On the L_p-theory of C_0-semigroups associated with second order elliptic operators. II

Vitali Liskevich
School of Mathematics
University of Bristol
Bristol BS8 1TW, UK

Zeev Sobol
School of Mathematics
University of Bristol
Bristol BS8 1TW, UK

Hendrik Vogt
Institut für Analysis
Fachrichtung Mathematik
Technische Universität Dresden
01062 Dresden, Germany

Abstract

We study positive C_0-semigroups on L_p associated with second order uniformly elliptic divergence type operators with singular lower order terms, subject to a wide class of boundary conditions. We obtain an interval $(p_{\min}, p_{\max})$ in the L_p-scale where these semigroups can be defined, including the case $2 \notin (p_{\min}, p_{\max})$. We present an example showing that the result is optimal. We also show that the semigroups are analytic with angles of analyticity and spectra of the generators independent of p, for the whole range of p where the semigroups are defined.

1 Introduction and main results

In this paper we continue to study the L_p-theory of second order elliptic differential operators on an open set $\Omega \subseteq \mathbb{R}^N$, $N \geqslant 3$, corresponding to the formal differential expression

$$\mathcal{L} = -\nabla \cdot (a\nabla) + b_1 \cdot \nabla + \nabla \cdot b_2 + V$$

with singular measurable coefficients $a\colon \Omega \to \mathbb{R}^N \otimes \mathbb{R}^N$, $b_1, b_2\colon \Omega \to \mathbb{R}^N$, $V\colon \Omega \to \mathbb{R}$. In [24] a quasi-contractive C_0-semigroup on $L_p := L_p(\Omega)$ is constructed, whose generator is associated with $\mathcal{L}$. In this paper we study the case of uniformly elliptic operators and show that, under some additional restrictions, the range of L_p-spaces in which one can associate a C_0-semigroup with $\mathcal{L}$, can be extended beyond the interval of quasi-contractivity. We also prove that the consistent semigroups associated with $\mathcal{L}$ on L_p are analytic with angles of analyticity and spectra of the generators independent of p.

The form associated with the above differential expression is

$$\tau(u,v) := \langle a\nabla u, \nabla v\rangle + \langle \nabla u, b_1 v\rangle - \langle b_2 u, \nabla v\rangle + \langle Vu, v\rangle \tag{1.1}$$

on a suitable domain $D(\tau)$ responding to the boundary conditions. (Here and in the sequel, $\langle f, g\rangle$ is defined as $\int_\Omega f(x)\cdot\overline{g}(x)\,dx$ whenever $f\cdot\overline{g} \in L_1$, for $f, g\colon \Omega \to \mathbb{C}$ or $f, g\colon \Omega \to \mathbb{C}^N$ measurable.)

Our main interest lies in the case when the semigroup associated with $\mathcal{L}$ can be defined on L_p for p from a proper subinterval of $[1,\infty)$. This case of the L_p-theory of second order elliptic operators has been extensively studied [2, 5, 12, 15, 18, 19, 20, 21]. However, most of the results are related to sectorial forms (especially to symmetric forms bounded below) and quasi-contractive semigroups. In [24] a general method of constructing positive C_0-semigroups on L_p corresponding to sesquilinear (not necessarily sectorial) forms in L_2 has been developed, and a precise condition for quasi-contractivity has been established.

It was first observed in [11] that the Schrödinger semigroup with $L_{N/2,weak}$-potential can be defined on L_p for certain p outside of the interval of quasi-contractivity. In [21] this result was extended to uniformly elliptic second order divergence type operators in $\mathbb{R}^N$ perturbed by a form bounded potential. Here we study a general second order differential expression $\mathcal{L}$ for a wide class of boundary conditions.

E.-M. Ouhabaz [17] was the first to establish analyticity of angle $\frac{\pi}{2}$ in $L_p(\mathbb{R}^N)$, $1 \leqslant p < \infty$, for symmetric semigroups satisfying Gaussian upper bounds. E. B. Davies [5] extended this result to a more general setting of metric spaces with polynomial volume growth. In [18] analyticity of angle $\frac{\pi}{2}$ was first shown for symmetric semigroups that are defined only for p from an interval in $[1,\infty)$, under the assumption of certain weighted estimates. In the present paper we prove analogous results for general uniformly elliptic second order operators. The result on p-independence of the spectrum we present here, which is an application of a criterion from [16], generalizes respective results from [8, 19, 20, 21].

The main tool of the present paper is a technique of weighted estimates analogous to that used in [19, 5, 18]. For further development of this technique with applications to L_p-theory we refer the reader to [14, 26].

We recall from [24] the following qualitative assumptions on the form τ.

(a) $a \in L_{1,loc}$, a is a.e. invertible with $a^{-1} \in L_{1,loc}$, and

$$|\operatorname{Im}\zeta^* a\zeta| \leqslant \alpha \operatorname{Re}\zeta^* a\zeta \quad \text{a.e. } (\zeta \in \mathbb{C}^N)$$

for some $\alpha \geqslant 0$, i.e., a is *uniformly sectorial* (ζ^* is the transpose of $\overline{\zeta}$). Let $a_s := \frac{a+a^\top}{2}$. Then

$$\tau_N(u,v) := \langle a\nabla u, \nabla v\rangle,\ D(\tau_N) := \left\{u \in W^1_{1,loc} \cap L_2;\ (\nabla u)^* a_s \nabla u \in L_1\right\}$$

defines a closed sectorial (non-symmetric) Dirichlet form in L_2. Let $\tau_a \subseteq \tau_N$ be a Dirichlet form.

(bV) The potentials $W_j := b_j^\top a_s^{-1} b_j$ ($j = 1,2$) and $|V|$ are τ_a-regular, i.e., $Q(W_j) \cap D(\tau_a)$ and $Q(|V|) \cap D(\tau_a)$ are dense in $D(\tau_a)$.

($Q(V)$ denotes the form domain of the multiplication operator V in L_2.)

We define the form τ on $D(\tau) := D(\tau_a) \cap Q(W_1 + W_2 + |V|)$ by (1.1).

As shown in [24], $D(\tau)$ is dense in $D(\tau_a)$, and the form $\tau + U_0 - U_0 \wedge m$ is sectorial and closed for all $U_0 \geqslant W_1 + W_2 + 2V^-$ and $m \in \mathbb{N}$.

In order to formulate the main result from [24] we need to introduce the following quadratic forms:

$$\begin{aligned}\tau_p(u) &:= \tfrac{4}{pp'}\langle a_s \nabla u, \nabla u\rangle + \tfrac{2}{p}\langle \nabla|u|, b_1|u|\rangle - \tfrac{2}{p'}\langle b_2|u|, \nabla|u|\rangle + \langle V|u|^2\rangle,\ 1 < p < \infty,\\ \tau_1(u) &:= 2\langle b_1\nabla|u|, |u|\rangle + \langle V|u|^2\rangle.\end{aligned}$$

on $D(\tau_p) := D(\tau)$ ($1 \leqslant p < \infty$).

The construction of the quasi-contractive C_0-semigroup on L_p, corresponding to the form τ, is given in the following theorem which is the main result in [24] (see [24, Thm. 1.1 and Cor. 4.4]).

Theorem 1.1. *Let assumptions* **(a)** *and* **(bV)** *be fulfilled. Let $U_0 \geqslant W_1 + W_2 + 2V^-$ be such that $Q(U_0) \cap D(\tau_a)$ is dense in $D(\tau_a)$, and $T_0 = T_{0,2}$ the C_0-semigroup associated with the form $\tau + U_0$ on L_2. Let I be the set of all $p \in [1,\infty)$ such that $\tau_p \geqslant -\omega_p$ for some $\omega_p \in \mathbb{R}$.*

(i) Then I is an interval in $[1,\infty)$, and T_0 extrapolates to a positive C_0-semigroup $T_{0,p}(t) = e^{-A_{0,p}t}$ on L_p, for all $p \in I$.

(ii) For all $p \in I$, the sequence of C_0-semigroups $T_{m,p}(t) = e^{-(A_{0,p} - U_0 \wedge m)t}$ strongly converges in L_p to a positive C_0-semigroup $T_p(t) = e^{-A_p t}$ satisfying $\|T_p(t)\| \leqslant e^{\omega_p t}$. For $p, q \in I$, the semigroups T_p and T_q are consistent.

