DEGENERATIONS OF IRRATIONAL TORIC VARIETIES

ELISA POSTINGHEL, FRANK SOTTILE, AND NELLY VILLAMIZAR

Abstract. An irrational toric variety X is an analytic subset of the simplex associated to a finite configuration of real vectors. The positive torus acts on X by translation, and we consider limits of sequences of these translations. Our main result identifies all possible Hausdorff limits of translations of X as toric degenerations using elementary methods and the geometry of the secondary fan of the vector configuration. This generalizes work of García-Puente et al., who used algebraic geometry and work of Kapranov, Sturmfels, and Zelevinsky, when the vectors were integral.

Dedicated to the memory of Andrei Zelevinsky

Introduction

A not necessarily normal complex projective toric variety $Y_{\mathcal{A}}$ is parametrized by monomials whose exponent vectors form a finite set $\mathcal{A}$ of integer vectors. The theory of toric varieties [2] elucidates many ways how the structure of $Y_{\mathcal{A}}$ is encoded in the point set $\mathcal{A}$. For example, the set $X_{\mathcal{A}}$ of nonnegative real points of $Y_{\mathcal{A}}$ is homeomorphic to the convex hull $\Delta_{\mathcal{A}}$ of $\mathcal{A}$ through a linear projection.

If we drop algebraicity, we may associate an irrational toric variety $X_{\mathcal{A}}$ to any finite set $\mathcal{A}$ of real vectors. This is the analytic subvariety of the standard $\mathcal{A}$-simplex $\triangle^{\mathcal{A}}$ parameterized by monomials with exponent vectors from $\mathcal{A}$, and it is homeomorphic to the convex hull $\Delta_{\mathcal{A}}$ of $\mathcal{A}$ in the same way as when $\mathcal{A}$ consists of integer vectors. Other aspects of the dictionary between toric varieties $Y_{\mathcal{A}}$ and sets of integer vectors $\mathcal{A}$ extend to irrational toric varieties $X_{\mathcal{A}}$ and finite sets of real vectors $\mathcal{A}$ [3].

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We further extend this dictionary. When $\mathcal{A} \subset \mathbb{Z}^n$, the torus $(\mathbb{C}^\times)^{\mathcal{A}}$ of the ambient projective space of the toric variety $Y_{\mathcal{A}}$ acts on it via translations. Kapranov, Sturmfels, and Zelevinsky [7, 8] identified the closure in the Hilbert scheme of the set of torus translations $\{w.Y_{\mathcal{A}} \mid w \in (\mathbb{C}^\times)^{\mathcal{A}}\}$ as the toric variety associated to the secondary fan of the set $\mathcal{A}$. The cones of this fan correspond to regular subdivisions of $\mathcal{A}$, which are described in [6, Ch. 7].

A consequence is that the only limiting schemes of torus translations of $Y_{\mathcal{A}}$ are one-parameter toric degenerations. Restricting to the nonnegative part $X_{\mathcal{A}}$ of $Y_{\mathcal{A}}$ and to positive translations $w.X_{\mathcal{A}}$ for $w \in \mathbb{R}^{\mathcal{A}}_>$ gives an identification between limiting positions of positive torus translations of $X_{\mathcal{A}}$ and toric degenerations of $X_{\mathcal{A}}$ by cosets of one-parameter subgroups of $\mathbb{R}^{\mathcal{A}}_>$. Here, limiting position is measured with respect to the Hausdorff metric on subsets of the nonnegative part of the projective space. In [5] this was used to identify all Hausdorff limits of Bézier patches in geometric modeling.

We extend this identification between Hausdorff limits of torus translates of $X_{\mathcal{A}}$ and toric degenerations from the case when $\mathcal{A}$ consists of integer vectors (and thus the methods of [7, 8] using algebraic geometry apply) to the case when $\mathcal{A}$ consists of any finite set of vectors, so that methods from algebraic geometry do not apply.

We do this through a direct argument, identifying all Hausdorff limits of torus translations of $X_{\mathcal{A}}$ with toric degenerations of $X_{\mathcal{A}}$. We state our main theorem.

Theorem 3.3. *Let $\{w_i \mid i \in \mathbb{N}\} \subset \mathbb{R}^{\mathcal{A}}_>$ be a sequence in the positive torus. Then there exists a subsequence $\{u_j \mid j \in \mathbb{N}\} \subset \{w_i \mid i \in \mathbb{N}\}$ and a toric degeneration $X(\mathcal{S}, w)$ such that*

$$\lim_{j\to\infty} u_j.X_{\mathcal{A}} \;=\; X(\mathcal{S}, w)$$

in the Hausdorff topology on subsets of the simplex $\triangle^{\mathcal{A}}$.

In particular, if a sequence of torus translates of $X_{\mathcal{A}}$ has a limit in the Hausdorff topology, then that limit is a toric degeneration. This shows that the space of torus translates of $X_{\mathcal{A}}$ is compactified by adding the toric degenerations.

This paper is organized as follows. In Section 1, we develop some technical results about sequences in cones, recall regular subdivisions and the secondary fan, and develop some of their properties. In Section 2, we define an irrational toric variety $X_{\mathcal{A}}$ associated to a configuration of points $\mathcal{A} \subset \mathbb{R}^d$, recalling some of its properties and identifying its torus translations, as well as recalling the Hausdorff metric and topology. In Section 3, we study toric degenerations of $X_{\mathcal{A}}$, relating them to the secondary fan, and (re)state our main theorem. Its proof occupies Section 4.

1. Cones and the secondary fan of a point configuration

We give some preliminaries from geometric combinatorics including technical results about sequences in cones, regular subdivisions, and the secondary fan. Write $\mathbb{N} = \{1, 2, \dots\}$ for the positive integers, $\mathbb{R}$ for the real numbers, $\mathbb{R}_\geq$ for the nonnegative real numbers, and $\mathbb{R}_>$ for the strictly positive real numbers. We use the standard notions of polyhedron, polytope, cone, face, etc. from geometric combinatorics, which may be found in any of [4, 6, 10]. For example, a polyhedron is an intersection of finitely many closed half-spaces, a polytope is a bounded polyhedron and it is also the convex hull of a finite set of points. Faces of a polyhedron arise as its intersections with hyperplanes bounding half-spaces containing it. The smallest affine space containing a polyhedron is its *affine span* and its *relative interior* is its interior as a subset of this affine span.

A *cone* is a polyhedron with half-spaces $\{x \mid \psi(x) \geq 0\}$ for ψ a linear form. The boundary hyperplane of such a half-space is the linear subspace $\psi^\perp := \{x \mid \psi(x) = 0\}$. The intersection of its boundary hyperplanes is the *lineality space* of the cone and it is *pointed* if its lineality space is the origin. Two faces ρ, σ of a pointed cone τ are *adjacent* if $\rho \cap \sigma$ is not the vertex of τ. The affine span of a cone σ coincides with its linear span, $\langle\sigma\rangle$.

A *polyhedral complex* is a finite collection Π of polyhedra that is closed under taking faces and such that the intersection of any two polyhedra in Π is a face of both (or empty). A polyhedral complex is a *triangulation* if it consists of simplices. A *fan* is a polyhedral complex consisting of cones.

1.1. **Sequences in cones.** We formulate two results about sequences in cones, Lemmas 1.3 and 1.4, that are used in an essential way in the formulation and proof of Theorem 3.3.

We will say that a sequence $\{v_i \mid i \in \mathbb{N}\}$ is *divergent* if it has no bounded subsequence, that is, if for all $M > 0$, there is an N such that if $i > N$, then $|v_i| > M$.

Lemma 1.1. *If ρ and σ are non-adjacent faces of a pointed cone τ and $\{r_i \mid i \in \mathbb{N}\} \subset \rho$ and $\{s_i \mid i \in \mathbb{N}\} \subset \sigma$ are divergent sequences, then $\{r_i - s_i \mid i \in \mathbb{N}\}$ is divergent.*

Proof. Let S be the unit sphere centered at the origin. As σ, ρ are not adjacent, there is a positive lower bound, δ, to the distance $|r-s|$ between any pair of points $r \in \rho \cap S$ and $s \in \sigma \cap S$.

Let $M > 0$. There exists N such that if $i > N$ then $|r_i|, |s_i| > M/\delta$, as $\{r_i\}$ and $\{s_i\}$ are divergent. Let $i > N$ and suppose that $|r_i| \geq |s_i|$ (otherwise interchange the sequences). Then

$$|r_i - s_i| \;=\; |s_i|\left|\frac{r_i}{|s_i|} - \frac{s_i}{|s_i|}\right| \;\geq\; |s_i|\left|\frac{r_i}{|r_i|} - \frac{s_i}{|s_i|}\right| \;>\; \frac{M}{\delta}\delta \;=\; M\,, \tag{1}$$

which completes the proof. □

The first inequality in (1) is elementary geometry: If u, v are unit vectors and $t \geq 1$, then

$$|tu - v| \;\geq\; |u - v|\,.$$

This is clear if $u = v$. Otherwise, let θ be the angle between u and v. Then in the triangle with vertices tu, u, v, the angle at u is $\frac{\pi}{2} + \frac{\theta}{2}$, which is obtuse. Then this inequality is just that the longest side of a triangle is opposite to its largest angle.

Let τ be a cone in $\mathbb{R}^n$, $\{v_i \mid i \in \mathbb{N}\}$ a sequence in τ, and σ a face of τ. We say that $\{v_i \mid i \in \mathbb{N}\}$ is *bounded with respect to σ* if there is a bounded set $B \subset \tau$ with $\{v_i \mid i \in \mathbb{N}\} \subset B + \sigma$. This is equivalent to the image of $\{v_i \mid i \in \mathbb{N}\}$ in $\mathbb{R}^n/\langle\sigma\rangle$ being bounded so that $\{v_i \mid i \in \mathbb{N}\}$ is bounded modulo $\langle\sigma\rangle$.

Lemma 1.2. *Let τ be a cone and suppose that $\{v_i \mid i \in \mathbb{N}\} \subset \tau$ is bounded with respect to two faces ρ and σ of τ. Then it is bounded with respect to their intersection $\rho \cap \sigma$.*

Proof. Suppose that $\{v_i \mid i \in \mathbb{N}\}$ is not bounded with respect to $\rho \cap \sigma$. Then neither ρ nor σ contains the other. Reducing modulo $\langle \rho \cap \sigma \rangle$ and replacing $\{v_i \mid i \in \mathbb{N}\}$ by a subsequence if necessary, we may assume that ρ and σ are non-adjacent faces of the pointed cone τ (with vertex $\rho \cap \sigma$), and that $\{v_i \mid i \in \mathbb{N}\}$ is divergent.

Since $\{v_i\}$ is bounded with respect to both ρ and σ, there is a bounded set $B \subset \tau$ and sequences $\{r_i \mid i \in \mathbb{N}\} \subset \rho$, $\{s_i \mid i \in \mathbb{N}\} \subset \sigma$, and $\{u_i \mid i \in \mathbb{N}\}, \{w_i \mid i \in \mathbb{N}\} \subset B$ such that

$$r_i + u_i \;=\; s_i + w_i \;=\; v_i \qquad \text{for all} \quad i \in \mathbb{N}\,. \tag{2}$$

The sequences $\{r_i\} \subset \rho$ and $\{s_i\} \subset \sigma$ are divergent (as $\{v_i\}$ is divergent, but $\{u_i\}, \{w_i\}$ are bounded), and so we have that $\{r_i - s_i \mid i \in \mathbb{N}\}$ is divergent, by Lemma 1.1. But by (2), $r_i - s_i = w_i - u_i \in B - B$, so $\{r_i - s_i \mid i \in \mathbb{N}\}$ is bounded, a contradiction. □

Lemma 1.3. *Let $\{v_i \mid i \in \mathbb{N}\} \subset \tau$ be a sequence in a cone τ. Then there is a face σ of τ and a subsequence $\{u_j \mid j \in \mathbb{N}\}$ which is bounded with respect to σ such that if ρ is a face of τ for which a subsequence of $\{u_j \mid j \in \mathbb{N}\}$ is bounded with respect to ρ, then $\sigma \subset \rho$.*

Proof. The set $\mathcal{B}$ of faces σ of τ for which $\{v_i \mid i \in \mathbb{N}\}$ has a subsequence that is bounded with respect to σ is nonempty (τ is one such face). It forms an order ideal, for if $\sigma \in \mathcal{B}$ and $\sigma \subset \rho$, then $\rho \in \mathcal{B}$. Let σ be a minimal element of $\mathcal{B}$ and $\{u_i \mid i \in \mathbb{N}\}$ be a subsequence that is bounded with respect to σ.

