_1(b_3 + 2b_6X) - 81b_1^2 - 2b_6^2X^2}{3X^2(9b_1 - 2b_3)}, \\
 b_{10} &= \left[18b_6X(-12b_1b_3 + 27b_1^2 + 2b_3^2) - 36b_3(-8b_1b_3 + 18b_1^2 + b_3^2) \right. \\
 &\quad \left. - 12b_3b_6^2X^2 + 4b_6^3X^3 \right] \left[9X^3(9b_1 - 2b_3)^2 \right]^{-1},
 \end{aligned}$$

and

$$v_1 = \frac{X(2b_6X - 9b_1)}{A_*[2A_*^2(9b_1 - 3b_3 + b_6X) + X(45b_1 - 12b_3 + 2b_6X)]} \quad (\text{D.22})$$

The three parameters b_1, b_3, b_6 are free.

³M-III). $9b_1 + 2b_2 = 0$ and $9b_1 - 2b_3 = 0$:

$$\begin{aligned}
 b_2 &= -\frac{9b_1}{2}, & b_3 &= \frac{9b_1}{2}, & b_4 &= -\frac{3b_1}{X}, \\
 b_6 &= \frac{9b_1}{2X}, & b_8 &= \frac{9b_1}{2X^2}, & b_9 &= -\frac{3b_1}{2X^2}, & b_{10} &= -\frac{b_1}{X^3}.
 \end{aligned}$$

We obtain that b_1 is free whereas there is no constraint on v_1 .

► $\ell_1 = 0$ and $T = 0$: ³M-IV

This case is characterized by the fact that v_1 and v_2 are totally fixed by

$$v_1 = \frac{X}{A_*(X + A_*^2)}, \quad v_2 = -\frac{1}{A_*(X + A_*^2)}. \quad (\text{D.23})$$

Furthermore, $\mathbf{A}\mathbf{b}_+ = 0$ fixes

$$b_3 = -b_2. \tag{D.24}$$

The four remaining equations give

$$\begin{aligned} b_2 &= -\frac{9}{2}b_1, & b_6 &= -3b_4 - \frac{9}{2X}b_1, \\ b_7 &= -3b_5 + \frac{27}{2X}b_1, & b_9 &= \frac{3b_1 - 2X(2b_4 + b_5)}{2X^2}, \end{aligned}$$

and $b_1, b_4, b_5, b_8, b_{10}$ are free.

- $\ell_1 \neq 0$ and $T \neq 0$: ³M-V

In that case, \mathbf{B} is invertible and \mathbf{A} reaches its maximal rank = 5. Hence, from $\mathbf{A}\mathbf{b}_+ = 0$ we get

$$b_2 = b_3 = b_5 = b_6 = b_7 = 0. \tag{D.25}$$

The four remaining equations $\mathbf{C}\mathbf{b}_+ + \mathbf{B}\mathbf{b}_- = 0$ give

$$b_8 = 0, \tag{D.26}$$

together with three equations. If $b_1 = 0$ all the other functions must be zero, therefore we assume $b_1 \neq 0$ and we obtain

$$b_9 = \frac{b_4^2}{3b_1}, \quad b_{10} = \frac{b_4^3}{27b_1^2}, \tag{D.27}$$

plus one relation between v_1 and v_2 :

$$A_*b_4(A_* + (A_*^2 + X)v_1 + (A_*^2 + X)^2v_2) = 3b_1(1 + 3A_*v_1 + A_*(A_*^2 + X)v_2). \tag{D.28}$$

As a conclusion, only two parameters, b_1 and b_4 , are free. One of the two components v_1 or v_2 of the eigenvector is also a free parameter.

This class possesses two more primary constraints of the form (2.35), hence there is only one scalar dof.

- $\ell_1 \neq 0$ and $T = 0$

v_1 is fixed by

$$v_1 = -\frac{A_*}{X + A_*^2} - (X + A_*^2)v_2. \tag{D.29}$$

Solving $\mathbf{A}\mathbf{b}_+ = 0$ leads to

$$b_2 = b_3 = b_6 = 0. \tag{D.30}$$

Furthermore, the equations $\mathbf{C}\mathbf{b}_+ + \mathbf{B}\mathbf{b}_- = 0$ give two branches:

³M-VI). $b_7 = 0$

and the component v_2 is fixed to

$$v_2 = \frac{X - 2A_*^2}{2A_*(A_*^2 + X)^2}. \quad (\text{D.31})$$

The components b_1, b_4, b_5, b_8, b_9 and b_{10} are free.

Also in this class, tensor modes are eliminated by the primary constraints (2.35), so only one scalar dof is left.