(iii) For all $p \in I \setminus \{1\}$ the form τ_p is closable, and for $u \in D(A_p)$ we have $|u|^{p/2} \operatorname{sgn} u \in D(\overline{\tau_p})$ and

$$\operatorname{Re}\langle A_p u, u|u|^{p-2}\rangle \geqslant \overline{\tau_p}(|u|^{p/2} \operatorname{sgn} u). \tag{1.2}$$

(iv) If, in addition, we assume that

$$\big|\mathrm{Im}\langle(b_1+b_2)u,\nabla u\rangle\big| \leqslant c_1\tau_p(u)+c_2\|u\|_2^2 \quad \big(u\in D(\tau)\big) \tag{1.3}$$

for some $p\in \mathring{I}$, $c_1\geqslant 0$, $c_2\in\mathbb{R}$, then T_p extends to an analytic semigroup on L_p for all $p\in\mathring{I}$ (the interior of I).

As shown in [24], the semigroup T_p does not depend on the choice of U_0. We say that the semigroup T_p is associated with the form τ.

In the rest of the paper we assume that $a\in L_\infty$. Moreover, we make the following assumption:

(BC) *For all $\varphi\in W^1_\infty$, if $u\in D(\tau_a)$ then $\varphi u\in D(\tau_a)$.*

The above assumption is a restriction on the type of boundary conditions. It holds in the case of Neumann boundary conditions, i.e. $\tau_a=\tau_N$, and one can easily see that it is also satisfied if $D(\tau_a)$ is an ideal of $D(\tau_N)$ ($u\in D(\tau_a)$, $v\in D(\tau_N)$ and $|v|\leqslant|u|$ imply that $v\in D(\tau_a)$). In particular, it is satisfied in case of Dirichlet boundary conditions. However, **(BC)** does not hold for periodic type boundary conditions.

Now we are ready to formulate the main result of this paper.

Theorem 1.2. *Let* **(a)**, **(bV)** *and* **(BC)** *hold, and let the interior $\mathring{I}=:(p_-,p_+)$ of the interval I defined in Theorem 1.1 be non-empty. Assume that*

(i) the matrix a is uniformly elliptic, i.e., there exists $\sigma\geqslant 1$ such that

$$\sigma^{-1}\,\mathrm{id}\leqslant a_s\leqslant \sigma\,\mathrm{id};$$

(ii) for some $p\in\mathring{I}$, (1.3) *holds and, for some $C\geqslant 0$,*

$$\big|\langle(b_1+b_2)|u|^2\rangle\big|\leqslant C\sqrt{(\tau_p+C)(u)}\,\|u\|_2 \quad \big(u\in D(\tau)\big); \tag{1.4}$$

(iii) $D(\tau_a)\subseteq L_{\frac{2N}{N-2}}$.

For $q\in I$, let T_q be the semigroup constructed in Theorem 1.1. Let $p_{\max}:=\frac{N}{N-2}p_+$, $p_{\min}:=\big(\frac{N}{N-2}p'_-\big)'$.

Then $T_q(t)\restriction_{L_{\infty,c}}$ extends to an analytic C_0-semigroup on L_p for all $p\in(p_{\min},p_{\max})$. The sector of analyticity and the spectrum of the generators are p-independent. For $p_{min}<p<q<p_{max}$, there exist constants $c_1,c_2>0$ such that

$$\|T_p(t)\|_{p\to q}\leqslant c_1 t^{-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}e^{c_2 t}. \tag{1.5}$$

In case $1\in I$ the assertions hold for all $p\in[1,p_{\max})$.

Remarks. 1. By [24, Prop. 4.1(b)], condition (1.4) holds in particular if, for some $C \geqslant 0$,

$$\left|\langle (b_1+b_2)|u|^2\rangle\right| \leqslant C\|u\|_{H^1}\|u\|_2 \quad (u \in H^1).$$

Note that it is much less restrictive to pose a condition on $\left|\langle (b_1+b_2)|u|^2\rangle\right|$ than on $\left\langle |b_1+b_2|\,|u|^2\right\rangle$.

2. Assumption (iii) of the theorem is in fact the Sobolev imbedding theorem which holds, for example, for Dirichlet boundary conditions or if the domain Ω satisfies the cone property or the extension property [1].

3. In Section 4 we present an example of a semigroup that cannot be extended to a wider interval in the L_p-scale than that obtained in Theorem 1.2. In this sense the result of Theorem 1.2 is sharp. For $b_1 = b_2 = 0$ the interval $(p_{\min}, p_{\max})$ was computed in [21].

As a direct consequence of Theorem 1.2 we obtain a variant of that theorem in which the interval $(p_{\min}, p_{\max})$ is more explicit.

Corollary 1.3. *Let assumptions* **(a)**, **(bV)** *and* **(BC)** *be fulfilled. Let $V_+, V_- \geqslant 0$ be τ_a-regular with $V_+ - V_- = V$, and $\tau_+ := \operatorname{Re}\tau_a + V_+$. Assume that the matrix a is uniformly elliptic, $D(\tau_a) \subseteq L_{\frac{2N}{N-2}}$, and*

$$(-1)^j\langle b_j u, \nabla u\rangle \leqslant \beta_j\tau_+(u) + B_j\|u\|_2^2, \quad \langle V_- u^2\rangle \leqslant \gamma\tau_+(u) + G\|u\|_2^2,$$
$$\langle |b_1+b_2|^2u^2\rangle \leqslant K(\tau_+(u) + \|u\|_2^2) \tag{1.6}$$

$(0 \leqslant u \in D(\tau)\cap Q(V_+),\ j = 1,2)$ for some constants $\beta_1, \beta_2, \gamma \geqslant 0$, $B_1, B_2, G, K \in \mathbb{R}$. Let I be the interval defined in Theorem 1.1.

Suppose that $(p_-, p_+) := \left\{p \in [1,\infty);\ \frac{4}{pp'} - \frac{2}{p}\beta_1 - \frac{2}{p'}\beta_2 - \gamma > 0\right\} \neq \emptyset$. Then $(p_-, p_+) \subseteq I$, and all the assertions of Theorem 1.2 hold with $p_{\max} := \frac{N}{N-2}p_+$, $p_{\min} = \left(\frac{N}{N-2}p_-'\right)'$.

Proof. The inclusion holds by [24, Cor. 4.5]. Condition (1.6) implies that assumption (ii) of Theorem 1.2 is fulfilled. Then, by Theorem 1.2, the assertion follows. □

The rest of the paper is organized as follows. In Section 2 we present an abstract result on weighted estimates which is a main tool in the proof the main theorem which is given in Section 3. Sharpness of the main result is shown in Section 4. In Section 5 we discuss L_p-theory for non-divergence type elliptic operators.

2 Technique of weighted estimates

In this section we are going to show the following theorem which contains an abstract statement needed for the proof of our main result and is useful in some other applications.

Theorem 2.1. *Let $1 \leqslant p \leqslant r_0 \leqslant q \leqslant \infty$, T an analytic semigroup of angle $\theta \in (0, \frac{\pi}{2}]$ on L_{r_0} satisfying*

$$\|e^{\xi x} T(t) e^{-\xi x}\|_{p \to q} \leqslant M t^{-\frac{N}{m}(\frac{1}{p}-\frac{1}{q})} e^{\mu|\xi|^m t + \omega t} \quad (t > 0,\ \xi \in \mathbb{R}^N) \tag{2.1}$$

for some $M, \mu > 0$, $m > 1$ and $\omega \in \mathbb{R}$. Then T extrapolates to an analytic semigroup of angle θ on L_r for all $r \in [p, q] \setminus \{\infty\}$, and the spectrum of the generators $-A_r$ is independent of r.

This theorem is a generalization of Theorem 2.3 in [9]. There the case $p = 1$, $q = \infty$ is treated by showing estimates on the integral kernels of powers of the resolvents $(\lambda + A)^{-1}$ for λ from some sector. In this case one can use Davies' trick to show that estimate (2.1) is equivalent to a Gaussian estimate of order m of the integral kernel of the semigroup (cf. [6]).

The main tools needed in the proof of the theorem are Stein interpolation and the following lemma on weighted estimates which is a refinement of Proposition 3.2 from [19].

Lemma 2.2. *Let $1 \leqslant p \leqslant q \leqslant \infty$, $\gamma > 0$. Let $B\colon L_{\infty,c} \to L_{1,loc}$ be a linear operator satisfying*

$$\|e^{\xi x} B e^{-\xi x}\|_{p \to q} \leqslant 1 \quad \textit{for all } \xi \in \mathbb{R}^N \textit{ with } |\xi| = \gamma.$$

Then $\|B\|_{r \to r} \leqslant c_N \gamma^{-N(\frac{1}{p}-\frac{1}{q})}$ for all $r \in [p, q]$, where the constant c_N depends only on the dimension N.

Proof. For $\gamma = 1$ the lemma is proved in [19], with $c_N = e^{\sqrt{N}} \|(e^{-|k|})_k\|_1$. (In fact, there the estimate $\|e^{\xi x} B e^{-\xi x}\|_{p \to q} \leqslant 1$ is assumed for all $|\xi| \leqslant 1$, but only $|\xi| = 1$ is used in the proof.) Using a rescaling argument, we now deduce the assertion for general γ.