By Lemma 1.2, if $\rho \subset \tau$ is a face and $\{u'_j \mid j \in \mathbb{N}\}$ is a subsequence of $\{u_i \mid i \in \mathbb{N}\}$ such that $\{u'_j \mid j \in \mathbb{N}\}$ is bounded with respect to ρ, then it is bounded with respect to $\rho \cap \sigma$. By the minimality of σ, $\rho \cap \sigma = \sigma$, so that $\sigma \subset \rho$. □

Lemma 1.4. *Let τ be a cone in $\mathbb{R}^n$, σ a face of τ, and $\{v_i \mid i \in \mathbb{N}\} \subset \tau$ a sequence that is bounded modulo $\langle \sigma \rangle$, and if $\rho \subset \tau$ is a face such that $\{v_i \mid i \in \mathbb{N}\}$ has a subsequence that is bounded modulo $\langle \rho \rangle$, then $\sigma \subset \rho$. Then for any $v \in \sigma$, all except finitely many elements of the sequence $\{v_i - v \mid i \in \mathbb{N}\}$ lie in τ.*

Proof. Suppose by way of contradiction that $v \in \sigma$ and $\{u_j \mid j \in \mathbb{N}\}$ is an infinite subset of $\{v_i \mid i \in \mathbb{N}\}$ such that $\{u_j - v \mid j \in \mathbb{N}\}$ is disjoint from τ. Without loss of generality, we may assume that $\langle \tau \rangle = \mathbb{R}^n$. Let Ψ be a finite irredundant collection of linear forms that define τ,

$$\tau \;=\; \{x \in \mathbb{R}^n \mid \psi(x) \geq 0 \quad \text{for all } \psi \in \Psi\}\,.$$

Since Ψ is irredundant and $\langle \tau \rangle = \mathbb{R}^n$, each form $\psi \in \Psi$ supports a facet $\psi^\perp \cap \tau$ of τ.

Since $\{u_j - v \mid j \in \mathbb{N}\} \cap \tau = \emptyset$, for each $j \in \mathbb{N}$ there is a form $\psi \in \Psi$ with $\psi(u_j - v) < 0$ so that $\psi(u_j) < \psi(v)$. Since Ψ is finite, there is a subsequence $\{w_k \mid k \in \mathbb{N}\}$ of $\{u_j \mid j \in \mathbb{N}\}$ and a form $\psi \in \Psi$ such that $\psi(w_k) < \psi(v)$ for all $k \in \mathbb{N}$. Since $\{w_k \mid k \in \mathbb{N}\} \subset \tau$, we have that $\psi(w_k) \in [0, \psi(v))$ for all $k \in \mathbb{N}$, and is thus bounded modulo the span $\psi^\perp$ of the facet $\rho := \psi^\perp \cap \tau$ of τ.

Since $\{v_i \mid i \in \mathbb{N}\}$ has a subsequence that is bounded modulo $\langle\rho\rangle$, we have $\sigma \subset \rho$. Since $v \in \sigma$ and $\rho \subset \psi^\perp$, we have $\psi(v) = 0$, which contradicts the inequality $0 \le \psi(u_j) < \psi(v)$ that we have for any j. This concludes the proof. □

1.2. **Regular subdivisions.** Fix a positive integer d and let $\mathcal{A} \subset \mathbb{R}^d$ be a finite set of points, which we assume affinely spans $\mathbb{R}^d$. We use elements of $\mathcal{A}$ throughout to index coordinates, variables, functions, etc. For example, $\mathbb{R}^{\mathcal{A}}$ is the space of real-valued functions on $\mathcal{A}$. This has a distinguished subspace $\mathrm{Aff}(\mathcal{A}) \simeq \mathbb{R}^{d+1}$, consisting of functions on $\mathcal{A}$ that are restrictions of affine functions on $\mathbb{R}^d$. For $z \in \mathbb{R}^{\mathcal{A}}$, we may write $z_{\mathbf{a}}$ for $z(\mathbf{a})$, its $\mathbf{a}$th coordinate.

For any subset $\mathcal{F} \subset \mathcal{A}$, extension by zero gives an inclusion $\mathbb{R}^{\mathcal{F}} \hookrightarrow \mathbb{R}^{\mathcal{A}}$ and restriction of functions $w \mapsto w|_{\mathcal{F}}$ gives a map $\mathbb{R}^{\mathcal{A}} \to \mathbb{R}^{\mathcal{F}}$.

For $\mathcal{F} \subset \mathcal{A}$, let $\Delta_{\mathcal{F}}$ be the convex hull of $\mathcal{F}$,

$$\Delta_{\mathcal{F}} \ := \ \Big\{\sum_{\mathbf{f}\in\mathcal{F}} \mu_{\mathbf{f}}\mathbf{f} \mid \mu_{\mathbf{f}} \ge 0 \ \text{ and } \ 1 = \sum_{\mathbf{f}\in\mathcal{F}} \mu_{\mathbf{f}}\Big\}\,.$$

A *polyhedral subdivision* $\mathcal{S}$ of $\mathcal{A}$ is a collection of subsets of $\mathcal{A}$, called *faces* of $\mathcal{S}$, such that the convex hulls $\{\Delta_{\mathcal{F}} \mid \mathcal{F} \in \mathcal{S}\}$ form a polyhedral complex $\Pi_{\mathcal{S}}$ which covers $\Delta_{\mathcal{A}}$. In particular, if $\mathcal{F}, \mathcal{G}$ are faces of a polyhedral subdivision $\mathcal{S}$ of $\mathcal{A}$, then $\mathcal{H} := \mathcal{F} \cap \mathcal{G}$ is also a face of $\mathcal{S}$ and $\Delta_{\mathcal{H}} = \Delta_{\mathcal{F}} \cap \Delta_{\mathcal{G}}$. A *facet* is a maximal face of $\mathcal{S}$, which is a face $\mathcal{F}$ that affinely spans $\mathbb{R}^d$ so that $\Delta_{\mathcal{F}}$ has dimension d. A *triangulation* of $\mathcal{A}$ is a polyhedral subdivision $\mathcal{S}$ in which every face $\Delta_{\mathcal{F}}$ of $\Pi_{\mathcal{S}}$ is a simplex with vertices $\mathcal{F}$. A polyhedral subdivision $\mathcal{S}$ is *regular* if there is a piecewise-affine concave function g on $\Delta_{\mathcal{A}}$ where the maximal domains on which g is affine are $\Delta_{\mathcal{F}}$ for facets $\mathcal{F}$ of $\mathcal{S}$. Such a concave function g is *strictly concave* on the subdivision $\mathcal{S}$.

Elements $\lambda \in \mathbb{R}^{\mathcal{A}}$ induce regular subdivisions. Let P_λ be the convex hull of the graph of λ,

$$P_\lambda \ := \ \mathrm{conv}\{(\mathbf{a}, \lambda(\mathbf{a})) \ \mid \ \mathbf{a} \in \mathcal{A}\}\,.$$

The *upper faces* of P_λ are those faces which have an outward-pointing normal vector with last coordinate positive. To an upper face F, let $\mathcal{F}(F)$ be those points $\mathbf{a}$ of $\mathcal{A}$ with $(\mathbf{a}, \lambda(\mathbf{a}))$ lying on F. Let $\mathcal{S}_\lambda$ be the collection of subsets $\mathcal{F}(F)$ of $\mathcal{A}$ where F ranges over the upper faces of P_λ. This forms a polyhedral subdivision of $\mathcal{A}$ as the upper faces of P_λ form a polyhedral complex whose projection to $\Delta_{\mathcal{A}}$ covers $\Delta_{\mathcal{A}}$, with the projection of an upper face F equal to $\Delta_{\mathcal{F}(F)}$.

Lastly, S_λ is regular—the upper faces of P_λ form the graph of the desired concave function, g_λ. Conversely, if $\mathcal{S}$ is a regular subdivision with strictly concave function g, then any $\lambda \in \mathbb{R}^{\mathcal{A}}$ satisfying $\lambda(\mathbf{a}) \le g(\mathbf{a})$ with equality if and only if $\mathbf{a}$ lies in some face of $\mathcal{S}$ has $\mathcal{S} = \mathcal{S}_\lambda$.

Example 1.5. Let $\mathcal{A} \subset \mathbb{R}^2$ be a 3×3 grid of nine points. Figure 1 shows three polyhedral subdivisions of $\mathcal{A}$ induced by elements $\lambda \in \mathbb{R}^{\mathcal{A}}$, together with the lifted points $\{(\mathbf{a}, \lambda(\mathbf{a})) \mid \mathbf{a} \in \mathcal{A}\}$ and the corresponding upper faces. All elements of $\mathcal{A}$ participate in the first two subdivisions, but the center element of $\mathcal{A}$ does not participate in the third, for it does not lie on an upper face.

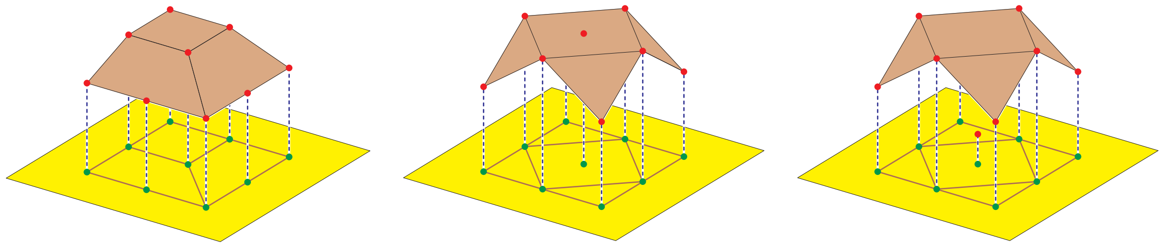


FIGURE 1. Three regular subdivisions

A subset $\mathcal{F}$ of $\mathcal{A}$ is a subset of a face of $\mathcal{S}_\lambda$ if and only if the restriction $\lambda|_{\mathcal{F}}$ of λ to $\mathcal{F}$ is an affine function whose extension to $\Delta_{\mathcal{F}}$ agrees with the restriction of g_λ to $\Delta_{\mathcal{F}}$, where g_λ is the strictly convex function whose graph is the upper faces of P_λ. The minimal subsets of $\mathcal{A}$ that are not contained in any face of $\mathcal{S}_\lambda$ are singletons $\{\mathbf{c}\}$ that do not participate in the subdivision and doubletons $\{\mathbf{a}, \mathbf{b}\}$ in which both $\mathbf{a}$ and $\mathbf{b}$ participate in the subdivision, but no face contains both, so that the interior of the line segment between the lifted points lies below the upper faces.

Lemma 1.6. *Let $\mathcal{S} = \mathcal{S}_\lambda$ be a regular subdivision of $\mathcal{A}$.*

(i) *If $\{\mathbf{a}, \mathbf{b}\} \subset \mathcal{A}$ is not a subset of any face of $\mathcal{S}$, then there is a facet $\mathcal{G}$ of $\mathcal{S}$, a point $p \in \Delta_{\mathcal{G}}$, and numbers $\beta_{\mathbf{a}}, \beta_{\mathbf{b}} > 0$ and $\alpha_{\mathbf{g}} \geq 0$ for $\mathbf{g} \in \mathcal{G}$ with*

$$p \;=\; \beta_{\mathbf{a}}\mathbf{a} + \beta_{\mathbf{b}}\mathbf{b} \;=\; \sum_{\mathbf{g}\in\mathcal{G}} \alpha_{\mathbf{g}}\mathbf{g}\,. \qquad \textit{where} \quad 1 \;=\; \beta_{\mathbf{a}} + \beta_{\mathbf{b}} \;=\; \sum_{\mathbf{g}\in\mathcal{G}} \alpha_{\mathbf{g}}\;. \tag{3}$$

(ii) *If $\mathbf{c} \in \mathcal{A}$ is not a member of any face of $\mathcal{S}$, then there is a facet $\mathcal{G}$ of $\mathcal{S}$ with $\mathbf{c} \in \Delta_{\mathcal{G}}$ and therefore an expression*

$$\mathbf{c} \;=\; \sum_{\mathbf{g}\in\mathcal{G}} \alpha_{\mathbf{g}}\mathbf{g} \qquad \textit{where} \quad \alpha_{\mathbf{g}} \geq 0 \textit{ and } 1 \;=\; \sum_{\mathbf{g}\in\mathcal{G}} \alpha_{\mathbf{g}}\,. \tag{4}$$

(iii) *If $\mathcal{G}$ is a facet of $\mathcal{S}$ and $\mathbf{d} \notin \mathcal{G}$, then there is an expression*

$$\mathbf{d} \;=\; \sum_{\mathbf{g}\in\mathcal{G}} \alpha_{\mathbf{g}}\mathbf{g} \qquad \textit{where} \quad 1 \;=\; \sum_{\mathbf{g}\in\mathcal{G}} \alpha_{\mathbf{g}}\,. \tag{5}$$

of $\mathbf{d}$ as an affine combination of points of $\mathcal{G}$.