³M-VII). $b_1 = b_4 = 0, \quad b_9 = -b_5/X$

and the component v_2 is fixed by

$$2A_*b_5(A_*^2 + X)^2 v_2 = X(b_5 + b_7) - 2A_*^2b_5. \quad (\text{D.32})$$

The components b_5, b_7, b_8 and b_{10} are free.

D.2 Non-minimally coupled theories

The resolution follows the same strategy as in the minimally coupled case. First, we write the generalised conditions in a form analogous to (D.14)

$$\begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{C} & \mathbf{B} \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{b}}_+ \\ \tilde{\mathbf{b}}_- \end{pmatrix} = \frac{X f_{3X}}{A_*} \Sigma, \quad (\text{D.33})$$

where Σ is a matrix given by

$$\Sigma = \begin{pmatrix} -\ell_2 \\ 2\ell_2/3 \\ v_2 \\ 2A_*v_2 \\ -2A_*v_2 \\ \ell_2 + A_*^2v_2 \\ -(\ell_2 + A_*^2v_2) \\ 0 \\ v_2 \\ 0 \end{pmatrix}. \quad (\text{D.34})$$

Hence, the solution for $\tilde{\mathbf{b}} = (\tilde{\mathbf{b}}_+, \tilde{\mathbf{b}}_-)$ is the sum of the general solution of the homogeneous equation (with $f_3 = 0$) and a particular solution. Again, we solve them according to whether ℓ_1 and T vanish or not.

When $\ell_1 = 0$ necessarily $v_1 = v_2 = 0$, which would imply in turn that $\ell_1 = 1$. This is an inconsistency, hence there is no solution when $\ell_1 = 0$.

► $\ell_1 \neq 0$ and $T \neq 0$: ${}^3\text{N-I}$

We need to assume $b_1 \neq 0$ otherwise $T = 0$, and we end up in the next class of theories.

$$\begin{aligned} b_2 &= -3b_1, & b_3 &= 2b_1, & b_6 &= -b_4, \\ b_5 &= \frac{2(f_{3X} - 3b_1)^2 - 2b_4f_{3X}X}{3b_1X}, & b_7 &= \frac{2b_4f_{3X}X - 2(f_{3X} - 3b_1)^2}{3b_1X}, \\ b_8 &= \frac{2(3b_1 + b_4X - f_{3X})((f_{3X} - 3b_1)^2 - b_4f_{3X}X)}{9b_1^2X^2}, \\ b_9 &= \frac{2b_4(3b_1 + b_4X - f_{3X})}{3b_1X}, & b_{10} &= \frac{2b_4(3b_1 + b_4X - f_{3X})^2}{9b_1^2X^2}, \end{aligned}$$

and b_1, b_4 and f_3 are free. Furthermore, the eigenvector is given by

$$v_1 = \frac{A_*(3b_1 + b_4X - f_{3X})}{A_*^2(b_4X - 3b_1 + f_{3X}) + A_*^4b_4 + f_{3X}X}, \quad (\text{D.35})$$

$$v_2 = -\frac{A_*b_4}{A_*^2(b_4X - 3b_1 + f_{3X}) + A_*^4b_4 + f_{3X}X}. \quad (\text{D.36})$$

This is the particular solution, now we need to find which homogeneous solution is compatible with it. To ensure the full theory to be degenerate, the eigenvectors of the homogeneous and the particular solutions must coincide. It is easy to show that it cannot be supplemented with any of the minimally coupled theories.

► $\ell_1 \neq 0$ and $T = 0$: ${}^3\text{N-II}$

We obtain

$$\begin{aligned} b_1 &= b_2 = b_3 = b_5 = b_7 = 0, \\ b_4 &= -b_6 = \frac{f_{3X}}{X}, & b_9 &= -\frac{2f_{3X}}{X^2}. \end{aligned}$$

Furthermore, the eigenvector is given by

$$v_1 = 0, \quad v_2 = -\frac{A_*}{(X + A_*^2)^2}. \quad (\text{D.37})$$

Now we study which homogeneous solution can be added to this particular one. It is possible to add only ${}^3\text{M-VII}$ where $b_5 + b_7 = 0$. Therefore, the full conditions for this class of theories are

$$\begin{aligned} b_1 &= b_2 = b_3 = 0, & b_7 &= -b_5, \\ b_4 &= -b_6 = \frac{f_{3X}}{X}, & b_9 &= -\frac{2f_{3X} + Xb_5}{X^2}. \end{aligned}$$

and b_5, b_8, b_{10} and f_3 are free.

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