Define the operator D_γ by $D_\gamma f(x) := f(\gamma x)$ for all $f\colon \Omega \to \mathbb{C}$ and all $x \in \Omega$. Then $\|D_\gamma f\|_r = \gamma^{-\frac{N}{r}} \|f\|_r$ for all $r \in [1, \infty]$, $f \in L_r$. Moreover, $D_\gamma \circ e^{\xi x} = e^{\gamma \xi x} \circ D_\gamma$ for all $\xi \in \mathbb{R}^N$. From the assumption we thus obtain, with $\tilde{B} := D_\gamma^{-1} B D_\gamma$,

$$\|e^{\xi x} \tilde{B} e^{-\xi x}\|_{p \to q} = \|D_\gamma^{-1} e^{\gamma \xi x} B e^{-\gamma \xi x} D_\gamma\|_{p \to q} \leqslant \gamma^{-N(\frac{1}{p}-\frac{1}{q})} \quad \text{for all } |\xi| = 1.$$

An application of the lemma in the known case $\gamma = 1$ completes the proof. □

It should be pointed out that Lemma 2.2 is of particular interest for *large* γ. Similar results have first been used in [6] and [18], the difference being that there a weighted norm estimate for all $\xi \in \mathbb{R}^N$ is assumed, not only for $|\xi| = \gamma$. Lemma 2.2 will be applied in form of the next corollary.

Corollary 2.3. *Let $B\colon L_{\infty,c} \to L_{1,loc}$ be a linear operator. Assume that*

$$\|e^{\xi x} B e^{-\xi x}\|_{p\to q} \leqslant M t^{-\frac{N}{m}(\frac{1}{p}-\frac{1}{q})} e^{\mu|\xi|^m t} \quad (\xi \in \mathbb{R}^N)$$

for some $1 \leqslant p \leqslant q \leqslant \infty$, $M, t, \mu > 0$. Then

(a) *$\|B\|_{r\to r} \leqslant M_1 := M e c_N \mu^{\frac{N}{m}(\frac{1}{p}-\frac{1}{q})}$ for all $r \in [p,q]$.*

(b) *For all $p \leqslant r \leqslant s \leqslant q$ we have*

$$\|e^{\xi x} B e^{-\xi x}\|_{r\to s} \leqslant M_1 t^{-\frac{N}{m}(\frac{1}{r}-\frac{1}{s})} e^{\mu_1|\xi|^m t} \quad (\xi \in \mathbb{R}^N),$$

with $\mu_1 = 2^m \mu$.

Proof. (a) By Lemma 2.2 we have, choosing $\gamma = (\mu t)^{-1/m}$:

$$\begin{aligned}\|B\|_{r\to r} &\leqslant c_N (\mu t)^{\frac{N}{m}(\frac{1}{p}-\frac{1}{q})} \cdot M t^{-\frac{N}{m}(\frac{1}{p}-\frac{1}{q})} e^1 \\ &= c_N \mu^{\frac{N}{m}(\frac{1}{p}-\frac{1}{q})} M e.\end{aligned}$$

(b) Let $\xi \in \mathbb{R}^N$. For $B_\xi := e^{\xi x} B e^{-\xi x}$ and $\xi_0 \in \mathbb{R}^N$ we have by assumption that

$$\|e^{\xi_0 x} B_\xi e^{-\xi_0 x}\|_{p\to q} \leqslant M t^{-\frac{N}{m}(\frac{1}{p}-\frac{1}{q})} e^{\mu|\xi+\xi_0|^m t}.$$

By (a) we conclude, noting $|\xi + \xi_0|^m \leqslant 2^m(|\xi|^m + |\xi_0|^m)$,

$$\|B_\xi\|_{r\to r} \leqslant M_1 e^{\mu 2^m |\xi|^m t}.$$

Riesz-Thorin interpolation between this inequality and the assumption of the corollary leads to the desired conclusion. □

Proposition 2.4. *Let T be a C_0-semigroup on L_q and assume that*

$$\|e^{\xi x} T(t) e^{-\xi x}\|_{p\to q} \leqslant M t^{-\frac{N}{m}(\frac{1}{p}-\frac{1}{q})} e^{\mu|\xi|^m t} \quad (t > 0,\ \xi \in \mathbb{R}^N)$$

for some $1 \leqslant p \leqslant q$, $M, \mu > 0$. Then T extrapolates to a C_0-semigroup on L_p.

Proof. It follows from Corollary 2.3(a) that T extrapolates to a bounded semigroup on L_p. Thus, it suffices to show that $T(t)f \to f$ in L_p as $t \to 0$, for all $f \in L_{\infty,c}$ with $\|f\|_p = 1$. By Corollary 2.3(b) (with $r = s = p$) we have

$$\|e^{\xi x} T(t) f\|_p \leqslant M_1 e^{\mu_1|\xi|^m t} \|e^{\xi x} f\|_p \quad (t \geqslant 0,\ \xi \in \mathbb{R}^N).$$

Let $t \leqslant 1$, $|\xi| = 1$. Then $\|e^{\xi x}T(t)f\|_p \leqslant M_1 e^{\mu_1}\|e^{|x|}f\|_p =: c < \infty$ since f has compact support. Let $R > 0$ and χ_ξ the characteristic function of the set $\{x \in \Omega;\ \xi x \geqslant R\}$. Then $\|\chi_\xi T(t)f\|_p \leqslant \|e^{\xi x - R}T(t)f\|_p \leqslant ce^{-R}$. Let K_R be the cube of edge length $2R$ centered at 0. Then, with e_j being the standard unit vectors of $\mathbb{R}^N$,

$$\|\chi_{\Omega\setminus K_R}T(t)f\|_p \leqslant \Big\| \sum_{j=1}^{N}(\chi_{e_j} + \chi_{-e_j})T(t)f\Big\|_p \leqslant 2Nce^{-R}.$$

For R so large that $\operatorname{supp} f \subseteq K_R$ it follows that

$$\begin{aligned}\|T(t)f - f\|_p &\leqslant \|\chi_{\Omega\cap K_R}T(t)f - f\|_p + \|\chi_{\Omega\setminus K_R}T(t)f\|_p \\ &\leqslant |\Omega \cap K_R|^{\frac{1}{p}-\frac{1}{q}}\|T(t)f - f\|_q + 2Nce^{-R},\end{aligned}$$

which proves the assertion. □

Remark. For $p > 1$ or in case T is positive, the above proposition follows directly from Corollary 2.3(a) and [28].

Until now we have used weighted estimates with weights of the form $\rho(x) = e^{\xi x}$. Generally, we call $\rho\colon \Omega \to (0,\infty)$ a *weight function* if $\rho, \rho^{-1} \in L_{\infty,loc}$. In the proof of Theorem 2.1 we need to extend the weighted estimate (2.1) from real to complex times. The next proposition serves this purpose. Comparable results are shown in [4], [18] and [9] by means of the Phragmen-Lindelöf theorem on a sector. But it seems to be more natural to use the Stein interpolation on a strip, similar to the proof of [6, Lemma 9] by means of the three lines theorem.

Proposition 2.5. *Let* $\rho\colon \Omega \to (0,\infty)$ *be a weight function,* $\theta \in (0, \frac{\pi}{2}]$, $S_\theta := \big\{0 \neq z \in \mathbb{C};\ |\arg z| < \theta\big\}$. *Let* $F\colon S_\theta \to \mathfrak{L}(L_p)$ *be a bounded continuous function, analytic in the interior of* S_θ*, satisfying the inequality*

$$\|\rho^\gamma F(t)\rho^{-\gamma}\| \leqslant Me^{\mu\gamma^m t} \quad (t > 0,\ \gamma \geqslant 0)$$

for some $M \geqslant 1$, $\mu > 0$, $m > 1$. *Then, for* $\alpha \in (0,\theta)$, *there exists* $\mu_\alpha > 0$ *such that*

$$\|\rho^\gamma F(z)\rho^{-\gamma}\| \leqslant M_1 e^{\mu_\alpha \gamma^m \operatorname{Re} z} \quad (z \in S_\alpha,\ \gamma \geqslant 0),$$

with $M_1 = \max\{\|F\|_\infty, M\}$.