In any of (i), (ii), *or* (iii), *if $\widetilde{\lambda}$ is the affine function whose restriction to the facet $\mathcal{G}$ agrees with λ, then*

$$\widetilde{\lambda}\Big(\sum_{\mathbf{g}\in\mathcal{G}} \alpha_{\mathbf{g}}\mathbf{g}\Big) \;=\; \sum_{\mathbf{g}\in\mathcal{G}} \alpha_{\mathbf{g}}\lambda(\mathbf{g})\,,$$

and we have the (respective) inequalities

$$\beta_{\mathbf{a}}\lambda(\mathbf{a}) + \beta_{\mathbf{b}}\lambda(\mathbf{b}) < \widetilde{\lambda}(p)\,, \quad \lambda(\mathbf{c}) < \widetilde{\lambda}(\mathbf{c})\,, \quad \textit{and} \quad \lambda(\mathbf{d}) < \widetilde{\lambda}(\mathbf{d})\,. \tag{6}$$

Proof. If $\{\mathbf{a},\mathbf{b}\}$ is not a subset of any face of $\mathcal{S}$, then the interior of the line segment they span meets the convex hull $\Delta_{\mathcal{F}}$ of some facet $\mathcal{G}$ of $\mathcal{S}$ in a point, p. This gives the expression (3). The first inequality of (6) expresses that the interior of the segment joining the lifted points $(\mathbf{a},\lambda(\mathbf{a}))$ and $(\mathbf{b},\lambda(\mathbf{b}))$ lies strictly below the upper hull of P_λ.

If $\mathbf{c}\in\mathcal{A}$ is not a member of a face of $\mathcal{S}$, then there is a facet $\mathcal{G}$ of $\mathcal{S}$ with $\mathbf{c}$ lying in $\Delta_{\mathcal{G}}$. Thus $\mathbf{c}$ is a convex combination of the points of $\mathcal{G}$, giving (4). Since $\mathbf{c}$ lies in no face of $\mathcal{S}$, the lifted point $(\mathbf{c},\lambda(\mathbf{c}))$ is below the upper hull of P_λ, which implies the middle inequality of (6) as $\widetilde{\lambda}(\mathbf{c})$ is the height of the point on the upper hull of P_λ above $\mathbf{c}$.

Finally, as $\mathcal{G}$ is a facet of $\mathcal{S}$, its points affinely span $\mathbb{R}^d$, so there is an expression of $\mathbf{d}$ as an affine combination of the points of $\mathcal{G}$ (5). The graph of the function $\widetilde{\lambda}$ is the hyperplane supporting the upper facet of the lifted polytope P_λ corresponding to $\mathcal{G}$. Then, if $\mathbf{a}\notin\mathcal{G}$, we have $\lambda(\mathbf{a})<\widetilde{\lambda}(\mathbf{a})$, and the third inequality of (6) is a special case of this. □

1.3. **Secondary fan of a point configuration.** For a regular subdivision $\mathcal{S}$ of a point configuration $\mathcal{A}\subset\mathbb{R}^d$, let $\sigma(\mathcal{S})\subset\mathbb{R}^{\mathcal{A}}$ be the (closure of) the set of all functions λ which induce $\mathcal{S}$. This forms a cone in $\mathbb{R}^{\mathcal{A}}$ which is full-dimensional if and only if $\mathcal{S}$ is a regular triangulation of $\mathcal{A}$. The collection of these cones forms the *secondary fan* $\Sigma_{\mathcal{A}}$ of the point configuration $\mathcal{A}$. Write $\mathcal{S}_\sigma$ for the subdivision corresponding to a cone σ of the secondary fan. The minimal cone of $\Sigma_{\mathcal{A}}$ is the linear space $\mathrm{Aff}(\mathcal{A})$, for adding an affine function ψ to a function λ does not change the subdivision, $\mathcal{S}_\lambda=\mathcal{S}_{\psi+\lambda}$, and elements of $\mathrm{Aff}(\mathcal{A})$ induce the trivial subdivision of $\mathcal{A}$ whose only facet is $\mathcal{A}$.

A polyhedral subdivision $\mathcal{S}$ of $\mathcal{A}$ *is refined* by another $\mathcal{S}'$ ($\mathcal{S}\prec\mathcal{S}'$) if for every face $\mathcal{F}'$ of $\mathcal{S}'$, there is a face $\mathcal{F}$ of $\mathcal{S}$ with $\mathcal{F}\supset\mathcal{F}'$. This refinement poset is equal to the poset of the cones of the secondary fan under inclusion. That is, $\mathcal{S}_\sigma$ is refined by $\mathcal{S}_\rho$ if and only if σ is a face of ρ. In particular, if $\{\mathbf{a}_1,\dots,\mathbf{a}_r\}$ is not a subset of any face of $\mathcal{S}_\sigma$ then it is not a subset of any face of $\mathcal{S}_\rho$ for any cone ρ of the secondary fan that contains σ.

Example 1.7. Let $\mathcal{A}=\{(0,0),(1,0),(1,1),(\frac{1}{2},\frac{3}{2}),(0,1)\}\subset\mathbb{R}^2$. Its convex hull is a pentagon.

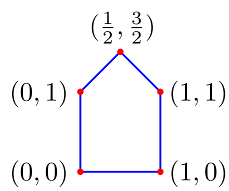


Figure 2 shows the poset of regular subdivisions of $\mathcal{A}$. For each, it gives the corresponding polyhedral subdivision of $\Delta_{\mathcal{A}}$ and functions λ inducing the subdivision. Working modulo $\mathrm{Aff}(\mathcal{A})$, we assume that a function $\lambda\in\mathbb{R}^{\mathcal{A}}$ takes value zero at the three points where the second coordinate is positive. The parameter r in the middle row is always positive.

Working modulo $\mathrm{Aff}(\mathcal{A})$ (using the parameters of Figure 2), the secondary fan of $\mathcal{A}$ is shown in Figure 3.

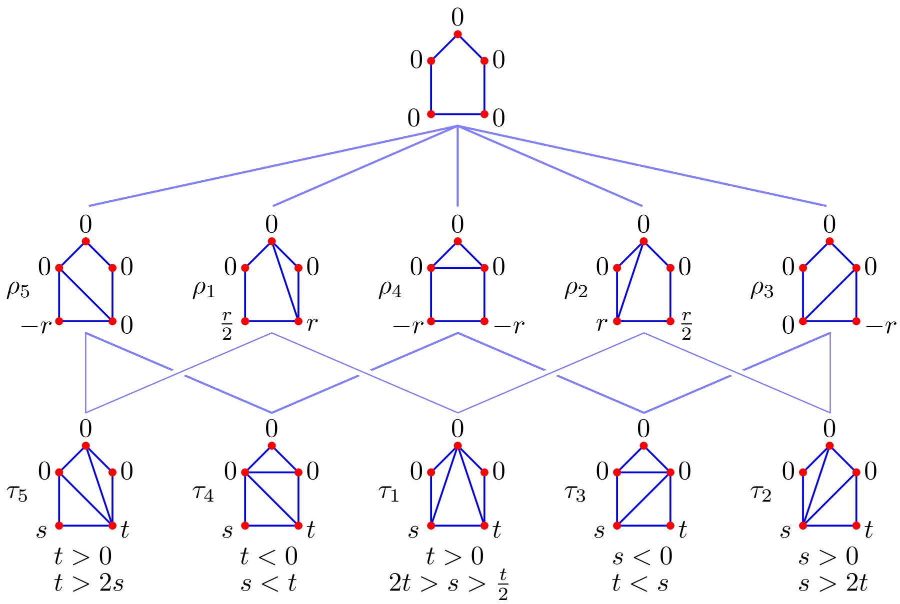


FIGURE 2. Poset of regular subdivisions of $\mathcal{A}$.

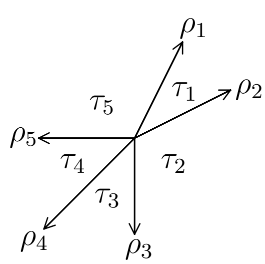


FIGURE 3. Secondary fan of $\mathcal{A}$.

2. Irrational Toric Varieties

Let $\mathcal{A} \subset \mathbb{R}^d$ be a finite set of vectors. We do not assume that $\mathcal{A}$ affinely spans $\mathbb{R}^d$. The *$\mathcal{A}$-simplex* $\triangle^{\mathcal{A}} \subset \mathbb{R}^{\mathcal{A}}_{\geq}$ is the convex hull of the basis vectors $\{e_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\}$,

$$\triangle^{\mathcal{A}} \;:=\; \{z \in \mathbb{R}^{\mathcal{A}}_{\geq} \mid \sum_{\mathbf{a}\in\mathcal{A}} z_{\mathbf{a}} = 1\}\,.$$

It is convenient to represent points of $\triangle^{\mathcal{A}}$ using homogeneous coordinates $[z_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}]$ where each $z_{\mathbf{a}} \geq 0$, not all coordinates are equal to zero, and we have

$$[z_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}] \;=\; [\gamma z_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}] \qquad \text{for all } \gamma \in \mathbb{R}_{>}\,.$$

The real torus $\mathbb{R}^{\mathcal{A}}_{>}$ acts on $\triangle^{\mathcal{A}}$ where for $w \in \mathbb{R}^{\mathcal{A}}_{>}$ and $z \in \triangle^{\mathcal{A}}$, we have

$$w.z \;:=\; [w_{\mathbf{a}} \cdot z_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}]\,.$$

When $\mathcal{F} \subset \mathcal{A}$, the simplex $\triangle^{\mathcal{F}}$ is a face of $\triangle^{\mathcal{A}}$ and all faces of $\triangle^{\mathcal{A}}$ arise in this way. The restriction map $z \mapsto z|_{\mathcal{F}}$ induces a rational map $\pi_{\mathcal{F}} \colon \triangle^{\mathcal{A}} \dashrightarrow \triangle^{\mathcal{F}}$ which is undefined on $\triangle^{\mathcal{A} \smallsetminus \mathcal{F}}$. On the remainder of $\triangle^{\mathcal{A}}$, we restrict $z \in \mathbb{R}^{\mathcal{A}}$ to $z|_{\mathcal{F}} \in \mathbb{R}^{\mathcal{F}}$ and then rescale $z|_{\mathcal{F}}$ to obtain a point in the simplex $\triangle^{\mathcal{F}}$.

The simplex $\triangle^{\mathcal{A}}$ is a compact metric space where we measure distance with the ℓ_1-metric from $\mathbb{R}^{\mathcal{A}}$. That is, if $y, z \in \triangle^{\mathcal{A}}$, then

$$d(y,z) \ := \ \sum_{\mathbf{a} \in \mathcal{A}} |y_{\mathbf{a}} - z_{\mathbf{a}}| \,.$$

Lemma 2.1. *Suppose that $\mathcal{F} \subset \mathcal{A}$ and $z \in \triangle^{\mathcal{A}} \smallsetminus \triangle^{\mathcal{A} \smallsetminus \mathcal{F}}$, so that the projection $\pi_{\mathcal{F}}(z)$ to $\triangle^{\mathcal{F}}$ is defined. Then*

$$d(z, \pi_{\mathcal{F}}(z)) \ = \ 2 \sum_{\mathbf{a} \in \mathcal{A} \smallsetminus \mathcal{F}} z_{\mathbf{a}} \,.$$

Proof. Set $y := \pi_{\mathcal{F}}(z)$, which is obtained by restricting $z \in \mathbb{R}^{\mathcal{A}}$ to $z|_{\mathcal{F}} \in \mathbb{R}^{\mathcal{F}}$ and then scaling to obtain a point in the simplex $\triangle^{\mathcal{F}}$. That is,

$$y_{\mathbf{a}} \ = \ \left\{ \begin{array}{cl} 0 & \text{if} \ \ \mathbf{a} \notin \mathcal{F} \\ \dfrac{z_{\mathbf{a}}}{\sum_{\mathbf{f} \in \mathcal{F}} z_{\mathbf{f}}} & \text{if} \ \ \mathbf{a} \in \mathcal{F} \end{array} \right. .$$

Note that if $\mathbf{f} \in \mathcal{F}$ then $y_{\mathbf{f}} \geq z_{\mathbf{f}}$. Then

$$\begin{aligned} d(y,z) \ &= \ \sum_{\mathbf{a} \in \mathcal{A} \smallsetminus \mathcal{F}} z_{\mathbf{a}} \ + \ \sum_{\mathbf{a} \in \mathcal{F}} \left(\frac{z_{\mathbf{a}}}{\sum_{\mathbf{f} \in \mathcal{F}} z_{\mathbf{f}}} - z_{\mathbf{a}} \right) \\ &= \ \sum_{\mathbf{a} \in \mathcal{A} \smallsetminus \mathcal{F}} z_{\mathbf{a}} \ + \ 1 \ - \ \sum_{\mathbf{a} \in \mathcal{F}} z_{\mathbf{a}} \ = \ 2 \sum_{\mathbf{a} \in \mathcal{A} \smallsetminus \mathcal{F}} z_{\mathbf{a}} \,, \end{aligned}$$

as $1 = \sum_{\mathbf{a} \in \mathcal{A} \smallsetminus \mathcal{F}} z_{\mathbf{a}} + \sum_{\mathbf{a} \in \mathcal{F}} z_{\mathbf{a}}$. ∎

The *Hausdorff distance* beween two closed subsets $X, Y \subset \triangle^{\mathcal{A}}$ is

$$d_H(X,Y) \ := \ \max\{\sup_{x \in X} \inf_{y \in Y} d(x,y) \,, \ \sup_{y \in Y} \inf_{x \in X} d(x,y)\} \,.$$

This endows the set of closed subsets of $\triangle^{\mathcal{A}}$ with the structure of a complete metric space, and the corresponding metric topology is the *Hausdorff topology*. If we have a sequence $\{X_i \mid i \in \mathbb{N}\}$ of subsets of $\triangle^{\mathcal{A}}$ and a subset X, then

$$\lim_{i \to \infty} X_i \ = \ X$$

if and only if X contains all accumulation points of the sequence $\{X_i \mid i \in \mathbb{N}\}$, and each point of X is a limit point of the sequence.