Proof. Fix $\gamma \geqslant 0$ and let $\varphi(z) := \exp\big(-\frac{\mu\gamma^m}{\sin\theta}e^{i(\frac{\pi}{2}-\theta z)}\big)$ for $0 \leqslant \operatorname{Re} z \leqslant 1$. Then $|\varphi(z)| = \exp\big(-\mu\gamma^m \frac{\sin\theta x}{\sin\theta}e^{\theta y}\big)$, where $z = x + iy$. We apply the Stein interpolation theorem to the function

$$G(z) := \varphi(z)\rho^{z\gamma}F(e^{i\theta(1-z)})\rho^{-z\gamma}.$$

For $\operatorname{Re} z = 0$ the function $z \mapsto e^{i\theta(1-z)}$ describes the upper ray of the boundary of S_θ, for $\operatorname{Re} z = 1$ it describes the positive real semi-axis. For $f, g \in L_{\infty,c}$, the function $z \mapsto \langle G(z)f, g\rangle$ is analytic, and we have

$$|\langle G(z)f,g\rangle| \leqslant |\varphi(z)| \|F(e^{i\theta(1-z)})\| \cdot \|\rho^{-z\gamma} f\|_p \|\rho^{z\gamma} g\|_{p'} \leqslant \|F\|_\infty \cdot c\|f\|_p \|g\|_{p'} < \infty,$$

where c depends on γ and on the supports of f and g, but not on z. The function φ is adapted to have $\|G(z)\| \leqslant M_1 = \max\{\|F\|_\infty, M\}$ for $\operatorname{Re} z = 0, 1$. We infer that $\|G(z)\| \leqslant M_1$ for all $0 \leqslant \operatorname{Re} z \leqslant 1$, so

$$\|\rho^{x\gamma} F(e^{i\theta(1-x)} e^{\theta y}) \rho^{-x\gamma}\| \leqslant M_1 / |\varphi(x+iy)| = M_1 \exp\big(\mu\gamma^m \tfrac{\sin\theta x}{\sin\theta} e^{\theta y}\big).$$

Choose now $x = 1 - \frac{\alpha}{\theta}$ and let $z := e^{i\theta(1-x)} e^{\theta y} = e^{i\alpha} e^{\theta y}$. Then

$$\|\rho^{x\gamma} F(z) \rho^{-x\gamma}\| \leqslant M_1 \exp\big(\mu\gamma^m \tfrac{\sin(\theta-\alpha)}{\sin\theta} \tfrac{\operatorname{Re} z}{\cos\alpha}\big).$$

Writing $\frac{\gamma}{x} = \frac{\theta}{\theta-\alpha}\gamma$ instead of γ we obtain the assertion with $\mu_\alpha = \mu(\frac{\theta}{\theta-\alpha})^m \frac{\sin(\theta-\alpha)}{\sin\theta\cos\alpha}$. $\square$

Proof of Theorem 2.1. Without restriction let $\omega = 0$. Observe that for the first assertion it suffices to consider the case $p = r \wedge r_0$, $q = r \vee r_0$, by Corollary 2.3(b). We confine ourselves to the case $r < r_0$ (so that $p = r$, $q = r_0$), the proof of the case $r > r_0$ being almost the same.

By Proposition 2.4, $T(t)\restriction_{L_{\infty,c}}$ extends to a C_0-semigroup on L_p. Let $0 < \alpha < \theta$. Note that the function $S_\alpha \ni z \mapsto \langle T(z)f, g\rangle$ is analytic for all $f, g \in L_{\infty,c}$ and that $L_{\infty,c}$ is dense in L_p and a norming subset of L_p^*. So we only have to show that $\|T(z)\restriction_{L_{\infty,c}}\|_{p\to p} \leqslant M_\alpha$ for $|\arg z| \leqslant \alpha$ to conclude the assertion by a slight modification of [10, Thm. III.1.12].

From assumption (2.1) and Corollary 2.3(b) we obtain that

$$\|e^{\xi x} T(t) e^{-\xi x}\|_{q\to q} \leqslant C e^{\mu_1 |\xi|^m t} \quad (t \geqslant 0,\ \xi \in \mathbb{R}^N).$$

Let $\alpha_1 := \frac{\alpha+\theta}{2}$, and $\delta > 0$ be such that $z - \delta \operatorname{Re} z \in S_{\alpha_1}$ for all $z \in S_\alpha$. For $z \in S_\alpha$, $\xi \in \mathbb{R}^N$ and $f \in L_{\infty,c}$ we obtain, taking into account Proposition 2.5 and assumption (2.1),

$$\begin{aligned}\|e^{\xi x} T(z) e^{-\xi x} f\|_q &= \|e^{\xi x} T(z - \delta \operatorname{Re} z) e^{-\xi x} e^{\xi x} T(\delta \operatorname{Re} z) e^{-\xi x} f\|_q \\ &\leqslant M_1 e^{\mu_{\alpha_1} |\xi|^m \operatorname{Re}(z - \delta \operatorname{Re} z)} M(\delta \operatorname{Re} z)^{-\frac{N}{m}(\frac{1}{p} - \frac{1}{q})} e^{\mu |\xi|^m \delta \operatorname{Re} z} \|f\|_p \\ &= M_2 (\operatorname{Re} z)^{-\frac{N}{m}(\frac{1}{p} - \frac{1}{q})} e^{\mu_2 |\xi|^m \operatorname{Re} z} \|f\|_p,\end{aligned}$$

with $\mu_2 = (1-\delta)\mu_{\alpha_1} + \delta\mu$. An application of Corollary 2.3(a) yields the first assertion.

The statement on p-independence of the spectra follows from [16, Sec. 5, 1.]. $\square$

In applications of Theorem 2.1 it is often hard to verify the weighted estimate (2.1) for $p = 1$. The next result serves the purpose to overcome this difficulty.

Proposition 2.6. *Let $r_0 \geqslant 1$, T a contractive C_0-semigroup on L_{r_0}, and $\rho > 0$ a weight function. Assume that*

$$\|\rho^\gamma T(t)\rho^{-\gamma}\|_{p\to q} \leqslant Mt^{-\alpha(\frac{1}{p}-\frac{1}{q})}e^{\mu\gamma^m t} \quad (t,\gamma>0)$$

for some $r_0 < p < q$, $M, \alpha, \mu > 0$, $m > 1$. Then there exist $M_1, \mu_1 > 0$ such that

$$\|\rho^\gamma T(t)\rho^{-\gamma}\|_{r_0\to q} \leqslant M_1 t^{-\alpha(\frac{1}{r_0}-\frac{1}{q})}e^{\mu_1\gamma^m t} \quad (t,\gamma>0).$$

Proof. For $0 < \theta \leqslant 1$ let $p_\theta := \big(\frac{\theta}{p} + \frac{1-\theta}{r_0}\big)^{-1}$, $q_\theta := \big(\frac{\theta}{q} + \frac{1-\theta}{r_0}\big)^{-1}$. By the Stein interpolation theorem, the assumption implies that

$$\|\rho^{\theta\gamma} T(t)\rho^{-\theta\gamma}\|_{p_\theta\to q_\theta} \leqslant M^\theta t^{-\theta\alpha(\frac{1}{p}-\frac{1}{q})}e^{\theta\mu\gamma^m t} \quad (t,\gamma>0). \tag{2.2}$$

Let $t,\gamma > 0$, define $\theta \in (0,1)$ by $q_\theta = p$ and let $\theta_k := \theta^k$, $t_k := \theta_k^m t$ ($k \in \mathbb{N}_0$) and $\beta := \alpha(\frac{1}{p} - \frac{1}{q})$. Then $p_{\theta_k} = q_{\theta_{k+1}}$ ($k \in \mathbb{N}_0$), and (2.2) yields

$$\|\rho^\gamma T(t_k)\rho^{-\gamma}\|_{q_{\theta_{k+1}}\to q_{\theta_k}} \leqslant M^{\theta_k} t_k^{-\theta_k\beta} e^{\theta_k\mu(\gamma/\theta_k)^m t_k} = M^{\theta_k}(\theta^{mk}t)^{-\theta_k\beta}e^{\theta_k\mu\gamma^m t}$$

for all $k \in \mathbb{N}_0$. We use this as a starting point for a Moser type iteration: for $f \in L_{\infty,c}$ we obtain by Fatou's lemma that

$$\begin{aligned}\|\rho^\gamma T(\tfrac{t}{1-\theta^m})\rho^{-\gamma}f\|_q &\leqslant \liminf_{n\to\infty}\Big\|\rho^\gamma T\Big(\sum_{k=0}^n t_k\Big)\rho^{-\gamma}f\Big\|_q \\ &\leqslant \liminf_{n\to\infty}\prod_{k=0}^n\big(M^{\theta_k}\theta^{-m\beta k\theta_k}t^{-\theta_k\beta}e^{\theta_k\mu\gamma^m t}\big)\cdot\|f\|_{q_{\theta_{n+1}}}.\end{aligned}$$

Set $r := \sum_{k=0}^\infty \theta_k (= \frac{1}{1-\theta})$ and $s := \sum_{k=0}^\infty k\theta_k (= \frac{\theta}{(1-\theta)^2})$, and note that $\sum_{k=0}^\infty \theta_k\beta = \alpha(\frac{1}{r_0} - \frac{1}{q})$. We conclude that

$$\|\rho^\gamma T(\tfrac{t}{1-\theta^m})\rho^{-\gamma}f\|_q \leqslant M^r\theta^{-m\beta s}t^{-\alpha(\frac{1}{r_0}-\frac{1}{q})}e^{r\mu\gamma^m t}\|f\|_{r_0}.$$

This yields the assertion with $M_1 = M^r\theta^{-m\beta s}(1-\theta^m)^{-\alpha(\frac{1}{r_0}-\frac{1}{q})}$ and $\mu_1 = (1-\theta^m)r\mu$. $\square$

The next extrapolation lemma is a modification of the result from [3] with literally the same proof.

Lemma 2.7. *Let $p_0 \leqslant p < q \leqslant p_1$. Let T be a semigroup satisfying $\|T(t)\|_{p_0 \to p_0} \leqslant C$, $\|T(t)\|_{p_1 \to p_1} \leqslant C$ and*

$$\|T(t)\|_{p \to q} \leqslant Ct^{-\alpha(\frac{1}{p}-\frac{1}{q})} \quad (t > 0).$$

Then there exists $C_1 > 0$ such that

$$\|T(t)\|_{p_0 \to p_1} \leqslant C_1 t^{-\alpha(\frac{1}{p_1}-\frac{1}{p_0})} \quad (t > 0).$$

In the next section we apply Theorem 2.1 via the following proposition.

Proposition 2.8. *Let $1 \leqslant p_0 < \infty$, T an analytic semigroup of angle $\theta \in (0, \frac{\pi}{2}]$ on L_{p_0} satisfying*

$$\|e^{\xi x} T(t) e^{-\xi x}\|_{p_0 \to p_0} \leqslant M e^{\mu |\xi|^m t + \omega t},$$
$$\|T(t)\|_{p \to q} \leqslant M t^{-\frac{N}{m}(\frac{1}{p}-\frac{1}{q})} e^{\omega t},$$

for all $t > 0$, $\xi \in \mathbb{R}^N$ and some $1 \leqslant p < q \leqslant \infty$. Then $T(t)\restriction_{L_{\infty,c}}$ extends to an analytic semigroup of angle θ on L_r for $r \in (p \wedge p_0, q \vee p_0) \cup \{p_0\}$, and the spectrum of the generators $-A_r$ is independent of r. If in addition T is L_{r_0}-contractive for some $1 \leqslant r_0 < p_0$, the same holds for $r \in [r_0, p_0]$.

Proof. Denote $\rho(x) = e^{\xi x}$. Without restriction let $\omega = 0$. By the Stein interpolation theorem the assumptions imply that, for all $\theta \in (0, 1)$,

$$\|e^{\xi x} T(t) e^{-\xi x}\|_{p_\theta \to q_\theta} \leqslant M t^{-\frac{N}{m}(\frac{1}{p_\theta}-\frac{1}{q_\theta})} e^{\theta^{1-m} \mu |\xi|^m t},$$

with $\frac{1}{p_\theta} = \frac{1-\theta}{p} + \frac{\theta}{p_0}$ and $\frac{1}{q_\theta} = \frac{1-\theta}{q} + \frac{s}{p_0}$. In the rest of the proof we distinguish between three cases.

Case 1. $p \leqslant p_0 \leqslant q$. The assertion follows directly from Theorem 2.1.

Case 2. $p_0 < p$. By Corollary 2.3(b) we have that

$$\|e^{\xi x} T(t) e^{-\xi x}\|_{q_\theta \to q_\theta} \leqslant M e^{\mu_\theta |\xi|^m t}.$$

Then Lemma 2.7 (applied to the semigroup $T_\xi(t) = e^{\xi x} T(t) e^{-\xi x}$) and Theorem 2.1 yield the assertion.

Case 3. $p_0 > q$. The proof is analogous to that of Case 2.

The last assertion is obtained in the same way, using Proposition 2.6. □

3 Proof of the main result

In this section we prove Theorem 1.2. In order to apply Proposition 2.8, we need to show appropriate weighted estimates for the semigroups T_p constructed in Theorem 1.1. Recall that the semigroups T_p are associated with the form τ defined in (1.1). We will establish estimates on the 'twisted semigroups' $e^{\xi x}T_p e^{-\xi x}$, for $\xi \in \mathbb{R}^N$, by studying the 'twisted form' τ_ξ which is formally defined by $\tau_\xi(u,v) = \tau(e^{-\xi x}u, e^{\xi x}v)$. We point out that it is a nontrivial technical problem to establish the relationship between τ_ξ and $e^{\xi x}T_p e^{-\xi x}$ (see, e.g., [20, Prop. 3.4]).

Throughout this section we assume that **(a)**, **(bV)** and **(BC)** are fulfilled and that $a \in L_\infty$. Let τ_a, τ, τ_p $(1 \leqslant p < \infty)$ be the forms defined in Section 1. Recall that

$$I = \big\{p \in [1,\infty);\, \tau_p \geqslant -\omega_p \text{ for some } \omega_p \in \mathbb{R}\big\}.$$

For a Lipschitz continuous function $\phi\colon \Omega \to \mathbb{R}$, we introduce the form

$$\begin{aligned}\tau_\phi(u,v) := \tau(u,v) &- \big\langle (a\nabla\phi)u, \nabla v\big\rangle + \big\langle \nabla u, (a^\top\nabla\phi)v\big\rangle \\ &- \big\langle \big[a_s\nabla\phi\cdot\nabla\phi + (b_1+b_2)\nabla\phi\big]u, v\big\rangle\end{aligned}$$

on $D(\tau_\phi) := D(\tau)$. It is straightforward that

$$\tau_\phi(u,v) = \tau(e^{-\phi}u, e^{\phi}v) \quad \big(u, v \in D(\tau_\phi) \text{ such that } e^{-\phi}u, e^{\phi}v \in D(\tau)\big).$$

The form τ_ϕ is of the same type as the form τ, with new lower order coefficients

$$\tilde{b}_1 = b_1 + a^\top\nabla\phi, \quad \tilde{b}_2 = b_2 + a\nabla\phi, \quad \tilde{V} = V - a_s\nabla\phi\cdot\nabla\phi - (b_1+b_2)\nabla\phi.$$

Since $a \in L_\infty$ and $\nabla\phi \in L_\infty$, it is easy to see that these new coefficients satisfy assumption **(bV)**.

Proposition 3.1. *Assume that* **(a)**, **(bV)** *and* **(BC)** *hold, and recall that* $a \in L_\infty$. *Let* $c > 0$, $0 < \varepsilon < \frac{1}{2}$, $p \in \mathring{I}$, *and* T_p *the positive* C_0*-semigroup on* L_p *associated with* τ. *Then there exists* $\mu > 0$ *such that, for all Lipschitz continuous* $\phi\colon \Omega \to \mathbb{R}$ *satisfying*

$$\big|\langle (b_1+b_2)\nabla\phi, u^2\rangle\big| \leqslant \varepsilon\tau_p(u) + c(1 + \|\nabla\phi\|_\infty^2)\|u\|_2^2 \quad \big(0 \leqslant u \in D(\tau)\big), \tag{3.1}$$

the following assertions hold.

The form τ_ϕ *is associated with a positive* C_0*-semigroup* $T_{\phi,p}(t) = e^{-tA_{\phi,p}}$ *on* L_p. *For all* $u \in D(A_{\phi,p})$ *we have* $|u|^{\frac{p}{2}}\operatorname{sgn} u \in D(\overline{\tau_p})$ *and*

$$\langle A_{\phi,p}u, |u|^{p-1}\operatorname{sgn} u\rangle \geqslant (1-2\varepsilon)\overline{\tau_p}\big(|u|^{\frac{p}{2}}\operatorname{sgn} u\big) - \mu(1 + \|\nabla\phi\|_\infty^2)\|u\|_p^p.$$

Further, $T_{\phi,p}(t)f = e^{\phi}T_p(t)e^{-\phi}f$ *for all* $f \in L_{\infty,c}$, $t \geqslant 0$. *In particular,*

$$\|e^{\phi}T_p(t)e^{-\phi}\|_{p\to p} \leqslant e^{\mu(1+\|\nabla\phi\|_\infty^2)t} \quad (t \geqslant 0).$$

For the proof of the proposition, we need the following technical lemma.