2.1. **Irrational Toric Varieties.** For $x \in \mathbb{R}_>$ and $a \in \mathbb{R}$, set $x^a := \exp(a\log(x))$. For $x \in \mathbb{R}^d_>$ and $\mathbf{a} \in \mathbb{R}^d$, we have the monomial $x^{\mathbf{a}} := x_1^{\mathbf{a}_1} \cdots x_d^{\mathbf{a}_d}$. The points $\mathcal{A} \subset \mathbb{R}^d$ define a map

$$\varphi_{\mathcal{A}} \;:\; \mathbb{R}^d_> \;\longrightarrow\; \triangle^{\mathcal{A}} \qquad \text{where} \qquad \varphi_{\mathcal{A}}(x) \;=\; [x^{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}]\,. \tag{7}$$

The *irrational toric variety* $X_{\mathcal{A}}$ is the closure of the image $X^\circ_{\mathcal{A}}$ of $\varphi_{\mathcal{A}}$ in $\triangle^{\mathcal{A}}$.

The convex hull $\Delta_{\mathcal{A}}$ of $\mathcal{A}$ is the image of $\triangle^{\mathcal{A}}$ under the map $\mathrm{taut}_{\mathcal{A}} \colon \mathbb{R}^{\mathcal{A}} \to \mathbb{R}^d$ defined by

$$\mathrm{taut}_{\mathcal{A}} \;:\; (z_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}) \;\longmapsto\; \sum_{\mathbf{a}\in\mathcal{A}} z_{\mathbf{a}}\mathbf{a}\,.$$

The following theorem of Birch from algebraic statistics [1, p. 168] identifies $X_{\mathcal{A}}$ with $\Delta_{\mathcal{A}}$.

Theorem 2.2. *The restriction of* $\mathrm{taut}_{\mathcal{A}}$ *to* $X_{\mathcal{A}}$ *is a homeomorphism* $\mathrm{taut}_{\mathcal{A}} \colon X_{\mathcal{A}} \to \Delta_{\mathcal{A}}$.

In particular this shows that the toric variety $X_{\mathcal{A}}$ has dimension equal to the dimension of the convex hull of $\mathcal{A}$. We call this restriction of $\mathrm{taut}_{\mathcal{A}}$ to $X_{\mathcal{A}}$ the *algebraic moment map*.

Homogeneous equations for $X_{\mathcal{A}}$ were described in [3, Prop. B.3] as follows. For every affine relation among the points of $\mathcal{A}$ with nonnegative coefficients

$$\sum_{\mathbf{a}\in\mathcal{A}} \alpha_{\mathbf{a}}\mathbf{a} \;=\; \sum_{\mathbf{a}\in\mathcal{A}} \beta_{\mathbf{a}}\mathbf{a} \qquad \text{where} \qquad \sum_{\mathbf{a}\in\mathcal{A}} \alpha_{\mathbf{a}} \;=\; \sum_{\mathbf{a}\in\mathcal{A}} \beta_{\mathbf{a}}\,, \tag{8}$$

with $\alpha_{\mathbf{a}}, \beta_{\mathbf{a}} \in \mathbb{R}_>$, we have the valid equation for points $z \in X_{\mathcal{A}}$,

$$\prod_{\mathbf{a}\in\mathcal{A}} z_{\mathbf{a}}^{\alpha_{\mathbf{a}}} \;=\; \prod_{\mathbf{a}\in\mathcal{A}} z_{\mathbf{a}}^{\beta_{\mathbf{a}}}\,. \tag{9}$$

Conversely, if $z \in \triangle^{\mathcal{A}}$ satisfies equation (9) for every affine relation (8), then $z \in X_{\mathcal{A}}$.

Given a point $w = (w_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}) \in \mathbb{R}^{\mathcal{A}}_>$ of the real torus, we have the translated toric variety $X_{\mathcal{A},w} := w.X_{\mathcal{A}}$, which is the closure of $X^\circ_{\mathcal{A},w} := w.X^\circ_{\mathcal{A}}$. Birch's Theorem still holds for $X_{\mathcal{A},w}$; it is mapped homeomorphically to $\Delta_{\mathcal{A}}$ by the algebraic moment map $\mathrm{taut}_{\mathcal{A}}$. We have the following description of the equations for $X_{\mathcal{A},w}$.

Proposition 2.3. *A point* $z \in \triangle^{\mathcal{A}}$ *lies in* $X_{\mathcal{A},w}$ *if and only if*

$$\prod_{\mathbf{a}\in\mathcal{A}} z_{\mathbf{a}}^{\alpha_{\mathbf{a}}} \cdot \prod_{\mathbf{a}\in\mathcal{A}} w_{\mathbf{a}}^{\beta_{\mathbf{a}}} \;=\; \prod_{\mathbf{a}\in\mathcal{A}} z_{\mathbf{a}}^{\beta_{\mathbf{a}}} \cdot \prod_{\mathbf{a}\in\mathcal{A}} w_{\mathbf{a}}^{\alpha_{\mathbf{a}}}\,,$$

for every affine relation (8) *among the points of* $\mathcal{A}$. *On* $X^\circ_{\mathcal{A},w}$, *we additionally have such equations coming from affine relations* (8) *where the numbers* $\alpha_{\mathbf{a}}, \beta_{\mathbf{a}}$ *are allowed to be negative.*

While the real torus $\mathbb{R}^{\mathcal{A}}_>$ acts on $X_{\mathcal{A}}$, it does not do so freely, as the image of $\mathbb{R}^{d+1}_>$ in $\mathbb{R}^{\mathcal{A}}_>$ under the map

$$(t_0, t_1, \ldots, t_d) \;\mapsto (t_0 t^{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A})\,,$$

is the stabilizer of $X_{\mathcal{A}}$. Under the coordinatewise logarithm map $\mathrm{Log} \colon \mathbb{R}^{\mathcal{A}}_> \to \mathbb{R}^{\mathcal{A}}$, this stabilizer subgroup is mapped to the subspace $\mathrm{Aff}(\mathcal{A})$ of affine functions on $\mathcal{A}$.

Lemma 2.4. *We have $X_{\mathcal{A},w} = X_{\mathcal{A},w'}$ for $w, w' \in \mathbb{R}^{\mathcal{A}}_{>}$ if and only if* $\mathrm{Log}(w) - \mathrm{Log}(w') \in \mathrm{Aff}(\mathcal{A})$.

The toric variety $X_{\mathcal{F}} \subset \triangle^{\mathcal{F}}$ is the image of the toric variety $X_{\mathcal{A}}$ under the map $\pi_{\mathcal{F}}$. This can be seen either from the definition (7) or from these equations for $X_{\mathcal{A}}$. Likewise, if $w \in \mathbb{R}^{\mathcal{A}}_{>}$ and $w|_{\mathcal{F}}$ is its restriction to $\mathcal{F}$, then $\pi_{\mathcal{F}}(X_{\mathcal{A},w}) = X_{\mathcal{F},w|_{\mathcal{F}}}$. We often write $X_{\mathcal{F},w}$ for $X_{\mathcal{F},w|_{\mathcal{F}}}$ to simplify notation. When $\Delta_{\mathcal{F}}$ has dimension d, the map $\pi_{\mathcal{F}} \colon X^{\circ}_{\mathcal{A},w} \to X^{\circ}_{\mathcal{F},w}$ is a bijection.

A consequence of the properties of $\mathrm{taut}_{\mathcal{A}}$ is a description of the boundary of $X_{\mathcal{A},w}$. Let F be a face of the polytope $\Delta_{\mathcal{A}}$ and $\mathcal{F} = \mathcal{A} \cap F$ be the points of $\mathcal{A}$ lying on F, which is also called a *face* of $\mathcal{A}$. Then the toric variety $X_{\mathcal{F},w}$ is equal to $X_{\mathcal{A},w} \cap \triangle^{\mathcal{F}}$. The collection of toric varieties $X_{\mathcal{F},w}$ where F ranges over the faces of $\Delta_{\mathcal{A}}$ forms the boundary of $X_{\mathcal{A},w}$. We have the decomposition of $X_{\mathcal{A},w}$ into disjoint subsets,

$$X_{\mathcal{A},w} \; = \; \bigsqcup_{\mathcal{F}} X^{\circ}_{\mathcal{F},w}\,, \qquad (10)$$

where the index ranges over all faces of $\mathcal{A}$. Each set $X^{\circ}_{\mathcal{F},w}$ is an orbit of $\mathbb{R}^{\mathcal{A}}_{>}$ acting on $X_{\mathcal{A},w}$.

3. Toric degenerations of irrational toric varieties

We describe all limits of the toric variety $X_{\mathcal{A}}$ under cosets of one-parameter subgroups of $\mathbb{R}^{\mathcal{A}}_{>}$, called *toric degenerations*. Each limit is a complex of toric varieties supported on a union of faces of $\triangle^{\mathcal{A}}$ which will be the geometric realization of a regular subdivision of $\mathcal{A}$. This uses essentially the same arguments as the proof of Theorem A.1 in [5] which was for the case when $\mathcal{A} \subset \mathbb{Z}^d$.

3.1. **Complexes of toric varieties.**

Let $\mathcal{S}$ be a polyhedral subdivision of $\mathcal{A}$. The *geometric realization* $|\mathcal{S}|$ of $\mathcal{S}$ is the union

$$\bigcup_{\mathcal{F} \text{ a face of } \mathcal{S}} \triangle^{\mathcal{F}}$$

of faces of $\triangle^{\mathcal{A}}$ corresponding to faces of the subdivision $\mathcal{S}$. The following is standard, it holds for more general simplicial complexes on $\mathcal{A}$.

Proposition 3.1. *The geometric realization $|\mathcal{S}|$ of a polyhedral subdivision $\mathcal{S}$ of $\mathcal{A}$ is defined in $\triangle^{\mathcal{A}}$ by*

$$\{z_{\mathbf{a}_1} z_{\mathbf{a}_2} \cdots z_{\mathbf{a}_r} \mid \{\mathbf{a}_1, \ldots, \mathbf{a}_r\} \text{ is not a subset of a face of } \mathcal{S}\}\,,$$

and minimally defined by

$$\{z_{\mathbf{a}} z_{\mathbf{b}} \mid \{\mathbf{a}, \mathbf{b}\} \text{ is not a subset of any face of } \mathcal{S}\} \bigcup \{z_{\mathbf{c}} \mid \mathbf{c} \text{ lies in no face of } \mathcal{S}\}\,.$$

For a polyhedral subdivision $\mathcal{S}$ of $\mathcal{A}$ the corresponding union of toric varieties

$$X(\mathcal{S}) \; := \bigcup_{\mathcal{F} \text{ a face of } \mathcal{S}} X_{\mathcal{F}} \qquad (11)$$

is the *complex of toric varieties corresponding to $\mathcal{S}$*. This is the union of toric varieties $X_{\mathcal{F}}$ for $\mathcal{F}$ a facet of $\mathcal{S}$ glued together along toric subvarieties corresponding to common faces. That is, if $\mathcal{G} = \mathcal{F} \cap \mathcal{F}'$, then $\Delta_{\mathcal{G}} = \Delta_{\mathcal{F}} \cap \Delta_{\mathcal{F}'}$ is a common face and $X_{\mathcal{G}} = X_{\mathcal{F}} \cap X_{\mathcal{F}'}$.

A point $w \in \mathbb{R}^{\mathcal{A}}_{>}$ of the positive torus acts on the complex of toric varieties (11) by translation, giving the translated complex,

$$X(\mathcal{S}, w) \;:=\; w.X(\mathcal{S}) \;=\; \bigcup_{\mathcal{F} \text{ a face of } \mathcal{S}} X_{\mathcal{F},w}\,. \tag{12}$$

A consequence of (12) and the decomposition (10) of $X_{\mathcal{A}}$ into disjoint orbits is the decomposition of $X(\mathcal{S}, w)$ into disjoint orbits,

$$X(\mathcal{S}, w) \;=\; \bigsqcup_{\mathcal{F} \text{ a face of } \mathcal{S}} X^{\circ}_{\mathcal{F},w}\,.$$

The union of $X^{\circ}_{\mathcal{F},w}$ where $\mathcal{F}$ ranges over the facets of $\mathcal{S}$ forms a dense open subset of the complex $X(\mathcal{S}, w)$ of toric varieties.

3.2. **Toric degenerations.** An element $\lambda \in \mathbb{R}^{\mathcal{A}}$ defines a one-parameter subgroup $\lambda(t)$ in $\mathbb{R}^{\mathcal{A}}$: for $t \in \mathbb{R}$ set $\lambda(t)_{\mathbf{a}} := \exp(t\lambda(\mathbf{a}))$. Given $w \in \mathbb{R}^{\mathcal{A}}$, we have the coset $w_{\lambda}(t) := \lambda(t) \cdot w$ and the corresponding family of translated toric varieties $X_{\mathcal{A}, w_{\lambda}(t)} = \lambda(t).X_{\mathcal{A},w}$.

Theorem 3.2. *Let $\lambda \in \mathbb{R}^{\mathcal{A}}$. For any $w \in \mathbb{R}^{\mathcal{A}}_{>}$, the family $\lambda(t).X_{\mathcal{A},w}$ of translated toric varieties has a limit as $t \to \infty$ in the Hausdorff topology on closed subsets of $\triangle^{\mathcal{A}}$, and*

$$\lim_{t\to\infty} \lambda(t).X_{\mathcal{A},w} \;=\; X(\mathcal{S}, w)\,,$$

where $\mathcal{S}$ is the regular subdivision of $\mathcal{A}$ induced by λ.

Proof. Our proof is in three steps. We first show that any accumulation point of $\{\lambda(t).X_{\mathcal{A},w}\}_t$ as t increases must be a subset of the geometric realization $|\mathcal{S}|$ of the regular subdivision induced by λ. Then we show that for each face $\mathcal{F}$ of $\mathcal{S}$, any accumulation point of $\lambda(t).X_{\mathcal{A},w}$ as $t \to \infty$ that lies in $\triangle^{\mathcal{F}}$ lies in $X_{\mathcal{F},w}$. We complete the proof by showing that every point of $X(\mathcal{S}, w)$ is a limit point of $\lambda(t).X_{\mathcal{A},w}$ as $t \to \infty$.

Let $y \in \triangle^{\mathcal{A}}$ be a point not in $|\mathcal{S}|$. We claim that y cannot be an accumulation point of $\{\lambda(t).X_{\mathcal{A},w}\}$ as $t \to \infty$. For this, we give an $\epsilon > 0$ such that for all sufficiently large t and any $z \in \lambda(t).X_{\mathcal{A},w}$, we have $d(y, z) > \epsilon$. We first define ϵ. By Proposition 3.1, either there are $\mathbf{a}, \mathbf{b} \in \mathcal{A}$ that do not both lie in any face of $\mathcal{S}$ and $y_{\mathbf{a}}y_{\mathbf{b}} \neq 0$, or else there is a $\mathbf{c} \in \mathcal{A}$ that lies in no face of $\mathcal{S}$ and $y_{\mathbf{c}} \neq 0$. In the first case, set $\epsilon := \frac{1}{2}\min\{y_{\mathbf{a}}, y_{\mathbf{b}}\}$ and in the second case, set $\epsilon := \frac{1}{2}y_{\mathbf{c}}$.

Suppose that we are in the first case. By Lemma 1.6 there is a relation (3) expressing a point p in the interior of the segment $\overline{\mathbf{a}, \mathbf{b}}$ as a convex combination of the points in a face $\mathcal{F}$ of $\mathcal{S}$. By Proposition 2.3 this gives a valid equation for $X_{\mathcal{A},w}$,

$$z_{\mathbf{a}}^{\beta_{\mathbf{a}}} z_{\mathbf{b}}^{\beta_{\mathbf{b}}} \cdot \prod_{\mathbf{f}\in\mathcal{F}} w_{\mathbf{f}}^{\alpha_{\mathbf{f}}} \;=\; w_{\mathbf{a}}^{\beta_{\mathbf{a}}} w_{\mathbf{b}}^{\beta_{\mathbf{b}}} \cdot \prod_{\mathbf{f}\in\mathcal{F}} z_{\mathbf{f}}^{\alpha_{\mathbf{f}}}\,.$$

For $z \in \lambda(t).X_{\mathcal{A},w}$ this becomes

$$z_{\mathbf{a}}^{\beta_{\mathbf{a}}} z_{\mathbf{b}}^{\beta_{\mathbf{b}}} \cdot \exp(t \sum_{\mathbf{f}\in\mathcal{F}} \alpha_{\mathbf{f}} \lambda(\mathbf{f})) \cdot \prod_{\mathbf{f}\in\mathcal{F}} w_{\mathbf{f}}^{\alpha_{\mathbf{f}}} \;=\; \exp(t(\beta_{\mathbf{a}}\lambda(\mathbf{a}) + \beta_{\mathbf{b}}\lambda(\mathbf{b}))) \cdot w_{\mathbf{a}}^{\beta_{\mathbf{a}}} w_{\mathbf{b}}^{\beta_{\mathbf{b}}} \cdot \prod_{\mathbf{f}\in\mathcal{F}} z_{\mathbf{f}}^{\alpha_{\mathbf{f}}} \,. \tag{13}$$

By (6), the difference of the two sums in the exponentials in (13),

$$\delta \;:=\; \big(\beta_{\mathbf{a}}\lambda(\mathbf{a}) + \beta_{\mathbf{b}}\lambda(\mathbf{b})\big) \;-\; \sum_{\mathbf{f}\in\mathcal{F}} \alpha_{\mathbf{f}}\lambda(\mathbf{f})\,, \tag{14}$$

is strictly negative. Then any point $z \in \lambda(t).X_{\mathcal{A},w}$ satisfies

$$z_{\mathbf{a}}^{\beta_{\mathbf{a}}} z_{\mathbf{b}}^{\beta_{\mathbf{b}}} \;=\; \exp(t\delta) \prod_{\mathbf{f}\in\mathcal{F}} z_{\mathbf{f}}^{\alpha_{\mathbf{f}}} \cdot \frac{w_{\mathbf{a}}^{\beta_{\mathbf{a}}} w_{\mathbf{b}}^{\beta_{\mathbf{b}}}}{\prod_{\mathbf{f}\in\mathcal{F}} w_{\mathbf{f}}^{\alpha_{\mathbf{f}}}} \;< \exp(t\delta) \cdot \frac{w_{\mathbf{a}}^{\beta_{\mathbf{a}}} w_{\mathbf{b}}^{\beta_{\mathbf{b}}}}{\prod_{\mathbf{f}\in\mathcal{F}} w_{\mathbf{f}}^{\alpha_{\mathbf{f}}}}\,,$$

as each component $z_{\mathbf{f}}$ of $z \in \triangle^{\mathcal{A}}$ lies in $[0,1]$. This inequality implies that if t is sufficiently large, so that the right hand side is sufficiently small, then, since $\beta_{\mathbf{a}}$ and $\beta_{\mathbf{b}}$ are positive, at least one among $z_{\mathbf{a}}$ and $z_{\mathbf{b}}$ is less than ϵ and thus $d(y,z) > \epsilon$.

A similar argument gives the same conclusion in the second case of $y_{\mathbf{c}} \neq 0$ with $\mathbf{c}$ not in any face of $\mathcal{S}$.

For the next step of the proof, recall that if $\mathcal{F}$ is a face of $\mathcal{S}$, then

$$\pi_{\mathcal{F}}(X_{\mathcal{A},w}) \;=\; X_{\mathcal{F},w}\,.$$

In fact, for every t we have $\pi_{\mathcal{F}}(\lambda(t).X_{\mathcal{A},w}) = X_{\mathcal{F},w}$. This is because the restriction of λ to $\mathcal{F}$ is affine, by the construction of $\mathcal{F}$ as a face of the polyhedral subdivision of $\mathcal{A}$ induced by λ. Thus $\mathrm{Log}(w|_{\mathcal{F}}) - \mathrm{Log}((\lambda(t)w)|_{\mathcal{F}}) = (t\lambda)|_{\mathcal{F}} \in \mathrm{Aff}(\mathcal{F})$, and so by Lemma 2.4,

$$X_{\mathcal{F},w} \;=\; X_{\mathcal{F},w|_{\mathcal{F}}} \;=\; X_{\mathcal{F},(\lambda(t)\cdot w)|_{\mathcal{F}}} \;=\; \pi_{\mathcal{F}}(\lambda(t).X_{\mathcal{A},w})\,.$$

Thus any accumulation point of $\{\lambda(t).X_{\mathcal{A},w}\}_t$ that lies in the face $\triangle^{\mathcal{F}}$ of $|\mathcal{S}|$ lies in $X_{\mathcal{F},w}$.

We complete the proof by showing that every point of $X(\mathcal{S},w)$ is a limit point of the family of translates $\{\lambda(t).X_{\mathcal{A},w}\}_t$. Recall that we have the decomposition into disjoint sets

$$X(\mathcal{S},w) \;=\; \bigsqcup_{\mathcal{F} \text{ a face of } \mathcal{S}_\lambda} X^{\circ}_{\mathcal{F},w}\,.$$

We show that for every face $\mathcal{F}$ of $\mathcal{S}$, every point of $X^{\circ}_{\mathcal{F},w}$ is a limit point of the family $\{\lambda(t).X_{\mathcal{A},w}\}_t$.

Since $\mathcal{F}$ is a face of $\mathcal{S}$, the convex hull of the graph of $\lambda|_{\mathcal{F}}$ is an upper face of the lifted polytope P_λ. Let $(\mathbf{v},1)$ be an outward-pointing normal vector to this face of P_λ. On P_λ the dot product with $(\mathbf{v},1)$ is maximized on this face, and its restriction to the points $\{(\mathbf{a},\lambda(\mathbf{a})) \mid \mathbf{a}\in\mathcal{A}\}$ is maximized on those points from $\mathcal{F}$. That is, the function $\mathcal{A}\to\mathbb{R}$

$$\mathbf{a} \;\longmapsto\; \mathbf{v}\cdot\mathbf{a} + \lambda(\mathbf{a})$$

is maximized on $\mathcal{F}$. Let δ be this maximum value.

Consider the action $*$ of $\mathbb{R}_>$ on $\mathbb{R}^d_>$ where, for $t \in \mathbb{R}_>$ and $x \in \mathbb{R}^d_>$ we have

$$(t * x)_i \ := \ t^{v_i} x_i \,.$$

Let $z \in X^\circ_{\mathcal{F},w}$. Then $z = w|_{\mathcal{F}}.\varphi_{\mathcal{F}}(x)$ for some $x \in \mathbb{R}^d_>$, and for $\mathbf{a} \in \mathcal{A}$ and $t \in \mathbb{R}_>$ we have

$$(w.\varphi_{\mathcal{A}}(t * x))_{\mathbf{a}} \ = \ w_{\mathbf{a}} \cdot t^{\mathbf{v}\cdot\mathbf{a}} x^{\mathbf{a}} \,.$$

Under the action of $\mathbb{R}_>$ on $\mathbb{R}^{\mathcal{A}}_>$ given by $(t.z)_{\mathbf{a}} = t^{\lambda(\mathbf{a})} z_{\mathbf{a}}$, we have

$$(t.w.\varphi_{\mathcal{A}}(t * x))_{\mathbf{a}} \ = \ w_{\mathbf{a}} \cdot t^{\mathbf{v}\cdot\mathbf{a}+\lambda(\mathbf{a})} x^{\mathbf{a}} \,.$$

As points of $\triangle^{\mathcal{A}}$, we have $t.w.\varphi_{\mathcal{A}}(t * x) = t^{-\delta}(t.w.\varphi_{\mathcal{A}}(t * x))$, whose $\mathbf{a}$-th coordinate is

$$t^{-\delta}(t.w.\varphi_{\mathcal{A}}(t * x))_{\mathbf{a}} \ = \ w_{\mathbf{a}} \cdot t^{\mathbf{v}\cdot\mathbf{a}+\lambda(\mathbf{a})-\delta} x^{\mathbf{a}}$$

Since $\mathbf{v} \cdot \mathbf{a} + \lambda(\mathbf{a}) - \delta \leq 0$ with equality only when $\mathbf{a} \in \mathcal{F}$, we see that

$$\lim_{t\to\infty} t.w.\varphi_{\mathcal{A}}(t * x) \ = \ \lim_{t\to\infty} t^{-\delta}(t.w.\varphi_{\mathcal{A}}(t * x)) \ = \ w|_{\mathcal{F}}.\varphi_{\mathcal{F}}(x) \ = \ z \,,$$

which completes the proof. □

This theorem shows that when $\mathcal{S}$ is the regular subdivision induced by the function λ, the complex of toric varieties $X(\mathcal{S}, w)$ is the limit of a sequence of translates of the irrational toric variety $X_{\mathcal{A},w}$ by elements of the one-parameter subgroup $\lambda(t)$.