Lemma 3.2. *Let $\phi \in L_{\infty,loc}$. Let τ, τ_ϕ be closed sectorial forms in L_2 with $D(\tau) = D(\tau_\phi)$, and A, A_ϕ the corresponding m-sectorial operators in L_2. Assume that*

$$\tau_\phi(u,v) = \tau(e^{-\phi}u, e^{\phi}v) \quad \big(u,v \in D(\tau_\phi) \text{ such that } e^{-\phi}u, e^{\phi}v \in D(\tau)\big).$$

Let $\lambda \in \rho(-A_\phi) \cap \rho(-A)$. Then

$$(\lambda + A_\phi)^{-1} f = e^{\phi}(\lambda + A)^{-1} e^{-\phi} f \quad (f \in L_{\infty,c})$$

if and only if

$$e^{\phi} v \in D(\tau) \quad (v \in (\lambda + A)^{-1} L_{\infty,c}).$$

Proof. The "only if" part is clear, so we prove the "if" part. Let $f \in L_{\infty,c}$, $u := e^{\phi}(\lambda + A)^{-1} e^{-\phi} f$. Then $e^{-\phi}u \in (\lambda + A)^{-1} L_{\infty,c} =: D \subseteq D(\tau)$ and hence $u \in D(\tau)$. For all $v \in D$ we have $e^{\phi}v \in D(\tau)$, so

$$\tau_\phi(u,v) = \tau(e^{-\phi}u, e^{\phi}v) = \langle e^{\phi} A e^{-\phi} u, v\rangle.$$

Moreover, since $D(\tau) = D(\tau_\phi)$, the closed graph theorem implies that D is a core for τ_ϕ. Thus we obtain that $u \in D(A_\phi)$ and $A_\phi u = e^{\phi} A e^{-\phi} u$, which implies the assertion. ☐

Proof of Proposition 3.1. In order to apply Theorem 1.1 we have to consider the symmetric form $\tau_{\phi,p}$ defined by

$$\begin{aligned} \tau_{\phi,p}(u) &:= \operatorname{Re} \tau_a(u) + \tfrac{2}{p}\langle \nabla|u|, \tilde b_1 |u|\rangle - \tfrac{2}{p'}\langle \tilde b_2 |u|, \nabla|u|\rangle + \langle \tilde V |u|^2\rangle \\ &= \tau_p(u) + \big\langle [(\tfrac{2}{p} a^\top - \tfrac{2}{p'} a)\nabla\phi]|u|, \nabla|u|\big\rangle - \big\langle \big[a_s \nabla\phi \cdot \nabla\phi + (b_1 + b_2)\cdot \nabla\phi\big]|u|^2\big\rangle \end{aligned}$$

on $D(\tau_{\phi,p}) := D(\tau_\phi)$. By assumption **(a)** we have $|\langle a\zeta, \eta\rangle| \leqslant (\alpha+1)|a_s^{1/2}\zeta|\cdot|a_s^{1/2}\eta|$ for all $\zeta, \eta \in \mathbb{C}^N$. A standard quadratic estimate shows that

$$\tau_{\phi,p}(u) \geqslant \tau_p(u) - \delta(\alpha+1)^2 \tau_a(|u|) - \big\langle \big[(1 + \tfrac{1}{\delta}) a_s \nabla\phi \cdot \nabla\phi + (b_1 + b_2)\nabla\phi\big]|u|^2\big\rangle$$

for all $\delta > 0,\ u \in D(\tau)$. By [24, Prop. 4.1(b)] there exist $\delta > 0,\ \omega \in \mathbb{R}$ such that

$$\delta(\alpha+1)^2 \operatorname{Re} \tau_a \leqslant \varepsilon \tau_p + \omega.$$

By (3.1) we thus obtain

$$\tau_{\phi,p} \geqslant \big(1 - 2\varepsilon\big)\tau_p - \omega - (1 + \tfrac{1}{\delta})\|a_s\|_\infty \|\nabla\phi\|_\infty^2 - c(1 + \|\nabla\phi\|_\infty^2).$$

An application of Theorem 1.1 completes the proof of the first two assertions.

Let now $U_\phi := (\alpha+1)^2\|a_s\|_\infty\|\nabla\phi\|_\infty^2 + W_1 + W_2 + |V|$. Then $U := 5U_\phi$ is τ_a-regular by assumption **(bV)**. Standard quadratic estimates show that

$$\operatorname{Re}\tau_\phi \geqslant \frac{1}{4}\operatorname{Re}\tau_a - 4U_\phi \tag{3.2}$$

and that $\tau+U$, $\tau_\phi+U$ are densely defined closed sectorial forms, with domains $D(\tau_a+U_\phi)$. For $m\in\mathbb{N}$ let $U_m := (U-m)^+$, and A_m, $A_{\phi,m}$ the m-sectorial operators associated with $\tau+U_m$, $\tau_\phi+U_m$, respectively. Due to Theorem 1.1(ii), the last assertion of the proposition will follow by passing to the limit in

$$e^{-tA_{\phi,m}}f = e^{\phi}e^{-tA_m}e^{-\phi}f \quad (f\in L_{\infty,c},\ t\geqslant 0).$$

This in turn is equivalent to

$$(\lambda+A_{\phi,m})^{-1}f = e^{\phi}(\lambda+A_m)^{-1}e^{-\phi}f \quad (m\in\mathbb{N},\ \lambda>m,\ f\in L_{\infty,c}).$$

Thus, by Lemma 3.2, it remains to show that

$$e^{\phi}v\in Q := D(\tau_a+U_\phi) \quad \text{for all } v\in D := (\lambda+A_m)^{-1}L_{\infty,c}. \tag{3.3}$$

For $n\in\mathbb{N}$ let $\phi_n := \phi\wedge n$. It is easy to see that $\tau_{\phi_n}+U_m$ is a densely defined closed sectorial form with domain Q. Let $A_{\phi_n,m}$ denote the m-sectorial operator associated with $\tau_{\phi_n}+U_m$. By (3.2) we estimate

$$\operatorname{Re}\tau_{\phi_n}+U_m \geqslant \frac{1}{4}\operatorname{Re}\tau_a - 4U_{\phi_n} + 5U_\phi - m \geqslant \frac{1}{4}\operatorname{Re}\tau_a + U_\phi - m \geqslant -m$$

Let $g\in L_{\infty,c}$, $v := (\lambda+A_m)^{-1}g$. Note that $\phi_n\in W^1_\infty$. Hence, by assumption **(BC)**, we conclude from Lemma 3.2 that

$$(\lambda+A_{\phi_n,m})^{-1}(e^{\phi}g) = e^{\phi_n}(\lambda+A_m)^{-1}e^{-\phi_n}(e^{\phi_n}g) = e^{\phi_n}v$$

for all $\lambda>m$ and sufficiently large $n\in\mathbb{N}$. Therefore,

$$(\tfrac{1}{4}\operatorname{Re}\tau_a+U_\phi)(e^{\phi_n}v) \leqslant \operatorname{Re}(\tau_{\phi_n}+U_m+\lambda)(e^{\phi_n}v) = \operatorname{Re}\langle e^{\phi}g, e^{\phi_n}v\rangle \leqslant \tfrac{1}{\lambda-m}\|e^{\phi}g\|_2^2.$$

This shows that $(e^{\phi_n}v)$ is a bounded sequence in Q. Moreover, $(|e^{\phi_n}v|)$ is pointwise increasing, and $e^{\phi_n}v\to e^{\phi}v$ a.e. as $n\to\infty$. Hence $e^{\phi}v\in L_2$ by monotone convergence, and $e^{\phi_n}v\to e^{\phi}v$ in L_2 by dominated convergence. We conclude that $e^{\phi}v\in Q$, i.e., (3.3) holds. □

Proof of Theorem 1.2. Let $p\in(p_-,p_+)$, $T_p(t)=e^{-A_pt}$ be the semigroup on L_p associated with the form τ. For $\xi\in\mathbb{R}^N$ let $\phi_\xi(x) := \xi\cdot x$. Then $\nabla\phi_\xi=\xi$. By assumption (ii) of the theorem and Euclid's inequality, we have

$$\left|\langle(b_1+b_2)\xi,u^2\rangle\right| \leqslant |\xi|\cdot\left|\langle(b_1+b_2)u^2\rangle\right| \leqslant \frac{1}{4}(\tau_p+C)(u) + C^2|\xi|^2\|u\|_2^2.$$

So we can apply Proposition 3.1 to the form τ_{ϕ_ξ} and obtain that

$$\|e^{\xi x}T_p(t)e^{-\xi x}\|_{p\to p} \leqslant e^{\mu(1+|\xi|^2)t} \quad (t \geqslant 0,\ \xi \in \mathbb{R}^N),$$

which verifies the first assumption of Proposition 2.8. Now we are going to establish an estimate on $\|T_p(t)\|_{p\to\frac{N}{N-2}p}$.