In [5, Th. 5.2] a weak converse of Theorem 3.2 was proved when $\mathcal{A} \subset \mathbb{Z}^d$: if a sequence of translates of $X_{\mathcal{A}}$ has a limit in the Hausdorff topology, then this limit is the complex of toric varieties $X(\mathcal{S}, w)$ for some regular subdivision $\mathcal{S}$ of $\mathcal{A}$ and $w \in \mathbb{R}^{\mathcal{A}}_>$.

We prove a stronger result. Every sequence of translates has a subsequence that converges in the Hausdorff topology to some complex $X(\mathcal{S}, w)$ of toric varieties. This implies that the set of translates of $X_{\mathcal{A}}$ is compactified by the set of toric degenerations of $X_{\mathcal{A}}$.

Theorem 3.3. *For every finite set $\mathcal{A} \subset \mathbb{R}^d$ and every sequence $\{w_i \mid i \in \mathbb{N}\}$ in the positive torus $\mathbb{R}^{\mathcal{A}}_>$, there is a subsequence $\{u_j \mid j \in \mathbb{N}\} \subset \{w_i \mid i \in \mathbb{N}\}$, a regular subdivision $\mathcal{S}$ of $\mathcal{A}$, and an element $w \in \mathbb{R}^{\mathcal{A}}_>$ such that*

$$\lim_{j\to\infty} u_j.X_{\mathcal{A}} \ = \ X(\mathcal{S}, w)$$

in the Hausdorff topology on subsets of the simplex $\triangle^{\mathcal{A}}$.

A proof of Theorem 3.3 is given in Section 4 and it follows the proof of Theorem 3.2. Given a sequence of elements $\{w_i \mid i \in \mathbb{N}\} \subset \mathbb{R}^{\mathcal{A}}_>$, in Subsection 4.1 we construct a subsequence $\{u_i \mid i \in \mathbb{N}\}$, a regular subdivision $\mathcal{S}$ of $\mathcal{A}$, and a point $w \in \mathbb{R}^{\mathcal{A}}_>$. This gives a complex $X(\mathcal{S}, w)$ of toric varieties that we show in Subsection 4.2 is the limit of the sequence $\{X_{\mathcal{A},u_i} \mid i \in \mathbb{N}\}$ of translates of $X_{\mathcal{A}}$.

4. Hausdorff limits of torus translates

Let $\{w_i \mid i \in \mathbb{N}\} \subset \mathbb{R}^{\mathcal{A}}_{>}$ be a sequence of elements of the positive torus. Consider the corresponding sequence of logarithms, $v_i := \mathrm{Log}(w_i)$ for $i \in \mathbb{N}$. We show the existence of a subsequence of $\{w_i \mid i \in \mathbb{N}\}$ (equivalently of $\{v_i \mid i \in \mathbb{N}\}$) so that the corresponding sequence of torus translates $X_{\mathcal{A},w_i}$ of $X_{\mathcal{A}}$ converges in the Hausdorff topology to a complex of toric varieties $X(w, \mathcal{S})$ for some regular subdivision $\mathcal{S}$ (which we construct) of $\mathcal{A}$ and $w \in \mathbb{R}^{\mathcal{A}}_{>}$ (which we also identify). For this, we will freely replace the sequence $\{v_i \mid i \in \mathbb{N}\}$ by subsequences throughout. Let us begin with an example.

Example 4.1. Suppose that the sequence $\{v_i \mid i \in \mathbb{N}\}$ has an accumulation point modulo $\mathrm{Aff}(\mathcal{A})$. Replacing $\{v_i \mid i \in \mathbb{N}\}$ by a subsequence, we may assume that it is convergent modulo $\mathrm{Aff}(\mathcal{A})$. Since convergent sequences are bounded, there is a bounded set $B \subset \mathbb{R}^{\mathcal{A}}$ with $\{v_i \mid i \in \mathbb{N}\} \subset \mathrm{Aff}(\mathcal{A}) + B$ and therefore a sequence $\{u_i \mid i \in \mathbb{N}\} \subset \mathrm{Aff}(\mathcal{A})$ and a convergent sequence $\{\overline{v}_i \mid i \in \mathbb{N}\} \subset B$ such that $v_i = u_i + \overline{v}_i$, for each $i \in \mathbb{N}$. Let v be the limit of the sequence $\{\overline{v}_i \mid i \in \mathbb{N}\}$.

Set $\overline{w}_i := \mathrm{Exp}(\overline{v}_i)$ and $w := \mathrm{Exp}(\overline{v})$. As $v_i - \overline{v}_i \in \mathrm{Aff}(\mathcal{A})$, we have $X_{\mathcal{A},w_i} = X_{\mathcal{A},\overline{w}_i}$. Then, as $\lim_{i\to\infty} \overline{w}_i = w$, the equations of Proposition 2.3 for $X_{\mathcal{A},w_i}$ show that

$$\lim_{i\to\infty} w_i.X_{\mathcal{A}} \;=\; \lim_{i\to\infty} \overline{w}_i.X_{\mathcal{A}} \;=\; w.X_{\mathcal{A}}\,,$$

In this case, the limit of torus translate is just another torus translate.

4.1. **The limiting set** $X(\mathcal{S}, w)$**.** We replace $\{v_i \mid i \in \mathbb{N}\}$ by a subsequence from which we determine a regular subdivision $\mathcal{S}$ of $\mathcal{A}$ and $w \in \mathbb{R}^{\mathcal{A}}$. These define a complex $X(\mathcal{S}, w)$ of toric varieties which we will show is the limit of the sequence of toric translates $X_{\mathcal{A},w_i}$ corresponding to that subsequence.

The secondary fan $\Sigma_{\mathcal{A}}$ consists of finitely many cones σ. Let $\tau \in \Sigma_{\mathcal{A}}$ be a cone which is minimal under inclusion such that $\tau \cap \{v_i \mid i \in \mathbb{N}\}$ is infinite. Thus $\{v_i \mid i \in \mathbb{N}\} \subset \tau$ and if $\sigma \subsetneq \tau$, then $\{v_i \mid i \in \mathbb{N}\} \cap \sigma$ is finite.

By Lemma 1.3, replacing $\{v_i \mid i \in \mathbb{N}\}$ by a subsequence if necessary, there is a face σ of τ for which $\{v_i \mid i \in \mathbb{N}\}$ is bounded with respect to σ, and σ is the minimal such face of τ. Let $\mathcal{S} := \mathcal{S}_\sigma$ be the regular subdivision of $\mathcal{A}$ corresponding to σ.

Since $\{v_i \mid i \in \mathbb{N}\}$ is bounded with respect to σ, there is a closed bounded set B (which we may assume lies in τ) with $\{v_i \mid i \in \mathbb{N}\} \subset \sigma + B$. This implies there are sequences $\{u_i \mid i \in \mathbb{N}\} \subset \sigma$ and $\{\overline{v}_i \mid i \in \mathbb{N}\} \subset B$ with

$$v_i \;=\; u_i + \overline{v}_i \qquad \text{for} \quad i \in \mathbb{N}\,.$$

Since B is closed and bounded, $\{\overline{v}_i \mid i \in \mathbb{N}\}$ has an accumulation point $v \in B$. We may replace $\{\overline{v}_i \mid i \in \mathbb{N}\}$ by a subsequence and assume that

$$\lim_{i\to\infty} \overline{v}_i \;=\; v\,,$$

and replace $\{u_i \mid i \in \mathbb{N}\}$ and $\{v_i \mid i \in \mathbb{N}\}$ by the corresponding subsequences. We define the vector w by $w_{\mathbf{a}} := \exp(v_{\mathbf{a}})$, and write $w = \mathrm{Exp}(v)$.

We show that $X(\mathcal{S},w)$ is the limit of the sequence of translations of the complex $X(\mathcal{S})$ of toric varieties by the sequence $\{w_i \mid i\in\mathbb{N}\}$.

Lemma 4.2. *In the Hausdorff topology we have*

$$\lim_{i\to\infty} X(\mathcal{S},w_i) \;=\; X(\mathcal{S},w)\,.$$

Proof. Set $\overline{w}_i := \mathrm{Exp}(\overline{v}_i)$. Let $\mathcal{F}$ be a face of $\mathcal{S}=\mathcal{S}_\sigma$. Then $\overline{v}_i - v_i = u_i \in \sigma$, and so the restriction of each u_i to $\mathcal{F}$ is an affine function. Thus $X_{\mathcal{F},w_i} = X_{\mathcal{F},\overline{w}_i}$, by Lemma 2.4. Since $\overline{v}_i \to v$ as $i\to\infty$, we have $\overline{w}_i \to w$ as $i\to\infty$, and thus

$$\lim_{i\to\infty} X_{\mathcal{F},w_i} \;=\; X_{\mathcal{F},w}\,. \tag{15}$$

This proves the lemma, by the definition (12), $X(\mathcal{S},w_i)$ and $X(\mathcal{S},w)$ are the union of $X_{\mathcal{F},w_i}$ and $X_{\mathcal{F},w}$ for $\mathcal{F}$ a face of $\mathcal{S}$, respectively. ☐

Example 4.3. For the point configuration $\mathcal{A} := \{(0,0),(1,0),(1,1),(\frac{1}{2},\frac{3}{2}),(0,1)\}$ in $\mathbb{R}^2$ of Example 1.7, consider the sequence $\{w_i | i\in\mathbb{N}\}\subset\mathbb{R}^{\mathcal{A}}_{>}$ where $w_i := \mathrm{Exp}(v_i)$ and

$$v_i \;:=\; \left(-i-\tfrac{1}{i}\,,\; i-1\,,\; i\,,\; -\tfrac{i}{2}\,,\; -i\right)\,.$$

An affine function on $\mathbb{R}^2$ is given by $(x,y)\mapsto a+bx+cy$, and so this sequence is equivalent modulo $\mathrm{Aff}(\mathcal{A})$ to any sequence of the form

$$\left(a_i-i-\tfrac{1}{i}\,,\; a_i+b_i+i-1\,,\; a_i+b_i+c_i+i\,,\; a_i+\tfrac{b_i}{2}+\tfrac{3c_i}{2}-\tfrac{i}{2}\,,\; a_i+c_i-i\right)\,.$$

Setting $a_i=0$, $b_i=-2i$, and $c_i=i$, we obtain the equivalent sequence

$$\widetilde{v}_i \;:=\; \left(-i-\tfrac{1}{i}\,,\; -i-1\,,\; 0\,,\; 0\,,\; 0\right)\,,$$

which lies in the plane used in Example 1.7 for representatives of $\mathbb{R}^{\mathcal{A}}$ modulo $\mathrm{Aff}(\mathcal{A})$.

In Figure 4, we show the coordinates of v_i and $\widetilde{v}_i$, together with the induced triangulation, which is the same for all $i>1$. Thus we see that each $\widetilde{v}_i$ and also v_i lies in the full-dimensional

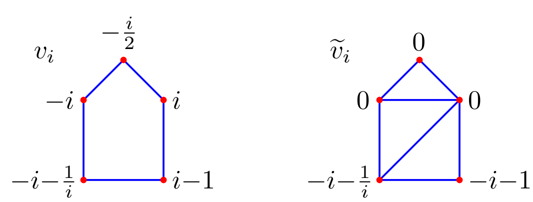


FIGURE 4. Triangulation induced by $\{v_i \mid i\in\mathbb{N}\}$.

cone τ_3 of the secondary fan $\Sigma_{\mathcal{A}}$ (see Figures 2 and 3). In the coordinates $\mathbb{R}^2$ for $\mathbb{R}^{\mathcal{A}}/\,\mathrm{Aff}(\mathcal{A})$, we have $\widetilde{v}_i = (-i-\frac{1}{i}, -i-1)$ and the rays ρ_3 and ρ_4 of τ_3 are generated by $e_3 := (0,-1)$ and $e_4 := (-1,-1)$, respectively. Writing $\widetilde{v}_i$ in this basis for $\mathbb{R}^2$ gives

$$\widetilde{v}_i \;:=\; (1-\tfrac{1}{i})e_3 \;+\; (i+\tfrac{1}{i})e_4\,.$$

Thus the images of $\{v_i\}$ in the quotients $\mathbb{R}^{\mathcal{A}}/\langle\rho_3\rangle = \mathbb{R}^2/\mathbb{R}e_3 \simeq \mathbb{R}e_4$ and $\mathbb{R}^{\mathcal{A}}/\langle\rho_4\rangle = \mathbb{R}^2/\mathbb{R}e_4 \simeq \mathbb{R}e_3$ are, respectively,

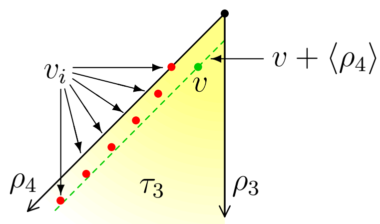
$$\{(i+\tfrac{1}{i})e_4 \mid i\in\mathbb{N}\} \qquad\text{and}\qquad \{(1-\tfrac{1}{i})e_3 \mid i\in\mathbb{N}\}\,.$$

The first is divergent while the second is bounded.

We need not replace $\{v_i\}$ by a subsequence and may take the cone σ to be ρ_4. If we set

$$u_i \;:=\; (-i-\frac{1}{i}\,,\,-i-\frac{1}{i}\,,\,0\,,\,0\,,\,0) \qquad\text{and}\qquad \overline{v}_i \;:=\; (0\,,\,-1+\frac{1}{i}\,,\,0\,,\,0\,,\,0)\,,$$

then $\widetilde{v}_i = u_i + \overline{v}_i$, where $u_i\in\rho_4$ and $\{\overline{v}_i \mid i\in\mathbb{N}\}$ is bounded in τ_3. Then $v=(0,-1,0,0,0)$ and thus $w := (1,\frac{1}{e},1,1,1)$. We display $\widetilde{v}_i$ for $i=1,\ldots,6$, $v+\langle\rho_4\rangle$, and v below.



We determine the limit of the sequence $\{X_{\mathcal{A},w_i} \mid i\in\mathbb{N}\}$. By Proposition 2.3, for $w\in\mathbb{R}^{\mathcal{A}}$, $X_{\mathcal{A},w}$ is defined by the vanishing of the five homogenous binomials,

$$\begin{gathered} w_{10}w_{01}z_{00}z_{11} - w_{00}w_{11}z_{10}z_{01}\,,\\ w_{00}w_{11}^3z_{10}^2z_{\frac12\frac32}^2 - w_{10}^2w_{\frac12\frac32}^2z_{00}z_{11}^3\,,\; w_{10}w_{01}^3z_{00}^2z_{\frac12\frac32}^2 - w_{00}^2w_{\frac12\frac32}^2z_{10}z_{01}^3\,,\\ w_{01}^2w_{11}z_{00}z_{\frac12\frac32}^2 - w_{00}w_{\frac12\frac32}^2z_{01}^2z_{11}\,,\; w_{01}w_{11}^2z_{10}z_{\frac12\frac32}^2 - w_{10}w_{\frac12\frac32}^2z_{01}z_{11}^2 \end{gathered} \tag{16}$$

Set $w_i = \mathrm{Exp}(v_i) = (e^{-i-\frac1i}, e^{i-1}, e^i, e^{-\frac12}, e^{-i})$ and consider the sequence $\{X_{\mathcal{A},w_i} \mid i\in\mathbb{N}\}$. We invite the reader to check that if $w\in\mathrm{Exp}(\mathrm{Aff}(\mathcal{A}))$, then the coefficients of the monomial terms in each binomial of (16) are equal and therefore $X_{\mathcal{A}} = X_{\mathcal{A},w}$. It follows that for each i, $X_{\mathcal{A},w_i}\simeq X_{\mathcal{A},\widetilde{w}_i}$, where

$$\widetilde{w}_i \;:=\; \mathrm{Exp}(\widetilde{v}_i) \;=\; (e^{-i-\frac1i}, e^{-i-1}, 1, 1, 1)\,.$$

Then $X_{\mathcal{A},\widetilde{w}_i}$ is defined by the binomials

$$\begin{gathered} e^{-i-1}z_{00}z_{11} - e^{-i-\frac1i}z_{10}z_{01}\,,\; e^{-i-\frac1i}z_{10}^2z_{\frac12\frac32}^2 - e^{-2i-2}z_{00}z_{11}^3\,,\; e^{-i-1}z_{00}^2z_{\frac12\frac32}^2 - e^{-2i-\frac2i}z_{10}z_{01}^3\,,\\ z_{00}z_{\frac12\frac32}^2 - e^{-i-\frac1i}z_{01}^2z_{11}\,,\; z_{10}z_{\frac12\frac32}^2 - e^{-i-1}z_{01}z_{11}^2 \end{gathered}$$

We may rewrite the first three as

$$e^{-1+\frac1i}z_{00}z_{11} - z_{10}z_{01}\,,\; z_{10}^2z_{\frac12\frac32}^2 - e^{-i-2+\frac1i}z_{00}z_{11}^3\,,\; z_{00}^2z_{\frac12\frac32}^2 - e^{-i+1-\frac2i}z_{10}z_{01}^3\,.$$

Then, if we let $i\to\infty$, these five binomials become one binomial and two monomials,

$$e^{-1}z_{00}z_{11} - z_{10}z_{01}\,,\; z_{10}^2z_{\frac12\frac32}^2\,,\; z_{00}^2z_{\frac12\frac32}^2\,.$$

The monomials define the subdivision $\mathcal{S}$ of $\mathcal{A}$ corresponding to the ray ρ_4, and the binomial defines the toric variety $X_{\mathcal{F},w}$, where $\mathcal{F}$ is the facet of the subdivision consisting of the points $\mathcal{A} \smallsetminus \{(\frac{1}{2}, \frac{3}{2})\}$. In particular, this computation implies that

$$\lim_{i\to\infty} X_{\mathcal{A},w_i} \;=\; X(\mathcal{S}, w)\,,$$

which shows the conclusion of Theorem 3.3 for this example.

4.2. **Hausdorff limits of translates.** We prove Theorem 3.3, that

$$\lim_{i\to\infty} X_{\mathcal{A},w_i} \;=\; X(\mathcal{S}, w)\,.$$

As with the proof of Theorem 3.2, we establish this limit in three steps:

(i) Any accumulation point of the sequence $\{X_{\mathcal{A},w_i} \mid i \in \mathbb{N}\}$ lies in the geometric realization $|\mathcal{S}|$ of the regular subdivision $\mathcal{S}$. (Lemma 4.4.)

(ii) For each face $\mathcal{F}$ of $\mathcal{S}$, any accumulation points of $\{X_{\mathcal{A},w_i} \mid i \in \mathbb{N}\}$ in $\triangle^{\mathcal{F}}$ lie in $X_{\mathcal{F},w}$. (Lemma 4.5.)

(iii) Every point of $X(\mathcal{S}, w)$ is a limit point of $\{X_{\mathcal{A},w_i} \mid i \in \mathbb{N}\}$. (Lemma 4.6.)

Lemma 4.4. *Let $y \in \triangle^{\mathcal{A}}$ be an accumulation point of $\{X_{\mathcal{A},w_i} \mid i \in \mathbb{N}\}$. Then $y \in |\mathcal{S}|$.*

Proof. We show that no point $y \notin |\mathcal{S}|$ of $\triangle^{\mathcal{A}}$ can be an accumulation point of $\{X_{\mathcal{A},w_i} \mid i \in \mathbb{N}\}$. Let $y \notin |\mathcal{S}|$. We will produce an $\epsilon > 0$ and an N such that if $i > N$ then $d(y, X_{\mathcal{A},w_i}) > \epsilon$.

By Proposition 3.1, either there are $\mathbf{a}, \mathbf{b} \in \mathcal{A}$ with $\{\mathbf{a}, \mathbf{b}\}$ not a subset of any face of $\mathcal{S}$ and $y_{\mathbf{a}} y_{\mathbf{b}} \neq 0$, or else there is a $\mathbf{c} \in \mathcal{A}$ that lies in no face of $\mathcal{S}$ and $y_{\mathbf{c}} \neq 0$. In the first case, set $\epsilon := \frac{1}{2}\min\{y_{\mathbf{a}}, y_{\mathbf{b}}\}$ and in the second case, set $\epsilon := \frac{1}{2} y_{\mathbf{c}}$.

Suppose that we are in the first case. Since $\{\mathbf{a}, \mathbf{b}\}$ is not a subset of any face of $\mathcal{S}_\sigma$ and σ is a face of τ, $S = S_\sigma$ is refined by S_τ, then $\{\mathbf{a}, \mathbf{b}\}$ is not a subset of any face of $\mathcal{S}_\tau$. By Lemma 1.6 there is a relation (3) expressing a point p in the interior of the segment $\overline{\mathbf{a}, \mathbf{b}}$ as a convex combination of the points in a facet $\mathcal{G}$ of $\mathcal{S}_\tau$. By Proposition 2.3 this gives the valid equation on points $z \in X_{\mathcal{A},w_i}$,

$$z_{\mathbf{a}}^{\beta_{\mathbf{a}}} z_{\mathbf{b}}^{\beta_{\mathbf{b}}} \;=\; \frac{w_i(\mathbf{a})^{\beta_{\mathbf{a}}} w_i(\mathbf{b})^{\beta_{\mathbf{b}}}}{\prod_{\mathbf{g}\in\mathcal{G}} w_i(\mathbf{g})^{\alpha_{\mathbf{g}}}} \cdot \prod_{\mathbf{g}\in\mathcal{G}} z_{\mathbf{g}}^{\alpha_{\mathbf{g}}}\,,$$

where we write $w_i(\mathbf{a})$ for $(w_i)_a$. As $0 \leq z_{\mathbf{g}} \leq 1$, $\alpha_{\mathbf{g}} \geq 0$, and $w_i = \mathrm{Exp}(v_i)$, we have

$$z_{\mathbf{a}}^{\beta_{\mathbf{a}}} z_{\mathbf{b}}^{\beta_{\mathbf{b}}} \;\leq\; \exp\bigl(\beta_{\mathbf{a}} v_i(\mathbf{a}) + \beta_{\mathbf{b}} v_i(\mathbf{b}) \;-\; \sum_{\mathbf{g}\in\mathcal{G}} \alpha_{\mathbf{g}} v_i(\mathbf{g})\bigr)\,.$$

It suffices to show that the exponential has limit 0, which is equivalent to

$$\lim_{i\to\infty} \bigl(\beta_{\mathbf{a}} v_i(\mathbf{a}) + \beta_{\mathbf{b}} v_i(\mathbf{b}) \;-\; \sum_{\mathbf{g}\in\mathcal{G}} \alpha_{\mathbf{g}} v_i(\mathbf{g})\bigr) \;=\; -\infty\,. \tag{17}$$

For then, as $0 < \beta_{\mathbf{a}}, \beta_{\mathbf{b}} < 1$, if i is large enough, then one of $z_{\mathbf{a}}$ or $z_{\mathbf{b}}$ is less than ϵ, which implies that $d(y, z) > \epsilon$ and thus $d(y, X_{\mathcal{A},w_i}) > \epsilon$.

To establish (17), consider the linear function φ defined for $\lambda \in \mathbb{R}^{\mathcal{A}}$ by

$$\varphi(\lambda) \ := \ \beta_{\mathbf{a}}\lambda(\mathbf{a}) + \beta_{\mathbf{b}}\lambda(\mathbf{b}) - \sum_{\mathbf{g}\in\mathcal{G}} \alpha_{\mathbf{g}}\lambda(\mathbf{g})\,.$$

Then the limit (17) is equivalent to

$$\lim_{i\to\infty} \varphi(v_i) \ = \ -\infty\,. \tag{18}$$

By the inequality (6), $\varphi(\lambda) < 0$ for λ in the relative interior of τ, and thus φ is nonpositive on τ. As $\mathcal{G}$ is a subset of a facet $\mathcal{F}$ of $\mathcal{S}_\sigma$, the inequality (6) shows that $\varphi(\lambda)$ is also negative for λ in the relative interior of σ.

If $\varphi(v_i)$ does not have the limit (18), then there is an M with $M < \varphi(v_i)$ for infinitely many v_i. Then M is negative, as $\varphi(v_i) < 0$ for all but finitely many $i \in \mathbb{N}$ since all but finitely many v_i lie in the relative interior of τ. Since φ is negative on the relative interior of the cone σ, there is some $v \in \sigma$ with $\varphi(v) = M$, and so there are infinitely many v_i with $\varphi(v) < \varphi(v_i)$. Such a v_i has $0 < \varphi(v_i - v)$, which implies that $v_i - v \notin \tau$. Consequently, infinitely many elements of $\{v_i - v \mid i \in \mathbb{N}\}$ do not lie in τ, which contradicts Lemma 1.4. This establishes the limit (17) and shows that there is some $N \in \mathbb{N}$ such that if $i > N$ then $d(y, X_{\mathcal{A},w_i}) > \epsilon$.