By [24, Prop. 4.1(b)], there exist $\varepsilon_p > 0$ and $C_p \in \mathbb{R}$ such that

$$\tau_p \geqslant \varepsilon_p \operatorname{Re}\tau_a + C_p. \tag{3.4}$$

Without restriction $C_p = 1$. Let $0 \leqslant f \in L_p$, $t \geqslant 0$, $u := e^{-A_pt}f$. Then $0 \leqslant u \in D(A_p)$ since T_p is positive and analytic.

By Theorem 1.1(iii), (3.4), and assumption (iii) of the theorem, there exists $\delta > 0$ such that

$$\langle A_pu, u^{p-1}\rangle \geqslant \varepsilon_p\tau_a(u^{p/2}) + \|u^{p/2}\|_2^2 \geqslant \delta\|u\|^p_{\frac{N}{N-2}p}.$$

Using the analyticity of T_p we obtain by Hölder's inequality that

$$\langle A_pu, u^{p-1}\rangle \leqslant \frac{C}{t}\|f\|_p^p,$$

with some $C > 0$ not depending on t. Combining the above two estimates we arrive at $\|u\|_{\frac{N}{N-2}p} \leqslant C_1t^{-\frac{1}{p}}\|f\|_p$, so that

$$\|T(t)\|_{p\to\frac{N}{N-2}p} \leqslant C_1t^{-\frac{1}{p}}. \tag{3.5}$$

Applying now Proposition 2.8 (note that $\frac{1}{p} = \frac{N}{2}(\frac{1}{p} - \frac{N-2}{Np})$), we infer the assertion of the theorem for $p \in (p_-, p_{\max})$ (and, in case $1 \in I$, for $p \in [1, p_{\max})$).

The $p\to q$ estimate (1.5) follows from (3.5) and Lemma 2.7.

Thus, in case $1 \in I$ the proof is complete while otherwise we obtain the assertions of the theorem only with $(p_-, p_{\max})$ in place of $(p_{\min}, p_{\max})$. In order to complete the proof in the case $1 \notin I$, one should repeat the arguments for the adjoint semigroup T^* which is associated with the form τ^* (see [24, Prop. 3.11]). $\square$

4 Sharpness of the main theorem

In this section we give an example of a semigroup for which the interval in the L_p-scale obtained in Corollary 1.3 cannot be extended.

Let $b\colon \Omega \to \mathbb{R}^N$, $V\colon \Omega \to \mathbb{R}$ be such that $H_0^1 \cap Q(|b|^2 + |V|)$ is dense in H_0^1. Define the form τ in L_2 by

$$\tau(u, v) = \langle\nabla u, \nabla v\rangle + \langle b\nabla u, v\rangle + \langle Vu, v\rangle$$

on $D(\tau) := H_0^1 \cap Q(|b|^2 + |V|)$.

Proposition 4.1. *Assume that τ is associated with a C_0-semigroup $e^{-A_p t}$ on L_p for some $p \geqslant 1$. Then*

$$D(A_p) \supseteq D_p := \left\{ u \in H_0^2 \cap W_p^2;\ |b|\,|\nabla u|, |b|^2 u, Vu \in L_2 \cap L_p \right\}$$

and $A_p \supseteq (-\Delta + b\nabla + V)\restriction_{D_p}$.

Proof. Define the operator $\mathcal{L}$ by $\mathcal{L}u = (-\Delta + b\nabla + V)u$, $D(\mathcal{L}) = D_p$. Then $\mathcal{L}$ acts in both L_2 and L_p. Let $U_0 := |b|^2 + 2|V|$, and let $e^{-A_0 t}$ be the semigroup on L_2 associated with the closed sectorial form $\tau + U_0$. Then $e^{-A_0 t}$ extrapolates to a C_0-semigroup $e^{-A_{0,p} t}$ on L_p.

It is easy to see that $D_p \subseteq D(\tau + U_0)$ and

$$(\tau + U_0)(u, v) = \langle (\mathcal{L} + U_0)u, v \rangle \quad \big(u \in D_p,\ v \in D(\tau)\big).$$

Hence $A_0 \supseteq \mathcal{L} + U_0$ and, moreover, $A_{0,p} \supseteq \mathcal{L} + U_0$ since $\mathcal{L} + U_0$ is an operator in L_p. By [27, Cor. 2.7] we conclude that $A_p \supseteq A_{0,p} - U_0 \supseteq \mathcal{L}$. $\square$

In the following we denote $r(x) := |x|$.

Corollary 4.2. *Let $\Omega = \mathbb{R}^N$, $b = c_1 r^{-1} \nabla r$, $V = c_2 r^{-2} + r^2$ and $u = r^{-\sigma} e^{-\frac{r^2}{2}}$, $\sigma \in \mathbb{R}$. Assume that τ is associated with a C_0-semigroup $e^{-A_p t}$ on L_p, for some $p \in [1, \infty)$ satisfying $p(\sigma + 2) < N$. Then $u \in D(A_p)$ and*

$$A_p u = \big(-(\sigma^2 - (N - 2 - c_1)\sigma - c_2) r^{-2} + N - c_1 - 2\sigma\big) u.$$

Proof. Note that Δu, $\frac{\nabla u}{r}$, $\frac{u}{r^2}$ and $r^2 u$ belong to $L_1 \cap L_\infty(\mathbb{R}^N \setminus B_\varepsilon)$ for all $\varepsilon > 0$, where $B_\varepsilon = \{x \in \mathbb{R}^N; |x| < \varepsilon\}$. Let $\varphi \in C^\infty(\mathbb{R}^N)$, $0 \leqslant \varphi \leqslant 1$, $\varphi(x) = 1$ for all $x \in B_1^c$, $\varphi(x) = 0$ for all $x \in B_{1/2}$. Let $\varphi_n(x) := \varphi(nx)$, $u_n := \varphi_n u$. Then, by Proposition 4.1, $u_n \in D(A_p)$ and

$$\begin{aligned} A_p u_n &= (-\Delta + b\nabla + V)u_n \\ &= \varphi_n(-\Delta + b\nabla + V)u - 2\nabla\varphi_n \cdot \nabla u + (b \cdot \nabla\varphi_n - \Delta\varphi_n)u. \end{aligned}$$

Since $\operatorname{supp}(1 - \varphi_n) \subseteq B_{\frac{1}{n}}$, we have $|\Delta\varphi_n(x)| \leqslant |\Delta\varphi|(nx) r^{-2}$ and $|\nabla\varphi_n(x)| \leqslant |\nabla\varphi|(nx) r^{-1}$. Moreover, $\Delta u, \frac{\nabla u}{r}, \frac{u}{r^2} \in L_p$ since $\sigma + 2 < \frac{N}{p}$. Hence $A_p u_n \to (-\Delta + b\nabla + V)u$ in L_p, by the dominated convergence theorem. So $u \in D(A_p)$ and $A_p u = (-\Delta + b\nabla + V)u$ since $u_n \to u$ in L_p and A_p is a closed operator. The second assertion now results from a direct computation. $\square$

Let now $b = \beta \frac{N-2}{2} r^{-1} \nabla r$ and $V = -\gamma \frac{(N-2)^2}{4} r^{-2} + r^2$ with $\beta < 2$, $0 < \gamma < (1 - \beta/2)^2$. Let $\mu := \sqrt{(1 - \beta/2)^2 - \gamma}$. Then by Corollary 1.3, τ is associated with a consistent family of C_0-semigroups $e^{-A_p t}$ on L_p, for all

$$p_{\min} := \frac{2N}{4 + (N-2)(1 - \frac{\beta}{2} + \mu)} < p < \frac{2N}{(N-2)(1 - \frac{\beta}{2} - \mu)} =: p_{\max}.$$

We are going to show that, for $q \notin (p_{\min}, p_{\max})$, the semigroup $e^{-A_p t}$ does *not* extrapolate to a C_0-semigroup on L_q. Let

$$\sigma := \frac{N}{p_{\max}} = \frac{N-2}{2}(1 - \frac{\beta}{2} - \mu), \quad p_0 := \frac{N}{\sigma + 2} = \frac{2N}{4 + (N-2)(1 - \frac{\beta}{2} - \mu)}.$$

Then $p_0 \in (p_{\min}, p_{\max})$. By Corollary 4.2, $u = r^{-\sigma} e^{-\frac{r^2}{2}}$ is an eigenfunction of A_p for $p \in (p_{\min}, p_0)$. Now assume that $e^{-A_p t}$ extrapolates to a semigroup on L_q, for some $q \geqslant p_{\max}$. Then, by (1.5) and Lemma 2.7, $e^{-A_p t}\colon L_p \to L_q$ for all $p \in (p_{\min}, p_{\max})$. In particular, $e^{-A_p t} u \in L_q$. This contradicts the fact that $e^{-A_p t} u = e^{ct} u \notin L_q$ (recall $\sigma = \frac{N}{p_{\max}} \geqslant \frac{N}{q}$). Considering the adjoint semigroup we show that $e^{-A_p t}$ does not extrapolate to a semigroup on L_q, for any $q \leqslant p_{\min}$.