Suppose that we are in the second case of $y_{\mathbf{c}} \neq 0$ with $\mathbf{c}$ not lying in any face of $\mathcal{S}$. As σ is a face of τ, $\mathcal{S} = \mathcal{S}_\sigma$ is refined by $\mathcal{S}_\tau$ and we see that $\mathbf{c}$ does not lie in any face of $\mathcal{S}_\tau$. By Lemma 1.6 there is a relation (4) expressing $\mathbf{c}$ as a convex combination of the points of a facet $\mathcal{G}$ of $\mathcal{S}_\tau$. By Proposition 2.3 this gives the valid equation on points $z \in X_{\mathcal{A},w_i}$,

$$z_{\mathbf{c}} \ = \ \frac{w_i(\mathbf{c})}{\prod_{\mathbf{g}\in\mathcal{G}} w_i(\mathbf{g})^{\gamma_{\mathbf{g}}}} \cdot \prod_{\mathbf{g}\in\mathcal{G}} z_{\mathbf{g}}^{\gamma_{\mathbf{g}}} \qquad \text{with} \quad 0 \leq \gamma_{\mathbf{g}} \leq 1\,.$$

As $0 \leq z_{\mathbf{g}} \leq 1$ and $w_i = \mathrm{Exp}(v_i)$, this implies that

$$z_{\mathbf{c}} \ \leq \ \exp\bigl(v_i(\mathbf{c}) - \sum_{\mathbf{g}\in\mathcal{G}} \gamma_{\mathbf{g}} v_i(\mathbf{g})\bigr)\,.$$

We complete the proof by showing that

$$\lim_{i\to\infty}\bigl(v_i(\mathbf{c}) - \sum_{\mathbf{g}\in\mathcal{G}} \gamma_{\mathbf{g}} v_i(\mathbf{g})\bigr) \ = \ -\infty\,.$$

Set $\varphi(\lambda) := \lambda(\mathbf{c}) - \sum_{\mathbf{g}\in\mathcal{G}} \gamma_{\mathbf{g}}\lambda(\mathbf{g})$, for $\lambda \in \mathbb{R}^{\mathcal{A}}$. This limit becomes $\lim_{i\to\infty}\varphi(v_i) = -\infty$, which is proved by the same arguments as for the limit (18). Thus in this second case there is a number $N \in \mathbb{N}$ such that if $i > N$, then $d(y, X_{\mathcal{A},w_i}) > \epsilon$. □

Lemma 4.5. *If $y \in |\mathcal{S}|$ is an accumulation point of $\{X_{\mathcal{A},w_i} \mid i \in \mathbb{N}\}$, then $y \in X(\mathcal{S}, w)$.*

Proof. Let $y \in |\mathcal{S}|$, so that $y \in \triangle^{\mathcal{F}}$ for some face $\mathcal{F}$ of $\mathcal{S}$. If y is also an accumulation point of $\{X_{\mathcal{A},w_i} \mid i \in \mathbb{N}\}$, then for all $\epsilon > 0$ and for all $N > 0$ there is an $i > N$ and point $z \in X_{\mathcal{A},w_i}$

with $d(y,z) < \frac{1}{3}\epsilon$. Since $y_{\mathbf{a}} = 0$ for $\mathbf{a} \in \mathcal{A} \smallsetminus \mathcal{F}$, we must have

$$\sum_{\mathbf{a}\in\mathcal{A}\smallsetminus\mathcal{F}} z_{\mathbf{a}} \;<\; \frac{1}{3}\epsilon\ ,$$

and so by Lemma 2.1, $d(z, \pi_{\mathcal{F}}(z)) < \frac{2}{3}\epsilon$, which implies that $d(y, \pi_{\mathcal{F}}(z)) < \epsilon$. As $\pi_{\mathcal{F}}(z) \in \pi_{\mathcal{F}}(X_{\mathcal{A},w_i}) = X_{\mathcal{F},w_i}$, this shows that y is an accumulation point of $\{X_{\mathcal{F},w_i} \mid i \in \mathbb{N}\}$. Since we have $\lim_{i\to\infty} X_{\mathcal{F},w_i} = X_{\mathcal{F},w}$ (15), we have $y \in X_{\mathcal{F},w}$. ☐

Lemma 4.6. *Every point of $X(\mathcal{S}, w)$ is a limit point of the sequence $\{X_{\mathcal{A},w_i} \mid i \in \mathbb{N}\}$.*

Proof. We prove that every $x \in X^{\circ}_{\mathcal{F},w}$ for $\mathcal{F}$ a facet of $\mathcal{S}$ is a limit point of $\{X_{\mathcal{A},w_i} \mid i \in \mathbb{N}\}$. This suffices, as the union of these sets,

$$X^{\circ}(\mathcal{S}, w) \;:=\; \bigsqcup_{\mathcal{F}\text{ a facet of }\mathcal{S}} X^{\circ}_{\mathcal{F},w}\,.$$

is a dense subset of $X(\mathcal{S}, w)$.

Let $\mathcal{F}$ be a facet of $\mathcal{S}$. For $\delta > 0$, define

$$B_\delta \;:=\; \{y \in \triangle^{\mathcal{F}} \mid y_{\mathbf{f}} \geq \delta \text{ for } \mathbf{f} \in \mathcal{F}\}\,.$$

Let $x \in X^{\circ}_{\mathcal{F},w}$ and $\epsilon > 0$. Since $x_{\mathbf{f}} \neq 0$ for $\mathbf{f} \in \mathcal{F}$, there is a $\delta > 0$ with $x_{\mathbf{f}} \geq 2\delta$ for $\mathbf{f} \in \mathcal{F}$. By Lemma 4.7 below, there is a number N_1 such that if $i > N_1$ and $y \in B_\delta \cap X_{\mathcal{F},w_i}$, there is a point $z \in X_{\mathcal{A},w_i}$ with $d(z,y) < \epsilon$. We showed (in (15)) that

$$\lim_{i\to\infty} X_{\mathcal{F},w_i} \;=\; X_{\mathcal{F},w}\,.$$

Thus there is a number $N \geq N_1$ such that if $i > N$, there is a point $y \in X_{\mathcal{F},w_i}$ with $d(x,y) < \min\{\epsilon, \delta\}$. Since $|x_{\mathbf{f}} - y_{\mathbf{f}}| < \delta$ and $x_{\mathbf{f}} \geq 2\delta$, we have $y_{\mathbf{f}} > \delta$ for all $\mathbf{f} \in \mathcal{F}$, and thus $y \in B_\delta$. As $i > N_1$, there is a point $z \in X_{\mathcal{A},w_i}$ with $d(y,z) < \epsilon$. Therefore $d(x,z) \leq 2\epsilon$, which shows that x is a limit point of $\{X_{\mathcal{A},w_i} \mid i \in \mathbb{N}\}$. ☐

Lemma 4.7. *Let $\mathcal{F}$ be a facet of $\mathcal{S}$ and $\delta, \epsilon > 0$. Then there exists a number N such that for every $i > N$ and $y \in B_\delta \cap X_{\mathcal{F},w_i}$ the point $z \in X_{\mathcal{A},w_i}$ with $\pi_{\mathcal{F}}(z) = y$ satisfies $d(y,z) < \epsilon$.*

Proof. Let $\mathbf{d} \in \mathcal{A} \smallsetminus \mathcal{F}$. As $\mathcal{F}$ is a facet of $\mathcal{S}_\sigma$ and $\mathcal{S}_\sigma$ is refined by $\mathcal{S}_\tau$, there is a facet $\mathcal{G}$ of $\mathcal{S}_\tau$ with $\mathcal{G} \subset \mathcal{F}$. By Lemma 1.6 there is a relation (5) expressing $\mathbf{d}$ as an affine combination of points of $\mathcal{G}$, and by Proposition 2.3 this gives the valid equation on points $x \in X^{\circ}_{\mathcal{A},w_i}$,

$$x_{\mathbf{d}} \;=\; \frac{w_i(\mathbf{d})}{\prod_{\mathbf{g}\in\mathcal{G}} w_i(\mathbf{g})^{\alpha_{\mathbf{g}}}} \prod_{\mathbf{g}\in\mathcal{G}} x_{\mathbf{g}}^{\alpha_{\mathbf{g}}}\,. \tag{19}$$

For each $\mathbf{d} \in \mathcal{A} \smallsetminus \mathcal{F}$, fix one such affine expression (5) for $\mathbf{d}$ in terms of a subset $\mathcal{G}$ of $\mathcal{F}$ that is a facet in $\mathcal{S}_\tau$, together with the corresponding equation (19) on $X^{\circ}_{\mathcal{A},w_i}$.

For $y \in B_\delta \cap X_{\mathcal{F},w_i}$, we have $y \in X^{\circ}_{\mathcal{F},w_i}$, so there is a unique $z \in X^{\circ}_{\mathcal{A},w_i}$ with $\pi_{\mathcal{F}}(z) = y$. We find z by first computing the number $y_{\mathbf{d}}$ satisfying (19) (with $y_{\mathbf{g}}$ substituted for $x_{\mathbf{g}}$) for each $\mathbf{d} \in \mathcal{A} \smallsetminus \mathcal{F}$. Then the point y' whose coordinates for $\mathbf{f} \in \mathcal{F}$ equal those of y and whose other coordinates are these $y_{\mathbf{d}}$ satisfies the equations (19) for $X^{\circ}_{\mathcal{A},w_i}$, but it is not a point of the standard simplex, $\triangle^{\mathcal{A}}$ for the sum of its coordinates exceeds 1. Dividing each coordinate of y' by this sum gives the point $z \in X^{\circ}_{\mathcal{A},w_i}$ lying in the simplex $\triangle^{\mathcal{A}}$ with $\pi_{\mathcal{F}}(z) = y$.

We extract from this discussion that the coordinate $z_{\mathbf{d}}$ of z is smaller than the coordinate $y_{\mathbf{d}}$ of y' that we computed from $y \in B_\delta \cap X_{\mathcal{F},w_i}$ and (19).

We show below that for every $\epsilon > 0$ there is a number N such that if $i > N$ and $y \in B_\delta \cap X_{\mathcal{F},w_i}$, then for each $\mathbf{d} \in \mathcal{A} \smallsetminus \mathcal{F}$ the number $y_{\mathbf{d}}$ that we compute from y and (19) is at most $\frac{1}{2|\mathcal{A}\smallsetminus\mathcal{F}|}\epsilon$. Then if $z \in X_{\mathcal{A},w_i}$ is the point which projects to y, we have

$$d(y,z) \;=\; 2\sum_{\mathbf{d}\in\mathcal{A}\smallsetminus\mathcal{F}} z_{\mathbf{d}} \;<\; 2\sum_{\mathbf{d}\in\mathcal{A}\smallsetminus\mathcal{F}} y_{\mathbf{d}} \;<\; 2\sum_{\mathbf{d}\in\mathcal{A}\smallsetminus\mathcal{F}} \frac{\epsilon}{2|\mathcal{A}\smallsetminus\mathcal{F}|} \;=\; \epsilon\,,$$

which will complete the proof. (The formula for $d(y,z)$ is from Lemma 2.1).

First fix $\mathbf{d} \in \mathcal{A} \smallsetminus \mathcal{F}$. For $y \in B_\delta$, the monomial from (19),

$$\prod_{\mathbf{g}\in\mathcal{G}} y_{\mathbf{g}}^{\alpha_{\mathbf{g}}}\,,$$

is defined (as $y_{\mathbf{g}} \geq \delta$) and is thus bounded on the compact set B_δ by some number, L. Then

$$y_{\mathbf{d}} \;\leq\; \frac{w_i(\mathbf{d})}{\prod_{\mathbf{g}\in\mathcal{G}} w_i(\mathbf{g})^{\alpha_{\mathbf{g}}}}\cdot L\,,$$

if $y \in B_\delta \cap X_{\mathcal{F},w_i}$. We will show that the coefficient of L has limit zero as $i \to \infty$. Since this holds for all $\mathbf{d} \in \mathcal{A} \smallsetminus \mathcal{F}$, there is a number N such that if $i > N$, then every number $y_{\mathbf{d}}$ is bounded by $\frac{1}{2|\mathcal{A}\smallsetminus\mathcal{F}|}\epsilon$, which will complete the proof.

Taking logarithms, this limit being zero is equivalent to

$$\lim_{i\to\infty}\bigl(v_i(\mathbf{d}) \;-\; \sum_{\mathbf{g}\in\mathcal{G}} \alpha_{\mathbf{g}} v_i(\mathbf{g})\bigr) \;=\; -\infty\,.$$

Define the linear function φ on $\mathbb{R}^{\mathcal{A}}$ by

$$\varphi(\lambda) \;:=\; \lambda(\mathbf{d}) \;-\; \sum_{\mathbf{g}\in\mathcal{G}} \alpha_{\mathbf{g}}\lambda(\mathbf{g})\,,$$

where $\lambda \in \mathbb{R}^{\mathcal{A}}$. Then our limit becomes $\lim_{i\to\infty}\varphi(v_i) = -\infty$, which is proved by the same arguments as for the limit (18). $\square$

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