Remark. In the case of Schrödinger semigroups, a similar example was given by Yu. Semenov (private communication).

5 Non-divergence type operators

In this section we consider the operator

$$A = -a\nabla^2 = -\sum_{j,k=1}^{N} a_{jk} \frac{\partial^2}{\partial x_j \partial x_k}$$

in $UC_b(\mathbb{R}^N)$, the space of bounded uniformly continuous functions, with $D(A) = UC_b^2(\mathbb{R}^N)$ (the functions and their first and second derivatives are in $UC_b(\mathbb{R}^N)$). We assume that (a_{jk}) is symmetric with smooth entries and that $\sigma^{-1}\,\mathrm{id} \leqslant a \leqslant \sigma\,\mathrm{id}$ for some $\sigma \geqslant 1$. It is well-known that the closure of $-A$ generates an analytic semigroup T of full angle (i.e., of angle $\frac{\pi}{2}$) on $UC_b(\mathbb{R}^N)$ (see, e.g., [13, Thm. 8.2.1]).

The semigroup operators $T(t)$ are integral operators with smooth integral kernels $p(t)$ satisfying

$$\int_{\mathbb{R}^N} p(t, x, y)dy = 1.$$

The adjoint semigroup T^* on L_1 is defined by

$$(T^*(t)f)(y) = \int_{\mathbb{R}^N} p(t, x, y) f(x) dx.$$

It was proved in [7] that there exist $q = q(\sigma, N) > \frac{N}{N-1}$ and $C = C(\sigma, N)$ such that

$$\sup_{x \in \mathbb{R}^N} \|p(t, x, \cdot)\|_q < C t^{-\frac{N}{2q'}},$$

which implies that

$$\|T^*(t)\|_{1\to q} \leqslant Ct^{-\frac{N}{2q'}}. \tag{5.1}$$

Now we introduce the 'weighted semigroups'. Let $\xi \in \mathbb{R}^N$, $\rho_\xi(x) := e^{\xi x}$. Then

$$(\rho_\xi^{-1}T(t)\rho_\xi f)(x) = \int_{\mathbb{R}^N} e^{-\xi x}p(t,x,y)e^{\xi y}f(y)dy$$

Using the maximum principle we see that (cf. [25])

$$\|\rho_\xi^{-1}T(t)\rho_\xi\|_{\infty\to\infty} \leqslant e^{\sigma|\xi|^2 t} \quad \text{so that} \quad \|\rho_\xi T^*(t)\rho_\xi^{-1}\|_{1\to 1} \leqslant e^{\sigma|\xi|^2 t}. \tag{5.2}$$

Estimates (5.1) and (5.2) allow us to apply Proposition 2.8.

Proposition 5.1. *Let $\alpha \in [0, \frac{\pi}{2})$. There exists a constant C_α depending only on α, N, σ such that*

$$\|e^{-\overline{A}z}\|_{p\to p} \leqslant C_\alpha \quad \big(p \in [N,\infty),\ |\arg z| \leqslant \alpha\big).$$

In particular, the family $e^{-\overline{A}t}\upharpoonright_{C_c}$ extends to an analytic semigroup of full angle on L_p.

Remark. For $\alpha = 0$, the above proposition was first proved in [22].

The estimate obtained in Proposition 5.1 is an a-priori estimate which carries over to semigroups associated with the non-divergence form operator A that are obtained by approximation by semigroups corresponding to operators with smooth coefficients. We stress, however, that the above result does not contribute to the problem of solvability of non-divergence type equations for non-smooth a.

At the same time, the main results of this paper can be applied to the problem of well-posedness of the abstract Cauchy problem in $L_p(\mathbb{R}^N)$ for an operator realization corresponding to the non-divergence type elliptic differential expression $A = -a\nabla^2$.

Assume that (a_{jk}) is uniformly elliptic. Set $b_{1,k} = \sum_{j=1}^N \partial_j a_{jk}$ $(k = 1,\ldots,N)$. Suppose that $b_1 \in L_{1,loc}$,

$$\begin{aligned}
&\|b_1 u\| \leqslant K\|u\|_{H^1} \text{ for some } K > 0,\ b_1 = b_{11} + b_{12},\\
&\langle |b_{11}|^2|u|^2\rangle \leqslant \beta\|a_s^{1/2}\nabla u\|_2^2 + C_\beta\|u\|_2^2 \text{ for some } \beta \in [0,4),\ C_\beta \geqslant 0,\\
&\operatorname{div} b_{12} \in L_{1,loc}(\mathbb{R}^N),\ (\operatorname{div} b_{12})^- \in L_\infty(\mathbb{R}^N).
\end{aligned}$$

Then $A = -\nabla(a\nabla) + b_1\nabla$. By Corollary 1.3 one can associate with A an analytic C_0-semigroup T_p on $L_p(\mathbb{R}^N)$, for all $p \in \big(\frac{2N}{2N-\sqrt{\beta}(N-2)}, \infty\big)$, with sector of analyticity independent of p. This result is a generalization of the corresponding result in [15] (for the case of a uniformly elliptic matrix (a_{jk})) in several directions: firstly, the interval of solvability in the L_p-scale is extended (and in fact is sharp, see Section 4); secondly, the conditions on b_1 are relaxed; and thirdly, as follows from Corollary 1.3, the sector of analyticity is p-independent.

6 Remark on higher order operators

In this short section we show that, employing Theorem 2.1, one can obtain a result similar to Theorem 1.2 for higher order (non-symmetric) operators from the class of superelliptic operators studied by E. B. Davies [6]. We sketch the construction of these operators below and refer the reader to [6] for details.

Let $m < \frac{N}{2}$, $H^m := W_2^m(\mathbb{R}^N)$. Let τ, with $D(\tau) = H^m$, be a closed sectorial form in L_2 which satisfies the Gårding inequality

$$\frac{1}{2}\|(-\Delta)^{m/2}f\|_2^2 \leqslant \operatorname{Re}\tau(f) \leqslant c\|(-\Delta)^{m/2}f\|_2^2 + c\|f\|_2^2, \tag{6.1}$$

for some $c > 0$ and all $f \in H^m$. Let $\mathcal{E}_m$ denote the set of all bounded real-valued C^∞-functions ϕ on $\mathbb{R}^N$ such that $\|D^\alpha\phi\|_\infty \leqslant 1$ for all α such that $1 \leqslant |\alpha| \leqslant m$. Given $\lambda \in \mathbb{R}$ and $\phi \in \mathcal{E}_m$, let

$$\tau_{\lambda\phi}(f, g) = \tau(e^{-\lambda\phi}f, e^{\lambda\phi}g) \quad (f, g \in H^m).$$

We assume that

$$|\tau_{\lambda\phi}(f) - \operatorname{Re}\tau(f)| \leqslant \frac{1}{4}\operatorname{Re}\tau(f) + k(1 + |\lambda|^{2m})\|f\|_2^2 \quad (f \in H^m), \tag{6.2}$$

for some $k > 0$ independent of λ and ϕ.

Proposition 6.1. *Let assumptions* (6.1) *and* (6.2) *hold. Then the analytic C_0-semigroup $T(t) = e^{-At}$ on L_2, associated with τ, extrapolates to an analytic semigroup $T_p(t) = e^{-A_p t}$ on L_p, for all $\frac{2N}{N+2m} \leqslant p \leqslant \frac{2N}{N-2m}$. The sector of analyticity of T_p and the spectrum $\sigma(A_p)$ are p-independent.*

Sketch of the proof. In order to apply Theorem 2.1 one needs to verify the estimate

$$\|e^{\lambda\phi}T(t)e^{-\lambda\phi}\|_{2\to\frac{2N}{N-2m}} \leqslant \frac{c}{\sqrt{t}}\, e^{\mu(|\lambda|^{2m}+1)t} \quad \big(t > 0,\ \lambda \in \mathbb{R},\ \phi \in \mathcal{E}_m\big), \tag{6.3}$$

for some $c, \mu > 0$ (see [6, Lemma 4]). It follows from (6.1) and (6.2) that $\tau_{\lambda\phi}$ is a closed sectorial form in L_2. By Lemma 3.2, the semigroup $e^{\lambda\phi}T(t)e^{-\lambda\phi}$ is associated with $\tau_{\lambda\phi}$. Now a simple modification of the arguments in [6, Lemmata 6, 7, 22] leads to estimate (6.3). □

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